Problem Sheet 8

discussion: week of Monday, 19.12.2022

8.1. **a)** Prove that the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

does not have a factorization $\mathbf{A} = \mathbf{L}\mathbf{U}$ with normalized lower triangular matrix $\mathbf{L}$ and upper triangular matrix $\mathbf{U}$.

b) Let $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

where the off-diagonal 1 are in the positions (i_1, i_2) and (i_2, i_1) (with $i_1 \neq i_2$). Show: The matrix $\mathbf{PA}$ is the matrix $\mathbf{A}$ with rows i_1 and i_2 interchanged. Furthermore, $\mathbf{P}^{-1} = \mathbf{P}^\top = \mathbf{P}$.

8.2. The lengths of the 3 edges (meeting at a corner) and the circumferences (orthogonal to the first and second edge) of a brick are measured. The measured values are:

edge 1: $26mm$, edge 2: $38mm$, edge 3: $55mm$
circumference $\perp$ edge 1: $188mm$, circumference $\perp$ edge 2: $163mm$.

Determine the edge lengths using the method of least squares.

8.3. The function $f(x) = \sin x$ is to be approximated by a polynomial of the form $\pi(x) = a_1 x + a_3 x^3$. To this end, the coefficients a_1, a_3 are determined using the least squares method by minimizing $\sum_{j=0}^{m}(\pi(x_j) - f(x_j))^2$, where $x_0, \ldots, x_m$ are given points.

Set up the least squares problem for a_1 and a_3. Write a program that computes the coefficients a_1, a_3 for the following 9 choices of knots x_j: the x_j are N randomly chosen points in the interval $[-1/N, 1/N]$ for $N = 2^n$, $n = 2, \ldots, 10$. Do the values a_1, a_3 converge to a limit as $N \to \infty$? Which limit do you expect?

8.4. The least squares method can also be used to fit the parameters of certain nonlinear problems. How would you determine the parameters C, k to fit given data (t_i, y_i), $i = 1, \ldots, N$, to the law $y(t) = Ce^{-kt}$? How do you proceed to determine C, α for the law $y(t) = Ct^\alpha$?

8.5. Let $\mathbf{Q}$ be an orthogonal matrix. Show:

a) $\mathbf{x}^\top \mathbf{y} = ((\mathbf{Q})\mathbf{x})^\top (\mathbf{Q}\mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$.

b) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$ and its QR-factorization $\mathbf{A} = \mathbf{QR}$. Show: If $\mathbf{A}$ has full rank (i.e., $\operatorname{rank}(\mathbf{A}) = n$), then the diagonal entries of $\mathbf{R}$ are non-zero.