Exam 1

25.11.2022, 10:00-11:30

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4
max. Points	6	9	7	8

Good luck!

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1. Consider the sets

$$A := \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}, \qquad B := \{x \in \mathbb{R}^3 : x_1x_2x_3 = 0\}.$$

For both A and B answer the following questions: Is it a vector space? If no, why not? If yes, what is its dimension? Demonstrate by giving a basis.

A is a vector space **(1pt)**. Its elements are of the form $x = (x_1, x_2, -x_1 - x_2)^T$, and any such vector is a linear combination of the vectors $b_1 = (1, 0, -1)^T$ and $b_2 = (0, 1, -1)^T$ **(1pt)**. Since b_1 and b_2 are linearly independent **(1pt)**, they form a basis of A, and so the dimension of A is 2 **(1pt)**.

B is not a vector space **(1pt)**. For example, $v_1 = (1, 0, 0)^T, v_2 = (0, 1, 1)^T \in B$, but $v_1 + v_2 = (1, 1, 1)^T \notin B$ **(1pt)**.

2. **a)** Formulate Schwarz's theorem.

b) Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Compute the partial derivatives of f at all points $(x, y) \in \mathbb{R}^2$.

c) Compute the mixed derivatives $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ at the point $(0, 0)$. What could have gone wrong in Schwarz's theorem?

a) *Theorem*: If the second order partial derivatives of f exist and they are continuous, then

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \textbf{(2pt)}.$$

b) If $(x, y) \neq (0, 0)$, then the chain rule gives

$$\frac{\partial f}{\partial x} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}, \qquad \frac{\partial f}{\partial y} = \frac{x(x^4 - 4x^2y^2 - y^2)}{(x^2 + y^2)^2}. \qquad \textbf{(2pt)}$$

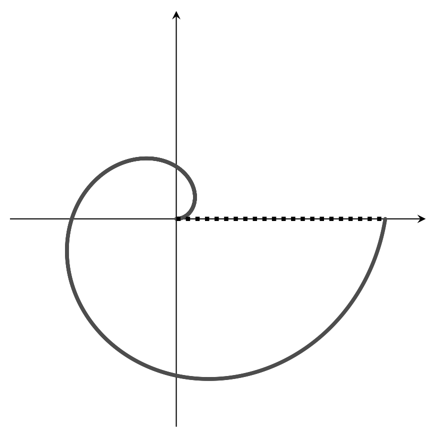
If $(x, y) = (0, 0)$, then

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0, \quad \text{and similarly} \quad \frac{\partial f}{\partial y} = 0 \qquad \textbf{(2pt)}.$$

c) From the above formula we have that $\frac{\partial f}{\partial x}(0,y) = -y$ and $\frac{\partial f}{\partial y}(x,0) = x$. Therefore $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -1$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 1$ **(2pt)**. Since the second order partial derivatives of f exists but they are not equal, the only way Schwarz's theorem can fail if they are not continuous **(1pt)**.

3. Take the so-called *Archimedes spiral* given in polar coordinates by

$$C = \{(\rho,\varphi) : \rho = \varphi,\, 0 \le \varphi \le 2\pi\}.$$



a) Calculate the area enclosed by C and the line connecting $(0,0)$ and $(2\pi,0)$.

b) Give a (one-variable) parametrisation of C.

c) Calculate the curvature of C at the origin.

a) We use the transformation formula with the usual polar transformation $\phi(\rho,\varphi) = (\rho\cos(\varphi), \rho\sin(\varphi))$, for which we have already seen many times that $|\det D\phi| = \rho$ **(1pt)**.

$$\begin{aligned}\text{Area} &= \int_S 1\, dx\, dy \\ &= \int_0^{2\pi}\int_0^{\varphi} \rho\, d\rho\, d\varphi \\ &= \int_0^{2\pi} \frac{\varphi^2}{2}\, d\varphi = \frac{8\pi^3}{6}. \qquad \textbf{(2pt)}\end{aligned}$$

Alternatively, one can use Green's theorem and compute the line integral.

b) The most natural parametrisation of C is

$$r(t) = (t\cos(t), t\sin(t)). \qquad \textbf{(1pt)}$$

c) Using r from above, the origin is $r(0)$. Differentiating gives

$$\begin{aligned} r'(t) &= (\cos(t) - t\sin(t), \sin(t) + t\cos(t)),\\ r''(t) &= (-2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t)). \qquad \textbf{(1pt)}\end{aligned}$$

In particular,

$$r'(0) = (1, 0), \qquad r''(0) = (0, 2),$$

so by the curvature formula **(1pt)**

$$\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = 2. \qquad \textbf{(1pt)}$$

4. Answer the following questions (Simply write true/false on your sheet) (**!** Wrong answers will lose points). Correct/incorrect answers are **+1/-1 pt**.

a) Which of the following vector differential identities hold? ($\psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function)

1. $\nabla \cdot (\nabla \times \psi) = 0$. True
2. $\nabla \cdot (\nabla \phi) = \Delta \phi$. True
3. $\nabla \times (\nabla \phi) = 0$. True
4. $\nabla \cdot (\phi \psi) = \phi(\nabla \cdot \psi) + (\nabla \phi) \cdot \psi$. True

b) Let $U \subset \mathbb{R}^2$ and $\psi : U \to \mathbb{R}^2$ be a continuously differentiable vector field. For a closed curve $C \subset U$ the line integral $\oint_C \psi \cdot ds$ is zero

1. if $\nabla \psi = 0$. False
2. if $\nabla \times \psi = 0$. False
3. if there exists a scalar potential for ψ. True
4. if $\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial x}$ and U is multiply connected. False