Sheet 9

Discussion of the sheet: Tue., 23.05.2023

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1. (Prager-Synge theorem) Let $\Omega \subset \mathbb{R}^d$. Consider the solution $u \in H^1_0(\Omega)$ of the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let σ be a vector field satisfying $\operatorname{div} \sigma = -f$. Show that for any $v \in H^1_0(\Omega)$ there holds

$$(\nabla(u-v), \nabla u - \sigma)_{L^2(\Omega)} = 0$$

and consequently

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} + \|\nabla u - \sigma\|^2_{L^2(\Omega)} = \|\nabla v - \sigma\|^2_{L^2(\Omega)}\,.$$

Note that the idea stems from a rewriting of the Poisson problem as $\nabla u = \sigma$, $\operatorname{div} \sigma = -f$, where the new variable σ can be seen as the flux of u. Prager-Synge theorem can then be used to define an error estimator by choosing $v = u_h$ and constructing σ as a FEM approximation of the flux (i.e. post-process ∇u_h to σ_h so that $\nabla \cdot \sigma_h = f$).

2. Go to gitlab.tuwien.ac.at/asc/praetorius/mooafem and download the MATLAB package MooAFEM (an object orientated adaptive FEM code). Run 'setup' in your Matlab to install it.

Go to the examples folder and run the codes 'adaptiveFEM.m' and 'convergenceRates.m' and explain: what is the PDE? on which domain? what kind of FEM space is used? how are the adaptive steps (error estimator, marking, refinement) implemented? where does most of the refinement take place? what do we see from the convergence rates?

3. Consider the advection equation. Determine the numerical domain of dependence for the forward time/backward space method. Argue that the CFL-condition is $\frac{k}{h} \leq 1$.

4. Write a program to solve the heat equation

$$\begin{aligned} \partial_t u - \partial_{xx} u &= f \qquad \text{in } \ (0,1) \times (0,T) \\ u(0,t) = u(1,t) &= 0 \qquad \forall t \in (0,T) \\ u(x,0) &= u_0(x) \end{aligned}$$

by using a semi-discretization by classical FEM in space and discretize the resulting ODE with the implicit Euler method. Take piecewise linear Lagrangian finite elements of order 1 on an uniform mesh of mesh.width h in space and take uniform time-steps of length k.

Test your program with the exact solution $u(x,t) = e^{-t}x(1-x)$ at $T = 1$ using $k = ch$ with different constants $c > 0$. For the error you can take the error measure $\sqrt{h}\sqrt{\sum_j |u(x_j,T) - u_j^T|^2}$, where you should see convergence of order h.

5. Modify your program from the previous example by using the explicit Euler method instead of the implicit Euler method. Take two choices of time steps k:

$$k = \frac{2.01}{\lambda_{\max}}, \qquad k = \frac{1.99}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of $M^{-1}A$ (can e.g. be computed in Matlab using `eigs`).

Plot your computed solutions, the error versus h, and $\lambda_{\max}$ versus h.

1. (Prager-Synge theorem) Let $\Omega \subset \mathbb{R}^d$. Consider the solution $u \in H_0^1(\Omega)$ of the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let σ be a vector field satisfying $\operatorname{div}\sigma = -f$. Show that for any $v \in H_0^1(\Omega)$ there holds

$$(\nabla(u-v), \nabla u - \sigma)_{L^2(\Omega)} = 0$$

and consequently

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} + \|\nabla u - \sigma\|^2_{L^2(\Omega)} = \|\nabla v - \sigma\|^2_{L^2(\Omega)}.$$

Note that the idea stems from a rewriting of the Poisson problem as $\nabla u = \sigma$, $\operatorname{div}\sigma = -f$, where the new variable σ can be seen as the flux of u. Prager-Synge theorem can then be used to define an error estimator by choosing $v = u_h$ and constructing σ as a FEM approximation of the flux (i.e. post-process ∇u_h to σ_h so that $\nabla \cdot \sigma_h = f$).

$$(\nabla(u-v), \nabla u - \sigma)_{L^2(\Omega)} = \int_\Omega \nabla(u-v)(\nabla u - \sigma)\,dx \quad [\int f'g = fg - \int fg']$$

$$= \underbrace{(u-v)(\nabla u - \sigma)}_{=0}\Big|_{\partial\Omega} - \int_\Omega (u-v)(\Delta u - \nabla\cdot\sigma)\,dx \quad \Big| \begin{array}{l} \Delta u = -f \\ -\nabla\cdot\sigma = +f \end{array}$$

$$= -\int_\Omega (u-v)(-f+f)\,dx = 0$$

and same goes for $(\nabla(v-u), \nabla u - \sigma)_{L^2(\Omega)}$

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} + \|\nabla u - \sigma\|^2_{L^2(\Omega)} =$$

check in $2\int \nabla(v-u)(\nabla u - \sigma)\,dx = 0$

$\|\nabla(v-u)\|^2_{L^2(\Omega)} = |-1|\,\|\nabla(v-u)\|^2_{L^2(\Omega)}$

$$= \int_\Omega \underbrace{(\nabla(v-u))^2}_{a} + \underbrace{2\nabla(v-u)(\nabla u - \sigma)}_{=0} + \underbrace{(\nabla u - \sigma)^2}_{b}\,dx \quad \Big| (a+b)^2 = a^2 + 2ab + b^2$$

$$= \int_\Omega (\nabla(v-u) + (\nabla u - \sigma))^2\,dx$$

$$= \|\nabla(v-u) + \nabla u - \sigma\|^2_{L^2(\Omega)} = \|\nabla v - \nabla u + \nabla u - \sigma\|^2_{L^2(\Omega)}$$

$$= \|\nabla v - \sigma\|^2_{L^2(\Omega)}$$

2. Go to gitlab.tuwien.ac.at/asc/praetorius/mooafem and download the MATLAB package MooAFEM (an object orientated adaptive FEM code). Run 'setup' in your Matlab to install it.

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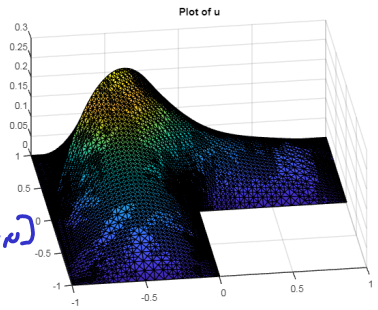
(1) We use 1. order Lagrangian elements

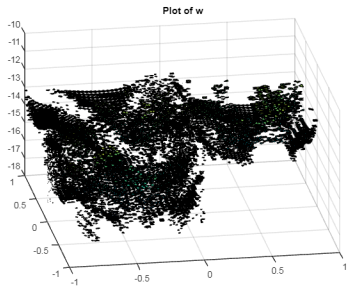


$\Omega = [-1,1] \times [0,1] + [-1,0] \times [-1,0]$

Bilinear form

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot a\,\varphi_j + b\,\nabla\varphi_j \cdot \varphi_i + c\,\varphi_j\,\varphi_i \,dx + \int_{\Gamma_R} r\,\varphi_j\,\varphi_i \,ds$$

(Robin)



$a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $c = x^2$, $r = 0$

Linear form

$$f_i = \int_\Omega f\,\varphi_i + f^{vec}\,\nabla\varphi_i \,dx + \int_{\Gamma_N} n\,\varphi_i \,ds + \int_{\Gamma_R} r\,\varphi_i \,ds$$

(Neumann, Robin)

$f = 1$, $n = 0$, $r = 0$, $f^{vec} =$ (complex function @(w) [w, w])

$$-\operatorname{div} A\nabla u + b\nabla u + c\cdot u = f - f^{vec}, \quad \Omega$$
$$u = 0, \quad \Gamma$$

$$\eta_T^2 = h_T^2 \cdot \| -\nabla a \cdot \nabla u + b \cdot \nabla u + c \cdot u - f + \operatorname{div}(f^{vec}) \|_{L^2(T)}^2 + h_T \| a \cdot \nabla u - f^{vec} \|_{L^2(\partial T)}^2$$

We use Dörfler marking and Newest Vertex Bisection

(2) We use high order Lagrangian elements

$\square$: $\Omega = [-1, 1] \times [0, 1] + [-1, 0] \times [-1, 0]$

Bilinear form

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot a\varphi_j + b\nabla\varphi_j \cdot \varphi_i + c\varphi_j\varphi_i \, dx + \int_{\Gamma_R} r\varphi_j\varphi_i \, ds$$

(Γ_R: robin)

$a = (1)$

Linear form

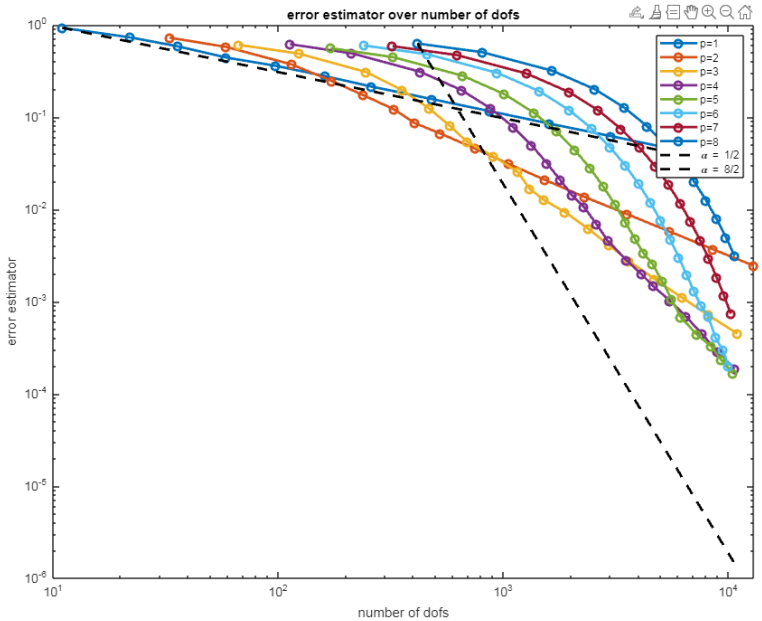
$$f_i = \int_\Omega f\varphi_i + f^{vec}\nabla\varphi_i \, dx + \int_{\Gamma_N} (\text{neumann})\varphi_i \, ds + \int_{\Gamma_R} (\text{robin})\varphi_i \, ds$$

$f = 0$, $f^{vec} = 0$, $r = 0$, h > Mesh Function (mesh, exact solution, ...)

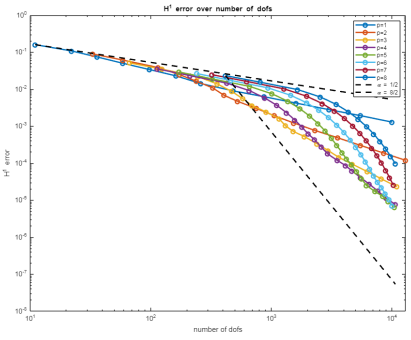
$\rightarrow -\Delta u = 1$

$\eta^2 = \left(\sum \eta_T^2\right)^{\frac{1}{2}}$

again Dörfler marking and NVB



From the convergence rates we see a $h^{-\frac{p}{2}}$ convergence for elements of order p



3. Consider the advection equation. Determine the numerical domain of dependence for the forward time/backward space method. Argue that the CFL-condition is $\frac{k}{h} \leq 1$.

Forward time / backward space:

$j = x$

$n = t$

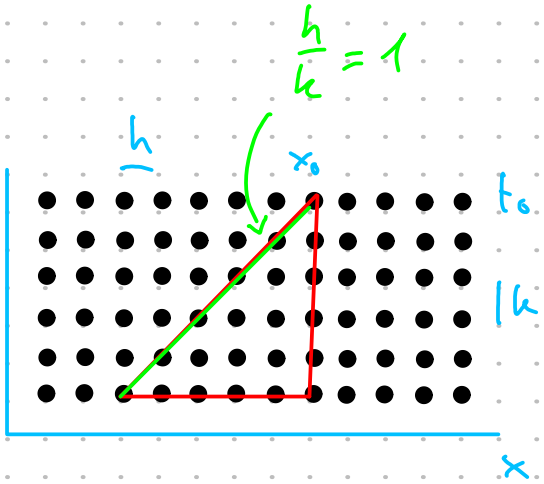
$$\frac{v_j^{n+1} - v_j^n}{k} + \frac{v_j^n - v_{j-1}^n}{h} = f(x_j, t_n)$$

We take $f = 0$

$$\frac{v_j^{n+1} - v_j^n}{k} = -\frac{v_j^n - v_{j-1}^n}{h}$$

$$\rightarrow v_j^{n+1} = -\frac{k}{h}\left(v_j^n - v_{j-1}^n\right) + v_j^n$$

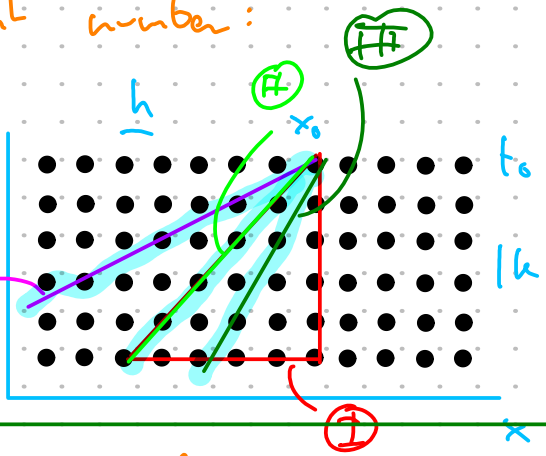
$$= \boxed{\left(1 - \frac{k}{h}\right) v_j^n} + \boxed{\frac{k}{h} v_{j-1}^n}$$



Dependency of the approximation v_h at x_0, t_0 on $v_h(x_0, t_0 - k)$ and $v_h(x_0 - h, t_0 - k)$ - gives us **numerical domain of dependance.**

For the domain of dependance, we need (look at the Courant number:

If $\frac{h}{k}$ was > 1:



(II) $\Rightarrow 1 = \frac{h}{k}$

(III) $h \Rightarrow \frac{1}{i} = \frac{h}{k}$

(IV) $h \Rightarrow 2 = \frac{h}{k}$

We see that (IV) $\notin$ (I) (hence we'd get unstable solutions)

whereas (II), (III) $\subseteq$ (I) $\rightarrow \frac{h}{k} \leq 1$