Exam

Date: Mo., 8.2.2021, 10:00

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
max. Points	11	10	15	15	9

Don't worry! Computational effort can be reduced by cleverness - Good luck!

1. **a)** Solve the ODE (find an equation for the function $y(x)$)

$$(2x^2y + 2y)y' + (2xy^2 + \cos x) = 0.$$

b) Compute a scalar potential for the vector field

$$\psi = \begin{pmatrix} 2xy^2 + \cos x \\ 2x^2y + 2y \end{pmatrix}.$$

c) With ψ from above, compute the value of the line integrals

$$\int_{C_1} \psi \, ds \qquad \text{and} \qquad \int_{C_2} \psi \, ds,$$

where C_1 is a straight line connecting the points $(0,0)$ and $(1,1)$ and C_2 is parametrized by $r(t) = (t, t^3)$ with $t \in [0,1]$.

2. **a)** Provide the general solution to the homogeneous ODE

$$u'''(x) - 3u''(x) + 2u'(x) = 0.$$

b) Solve the inhomogeneous ODE

$$u'''(x) - 3u''(x) + 2u'(x) = e^x.$$

c) Argue whether the solution space of the homogeneous and inhomogeneous ODE are vector spaces. In case of a vector space, provide a basis and its dimension.

3. **a)** Formulate Stokes theorem.

b) Sketch the surface $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + z^2 = 1, 0 \leq y \leq 1\}$, compute the normal vector to the surface and parametrize its boundary.

c) Verify the theorem for S by computing both the surface and the line integrals for the vector field

$$\psi = \begin{pmatrix} zy \\ x^2y \\ 1 \end{pmatrix}.$$

hints: $\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt = \pi, \qquad \int_0^{2\pi} \sin t \cos t \, dt = 0$

4. **a)** Compute the Fourier series of the periodic continuation of the function

$$f(x) = -3x \qquad x \in [-1, 1].$$

Is the value of the Fourier series equal to $f(x)$ everywhere? If not, which value does it have?

b) Draw the solution of the wave equation with $c = 0.5$ and initial data

$$u(x,0) = \begin{cases} 2 & \text{if } x \in [0,1] \\ 0 & \text{else} \end{cases} \qquad\qquad u_t(x,0) = 0$$

at $t = 1$ and $t = 4$.

5. Answer the following questions (Simply write true/false on your sheet).

a) A function $f(z) = u(x,y) + iv(x,y)$ is complex differentiable, if

1. Real and imaginary part are differentiable in $\mathbb{R}^2$.
2. The limit of the complex difference quotient exists and is unique regardless of the direction taken.
3. The equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ hold and u, v are cont. differentiable.
4. The limit of the complex difference quotient exists in directions parallel to the real and imaginary axis.

b) A line integral $\oint_C \psi \cdot ds$ over a closed curve $C \subset \mathbb{R}^2$ for a continuously differentiable vector field ψ is zero,

1. always.
2. if ψ is a gradient field.
3. if, $\operatorname{curl} \psi = 0$.
4. if, $\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial x}$ and the region bounded by C is simply connected.

c) An ODE $u' = f(x,u)$

1. is always solvable.
2. has a unique solution, if f is continuous.
3. may not have a solution, but if it does, it is unique, provided f is continuous.
4. has a solution, if f is continuous, but sometimes the solution is not unique.

d) Which of the following vector identities hold? (ψ vector field, ϕ scalar function)

1. $\operatorname{div}(\phi\psi) = \phi \operatorname{div}(\psi) + \nabla\phi \cdot \psi$.
2. $\nabla \operatorname{div} \psi = \Delta\psi$.
3. $\operatorname{div}(\operatorname{curl} \psi) = 0$.