Sheet 8

Discussion of the sheet: Tue., 16.05.2023

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1. (1D-FEM with graded mesh) Consider

$$\begin{aligned} -u'' &= f \qquad\qquad \text{in } (0,1) \\ u(0) &= u(1) = 0, \end{aligned}$$

where f is such that the exact solution of the equation is given by $u(x) = x^{2/3}(1-x)$.

Use your 1D-FEM code to approximate this solution and plot the error on equidistant grids of different mesh-widths h in the energy norm.

Modify your code such that the grid points are given by

$$x_i = \left(\frac{i}{N}\right)^{\beta}, \qquad i = 0, \dots, N$$

for some $\beta > 0$. Try different values of β. What is the best convergence behavior you can achieve?

2. It might seem that the previous exercise falls outside of our usual theory: show that $f \notin L^2(0,1)$. However, in the abstract formulation we need less: we only need that

$$v \mapsto \ell(v) := \int_0^1 v(x) f(x)\, dx$$

is a continuous linear operator on $H_0^1(0,1)$. Show that this property does hold.

3. (Shift theorem, positive example). The aim of this exercise is to "show" a shift theorem on a square. Let $\Omega = (0,\pi)^2$ consider the Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions.

a) Take a *finite* Fourier-series: for some $N < \infty$, take a function u_N to be of the form $u_N := \sum_{k,\ell=1}^{N} a_{k,\ell} \sin(kx)\sin(\ell y)$. Argue that u_N fulfills the boundary conditions. Moreover, compute the partial derivatives of second order and show

$$|u_N|^2_{H^2(\Omega)} \le C \sum_{k,\ell=1}^{N} (k^4 + \ell^4) a_{k,\ell}^2.$$

Hint: you can use Parseval's identity: for any Fourier series $v_N = \sum_{k,\ell=1}^{N} b_{k,\ell} \sin(kx)\sin(\ell y)$, there holds $\|v_N\|^2_{L^2(\Omega)} = \frac{\pi^2}{4} \sum_{k,\ell=1}^{N} b_{k,\ell}^2$ (and the same holds when $\sin(\cdot)$ is replaced by $\cos(\cdot)$).

b) Do the same to show

$$\sum_{k,\ell=1}^{N} (k^4 + \ell^4) a_{k,\ell}^2 \leq C' \left\| -\Delta u_N \right\|_{L^2(\Omega)}^2 .$$

c) Combine the previous steps to obtain $|u_N|_{H^2(\Omega)} \leq C'' \left\| \Delta u_N \right\|_{L^2(\Omega)} = C'' \left\| f_N \right\|_{L^2(\Omega)}$, where $-\Delta u_N = f_N$.

One can finish the proof of the shift theorem for this particular Ω by a standard density argument: we know that there is a sequence of functions with Fourier series converging to u, that is, $u_N \to u$ in H^2, which shows estimate for u.

4. (Shift theorem, negative example) Let Ω be given in polar coordinates as the sector of the unit disk

$$\Omega = \{(r\cos\varphi, r\sin\varphi) \ : \ 0 < r < 1, 0 < \varphi < \frac{3}{2}\pi\}.$$

Verify that Ω is not convex. Show that the function (in polar coordinates)

$$u = r^{2/3} \sin(2\varphi/3)$$

satisfies $\Delta u \in L^2(\Omega)$, but $\partial_{xx} u \notin L^2(\Omega)$, and therefore $u \notin H^2(\Omega)$.

From here it is easy to construct a function v that is even in H_0^1 that fails the shift theorem: take $v = (1 - r^2)u$.

5. Write an adaptive FEM code in 1D to solve

$$\begin{aligned} -u'' &= f \quad \text{ in } (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

Use piecewise linear Lagrangian finite elements. For the `SOLVE` step, e.g., use your FEM solver previous exercises (note that for this example your code needs to handle different element sizes). For the `ESTIMATE` step compute the residual error indicator from the lecture notes and for the `MARK` step implement Dörfler marking. The `REFINE` step in 1D is easy, as only all marked elements have to be halved.

Test your code with right-hand side such that the exact solution is $u(x) = x^{2/3}(1-x)$ (as in Example 1). Plot the error over the number of elements as well as the error indicator over the number of elements for marking parameters $\theta = 0.25$ and $\theta = 1$. Compare with the results in Example 1.

1. (1D-FEM with graded mesh) Consider

$$-u'' = f \qquad \text{in } (0,1)$$
$$u(0) = u(1) = 0,$$

where f is such that the exact solution of the equation is given by $u(x) = x^{2/3}(1-x)$.

Use your 1D-FEM code to approximate this solution and plot the error on equidistant grids of different mesh-widths h in the energy norm.

Modify your code such that the grid points are given by

$$x_i = \left(\frac{i}{N}\right)^{\beta}, \qquad i = 0, \dots, N$$

for some $\beta > 0$. Try different values of β. What is the best convergence behavior you can achieve?

$$u = x^{2/3}(1-x) = x^{2/3} - x^{5/3}$$

$$\boxed{u' = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3}}$$

$$u'' = -\frac{2}{9}x^{-4/3} - \frac{10}{9}x^{-1/3}$$

$$f = -u'' = \frac{2}{9}x^{-4/3} + \frac{10}{9}x^{-1/3}$$

$$\boxed{f = \frac{2}{9}\left(x^{-4/3} + 5x^{-1/3}\right)}$$

energy norm: $\|u - u_N\|_E$

From the past we know

$$\|u\|_E := \sqrt{a(u,u)} = \sqrt{\int_\Omega u'^2 dx} = \|u'\|_{L^2(\Omega)}$$

$$\|u_N\|_E^2 = u^T f$$

1. **(Energy norm)** Let $a(\cdot,\cdot)$ be a coercive, symmetric bilinear form on a vector space V and $l(\cdot)$ be a continuous linear form.

a) Show that $a(\cdot,\cdot)$ is an inner-product. The induced norm

$$\|u\|_E := \sqrt{a(u,u)}$$

is called the **energy norm**.

b) Now, take a finite dimensional subspace $V_N \subset V$, a basis $\{\varphi_i : i = 1, \dots, N\}$. Define the stiffness matrix and load vector

$$\mathbf{A}_{ij} = a(\varphi_j, \varphi_i) \qquad \mathbf{f}_j = l(\varphi_j).$$

Show that the energy of the FEM solution u_N satisfying $a(u_N, v_N) = l(v_N)$ for all $v_N \in V_N$ can be computed as

$$\|u_N\|_E^2 = \mathbf{u}^T\mathbf{A}\mathbf{u} = \mathbf{u}^T\mathbf{f},$$

where $\mathbf{u}$ is the solution of the linear system $\mathbf{A}\mathbf{u} = \mathbf{f}$.

c) Let u be the solution of $a(u,v) = l(v)$ for all $v \in V$. Use Galerkin orthogonality to show

$$\|u - u_N\|_E^2 = \|u\|_E^2 - \|u_N\|_E^2.$$

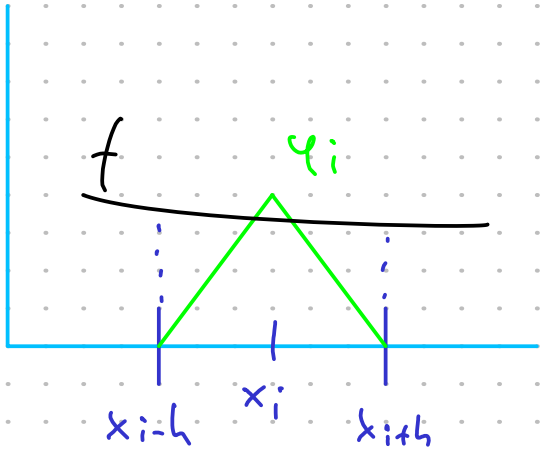
Error in the energy norm

$$\|u - u_N\|_E^2 = \|u\|_E^2 - \|u_N\|_E^2$$
$$= \|u'\|_{L^2}^2 - u^T f$$

$$\|u'\|_{L^2}^2 = \int_0^1 \left(\frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3}\right)^2 dx$$

load vector f $\quad f := l(\varphi_j) = \int_0^1 f \cdot \varphi_j \, dx$

$$f = \begin{pmatrix} \int_0^1 f \varphi_2 \, dx \\ \int_0^1 f \varphi_3 \, dx \\ \vdots \\ \int_0^1 f \varphi_N \, dx \end{pmatrix}$$

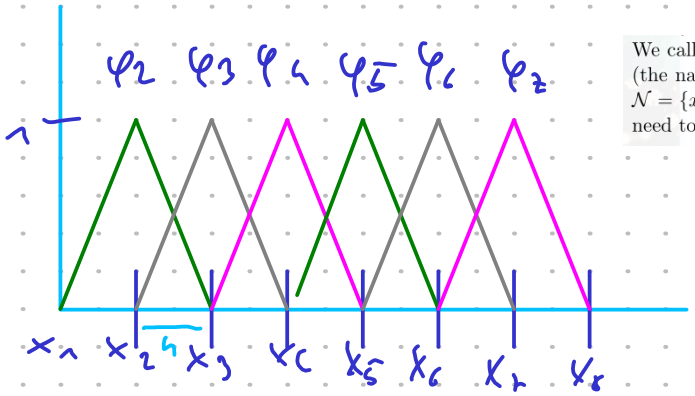


for all $j = 2, \dots, N$. We define the so called **stiffness matrix** $\mathbf{A} = (A_{ij}) \in \mathbb{R}^{N-1 \times N-1}$ and the **load vector** $\mathbf{f} = (f_j) \in \mathbb{R}^{N-1}$ as

$$\begin{aligned} A_{ij} &:= \int_\Omega \varphi_j' \cdot \varphi_i' \, dx = a(\varphi_j, \varphi_i), \\ f_j &:= \int_\Omega f \varphi_j \, dx = l(\varphi_j). \end{aligned}$$

Stiffness matrix $A := a(\varphi_j, \varphi_i) = \int_0^1 \varphi_j' \cdot \varphi_i' \, dx$

$$a(\varphi_j, \varphi_i) = \begin{cases} 0 & , |i-j| > 1 \\ \Delta x_i \cdot \left(-\frac{1}{\Delta x_i}\right)^2 & , |i-j| = 1 \\ \left(\frac{1}{\Delta x_{i-1}}\right)^2 \cdot \Delta x_{i-1} + \left(\frac{1}{\Delta x_i}\right)^2 \cdot \Delta x_i & , i = j \end{cases}$$



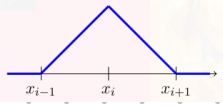
We call the subintervals T_i **elements** and its collection $\mathcal{T} := \{T_i \;\; i = 1, \dots, N\}$ a **triangulation** (the name is motivated from higher space dimensions) or **mesh**. The collection of the endpoints $\mathcal{N} = \{x_j \; : \; j = 1, \dots, N+1\}$ is called the **nodes** of the mesh. We note that the nodes x_j do not need to be equidistant and the lengths $h_i = x_{i+1} - x_i$ are called **local mesh-widths**.

As the space of linear polynomials is finite dimensional with dimension 2, we obtain that the space V_h is finite dimensional as well. In fact, a function $v_h \in V_h$ is uniquely defined by its values $v(x_j)$ in the nodes $x_j \in \mathcal{N}$. Consequently, one can obtain a so called **nodal basis** $\{\varphi_i \; : \; i = 1, \dots, N+1\}$ of V_h by functions characterized by

$$\varphi_i(x_j) = \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} . \tag{1.7}$$

These functions are called **hat functions** and can be written as

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}}, & x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h_i}, & x \in (x_i, x_{i+1}) \\ 0, & \text{otherwise.} \end{cases}$$



→ equidistant grid

$$a(\varphi_j, \varphi_i) = \begin{cases} 0 & , |i-j| > 1 \\ 1/h & , |i-j| = 1 \\ 2/h & , i = j \end{cases}$$

$$\hookrightarrow A = \begin{pmatrix} 2/h & -1/h & & \\ -1/h & 2/h & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

2. It might seem that the previous exercise falls outside of our usual theory: show that $f \notin L^2(0,1)$. However, in the abstract formulation we need less: we only need that

$$v \mapsto \ell(v) := \int_0^1 v(x) f(x)\, dx$$

is a continuous linear operator on $H_0^1(0,1)$. Show that this property does hold.

$$f = \frac{2}{9}\left(x^{-\frac{4}{3}} + 5x^{-\frac{1}{3}}\right)$$

$$\|f\|_{L^2}^2 = \int_0^1 \left(\frac{2}{9}\left(x^{-\frac{4}{3}} + 5x^{-\frac{1}{3}}\right)\right)^2 dx$$

$$\frac{4}{81}\int_0^1 x^{-\frac{8}{3}} + 10x^{-\frac{5}{3}} + 25x^{-\frac{2}{3}}\, dx$$

$$= \frac{4}{81}\left(-\frac{3}{5}x^{-\frac{5}{3}} - 15x^{-\frac{2}{3}} + 75x^{\frac{1}{3}}\right)_0^1$$

$\to \infty$ $\quad$ $\to \infty$

$$\left| x^{-\frac{5}{3}} \right|_0^1 = \frac{1}{1} - \frac{1}{0}$$

$$\boxed{\to \infty \;\Rightarrow\; f \notin L^2(0,1)}$$

Linearity of $\ell(v)$:

$$\ell(\alpha(v_1+v_2)) = \int_0^1 \alpha(v_1+v_2) f\, dx = \alpha\int_0^1 v_1 f\, dx + \alpha\int_0^1 v_2 f\, dx \checkmark$$

$\ell(v)$ is continuous if it is bounded

$$\ell(v) = \int_0^1 v(x) f(x)\, dx \underset{C.S.}{=} -\int_0^1 v'(x) F(x)\, dx$$

$$\leq \|v'\|_{L^2(\Omega)} \|F\|_{L^2(\Omega)}$$

$$\leq \|v\|_{H_0^1(\Omega)} \|F\|_{L^2(\Omega)}$$

with $F(x) = \int f(x)$

$\|v\|_{H_0^1(\Omega)} = \|v\|_{L^2(\Omega)} + \|v'\|_{L^2(\Omega)}$

$\to \|v\|_{H_0^1(\Omega)} \geq \|v'\|_{L^2(\Omega)}$

as $v, f \in H_0^1(\Omega) < \infty$ $\quad$ Not sure if this is enough

wrong

Is $l(v)$ bounded? If yes, continuous

$$\int_0^1 f(x)\,v(x)\,dx = \underbrace{\int_0^1 v(x)\,\frac{2}{9}\,x^{-\frac{4}{3}}}_{②} + \underbrace{\int_0^1 \frac{5}{9}\,x^{-\frac{1}{3}}\,v(x)\,dx}_{①} = \star$$

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① $$\int_0^1 \frac{5}{9}\,x^{-\frac{1}{3}}\,v(x)\,dx \le \frac{5}{9}\,\|x^{-\frac{1}{3}}\|_{L^2}\,\|v(x)\|_{L^2}$$

$$= \frac{5}{9}\int_0^1 x^{-\frac{2}{3}}\,dx\,\|v(x)\|_{L^2}$$

$$\boxed{= \frac{5}{3}\,\|v(x)\|_{L^2}} \checkmark$$

② $$\int_0^1 \frac{2}{9}\,\underbrace{x^{-\frac{4}{3}}}_{f'}\,\underbrace{v(x)}_{g}\,dx$$

$[\int f'g = fg - \int fg']$

$$= \frac{2}{9}\left(\underbrace{\left(-3x^{-\frac{1}{3}}v(x)\right)\Big|_0^1}_{0} + \int_0^1 3\,x^{-\frac{1}{3}}\,v'(x)\,dx\right)$$

$$\le \frac{2}{3}\,\|x^{-\frac{1}{3}}\|_{L^2}\,\|v'(x)\|_{L^2}$$

$$= \frac{2}{3}\int_0^1 x^{-\frac{2}{3}}\,dx\,\|v'(x)\|_{L^2}$$

$$\boxed{= 2\,\|v'(x)\|_{L^2}} \checkmark$$

$$\star = 2\,\|v'(x)\|_{L^2} - \frac{5}{3}\,\|v(x)\|_{L^2}$$

$$\le \|v\|_{H_0^1(\Omega)}\left(2 - \frac{5}{3}\right)$$

$$\boxed{= \frac{1}{3}\,\|v\|_{H_0^1(\Omega)} < \infty}$$

$\|v\|_{H_0^1(\Omega)} = \|v\|_{L^2(\Omega)} + \|v'\|_{L^2(\Omega)}$

$\rightarrow \|v\|_{H_0^1(\Omega)} \ge \|v'\|_{L^2(\Omega)}$

$\rightarrow \|v\|_{H_0^1(\Omega)} \ge \|v\|_{L^2(\Omega)}$

$\rightarrow l(v)$ is bounded $\rightarrow$ continuous

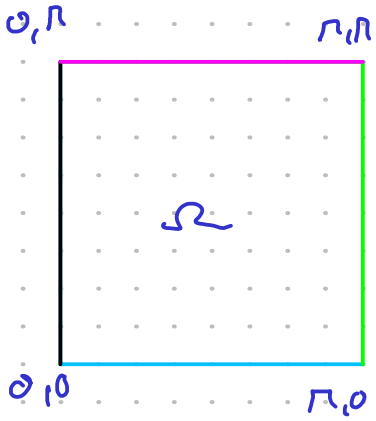
3. (Shift theorem, positive example). The aim of this exercise is to "show" a shift theorem on a square. Let $\Omega = (0,\pi)^2$ consider the Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions.

a) Take a *finite* Fourier-series: for some $N < \infty$, take a function u_N to be of the form $u_N := \sum_{k,\ell=1}^{N} a_{k,\ell} \sin(kx)\sin(\ell y)$. Argue that u_N fulfills the boundary conditions. Moreover, compute the partial derivatives of second order and show

$$|u_N|^2_{H^2(\Omega)} \leq C \sum_{k,\ell=1}^{N} (k^4 + \ell^4) a_{k,\ell}^2.$$

Hint: you can use Parseval's identity: for any Fourier series $v_N = \sum_{k,\ell=1}^{N} b_{k,\ell} \sin(kx)\sin(\ell y)$, there holds $\|v_N\|^2_{L^2(\Omega)} = \frac{\pi^2}{4} \sum_{k,\ell=1}^{N} b_{k,\ell}^2$ (and the same holds when $\sin(\cdot)$ is replaced by $\cos(\cdot)$).

a) BC: $\boxed{u_h(0,y)} = \boxed{u_h(x,0)} = \boxed{u_h(\pi,y)} = \boxed{u_h(x,\pi)} = 0$



$$u_h(0,y) = \sum_{k,l=1}^{N} a_{k,l} \underbrace{\sin(k\cdot 0)}_{=0} \sin(ly) = 0 \checkmark$$

$$u_h(\pi,y) = \sum_{k,l=1}^{N} a_{k,l} \underbrace{\sin(k\cdot \pi)}_{=0} \sin(ly) = 0 \checkmark$$

$$u_h(x,0) = \sum_{k,l=1}^{N} a_{k,l} \sin(kx) \underbrace{\sin(l\cdot 0)}_{=0} = 0 \checkmark$$

$$u_h(x,\pi) = \sum_{k,l=1}^{N} a_{k,l} \sin(kx) \underbrace{\sin(l\cdot \pi)}_{=0} = 0 \checkmark$$

u_h fulfills the boundary conditions

Partial derivatives:

$$\frac{\partial}{\partial x} u_h = \sum_{k,l=1}^{N} k\, a_{k,l} \cos(kx) \sin(ly)$$

$$\frac{\partial}{\partial y} u_h = \sum_{k,l=1}^{N} l\, a_{k,l} \sin(kx) \cos(ly)$$

$$\frac{\partial^2}{\partial x^2} u_n = \sum_{k,l=1}^{N} k^2 a_{k,l} (-1) \sin(kx) \sin(ly)$$

$$\frac{\partial^2}{\partial y^2} u_n = \sum_{k,l=1}^{N} l^2 a_{k,l} (-1) \sin(kx) \sin(ly)$$

$$\frac{\partial^2}{\partial x \partial y} u_n = \frac{\partial^2}{\partial y \partial x} u_n = \sum_{k,l=1}^{N} k \cdot l \, a_{k,l} \cos(kx) \cos(ly)$$

Show: $|u_N|^2_{H^2(\Omega)} \leq C \sum_{k,\ell=1}^{N} (k^4 + \ell^4) a^2_{k,\ell}$.

$$|u_n|^2_{H^2(\Omega)} = \sum_{|\alpha|=2} \|D^\alpha u_n\|^2_{L^2(\Omega)}$$

Sobolev seminorm ... highest order derivatives only

$$= \left\|\frac{\partial^2}{\partial x^2} u_n\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial^2}{\partial y^2} u_n\right\|^2_{L^2(\Omega)} + 2\left\|\frac{\partial^2}{\partial x \partial y} u_n\right\|^2_{L^2(\Omega)}$$

$\boxed{\|v_N\|^2_{L^2(\Omega)} = \frac{\pi^2}{4} \sum_{k,\ell=1}^{N} b^2_{k,\ell}}$ for any $v_N = \sum_{k,\ell=1}^{N} b_{k,\ell} \sin(kx) \sin(\ell y)$, → throw cos, sin away

$$= + \frac{\pi^2}{4} \sum_{k,l=1}^{N} a_{k,l}^2 k^4 + \frac{\pi^2}{4} \sum_{k,l=1}^{N} a_{k,l}^2 l^4 + 2 \cdot \frac{\pi^2}{4} \sum_{k,l=1}^{N} k^2 \cdot l^2 a_{k,l}^2$$

Minus get squared away

Each term can be bounded by $\sum_{k,l=1}^{N} (k^4 + l^4) a^2_{k,l}$

$$\leq \left(\frac{\pi^2}{4} \sum_{k,l=1}^{N} a_{k,l}^2 (k^4 + l^4)\right)(1 + 1 + 2)$$

$$\boxed{= \underbrace{\pi^2}_{C} \sum_{k,l=1}^{N} (k^4 + l^4) a_{k,l}^2}$$

b) Do the same to show

$$\sum_{k,\ell=1}^{N} (k^4+\ell^4)a_{k,\ell}^2 \le C' \|-\Delta u_N\|^2_{L^2(\Omega)}.$$

$$\Delta f = \sum_{i=1}^{\dim} \frac{\partial^2 f}{\partial x_i}$$

homogenity

$$\|-\Delta u_n\|^2_{L^2(\Omega)} = |-1|\,\|\Delta u_n\|^2_{L^2(\Omega)} = \left\| \frac{\partial^2}{\partial x^2} u_n + \frac{\partial^2}{\partial y^2} u_n \right\|^2_{L^2(\Omega)}$$

$$= \int_\Omega \left(\frac{\partial^2}{\partial x^2} u_n + \frac{\partial^2}{\partial y^2} u_n\right)^2 dx = \int_\Omega \left(\frac{\partial^2}{\partial x^2} u_n\right)^2 + 2\frac{\partial^2}{\partial x^2} u_n \cdot \frac{\partial^2}{\partial y^2} u_n + \left(\frac{\partial^2}{\partial y^2} u_n\right)^2 dx$$

$$\int \frac{\partial^2}{\partial x^2} u_n \cdot \frac{\partial^2}{\partial y^2} u_n = \sum_{k,l,m,n=1}^{N} k^2 n^2 a_{k,l} a_{mn} \int_0^\pi \sin(kx)\sin(mx)\,dx \int_0^\pi \sin(ly)\sin(ny)\,dy$$

$$\sin x \, \sin y = \frac{1}{2}\Big(\cos(x-y) - \cos(x+y)\Big)$$

$$= \sum_{k,l,m,n=1}^{N} k^2 n^2 a_{k,l} a_{mn} \int_0^\pi \frac{1}{2}\Big(\cos((k-m)x) - \cos((k+m)x)\Big)dx \int_0^\pi \frac{1}{2}\Big(\cos((l-n)y) - \cos((l+n)y)\Big)dy$$

$$= \frac{1}{4} \sum_{\substack{k,l,m,n=1 \\ k\neq m,\, l\neq n}}^{N} k^2 n^2 a_{k,l} a_{mn} \left(\int_0^\pi \frac{1}{2}\Big(\cos((k-m)x) - \cos((k+m)x)\Big)dx \int_0^\pi \frac{1}{2}\Big(\cos((l-n)y) - \cos((l+n)y)\Big)dy\right)$$

$$+ \frac{1}{4} \sum_{k,l=1}^{N} k^2 l^2 a_{k,l}^2 \left(\int_0^\pi 1 - \cos(2kx)\,dx \int_0^\pi 1 - \cos(2ly)\,dy\right) \Bigg|_{\substack{k=m \\ l=n}}$$

$$= \frac{1}{4} \sum_{\substack{k,l,m,n=1 \\ k\neq m,\, l\neq n}}^{N} k^2 n^2 a_{k,l} a_{mn} \left(\frac{1}{k-m} \underbrace{\sin((k-m)x)}_{0} - \frac{1}{k+m} \underbrace{\sin((k+m)x)}_{0}\Big|_0^\pi\right)\left(\frac{1}{l-n} \underbrace{\sin((l-n)y)}_{0} - \frac{1}{l+n} \underbrace{\sin((l+n)y)}_{0}\Big|_0^\pi\right)$$

$$+ \frac{1}{4} \sum_{k,l=1}^{N} k^2 l^2 a_{k,l}^2 \Big(\underbrace{x - \frac{1}{2k}\sin(2kx)\Big|_0^\pi}_{(\pi-0)-(0-0)}\Big)\Big(\underbrace{y - \frac{1}{2l}\sin(2ly)\Big|_0^\pi}_{(\pi-0)-(0-0)}\Big) \Bigg|_{\substack{k=m \\ l=n}}$$

$$= \frac{1}{4} \sum_{k,l=1}^{N} k^2 l^2 a_{k,l}^2 \pi^2$$

$$= \frac{\pi^2}{4} \sum_{k,l=1}^{N} k^2 l^2 a_{k,l}^2$$ only terms with $k=m, l=n$ matter

$$= \left\|\frac{\partial^2}{\partial x^2} u_n\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial^2}{\partial y^2} u_n\right\|^2_{L^2(\Omega)} + 2\left\|\frac{\partial^2}{\partial x \partial y} u_n\right\|^2$$

$$= \underline{\left\| \frac{\partial^2}{\partial x^2} u_n \right\|^2_{L^2(\Omega)}} + \underline{\left\| \frac{\partial^2}{\partial y^2} u_n \right\|^2_{L^2(\Omega)}} + 2 \underline{\left\| \frac{\partial^2}{\partial x \partial y} u_n \right\|^2}$$

$$= + \frac{\pi^2}{4} \sum_{k,l=1}^{N} a_{k,l}^2 k^4 + \frac{\pi^2}{4} \sum_{k,l=1}^{N} a_{k,l}^2 l^4 + 2 \cdot \frac{\pi^2}{4} \sum_{k,l=1}^{N} k^2 \cdot l^2 a_{k,l}^2$$

Minus get squared away

$$= \frac{\pi^2}{4} \sum_{k,l=1}^{N} (k^4 + l^4) a_{k,l}^2 + \underbrace{2 \cdot \frac{\pi^2}{4} \sum_{k,l=1}^{N} k^2 \cdot l^2 a_{k,l}^2}_{>0}$$

$$\geq \frac{\pi^2}{4} \sum_{k,l=1}^{N} (k^4 + l^4) a_{k,l}^2$$

$$\Rightarrow \boxed{\underbrace{\frac{4}{\pi^2}}_{C'} \left\| -\Delta u_n \right\|^2_{L^2(\Omega)} \geq \sum_{k,l=1}^{N} (k^4 + l^4) a_{k,l}^2}$$

c) Combine the previous steps to obtain $|u_N|_{H^2(\Omega)} \leq C'' \|\Delta u_N\|_{L^2(\Omega)} = C'' \|f_N\|_{L^2(\Omega)}$, where $-\Delta u_N = f_N$.

$$|u_n|_{H^2(\Omega)} \leq C \sum_{k,l=1}^{N} (k^4 + l^4) a_{k,l}^2 \leq C'' \left\| -\Delta u_n \right\|^2_{L^2(\Omega)} \quad \Big| -\Delta u_n = f_N$$

$$= C'' \left\| f_N \right\|_{L^2(\Omega)}$$

4. (Shift theorem, negative example) Let Ω be given in polar coordinates as the sector of the unit disk

$$\Omega = \{(r\cos\varphi, r\sin\varphi) \ : \ 0 < r < 1, 0 < \varphi < \frac{3}{2}\pi\}.$$

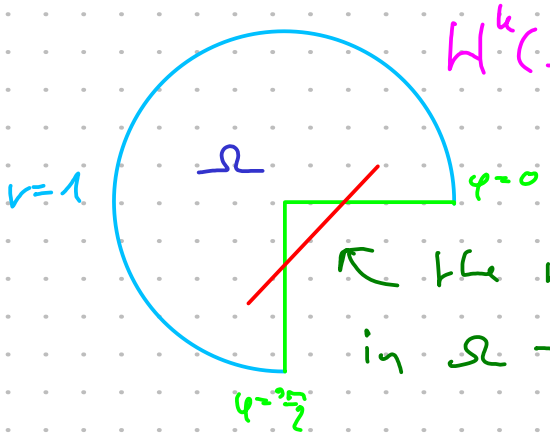
Verify that Ω is not convex. Show that the function (in polar coordinates)

$$u = r^{2/3}\sin(2\varphi/3)$$

satisfies $\Delta u \in L^2(\Omega)$, but $\partial_{xx}u \notin L^2(\Omega)$, and therefore $u \notin H^2(\Omega)$.

From here it is easy to construct a function v that is even in H_0^1 that fails the shift theorem: take $v = (1-r^2)u$.

$$H^k(\Omega) = \{v \in L^1_{loc}(\Omega) : \|v\|_{H^k(\Omega)} < \infty\}$$



the red line is not fully contained in Ω → $\boxed{\text{not convex}}$

$$\Delta v = \frac{\partial^2}{\partial r^2}v + \frac{1}{r}\frac{\partial}{\partial r}v + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}v$$

Laplace in polar coordinates

$$= \frac{\partial}{\partial r}\left(\frac{2}{9}r^{-\frac{1}{3}}\sin\left(\frac{2\varphi}{3}\right)\right) + \frac{\partial}{\partial \varphi}\left(r^{-\frac{4}{3}}\cos\left(\frac{2\varphi}{3}\right)\frac{2}{3}\right) + \frac{2}{9}r^{-\frac{4}{3}}\sin\left(\frac{2\varphi}{3}\right)$$

$$= \left(\frac{4}{9}r^{-\frac{4}{3}}\sin\left(\frac{2\varphi}{3}\right)\right) + \left(r^{-\frac{4}{3}}(-1)\sin\left(\frac{2\varphi}{3}\right)\frac{4}{9}\right)$$

$$= 0$$

$$\|\Delta v\|^2_{L^2(\Omega)} = \int_\Omega 0^2 \, dr\, d\varphi = \boxed{0 < \infty}$$

$$\boxed{\|\Delta v\|^2_{L^2(\Omega)} \in L^2(\Omega)}$$

$$\nabla_{r,\varphi} = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$ | nabla in polar coords

$$\nabla_{x,y} = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}$$ | nabla in cartesian

$$u = r^{2/3}\sin(2\varphi/3)$$

$$\frac{\partial u}{\partial r} = \frac{2}{3} r^{-\frac{1}{3}} \sin\left(\frac{2\varphi}{3}\right)$$

$$\frac{\partial u}{\partial \varphi} = \frac{2}{3} r^{\frac{2}{3}} \cos\left(\frac{2\varphi}{3}\right)$$

With $x = r\cos\varphi$, $y = r\sin\varphi$, we get the derivative in regards to x, y by deriving over r, φ

$$\hat{\mathbf{x}} = \cos\varphi\hat{\rho} - \sin\varphi\hat{\varphi}$$
$$\hat{\mathbf{y}} = \sin\varphi\hat{\rho} + \cos\varphi\hat{\varphi}$$

$$\partial_{x,y} u = \nabla_{r,\varphi} u(r,\varphi) = \frac{\partial u}{\partial r}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} + \frac{1}{r}\frac{\partial u}{\partial \varphi}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$= \frac{2}{3} r^{-\frac{1}{3}} \sin\left(\frac{2\varphi}{3}\right)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} - \frac{2}{3} r^{-\frac{1}{3}} \cos\left(\frac{2\varphi}{3}\right)\sin\varphi$$

We are only interested in x:

$$\partial_x u = \frac{2}{3} r^{-\frac{1}{3}}\left(\sin\left(\frac{2\varphi}{3}\right)\cos\varphi - \cos\left(\frac{2\varphi}{3}\right)\sin\varphi\right)$$

$$\left[\frac{\partial u_x}{\partial r} = -\frac{2}{9} r^{-\frac{4}{3}}\left(\sin\left(\frac{2\varphi}{3}\right)\cos\varphi - \cos\left(\frac{2\varphi}{3}\right)\sin\varphi\right)\right.$$ | $\sin(x-y) = \sin x\cos y - \cos x\sin y$ |

$$\left.= -\frac{2}{9} r^{-\frac{4}{3}} \sin\left(-\frac{\varphi}{3}\right)\right]$$

$$\left[\frac{1}{r}\frac{\partial u_x}{\partial \varphi} = \frac{2}{3} r^{-\frac{4}{3}}\left(\frac{2}{3}\cos\left(\frac{2\varphi}{3}\right)\cos\varphi - \sin\left(\frac{2\varphi}{3}\right)\sin\varphi + \frac{2}{3}\sin\left(\frac{2\varphi}{3}\right)\sin\varphi - \cos\left(\frac{2\varphi}{3}\right)\cos\varphi\right)\right.$$

$$= -\frac{2}{9} r^{-\frac{4}{3}}\left(\cos\left(\frac{2\varphi}{3}\right)\cos\varphi + \sin\left(\frac{2\varphi}{3}\right)\sin\varphi\right)$$ | $\cos(x-y) = \cos x\cos y + \sin x\sin y$

$$= -\frac{2}{9} r^{-\frac{4}{3}}\left(\cos\left(\varphi - \frac{2\varphi}{3}\right)\right) = -\frac{2}{9} r^{-\frac{4}{3}} \cos\left(\frac{\varphi}{3}\right)$$

$$\partial_{xx} u = -\frac{2}{9} r^{-\frac{4}{3}} \sin\left(-\frac{\varphi}{3}\right)\cos\varphi + \frac{2}{9} r^{-\frac{4}{3}} \cos\left(\frac{\varphi}{3}\right)\sin\varphi$$

$$= \frac{2}{9} r^{-\frac{4}{3}} \sin\left(\frac{\varphi}{3}\right)\cos\varphi + \frac{2}{9} r^{-\frac{4}{3}} \cos\left(\frac{\varphi}{3}\right)\sin\varphi$$

$$= \frac{2}{9} r^{-\frac{4}{3}} \sin\left(\frac{4\varphi}{3}\right)$$

$$\|\partial_{xx} u\|_{L^2}^2 = \int_\Omega \left(\frac{2}{9} r^{-\frac{8}{3}} \sin\left(\frac{4\varphi}{3}\right)\right)^2 dr\, d\varphi$$

$$= \int_{r=0}^{1} \int_{\varphi=0}^{\frac{3\pi}{2}} \frac{4}{81} \underbrace{\boxed{r^{-\frac{8}{3}}}}_{\to\infty} \underbrace{\sin^2\left(\frac{4\varphi}{3}\right)}_{>0\ <\infty} dr\, d\varphi$$

$$\int_0^1 x^{-\frac{8}{3}} dx \approx x^{-\frac{5}{3}} \Big|_0^1 = \frac{1}{1} - \frac{1}{0} \to \infty$$

$$\boxed{\to \infty \quad \lightning}$$

$$\boxed{\partial_{xx} u \notin L^2(\Omega) \Rightarrow u \notin H^2(\Omega)}$$

as

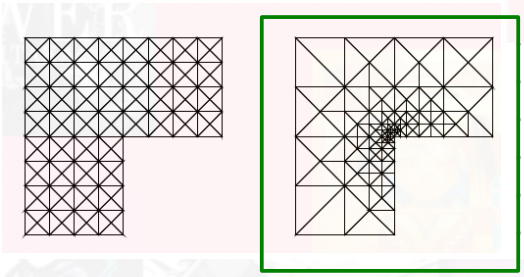
$$\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 + \|\partial_y u\|_{L^2}^2 + \|\partial_{x^2} u\|_{L^2}^2 + \|\partial_{y^2} u\|_{L^2}^2 + 2\|\partial_{xy} u\|_{L^2}^2$$

5. Write an adaptive FEM code in 1D to solve

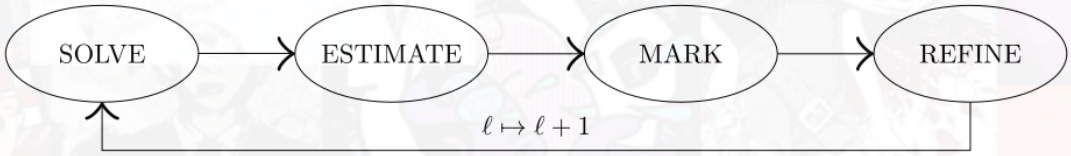
$$-u'' = f \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0.$$

Use piecewise linear Lagrangian finite elements. For the `SOLVE` step, e.g., use your FEM solver previous exercises (note that for this example your code needs to handle different element sizes). For the `ESTIMATE` step compute the residual error indicator from the lecture notes and for the `MARK` step implement Dörfler marking. The `REFINE` step in 1D is easy, as only all marked elements have to be halved.

Test your code with right-hand side such that the exact solution is $u(x) = x^{2/3}(1-x)$ (as in Example 1). Plot the error over the number of elements as well as the error indicator over the number of elements for marking parameters $\theta = 0.25$ and $\theta = 1$. Compare with the results in Example 1.



already done



- SOLVE: Compute your FEM-solution on the current mesh $\mathcal{T}_\ell$.
- ESTIMATE: Compute for each $T \in \mathcal{T}_\ell$ a quantity $\eta_\ell(T)$ that somewhat represents the error on T (note that the true solution in general is unknown, so the true error can not be computed!).
- MARK: Mark the elements in $\mathcal{T}_\ell$ with largest indicators for refinement.
- REFINE: Bisect the (at least) the marked elements.

We don't have the true solution, so estimate error as

The usual error estimators are defined as sum over element contributions:

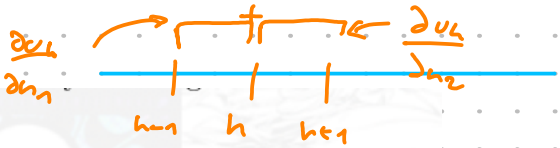
$$\eta^2(u_h, f) = \sum_{T \in \mathcal{T}} \eta_T^2(u_h, f),$$

makes statement about local error

Algorithmus 5.4:

```
Input:  initial mesh T_0, tolerance TOL, marking parameter θ ∈ (0,1]
ℓ := 0
Loop over ℓ
      compute FEM solution u_ℓ on T_ℓ ✓
      compute local error contributions η_T(u_h, f) ✓
      sum the local contributions η² = Σ_T η_T²(u_h, f) ✓
      if η ≤ TOL then stop ✓
      choose a marking set M_ℓ ⊂ T_ℓ ✓
      generate the mesh T_{ℓ+1} by refining all elements in M_ℓ (and maybe some more
        to remove hanging nodes) ✓
      ℓ := ℓ + 1
end
```

For the ESTIMATE step compute the residual error indicator



We define the normal-jump term

$$\left[\frac{\partial u_h}{\partial n}\right] := \frac{\partial u_h}{\partial n_1}|_{T_1} + \frac{\partial u_h}{\partial n_2}|_{T_2}.$$

The residual error estimator is

$$\eta^{res}(u_h, f)^2 := \sum_T \eta_T^{res}(u_h, f)^2$$

square over elements

with the element contributions

$$\eta_T^{res}(u_h, f)^2 := h_T^2 \|f + \Delta u_h\|^2_{L^2(T)} + \sum_{\substack{E:E\subset T \\ E\subset\Omega}} h_E \left\| \left[\frac{\partial u_h}{\partial n}\right] \right\|^2_{L^2(E)}.$$

use mid points and integrate over size

use array of widths for bins

bumpy does the rest

for the MARK step implement Dörfler marking.

Marking → use Dörfler marking

The first idea is to select elements such that the corresponding local error indicators make up (at least) a fixed percentage of the total error indicator, which is called **Dörfler marking**. More precisely, for $\theta \in (0, 1)$, one seeks a minimal set $\mathcal{M}_\ell$ such that

$$\theta\eta^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_T^2(u_h, f).$$

A way to obtain this minimal set is to sort all element indicators $\eta_T^2(u_h, f)$ from largest to smallest and add elements to the sum on the right-hand side until the criterium is fulfilled.

We have an array of error indicators

$[e_1, e_2, \ldots]$ etc.

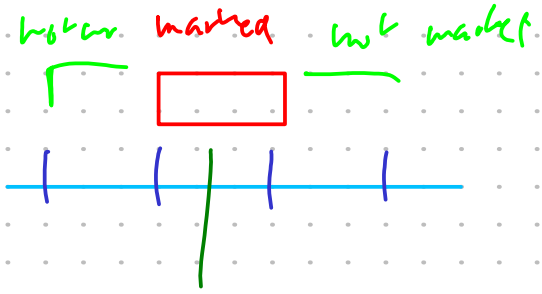
We check if the cumulative sum of these elements is $\geq \theta \cdot \eta^2$ total error indicator

If not → add more elements to the cumulative sum (sorted from largest to smallest → refine the "worst" elements first)

If yes → mark and sort

Consequence: $\theta = 1 \rightarrow$ all elements are always refined (uniform refinement)

$\rightarrow$ Refinement: bisect element



bisect into two elements

Hence, list of marked indices

for each marked element

NOTE: For each marked element we add in total 1 element to the list

Consequence: When calculating indices in python, add i to all indices (for i in range(...)) to account for this