Sheet 4

Discussion of the sheet: Tue., 18.04.2023

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1. Let $u : (0,1) \to \mathbb{R}$ be a function that is continuously differentiable on $(0, 1/2)$ and $(1/2, 1)$ but such that $\lim_{x\uparrow 1/2} u(x) \neq \lim_{x\downarrow 1/2} u(x)$. Draw an example for such a function. Calculate the generalised/distributional derivative of u using the definition (that is, without just refering to the script or Wikipedia for the answer).

2. Show that the bilinear form $A(\cdot,\cdot)$ corresponding to the Poisson problem with mixed boundary conditions (Section 3.2.2 in the lecture notes) is indeed a continuous and coercive bilinear form on $H^1_D(\Omega)$. Moreover, show that the linear form $f(\cdot)$ given in Section 3.2.2 is a bounded linear form on $H^1_D(\Omega)$.

3. (**Fourier description of Sobolev spaces**) In some special domains, Sobolev spaces have a nice description in terms of Fourier series. Take $\Omega = (0,1)$ and recall that any $f \in L^2(\Omega)$ can be written in a complex Fourier series

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{i2\pi kx}, \qquad c_k = \int_0^1 f(y)e^{i2\pi ky}dy.$$

a) Show that for any $m \geq 1$, the quantity

$$|f|^2_{\tilde{H}^m} = \sum_{k\in\mathbb{Z}} k^{2m}|c_k|^2$$

is equivalent to the seminorm $|f|_{H^m}$ (that is, for some constant C, one has $|f|_{\tilde{H}^m} \leq C|f|_{H^m}$ and $|f|_{H^m} \leq C|f|_{\tilde{H}^m}$ for all $f \in H^m$).

b) Based on the above observation, give a 1 line proof of the Poincaré inequality.

c) Based on the above, how would you define $H^{1/2}(\Omega)$? How about $H^{-1}(\Omega)$?

d) With the preceding definitions, for which $s \in \mathbb{R}$ does the Dirac-delta at $1/2$ belong to $H^s(\Omega)$?

4. (**Hölder inequality and generalised Poincaré**) The Hölder inequality states that if $1/p + 1/q = 1$, then

$$\Big|\int_\Omega f(x)g(x)\,dx\Big| \leq \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p}\Big(\int_\Omega |g(x)|^q\,dx\Big)^{1/q},$$

which is just Cauchy-Schwartz if $p = q = 2$. Use this to prove the following general form of Poincaré inequality in 1D, for arbitrary $p \in [1,\infty)$:

$$\int_0^1 |v(x)|^p dx \leq C_p \int_0^1 |v'(x)|^p dx$$

with some constant C_p, for all v in the more general Sobolev space

$$W_0^{1,p} = \left\{ v : \int_0^1 |v(x)|^p dx < \infty, \int_0^1 |v'(x)|^p dx < \infty, v(0) = v(1) = 0 \right\}.$$

Hint: Use the fundamental theorem of calculus $v(x) - v(0) = \int_0^x v'(y)dy$.

5. Let $\overline{H}^1(\Omega) := \{u \in H^1(\Omega) : \int_\Omega u \, dx = 0\}$. Show that $\overline{H}^1(\Omega)$ is a complete subspace of $H^1(\Omega)$. As a consequence, show that for any $f \in L^2(\Omega)$, the weak form of the pure Neumann problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ \nabla u \cdot n &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

has a unique solution in $\overline{H}^1(\Omega)$.

(*Optional:* What happened to the compatibility condition $\int_\Omega f = 0$? How come we don't need it?)

Are we supposed to prove something?

6. We now aim to implement the pure Neumann problem in 1D: Let $\Omega = (0, 1)$ and $\mathcal{T}$ be a uniform mesh with $N + 1$ points. Take V_N to be the space of piecewise linear functions and $\overline{V}_N = V_N \cap \overline{H}^1$. Therefore the FEM reads as: $u_N \in \overline{V}_N$ s.t.

$$A(u_N, v_N) = \int_\Omega u_N' v_N' \, dx = \int_\Omega f v_N \, dx = \ell(v_N) \qquad \forall v_N \in \overline{V}_N.$$

The idea is now to take the hat functions $\{\varphi_i : i = 1, \dots, N+1\}$ as basis of V_N and enforce the condition $\int_0^1 u_N dx = 0$ separately.

Define the stiffness matrix $\mathbf{A} \in \mathbb{R}^{(N+1)\times(N+1)}$ given as $\mathbf{A}_{ij} = A(\varphi_i, \varphi_j)$ and the load vector $\mathbf{l}$ given as $\mathbf{l}_i = \ell(\varphi_i)$. Show that, provided $\sum_i \mathbf{l}_i = 0$, writing $u_N = \sum_i \mathbf{u}_i \varphi_i$ the FEM formulation leads to the linear system of equations

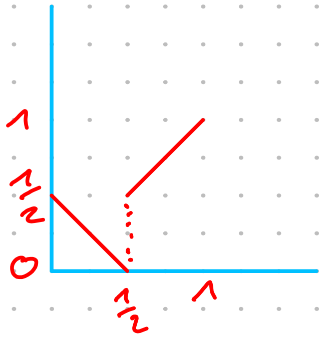
$$\begin{pmatrix} \mathbf{A} \\ \mathbf{P}^T \end{pmatrix} \cdot \mathbf{u} = \begin{pmatrix} \mathbf{l} \\ 0 \end{pmatrix},$$

where $\mathbf{P} \in \mathbb{R}^{N+1}$ is given as $\mathbf{P}_i := \int_\Omega \varphi_i dx$. Note that this implies that we get a solution to a symmetric $(N+2) \times (N+2)$ system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{l} \\ 0 \end{pmatrix}.$$

1. Let $u:(0,1)\to\mathbb{R}$ be a function that is continuously differentiable on $(0,1/2)$ and $(1/2,1)$ but such that $\lim_{x\uparrow 1/2} u(x) \neq \lim_{x\downarrow 1/2} u(x)$. Draw an example for such a function. Calculate the generalised/distributional derivative of u using the definition (that is, without just refering to the script or Wikipedia for the answer).

$$u = \begin{cases} -x+\frac{1}{2}, & x\in\left(0,\frac{1}{2}\right)\\ x, & x\in\left(\frac{1}{2},1\right)\end{cases}$$



$$\lim_{x\uparrow\frac{1}{2}} u = \lim_{x\uparrow\frac{1}{2}}\left(-\frac{1}{2}+\frac{1}{2}\right)=0$$

$$\lim_{x\downarrow\frac{1}{2}} u = \lim_{x\downarrow\frac{1}{2}} x = \frac{1}{2}$$

Generalized derivative $\alpha=1$

$$g(\varphi) = (-1)^{|\alpha|}\, u(D^\alpha\varphi) = -u(\varphi') = -\langle u,\varphi'\rangle_{D'\times D} = -\int_0^1 u\varphi'\,dx$$

$$\overset{!}{=} \langle g,\varphi\rangle_{D'\times D} = \int_0^1 g\varphi\,dx$$

$$\int_0^1 u\varphi'\,dx = \int_0^{\frac{1}{2}}\Big(\underbrace{-x+\tfrac{1}{2}}_{f}\Big)\underbrace{\varphi'}_{g'}\,dx + \int_{\frac{1}{2}}^1 \underbrace{x}_{f}\underbrace{\varphi'}_{g'}\,dx = \left[\int fg' = fg - \int f'g\right]$$

$$= \left(-x+\tfrac{1}{2}\right)\varphi\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}(-1)\varphi\,dx + x\varphi\Big|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1}\varphi\,dx$$

$$= \underbrace{\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)}_{=0}\varphi\left(\tfrac{1}{2}\right) - \underbrace{\tfrac{1}{2}\varphi(0)}_{=0} + \int_0^{\frac{1}{2}}\varphi\,dx + \underbrace{\varphi(1)}_{=0} - \tfrac{1}{2}\varphi\left(\tfrac{1}{2}\right) - \int_{\frac{1}{2}}^{1}\varphi\,dx$$

$$= \int_0^{\frac{1}{2}}\varphi\,dx - \int_{\frac{1}{2}}^{1}\varphi\,dx - \tfrac{1}{2}\varphi\left(\tfrac{1}{2}\right) \qquad \left[\varphi(x) = \int\varphi\,\delta(x-x)\,dx\right.$$

$$= \int_0^1 \left(1-2H\left(x-\tfrac{1}{2}\right)\right)\varphi - \delta(x-1/2)\,\varphi\,dx$$

$$-\int_0^1 u\varphi'\,dx = \int_0^1 \underbrace{\left[\delta\left(x-\tfrac{1}{2}\right)+1-2H\left(x-\tfrac{1}{2}\right)\right]}_{\text{generalized derivative}}\varphi\,dx = \int_0^1 g\varphi\,dx$$

2. Show that the bilinear form $A(\cdot,\cdot)$ corresponding to the Poisson problem with mixed boundary conditions (Section 3.2.2 in the lecture notes) is indeed a continuous and coercive bilinear form on $H^1_D(\Omega)$. Moreover, show that the linear form $f(\cdot)$ given in Section 3.2.2 is a bounded linear form on $H^1_D(\Omega)$.

1 2 3

Coercive: $A(v,v) \geq \alpha_1 \|v\|_V^2$

Continuous: $A(u,v) \leq \alpha_2 \|u\|_V \|v\|_V$

$$\left[f(v) = \int_\Omega f v \, dx + \int_{\Gamma_N} u_N \, tr(v) \, ds + \int_{\Gamma_R} u_R \, tr(v) \, ds \right]$$

Poisson with mixed boundary conditions:

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \alpha \, tr(u) \, tr(v) \, ds$$

Continuity: 1

$$|A(u,v)| = \left| \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \alpha \, tr(u) \, tr(v) \, ds \right|$$

triangle ineq

$$\leq \left| \int_\Omega \underbrace{\nabla u \nabla v}_{L^2 \text{ inner prod}} dx \right| + \left| \int_{\Gamma_R} \alpha \underbrace{tr(v) tr(v)}_{L^2 \text{ inner prod}} ds \right|$$

Cauchy-Schwarz $|(f,g)_{L^2}| \leq \|f\|_{L^2} \|g\|_{L^2}$

$$\leq \|\nabla u\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)} + \|\alpha\|_\infty \|tr(u)\|_{L^2(\Gamma_R)} \|tr(v)\|_{L^2(\Gamma_R)}$$

Theorem 3.13: $\|tr(v)\|_{L^2} \leq C \|v\|_{H^1(\Omega)}$

$\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ so its bigger/eq

$$\leq \|u\|_{H^1_D(\Omega)} \|v\|_{H^1_D(\Omega)} + \|\alpha\|_\infty C_1 C_2 \|u\|_{H^1_D(\Omega)} \|v\|_{H^1_D(\Omega)}$$

$$\boxed{= \underbrace{(1 + \|\alpha\|_\infty C_1 C_2)}_{=: \alpha} \|u\|_{H^1_D(\Omega)} \|v\|_{H^1_D(\Omega)}}$$

Coercivity: $A(v,v) \geq \alpha_1 \|v\|_V^2$ ②

$$\|v\|^2_{H^1_D(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

Poincaré–Friedrich: $\|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}$

$$\leq C\|\nabla v\|_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

$$= (C+1)\|\nabla v\|_{L^2(\Omega)}$$

$$\leq (C+1)\|\nabla v\|^2_{L^2(\Omega)} + (C+1)\|\sqrt{\alpha}\, tr(v)\|^2_{L^2(\Gamma_R)}$$

$$= (C+1)\left(\int_\Omega \nabla v \nabla v \, dx + \int_{\Gamma_R} \alpha \, tr(v) \, tr(v) \, ds\right)$$

$$= (1+c)\, A(v,v)$$

$$\rightarrow \|v\|^2_{H^1_D(\Omega)} \leq (1+c)\, A(v,v)$$

$$\boxed{A(v,v) \geq \underbrace{\frac{1}{1+c}}_{\alpha} \|v\|^2_{H^1_D(\Omega)}}$$

3

$$f(v) = \int_{\Omega} f v \, dx + \int_{\Gamma_N} u_N \, tr(v) \, ds + \int_{\Gamma_R} u_R \, tr(v) \, ds$$

$$|f(v)| \leq \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} + \|u_N\|_{L^2(\Gamma_N)} \|tr(v)\|_{L^2(\Gamma_N)}$$

$$+ \|u_R\|_{L^2(\Gamma_R)} \|tr(v)\|_{L^2(\Gamma_R)}$$

Theorem 3.13 (trace inequality)

$+ \|v\|_{L^2} \leq \|v\|_{H^1_D}$

$$\leq \|v\|_{H^1_D(\Omega)} \|f\|_{L^2(\Omega)} + \|u_N\|_{L^2(\Gamma_N)} C_1 \|v\|_{H^1_D(\Omega)}$$

$$+ \|u_R\|_{L^2(\Gamma_R)} C_2 \|v\|_{H^1_D(\Omega)}$$

$$\boxed{= \Big(\|f\|_{L^2(\Omega)} + \underbrace{C_1 \|u_N\|_{L^2(\Gamma_N)} + C_2 \|u_R\|_{L^2(\Gamma_R)}}_{\geq 0} \Big) \|v\|_{H^1_D(\Omega)}}$$

3. (**Fourier description of Sobolev spaces**) In some special domains, Sobolev spaces have a nice description in terms of Fourier series. Take $\Omega = (0,1)$ and recall that any $f \in L^2(\Omega)$ can be written in a complex Fourier series

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is equivalent to the seminorm $|f|_{H^m}$ (that is, for some constant C, one has $|f|_{\tilde{H}^m} \leq C|f|_{H^m}$ and $|f|_{H^m} \leq C|f|_{\tilde{H}^m}$ for all $f \in H^m$).

b) Based on the above observation, give a 1 line proof of the Poincaré inequality.

c) Based on the above, how would you define $H^{1/2}(\Omega)$? How about $H^{-1}(\Omega)$?

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$$\Big| \int_\Omega f(x)g(x)\,dx \Big| \le \Big(\int_\Omega |f(x)|^p\,dx \Big)^{1/p} \Big(\int_\Omega |g(x)|^q\,dx \Big)^{1/q}, \qquad (1)$$

which is just Cauchy-Schwartz if $p = q = 2$. Use this to prove the following general form of Poincaré inequality in 1D, for arbitrary $p \in [1, \infty)$:

To prove $$\int_0^1 |v(x)|^p dx \le C_p \int_0^1 |v'(x)|^p dx$$

with some constant C_p, for all v in the more general Sobolev space

$$W_0^{1,p} = \Big\{ v : \int_0^1 |v(x)|^p dx < \infty, \int_0^1 |v'(x)|^p dx < \infty, v(0) = v(1) = 0 \Big\}.$$

Hint: Use the fundamental theorem of calculus $v(x) - v(0) = \int_0^x v'(y)dy$.

Let $f(x) = |v'(x)|$, $g(x) = 1$, $\Omega = [0,1]$

in (1)

$$\Big| \int_\Omega |v'(x)|\,dx \Big| \le \Big(\int_\Omega |v'(x)|^p dx \Big)^{1/p} \quad \Big|\ ^{\wedge} p$$

$$\Big| \int_\Omega |v'(x)|\,dx \Big|^p \le \int_\Omega |v'(x)|^p dx$$

Close but not quite

5. Let $\overline{H}^1(\Omega) := \{u \in H^1(\Omega) \ : \ \int_\Omega u \, dx = 0\}$. Show that $\overline{H}^1(\Omega)$ is a complete subspace of $H^1(\Omega)$. As a consequence, show that for any $f \in L^2(\Omega)$, the weak form of the pure Neumann problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \nabla u \cdot n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

2

has a unique solution in $\overline{H}^1(\Omega)$.

(*Optional:* What happened to the compatibility condition $\int_\Omega f = 0$? How come we don't need it?)

1

Complete subspace: every Cauchy sequence of points in space M has a limit that is also in M

$(v_n)_{n\in\mathbb{N}} \subseteq \overline{H}^1(\Omega)$ ... Cauchy sequence

To show: $\exists v \in \overline{H}^1(\Omega): \lim_{n\to\infty} v_n = v$

i) pick $v_n \in \overline{H}^1(\Omega) \in H^1(\Omega)$

H^1 is a Banachspace - so its complete & normed

$\longrightarrow \exists v \in H^1(\Omega): \lim_{n\to\infty} v_n = v$

$\boxed{\longrightarrow v \in H^1(\Omega)}$

ii) show $v \in \overline{H}^1(\Omega)$ (i.e. show $\int_\Omega v\,dx = 0$) $\quad |\cdot| = \sqrt{(\cdot)^2}$

$$\left|\int_\Omega v\,dx\right| = \left|\int_\Omega v\,dx - \underbrace{\int_\Omega v_n\,dx}_{=0\ [v_n \in \overline{H}^1(\Omega)]}\right| = \left|\int_\Omega v - v_n\,dx\right| = \sqrt{\left(\int_\Omega v - v_n\,dx\right)^2}$$

$\left|\left(\int_\Omega v\,dx\right)^2 \leq C \int_\Omega |v|^2\,dx\right|$

$$\leq \sqrt{C}\sqrt{\int_\Omega |v - v_n|^2\,dx} = C\|v - v_n\|_{L^2}$$

$$\leq C\|v - v_n\|_{H^1} = 0 \quad \uparrow\ n\to\infty$$

$\longrightarrow \left|\int_\Omega v\,dx\right| = 0 \longrightarrow \boxed{\int_\Omega v\,dx = 0}$

② $-\Delta u = f \quad | \cdot v \mid \int$

$$-\int_\Omega \Delta u\, v\, dx = \int_\Omega f v\, dx$$ | Integration by parts S. 29

$$-\int_{\partial\Omega} \underbrace{\nabla u \cdot n}_{=0 \text{ (boundary conditions)}}\, tr(v)\, ds + \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx$$

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f(v)\, dx = l(v)$$

Lax-Milgram: A, cont. coercive and l bounded?

A continuous?

$$|A(u,v)| = \left|\int_\Omega \nabla u\, \nabla v\, dx\right| \overset{C.S.}{\leq} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$
$$\leq \|u\|_{\bar{H}^1(\Omega)} \|v\|_{\bar{H}^1(\Omega)} \checkmark$$

A coercive?

$$\|u\|^2_{\bar{H}^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$$

Poincaré-Inequality (Lemma 3.10): $\|u\|_{L^2(\Omega)} \leq C\left(\|\nabla u\|_{L^2(\Omega)} + \left|\int_\Omega u\, dx\right|\right)$

$$\leq C^2\Big(\|\nabla u\|_{L^2(\Omega)} + \underbrace{\int_\Omega u\, dx}_{=0,\ u \in \bar{H}^1(\Omega)}\Big) + \|\nabla u\|^2_{L^2(\Omega)}$$

$$= (1 + C^2)\|\nabla u\|_{L^2(\Omega)} = (1+C)\, A(u,u) \rightarrow A(u,u) \geq \frac{1}{1+C}\|u\|^2_{\bar{H}^1} \checkmark$$

l bounded?

$|l(v)| \leq \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$ ✓

By Lax-Milgram there exists a unique solution

6. We now aim to implement the pure Neumann problem in 1D: Let $\Omega = (0,1)$ and $\mathcal{T}$ be a uniform mesh with $N+1$ points. Take V_N to be the space of piecewise linear functions and $\overline{V}_N = V_N \cap \overline{H}^1$. Therefore the FEM reads as: $u_N \in \overline{V}_N$ s.t.

$$A(u_N, v_N) = \int_\Omega u_N' v_N' \, dx = \int_\Omega f v_N \, dx = \ell(v_N) \qquad \forall v_N \in \overline{V}_N.$$

The idea is now to take the hat functions $\{\varphi_i \,:\, i = 1, \dots, N+1\}$ as basis of V_N and enforce the condition $\int_0^1 u_N dx = 0$ separately.

Define the stiffness matrix $\mathbf{A} \in \mathbb{R}^{(N+1)\times(N+1)}$ given as $\mathbf{A}_{ij} = A(\varphi_i, \varphi_j)$ and the load vector $\mathbf{l}$ given as $\mathbf{l}_i = \ell(\varphi_i)$. Show that, provided $\sum_i \mathbf{l}_i = 0$, writing $u_N = \sum_i \mathbf{u}_i \varphi_i$ the FEM formulation leads to the linear system of equations

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{P}^T \end{pmatrix} \cdot \mathbf{u} = \begin{pmatrix} \mathbf{l} \\ 0 \end{pmatrix}, \qquad 1$$

where $\mathbf{P} \in \mathbb{R}^{N+1}$ is given as $\mathbf{P}_i := \int_\Omega \varphi_i dx$. Note that this implies that we get a solution to a symmetric $(N+2)\times(N+2)$ system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{l} \\ 0 \end{pmatrix}.$$

1

$$l(v_N) = l\left(\sum_{i=1}^{N+1} v_i \varphi_i\right) = \sum_{i=1}^{N+1} v_i\, l(\varphi_i) = \sum_{i=1}^{N+1} v_i\, l_i$$

$$\|$$

$$A(u_N, v_N) = \sum_{j=1}^{N+1} u_j \sum_{i=1}^{N+1} v_i\, A(\varphi_j, \varphi_i) = \sum_{i=1}^{N+1} v_i \sum_{j=1}^{N+1} u_j A_{ij}$$

KVG:

$$\boxed{\sum_{j=1}^{N+1} u_j A_{ij} = l_i \iff Au = l}$$

$$p^T u = \sum_i p_i u_i = \sum_i \underbrace{\int_0^1 \varphi_i(x)\,dx}_{p_i \text{ (Definition)}}\, u_i$$

$\in \mathbb{R}$, just a number

$u_N \in \overline{H}^1 \to \int_0^1 u_N dx = 0$

$$= \int_0^1 \sum_i u_i \varphi_i(x)\, dx = \int_0^1 u_N \, dx = 0$$

$$\boxed{\to p^T u = 0}$$

$$\boxed{\to \begin{pmatrix} A \\ p^T \end{pmatrix} \cdot u = \begin{pmatrix} l \\ 0 \end{pmatrix}}$$