Sheet 2

Discussion of the sheet: Tue./Wed., 18-19 October

This exercise sheet is concerned with the topic

- Vector spaces (continued)
- Differentiation

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1. Let C be the space of continuous functions on $[0,1]$ with the inner product

$$(p,q)_{L^2} = \int_0^1 p(x)q(x)dx. \tag{1}$$

Verify that the elements $\{1, x, x^2, x^3\}$ are linearly independent and use the Gram-Schmidt process to produce an orthonormal set from them.

2. Continuing the previous example: we have constructed an orthonormal basis of P_3, the space of polynomials with maximal degree 3 equipped with the inner product (1). P_3 is a finite dimensional (and therefore closed) subspace of C. Compute the best approximation of the function $f(x) = \sin(\pi x)$ from P_3.

3. Compute the partial derivatives of first and second order for the functions

a) $f(x,y) = x^2 y$

b) $f(x,y) = \sin(x/y)$

c) $f(x,y,z) = (x^2 + y^2 + z^2)^{1/2}$

Check Schwarz' theorem for these examples.

4. Compute the gradient ∇f and the directional derivative $\partial_\nu f$ in the direction $\nu = (1,1)^T$ at the point $(0,0)$ of the function

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Does the equality $\partial_\nu f = \nabla f \cdot \nu$ hold? Is f totally differentiable?

5. (Taylor approximation)

With the fundamental theorem of calculus, we may write

$$f(x) = f(a) + \int_a^x f'(y)dy. \tag{2}$$

Replacing $f'(y)$ by $f'(a)$, we get the familiar approximation

$$f(x) \approx f(a) + (x-a)f'(a). \tag{3}$$

Replacing f by its derivative, we get in the same way

$$f'(x) \approx f'(a) + (x-a)f''(a),$$

and inserting this into (2), we compute

$$f(x) \approx f(a) + \int_a^x f'(a) + (y-a)f''(a)dy = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a),$$

which is a quadratic approximation. Again doing the same for the second derivative f'' and higher derivatives gives the general formula for the **Taylor polynomial** (of degree n)

$$f(x) \approx \sum_{j=0}^{n} \frac{1}{j!}(x-a)^j f^{(j)}(a).$$

Compute the Taylor polynomial of degree 3 for the function $f(x) = \sin(\pi x)$ at $a = 0$. Plot and compare the Taylor polynomial with the best degree 3 approximation computed in Example 2.

Taylor polynomials can also be computed in functions defined in $\mathbb{R}^d$. E.g., for a function $f : \mathbb{R}^2 \to \mathbb{R}$ the Taylor polynomial of degree 2 reads as

$$\begin{aligned} f(x,y) \approx\ & f(a,b) + (x-a)\partial_x f(a,b) + (y-b)\partial_y f(a,b) + \\ & \frac{1}{2}\big((x-a)^2\partial_{xx} f(a,b) + 2(x-a)(y-b)\partial_{xy} f(a,b) + (y-b)^2\partial_{yy} f(a,b)\big) \end{aligned}$$

Compute the Taylor polynomial of degree 2 of the function $f(x,y) = \exp(x+y)$ at the point $(a,b) = (1,1)$.

6. Use Taylor approximation of degree $n = 1$ (that is, the simple formula (3)) to find approximations to $17^{1/4}$ and $26^{1/3}$ (what is a good choice of a?).