1. Some special nonlinear differential equations can be transformed to linear equations. Such examples are the *Bernoulli equations*, which have the form

$$u'(x) + P(x)u(x) = Q(x)u^n(x), \quad (1)$$

where $n \neq 0, 1$ and P, Q are given functions.

a) Show that the transformation $v = u^{1-n}$ transforms the Bernoulli equation into the linear ODE

$$v'(x) + (1-n)P(x)v(x) = (1-n)Q(x).$$

b) Use this to solve the Bernoulli equation

$$u' + 2xu = u^2.$$

a) $v = u^{1-n}$

$$v' = (1-n)u^{-n}u' \rightarrow u' = \frac{v'}{(1-n)}u^n$$

insert u' into (1):

$$\frac{v'}{(1-n)}u^n + Pu = Qu^n \quad \Big| \, u^{-n} \quad \Big| \cdot (1-n)$$

$$v' + (1-n)Pu^{1-n} = (1-n)Q \quad [v = u^{1-n}]$$

$$\boxed{v' + (1-n)Pv = (1-n)Q} \quad (2)$$

b) $u' + 2xu = u^2 \rightarrow P(x) = 2x, \; Q(x) = 1, \; n = 2, \; v = u^{-1}$

insert into (2): $v' + (-1)2xv = (-1)$

$v' - 2xv = 0$

hom: $v' = 2xv \quad f(x) = 2x \quad g(v) = v$

$$\int f(x)\,dx = \int \frac{1}{v}\,dv \Rightarrow x^2 + C = \ln v$$

$$\boxed{v_H = c\,e^{x^2}}$$ $\rightarrow v_p$: variation of constants

We finish this section with a general formula for the computation of particular solutions for systems

$$u'(x) = a(x)u(x) + g(x),$$

$$v' = \underbrace{2x}_{a(x)}\underbrace{v}_{v(x)} - \underbrace{1}_{g(x)}$$

The general idea behind the variation of constant formula is that a particular solution should have a similar structure, i.e., look like

$$u_p(x) = U(x) \cdot c(x),$$

$$v_p = e^{x^2} c(x) \quad \to \quad U = e^{x^2}, \quad U^{-1} = e^{-x^2}$$

$$v' = 2xe^{x^2}c(x) + c'(x)e^{x^2}$$

$\to$ insert v and v' into (2):

$$\cancel{2xe^{x^2}c(x)} + c'(x)e^{x^2} = \cancel{2xe^{x^2}c(x)} - 1$$

$$c'(x) = -e^{-x^2} \quad \Big| \int$$

$$\to \quad c(x) = -\int e^{-x^2} dx$$

For first order linear equations (i.e. u, a, g are scalar functions), this formula reduces to

$$u_p(x) = e^{A(x)} \int e^{-A(y)} g(y) dy,$$

where $A(x) = \int a(x)dx$.

$$A(x) = \int a(x)\,dx = \int 2x\,dx = x^2 \; [+C]$$

$$v_p = e^{x^2} \int e^{-x^2}(-1)\,dx$$

$$\to v = v_H + v_p$$

$$v = ce^{x^2} - e^{x^2}\int e^{-x^2} dx$$

$$v = e^{x^2}\left(c - \int e^{-x^2} dx\right)$$

$$\boxed{[v = u^{-1}] \qquad u = v^{-1} = \left[e^{x^2}\left(c - \int e^{-x^2} dx\right)\right]^{-1}}$$

2. Use the variation of constants formula to solve the ODE

$$u'' - \frac{3}{x}u' - \frac{5}{x^2}u = \log(x).$$

In order to derive the general solution for the homogeneous equation, make the ansatz $u(x) = x^\alpha$ and compute $\alpha \in \mathbb{N}$.

hom: $u(x) = x^\alpha$

$u'(x) = \alpha x^{\alpha-1}$

$u''(x) = \alpha(\alpha-1)x^{\alpha-2}$

$\rightarrow \alpha(\alpha-1)x^{\alpha-2} - \frac{3}{x}\alpha x^{\alpha-1} - \frac{5}{x^2}x^\alpha = 0$

$(\alpha^2-\alpha)x^{\alpha-2} - 3\alpha x^{\alpha-2} - 5x^{\alpha-2} = 0$

$\left((\alpha^2-\alpha) - 3\alpha - 5\right)x^{\alpha-2} = 0$

$\alpha^2 - 4\alpha - 5 = 0$

$2 \pm \sqrt{4+5} = 2 \pm 3$

$\alpha_1 = 5, \quad \alpha_2 = -1 \; [\notin \mathbb{N}]$

$$\boxed{u_H = c_1 x^5 + c_2 x^{-1}}$$

$v_1 = u, \; v_2 = u'$

$\rightarrow v_1' = v_2$

$v_2' = \frac{3}{x}v_2 + \frac{5}{x^2}v_1 + \log(x)$

Example. We want to solve the inhomogeneous ODE

$$u''(x) + u'(x) = e^{-x}$$

with the variation of constants formula. Using the characteristic equation $\lambda^2 + \lambda = 0$, we obtain the general solution $c_1 + c_2 e^{-x}$ of the homogeneous equation. Now, we rewrite the ODE into a first order system by introducing $v_1 := u$ and $v_2 := u'$. Then,

$$v_1' = v_2$$
$$v_2' = -v_2 + e^{-x}$$

or in matrix notation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix}.$$

$$\overbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}'}^{u'} = \overbrace{\begin{pmatrix} 0 & 1 \\ \frac{5}{x^2} & \frac{3}{x} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}^{a(x)u(x)} + \overbrace{\begin{pmatrix} 0 \\ \log(x) \end{pmatrix}}^{g(x)}$$

$$U = \begin{pmatrix} u_H \\ u_H' \end{pmatrix} = \begin{pmatrix} x^5 & x^{-1} \\ 5x^4 & -x^{-2} \end{pmatrix}$$

$$U^{-1}: \left(\begin{array}{cc|cc} x^5 & x^{-1} & 1 & 0 \\ 5x^4 & -x^{-2} & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} x^4 & x^{-2} & x^{-1} & 0 \\ 5x^4 & -x^{-2} & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} x^4 & x^{-2} & x^{-1} & 0 \\ 0 & -6x^{-2} & -5x^{-1} & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} x^4 & x^{-2} & x^{-1} & 0 \\ 0 & x^{-2} & \frac{5}{6}x^{-1} & -\frac{1}{6} \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} x^4 & 0 & x^{-1} - \frac{5}{6}x^{-1} & \frac{1}{6} \\ 0 & x^{-2} & \frac{5}{6}x^{-1} & -\frac{1}{6} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & x^{-5}\left(1-\frac{5}{6}\right) & \frac{x^{-4}}{6} \\ 0 & 1 & \frac{5}{6}x & -\frac{x^2}{6} \end{array}\right)$$

$$\boxed{\to U^{-1} = \frac{1}{6}\begin{pmatrix} x^{-5} & x^{-4} \\ 5x & -x^2 \end{pmatrix}}$$

$$v_p(x) = U(x)\int U^{-1}(y)g(y)dy$$

$$v_p = \begin{pmatrix} x^5 & x^{-1} \\ 5x^4 & -x^{-2} \end{pmatrix} \int \frac{1}{6}\begin{pmatrix} y^{-5} & y^{-4} \\ 5y & -y^2 \end{pmatrix} \begin{pmatrix} 0 \\ \log(y) \end{pmatrix} dy$$

$$= \begin{pmatrix} x^5 & x^{-1} \\ 5x^4 & -x^{-2} \end{pmatrix} \frac{1}{6} \int \begin{pmatrix} y^{-4}\log(y) \\ -y^2\log(y) \end{pmatrix} dy \qquad \left[\int f'g = fg - \int fg'\right]$$

$$\int \overset{f}{y^{-4}}\,\overset{g'}{\log y}\, dy = -\frac{y^{-3}}{3}\log y - \int -\frac{y^{-3}}{3}\frac{1}{y}\, dy = -\frac{y^{-3}}{3}\log y + \int \frac{y^{-4}}{3}\, dy$$

$$= -\frac{y^{-3}}{3}\log y - \frac{y^{-3}}{9}$$

$$\int \overset{f}{-y^2}\,\overset{g'}{\log y} = -\frac{y^3}{3}\log y - \int -\frac{y^3}{3}y^{-1}\, dy = -\frac{1}{3}y^3\log y + \frac{1}{6}\frac{y^3}{3}$$

$$= \begin{pmatrix} x^5 & x^{-1} \\ 5x^4 & -x^{-2} \end{pmatrix} \frac{1}{6} \begin{pmatrix} -\frac{1}{3} x^{-3}\left(\log x + \frac{1}{3}\right) \\ -\frac{x^3}{3}\left(\log x - \frac{1}{3}\right) \end{pmatrix}$$

$$= -\frac{1}{18} \begin{pmatrix} x^5 x^{-3}\left(\log x + \frac{1}{3}\right) + x^{-1} x^{3}\left(\log x - \frac{1}{3}\right) \\ 5x^4 x^{-3}\left(\log x + \frac{1}{3}\right) - x^{-2} x^{3}\left(\log x - \frac{1}{3}\right) \end{pmatrix}$$

$$= -\frac{1}{18} \begin{pmatrix} x^2\left(\log x + \frac{1}{3}\right) + x^2\left(\log x - \frac{1}{3}\right) \\ 5x\left(\log x + \frac{1}{3}\right) - x\left(\log x - \frac{1}{3}\right) \end{pmatrix}$$

$$= -\frac{1}{18} \begin{pmatrix} 2x^2 \log x \\ 4x \log x + \frac{6}{3} x \end{pmatrix} = \begin{pmatrix} -\frac{1}{9} x^2 \log x \\ -\frac{2}{9} x \log x - \frac{1}{9} x \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + v_p$$

with $v_1 = u$

$$\boxed{u = c_1 x^5 + c_2 x^{-1} - \frac{1}{9} x^2 \log x}$$

3. (Picard iteration) The Picard-Lindelöf/Cauchy-Lipschitz theorem in fact gives more than what we discussed in the lecture: it also provides a way to approximate solutions of the initial value problem

$$u' = f(t,u)$$
$$u(0) = u_0.$$

This is the so-called *Picard iteration*, defined as

$$\varphi_0(t) = u_0$$
$$\varphi_n(t) = u_0 + \int_0^t f(s, \varphi_{n-1}(s))\, ds \qquad n \geq 1.$$

The produced sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges (provided the conditions in the Picard-Lindelf theorem hold) to the solution u of the ODE.

a) Compute the first 3 elements in the Picard iteration for

$$u'(t) = tu(t), \qquad u(0) = 1.$$

To which function does the sequence converge?

b) Compute the first 4 elements in the Picard iteration for

$$u'(t) = u^2(t), \qquad u(0) = 1.$$

To which function does the sequence seem to converge?

a) $u' = tu \quad , \quad u := u(t) \quad , \quad f(t,u) = tu$

$u(0) = u_0 = 1 \rightarrow \boxed{\varphi_0(t) = 1}$

$\varphi_1(t) = 1 + \int_0^t (s \cdot 1)\, ds = 1 + \frac{s^2}{2}\Big|_0^t = \boxed{1 + \frac{t^2}{2} = \varphi_1(t)}$

$\varphi_2(t) = 1 + \int_0^t \left(s \cdot \left(1 + \frac{s^2}{2}\right)\right) ds$

$= 1 + \int_0^t s\, ds + \int_0^t \frac{s^3}{2}\, ds$

$\boxed{= 1 + \frac{t^2}{2} + \frac{t^4}{8} = \varphi_2(t)}$

$\left[e^x = \sum_{i=0}^{\infty} \frac{x^n}{n!}\right]$

$\left[e^{\frac{t^2}{2}} = 1 + \frac{t^2}{2} + \frac{\frac{t^4}{4}}{2} + \dots\right]$

$\boxed{\text{converges to } e^{\frac{t^2}{2}}}$

b) $u' = u^2(t) \qquad u(0) = u_0 = 1$

$f(t,u) = u^2(t)$

$\boxed{\varphi_0 = 1}$

3. (Picard iteration) The Picard-Lindelöf/Cauchy-Lipschitz theorem in fact gives more than what we discussed in the lecture: it also provides a way to approximate solutions of the initial value problem

$$u' = f(t,u)$$
$$u(0) = u_0.$$

This is the so-called *Picard iteration*, defined as

$$\varphi_0(t) = u_0$$
$$\varphi_n(t) = u_0 + \int_0^t f(s, \varphi_{n-1}(s))\,ds \qquad n \geq 1.$$

The produced sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges (provided the conditions in the Picard-Lindelf theorem hold) to the solution u of the ODE.

$\varphi_1 = 1 + \int_0^t 1\,ds = \boxed{1 + t = \varphi_1(t)}$

$\varphi_2 = 1 + \int_0^t (1+s)^2\,ds = 1 + \int_0^t 1 + 2s + s^2\,ds$

$\boxed{\varphi_2(t) = 1 + t + t^2 + \frac{t^3}{3}}$

$\varphi_3 = 1 + \int_0^t \left(1 + t + t^2 + \frac{t^3}{3}\right)^2$

$\left(1 + t + t^2 + \frac{t^3}{3}\right)\left(1 + t + t^2 + \frac{t^3}{3}\right) =$

$1 + t + t^2 + \frac{t^3}{3} + t + t^2 + t^3 + \frac{t^4}{3} + t^2 + t^3 + t^4 + \frac{t^5}{3}$

$+ \frac{t^3}{3} + \frac{t^4}{3} + \frac{t^5}{3} + \frac{t^6}{9} = 1 + 2t + 3t^2 + \frac{8}{3}t^3 + \frac{5}{3}t^4 + \frac{2}{3}t^5 + \frac{t^6}{9}$

$\varphi_3 = 1 + \int_0^s 1 + 2t + 3t^2 + \frac{8}{3}t^3 + \frac{5}{3}t^4 + \frac{2}{3}t^5 + \frac{t^6}{9}\,ds$ ← s and t should be switched here but too much writing

$\varphi_3 = 1 + t + t^2 + t^3 + \frac{8}{12}t^4 + \frac{5}{15}t^5 + \frac{2}{18}t^6 + \frac{t^7}{63}$

Seems to converge to $\boxed{\sum_{i=0}^{\infty} t^i = \frac{1}{1-t}}$

4. Consider the function

$$f(u) = \begin{cases} 1 & \text{if } u \leq 0 \\ -1, & \text{if } u > 0. \end{cases}$$

a) Show that the initial value problem

$$\begin{aligned} u' &= f(u) \\ u(0) &= 1 \end{aligned}$$

does not have a solution.

b) Check why Peano's theorem doesn't apply.

a) $u' = \begin{cases} 1, & u \leq 0 \\ -1, & u > 0 \end{cases}, \quad u(0) = 1$

$u \leq 0$:

Ansatz: $u = x + c \rightarrow u' = 1$ $\quad u(0) = 1 \rightarrow c = 1$

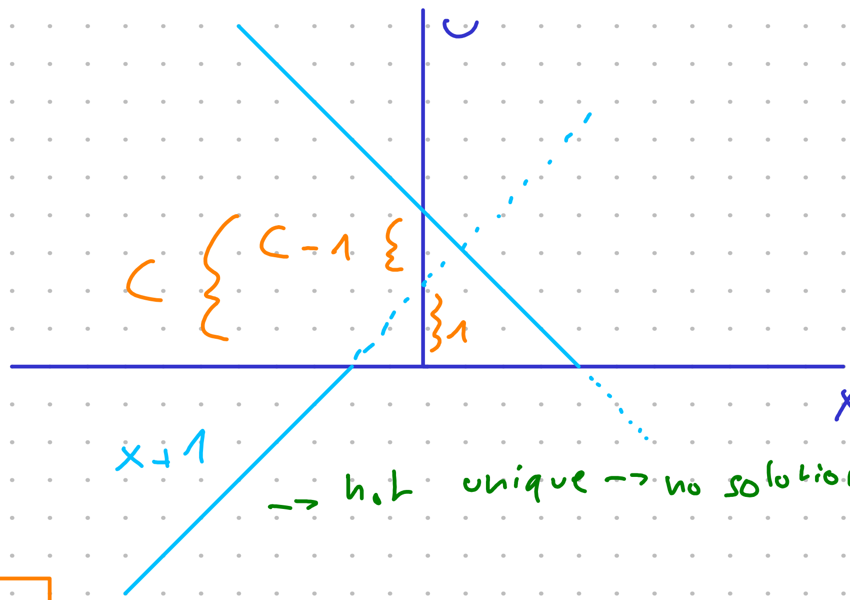
$\rightarrow \underline{u = x + 1}$

$u > 0$:

Ansatz: $\underline{u = -x + c}$

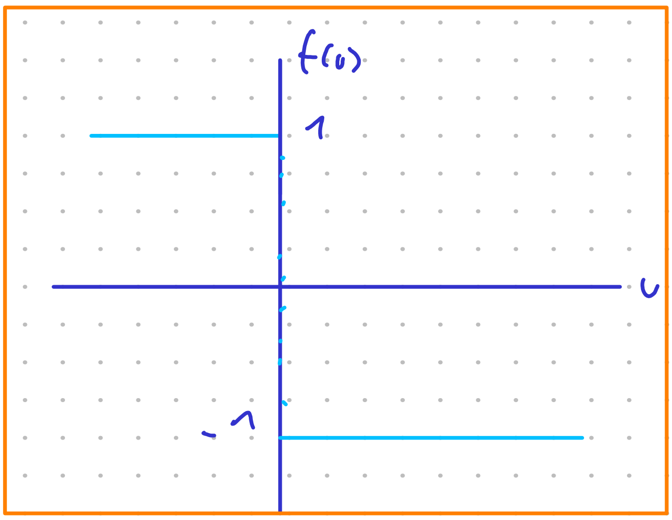
$u' = -1$

$\rightarrow u = \begin{cases} \underline{x + 1, \; u \leq 0} \\ \underline{-x + C, \; u > 0} \end{cases} \rightarrow$



$\rightarrow$ n.t unique $\rightarrow$ no solution

b)



$\boxed{\text{f(u) is not continuous, hence Peano's theorem does not apply.}}$

Continuing with the equation from the previous exercise, let us test some approximation methods.

a) Compute the first 4 elements in the Picard iteration, and see if it looks convergent.

b) Another approximation method for ODEs is the *Euler method*, defined as follows. For each n, we define not a continuous function u, but rather a function φ_n on gridpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots$, inductively as follows:

$$\varphi_n(0) = u_0, \qquad \varphi_n(\tfrac{k+1}{n}) = \varphi(\tfrac{k}{n}) + \frac{f\left(\frac{k}{n}, u(\frac{k}{n})\right)}{n}, \; k \geq 0.$$

Compute the Euler approximation of the ODE from the previous exercise with $n = 2, 4, 6$. Does this look convergent?

$$f(u) = \begin{cases} 1 & \text{if } u \leq 0 \\ -1, & \text{if } u > 0. \end{cases}$$

3. (Picard iteration) The Picard-Lindelöf/Cauchy-Lipschitz theorem in fact gives more than what we discussed in the lecture: it also provides a way to approximate solutions of the initial value problem

$$u' = f(t, u)$$
$$u(0) = u_0.$$

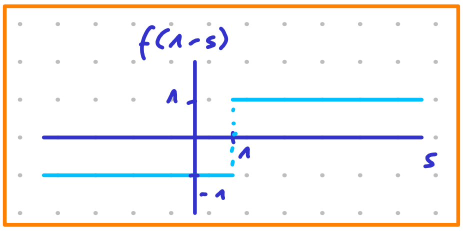
This is the so-called *Picard iteration*, defined as

$$\varphi_0(t) = u_0$$
$$\varphi_n(t) = u_0 + \int_0^t f(s, \varphi_{n-1}(s))\, ds \qquad n \geq 1.$$

The produced sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges (provided the conditions in the Picard-Lindelf theorem hold) to the solution u of the ODE.

$u_0 = 1 = \varphi_0 \to \boxed{\varphi_0 = 1}$

$$\boxed{\varphi_1 = 1 + \int_0^t f(1)\, ds = 1 + \int_0^t -1\, ds = 1 - t}$$



$$\varphi_2 = 1 + \int_0^t f(1-s)\, ds = ?$$

if $t < 1$:

$$\varphi_2 = 1 + t\underbrace{(-1)}_{} = 1 - t$$

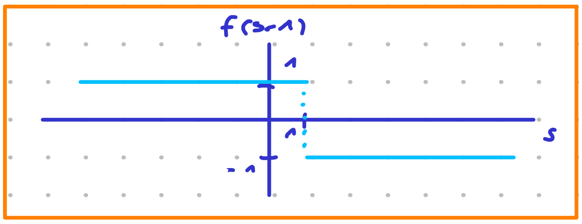
$1 - s > 0 \Rightarrow f(1-s) = -1$

if $t \geq 1$:

$$\varphi_2 = 1 + \int_0^1 f(1-s)\, ds + \int_1^t f(1-s)\, ds$$

$$\varphi_2 = 1 + (-1) + (t-1) = t - 1$$

$$\boxed{\varphi_2 = \begin{cases} 1 - t, & t < 1 \\ t - 1, & t \geq 1 \end{cases}}$$



$$\varphi_3 = \begin{cases} 1 + \int_0^t f(1-s)\, ds, & t < 1 \\ 1 + \int_0^t f(s-1)\, ds, & t \geq 1 \end{cases} = \begin{cases} 1 - t, & t < 1 \\ ? \end{cases}$$

if $t \geq 1$:

$$1 + \int_0^1 f(s-1) + \int_1^t f(s-1) = 1 + 1 + \left(\int_1^t -1\, ds \right)$$

$$= 2 + (-(t-1)) = 3 - t$$

$$\varphi_3 = \begin{cases} 1-t, & t<1 \\ 3-t, & t\geq 1 \end{cases}$$

Does not look convergent for $t \geq 1$ (?)
For $t<1$: $1-t$

b) Another approximation method for ODEs is the *Euler method*, defined as follows. For each n, we define not a continuous function u, but rather a function φ_n on gridpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots$, inductively as follows:

$$\varphi_n(0) = u_0, \qquad \varphi_n(\tfrac{k+1}{n}) = \qquad k \geq 0. : \varphi_n\left(\frac{k}{n}\right) + \frac{f\left(\frac{k}{n}, \varphi_n\left(\frac{k}{n}\right)\right)}{n}$$

Compute the Euler approximation of the ODE from the previous exercise with $n = 2, 4, 6$. Does this look convergent?

$$\varphi_2 = \varphi_4 = \varphi_6 = u_0 = 1 \qquad f(u) = \begin{cases} 1 & \text{if } u \leq 0 \\ -1, & \text{if } u > 0. \end{cases}$$

$n=2$

$$\varphi_2\left(\frac{1}{2}\right) = \varphi_2(0) + \frac{f(0, \varphi_2(0))}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\varphi_2(1) = \varphi_2\left(\frac{1}{2}\right) + \frac{f\left(\frac{1}{2}, \varphi_2\left(\frac{1}{2}\right)\right)}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\varphi_2\left(\frac{3}{2}\right) = \varphi_2(1) + \frac{f(1, \varphi_2(1))}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

$$\varphi_2(2) = \varphi_2\left(\frac{3}{2}\right) + \frac{f\left(\frac{3}{2}, \varphi_2\left(\frac{3}{2}\right)\right)}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

...

$t \geq 1$: alternating between 0 and $1/2$

$n=4$:

$$\varphi_4\left(\tfrac{1}{4}\right) = \varphi_4(0) + \frac{f(0,\varphi_4(0))}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\varphi_4\left(\tfrac{1}{2}\right) = \varphi_4\left(\tfrac{1}{4}\right) + \frac{f(\frac{1}{4},\varphi_4(\frac{1}{4}))}{4} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\varphi_4\left(\tfrac{3}{4}\right) = \ldots = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\varphi_4(1) = \frac{1}{4} - \frac{1}{4} = 0$$

$$\varphi_5\left(\tfrac{5}{4}\right) = 0 + \frac{f(0,\varphi_4(0))}{4} = 0 + \frac{1}{4} = \frac{1}{4}$$

$\boxed{t \geq 1\text{: alternating between } 0 \text{ and } \tfrac{1}{4}}$

$n=6$:

$$\varphi_6\left(\tfrac{1}{6}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\varphi_6\left(\tfrac{2}{6}\right) = \frac{5}{6} - \frac{1}{6} = \frac{4}{6}$$

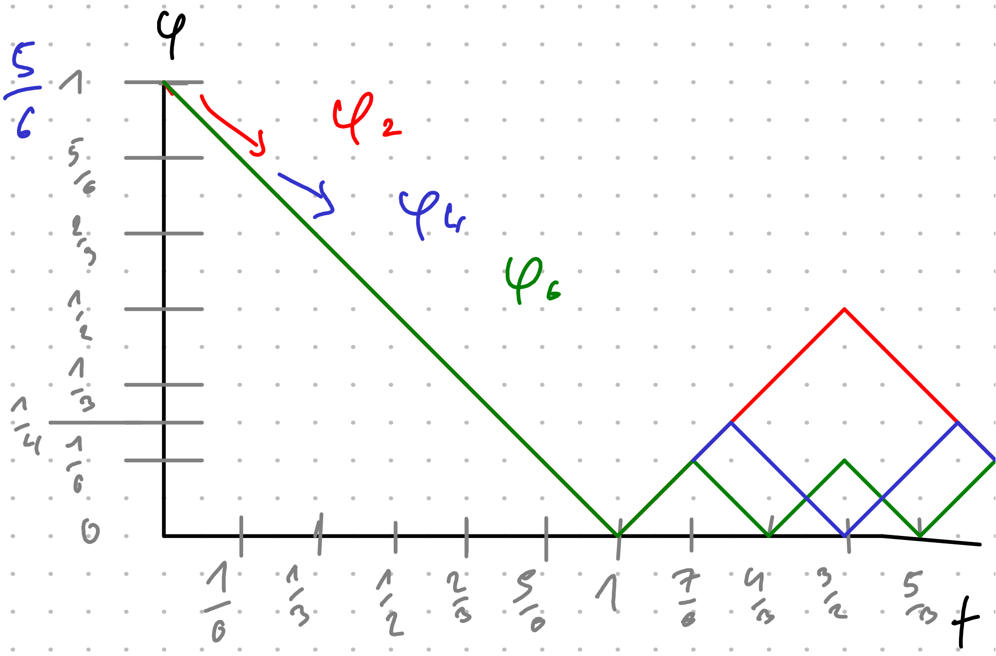
$$\varphi_6\left(\tfrac{3}{6}\right) = \frac{3}{6}$$

$$\varphi_6\left(\tfrac{4}{6}\right) = \frac{2}{6}$$

$$\varphi_6\left(\tfrac{5}{6}\right) = \frac{1}{6}$$

$$\varphi_6(1) = 0$$

$$\varphi_6\left(\tfrac{7}{6}\right) = \frac{1}{6}$$



$\boxed{t \geq 1\text{: alternating between } 0 \text{ and } \tfrac{1}{6}}$

Convergence:

$\boxed{\rightarrow 1 - t \text{ for } t < 1 \text{ and } 0 \text{ for } t \geq 1}$