UE 10

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10.1) $u_i' = \frac{u_{i+1} - u_{i-1}}{2h}$ or if we are on an edge: $\frac{u_{i+1} - u_i}{h}$, $\frac{u_i - u_{i-1}}{h}$

$$f(u_i) = \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + u_i^3 - 1 = 0$$

$$f'(u_i) = \frac{f(u_i) - f(u_{i-1})}{u_i - u_{i-1}}$$

$$u_{i,n+1} = u_{i,n} - \frac{f(u_{i,n})}{f'(u_{i,n})}$$

$$\lambda \|v_1\|_2^2 + \lambda\|v\| + \lambda\|v\| + \Theta\|\varepsilon_l\|_2 =$$
$$= 3\lambda$$

$$|\lambda_n - \tilde{\lambda}_l| \le C \left|\frac{\lambda_n - \lambda}{\lambda_{n-1} - \lambda}\right|^l$$

$$\vec{x}_0 = \sum_i \alpha_i \vec{v}_i \quad \rightarrow (\vec{A} - \lambda)\, x$$

10.4) $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad x_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad x_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad x_0^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eigenvalues of A: $\lambda_1 = 2$, $\lambda_2 = -2$

$|\lambda_1| = |\lambda_2| \rightarrow \left|\frac{\lambda_1}{\lambda_2}\right| = 1$ → no convergence

eigenvectors of A: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x_0^{(1)}$:
$$x_0 = \frac{x_0}{\|x_0\|_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$x_1 = \frac{A x_0}{\|A x_0\|} = \frac{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\tilde{\lambda}_1 = (1 \;\; 0) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2$$

yes it converges

$x_0^{(2)}$:
$$x_0 = \frac{x_0}{\|x_0\|_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$x_1 = \frac{\begin{pmatrix} 0 \\ -2 \end{pmatrix}}{2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\tilde{\lambda}_1 = (0 \; -1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (0 \; -1) \begin{pmatrix} 0 \\ 2 \end{pmatrix} = -2$$
$$x_2 = \frac{\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\tilde{\lambda}_2 = (0 \;\; 1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2$$

no convergence (yes because just a factor)

$x_0^{(3)}$:
$$x_0 = \frac{x_0}{\|x_0\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$x_1 = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\|A x_0\|} = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ -2 \end{pmatrix}}{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda_1 = \frac{1}{2}(1 \; -1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(1 \; -1) \begin{pmatrix} 2 \\ +2 \end{pmatrix} = 0$$
$$x_2 = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = \frac{1}{2}(1 \;\; 1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

no convergence

10.5)

10.1) $f(x) = \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + u_i^3 - 1$

$$f'(x) = \frac{1}{h^2}\begin{pmatrix} 2 + 3u_i^2 h^2 - 1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & \end{pmatrix}$$

$$f(x) = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ 0 & -1 & 2 & -1 \\ & & \ddots & \ddots \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} + \begin{pmatrix} u_1^3 \\ \vdots \\ u_N^3 \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

10.2) just 1 func. eval. because we have previously calculated $F(x_{n-1})$

10.3) $(A - \lambda I)^l x_l = x_0 \quad \Rightarrow x_l = ((A - \lambda I)^{-1})^l x_0$

$$\tilde{v}_i = \frac{1}{\lambda_i - \lambda}$$
$$x_0 = \sum \alpha_i v_i$$

$$= \sum_1^n \alpha_i \left(\frac{1}{\lambda_i - \lambda}\right)^l v_i$$
$$= \frac{\alpha_n}{\lambda_n - \lambda}\Bigg(v_n + \underbrace{\sum_1^{n+1} \frac{\alpha_i}{\alpha_n}\left(\frac{\lambda_n - \lambda}{\lambda_i - \lambda}\right)^l v_i}\Bigg)$$

$$\|\cdot\|_2 \le \sum \left|\frac{\alpha_i}{\alpha_n}\right| \left|\left(\frac{\lambda_n - \lambda}{\alpha_i - \alpha}\right)\right|^l \cdot \|v_i\|_2 \le \underbrace{C_1}_{>0} \left|\frac{\lambda_n - \lambda}{\lambda_{n-1} - \lambda}\right|^l$$

$$\tilde{\lambda}_l = x_l^H A x_l = \frac{x_l^H A x_l}{\|x_l\|^2} = \frac{(v_n + \varepsilon_l)^H A (v_n + \varepsilon_l)}{\|v_n + \varepsilon_l\|^2} =$$
$$= \frac{x_n \|v_n\|^2 + \Theta(\varepsilon_l)}{\|v_n\|^2 + \Theta(\varepsilon_l)} = \lambda_n + \alpha(n,n)$$

$$|\lambda_n - \lambda_l| \le C_2 \|\varepsilon_l\|$$

$$\underbrace{v_n^T A v_n} + v_n^T A \varepsilon_l + \varepsilon_l^T A v_n + \varepsilon_l^T A \varepsilon_l$$
$$\lambda_n \|v_n\|^2 + C\|\varepsilon\| + C\|\varepsilon\| + C\|\varepsilon\|^2$$
$$= \lambda_n \|v_n\|^2 + \Theta\|\varepsilon\|$$