Sheet 2

Discussion of the sheet: Tue., 21.03.2023

This exercise sheet is concerned with the topics

- Weak formulation
- Bilinear forms

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1. **a)** Implement last week's first exercise in MATLAB, with $N = 4, 8, 16, 32$. Is there any change in the error in the nodal values, i.e., $|u(x_j) - u_j|$ for $j \in \{2, \ldots, N\}$?

b) Change the right-hand side of the equation from $1 + x$ to $2x^2 + 3x - 4/3$. Find again the exact solution and run the code with the new right-hand side. Calculate the errors

$$\max_{j \in \{2,\ldots,N\}} |u(x_j) - u_j|$$

for each $N = 4, 8, 16, 32$.

2. (**In very special cases the FEM and FDM are actually equivalent**) Consider the 1-dimensional Poisson equation with Dirichlet boundary conditions. Take an equidistant grid $x_i = (i-1)/N$. Show that the finite difference method and the finite element method (with V_h being the space of piecewise linear continuous functions as in the introductory example) yield the same approximate function.

3. Give a compatibility condition on the function g so that the Poisson equation with Neumann boundary conditions

$$\begin{aligned} -u'' &= f \qquad\qquad \text{in } (0,1) \\ u'(0) &= g(0) \\ u'(1) &= g(1) \end{aligned}$$

has a solution. What additional constraint can be added for the uniqueness of the solution u?

4. Derive a weak formulation for the problem

$$\begin{aligned} -u'' + u &= f \qquad \text{in } (0,1) \\ u(0) - 2u'(0) &= 0 \\ u(1) + 2u'(1) &= 0 \end{aligned}$$

with Robin boundary conditions.

5. Let $b, c > 0$. Find the bilinear form on H_0^1 associated to the equation

$$\begin{aligned} -u'' + bu' + cu = f \qquad & \text{in } (0,1) \\ u(0) = u(1) = 0 & \end{aligned}$$

Is it symmetric? Is it coercive?

6. **a)** Let $X = \{u \in C^4([0,1]) \ : \ u(0) = u(1) = u'(0) = u'(1) = 0\}$. Show that

$$A(u,v) = \int_0^1 \Delta^2(u) v dx$$

is a symmetric bilinear form on $X \times X$. (Here, Δ^2 denotes the operator $\Delta(\Delta u)$.)

b) Let $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Is

$$A(u,v) = u^T T v$$

a coercive bilinear form on $\mathbb{R}^2 \times \mathbb{R}^2$ (with the Euclidean norm)?