# Problem Sheet 6

discussion: week of Monday, 5.12.2022

*Note that this week there are only 3 examples, the remaining time of the exercise is used to discuss the correct solutions to the first exam.*

**6.1.** Consider the three recursions

- $x_{n+1} = \frac{x_n}{3}$ with $x_0 = 1$,
- $y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1}$ with $y_0 = 1, y_1 = \frac{1}{3}$,
- $z_{n+1} = \frac{10}{3}z_n - z_{n-1}$ with $z_0 = 1, z_1 = \frac{1}{3}$.

These recursions in explicit form lead to the sequence $(\frac{1}{3})^n$. Write a program (in `matlab`/`python`) that realizes each recursion and computes the absolute and relative errors between $\left(\frac{1}{3}\right)^N$ and $x_N, y_N, z_N$ for different values of $N$. Also try your program with slightly perturbed initial values, i.e., $x_0 = y_0 = z_0 = 1 + 10^{-14}$ and $y_1 = z_1 = \frac{1}{3} + 10^{-14}$. What do you observe?

*Note:* `matlab` has the possibility to compute results in higher precision (which you can use to compute $\left(\frac{1}{3}\right)^N$) using the `vpa` library (use `help vpa` for a documentation).

**6.2.** Consider the function

$$\varphi(x) = \sqrt{x+1} - \sqrt{x}$$

**a)** Is the evaluation of $\varphi$ well-conditioned for large $x$? Consider relative conditioning.

**b)** Formulate a stable numerical realization of $\varphi$ (*Hint:* You may use that a stable realization of $\sqrt{\cdot}$ is available.)

**6.3.** The sequence $u_k$, $k = 0, 1, \ldots$, given by[1]

$$u_1 := 2, \qquad u_{k+1} = 2^k \sqrt{2\left(1 - \sqrt{1 - (2^{-k}u_k)^2}\right)} \tag{1}$$

converges to the number $\pi = 3.1415\ldots$.

**a)** Compute (in `matlab`/`python`) the first 30 members of the sequence and the absolute error $|\pi - u_k|$. When is the error minimal?

**b)** Explain why you should expect that the error grows for $k \geq k_0$ for some $k_0$.

[1] The $u_k$ correspond to the circumference of regular polygons with $2^k$ edges; this method of approximating $\pi$ is due to Archimedes