Applied Mathematics Foundations, exercise **WS 2022** **M. Gerencsér, A. Miraci**

Sheet 1

Discussion of the sheet: Tue./Wed.,

This exercise sheet is concerned with the topic

- Vector spaces

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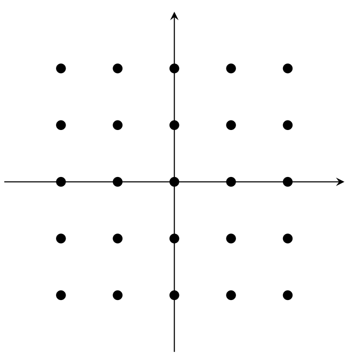
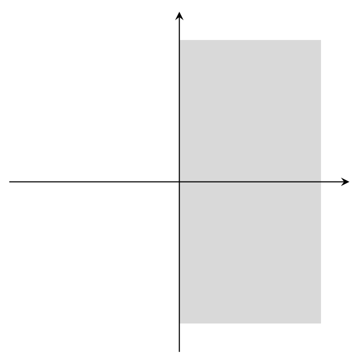
1. Show that the set $\mathbb{S} := \{B \in \mathbb{R}^{2\times 2} \,:\, B \text{ symmetric } \}$ of all real symmetric $\mathbb{R}^{2\times 2}$-matrices is a vector space. Find 3 linearly independent vectors in $\mathbb{S}$. Can you find a fourth one?

Remark: A matrix $B \in \mathbb{R}^{2\times 2}$ is symmetric if $B = B^T$, where B^T is the transposed matrix.

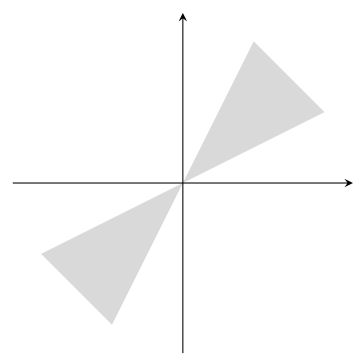
2. Show that none of the three sets below are vector spaces.



$\{x \in \mathbb{R}^2 : x_1 \text{ and } x_2 \text{ are integers}\}$



$\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \in \mathbb{R}\}$



$\{x \in \mathbb{R}^2 : \frac{x_1}{x_2} \in [\frac{1}{2}, 2]\} \cup \{(0,0)\}$

3. Show that the following mappings are norms on $\mathbb{R}^n$:

a) $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$.

b) $\|x\|_1 := |x_1| + \dots + |x_n| = \sum_{i=1}^n |x_i|$.

Draw the sets $\{x \in \mathbb{R}^2 \,:\, \|x\|_1 \leq 1\}$, $\{x \in \mathbb{R}^2 \,:\, \|x\|_2 \leq 1\}$, and $\{x \in \mathbb{R}^2 \,:\, \|x\|_\infty \leq 1\}$.

4. Lorentz-Minkowski space: In relativistic physics a way to describe space-time is to actually use $\mathbb{R}^n$ ($n = 1, 2, 3$) in space and $\mathbb{R}$ in time, e.g. the space $\mathbb{R}^{n+1}$ with a special 'inner'-product, the so-called Minkowski product. For $u = (t, x_1, \dots, x_n)$ and $v = (s, y_1, \dots, y_n)$ this is defined as

$$(u, v) = c^2 ts - x_1 y_1 - \dots - x_n y_n,$$

where $c > 0$ is the speed of light.

Even though it is often called a scalar product, not all three axioms in the definition of an inner product hold. Check which ones do or do not hold!

Vectors u with $(u,u) > 0$ are called **timelike**, vectors with $(u,u) < 0$ are called **spacelike** and vectors with $(u,u) = 0$ are called **lightlike**.

Take $n = 1$ and $n = 2$ and draw the sets of timelike, spacelike and lightlike vectors.

5. Let $\big(V, (\cdot,\cdot)\big)$ be an inner product space and let $\{v_1, \ldots, v_n\} \subset V$ be an orthonormal set. Suppose that $w \in V$ is a linear combination of the v_i-s, that is, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = w.$$

Show that one can compute the coefficients by the formula $\alpha_i = (w, v_i)$. (Hint: what happens when taking the inner product with v_i of both sides of the equality?) From this formula, prove the statement from the lecture: *"Every orthonormal set of vectors is also linearly independent."*

6. Take the vectors

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \; v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \quad w = \begin{pmatrix}1\\2\end{pmatrix}.$$

Show that $\{v_1, v_2\}$ is and orthonormal set in $\mathbb{R}^2$ and compute the coefficients α_1, α_2 from the previous exercise.