Problem Sheet 4

discussion: week of Monday, 21.11.2022

4.1. **a)** Show that the weights of the Newton-Cotes formulas satisfy $\sum_{i=0}^{n} w_i = 1(=$ length of the interval $[0,1])$. (*Hint:* apply the quadrature formula to a suitable function f.)

b) Show the symmetry property $w_{n-i} = w_i$, $i = 0, \ldots, n$.
(*Hint:* use the symmetry of the points, i.e., $x_j = 1 - x_{n-j}$.)

c) Let $n = 2m$ be even. Consider the function $f = (x - 1/2)^{n+1}$, which is antisymmetric with respect to $1/2$. Show: $\int_0^1 f(x)\,dx = 0 = \widehat{Q}_n^{cNC}(f) = \widehat{Q}_n^{oNC}(f)$. Conclude that the quadrature formulas $\widehat{Q}_n^{cNC}$ and $\widehat{Q}_n^{oNC}$ are exact for polynomials of degree $n+1$.
In particular, the midpoint rule is exact for polynomials in $\mathcal{P}_1$, and the Simpson rule is exact for polynomials in $\mathcal{P}_3$.

4.2. Let C, $\alpha > 0$ and consider the function $h \mapsto f(h) = Ch^{\alpha}$. Why is this function a straight line in a `loglog`-plot? What is its slope? Other popular plotting schemes are, `semilogx` and `semilogy`. Which one would you use to plot functions of the form $N \mapsto Ce^{-bN}$? How would you proceed if you suspect that a function $h \mapsto f(h)$ has the form $f(h) = Ce^{-b/h}$? What if you suspect $f(h) = Ce^{-b/h^2}$?

4.3. We wish to show that the extrapolation of the composite trapezoidal rule is the composite Simpson rule. To that end, let $T(h)$ be the composite trapezoidal rule with step size $h = (b-a)/N$ and $S(h)$ be the composite Simpson rule with step size $h = (b-a)/N$. Use Romberg extrapolation with step sizes $h_i = (b-a)2^{-i}$, $i = 0, 1, \ldots,$

a) Extrapolation of the composite trapezoidal rule (with step size h) has in column $m = 0$ the values $T(h_i)$. Show: in column $m = 1$ of the Neville scheme are the values

$$N_i := T(h_{i+1}) + \frac{1}{3}(T(h_{i+1}) - T(h_i))$$

b) Show: $N_i = S(h_i)$.

4.4. Write a program with signature $I = \texttt{adapt}(f, a, b, \tau, h_{min})$ that realizes an adaptive quadrature for $\int_a^b f(x)\,dx$. The quadrature should be based on the Simpson rule. τ is the desired (absolute) accuracy and h_{min} the minimal interval length. To estimate the accuracy, compare the value of the Simpson rule $S_{\{a,b\}}(f)$ for the integration on $[a,b]$ with the value $S_{\{a,m\}}(f) + S_{\{m,b\}}(f)$ with $m = (a+b)/2$. Use your algorithm for the integration of the function

$$\begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}$$

over $[0,1]$. Use $\tau = h_{min} = 2^{-j}$, $j = 0, \ldots, 10$. Plot the error versus τ. What convergence do you observe? Why was $h_{min} = \tau$ chosen?

4.5. Develop an adaptive algorithm for the integration of functions over the rectangle $[a_x, b_x] \times [a_y, b_y]$. Base your algorithm on the midpoint rule, i.e., $Q_{[a,b]\times[c,d]}(f) = (b-a)(d-c)f((a+b)/2, (c+d)/2)$. *Hint:* adapt the ideas of the 1d-adaptive algorithm of Problem 4.4. Test your adaptive algorithm for the integration over $[0,1]^2$ of the following functions:

$$f_1(x,y) = x^2 \quad \text{and} \quad f_2(x,y) = \begin{cases} 0 & x < y \\ 1 & x \geq y \end{cases}$$

Use the tolerances $\tau = 2^{-i}$, $i = 0, \ldots, 15$, and make a convergence plot (quadrature error versus tolerance) in `loglog` scale.