Euler-McLaurin formula

The expansion

$$T(h) = \int_a^b f(x)\,dx + c_1 h^2 + c_2 h^4 + \cdots \tag{1}$$

is the Euler-McLaurin formula and obtained by repeated integration by part as we now show.

- For simplicity of notation, we consider

$$\int_0^N \tilde{f}(x)\,dx = \sum_{j=0}^{N-1} \int_j^{j+1} \tilde{f}(x)\,dx$$

- Integration by parts gives

$$\int_j^{j+1} \tilde{f}(x)\,dx = \Big[(x-j-1/2)\tilde{f}(x)\Big]_j^{j+1} - \int_j^{j+1} (x-j-1/2)\tilde{f}'(x)\,dx$$

- Hence

$$\frac{1}{2}\tilde{f}(j) + \frac{1}{2}\tilde{f}(j+1) = \int_j^{j+1} \tilde{f}(x)\,dx + \int_j^{j+1} \varpi_1(x)\tilde{f}'(x)\,dx, \qquad \varpi_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$$

- Summation yields

$$\sum_{j=0}^{N-1} \frac{1}{2}\left(\tilde{f}(j) + \tilde{f}(j+1)\right) = \int_0^N \tilde{f}(x)\,dx + \int_0^N \varpi_1(x)\tilde{f}'(x)\,dx$$

- Let $\varpi_2(x)$ be an antiderivative of ϖ_1, i.e., $\varpi_2' = \varpi_1$. Note that ϖ_2 is 1-periodic since ϖ_1 is 1-periodic and $\int_0^1 \varpi_1(x)\,dx = 0$. This implies $\varpi_2(0) = \varpi_2(j) = \varpi_2(j+1)$ for all j. We select the integration constant of ϖ_2 such that $\int_0^1 \varpi_2(x)\,dx = 0$.

- A two-fold integration by parts yields with an antiderivative $\varpi_3(x)$ of $\varpi_2(x)$

$$\int_j^{j+1} \varpi_1(x)\tilde{f}'(x)\,dx =$$

$$\varpi_2(0)\left(\tilde{f}'(j+1) - \tilde{f}'(j)\right) - \varpi_3(0)\left(\tilde{f}''(j+1) - \tilde{f}''(j)\right) + \int_j^{j+1} \varpi_3(x)\tilde{f}'''(x)\,dx$$

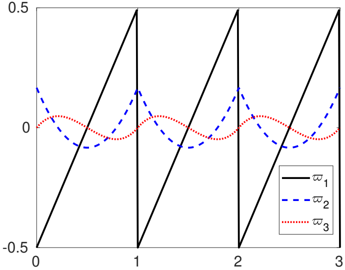
- again, one selects the integration constant of ϖ_3 such that $\int_0^1 \varpi_3(x)\,dx = 0$.
- repeating the process yields

$$\frac{1}{2}\left(\tilde{f}(j)+\tilde{f}(j+1)\right) = \int_j^{j+1} \tilde{f}(x)\,dx + \sum_{s=1}^{m-1}(-1)^{s+1}\varpi_{s+1}(0)\left(\tilde{f}^{(s)}(j+1)-\tilde{f}^{(s)}(j)\right) + (-1)^{m+1}\int_j^{j+1}\varpi_m(x)\tilde{f}^{(m)}(x)\,dx$$

- summation over j yields

$$\sum_{j=0}^{N}\frac{1}{2}(\tilde{f}(j)+\tilde{f}(j+1)) = \int_0^N \tilde{f}(x)\,dx + \sum_{s=1}^{m-1}(-1)^{s+1}\varpi_{s+1}(0)\left(\tilde{f}^{(s)}(N)-\tilde{f}^{(s)}(0)\right) + (-1)^{m+1}\int_0^N \varpi_m(x)\tilde{f}^{(m)}(x)\,dx \tag{2}$$

- The values $\varpi_s(0)$ are the $\rightarrow$ Bernoulli numbers



- Exercise: Given a function f on $(0,1)$, define $\tilde{f}(x) := f(hx)$ on $(0,N)$ and derive the expansion (1) from (2).

Romberg extrapolation

Romberg extrapolation for $\int_0^1 f(x)\,dx$ with f smooth

Procedure:

1. compute the values of the trapezoidal rule $T_{i0} := T(h_i)$ for $h_i = 2^{-i}$
2. evaluate the interpolating polynomial $P_{i,n} \in \mathcal{P}_n$ for the data $(h_{i+j}^2, T(h_{i+j}))$, $j = 0, \ldots, n$, at $h = 0$: $T_{in} := P_{i,n}(0)$.
   This is achieved with the Neville scheme:

$$
\begin{aligned}
T_{i0} &= T(h_i) \\
T_{ij} &= P_{i,j}(0) = P_{i+1,j-1}(0) - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[P_{i+1,j-1}(0) - P_{i,j-1}(0)\right] \\
&= T_{(i+1)\,(j-1)} - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[T_{(i+1)\,(j-1)} - T_{i\,(j-1)}\right], \qquad j \geq 1
\end{aligned}
$$

Romberg extrapolation for $\int_0^1 e^x \, dx$

h	h^2	$n=0$	$n=1$	$n=2$	$n=3$
1	1	1.859140914229523	1.718861151876593	1.718282687924754	1.718281828794499
2^{-1}	2^{-2}	1.753931092464825	1.718318841921747	1.718281842218437	1.718281828460412
2^{-2}	2^{-4}	1.727221904557517	1.718284154699897	1.718281828675358	1.718281828459105
2^{-3}	2^{-6}	1.720518592164302	1.718281974051892	1.718281828462428	1.718281828459017
2^{-4}	2^{-8}	1.718841128579994	1.718281837561771	1.718281828459097	1.718281828459077
2^{-5}	2^{-10}	1.718421660316327	1.718281829028016	1.718281828459049	1.718281828459047
2^{-6}	2^{-12}	1.718316786850094	1.718281828494605	1.718281828478246	
2^{-7}	2^{-14}	1.718290568083478	1.718281828461267		
2^{-8}	2^{-16}	1.718284013366820			
1	1	1.41_{-01}	5.79_{-04}	8.59_{-07}	3.35_{-10}
2^{-1}	2^{-2}	3.56_{-02}	3.70_{-05}	1.40_{-08}	1.37_{-12}
2^{-2}	2^{-4}	8.94_{-03}	2.33_{-06}	2.16_{-10}	5.95_{-14}
2^{-3}	2^{-6}	2.24_{-03}	1.46_{-07}	3.38_{-12}	-2.80_{-14}
2^{-4}	2^{-8}	5.59_{-04}	9.10_{-09}	5.20_{-14}	3.20_{-14}
2^{-5}	2^{-10}	1.40_{-04}	5.69_{-10}	3.77_{-15}	1.78_{-15}
2^{-6}	2^{-12}	3.50_{-05}	3.56_{-11}	1.92_{-11}	
2^{-7}	2^{-14}	8.74_{-06}	2.22_{-12}		
2^{-8}	2^{-16}	2.18_{-06}			
expected		$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$