Problem Sheet 11

discussion: week of Monday, 23.01.2023

11.1. Implement the CG method from the lecture to iteratively solve a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{n\times n}$. Stop the iteration, once the residual in the ℓ-th step is smaller than a prescribed tolerance.

Compare your CG implementation with a direct solver (e.g in MATLAB $\mathbf{A}\backslash\mathbf{b}$) by plotting computational times over matrix size n for stopping tolerances 10^{-1}, 10^{-4} and 10^{-8}. In MATLAB you can, e.g., use `A = gallery('poisson',n)` as test matrix.

11.2. Let $\mathbf{A} \in \mathbb{R}^{N\times N}$ be positive definite (but not necessarily be symmetric). The *minimal residual method* defines $\mathbf{x}_{\ell+1}$ from $\mathbf{x}_\ell$ by minimizing the function $\phi(\mathbf{x}) := \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$ on the line $\{\mathbf{x}_\ell + t\mathbf{r}_\ell \,|\, t \in \mathbb{R}\}$ with $\mathbf{r}_\ell = \mathbf{A}\mathbf{x}_\ell - \mathbf{b}$.

a) Formulate the method, i.e., compute α_ℓ such that α_ℓ is the minimizer of

$$t \mapsto \|\mathbf{A}(\mathbf{x}_\ell + t\mathbf{r}_\ell) - \mathbf{b}\|_2^2.$$

b) Define

$$\mu := \frac{1}{2}\lambda_{min}(\mathbf{A} + \mathbf{A}^T), \qquad \sigma := \|\mathbf{A}\|_2,$$

where $\lambda_{min}(\mathbf{A} + \mathbf{A}^T)$ is the smallest eigenvalue of $\mathbf{A} + \mathbf{A}^T$. Show: $\mu \le \sigma$.

c) Show that the *minimal residual method* converges by showing

$$\|\mathbf{r}_{\ell+1}\|_2^2 \le \left(1 - \frac{\mu^2}{\sigma^2}\right)\|\mathbf{r}_\ell\|_2^2$$

11.3. Implement the explicit and implicit Euler method to solve the general linear differential equation

$$y'(t) = f(t)y(t) + g(t), \qquad y(0) = y_0.$$

Here, f, g are given functions and y_0 is a given value. Compare both methods for the ODE

$$y'(t) = \lambda y(t), \qquad y(0) = 1,$$

which has the exact solution $y(t) = \exp(\lambda t)$. Take the different values $\lambda = 1, 10, -1, -10$ and plot the error $|y(1) - y_N|$ (in loglog-scale) over $N = 2^i$, $i = 1, \ldots, 10$. What do you observe?

11.4. Formulate the Newton method to compute y_{i+1} for given y_i for the implicit Euler method to solve $y' = f(t, y(t))$.

11.5. The solution of $y'(t) = f(t)$, $y(t_0) = 0$ is given by $y(t) = \int_{t_0}^t f(\tau)\,d\tau$. Hence, for right-hand sides of the form $f(t, y) = f(t)$, a Runge-Kutta method results in a quadrature formula. Which quadrature formula is obtained for RK4?