8.1. **a)** Prove that the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

does not have a factorization $\mathbf{A} = \mathbf{LU}$ with normalized lower triangular matrix $\mathbf{L}$ and upper triangular matrix $\mathbf{U}$.

b) Let $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

where the off-diagonal 1 are in the positions (i_1, i_2) and (i_2, i_1) (with $i_1 \neq i_2$). Show: The matrix $\mathbf{PA}$ is the matrix $\mathbf{A}$ with rows i_1 and i_2 interchanged. Furthermore, $\mathbf{P}^{-1} = \mathbf{P}^\top = \mathbf{P}$.

a) $L = \begin{pmatrix} 1 & 0 \\ l_{10} & 1 \end{pmatrix}$, $U = \begin{pmatrix} u_{00} & u_{01} \\ 0 & u_{11} \end{pmatrix}$

$u_{00} = a_{00} = 0$

$u_{01} = a_{01} = 1$

$$\boxed{0 \cdot l_{10} = 3 \quad ↯}$$

b) $(PA)_{ij} = \sum_{k=0}^{n-1} P_{ik} A_{kj}$

look at k column where $P_{ij} \neq 0$

$\Rightarrow (PA)_{ij} = P_{ik} A_{kj}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 16 & 17 & 18 & 19 & 20 \\ 11 & 12 & 13 & 14 & 15 \\ 6 & 7 & 8 & 9 & 10 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$$

$(PA)_{10} = P_{14} \cdot A_{40}$

$(PA)_{11} = P_{14} \cdot A_{41}$

...

$$\boxed{\begin{aligned} (PA)_{i_1,j} &= \underbrace{P_{i_1,i_2}}_{1} A_{i_2,j} = A_{i_2 j} \quad ✓ \\ (PA)_{i_2,j} &= \underbrace{P_{i_2,i_1}}_{1} A_{i_1 j} = A_{i_1 j} \quad ✓ \end{aligned}}$$

$P^{-1} \stackrel{?}{=} P$

$$\left(\begin{array}{ccccc|ccccc} 1&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1 \end{array}\right)\begin{array}{l} \\ \leftarrow r_{i_1} \\ \\ \leftarrow r_{i_2} \\ \\ \end{array}$$

$\boxed{\text{Inverse can be computed by row operation} \quad r_{i_1} \Longleftrightarrow r_{i_2}}$

general: $\left(\begin{array}{ccc|c} \mathbb{1}_{n_1} & & \\ 0 & \cdots & 1 & \\ \vdots & \mathbb{1}_{n_2} & \vdots & \mathbb{1}_{\Sigma(n_i)+2} \\ 1 & \cdots & 0 & \\ & & \mathbb{1}_{n_3} & \end{array}\right)$

$r_{i_1} \Longleftrightarrow r_{i_2}$ holds ✓

$P^T \stackrel{?}{=} P$

Def: $(P^T)_{ij} = P_{ji}$

$i \neq i_1 \land i \neq i_2$

$(P^T)_{ij} = P_{ji} = \delta_{ji} \rightarrow$ $\boxed{\begin{array}{l} P_{ii} = 1,\ P_{ij} = 0,\ P_{ji} = 0 \\ (P^T)_{ii} = 1,\ (P^T)_{ij} = 0,\ (P^T)_{ji} = 0 \end{array}}$

$i = i_1 \lor i = i_2$:

$(P^T)_{i_1 j} = P_{j i_1} \rightarrow P_{j i_1} = \delta_{j i_2}$ $\boxed{\begin{array}{l} \rightarrow P_{i_2 i_1} = 1,\ \text{rest } 0 \\ \hookrightarrow (P^T)_{i_1 i_2} = 1,\ \text{rest } 0 \end{array}}$

$(P^T)_{i_2 j} = P_{j i_2} \rightarrow P_{j i_2} = \delta_{j i_1}$ $\boxed{\begin{array}{l} \rightarrow P_{i_1 i_2} = 1,\ \text{rest } 0 \\ \hookrightarrow (P^T)_{i_2 i_1} = 1,\ \text{rest } 0 \end{array}}$

$\rightarrow P^T = P$ ✓

8.2. The lengths of the 3 edges (meeting at a corner) and the circumferences (orthogonal to the first and second edge) of a brick are measured. The measured values are:

edge 1: $26mm$, edge 2: $38mm$, edge 3: $55mm$
circumference $\perp$ edge 1: $188mm$, circumference $\perp$ edge 2: $163mm$.

Determine the edge lengths using the method of least squares.

$$A \, x = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \text{edge 1} \\ \text{edge 2} \\ \text{edge 3} \end{pmatrix} = \begin{pmatrix} 26\,mm \\ 38\,mm \\ 55\,mm \\ 188\,mm \\ 163\,mm \end{pmatrix}$$

Hence, $\mathbf{x}$ satisfies the *normal equations*

$$\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b} \quad \Big| \cdot \to (A^T A)^{-1}$$

$$x = (A^T A)^{-1} A^T b$$

1) Transpose A
2) Calculate $A^T A$
3) Invert $A^T A$
4) Calculate $A^T b$
5) Calculate $(A^T A)^{-1} A^T b$

8.3. The function $f(x) = \sin x$ is to be approximated by a polynomial of the form $\pi(x) = a_1 x + a_3 x^3$. To this end, the coefficients a_1, a_3 are determined using the least squares method by minimizing $\sum_{j=0}^{m}(\pi(x_j) - f(x_j))^2$, where $x_0, \ldots, x_m$ are given points.

Set up the least squares problem for a_1 and a_3. Write a program that computes the coefficients a_1, a_3 for the following 9 choices of knots x_j: the x_j are N randomly chosen points in the interval $[-1/N, 1/N]$ for $N = 2^n$, $n = 2, \ldots, 10$. Do the values a_1, a_3 converge to a limit as $N \to \infty$? Which limit do you expect?

From Taylor series we know for

$$\sin(x) = x - \frac{x^3}{6} + \ldots$$

1) Calculate random x_j on the Interval $[-\frac{1}{N}, \frac{1}{N}]$, $N = 2^n, n = 2, \ldots, 10$

$$\begin{pmatrix} x_j & x_j^3 \\ \cdots & \\ & \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sin(x_j) \\ \cdots \\ \end{pmatrix}$$

Hence, $\mathbf{x}$ satisfies the *normal equations*

$$\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b} \quad \Big| \cdot \to (A^T A)^{-1}$$

$$x = (A^T A)^{-1} A^T b$$

1) Transpose A
2) Calculate $A^T A$
3) Invert $A^T A$
4) Calculate $A^T b$
5) Calculate $(A^T A)^{-1} A^T b$

a_1 converges to 1

a_3 converges to $-\frac{1}{6}$

8.4. The least squares method can also be used to fit the parameters of certain nonlinear problems. How would you determine the parameters C, k to fit given data (t_i, y_i), $i = 1, \dots, N$, to the law $y(t) = Ce^{-kt}$? How do you proceed to determine C, α for the law $y(t) = Ct^{\alpha}$?

$$y(t) = Ce^{-kt}$$

Use transformation to linearize: $\mid \ln$

$$\ln(y(t)) = \ln C - kt$$

$$\left.\begin{aligned} \ln C - kt_1 &= \ln(y_1) \\ &\vdots \\ \ln C - kt_N &= \ln(y_N) \end{aligned}\right\} \begin{pmatrix} 1 & -t_1 \\ \dots \\ 1 & -t_N \end{pmatrix} \begin{pmatrix} \ln C \\ k \end{pmatrix} = \begin{pmatrix} \ln(y_1) \\ \dots \\ \ln(y_N) \end{pmatrix}$$

We can solve for $\underbrace{\ln(C)}_{\Rightarrow s}$, k using least squares

and get C using

$$\ln C = s \quad \mid e$$

$$C = e^{s}$$

$$y(t) = Ct^{\alpha}$$

same procedure: $\ln(y(t)) = \ln C + \alpha \ln(t)$

$$\ln C + \alpha \ln t_i = \ln(y_i), \quad i = 1, \dots N$$

$$\begin{pmatrix} 1 & + \ln t_1 \\ \dots \\ 1 & + \ln t_N \end{pmatrix} \begin{pmatrix} \ln C \\ \alpha \end{pmatrix} = \begin{pmatrix} \ln(y_1) \\ \dots \\ \ln(y_N) \end{pmatrix} \rightarrow \text{Solve for } \underbrace{\ln C}_{s}, \alpha \text{ and}$$

$$\ln C = s \quad \mid e$$

$$C = e^{s}$$

8.5. Let $\mathbf{Q}$ be an orthogonal matrix. Show:

a) $\mathbf{x}^\top\mathbf{y} = ((\mathbf{Q})\mathbf{x})^\top(\mathbf{Qy})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

b) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m > n$ and its QR-factorization $\mathbf{A} = \mathbf{QR}$. Show: If $\mathbf{A}$ has full rank (i.e., $\text{rank}(\mathbf{A}) = n$), then the diagonal entries of $\mathbf{R}$ are non-zero.

a) $x^T y = (Qx)^T (Qy)$ $\quad$ [$(AB)^T = B^T A^T$]

$$= x^T \underbrace{Q^T Q}_{1} y$$

[since Q is orthogonal $Q^{-1} = Q^T$ $(Q^{-1}Q = \mathbb{1})$ and thus $Q^T Q = I$]

$$\boxed{= x^T y}$$

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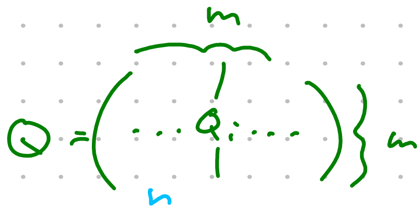
b) $R \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{m\times m}$ [$Q \in \mathcal{O}_m$]



$A = QR \rightarrow \text{rank}(A) = \text{rank}(QR)$

$\text{rank}\, Q = m$

$\boxed{\text{if } R_{ii} \neq 0 \rightarrow \text{rank}\, R = n}$, else $\text{rank}\, R < n$

Since $m \geq n$ and

[The rank of the product of two matrices equals the lower of the two ranks.]

$\rightarrow \text{rank}(QR) = \text{rank}\, R \leq n$

Thus, since $\text{rank}\, A = n \rightarrow \underbrace{\text{rank}(QR) = n = \text{rank}\, R}$

R_{ii} must be non-zero.