Sheet 7

Discussion of the sheet: Tue./Wed., December 13/14

This exercise sheet is concerned with the topics

- Fourier transform

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1. Let $f(t) = \exp(-|t|)$. Compute the Fourier transformation of f. Use the Fourier inversion formula to show

$$\frac{\pi}{2}\exp(-|t|) = \int_0^\infty \frac{\cos(\omega t)}{1+\omega^2} d\omega.$$

2. Use the definition of the Fourier transformation to prove the properties

a) $\mathcal{F}[f(at)](\omega) = \frac{1}{a}\mathcal{F}[f(t)]\left(\frac{\omega}{a}\right)$,

b) $\mathcal{F}[f(t+a)](\omega) = e^{ia\omega}\mathcal{F}[f(t)](\omega)$.

3. (Heisenberg's uncertainty principle) In quantum mechanics, particles are described by complex-valued functions. Given such a function $f : \mathbb{R} \to \mathbb{C}$, the position of a particle is random with density function $|f|^2$, that is, the probability of being in any region A is $\int_A |f(x)|^2\,dx$. In particular, we assume $\int_{\mathbb{R}} |f(x)|^2\,dx = 1$. By Plancherel's identity we know that we also have $\int_{\mathbb{R}} |\hat{f}(\omega)|^2\,d\omega = 1$, so $|\hat{f}|^2$ is also a probability density – of the momentum of the particle. Heisenberg's uncertainty principle simply says that the variance of both the position and the momentum can not be both small:

$$\int_{\mathbb{R}} x^2|f(x)|^2\,dx \int_{\mathbb{R}} \omega^2|\hat{f}(\omega)|^2\,d\omega \geq \frac{1}{4}.$$

We can prove this inequality in one (long) line, explain each step ($\Re$ is the real part of a complex number):

$$\begin{aligned}1 &= \int_{\mathbb{R}} |f(x)|^2\,dx = -\int_{\mathbb{R}} x(|f(x)|^2)'\,dx = -2\Re\int_{\mathbb{R}} xf(x)\bar{f}'(x)\,dx \leq 2\int_{\mathbb{R}} |x||f(x)||f'(x)|\,dx \\ &\leq 2\Big(\int_{\mathbb{R}} x^2|f(x)|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}} |f'(x)|^2\,dx\Big)^{1/2} = 2\Big(\int_{\mathbb{R}} x^2|f(x)|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}} \omega^2|\hat{f}(\omega)|^2\,d\omega\Big)^{1/2}.\end{aligned}$$

4. Let

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

a) Find $\mathcal{F}[f]$,

b) Compute the convolution $f \star f$,

c) Without any further computation, find $\mathcal{F}[f \star f]$.

5. Using the function from the previous exercies, prove the following identities:

a) $\int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} d\omega = \pi$,

b) $\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$.