Problem Sheet 5

discussion: week of Monday, 28.11.2022

5.1. (*Sinc quadrature:*) For certain integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ the simple trapezoidal rule works astonishingly well: Define the quadrature rule

$$Q^N(f) := h \sum_{i=-N}^{N} f(x_i), \qquad h := \frac{1}{\sqrt{N}}, \qquad x_i := ih.$$

Apply this rule to the integrand $f(x) = e^{-x^2} \sin^2(x)$. Plot the error versus N in a suitable scale.

5.2. **a)** Write a program with signature $y = \texttt{composite\_gauss}(n, L, q)$ that realizes a composite Gauss rule for integration over $(0, 1)$. The composite Gauss rule uses n points for each of the L subintervals that are given by

$$(0, q^{L-1}), (q^{L-1}, q^{L-2}), (q^{L-2}, q^{L-3}) \cdots, (q, 1)$$

Check your program with $f(x) = x^m$, $m = 0, 1, 2$. *Hint:* Gauss points and weights can be obtained by `numpy.polynomial.legendre.leggauss` or `gauleg.m` (see TUWEL).

b) Use your routine `composite_gauss` for $n = L = 1, \ldots, 20$ and the three choices $q \in \{0.5, 0.15, 0.05\}$ and the integrand

$$f(x) = x^{0.1} \log x.$$

(The exact integral is $\int_0^1 f(x)\,x = -1/1.1^2 \approx -0.82644$.) Plot semilogarithmically (`semilogy`) the quadrature error versus n for these 3 values of q. Which choice of q is the best one?

c) Fit (using `polyfit`) the error curves to the law Ce^{-bn}.

5.3. Give an explicit error bound (in dependence on n) for the Gaussian quadrature error

$$\left| \int_{-1}^{1} f(x)\,dx - Q_n^{Gauss}(f) \right| \qquad \text{with } f(x) = (4 - x^2)^{-1}.$$

5.4. (*Transformation techniques:*) We seek a quadrature formula for

$$\int_1^{\infty} f(x)\,dx.$$

Consider the specific case $f(x) = \log x/(x^\pi)$ with

$$\int_1^{\infty} \frac{\log x}{x^\pi}\,dx = \frac{1}{\pi^2 - 2\pi + 1}.$$

a) One possibility is to transform the integral to an integration over $(0, 1)$ using a suitable substitution. Formulate such a transformation. The transformed problem can then be treated with the quadrature formula of Problem 5.2.

b) Another option is the substitution $x = e^y$. One obtains an integral of the form

$$\int_{y=0}^{\infty} F(y)\,dy,$$

where the integrand decays rapidly so that the integral $\int_{y=0}^{\infty} F(y)\,dy$ can be approximated well by $\int_{y=0}^{L} F(y)\,dy$. Again, the integral can be computed with a composite Gauss rule with n points per subinterval where the L subintervals are given by

$$(0, Lq^{L-1}), (Lq^{L-1}, Lq^{L-2}), \cdots, (Lq, L)$$

Generate the composite quadrature rule using your program of Problem 5.2.

c) Plot the error using `semilogy` for both methods with $n = L = 1, \dots, 20$. Choose $q = 0.15$.

5.5. Consider quadrature rules Q^{2D} on the square $S = [0,1]^2$.

a) Show: the midpoint rule $Q(F) = F(0.5, 0.5)$ is exact for polynomials of the form $F(x,y) = a + bx + cy$.

b) Given $p \in \mathbb{N}_0$, give a quadrature formula Q^{2D} that is exact for polynomials of the form $F(x,y) = \sum_{i,j=0}^{p} a_{ij} x^i y^j$.