AMF UE2 $\{1, x, x^2, x^3\}$

1) $\alpha_0 1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = 0$

-> all coeff. must be 0!

$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ -> lin. indep.

$w_1 = 1$

$v_1 = \frac{1}{\|1\|}$, inner prod. indues norm -> $\|1\| = \sqrt{\int_0^1 1 \cdot 1 \, dx}$
$= 1$

$= \underline{1}$

$w_2 = x - (1, x) 1$, $(1, x) = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$= x - \frac{1}{2}$ = $x - \frac{1}{2}$ $\|\sqrt{(w_2, w_2)}\| = \int_0^1 x^2 - x + \frac{1}{4} \, dx$

$= \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{4}{12} - \frac{3}{12} = \sqrt{\frac{1}{12}}$

$v_2 = \frac{w_2}{\|w_2\|} = (x - \frac{1}{2})\sqrt{12}$

$= (x - \frac{1}{2})\sqrt{4 \cdot 3} = (2x - 1)\sqrt{3}$

$w_3 = x^2 - \underbrace{(1, x^2)}_{\frac{1}{3}} 1 - ((2x-1)\sqrt{3}, x^2)(2x-1)\sqrt{3}$

$= x^2 - \frac{1}{3} - \frac{1}{2}\frac{\sqrt{3}}{\sqrt{3}}(2x - 1)$ | $\sqrt{3}\int_0^1 2x^3 - x^2 \, dx = \sqrt{3}\left(\frac{2x^4}{4} - \frac{x^3}{3}\right)_0$

$= \sqrt{3}\left(\frac{1}{2} - \frac{1}{3}\right)$

$= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$

$= \sqrt{3} \, \frac{1}{6} = \sqrt{\frac{3}{36}}$

$= \sqrt{\frac{1}{12}}$

$= \frac{1}{2}\frac{1}{\sqrt{3}}$

$v_3 = \frac{w_3}{\|w_3\|}$; $\|w_3\| = \sqrt{\int_0^1 (x^2 - x + \frac{1}{6})(x^2 - x + \frac{1}{6}) \, dx}$

$= \int_0^1 x^4 - x^3 + \frac{x^2}{6} - x^3 + x^2 - \frac{x}{6} + \frac{x^2}{6} - \frac{x}{6} + \frac{1}{36} \, dx$

$= \int x^4 - 2x^3 + \frac{4x^2}{3} - \frac{x}{3} + \frac{1}{36} \, dx$

$= \frac{x^5}{5} - \frac{2x^4}{4} + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\Big|_0^1 = \frac{1}{5} - \frac{1}{2} + \frac{2}{3} - \frac{1}{6} + \frac{1}{36}$

WA said: $\sqrt{\frac{1}{180}} = \frac{\sqrt{5}}{30}$

$v_3 = (x^2 - x + \frac{1}{6})\frac{30}{\sqrt{5}} = (6x^2 - 6x + 1)\sqrt{5}$

$$w_4 = x^3 - (1, x^3)1 - (\sqrt{3}(2x-1), x^3)\sqrt{3}(2x-1)$$
$$- (\sqrt{5}(6x^2-6x+1), x^3)\sqrt{5}(6x^2-6x+1)$$

-> wolfram ...

$$= x^3 - \frac{1}{4} - 3 \cdot \frac{3}{20}(2x-1)$$
$$- 5 \frac{1}{20}(6x^2-6x+1)$$

$$= x^3 - \frac{1}{4} - \frac{9}{20}(2x+1) - \frac{1}{4}(6x^2-6x+1)$$
$$= x^3 - \frac{1}{4} - \frac{18x}{20} - \frac{9}{20} - \frac{6x^2}{4} - \frac{6x}{4} - \frac{1}{4}$$
$$= x^3 - \frac{1}{4} - \frac{9x}{10} - \frac{9}{20} - \frac{3}{2}x^2 - \frac{3}{2}x - \frac{1}{4}$$
$$= x^3 - \frac{19}{20} - \frac{24x}{10} - \frac{3}{2}x^2$$
$$= x^3 - \frac{3}{2}x^2 - \frac{12}{5}x - \frac{19}{20}$$

$$v_4 = \frac{w_4}{\|w_4\|}, \quad \|w_4\| = \sqrt{\frac{1}{2800}}$$

$$v_4 = \frac{140}{\sqrt{7}}\left(x^3 - \frac{1}{4} - \frac{9}{20}(2x-1) - \frac{1}{4}(6x^2-6x+1)\right)$$

$$= \sqrt{7}(20x^3 - 30x^2 + 12x - 1)$$

2) Approx. $\sin(\pi x)$

$$f = \sum_{i=1}^{4} (f, v_i) v_i = \sum_{i=1}^{4} f_i$$

$v_1 = 1$

$v_2 = \sqrt{3}(2x-1)$

$v_3 = \sqrt{5}(6x^2-6x+1)$

$v_4 = \sqrt{7}(20x^3-30x^2+12x-1)$

$$f_1 = \int_0^1 f(x) \cdot 1 \, dx = -\cos(\pi x)\Big|_0^1 = - (-1) = \frac{2}{\pi}$$

$$f_2 = 3 \underbrace{\int_0^1 \left(2\sin(\pi x)x - \sin(\pi x)\right) dx}_{} \; (2x-1)$$

$$\int_0^1 2 \sin(\pi x)\pi = -x\cos(\pi x) - \int -\cos(\pi x) = \sin \pi x \Big|_0^1 = 0$$

$$= -x \cos \pi x$$

$$f_2 = 3\left(2(-x\cos \pi x) + 2\right)(2x-1)$$

... WOLFRAM

$$f = \underset{f_1}{\frac{2}{\pi}} + \underset{f_2}{0} + \underset{f_3}{\frac{10(\pi^2-12)(6x^2-6x+1)}{\pi^3}} + \underset{f_4}{0}$$

$$f = \frac{2}{\pi} + \frac{10(\pi^2-12)(6x^2-6x+1)}{\pi^3}$$

3) a) $f(x,y) = x^2 y$

$$\frac{\partial f}{\partial y} = x^2 \qquad \frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x$$

$$\frac{\partial^2 f}{\partial y^2}\ 0 \qquad \frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x$$

$\boxed{\text{CS} \\ \checkmark}$

b) $f(x,y) = \sin\left(\frac{x}{y}\right)$

$$\frac{\partial f}{\partial y} = -\frac{x}{y^2}\cos\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = 2\frac{x}{y^3}\cos\left(\frac{x}{y}\right) + \left(\frac{x}{y^2}\ \cancel{\cdot}\ \sin\left(\frac{x}{y}\right) = \frac{2x}{y^3}\cos\left(\frac{x}{y}\right) - \frac{x}{y^2}\sin\left(\frac{x}{y}\right)$$

$$\frac{\partial f}{\partial x} = \frac{1}{y}\cos\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \cdot -\frac{1}{y^2}\sin\left(\frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\frac{1}{y^2}\cos\left(\frac{x}{y}\right) + \left(\frac{x}{y^3}\sin\left(\frac{x}{y}\right)\right.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{y^2}\cos\left(\frac{x}{y}\right) + \left(\frac{1}{y^2}\sin\left(\frac{x}{y}\right)\right.$$

$\boxed{\text{CS} \\ \checkmark}$

c) $f(x,y,z) = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} - \frac{x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}2x, \quad \frac{\partial^2 f}{\partial x^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}2y, \quad \frac{\partial f}{\partial y^2} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}2z \quad \frac{\partial f}{\partial z^2} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial x \partial y} = -\frac{xy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial x \partial z} = \frac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}}; \quad \frac{\partial f}{\partial y \partial z} = \frac{-yz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

4)

$$f(x,y) = \begin{cases} x^3(x^2+y^2)^{-1} & (x,y) \neq (0,0) \\ 0 & \text{sonst} \end{cases}$$

$$\frac{\partial f}{\partial x} = 3x^2(x^2+y^2)^{-1} - x^2(x^2+y^2)^{-2}\,2x$$

$$= \frac{3x^2}{(x^2+y^2)} - \frac{2x^3}{(x^2+y^2)^2}$$

$$= \frac{3x^2(x^2+y^2) - 2x^3}{(x^2+y^2)^2} = \frac{3x^4+3x^2y^2-2x^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = -x^3(x^2+y^2)^{-2}\,2y$$

$$= -\frac{2x^3y}{(x^2+y^2)^2}$$

at $(0,0)$ $\quad \frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$, $v = \binom{1}{1}$

$$\frac{\partial f}{\partial v}(0,0) = \lim_{h\to 0} \frac{f(0+h,0+h) - f(0,0)}{h} = \frac{h^3}{2h^2}\frac{1}{h} = \underline{\underline{\frac{1}{2}}}$$

$$\partial_v f \overset{?}{=} \nabla f \cdot v$$

$$\frac{1}{2} = 0\cdot 1 + 0 \cdot 1$$

$\frac{1}{2} = 0$ ↯ Not tot. diffable because not continuous in $0,0$.

5) Taylor-Approx of $f(x) = \sin(\pi x)$ at $a=0$

$$f(x;0) = \sin(0) + \cos(0)(x-0) - \frac{\sin(0)}{2}(x-0)^2 - \frac{\cos(0)}{6}(x-0)^3$$

$$= 0 + x - 0 - \frac{1}{6}x^3$$

Approx of $f(x,y) = e^{x+y}$ at $(a,b) = (1,1)$

$$f(x,y;1,1) = e^2 + (x-1)e^{x+y} + (y-1)e^{x+y} +$$
$$\frac{1}{2}\left((x-1)^2 e^{x+y} + 2(x-1)(y-1)e^{x+y} + (y-1)e^{x+y} \right)$$

6)a) $f(x) = \sqrt[4]{x}$ $\quad f(x;a) \approx f(a) + (x-a)\frac{1}{4}x^{-\frac{3}{4}}$

$f(17) = \sqrt[4]{17} \to a = 16$ | $\leftarrow$ mit $a = 17$ $\to P(17) = f(17)$

$$f(16) + (x-16)\frac{1}{4}x^{-\frac{3}{4}}$$
$$\sqrt[4]{16} + \frac{1}{4}(x-16)x^{-\frac{3}{4}} \to P(17) = 2{,}02986$$
$$17^{\frac{1}{4}} = 2{,}0305$$

Approx $26^{\frac{1}{3}}$

$f(x) = \sqrt[3]{x}$ $\quad f(x;a) \approx f(a) + (x-a)\frac{1}{3}x^{-\frac{2}{3}}$

$f(26) = \sqrt[3]{26} \to a = 25$

$$f(25) + (x-25)\frac{1}{3}x^{-\frac{2}{3}}$$
$$\sqrt[3]{25} + (x-25)\frac{1}{3}x^{-\frac{2}{3}} \to P(26) = 2{,}961998$$
$$26^{\frac{1}{3}} = 2{,}96249$$