Sheet 9

Discussion of the sheet: Tue./Wed., January 10/11.

This exercise sheet is concerned with the topics

- (mostly) nonlinear ODEs

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1. Some special nonlinear differential equations can be transformed to linear equations. Such examples are the *Bernoulli equations*, which have the form

$$u'(x) + P(x)u(x) = Q(x)u^n(x),$$

where $n \neq 0, 1$ and P, Q are given functions.

a) Show that the transformation $v = u^{1-n}$ transforms the Bernoulli equation into the linear ODE

$$v'(x) + (1-n)P(x)v(x) = (1-n)Q(x).$$

b) Use this to solve the Bernoulli equation

$$u' + 2xu = u^2.$$

2. Use the variation of constants formula to solve the ODE

$$u'' - \frac{3}{x}u' - \frac{5}{x^2}u = \log(x).$$

In order to derive the general solution for the homogeneous equation, make the ansatz $u(x) = x^\alpha$ and compute $\alpha \in \mathbb{N}$.

3. (Picard iteration) The Picard-Lindelöf/Cauchy-Lipschitz theorem in fact gives more than what we discussed in the lecture: it also provides a way to approximate solutions of the initial value problem

$$\begin{aligned} u' &= f(t, u) \\ u(0) &= u_0. \end{aligned}$$

This is the so-called *Picard iteration*, defined as

$$\begin{aligned} \varphi_0(t) &= u_0 \\ \varphi_n(t) &= u_0 + \int_0^t f(s, \varphi_{n-1}(s))\, ds \qquad n \geq 1. \end{aligned}$$

The produced sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges (provided the conditions in the Picard-Lindelf theorem hold) to the solution u of the ODE.

a) Compute the first 3 elements in the Picard iteration for

$$u'(t) = tu(t), \qquad u(0) = 1.$$

To which function does the sequence converge?

b) Compute the first 4 elements in the Picard iteration for

$$u'(t) = u^2(t), \qquad u(0) = 1.$$

To which function does the sequence seem to converge?

4. Consider the function

$$f(u) = \begin{cases} 1 & \text{if } u \leq 0 \\ -1, & \text{if } u > 0. \end{cases}$$

a) Show that the initial value problem

$$\begin{aligned} u' &= f(u) \\ u(0) &= 1 \end{aligned}$$

does not have a solution.

b) Check why Peano's theorem doesn't apply.

5. Continuing with the equation from the previous exercise, let us test some approximation methods.

a) Compute the first 4 elements in the Picard iteration, and see if it looks convergent.

b) Another approximation method for ODEs is the *Euler method*, defined as follows. For each n, we define not a continuous function u, but rather a function φ_n on gridpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots$, inductively as follows:

$$\varphi_n(0) = u_0, \qquad \varphi_n(\tfrac{k+1}{n}) = \varphi(\tfrac{k}{n}) + \frac{f\left(\frac{k}{n}, u(\frac{k}{n})\right)}{n}, \ k \geq 0.$$

Compute the Euler approximation of the ODE from the previous exercise with $n = 2, 4, 6$. Does this look convergent?