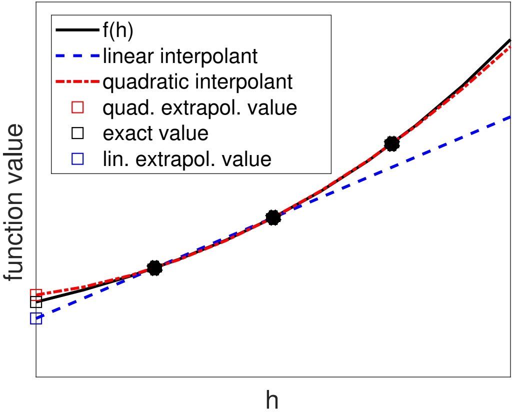
Extrapolation

goal: evaluate $h \mapsto f(h)$ at $h = 0$

procedure: given $n+1$ data pairs $(h_i, f(h_i))$, $i = 0, \ldots, n$:

find the interpolating polynomial $p \in \mathcal{P}_n$ and approximate $f(0) \approx p(0)$.



Numerical differentiation with one-sided difference quotients

$$D(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$h_i = 2^{-i}$$

$$\begin{aligned} D_{i0} &= D(h_i) \\ D_{ij} &= D_{(i+1)\,(j-1)} \\ &\quad - \frac{h_{i+j}}{h_{i+j} - h_i}\left[D_{(i+1)\,(j-1)} - D_{i\,(j-1)}\right] \end{aligned}$$

h	$m = 0$	$m = 1$	$m = 2$	$m = 3$
h_0	$D(h_0) = D_{00}$	D_{01}	D_{02}	D_{03}
h_1	$D(h_1) = D_{10}$	D_{11}	D_{12}	D_{13}
h_2	$D(h_2) = D_{20}$	D_{21}	D_{22}	D_{23}
h_3	$D(h_3) = D_{30}$	D_{31}	D_{32}	D_{33}
h_4	$D(h_4) = D_{40}$	D_{41}	D_{42}	$\vdots$
h_5	$D(h_5) = D_{50}$	D_{51}	$\vdots$	
h_6	$D(h_6) = D_{60}$	$\vdots$		
$\vdots$	$\vdots$			

Example

$f(x) = e^x$ and $x_0 = 0$ with one-sided difference quotients

Neville scheme:

h	$m = 0$	$m = 1$	$m = 2$	$m = 3$
1	1.71828182845905	0.87660325434147	1.00747997135508	0.99982039920503
2^{-1}	1.29744254140026	0.97476079210167	1.00077784572378	0.99999046433634
2^{-2}	1.13610166675097	0.99427358231826	1.00008888700977	0.99999944973780
2^{-3}	1.06518762453461	0.99863506083689	1.00001062939680	0.99999996693993
2^{-4}	1.03191134268575	0.99966673725682	1.00000129974704	0.99999999797395
2^{-5}	1.01578903997129	0.99991765912448	1.00000016069559	0.99999999987449
2^{-6}	1.00785334954789	0.99997953530281	1.00000001997713	
2^{-7}	1.00391644242535	0.99999489880855		
2^{-8}	1.00195567061695			

Example (cont.)

error ($f'(0) - D_{jm}$) and reduction factor (quotient of subsequent errors):

h	$m=0$	ρ	$m=1$	ρ	$m=2$	ρ	$m=3$	ρ
1	7.18_{-1}	—	-1.23_{-1}	—	7.48_{-3}	—	-1.78_{-4}	—
2^{-1}	2.97_{-1}	0.41	-2.52_{-2}	0.20	7.78_{-4}	0.10	-9.54_{-6}	0.053
2^{-2}	1.36_{-1}	0.46	-5.73_{-3}	0.23	8.89_{-5}	0.11	-5.50_{-7}	0.058
2^{-3}	6.52_{-2}	0.48	-1.37_{-3}	0.24	1.06_{-5}	0.12	-3.31_{-8}	0.060
2^{-4}	3.19_{-2}	0.49	-3.33_{-4}	0.24	1.30_{-6}	0.12	-2.03_{-9}	0.061
2^{-5}	1.58_{-2}	0.49	-8.23_{-5}	0.25	1.61_{-7}	0.12	-1.26_{-10}	0.062
2^{-6}	7.85_{-3}	0.50	-2.05_{-5}	0.25	1.99_{-8}	0.12		
2^{-7}	3.92_{-3}	0.50	-5.10_{-6}	0.25				
2^{-8}	1.96_{-3}	0.50						
expected:	$O(h)$		$O(h^2)$		$O(h^3)$		$O(h^4)$	
		0.5		0.25		0.125		0.0625

Numerical differentiation of $f(x) = |x|^{3/2}$ at $x = 0$

extrapolation error for one-sided difference quotients

h	m=0	m=1	m=2	m=3	m=4	m=5	m=6	m=7
2^{0}	1.00_{0}	4.14_{-1}	2.52_{-1}	1.68_{-1}	1.15_{-1}	8.06_{-2}	5.66_{-2}	3.99_{-2}
2^{-1}	7.07_{-1}	2.93_{-1}	1.79_{-1}	1.19_{-1}	8.17_{-2}	5.70_{-2}	4.00_{-2}	2.82_{-2}
2^{-2}	5.00_{-1}	2.07_{-1}	1.26_{-1}	8.40_{-2}	5.77_{-2}	4.03_{-2}	2.83_{-2}	
2^{-3}	3.54_{-1}	1.46_{-1}	8.93_{-2}	5.94_{-2}	4.08_{-2}	2.85_{-2}		
2^{-4}	2.50_{-1}	1.04_{-1}	6.31_{-2}	4.20_{-2}	2.89_{-2}			
2^{-5}	1.77_{-1}	7.32_{-2}	4.46_{-2}	2.97_{-2}				
2^{-6}	1.25_{-1}	5.18_{-2}	3.16_{-2}					
2^{-7}	8.84_{-2}	3.66_{-2}						
2^{-8}	6.25_{-2}							
error	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$		

reason for "failure" of extrapolation: The function $D(h) = h^{-1}(|h|^{3/2} - 0) = \sqrt{|h|}$ is not "smooth" at $h = 0$.