Problem Sheet 10

discussion: week of Monday, 16.01.2023

10.1. Consider the for $N \in \mathbb{N}$ the system of equations

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + u_i^3 = 1, \qquad i = 1, \dots, N-1, \qquad u_0 = u_N = 0.$$

Formulate Newton's method for its solution and program it in `matlab`/`python`. Estimate the error (in the $\|\cdot\|_2$-norm) by considering the difference of two consecutive iterates. Plot the error versus the iteration number.

Remark: the above system of equations results from the numerical approximation of the "boundary value problem"

$$-u''(x) + (u(x))^3 = 1, \qquad x \in (0,1), \qquad u(0) = u(1) = 0.$$

The values u_i are approximations to the true values $u(x_i)$ with $x_i = ih$, $i = 1, \dots, N-1$.

10.2. The *secant method* (i.e., Broyden's method in 1D) to find the zero x^* of $F(x) = 0$ is defined as follows given initial points x_0, x_1:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{F(x_n) - F(x_{n-1})} F(x_n), \qquad n = 1, 2, \dots$$

(If $F(x_n) = F(x_{n-1})$ the difference quotient is formally replaced with $F'(x_n)$.) Let $F(x) = 2 - x^2 - e^x$.

1. Compute, using Newton's method the positive zero x^* of F to machine precision.
2. Compute the zero x^* with the secant method. Set $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$. Plot for $n \in \{1, \dots, 8\}$ and $x_0 = 2.5$ the error $|x^* - x_n|$ versus the step number n. Also plot the *numerical convergence order* $p_n = \log(|x^* - x_{n+1}|)/\log(|x^* - x_n|)$ versus n. What convergence order do you observe?
3. Compare the *efficiency* of the secant method with that of the Newton method by comparing accuracy versus number of function evaluations. To that end, assume that a Newton step costs 3 function evaluations (this is realistic assuming that F' is approximated with a difference quotient) and plot achieved accuracy versus number of function evaluations. Which method is more efficient?

10.3. Show the following convergence result for the inverse iteration with shift: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be diagonalizable and $\lambda \in \mathbb{R}$. Let the eigenvalues of $\mathbf{A}$ be numbered such that $|\lambda_1 - \lambda| \geq |\lambda_2 - \lambda| \geq \cdots |\lambda_{n-1} - \lambda| > |\lambda_n - \lambda|$. Then there exists $C > 0$ such that there holds for the approximations $\widetilde{\lambda}_\ell$ of the inverse iteration:

$$|\lambda_n - \widetilde{\lambda}_\ell| \leq C \left| \frac{\lambda_n - \lambda}{\lambda_{n-1} - \lambda} \right|^\ell, \qquad \ell = 0, 1, \dots,$$

10.4. Consider the vector iteration ("power method") for the matrix $\mathbf{A}$ and the following three initial vectors $x_0^{(j)}$, $j = 0, 1, 2$:

$$\mathbf{A} = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}, \qquad \mathbf{x}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_0^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Discuss the behavior of the vector iteration. Do the eigenvalue approximations $\widetilde{\lambda}_\ell$ and the iterates $\mathbf{x}_\ell$ converge? If so, what do they converge to?

10.5. Use power iteration and inverse iteration (take starting values of your choice) to compute approximations $\widetilde{\lambda}_\ell$ ($\ell = 0, 1, \dots$ are the number of steps in the interations) to both eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 5 & 6 \end{pmatrix}.$$

Plot the error between the computed approximations and the true eigenvalues for $\ell = 0, 1, \dots$.