Exam 1

Fri., 26.11.2021, 14:00–15:30, Zoom

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
max. Points	6	5	6	9	4

Good luck!

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1. **a)** Consider matrices $M \in \mathbb{R}^{2\times 2}$. Lower triangular matrices are characterized by the condition $M_{12} = 0$.

Show that the space of real lower triangular matrices $M \in \mathbb{R}^{2\times 2}$ is a vector space. Give a basis for this space.

b) Consider the space V of polynomials of degree 4 that have no real zeros, i.e.,

$$V := \{p \in P_4 \ : p(x) \neq 0 \ \ \forall x \in \mathbb{R}\}.$$

Is this space a vector space? Explain your answer.

2. Consider the function

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Argue that the function is differentiable for every $(x,y) \neq (0,0)$ and compute the partial derivatives of f. Do the partial derivatives at $(0,0)$ exist?

Does every directional derivative of f exist?

3. **a)** Is the vector field

$$\psi = \begin{pmatrix} -y^2 \sin x + 3x^2 \\ 2y\cos x - \exp(y) \end{pmatrix}$$

a gradient field. If so, compute a scalar potential for this vector field.

b) With ψ from above, compute the value of the line integral

$$\int_C \psi \, ds,$$

where C is a curve starting at $(0,0)$ going along the parabola $y = x^2$ to the point $(1,1)$, then going to a straight line to $(1,0)$ and finally going along a half circle in the lower half plane (i.e. $y < 0$) to the point $(-1,0)$.

4. **a)** Formulate Gauss theorem.

b) Sketch the region V covered by the cone $x^2 + y^2 = z^2$, $z \in [0, 1]$. Compute the (outer) normal vector to both surfaces.

c) Verify the theorem for V by computing both the volume and surface and integrals for the vector field

$$\psi = \begin{pmatrix} x+y \\ -x \\ 2xz \end{pmatrix}.$$

hint: $\int_0^{2\pi} \sin t \cos t dt = 0$.

5. Answer the following questions (Simply write true/false on your sheet).

a) A function $f : \mathbb{R}^2 \to \mathbb{R}$

1. is totally differentiable, if all partial derivatives exist.
2. has all directional derivatives, if it is totally differentiable.
3. is Riemann integrable, if it is continuous.
4. has a finite integral, if it is bounded.

b) Which of the following vector identities hold? (ψ vector field, ϕ scalar function)

1. $\operatorname{curl}(\nabla\phi) = 0$.
2. $\nabla \operatorname{div} \psi = \Delta\psi$.
3. $\operatorname{div}(\phi\psi) = \phi \operatorname{div}(\psi) + \nabla\phi \cdot \psi$.

c) A line integral $\oint_C \psi \cdot ds$ over a closed curve $C \subset \mathbb{R}^2$ for a continuously differentiable vector field ψ is zero,

1. always.
2. if, $\operatorname{curl} \psi = 0$.
3. if, there exists a scalar potential for ψ.
4. if, $\frac{\partial\psi_1}{\partial y} = \frac{\partial\psi_2}{\partial x}$ and the region bounded by C is simply connected.