TU Wien
Institute for Logic and Computation
Algorithms and Complexity Group

Heuristic Optimization Techniques, WS 2023

October 19, 2023

Exercise Sheet 1

These exercises have to be solved alone. Hand in a pdf-file with your answers/solutions in a well-readable and understandable form **and** mark the questions you have solved in TUWEL by **Sunday, 5$^{\text{th}}$ November 2023, 23:55**. The exercise sheet contains six tasks à two points. In the interview session, for which you also need to register in TUWEL by Sunday, 5$^{\text{th}}$ November 2023, questions related to your solutions and more generally to the topics of the respective lecture units will be asked. You may get up to eight additional points in this interview.

Q: (8P) Prepare yourself for questions about the following topics:

- Introduction
- Construction heuristics
- Beam search
- PILOT method
- Local search
- Representation and neighborhoods
- Delta evaluation
- GRASP
- Variable neighborhood search

1. (2P) Consider a problem in which solutions are represented by binary vectors $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ of length n and the *1-flip neighborhood structure* is used in a local search with the *first improvement step function.*

   Although such a local search will in practice typically terminate at a local optimum after a reasonably small number of iterations, it may in the worst case require an exponential number.

   Construct a function $f(x)$ for which this would be the case when starting the local search from the initial solution $x = (0, \ldots, 0)$. Assume each neighborhood is searched by considering the flips of the variables $x_1, \ldots, x_n$ in this order.

   Hint: Start with a fixed small n and try to assign concrete values to all possible solutions in order to maximize the number of needed iterations. Then try to generalize your construction scheme.

2. (2P) Given is the symmetric TSP on the node set $V = \{1, 2, \ldots, n\}$. Assume node 1 is the start and end point of a tour. Let $s = (\pi_1 = 1, \pi_2, \ldots, \pi_k)$ be an arbitrary partial solution in permutation notation containing $k < n$ pairwise different nodes $\pi_1 = 1$, $\pi_2 \in V$, …, $\pi_k \in V$ that are already visited in this order.

   We apply beam search with $f(\pi) = g(\pi) + h(\pi)$ (see slides), where

   - $g(\pi) = d_{1\pi_2} + d_{\pi_2\pi_3} + \ldots + d_{\pi_{k-1}\pi_k}$ and
   - $h(\pi) = 0$

   The value $f(s)$ of a partial solution s is thus the path length from the first node to the last so far visited node of π ($g(s)$) and we do not count in any estimate for the further cost-to-go ($h(s) = 0$).

   In general, we may expect that a larger beam width will lead to a better or at least equally good solution in comparison to a smaller beam width. This, however, is not always the case. Show this under the above conditions by providing an appropriate concrete TSP instance, select two suitable values for β, and solve the instance with your selected beam widths.

3. (2P) In the bin packing problem, a set of items $I = \{1, 2, \ldots, n\}$ is given. Each item $i \in I$ has a weight $z_i > 0$. Moreover, bins with a fixed capacity $C > 0$ are given. The goal is to pack all the items in as few bins as possible such that the capacity of each bin is not exceeded.

   How can a solution be represented? What local search neighborhood structures do you think make sense? Describe them in terms of the respective move operators that define them.

   Design a representation such that your neighborhoods can be efficiently searched. In particular: How can delta evaluation be realized when applying the corresponding move operators to a solution?

   Illustrate your representation, neighborhood structures, and ideas for delta evaluation by means of a small example.

4. Consider the *Quadratic Unconstrained Binary Optimization* problem (QUBO) asking for a given real symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a vector of binary decision variables $x = (x_i)_{i=1}^n \in \{0,1\}^n$, to find

$$\max_{x \in \{0,1\}^n} x^\mathsf{T} A x.$$

   (a) (2P) Design your own greedy construction heuristic which tries to come up with a solution of reasonably good quality. In a second step, provide a randomized version of your construction.

   (b) (2P) As preparation for a local search procedure: What is your solution representation (data structures etc.)? How do you efficiently evaluate the objective function and how do you update it when your current solution is subject to a 1-flip move?

   (c) (2P) Consider the following problem $\mathcal{P}$: Given a set of numbers $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{N}$, find a "minimally imbalanced" bipartition, i.e., find

$$\min_{\substack{E,F \subseteq A \\ E \cap F = \emptyset \\ E \cup F = A}} |\sum_{e \in E} e - \sum_{f \in F} f|.$$

   Show that each instance of $\mathcal{P}$ can be reduced to a suitable QUBO instance whose optimal solution can be translated to an optimal solution of $\mathcal{P}$. Furthermore, design a General Variable Neighborhood Search (GVNS) procedure with the following specifications: Provide three neighborhood structures for the shaking moves and three neighborhood structures for the Variable Neighborhood Descent (VND), where the latter ones rely on the found QUBO representation of the problem $\mathcal{P}$. For each collection of neighborhood structures state which ones contain nested/disjoint neighborhood structures.