Exam - Solutions and Point Distribution

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
max. Points	12	14	13	12	9

The point distributions for each subexample are written below.

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1. **a)** With $A(x,y) = y(1+xy)$ and $B(x,y) = -x$ the ODE is inexact, since

$$\frac{\partial A}{\partial y} = 1 + 2xy \neq -1 = \frac{\partial B}{\partial x}.$$

An integrating factor can be found in the form $\mu = \mu(y)$ as

$$\mu(y) = \frac{1}{y^2},$$

which leads to the equivalent, exact ODE

$$-\frac{x}{y^2}y' + \frac{1}{y} + x = 0.$$

Therefore, it can be solved by integrating A with respect to x and B with respect to y and comparing the functions (and constants of integration), which gives the equation

$$\frac{x}{y} + \frac{x^2}{2} = C,$$

which can be rearranged for y.

b) By a) we have a gradient field and the scalar potential

$$\phi = \frac{x}{y} + \frac{x^2}{2}.$$

c) There are many possible solutions. Applying the divergence to the equation gives

$$\Delta\phi = 0,$$

which motivates, e.g., the choice $\phi = x^2 - y^2$. Then, one has to compute a ψ with $\operatorname{curl}\psi = (2x, -2y, 0)^T = \nabla\phi$, which is, $\psi = (0, 0, 2xy)^T$.

Point distribution: a) 7, b) 2, c) 3

2. **a)** The characteristic equation is

$$\lambda^2 + 1 = 0.$$

Therefore $\lambda = \pm i$ and the real valued general solution is

$$u(x) = C_1 \sin(x) + C_2 \cos(x)$$

b) As in a), we get the general solution

$$u(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x).$$

Inserting the boundary condition $u'(0) = 0$ gives $C_1 = 0$. Inserting the secod condition $u(1) = 0$ then gives

$$C_2 \cos(\sqrt{\lambda}) = 0 \qquad \Longrightarrow \qquad \sqrt{\lambda} = \frac{\pi}{2} + k\pi \quad k \in \mathbb{N}_0.$$

c) We make the ansatz

$$T(x,t) = u(x)v(t).$$

Plugging this into the equation gives $u(x)v_t(t) = u_{xx}(x)v(t)$ and after division with αuv, we arrive at

$$\frac{u_{xx}(x)}{u(x)} = \frac{v_t(t)}{v(t)} = \text{const} =: -\lambda.$$

From this equation, we deduce the ODEs

$$v'(t) = -\alpha\lambda v(t) \qquad \Longrightarrow \qquad v(t) = c_1 e^{-\lambda t}$$

and

$$u''(x) = -\lambda u(x) \qquad u'(0) = 0, u(1) = 0.$$

This has been solved in b) for $\lambda > 0$ as

$$u(x) = C_k \cos\left((\frac{\pi}{2} + k\pi)x \right)$$

and the cases $\lambda \leq 0$ provide only the zero solution. Inserting this in the ansatz gives

$$T(x,t) = \sum_{k=1}^{\infty} C_k e^{-\left(\frac{\pi}{2}+k\pi\right)^2 t} \cos\left((\frac{\pi}{2} + k\pi)x \right).$$

Finally, C_k can be computed as Fourier coefficients from

$$C_k = 2\int_0^1 T_0(x) \cos\left((\frac{\pi}{2} + k\pi)x \right) dx = 2\int_0^1 \cos\left((\frac{\pi}{2} + k\pi)x \right) dx = \frac{\sin\left(\frac{\pi}{2} + k\pi\right)}{(\frac{\pi}{2} + k\pi)} = \frac{(-1)^k}{\frac{\pi}{2} + k\pi}$$

Point distribution: a) 2, b) 4, c) 8

3. **a)** Let $V \subset \mathbb{R}^n$ be a bounded region and ∂V its boundary. Let ψ be a continuously differentiable vector field defined on V. Then,

$$\int_V \operatorname{div} \psi \, dV = \int_{\partial V} \psi \cdot dS.$$

b) Its a paraboloid with apex at $z = 4$. The normal vectors are $n_1 = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$ (by p.56 lecture notes) and $n_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

c) We have $\operatorname{div}\psi = 1$. Therefore, with cylindrical coordinates to describe V (transformation theorem with determinante r), we obtain

$$\int_V \operatorname{div}\psi \; dV = \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r \; dz d\varphi dr = 2\pi \int_0^1 4r - r^3 \; dr = 8\pi.$$

The surface integral has two summands (S_1 beeing the surface $z = 4 - x^2 - y^2$ and S_2 being the surface $z = 0$)

$$\int_{\partial V} \psi \cdot dS = \int_{S_1} \psi \cdot dS + \int_{S_2} \psi \cdot dS.$$

With the normal vector n_1, and a description of the surface in polar coordinates, we get

$$\int_{S_1} \psi \cdot dS = \int_G \begin{pmatrix} y \\ -x \\ 4 - x^2 - y^2 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix} dA = \int_G 4 - x^2 - y^2 dA = \int_0^2 \int_0^{2\pi} 4r - r^3 dr = 8\pi.$$

The second integral is zero, since

$$\int_{S_1} \psi \cdot dS = \int_G \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA = 0.$$

Point distribution: a) 2, b) 3, c) 8

4. **a)** The vector field is a gradient field, therefore the line integral is path independent and one can choose the path $r(t) = (t, 0)$ to connect $(0, 0)$ and (1,0), which gives

$$\int_C \psi \; ds = \int_0^1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = -1.$$

b) We have with integration by parts

$$\begin{aligned} \sqrt{2\pi}\widehat{f}(\omega) &:= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_0^{\infty} t e^{-i\omega t - t} dt = t \frac{e^{-i\omega t - t}}{-i\omega - 1}\Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-i\omega t - t}}{i\omega + 1} dt \\ &= \int_0^{\infty} \frac{e^{-i\omega t - t}}{i\omega + 1} dt = -\frac{1}{(i\omega + 1)^2}. \end{aligned}$$

Using the convolution theorem, we get $\widehat{f \star f} = \sqrt{2\pi}\widehat{f}\widehat{f} = \frac{1}{\sqrt{2\pi}} \frac{1}{(i\omega+1)^4}$.

Point distribution: a) 5, b) 5 (Fourier Transform)+2 (Convolution)

5. Answer the following questions (Simply write true/false on your sheet).

a) Which of the following vector identities hold? (ψ vector field, ϕ scalar function)

1. $\nabla \operatorname{div}\psi = \operatorname{curl}\operatorname{curl}\psi$. FALSE
2. $\operatorname{div}(\nabla\psi) = \Delta\psi$. TRUE

3. $\operatorname{curl}(\phi\psi) = \phi \operatorname{curl}(\psi) + \nabla\phi \times \psi$. TRUE

b) An orthonormal set in a vector space

1. is always linearly independent. TRUE
2. is always a basis of the vector space. FALSE
3. can be constructed from a basis of the vector space. TRUE
4. always contains vectors of length 1. TRUE

c) The Fourier series for a function f

1. is a continuous function. TRUE
2. is always finite at every point. FALSE
3. is the same as the value of f everywhere for periodic functions. FALSE
4. gives the mean value at jump points of f. TRUE

d) Analytic functions

1. are infinitely many times differentiable. TRUE
2. can have finitely many singularities. FALSE
3. are everywhere complex differentiable. TRUE
4. can have a Laurent series with non zero principal part. FALSE

Point distribution: a) 3, b) 2, c) 2, d) 2, every wrong or missing answer deduces 1 point