Problem Sheet 3

discussion: week of Monday, 07.11.2022

Hint: one way to evaluate interpolating polynomials is to reuse the Neville scheme that you have programmed already in Exercise 2.3a. This could also be used to compute the Lagrange interpolation polynomials ℓ_i.

3.1. Consider the function $f(x) = (4 - x^2)^{-1}$.

Provide an upper bound for $\min_{q \in \mathcal{P}_n} \|f - q\|_{\infty,[-1,1]}$ by selecting q to be the Taylor polynomial of f about a suitable point.

Plot semilogarithmically (`semilogy`) the error $\|f - I_n^{Cheb} f\|_{\infty,[-1,1]}$ versus $n \in \{1, \dots, 20\}$, where $I_n^{Cheb} f$ is the Chebyshev interpolant of degree n. Herby, compute the quantity $\|f - I_n^{Cheb} f\|_{\infty,[-1,1]}$ approximatively by evaluating $f - I_n^{Cheb} f$ in 100 uniformly distributed points in the interval $[-1, 1]$ and take the maximum of these values. Include in your graph also the error $\|f - T_{2n} f\|_{\infty,[-1,1]}$ where $T_{2n} f$ is the Taylor polynomial of f about $x = 0$ of degree $2n$. Compare the approximation quality of the Taylor polynomial with that of the Chebyshev interpolant.

3.2. For $n \in \{10, 20, 40\}$ consider interpolation on the interval $[-5, 5]$ in $n+1$ points. Compare the uniformly distributed points $x_i^{unif} := 5(-1 + 2\frac{i}{n})$, $i = 0, \dots, n$ and the Chebyshev points $x_i^{Cheb} = 5\cos(\frac{i+0.5}{n+1}\pi)$, $i = 0, \dots, n$.

a) For the function $f(x) = (1 + x^2)^{-1}$, plot the two interpolating polynomials on $[-5, 5]$.

b) Investigate numerically the Lebesgue constant

$$\Lambda_n := \max_{x \in [-5,5]} \sum_{i=0}^{n} |\ell_i(x)|,$$

for both the uniform interpolation point distribution and the Chebyshev points. To that end, plot Λ_n versus n in semilogarithmic scale (`semilogy`).

c) For the uniform point distribution, one can in fact show that $\Lambda_n \approx Ce^{bn}$ for some C, $b > 0$. Determine C and b from your data as follows by taking the logarithm $\log \Lambda_n \approx \log C + bn$ and fit your data for the values for $n \in \{20, 40\}$. *Hint:* you can let `polyfit` do the work for you to compute $\log C$ and b or you solve a 2×2 system.

3.3. (lazy change of basis for polynomials) Let two bases $\{p_0(x), \dots, p_n(x)\}$ and $\{q_0(x), \dots, q_n(x)\}$ of the space $\mathcal{P}_n$ of polynomials of degree n be given. Let interpolation points x_i, $i = 0, \dots, n$, be given. Define the matrices $\mathbf{G}$ and $\mathbf{H}$ by

$$\mathbf{G}_{ij} = p_j(x_i), \qquad \mathbf{H}_{ij} = q_j(x_i).$$

Show: the matrix $\mathbf{G}^{-1}\mathbf{H}$ realizes the change of basis, i.e.,

$$\mathbf{c} = \mathbf{G}^{-1}\mathbf{H}\mathbf{d}$$

implies $\sum_i \mathbf{c}_i p_i = \sum_i \mathbf{d}_i q_i$.

3.4. Write a program that realizes the composite trapezoidal rule for integration on $[a, b]$. The rule is based on a subdivision of $[a, b]$ into N subintervals of length $h = (b - a)/N$. Input are a function handle for f, N, and a, b.

Consider, for $[a, b] = [-1, 1]$ the 5 integrands

$$f_1(x) = x^2, \qquad f_2(x) = |x|, \qquad f_3(x) = \begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}, \qquad f_4(x) = \sin(\pi x), \qquad f_5(x) = \sin(4\pi x).$$

Plot in `loglog`-scale the quadrature error versus h for $h = (b - a)2^{-i}$, $i = 1, 2, \dots, 20$. What do you observe? Explain your observations for the functions f_1, f_2, f_3.

3.5. Explain the convergence behavior in Exercise 3.4 for the integrand f_4. You may use Euler's formula $e^{\mathbf{i}z} = \cos z + \mathbf{i} \sin z$ in the form $\sin z = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}z} - e^{-\mathbf{i}z})$ with the imaginary unit $\mathbf{i}$ and the geometric series

$$\sum_{j=0}^{N-1} q^j = \frac{q^N - 1}{q - 1}, \qquad q \neq 1.$$

*H*int: what is is the exact value of the integral?