Sheet 1

Discussion of the sheet: Tue., 14.03.2023

This exercise sheet is concerned with the topics

- 1D finite differences
- 1D Sobolev spaces
- weak formulations

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1. Write down the 1D finite difference scheme to the Poisson equation

$$\begin{aligned} -u'' &= 1 + x \qquad\qquad \text{in } (0,1) \\ u(0) &= u(1) = 0 \end{aligned}$$

on an equidistant grid of mesh-width h=$1/N$. Solve this ODE exactly, draw the solution and the finite difference approximation for $N = 4$ grid points. What is the error in the nodal values, i.e., $|u(x_j) - u_j|$ for $j \in \{0, \dots, N\}$?

2. (Consistency error of 1D-FD) Assume that $u \in C^4([0,1])$ (i.e. 4-times continuously differentiable). Show that for h sufficiently small and a constant $C > 0$, there holds

$$\left| \frac{1}{h^2} \left(u(x+h) - 2u(x) + u(x-h) \right) - u''(x) \right| \leq Ch^2.$$

What is the error if we use the one-sided approximation twice, i.e.

$$u''(x) \approx \frac{u'(x+h) - u'(x)}{h} \approx \frac{\frac{u(x+2h)-u(x+h)}{h} - \frac{u(x+h)-u(x)}{h}}{h} = \frac{u(x+2h) - 2u(x+h) - u(x)}{h^2}?$$

hint: Use Taylor expansion.

3. Show that the space $H^1(0,1)$ is a vector space. Moreover, show that

$$\|u\|^2_{H^1(0,1)} := \|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)}$$

is a norm on $H^1(0,1)$ and

$$(u,v)_{H^1} := (u',v')_{L^2} + (u,v)_{L^2} = \int_0^1 u'v' + uv \, dx$$

is an inner product on $H^1(0,1)$.

4. Show that a weak solution to the 1D Poisson equation

$$\begin{aligned} -u'' &= f && \text{in } (0,1) \\ u &= 0 && \text{on } \{0,1\} \end{aligned}$$

is also a classical (strong) solution, if additionally $u \in C^2([0,1])$.

5. For each of the following classes of functions, find a PDE that is satisfied by u for *all* choices of functions f, g (i.e. f, g should not appear in the PDE)

a) $u(x,y) = f(x) + g(y)$,

b) $u(x,y) = f(x+y)$,

c) $u(x,y) = f(x^2 - y^2)$.