1. **a)** Implement last week's first exercise in MATLAB, with $N = 4, 8, 16, 32$. Is there any change in the error in the nodal values, i.e., $|u(x_j) - u_j|$ for $j \in \{0, \ldots, N\}$?

b) Change the right-hand side of the equation from $1 + x$ to $2x^2 + 3x - 4/3$. Find again the exact solution and run the code with the new right-hand side. Calculate the errors

$$\max_{j\in\{0,\ldots,N\}} |u(x_j) - u_j|$$

for each $N = 4, 8, 16, 32$.

$$-u'' = 1 + x \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0$$

$$-u'' = 2x^2 + 3x - 4/3$$

$$-u' = \frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{4}{3}x + C_1$$

$$-u = \frac{1}{6}x^4 + \frac{1}{2}x^3 - \frac{4}{6}x^2 + C_1 x + C_2$$

$$u(0) = 0 \to C_2 = 0$$

$$u(1) = 0 \to \frac{1}{6} + \frac{3}{6} - \frac{4}{6} + C_1 = 0$$
$$\to C_1 = 0$$

$$u = -\frac{1}{6}x^4 - \frac{1}{2}x^3 + \frac{2}{3}x^2$$

(Rest in code)

2. (**In very special cases the FEM and FDM are actually equivalent**) Consider the 1-dimensional Poisson equation with Dirichlet boundary conditions. Take an equidistant grid $x_i = (i-1)/N$. Show that the finite difference method and the finite element method (with V_h being the space of piecewise linear continuous functions as in the introductory example) yield the same approximate function.

1D Poisson: $-u'' = f$ in $(0,1)$

$$u(0) = u(1) = c$$

$$x_i = \frac{i-1}{N}$$

FD: $u'' = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1})$

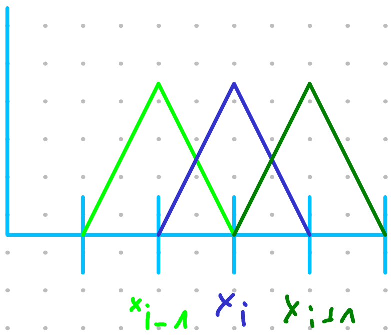
$\rightarrow -\frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) = f$

solve $Au = f$

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots \\ & \ddots & -1 & 2 & -1 \\ 0 & \ddots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ f_N \end{pmatrix}$$

FEM:

nodal basis: $\varphi_i(x_j) := \delta_{ij}$



$$\varphi_i(x) = \begin{cases} \frac{1}{h}(x - x_{i-1}) & x_{i-1} < x < x_i \\ \frac{1}{h}(x_{i+1} - x) & x_i < x < x_{i+1} \\ 0 & \text{else} \end{cases}$$

$$\varphi_i'(x) = \begin{cases} \frac{1}{h} & x_{i-1} < x < x_i \\ -\frac{1}{h} & x_i < x < x_{i+1} \\ 0 & \text{else} \end{cases}$$

stiffness matrix $A_{ij} = \int_\Omega \varphi_j' \varphi_i' \, dx = a(\varphi_j, \varphi_i)$

load vector $f_j = \int_\Omega f \varphi_j \, dx = l(\varphi_j)$

$$\left[\begin{array}{c} h = \frac{1}{N} \quad x_i := \frac{i-1}{N} \\ x_i = (i-1)h \\ x_{i-1} = (i-2)h \\ x_{i+1} = (i)h \end{array} \right]$$

$$A_{ii} = \int (\varphi_i')^2 \, dx = \int_{x_{i-1}}^{x} \frac{1}{h^2} \, dx + \int_{x}^{x_{i+1}} \frac{1}{h^2} \, dx$$

$$= \frac{x}{h^2} \Big|_{x_{i-1}=(i-2)h}^{x_i=(i-1)h} + \frac{x}{h^2} \Big|_{x_i=(i-1)h}^{x_{i+1}=ih} = \frac{h}{h^2}(i-1-i+2) + \frac{h}{h^2}(i-i+1)$$

$$\rightarrow A_{ii} = \frac{2}{h}$$

$$A_{i,i+1} = \int \varphi_i' \varphi_{i+1}' = \int_{x_{i-1}}^{x_i} \frac{1}{h} \cdot 0 \, dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{x}{h^2} \Big|_{x_i=(i-1)h}^{x_{i+1}=ih} = -\frac{h}{h^2}(i-i+1$$

$$\rightarrow A_{i,i+1} = -\frac{1}{h}$$

$$A_{i,i-1} = \int \varphi_i' \varphi_{i-1}' = \int_{x_{-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right) \cdot 0 \, dx = -\frac{x}{h^2} \Big|_{x_{i-1}=(i-2)h}^{x_i=(i-1)h} = -\frac{h}{h^2}(i-1-i+2)$$

$$\rightarrow A_{i,i-1} = -\frac{1}{h}$$

$$A_{ij} = 0, \quad j > i+1$$

$$f_j = \int_\Omega f \begin{cases} \frac{1}{h}\left(x - (i-2)h\right) & x_{i-1} < x < x_i \\ \frac{1}{h}\left(ih - x\right) & x_i < x < x_{i+1} \\ 0 & \text{else} \end{cases}$$

$$f_j = \begin{cases} f\left(\frac{x^2}{2h} - (i-2)x\right)\Big|_{x_{i-1}=(i-2)h}^{x_i=(i-1)h} \\ f\left(ix - \frac{x^2}{2h}\right)\Big|_{x_i=(i-1)h}^{x_{i+1}=ih} \\ 0 \quad \text{else} \end{cases} = \begin{cases} f\left(\frac{(i-1)^2 h^2}{2h} - (i-2)(i-1)h - \left(\frac{(i-2)^2 h^2}{2h} - (i-2)^2 h\right)\right) \\ f\left(i^2 h - \frac{i^2 h^2}{2h} - \left((i-1)hi - \frac{(i-1)^2 h^2}{2h}\right)\right) \\ 0 \end{cases}$$

$$f_j \propto h$$

Divide by h:

$$\tilde{A} u = f \Rightarrow \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & \ddots\ 0 \\ 0 & \ddots & \ddots & \ddots & \\ & \ddots & -1 \quad 2 & -1 \\ 0 & \ddots\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

3. Give a compatibility condition on the function g so that the Poisson equation with Neumann boundary conditions

$$\begin{aligned} -u'' &= f \qquad \text{in } (0,1) \\ u'(0) &= g(0) \\ u'(1) &= g(1) \end{aligned}$$

(1)

has a solution. What additional constraint can be added for the uniqueness of the solution u?

(2)

(1) $-u'' = f \;\rightarrow\; -\int_0^1 u'' v\,dx = \int_0^1 f v\,dx \qquad \left[\int u' v = uv - \int u v'\right]$

$\rightarrow\; -\int_0^1 u'' v\,dx = -u'v\Big|_0^1 + \int_0^1 u'v'\,dx = \int_0^1 f v\,dx$

$-u'(1)v(1) + u'(0)v(0) = \int_0^1 fv - u'v'\,dx$

$\left[\text{let } v = 1 \in H_1 \Rightarrow v' = 0\right]$

$u'(0) - u'(1) = \boxed{g(0) - g(1) = \int_0^1 f\,dx}$ compatibility condition

In contrast to the Dirichlet problem (where the boundary conditions are incorporated into the vector space), the boundary conditions are "hidden" in the bilinear form.
However, this formulation may not be solvable! E.g. testing with the constant function $1 \in H^1(\Omega)$ gives

$$0 = \int_0^1 f\,dx,$$

which may not hold for arbitrary functions f. Moreover, provided $\int f\,dx = 0$, solutions may not be unique as, if u is a weak solution, $u + c$ for an arbitrary constant $c \in \mathbb{R}$ is a solution as well. In order to guarantee unique solvability, additional conditions need to be imposed. A common conditions is to only look for functions u with vanishing mean, i.e., $\int_0^1 u\,dx = 0$.

(2) let $g(x) = \text{const}$

$\Rightarrow \int_0^1 f(x)\,dx = 0$

If u is a weak solution $u + c$ is too

Hence, add another condition by, e.g., constraining a point: $\boxed{u(a) = b}$

4. Derive a weak formulation for the problem

$$\begin{aligned} -u'' + u &= f \qquad \text{in } (0,1) \\ u(0) - 2u'(0) &= 0 \\ u(1) + 2u'(1) &= 0 \end{aligned}$$

with Robin boundary conditions.

$$-u'' + u = f \quad \rightarrow \quad \int_\Omega -u''v + uv \, dx = \int_\Omega fv \, dx$$

$$\left[\int u'v = uv - \int uv' \right]$$

$$-u'v\Big|_0^1 + \int_\Omega u'v' + uv \, dx = \int_\Omega fv \, dx$$

$$= \left(-u'(1)v(1) + u'(0)v(0) \right) + \int_\Omega u'v' + uv \, dx = a(u,v)$$

$$u(0) - 2u'(0) = 0 \rightarrow u'(0) = \frac{1}{2} u(0)$$

$$u(1) + 2u'(1) = 0 \rightarrow u'(1) = -\frac{1}{2} u(0)$$

$$= \left(\frac{1}{2} u(1)v(1) + \frac{1}{2} u(0)v(0) \right) + \int_\Omega u'v' + uv \, dx$$

$\rightarrow$ weak formulation

$$\boxed{a(u,v) = \int_\Omega u'v' + uv \, dx + \frac{1}{2}\left(u(1)v(1) + u(0)v(0) \right) = \int_\Omega fv \, dx = l(v)}$$

5. Let $b, c > 0$. Find the bilinear form on H_0^1 associated to the equation

$$-u'' + bu' + cu = f \qquad \text{in } (0,1)$$
$$u(0) = u(1) = 0$$

Is it symmetric? Is it coercive?

$$\int_0^1 -u''v + bu'v + cuv \, dx = \int_0^1 fv \, dx$$

$$\left[\int u'v = \int uv - \int uv' \right]$$

$0 \; [v(0) = v(1) = 0]$

$$-\cancel{u'}v\Big|_0^1 + \int u'v' + bu'v + cuv \, dx = \int_0^1 fv \, dx$$

$$\rightarrow \boxed{a(u,v) = \int u'v' + bu'v + cuv \, dx} = \int_0^1 fv \, dx = l(v)$$

bilinear form

Definition 2.2. *Let V be a vector space. A* **bilinear form** *$A(\cdot,\cdot)$ on V is a mapping $A : V \times V \to \mathbb{R}$ which is linear in u and in v.*
A bilinear form is called **symmetric** *if $A(u,v) = A(v,u)$ for all $u, v \in V$.*

$$\int u'v' + \underline{bu'v} + cuv \, dx \overset{?}{=} \int v'u' + \underline{bv'u} + cvu \, dx$$

$\rightarrow bu'v \neq bv'u \leftarrow \; \rightarrow$ not symmetric

Definition 2.5. *A bilinear form $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called* **coercive** *(or elliptic), if there is a constant $\alpha_1 \in \mathbb{R}$ such that*

$$A(u,u) \geq \alpha_1 \|u\|_V^2 \qquad \forall u \in V. \tag{2.3}$$

$A(\cdot,\cdot)$ is called **continuous***, if there is a constant $\alpha_2 \in \mathbb{R}$ such that*

$$A(u,v) \leq \alpha_2 \|u\|_V \, \|v\|_V \qquad \forall u, v \in V. \tag{2.4}$$

$$\left[\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 \right]$$

$$a(u,u) = \int_0^1 u'u' + bu'u + cuu \, dx$$

$$= \|u'\|_{L^2}^2 + c\|u\|_{L^2}^2 \boxed{+ \int_0^1 bu'u \, dx}$$

$$\left[\int_0^1 buu' \, dx = \left| \begin{array}{l} y = u(x), \; \frac{dy}{dx} = u' \\ u'(x)\,dx = dy \end{array} \right| = \int_{u(0)}^{u(1)} by \, dy = b \frac{y^2}{2} \Big|_{u(0)}^{u(1)} = \frac{b}{2}\Big(\underbrace{u(1)^2}_{0} - \underbrace{u(0)^2}_{0}\Big) = 0 \right]$$

$$a(u,u) = \|u'\|_{L^2}^2 + c\|u\|_{L^2}^2$$

$$\|u'\|_{L^2}^2 + c\|u\|_{L^2}^2 \overset{c>0}{\geq} \|u'\|_{L^2}^2$$

For coercivity, we employ the Poincaré-Friedrich inequality (denoting the constant there by C_P) to derive

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (C_P^2 + 1)\|\nabla u\|_{L^2(\Omega)}^2 = (C_P^2 + 1)A(u,u).$$

$$\left[\|u\|_{H^1}^2 \leq \alpha \|u'\|_L^2 \right]$$

$$a(u,u) \geq \|u'\|_{L^2}^2 \geq \underbrace{\frac{1}{\alpha}}_{:=\alpha} \|u\|_{H^1}$$

$$a(u,u) \geq \alpha \|u\|_{H^1}$$

6. **a)** Let $X = \{u \in C^4([0,1]) \,:\, u(0) = u(1) = u'(0) = u'(1) = 0\}$. Show that

$$A(u,v) = \int_0^1 \Delta^2(u) v \, dx$$

is a symmetric bilinear form on $X \times X$. (Here, Δ^2 denotes the operator $\Delta(\Delta u)$.)

b) Let $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Is

$$A(u,v) = u^T T v$$

a coercive bilinear form on $\mathbb{R}^2 \times \mathbb{R}^2$ (with the Euclidean norm)?

> **Definition 2.2.** *Let V be a vector space. A* **bilinear form** *$A(\cdot,\cdot)$ on V is a mapping $A : V \times V \to \mathbb{R}$ which is linear in u and in v.*
> *A bilinear form is called* **symmetric** *if $A(u,v) = A(v,u)$ for all $u, v \in V$.*

a)

1D: $\Delta u = u'' \Rightarrow \Delta^2 u = u^{(4)}$

Linearity:

1) $A(u+v, w) \overset{?}{=} A(u,w) + A(v,w)$

$$\boxed{A(u+v,w) =} \int_0^1 (u+v)^{(4)} w \, dx = \int_0^1 \left(u^{(4)} + v^{(4)}\right) w \, dx$$

$$= \int_0^1 u^{(4)} w \, dx + \int_0^1 v^{(4)} w \, dx$$

$$\boxed{= A(u,w) + A(v,w)}$$

2) $A(\alpha u, \beta v) = \alpha \beta A(u,v)$

$$\boxed{A(\alpha u, \beta v) =} \int_0^1 (\alpha u)^{(4)} \beta v = \int_0^1 \alpha u^{(4)} \beta v$$

$$= \alpha \beta \int_0^1 u^{(4)} v$$

$$\boxed{= \alpha \beta A(u,v)}$$

probably not needed to show for exercise

Symmetry: $A(u,v) \overset{?}{=} A(v,u)$

$$A(u,v) = \int_0^1 u^{(4)} v \, dx \qquad \left[\int u'v = uv - \int u v' \right]$$

$$= u^{(3)} v \Big|_0^1 - \int u^{(3)} v^{(1)} dx$$

$$= u^{(3)} v \Big|_0^1 - u^{(2)} v^{(1)} \Big|_0^1 + \int_0^1 u^{(2)} v^{(2)} dx$$

$$= u^{(3)} v \Big|_0^1 - u^{(2)} v^{(1)} \Big|_0^1 + u^{(1)} v^{(2)} \Big|_0^1 - \int_0^1 u^{(1)} v^{(3)} dx$$

$$= u^{(3)} v \Big|_0^1 - u^{(2)} v^{(1)} \Big|_0^1 + u^{(1)} v^{(2)} \Big|_0^1 - u v^{(3)} \Big|_0^1 + \int_0^1 u v^{(4)}$$

With $u(0) = u'(0) = u(1) = u'(1) = 0$

and $v(0) = v'(0) = v(1) = v'(1) = 0$

$$= \cancel{u^{(3)} v \Big|_0^1} - \cancel{u^{(2)} v^{(1)} \Big|_0^1} + \cancel{u^{(1)} v^{(2)} \Big|_0^1} - \cancel{u v^{(3)} \Big|_0^1} + \int_0^1 u v^{(4)}$$

$$= \int_0^1 u v^{(4)} dx = A(v,u)$$

$\rightarrow$ It is a symmetric bilinear form

b) Let $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Is

$$A(u, v) = u^T T v$$

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> $$A(u,v) \leq \alpha_2 \|u\|_V \, \|v\|_V \qquad \forall\, u, v \in V. \tag{2.4}$$

$$A(u,u) = (u_1 \quad u_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= (u_1 \quad u_2)\begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} = u_1 u_2 - u_2 u_1 = 0$$

$$0 \geq \alpha \underbrace{\|u\|_2^2}_{\geq 0}$$

Let $\alpha = 0$:

$0 \geq 0$ ✓ Coercive

from lecture

$\alpha \overset{!}{>} 0$

Then

$$0 \overset{?}{\geq} \underbrace{\alpha}_{>0} \underbrace{\|u\|_2^2}_{\geq 0} \quad ⚡$$

not coercive (must hold $\forall u \in V$)

Linearity:

1) $A(u+v, w) = (u_1+v_1 \quad u_2+v_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$= (u_1+v_1 \quad u_2+v_2)\begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}$$

$$= (u_1+v_1)w_2 - (u_2+v_2)w_1$$

$$= u_1 w_2 - u_2 w_1 + v_1 w_2 - v_2 w_1$$

$$= A(u,w) + A(v,w) \quad ✓$$

2) $A(\alpha u, \beta v) = \begin{pmatrix} \alpha u_1 & \alpha u_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta v_1 \\ \beta v_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha u_1 & \alpha u_2 \end{pmatrix} \begin{pmatrix} \beta v_2 \\ -\beta v_1 \end{pmatrix}$

$= \alpha u_1 \beta v_2 - \alpha u_2 \beta v_1$

$= \alpha \beta (u_1 v_2 - u_2 v_1)$

$= \alpha \beta A(u,v)$ ✓ Linear

-> Is coercive bilinear