Project 1 : Basic Methods

1. Programming:

The aim of this exercise is to implement and test the Richardson, Jacobi, Gauß-Seidel, SOR-method and SSOR-method to solve the system

$$Ax = b \tag{1}$$

with a given starting vector $x_0 \in \mathbb{R}^n$.

a) Write for each method a function (e.g. `SOR(A,b,x0,`$k_{\max}$`,`ω`)`) that realize $k_{\max}$- number of steps of the iterative method and return a vector $x_{k_{\max}}$ that approximates the exact solution x of (1).

b) Compare the implemented methods (and choose different parameters ω for the SOR-method and SSOR-method) using different problem sizes $n = \{8, 16, 32\}$. Plot the errors of each method over the number of iteration steps (use a semi logarthmic plot `semilogy`). You may use a direct solver for calculating the errors. As test cases you should take:

- The 1D-Poisson problem on the domain $\Omega = (0, 1)$ with homogeneous boundary conditions. The discretized system is given as

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} \tag{2}$$

  and

$$b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{3}$$

  where $h = \frac{1}{n+1}$.

- Finite difference discretization of the 2D-Poisson problem $-\Delta u = f$ (using the classic 5-point stencil) on the domain $\Omega = (0, 1)^2$ with constant force $f = 1$. Use homogeneous dirichlet boundary conditions, except for the "left" boundary where $u = y(1 - y)$ on $\Gamma = \{(0, y) \subset \partial\Omega\}$.

What do you observe?

c) Given a tolerance `tol`, in order to control the error of the iterative method, you would rather stop your iteration if the error satisfies $\|x - x_k\|_2 \leq$ `tol`. As you usually don't know the exact solution x, think about a different stopping criterium that only uses computable quantities. Modify your codes accordingly and test your methods for various tolerances.

2. Theory:

a) Convergence:

- Take the matrix A from (2) for $\alpha = 2$ and verify that the eigenvalues and eigenvectors of A are given by

$$\lambda_j = \alpha - 2\cos\left(\frac{\pi}{n+1}j\right)$$
$$v_j = \left(\sin\left(\frac{\pi}{n+1}j\right), \sin\left(2\frac{\pi}{n+1}j\right), \dots, \sin\left(n\frac{\pi}{n+1}j\right)\right)^T.$$

  Will the Jacobi and Gauß-Seidel iteration converge for this matrix? If it does, what is the corresponding convergence factor? For which $\omega \in \mathbb{R}$ will the SOR method converge?

- Let $\omega \in (0, 1]$ and A be SPD. Show that the SSOR method converges (note: showing steps (a) and (b) of Exercise 5.2 might be helpful).

b) Effort:

Let $A \in \mathbb{R}^{n\times n}$ be sparse, i.e., it has at most $C_A n$ non-zero entries, where $C_A > 0$. In this exercise, we want to give bounds for the computational effort (i.e., the number of arithmetic operations needed) of one step of the Jacobi, Gauß-Seidel and SOR-method. Assume that Nb has been precomputed and therefore does not influence the effort in each step.

- Show that, for the Jacobi and Gauß-Seidel method, the effort can be bounded by

$$\text{effort(Jacobi)} \leq 2(C_A - 1)n, \qquad \text{effort(GS)} \leq 2(C_A - 1)n.$$

- What is the minimal effort for the SOR-method?