Exam 2

31.01.2023, 10:00-11:30

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4
max. Points	8	7	7	8

Good luck!

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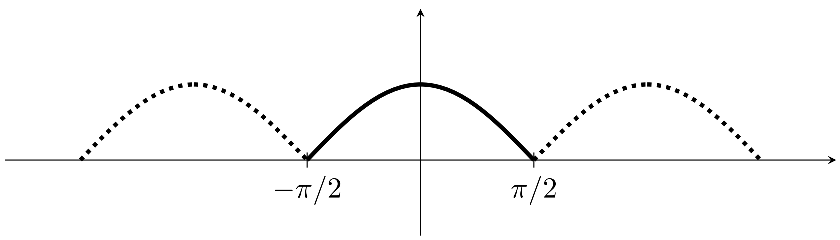
1. Consider the so-called *semi-sine wave* given by $f(x) = \cos(x)$ on $I = [-\pi/2, \pi/2]$, pictured below with its periodic extension:



a) Compute the real Fourier series of f on I.

b) Substitute $x = \pi/2$ in the Fourier series. What identity do we get?

a) Since $L = \pi$, the basis functions are given by $\sin(2kx)$, $k \geq 1$, and $\cos(2kx)$, $k \geq 0$. (**1pt**) We are therefore looking for a Fourier series in the form

$$f(x) = \cos(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx).$$

Since f is an even function, the sine coefficients all vanish, i.e. $b_k = 0$. (**1pt**) For the cosine coefficients, the 0-th term gives:

$$a_0 = \frac{2}{L}\int_I f(x)\,dx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = \frac{4}{\pi}. \quad (\mathbf{1pt})$$

For a_k, $k \geq 1$, we use integration by parts twice:

$$\begin{aligned}
a_k &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \underset{u'}{\cos(x)}\underset{v}{\cos(2kx)}\,dx = \frac{2}{\pi}\Big(\big[\sin(x)\cos(2kx)\big]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \underset{u}{\sin(x)}(-2k\underset{v'}{\sin(2kx)})\,dx\Big) \\
&= \frac{2}{\pi}\left(\begin{cases} 2 & \text{if } k \text{ even} \\ -2 & \text{if } k \text{ odd} \end{cases} + 2k\int_{-\pi/2}^{\pi/2} \underset{u'}{\sin(x)}\underset{v}{\sin(2kx)}\,dx\right) \\
&= \frac{2}{\pi}\left(\begin{cases} 2 \\ -2 \end{cases} + 2k\Big(\big[-\cos(x)\sin(2kx)\big]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} (-\underset{u}{\cos(x)})(2k\underset{v'}{\cos(2kx)})\,dx\Big)\right) \\
&= \begin{cases} 4/\pi \\ -4/\pi \end{cases} + 4k^2 a_k.
\end{aligned}$$

Comparing the two sides, we get $a_k = \pm\frac{4}{\pi(1-4k^2)}$. **(3pt)**
Alternatively, we can write $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$, so

$$a_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4}(e^{ix} + e^{-ix})(e^{i2kx} + e^{-i2kx})\, dx.$$

Expanding the brackets, we only have to integrate exponential functions, which is very easy. For example,

$$\int_{-\pi/2}^{\pi/2} e^{i(2k+1)x}\, dx = \left[\frac{e^{i(2k+1)x}}{i(2k+1)}\right]_{x=-\pi/2}^{\pi/2} = \frac{1}{i(2k+1)}\left(e^{ik\pi}e^{i\pi/2} - e^{-ik\pi}e^{-i\pi/2}\right).$$

Remembering that $e^{ik\pi}$ is either $+1$ (if k is even) or -1 (if k is odd), and that $e^{i\pi/2} = i$, $e^{-i\pi/2} = -i$, we get that the value of this integral is $\pm\frac{2}{2k+1}$. Adding it up with the other similar integrals, we get the same expression for a_k as before.

b) The left-hand side of the equality is $\cos(\pi/2) = 0$. The right-hand side of the equality is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2k(\pi/2)) = \frac{2}{\pi} + \sum_{k=1}^{\infty}(-1)^k \frac{4\cos(k\pi)}{\pi(1-4k^2)} = \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{4}{\pi(1-4k^2)}.$$

Rearranging that this equals 0, we get the identity

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{2}. \quad \textbf{(2pt)}$$

2. **a)** Consider the homogeneous ODE

$$u'(x) = -\frac{1}{x}u(x)$$

for $x \geq 1$. Find all solutions and argue that they are indeed all of them. Find a solution that satisfies the initial condition $u(1) = 1$.

b) Write the variation of constants formula for the inhomogeneous ODE

$$u'(x) = -\frac{1}{x}u(x) + \log x.$$

(You don't have to evaluate the integral).

a) We can use the method for separable ODEs. Alternatively, we can remember from the exercises that when division by x is involved, it is a good idea to make the ansatz $u(x) = x^\alpha$. Alternatively, the equation is simple enough that one can find by trial and error one that $u(x) = \frac{c}{x}$ is a solution to the ODE for all $c \in \mathbb{R}$. **(2pt)** The equation is first order, so the solutions form a 1-dimensional vector space, so these are all the solutions. **(1pt)** The initial condition is satisfied if $c = 1$, which gives the solution $u(x) = \frac{1}{x}$. **(1pt)**

b) Take the homogeneous solution $U(x) = \frac{1}{x}$ from a). The variation of constants formula tells us that a particular solution to the inhomogeneous equation is given by

$$u_p(x) = U(x) \int_1^x (U(y))^{-1} \log(y)\, dy = \frac{1}{x} \int_1^x y \log y \, dy. \qquad (\mathbf{3pt})$$

3. **a)** Show that any holomorphic function on $\mathbb{C}$ that takes only real values is constant.

b) Formulate the theorem on Cauchy's integral formula.

c) Using the formula, show that any holomorphic function on $\mathbb{C}$ that is bounded in magnitude is constant.

a) If $f = u + iv$ is a holomorphic function that takes only real values, then on one hand $v = 0$, on the other hand, by the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Therefore $\nabla u = 0$, so u is constant, and therefore f is also a constant. (**2pt**)

b) *Theorem:* Let C be a closed simple curve in $\mathbb{C}$, let f be holomorphic on the region bounded by C, and let z_0 be a point in this region. Then the following formula holds:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \qquad (\mathbf{2pt})$$

c) Suppose that f is a bounded holomorphic function: for some M, for all z, $|f(z)| \leq M$. From the more general form of Cauchy's integral formula (which is just differentating the first one), we have

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz.$$

Now we can use the trick from residue calculus. We can bound a line integral by the length of the line times the magnitude of the function on the line. In particular, if C is a circle of radius R around z_0, the length is $2\pi R$ and the maximal magnitude of the function $\frac{f(z)}{(z-z_0)^2}$ is $\frac{M}{R^2}$. So

$$|f'(z_0)| \leq \frac{M}{R},$$

letting $R \to \infty$ we get that $f'(z_0) = 0$, and since z_0 was arbitrary, $f' \equiv 0$, and therefore f is constant. (**3pt**)

4. Answer the following questions (Simply write true/false on your sheet). (**!** Wrong answers will lose points). Correct/incorrect answers are **+1/-1 pt**.

a) The convolution operation $*$ satisfies (for all continuous and integrable $f, g, h : \mathbb{R} \to \mathbb{R}$):

1. $f * g = g * f$. True
2. $f * (g * h) = (f * g) * h$. True

3. $\delta * f = \delta$, where δ is the Dirac-delta. False
4. $f * f \geq 0$. False

b) Consider the heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t T(t,x) & = \partial_{xx} T(t,x) \qquad t > 0, x \in (0,1), \\ T(t,0) & = T(t,1) = 0 \qquad t > 0 \end{cases} \tag{1}$$

1. Equation (1) is an elliptic partial differential equation. False
2. Equation (1) has infinitely many solutions. True
3. Complementing (1) with the condition $T(1,x) = x(x-1)$, the equation has a unique solution. False
4. Solutions of (1) tend to 0 as $t \to \infty$. True