Exam - Solutions and Point Distribution

Example	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
max. Points	10	11	15	15	9

The point distributions for each subexample are written below.

1. **a)** The characteristic equation for the ODE is

$$\lambda^3 + 2\lambda^2 - 3\lambda = 0.$$

with solutions $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -3$, which gives the general solution

$$u_h(x) = C_1 + C_2 e^x + C_3 e^{-3x}.$$

b) Since e^x is already in the homogeneous solution space, the correct ansatz for the particular solution is

$$u_p(x) = \alpha x e^x$$

and plugging that into the equation gives $\alpha = 1/4$.

c) By Theorem 6.1 we have that the solution space of the homogeneous ODE is a vector space. A basis is $\{1, e^x, e^{-3x}\}$ and the dimension is 3.

Adding two solutions of the inhomogeneous problem, however, produces a function where the particular solution part is not $1/4xe^x$, hence it is not closed with respect to $+$ and consequently no vector space.

Point distribution: a) 3, b) 3, c) 4 (1 point for each question)

2. **a)** The ODE (with $A = 2xy^2 + \cos x$ and $B = 2x^2y + y$) is exact, since

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

Therefore, it can be solved by integrating A with respect to x and B with respect to y and comparing the functions (and constants of integration), which gives the equation

$$x^2y^2 + \sin x + y^2 = C,$$

which can be rearranged for y.

b) The computation of the scalar potential is identical to the previous example, hence, $\psi = \nabla\phi$ with

$$\phi = x^2y^2 + \sin x + y^2.$$

c) For gradient fields, the line integral is path independent (and C_1 and C_2 have the same starting and endpoint), hence,

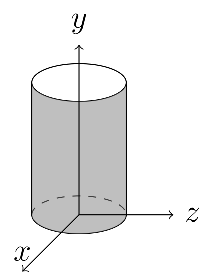
$$\int_{C_1} \psi \, ds = \int_{C_2} \psi \, ds = \phi(1,1) - \phi(0,0) = 2 + \sin 1.$$

Point distribution: a) + b) 5+2 or 2+5 (depending on where the computational work is done), c) 4 (2 points for each integral)

3. **a)** Let $S \subset \mathbb{R}^3$ be an open surface with boundary ∂S. Let ψ be a continuously differentiable vector field. Then,

$$\int_S \operatorname{curl}\psi \cdot dS = \int_{\partial S} \psi \cdot ds.$$

b) Cylinder mantle, rotational axis y



Parametrization with cylindric coordinates

$$r(u,v) = \begin{pmatrix} \cos u \\ v \\ \sin u \end{pmatrix} \qquad u \in [0, 2\pi], v \in [0,1]$$

produces the <u>outer</u> normal vector

$$n = -\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{pmatrix} \cos u \\ 0 \\ \sin u \end{pmatrix}$$

The boundary $\partial S = C_1 \cup C_2$ consists of two curves (bottom and top) parametrized by

$$r_1(t) = \begin{pmatrix} \cos t \\ 1 \\ \sin t \end{pmatrix}, \qquad r_2(t) = \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix} \qquad t \in [0, 2\pi].$$

c) We have

$$\operatorname{curl}\psi = \begin{pmatrix} 0 \\ y \\ 2xy - z \end{pmatrix}$$

and

$$\int_S \operatorname{curl}\psi \cdot dS = \int_0^1 \int_0^{2\pi} \operatorname{curl}\psi \cdot n du dv = \int_0^1 \int_0^{2\pi} 2\cos u \sin u - \sin^2 u \, du = -\pi.$$

For the line integrals, we have

$$\int_{C_1} \psi \cdot ds = \int_0^{2\pi} \psi(r_1(t)) \cdot r_1(t)' dt = \int_0^{2\pi} -\sin^2 t \, dt = -\pi$$

and

$$\int_{C_2} \psi \cdot ds = \int_0^{2\pi} \psi(r_2(t)) \cdot r_2(t)' dt = \int_0^{2\pi} 0 dt = 0.$$

Point distribution: a) 2, b) 5 (1 for drawing, 2 for normal vector, 1 for each boundary param.), c) 8 (4 for surface, 2 for each line integral)

4. **a)** We have an odd function, consequently, we have

$$a_j = 0 \qquad \forall j \in \mathbb{N}_0.$$

The b_j are computed with integration by parts as

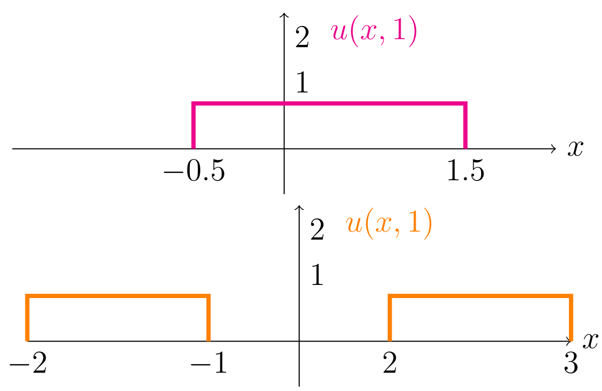
$$b_j = \int_{-1}^{1} -2x\sin(\pi jx)dx = -2x\frac{cos(\pi jx)}{\pi j} + \int_{-1}^{1} 2\frac{cos(\pi jx)}{\pi j}dx = \frac{4}{\pi j}(-1)^j.$$

Consequently, the Fourier series is

$$F(x) = \sum_{j=1}^{\infty} \frac{4}{\pi j}(-1)^j \sin(\pi jx).$$

By Dirichlet's theorem, the value of F is equal to the value of the periodic continuation of f at every point but on the jump points (which are the points $1 + 2k$ with $k \in \mathbb{Z}$). There, the value is the mean value of the left and right of the jump, hence, 0.

b) We have that $ct = 0.5$ or $ct = 2$ and using d'Alembert's formula gives:



Point distribution: a) 9 (6 for coefficients, 3 for theory question), b) 6 (3 for each drawing)

5. Answer the following questions (Simply write true/false on your sheet).

a) Which of the following vector identities hold? (ψ vector field, ϕ scalar function)

1. $\mathrm{curl}(\nabla\psi) = 0$. TRUE
2. $\nabla\,\mathrm{div}\,\psi = \Delta\psi$. FALSE
3. $\mathrm{div}(\phi\psi) = \phi\,\mathrm{div}(\psi) + \nabla\phi\cdot\psi$. TRUE

b) A function $f(z) = u(x,y) + iv(x,y)$ is complex differentiable, if

1. Real and imaginary part are differentiable in $\mathbb{R}^2$. FALSE
2. The limit of the complex difference quotient exists in directions parallel to the real and imaginary axis. FALSE
3. The equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ hold and u, v are cont. differentiable. TRUE
4. The limit of the complex difference quotient exists and is unique regardless of the direction taken. TRUE

c) A line integral $\oint_C \psi \cdot ds$ over a closed curve $C \subset \mathbb{R}^2$ for a continuously differentiable vector field ψ is zero,

1. always. FALSE
2. if, $\operatorname{curl} \psi = 0$. FALSE
3. if, there exists a scalar potential for ψ. TRUE
4. if, $\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial x}$ and the region bounded by C is simply connected. TRUE

d) An ODE $u' = f(x, u)$

1. is always solvable. FALSE
2. has a unique solution, if f is continuous. FALSE
3. has a solution, if f is continuous, but sometimes the solution is not unique. TRUE
4. may not have a solution, but if it does, it is unique, provided f is continuous. FALSE

Point distribution: a) 3, b) 2, c) 2, d) 2, every wrong answer deduces 1 point