1) $S := \{ B \in \mathbb{R}^{2\times 2},\ B = B^T \}$

i) $x + y \overset{?}{=} y + x$

$\begin{pmatrix} x & z \\ z & y \end{pmatrix} + \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} + \begin{pmatrix} x & z \\ z & y \end{pmatrix}$

$\begin{pmatrix} x+a & z+c \\ z+c & y+b \end{pmatrix} = \begin{pmatrix} x+a & c+z \\ c+z & b+y \end{pmatrix} \in S \square$

ii) $\begin{pmatrix} x & z \\ z & y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \quad \square$ $\quad \alpha \begin{pmatrix} x & z \\ z & y \end{pmatrix} = \begin{pmatrix} \alpha x & \alpha z \\ \alpha z & \alpha y \end{pmatrix} \in S$

iii) $\alpha \begin{pmatrix} \beta x & \beta z \\ \beta z & \beta y \end{pmatrix} = (\alpha \beta) \begin{pmatrix} x & z \\ z & y \end{pmatrix} \quad \square$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & z \\ z & y \end{pmatrix} = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \quad \square$

$\alpha \left(\begin{pmatrix} x & z \\ z & y \end{pmatrix} + \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right) = \alpha \begin{pmatrix} x & z \\ z & y \end{pmatrix} + a \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$(\alpha + \beta) \begin{pmatrix} x & z \\ z & y \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)x & (\alpha+\beta)z \\ (\alpha+\beta)z & (\alpha+\beta)y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta x & \alpha z + \beta z \\ \alpha z + \beta y & \alpha y + \beta y \end{pmatrix}$

$= \begin{pmatrix} \alpha x & \alpha z \\ \alpha z & \alpha y \end{pmatrix} + \begin{pmatrix} \beta x & \beta z \\ \beta z & \beta y \end{pmatrix} \quad \square$

l.v? $\quad x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ x_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

No 4th vector due to symmetry

2) i) For $\forall \alpha, \beta \in \mathbb{R}$, $\alpha x \notin S$ Ex: $\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\alpha = 1{,}1$

$\begin{pmatrix} 1{,}1 x_1 \\ 1{,}1 x_2 \end{pmatrix} \notin S$

ii) $\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 > 0$, $\alpha = -1$

$\begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \notin S$

iii) $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, $\quad \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \notin S$

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3) a) $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$ Norm on $\mathbb{R}^n$?

i) $\|\alpha x\|_\infty = \max\{|\alpha x_1|, \dots, |\alpha x_n|\}$

$= \max\{|\alpha||x_1|, \dots, |\alpha||x_n|\}$

$= |\alpha| \max\{|x_1|, \dots, |x_n|\}$

$= |\alpha| \|x\|_\infty$

ii) $\|x+y\|_\infty \overset{?}{\leq} \|x\|_\infty + \|y\|_\infty$

$\max\{|x_1+y_1|, \dots, |x_n+y_n|\} \leq |x_{max}| + |y_{max}|$

& $\forall z =$

if $\underbrace{x_n = x_{max} \wedge y_n = y_{max}}_{\text{n ou-}}$ $=$, else $\leq$

iii) $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

$|x_{max}| \geq 0$ □

b) $\|x\|_1 = \sum_{i=1}^n |x_i|$ Norm on $\mathbb{R}^n$?

i) $\|\alpha x\|_1 = \sum_{i=1}^n |\alpha x_i| = \sum_{i=1}^n |\alpha||x_i| = |\alpha| \sum_{i=1}^n |x_i| = |\alpha| \|x\|_1$

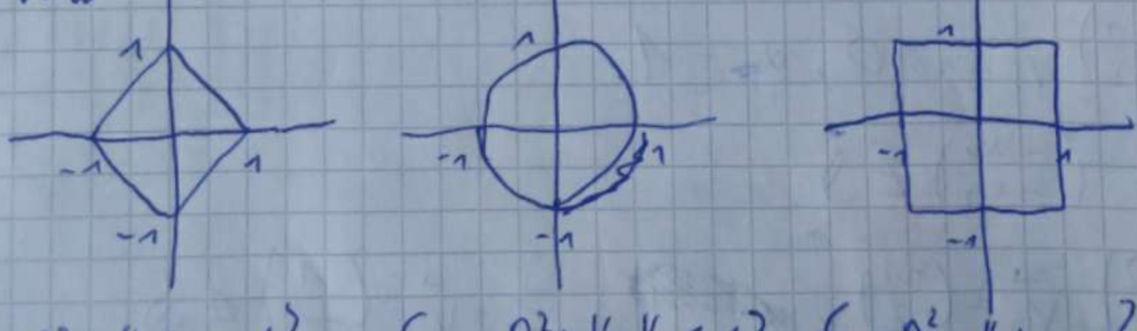
ii) $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$

$\sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|$

if $\forall x_i, y_i \geq 0$, then $=$, else $<$

iii) $\|x\|_1 = \sum_{i=1}^n |x_i| \geq 0$ □ (0 if $\forall x_i = 0$)

Draw



$\{x \in \mathbb{R}^2 : \|x\|_1 \leq 1\}$ $\{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$ $\{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$

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4) $u = \begin{pmatrix} t \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$ $v = \begin{pmatrix} s \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$(u,v) = c^2 ts - x_1 y_1 - \dots - x_n y_n \quad , \quad c > 0$$

i) $(u,v) \overset{?}{=} \overline{(v,u)}$

$$c^2 ts - x_1 y_1 - \dots - x_n y_n = c^2 st - y_1 x_1 - \dots - x_n y_n \quad \square$$

ii) $(\alpha u_1 + \beta_2 u_2, v) = c^2 (t_1 + t_2) s - (\alpha x_{1u_1} + \beta x_{1u_2}) y_1 - \dots - (\alpha x_{nu_1} + \beta x_{nu_2}) y_n$

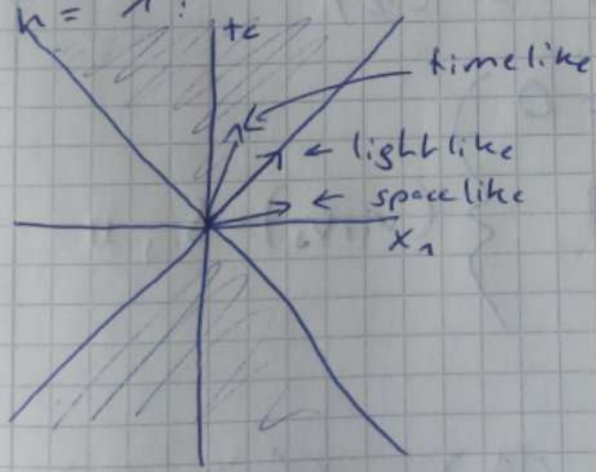
$$= c^2 t_1 s - \alpha x_{1u_1} y_1 - \dots - \alpha x_{nu_1} y_n$$
$$+ c^2 t_2 s - \beta x_{1u_2} y_1 - \dots - \beta x_{nu_2} y_n = \alpha (u_1, v) + \beta (u_2, v) \quad \square$$

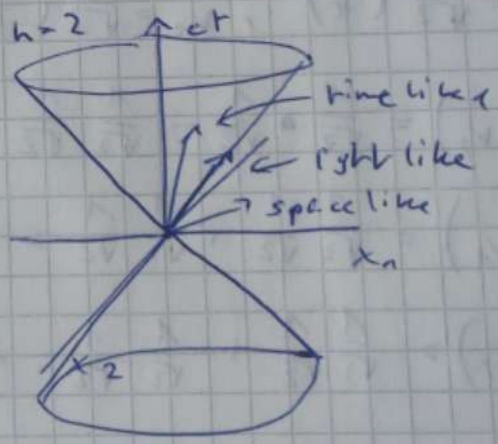
iii) $(u,u) \overset{?}{\geq} 0$

Let $t = 0$, $x_i > 0$

then $(u,u) = 0 - x_1^2 - \dots - x_n^2 < 0 \; ! \; ⚡$

$n = 1$:





5) $\alpha_1 v_1 + \dots + \alpha_n v_n = w$

$$\alpha_i \overset{?}{=} (w, v_i)$$

$$\alpha_i = (\alpha_1 v_1 + \dots + \alpha_n v_n / v_i)$$
$$= \alpha_1 (v_1, v_i) + \dots + \alpha_n (v_n, v_i)$$

$\{v_1, \dots, v_n\}$ are orthonormal:

$$\alpha_i (v_i, v_j) = \alpha_i \delta_{ij}$$

Thus

$$\alpha_i = \delta_{1i}\alpha_1 + \dots + \delta_{ii}\alpha_i + \dots + \delta_{ni}\alpha_n$$

$$\alpha_i = \alpha_i \quad \square$$

Since for each coefficient we only retain one vector, the solution for a given linear combination is unique $\rightarrow \{v_1, \dots, v_n\}$ are lin. indep.

6) $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $w = \begin{pmatrix}1\\2\end{pmatrix}$

$$\left.\begin{aligned}
(v_1, v_2) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0 \\
(v_1, v_1) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 \\
(v_2, v_2) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} = 1
\end{aligned}\right\} (v_i, v_j) = \delta_{ij} \quad \square$$

$$\alpha_1 v_1 + \alpha_2 v_2 = w \rightarrow \alpha_1 \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} + \alpha_2 \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\alpha_1 = \sqrt{2} - \alpha_2$$

$$\alpha_1 - \alpha_2 = 2\sqrt{2} \rightarrow \sqrt{2} - \alpha_2 - \alpha_2 = 2\sqrt{2}$$

$$-\sqrt{2} = 2\alpha_2 \rightarrow \boxed{\alpha_2 = -\frac{\sqrt{2}}{2}}$$

$$\boxed{\alpha_1 = \sqrt{2} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}}$$