1. (Hermite finite elements) The lowest order Hermite finite element in 1D is given by (T, V_T, Ψ_T), where $T = (0, 1)$, $V_T = P^3(T)$ (polynomials of degree ≤ 3) and the functionals in Ψ_T as

$$\psi_1 : v \mapsto v(0), \quad \psi_2 : v \mapsto v'(0), \quad \psi_{\cancel{2}\,3} : v \mapsto v(1), \quad \psi_{\cancel{2}\,4} : v \mapsto v'(1)$$

Provide a nodal basis for this finite element! Consider a finite element complex based on this reference element. What is the maximal regularity you can expect?

Nodal basis $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ found by $\psi_i(\varphi_j) = \delta_{ij}$

$P^3(x) = ax^3 + bx^2 + cx + d$

$$\rightarrow \psi_1(\varphi_i) = \varphi_i(0) = \begin{cases} 1, & i=1 \\ 0, & \text{else} \end{cases} \quad \mathrm{I}$$

$$\rightarrow \psi_2(\varphi_i) = (\varphi_i)'(0) = \begin{cases} 1, & i=2 \\ 0, & \text{else} \end{cases} \quad \mathrm{II}$$

$$\rightarrow \psi_3(\varphi_i) = \varphi_i(1) = \begin{cases} 1, & i=3 \\ 0, & \text{else} \end{cases} \quad \mathrm{III}$$

$$\rightarrow \psi_4(\varphi_i) = (\varphi_i)'(1) = \begin{cases} 1, & i=4 \\ 0, & \text{else} \end{cases} \quad \mathrm{IV}$$

$$\varphi_1: \mathrm{I} \ (ax^3 + bx^2 + cx + d)_{x=0} = 1$$

$$\mathrm{II} \ (3ax^2 + 2bx + c)_{x=0} = 0$$

$$\mathrm{III} \ (ax^3 + bx^2 + cx + d)_{x=1} = 0$$

$$\mathrm{IV} \ (3ax^2 + 2bx + c)_{x=1} = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$a=2,\ b=-3,\ c=0,\ d=1$

$$\boxed{\rightarrow \varphi_1 = 1-3x^2+2x^3}$$

φ_2:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$a=1,\ b=-2,\ c=1,\ d=0$

$$\boxed{\underline{\varphi_2} = x-2x^2+x^3}$$

φ_3:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$a=-2,\ b=-1,\ c=0,\ d=0$

$$\boxed{\varphi_2 = \quad -x^2-2x^3}$$

φ_4:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$a = 1,\ b = -1,\ c = 0,\ d = 0$

$$\boxed{\varphi_2 = -x^2 + x^3}$$

Nodal basis:

$$\{1 - 3x^2 + 2x^3,\ x - 2x^2 + x^3,\ -x^2 - 2x^3,\ -x^2 + x^3\}$$

Definition 4.5. *The* **finite element space** *(FEM-space) is given by*

$$V_{\mathcal{T}} := \{v = I_{\mathcal{T}} w : w \in C^m(\overline{\Omega})\}$$

We say that $V_{\mathcal{T}}$ has **regularity** *r, if $V_{\mathcal{T}} \subset C^r(\overline{\Omega})$. If $V_{\mathcal{T}} \not\subseteq C(\Omega)$, the regularity is defined as -1.*

The maximum regularity is 1, since the values of the derivatives at the nodes are set

2. **a)** Find the nodal basis for the 1D finite element defined by

- $T = [-1, 1]$;
- $V_T = P^k(T)$ for fixed $k \in \mathbb{N}$;
- $\psi_T = \{\psi_j : v \mapsto \int_T v(x)P_j(x)\, dx,\ j = 0, \dots, k\}$.

Here, P_j denote the Legendre Polynomials.

b) Consider the quadrilateral Lagrangian finite element

- $T = [0, 1]^2$;
- $V_T = Q^1(T)$;
- ψ_T are point evaluation functionals at the vertices of T.

Provide a nodal basis for this finite element.

Legendre-Polynomials fulfill $(P_n, P_m)_{L^2(-1,1)} = 0 \quad \forall n \neq m$

$$\boxed{\begin{aligned}&\int_{-1}^{1} P_m P_n = \frac{2}{2n+1}\delta_{mn} = (P_m, P_n)_{L^2(-1,1)} \\ &\|P_i\|_{L^2}^2 = \frac{2}{2i+1}\end{aligned}}$$

Must hold $\psi_j(\varphi_i) = \delta_{ij}$

Try Basis elements: $\varphi_i = \dfrac{P_i}{\|P_i\|_{L^2}^2} \longrightarrow$

$$\longrightarrow \psi_j(\varphi_i) \overset{!}{=} \delta_{ij} \overset{?}{=} (\varphi_i, P_j)_{L^2}$$

$$= \int_{-1}^{1} \varphi_i P_j \, dx$$

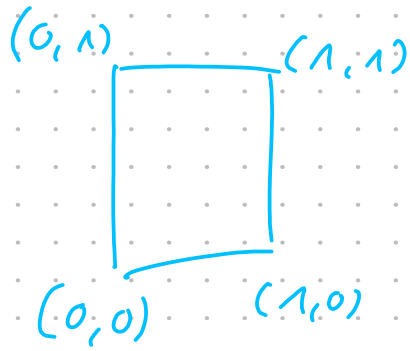
$$= \int_{-1}^{1} \frac{1}{\|P_i\|_{L^2}} P_i P_j \, dx = \cancel{\frac{2i+1}{2}} \cancel{\frac{2}{2i+1}} \delta_{ij} \checkmark$$

b) Consider the quadrilateral Lagrangian finite element

- $T = [0,1]^2$;
- $V_T = Q^1(T)$; $Q^p := \mathrm{span}\{x^i y^j : 0 \le i \le p,\ 0 \le j \le p\}$ if T is a quadrilateral.
- ψ_T are point evaluation functionals at the vertices of T.

Provide a nodal basis for this finite element.

$$Q^1(T) = \{x, y, xy\}$$



$\psi_1(\varphi_i) = \varphi_i(0,0)$

$\psi_2(\varphi_i) = \varphi_i(1,0)$

$\psi_3(\varphi_i) = \varphi_i(1,1)$

$\psi_4(\varphi_i) = \varphi_i(0,1)$

$$\boxed{\varphi_i = a_i xy + b_i x + c_i y + d}$$

$\psi_1(\varphi_1) = d_1 \overset{!}{=} 1$, $\psi_2(\varphi_1) = b_1 + 1 \overset{!}{=} 0$, $\psi_3(\varphi_1) = a_1 + b_1 + c_1 + 1 \overset{!}{=} 0$

$\psi_4(\varphi_1) = c_1 + 1 \overset{!}{=} 0$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right) \to \begin{array}{l} a = 1 \\ b = -1 \\ c = -1 \\ d = 1 \end{array}$$

$$\boxed{\varphi_1 = xy - x - y + 1}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \to \begin{array}{l} a = -1 \\ b = 1 \\ c = 0 \\ d = 0 \end{array}$$

$$\boxed{\varphi_2 = x - xy}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \to \begin{array}{l} a = 1 \\ b = 0 \\ c = 0 \\ d = 0 \end{array}$$

$$\boxed{\varphi_3 = xy}$$

$$
\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)
$$

$$
\to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \to \begin{array}{l} a = -1 \\ b = 0 \\ c = 1 \\ d = 0 \end{array}
$$

$$\boxed{\varphi_1 = y - xy}$$

Nodal basis:

$$\{xy - x - y + 1,\ x - xy,\ y - xy,\ xy\}$$

3. Let T be a triangle with vertices $(t_1, s_1), (t_2, s_2), (t_3, s_3)$ and K be a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Show that the matrix

$$\mathbf{M} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

maps the triangle T to the triangle K in the sense that $\mathbf{M} \cdot \begin{pmatrix} t_i \\ s_i \\ 1 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}$.

Provide an affine transformation between two rectangles. Can an affine transformation between two quadrilaterals always be found?

$$\begin{pmatrix} t_i \\ s_i \\ 1 \end{pmatrix} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

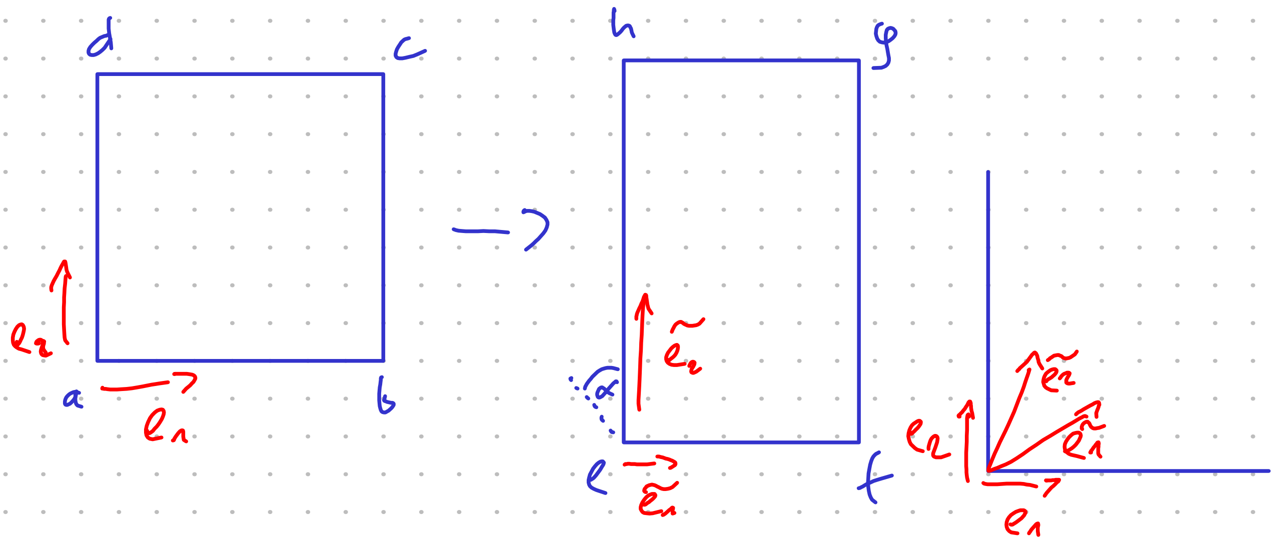
$$\rightarrow M \cdot \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{pmatrix}}_{\mathbb{1}} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$=$$

Provide an affine transformation between two rectangles. Can an affine transformation between two quadrilaterals always be found?

> **Definition 4.2.** *Two finite elements* (T, V_T, Ψ_T) *and* $(\widehat{T}, V_{\widehat{T}}, \Psi_{\widehat{T}})$ *are called* **equivalent**, *if there exists an invertible function* F *such that*
>
> - $T = F(\widehat{T})$
> - $V_T = \{\hat{v} \circ F^{-1} : \hat{v} \in V_{\widehat{T}}\}$
> - $\Psi_T = \{\psi_i^T : V_T \to \mathbb{R} : v \to \psi_i^{\hat{T}}(v \circ F)\}$
>
> *Two elements are called affine equivalent, if* F *is an affine-linear function, i.e.,* $F = A \cdot x + b$.



$$T = T_\alpha \cdot T_s$$ rotate and scale

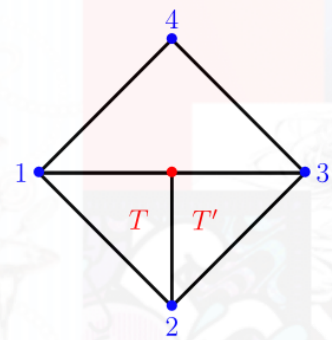
$$T_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$T_s = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\boxed{T = \begin{pmatrix} a\cos\alpha & -b\sin\alpha \\ a\sin\alpha & b\cos\alpha \end{pmatrix}}$$

A affine transformation for general quadrilaterals can only be found by chance (i.e. the 4th point lining up randomly) [ not possible]

4. We now consider a non-regular mesh which contains a so called hanging node. The easiest triangulation $\mathcal{T}$ containing such a node is depicted in the following:



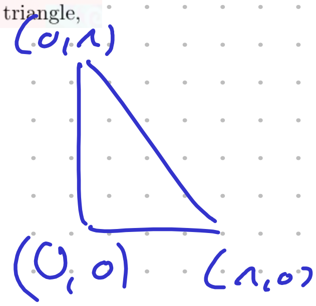
We now consider Lagrangian finite elements of order 1 and take the FEM space $V_{\mathcal{T}} = \mathcal{P}^1(\mathcal{T})$ (i.e. the space piecewise affine linear functions).

a) Argue that, if the nodal values of a function in $V_{\mathcal{T}}$ at the blue dots are known, then the value at the red dot is already determined. (Therefore, the red dot does not induce a global basis function!) Consequently, $V_{\mathcal{T}}$ has dimension 4.

b) Compute the connectivity matrices for the elements T and T'.

$P^p := \operatorname{span}\{x^i y^j : 0 \le i, 0 \le j, i + j \le p\}$ if T is a triangle,

$P^1 := \{x, y\}$



$\varphi_i = ax + by + c$

$\psi_1(\varphi_i) = \varphi_i(0,0)$

$\psi_2(\varphi_i) = \varphi_i(1,0)$

$\psi_3(\varphi_i) = \varphi_i(0,1)$

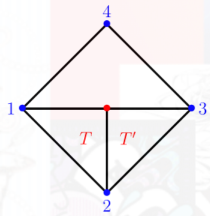
$\varphi_1: \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \varphi_1 = 1 - x - y$

$\varphi_2: \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \varphi_2 = x$

φ_3: analogous: $\varphi_3 = y$

The nodal basis has dim 3, so red point is determined

4. We now consider a non-regular mesh which contains a so called hanging node. The easiest triangulation $\mathcal{T}$ containing such a node is depicted in the following:



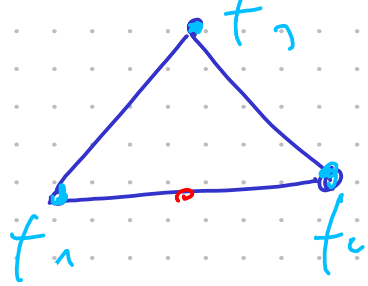
We now consider Lagrangian finite elements of order 1 and take the FEM space $V_{\mathcal{T}} = \mathcal{P}^1(\mathcal{T})$ (i.e. the space piecewise affine linear functions).

a) Argue that, if the nodal values of a function in $V_{\mathcal{T}}$ at the blue dots are known, then the value at the red dot is already determined. (Therefore, the red dot does not induce a global basis function!) Consequently, $V_{\mathcal{T}}$ has dimension 4.

b) Compute the connectivity matrices for the elements T and T'.

The value of an arbitrary point $f(x,y)$ is

$f(x,y) = ax + by + c$



The value of 3 blue points is fixed:

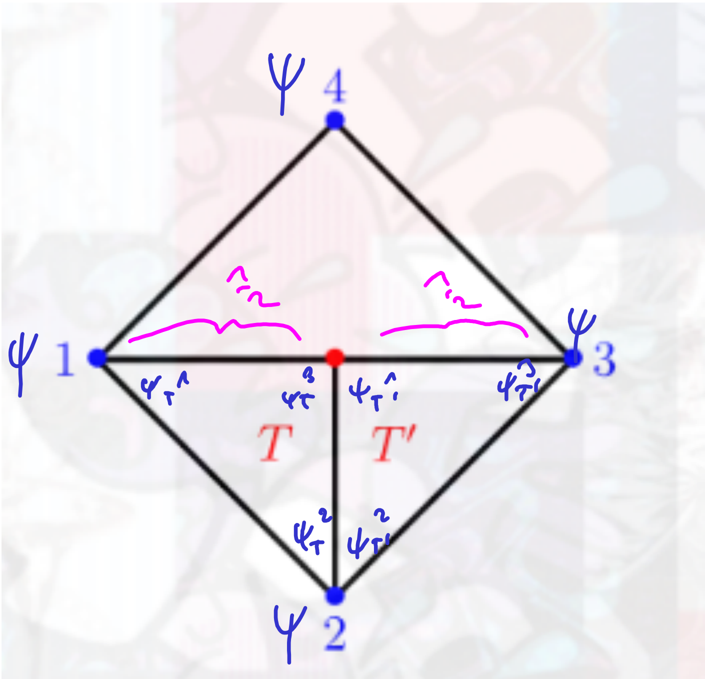
$$a x_1 + b y_1 + c = f_1$$
$$a x_2 + b y_2 + c = f_2$$
$$a x_3 + b y_3 + c = f_3$$

This system can be solved for, a, b, c,
After that the value at the red dot can be evaluated in

$f(x,y) = ax + by + c$

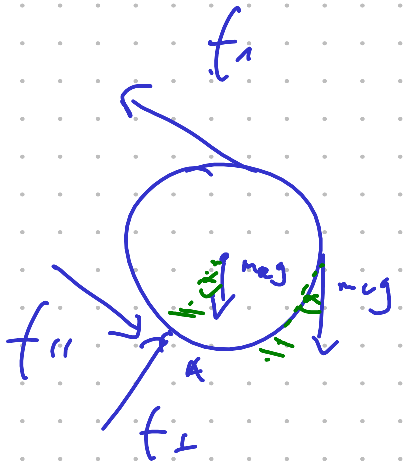
by plugging in x & y

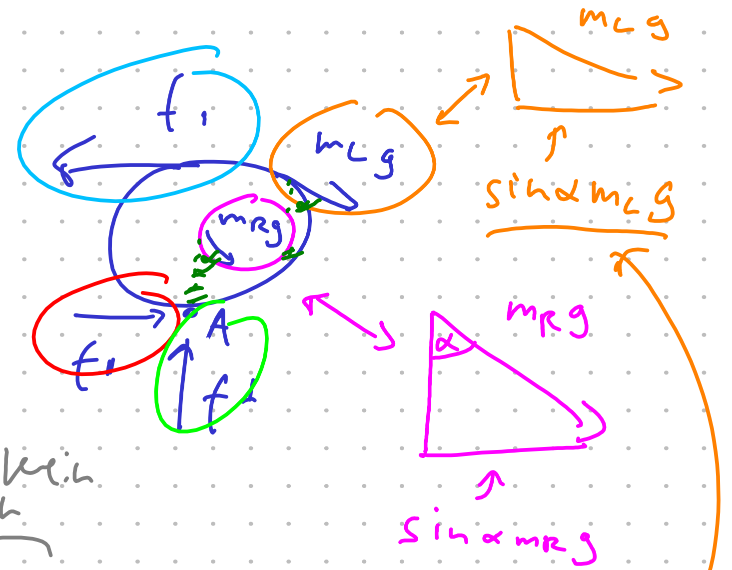
b) Compute the connectivity matrices for the elements T and T'.



$$T: \begin{pmatrix} \varphi_T^1 \\ \varphi_T^2 \\ \varphi_T^3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}}_{C_T} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

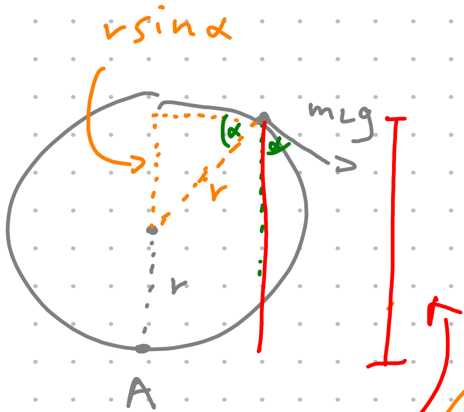
$$T': \begin{pmatrix} \varphi_{T'}^1 \\ \varphi_{T'}^2 \\ \varphi_{T'}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{C_{T'}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$





$$M_A^{\curvearrowleft} = 2rf_1 - \underbrace{0 \pm 0}_{\text{bewirken kein Moment}} - r\sin\alpha\, m_R g$$

$$-(r + r\sin\alpha)\, m_L G \sin\alpha$$



$$r + r\sin\alpha \longrightarrow M_L^{\curvearrowleft} = -(r + r\sin\alpha)\, m_L g \sin\alpha$$