Problem Sheet 7

discussion: week of Monday, 12.12.2022

7.1. **a)** Compute the number of additions and multiplications in Algorithms 4.2 and 4.3 (forward and backward substitution).

b) Show that the product $\mathbf{L}_1\mathbf{L}_2$ of two lower triangular matrices $\mathbf{L}_1, \mathbf{L}_2$ is again a lower triangular matrix. Also show that the inverse of a (invertible) lower triangular matrix is lower triangular.

7.2. **a)** Explain Crout's algorithm from Chapter 4.3.1 in the lecture notes.

b) Modify the algorithm to compute a Cholesky factorization

$$\mathbf{C}^\top \mathbf{C} = \mathbf{A}$$

and realize your algorithm in Matlab/Python.

7.3. ("arrowhead matrix") Let $n = 10$, $\mathbf{e} = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$, $\mathbf{b} = (1, 0, \dots, 0)^\top \in \mathbb{R}^n$. Consider the matrix $\mathbf{A} = 10\mathbf{I} + \mathbf{b}\mathbf{e}^\top + \mathbf{e}\mathbf{b}^\top$ and the matrix $\widetilde{\mathbf{A}} := \mathbf{A}(n : -1 : 1, n : -1, 1)$ that is obtained from $\mathbf{A}$ by reversing the numbering of the rows and columns.

Use the commands `spy` (`matplotlib.pyplot.spy`) and `lu` (`scipy.linalg.lu`) to visualize the sparsity patterns of $\mathbf{A}$, $\widetilde{\mathbf{A}}$ and the corresponding factors $\mathbf{L}$, $\mathbf{U}$ of the LU-factorization. What do you observe? Which variant of the numbering is to be preferred from a cost (i.e., number of floating point operations or storage requirement) point of view?

7.4. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix}$$

a) Compute by hand the LU-factorization of $\mathbf{A}$. Calculate $\mathbf{A}^{-1}$, $\mathbf{L}^{-1}$, $\mathbf{U}^{-1}$ and compute the three condition numbers $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{L}) = \|\mathbf{L}\|_\infty \|\mathbf{L}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{U}) = \|\mathbf{U}\|_\infty \|\mathbf{U}^{-1}\|_\infty$. Here, $\|\cdot\|_\infty$ is the row-sum norm.

b) Repeat the calculation of a) for the matrix $\widetilde{\mathbf{A}}$ that is obtained from $\mathbf{A}$ by interchanging the two rows. What do you observe?

7.5. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a tridiagonal matrix of the form

$$\mathbf{A} = \begin{pmatrix} d_1 & e_1 & & & \\ c_2 & d_2 & e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{n-1} \\ & & & c_n & d_n \end{pmatrix}$$

Assume that $\mathbf{A}$ has an LU-factorization.

a) Show: the factors $\mathbf{L}$ and $\mathbf{U}$ have the form

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_n & 1 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} u_1 & f_1 & & & \\ & u_2 & f_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & f_{n-1} \\ & & & & u_n \end{pmatrix}$$

Hint: do the induction step of Thm. 4.17 of the notes.

b) Formulate an algorithm that computes the l_i and the u_i for $i = 2, \dots, n$ and realize your algorithm in Matlab/Python. Input are the vectors $\mathbf{d}$, $\mathbf{e}$, $\mathbf{c}$ (i.e., the diagonals of $\mathbf{A}$), output are the vectors $\mathbf{l}$, $\mathbf{u}$, and $\mathbf{f}$ (i.e., the diagonals of $\mathbf{L}$ and $\mathbf{U}$).