Exam

Date: Mo., 25.2.2021, 10:00

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
max. Points	12	14	13	12	9

Don't worry! Computational effort can be reduced by cleverness - Good luck!

1. **a)** Solve the ODE

$$(-x)y' + y(1+xy) = 0.$$

b) Argue whether the vector field

$$\psi = \begin{pmatrix} \frac{1}{y} + x \\ -\frac{x}{y^2} \end{pmatrix}$$

is a gradient field. If so, compute a scalar potential.

c) Consider the space $\mathbb{R}^3$. Find a vector field ψ and a non-constant scalar function ϕ such that

$$\operatorname{curl} \psi = \nabla \phi.$$

2. **a)** Provide the general real valued solution to the ODE

$$u''(x) + u(x) = 0.$$

b) Find values for $\lambda > 0$ such that the ODE

$$u''(x) + \lambda u(x) = 0 \qquad u'(0) = 0, u(1) = 0$$

with given boundary conditions has a non-trivial (i.e. non zero) solution.

c) Use separation of variables to solve the heat equation

$$\begin{aligned} T_t &= T_{xx} && \text{in } (0,1) \\ T_x(0,t) &= 0, \ \ T(1,t) = 0 \\ T(x,0) &= 1. \end{aligned}$$

3. **a)** Formulate Gauss theorem.

b) Sketch the region V covered by the surfaces $z = 4 - x^2 - y^2$ and $z = 0$. Compute the (outer) normal vector to both surfaces.

c) Verify the theorem for V by computing both the volume and surface and integrals for the vector field

$$\psi = \begin{pmatrix} y \\ -x \\ z \end{pmatrix}.$$

hint: cylindrical/polar coordinates can be helpful

4. **a)** With $\psi = \begin{pmatrix} -\cos(y) \\ 2y + x\sin(y) \end{pmatrix}$ compute the value of the line integral

$$\int_C \psi \, ds,$$

where C is the curve starting at $(0,0)$ moving along $r(t) = (t\cos(\pi t), t)$ to the point $(-1,1)$ then going along a straight line to the point $(-1,0)$ and finally going along a half circle in the upper plane to the point $(1,0)$. Draw the curve!

b) Compute the Fourier transformation of the function

$$f(x) = \begin{cases} te^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

hint: $\lim_{t\to\infty} te^{-t} = 0$.
What is the Fourier transformation of $f \star f$?

5. Answer the following questions (Simply write true/false on your sheet).

a) Which of the following vector identities hold? (ψ vector field, ϕ scalar function)

1. $\nabla \operatorname{div} \psi = \operatorname{curl} \operatorname{curl} \psi$.
2. $\operatorname{div}(\nabla \psi) = \Delta \psi$.
3. $\operatorname{curl}(\phi\psi) = \phi \operatorname{curl}(\psi) + \nabla\phi \times \psi$.

b) An orthonormal set in a vector space

1. is always linearly independent.
2. is always a basis of the vector space.
3. can be constructed from a basis of the vector space.
4. always contains vectors of length 1.

c) The Fourier series for a function f

1. is a continuous function.
2. is always finite at every point.
3. is the same as the value of f everywhere for periodic functions.
4. gives the mean value at jump points of f.

d) Analytic functions

1. are infinitely many times differentiable.
2. can have finitely many singularities.
3. are everywhere complex differentiable.
4. can have a Laurent series with non zero principal part.