Sheet 6

Discussion of the sheet: Tue., 02.05.2023

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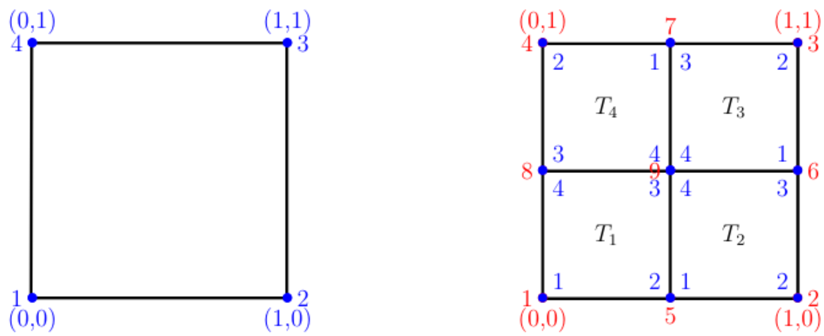
1. We now assemble a FEM system on four quadrilateral elements on $\Omega = [0, 1]^2$ by hand.



Take quadrilateral Lagrangian finite elements on the reference element $\Omega = [0, 1]^2$ as depicted on the left-hand side. Take the nodal basis of Sheet 5/Example 3b.

Moreover, let $\mathcal{T}_h$ be a mesh consisting of 4 quadrilateral elements T_1, T_2, T_3, T_4 depicted on the right-hand side that induces a global FEM space $V_{\mathcal{T}}$. The global numbering of the FEM basis $\{\varphi_i \,:\, i = 1, \dots, 9\}$ is depicted in red, the local numberings on each element in blue.

a) Compute the nodal basis functions $\varphi_1, \varphi_2, \varphi_5, \varphi_6, \varphi_9$.

b) Determine the affine linear map $F_2 : \widehat{T} \to T_2$ that maps $F_2(0, 0) = v_5$, $F_2(1, 0) = v_2$, $F_2(1, 1) = v_6$, $F_2(0, 1) = v_9$ (here v_i denote the vertices of the mesh numbered as depicted). Moreover, compute DF_2 and $\det(DF_2)$.

c) Compute $\int_\Omega \nabla\varphi_9 \cdot \nabla\varphi_9 dx$.

2. We continue with the previous setting. Let

$$a(u, v) := \int_\Omega 6\nabla u \cdot \nabla v \; dx \qquad l(v) := \int_\Omega 16 v dx$$

and consider the problem: find $u_h \in V_{\mathcal{T}}$ such that $a(u_h, v_h) = l(v_h)$ for all $v_h \in V_{\mathcal{T}}$.

a) Compute the element stiffness matrix and the element load vector for the element T_2 of the triangulation. Note: here, all element stiffness matrices are the same!

b) Compute the connectivity matrix C_{T_2} for the element T_2.

c) Solve the problem with homogeneous Dirichlet boundary conditions (this leads to a 1×1-"matrix").

3. We now aim to show that 1D-Lagrange finite elements reproduce exact nodal values. Let $\Omega = (0,1)$ and $\mathcal{T}_h$ a mesh of intervals with nodes $\{x_i \;:\; i = 1, \ldots, n\}$. Consider the 1D Poisson equation in the weak form, i.e., seek $u \in H_0^1(\Omega)$ such that

$$\int_\Omega u'v' \, dx = \int_\Omega fv \, dx.$$

Denote the finite element approximation with P^1-Lagrangian finite elements by u_h.

a) Define $G_i(x) = \begin{cases} (1-x_i)x & 0 \leq x \leq x_i \\ x_i(1-x) & x_i \leq x \leq 1 \end{cases}$ and show that for any $w \in H_0^1(\Omega)$ there holds

$$\int_\Omega G_i'(x) w'(x) \, dx = w(x_i).$$

b) Use this to show that $u(x_i) = u_h(x_i)$ by inserting the difference $u - u_h$ into the weak formulation and using G_i as test function (Galerkin orthogonality).

4. Let $T \subset \mathbb{R}^2$ be a triangle. Construct the so called polynomial bubble function $b_T : T \to \mathbb{R}$. This function is characterized by the conditions

- $0 \leq b_T \leq 1$, $\max b_T = 1$,
- $b_T \in P^k(T)$ for some $k \in \mathbb{N}$.
- $b_T = 0$ on ∂T.

What is the minimal polynomial degree k that is possible?

5. Write a code that evaluates the Legendre polynomials up to a fixed degree k at a point $x \in [-1,1]$ (i.e. a function `evalLegendre(x,k)`) by using the recursion formula

$$nL_n(x) = (2n-1)xL_{n-1}(x) - (n-1)L_{n-2}(x) \qquad 2 \leq n \leq k$$

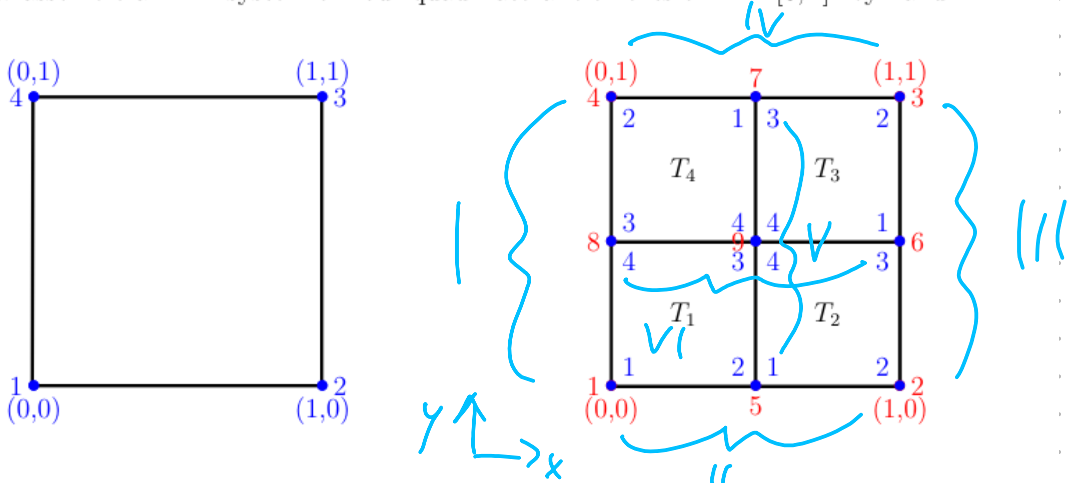
with $L_0 = 1$ and $L_1 = x$. Do the same for the integrated Legendre polynomials N_i. Plot both polynomials!

Moreover, write a code that computes

$$\int_{-1}^{1} f(x) N_i(x) dx$$

using Gaussian quadrature (e.g. use the provided routine `gauleg(m)` that provides m quadrature points and weights on $[-1,1]$). Test you code for different functions f, i, and different quadrature orders m.

1. We now assemble a FEM system on four quadrilateral elements on $\Omega = [0,1]^2$ by hand.



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c) Compute $\int_\Omega \nabla\varphi_9 \cdot \nabla\varphi_9 dx$.

5.3b) $N_1 = (1-x)(1-y)$, $N_2 = x(1-y)$, $N_3 = y(1-x)$, $N_4 = xy$

line I: $x = 0 \to x = 0$

line II: $y = 0 \to y = 0$

line III: $x = 1 \to x - 1 = 0$

line IV: $y = 1 \to y - 1 = 0$

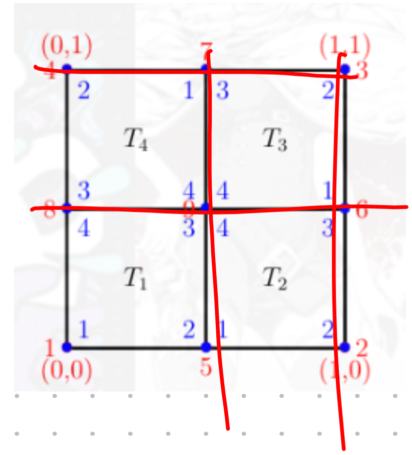
line V: $x = \frac{1}{2} \to x - \frac{1}{2} = 0$

line VI: $y = \frac{1}{2} \to y - \frac{1}{2} = 0$

φ_1: $\varphi_1(0,0) = 1$, $\varphi_1(0, \frac{1}{2}) = 0$, $\varphi_1(\frac{1}{2}, 0) = 0$, $\varphi_1(\frac{1}{2}, \frac{1}{2}) = 0$

Combine lines to calculate φ! :

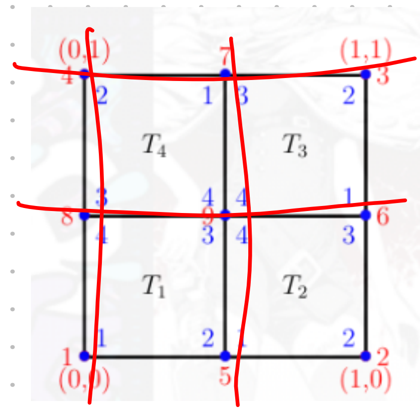
$$\varphi_1 = (y-1)(x-\tfrac{1}{2})(x-1)(y-\tfrac{1}{2})\cdot C$$



$$\varphi_1(0,0) = 1 \to \tfrac{1}{4}\cdot C = 1 \to C = 4$$

$$\boxed{\to \varphi_1 = 4(y-1)(x-\tfrac{1}{2})(x-1)(y-\tfrac{1}{2})}$$

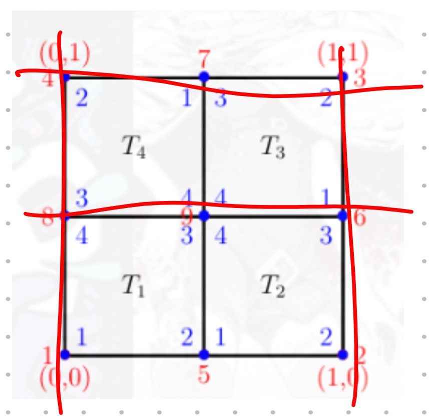
$$\varphi_2 = x(y-1)(x-\tfrac{1}{2})(y-\tfrac{1}{2})\cdot C$$



$$\varphi_2(1,0) = 1 \to \tfrac{1}{4} C = 1 \to C = 4$$

$$\boxed{\varphi_2 = x(y-1)(x-\tfrac{1}{2})(y-\tfrac{1}{2})\cdot 4}$$

$$\varphi_5 = x(x-1)(y-1)(y-\tfrac{1}{2})\cdot C$$

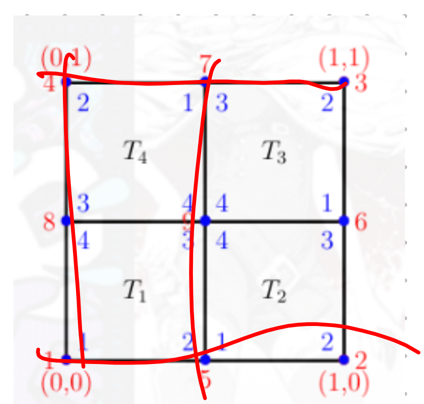


$$\varphi_5(\tfrac{1}{2},0) = 1 \to \tfrac{1}{2}(-\tfrac{1}{2})(-1)(-\tfrac{1}{2})\,C = 1$$

$$\to -\tfrac{1}{8}C = 1 \to C = 8$$

$$\boxed{\varphi_5 = x(x-1)(y-1)(y-\tfrac{1}{2})\cdot 8}$$

$$\varphi_6 = xy(y-1)(x-\tfrac{1}{2})\cdot C$$



$$\varphi_6(1,\tfrac{1}{2}) = 1 \to \tfrac{1}{2}(-\tfrac{1}{2})(\tfrac{1}{2})\,C = 1$$

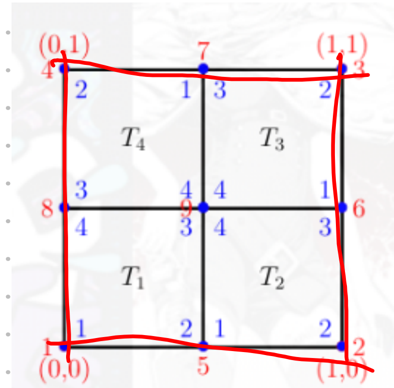
$$-\tfrac{1}{8}C = 1 \to C = 8$$

$$\boxed{\varphi_6 = xy(y-1)(x-\tfrac{1}{2})\cdot 8}$$

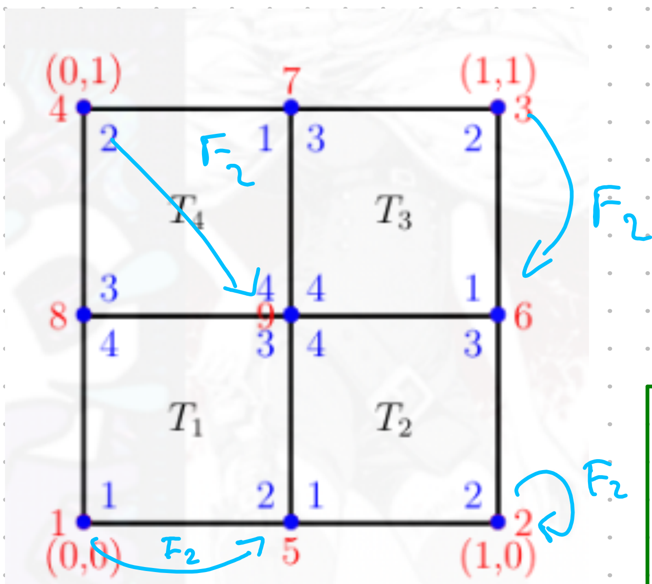
$$\varphi_9 = xy(x-1)(y-1)\,c$$

$$\varphi_9\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 1 \rightarrow \frac{1}{16}c = 1 \rightarrow c = 16$$

$$\boxed{\varphi_9 = xy(x-1)(y-1)\,16}$$



b) Determine the affine linear map $F_2 : \widehat{T} \to T_2$ that maps $F_2(0,0) = v_5$, $F_2(1,0) = v_2$, $F_2(1,1) = v_6$, $F_2(0,1) = v_9$ (here v_i denote the vertices of the mesh numbered as depicted). Moreover, compute DF_2 and $\det(DF_2)$.



$$F_2(x) = \begin{pmatrix} 1-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ 0-0 & \frac{1}{2}-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}}$$

$$\boxed{DF_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \det(DF_2) = \frac{1}{4}}$$

c) Compute $\int_\Omega \nabla\varphi_9 \cdot \nabla\varphi_9 dx$.

$$\nabla\varphi_9 = \begin{pmatrix} 16y(2x-1)(y-1) \\ 16x(x-1)(2y-1) \end{pmatrix}$$

$$\nabla\varphi_9 \cdot \nabla\varphi_9 = \underbrace{256y^2(2x-1)^2(y-1)^2 + 256x^2(x-1)^2(2y-1)^2}_{\star}$$

$$\boxed{\int_\Omega \star} = \int_0^1\int_0^1 \star \, dy\, dx = \ast = \boxed{\frac{256}{45}}$$

2. We continue with the previous setting. Let

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and consider the problem: find $u_h \in V_\mathcal{T}$ such that $a(u_h, v_h) = l(v_h)$ for all $v_h \in V_\mathcal{T}$.

a) Compute the element stiffness matrix and the element load vector for the element T_2 of the triangulation. Note: here, all element stiffness matrices are the same!

b) Compute the connectivity matrix C_{T_2} for the element T_2.

c) Solve the problem with homogeneous Dirichlet boundary conditions (this leads to a 1×1-"matrix").

$$A_{T_2}(\varphi_i, \varphi_j) = 6 \cdot \int_{T_2} \nabla\varphi_i \nabla\varphi_j \, dx$$

Use transformation theorem on the reference element:

T ... transpose

$$A^{T_2}_{kl} = 6 \int_{\hat{T}} \nabla N_l \, (DF_2)^{-1} (DF_2)^{-T} \nabla N_k \, \underbrace{|\det(DF_2)|}_{1/4} \, dx$$

$$N_1 = (1-x)(1-y) = 1-x-y+xy, \quad N_2 = x(1-y) = x-xy, \quad N_3 = xy, \quad N_4 = y(1-x) = y-xy,$$

$$\nabla N_1 = (y-1, x-1), \quad \nabla N_2 = (1-y, -x), \quad \nabla N_3 = (y, x), \quad \nabla N_4 = (-y, 1-x),$$

$$(DF_2)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = (DF_2)^{-T}$$

$$A^{T_2}_{11} = \frac{6}{4} \int_0^1 \int_0^1 \begin{pmatrix} y-1 \\ x-1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} y-1 \\ x-1 \end{pmatrix} dx\,dy$$

$$= 6 \int_0^1 \int_0^1 \begin{pmatrix} y-1 \\ x-1 \end{pmatrix} \begin{pmatrix} y-1 \\ x-1 \end{pmatrix} dx\,dy$$

$$= 6 \int_0^1 \int_0^1 (y-1)^2 + (x-1)^2 \, dx\,dy = 4$$

$$A^{T_2}_{12} = 6 \int_0^1 \int_0^1 (y-1)(1-y) + (x-1)(-x) \, dx\,dy = -1$$

$$A^{T_2}_{13} = 6 \int_0^1 \int_0^1 (y-1)(y) + (x-1)(x) \, dx\,dy = -2$$

$$A^{T_2}_{14} = 6\int_0^1\int_0^1 (y-1)(-y) + (x-1)(x)\,dx\,dy = -1$$

$$A^{T_2}_{22} = 4,\quad A^{T_2}_{23} = -1,\quad A^{T_2}_{24} = -2,\quad A^{T_2}_{33} = 4,\quad A^{T_2}_{34} = -1,\quad A^{T_2}_{44} = 4$$

$$\boxed{A^{T_2} = \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}}$$

element load vector

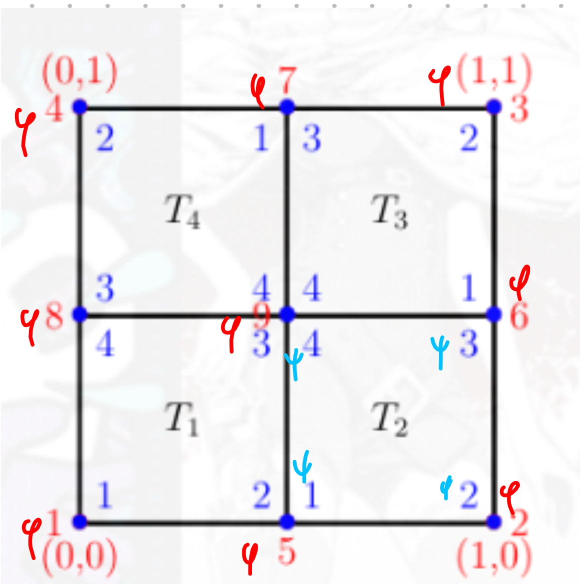
$$f_i^T = \int_{\hat{T}} \underbrace{\hat{f}}_{16} N_i \, |\det(D_F)| \, dx$$

$$f_1^{T_2} = \frac{16}{4}\int_0^1\int_0^1 (1-x)(1-y)\,dx\,dy = 1$$

$$f_2^{T_2} = 1,\quad f_3^{T_2} = 1,\quad f_4^{T_2} = 1$$

$$\boxed{f^{T_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

b) Compute the connectivity matrix C_{T_2} for the element T_2.



$$\begin{pmatrix} \varphi_1^{T_2} \\ \varphi_2^{T_2} \\ \varphi_3^{T_2} \\ \varphi_4^{T_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C^t_{T_2}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \end{pmatrix}$$

c) Solve the problem with homogeneous Dirichlet boundary conditions (this leads to a 1×1-"matrix").

Global stiffness matrix and load vector

$$A = \sum_{T \in \mathcal{T}} C_T A C_T^T$$

$$\begin{pmatrix} 4 & -1 & -2 & -1 \\ & 4 & -1 & -2 \\ -1 & & 4 & -1 \\ -2 & -1 & & 4 \\ -1 & -2 & -1 & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 4 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 4 \\ -2 & -1 & 4 & -1 \end{matrix} \qquad \begin{matrix} 4 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 \\ -2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 \end{matrix} \quad \ldots \quad A = \begin{pmatrix} 4 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\ -1 & 0 & 12 & -1 & -2 & -1 & -3 & 0 & -4 \\ 0 & 0 & -1 & 4 & 0 & 0 & -1 & 0 & -2 \\ -1 & -1 & -2 & 0 & 8 & -2 & 0 & 0 & -2 \\ 0 & -1 & -1 & 0 & -2 & 8 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -4 & -2 & -2 & -2 & -2 & 0 & \underline{16} \end{pmatrix}$$

$$f = \sum_{T \in \mathcal{T}} C_T f_T = (1, 1, 3, 1, 2, 2, 2, 0, \underline{4})^T$$

entry corresponding to φ_9

after removing Dirichlet entries

lines and colums corresponding to nodal functions with nodes on the dirichlet boundary can be omitted

$$A = [16], \quad f = [4] \rightarrow u = A^{-1} f = \frac{1}{16} \cdot 4$$

$$\boxed{u = \frac{1}{4}}$$

3. We now aim to show that 1D-Lagrange finite elements reproduce exact nodal values. Let $\Omega = (0,1)$ and $\mathcal{T}_h$ a mesh of intervals with nodes $\{x_i \; : \; i = 1, \dots, n\}$. Consider the 1D Poisson equation in the weak form, i.e., seek $u \in H_0^1(\Omega)$ such that

$$\int_\Omega u'v' \, dx = \int_\Omega fv \, dx.$$

Denote the finite element approximation with P^1-Lagrangian finite elements by u_h.

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$$G_i' = \begin{cases} (1-x_i), & 0 \le x \le x_i \\ -x_i, & x_i \le x \le 1 \end{cases}$$

$$\int_\Omega G_i'(x) w'(x) \, dx = w(x_i).$$

b) Use this to show that $u(x_i) = u_h(x_i)$ by inserting the difference $u - u_h$ into the weak formulation and using G_i as test function (Galerkin orthogonality).

just a number

just a number

$$\int_\Omega G_i' w'(x) = \int_0^{x_i} (1-x_i) \cdot w'(x) \, dx + \int_{x_i}^{1} -x_i \, w'(x) \, dx$$

$$= (1-x_i) \int_0^{x_i} w'(x) \, dx - x_i \int_{x_i}^{1} w'(x) \, dx$$

$$= (1-x_i) \left[w(x) \Big|_0^{x_i} \right] - x_i \left[w(x) \Big|_{x_i}^{1} \right]$$

$$= (1-x_i)\left(w(x_i) - \cancel{w(0)}\right) - x_i\left(\cancel{w(1)} - w(x_i)\right)$$

$$= w(x_i) - x_i \cancel{w(x_i)} + x_i \cancel{w(x_i)}$$

$$\boxed{= w(x_i)}$$

Can not use integration by parts here because w & G_i are not in H^2 → not enough differentiability. However, we know the derivatives of G_i → fundamental theorem of calculus

b) Use this to show that $u(x_i) = u_h(x_i)$ by inserting the difference $u - u_h$ into the weak formulation and using G_i as test function (Galerkin orthogonality).

$$\int_0^1 G_i'(x)\, w'(x)\, dx = w(x_i)$$

Galerkin orthogonality: $A(u-u_h, v_h) = 0$

$$A(u,v) = \int_\Omega u'v'\, dx$$

$$\rightarrow A(u-u_h, G_i) = \int_\Omega \underbrace{(u-u_h)'}_{w(x)'} G_i'\, dx$$

$$= u(x_i) - u_h(x_i) = 0$$

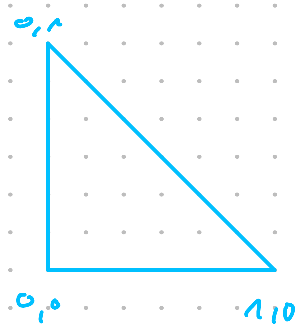
$$\boxed{\rightarrow u(x_i) = u_h(x_i)}$$

4. Let $T \subset \mathbb{R}^2$ be a triangle. Construct the so called polynomial bubble function $b_T : T \to \mathbb{R}$. This function is characterized by the conditions

- $0 \le b_T \le 1$, $\max b_T = 1$,
- $b_T \in P^k(T)$ for some $k \in \mathbb{N}$.
- $b_T = 0$ on ∂T.

What is the minimal polynomial degree k that is possible?

reference triangle



$$b_{\hat T}(x,y) = C \cdot xy(1-x-y)$$

$$\to b_{\hat T} \in P^3(T)$$

$$\to b_{\hat T}\big|_{\partial \hat T} = 0 \ \checkmark$$

$\max b_{\hat T} \overset{!}{=} 1 \to$ find C so that this holds

$$\nabla b_{\hat T} = 0 \quad = \nabla\left(C \cdot (xy - x^2y - y^2x)\right)$$

$$= C \begin{pmatrix} y - 2xy - y^2 \\ x - 2yx - x^2 \end{pmatrix} = 0$$

4 possibilities:
$$\left.\begin{array}{l} x=0,\ y=0 \\ x=1,\ y=0 \\ x=0,\ y=1 \end{array}\right\} \text{covered by } b_{\hat T}\big|_{\partial \hat T} = 0$$
$$x = \tfrac{1}{3},\ y = \tfrac{1}{3}$$

Check if $\left(\frac{1}{3}, \frac{1}{3}\right)$ is indeed a maximum

$$H_{bf}(x,y) = \begin{pmatrix} -2y & 1-2x-2y \\ 1-2y-2x & -2x \end{pmatrix}$$

$$H_{bf}\left(\tfrac{1}{3}, \tfrac{1}{3}\right) = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$\det\left(H_{bf}\left(\tfrac{1}{3}, \tfrac{1}{3}\right)\right) = \left(-\frac{2}{3}-\lambda\right)^2 - \frac{1}{9} = 0$$

$$= \left(-\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right) - \frac{1}{9} = 0$$

$$= \frac{4}{9} + \frac{4}{3}\lambda + \lambda^2 - \frac{1}{9} = 0$$

$$= \lambda^2 + \frac{12}{9}\lambda + \frac{3}{9} = 0$$

$$\lambda_{1,2} = -\frac{6}{9} \pm \sqrt{\frac{36}{81} - \frac{27}{81}}$$

$$\lambda_{1,2} = -\frac{6}{9} \pm \frac{3}{9} \rightarrow \underbrace{\lambda_1 = -\frac{1}{3},\ \lambda_2 = -1}_{\text{negative definite}}$$

$\rightarrow$ maximum

$$b_{\hat{T}}\left(\tfrac{1}{3},\tfrac{1}{3}\right) \overset{!}{=} 1 \rightarrow c \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}\left(1-\tfrac{1}{3}-\tfrac{1}{3}\right) = 1$$

$$\frac{c}{27} = 1 \rightarrow c = 27$$

$$\boxed{b_{\hat{T}}(x,y) = 27 \cdot x \cdot y \cdot (1-x-y)}$$

Using an affine linear map, this function can be applied to any 3 points in 2D space without changing the polynomial degree

i.e. $b_T(x) = b_{\hat{T}}(F_T^{-1}(x)) \in P^3(T)$