UE 9

Freitag, 6. Januar 2023 15:03

9.1) $|x|' = \frac{x}{|x|}$ $\qquad \left(|x|^{3/2}\right)' = \frac{3}{2}|x|^{1/2}\frac{x}{|x|} = \frac{3}{2}\frac{x}{\sqrt{|x|}}$

nodes: $f_1: 0$

$f_2: \ln(2)$

$f_3: 0$

$f_4: 1$

$$2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 + \frac{\frac{1}{2.1}-1}{\frac{1}{2.1^2}} = 2.1 + 2.1 - 2.1^2 = 4.2 - 4.41 = 0.21$$

$$\phi^{Newton}(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f4: \phi^{Newton}(x_n) = x_n - \boxed{\frac{\frac{1}{x_n}-1}{-\frac{1}{x_n^2}}} = x_n + x_n - x_n^2 = 2x_n - x_n^2$$

overshoots the zero

$|x_{n+1}| > |x_n|$ for all n

$$\|x^* - x_{n+1}\| \leq C\|x^* - x_n\|^2 \leq C^3\|x^* - x_{n-1}\|^4 \leq C^7\|x^* - x_{n-2}\|^8$$

quadratic convergence: $|x^* - x_{n+1}| \approx C|x^* - x_n|^2$

from plot: quadratic convergence given for: f2, (f4)

not given for: f1, f3

Theorem 6.8: if $\phi'(x^*) = 0$ ... faster conv. than linear

$$f2: \phi(x) = x - \frac{e^x - 2}{e^x} = x - 1 + 2e^{-x}$$

$$\phi'(x) = 1 - 2e^{-x}$$

$$\phi'(\ln(2)) = 1 - 2e^{-\ln 2} = 1 - \frac{2}{2} = 0$$

$$f1: \phi(x_n) = x_n - \frac{x_n^2}{2x_n} = x_n - \frac{x_n}{2} = \frac{x_n}{2}$$

$$\rightarrow \phi(x^*) = \phi(0) = 0 = x^*$$

$$\phi'(x_n) = \frac{1}{2} \quad \rightarrow \phi'(x^*) = \frac{1}{2} \neq 0$$

$$f3: \phi(x) = x - \frac{|x|^{3/2}}{\frac{3}{2}\frac{x}{\sqrt{|x|}}} = x - \frac{2}{3}\frac{|x|^{3/2}\cdot|x|^{1/2}}{x} = x - \frac{2}{3}\frac{|x|^2}{x} = x - \frac{2}{3}x = \frac{1}{3}x$$

$$\phi'(x) = \frac{1}{3} \quad \rightarrow \phi'(x^*) = \frac{1}{3} \neq 0$$

9.2) $f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3x_1 - \cos(x_2x_3) - \frac{3}{2} \\ 4x_1^2 - 625x_2^2 + 2x_3 - 1 \\ 20x_3 + e^{-x_1x_2} + 9 \end{pmatrix}$

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 3 & x_3\sin(x_2x_3) & x_2\sin(x_2x_3) \\ 8x_1 & -1250x_2 & 2 \\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20 \end{pmatrix}$$

Remark 6.12 *In practice the Newton step is* not *realized by computing the inverse* $(f')^{-1}$ *but by solving a linear system:*

1. *compute* $f'(x_n)$ *and the residual* $f(x_n)$
2. *compute the correction by solving the linear system* $f'(x_n)\delta = f(x_n)$
3. *perform the update* $x_{n+1} := x_n - \delta$ ∎

$$\phi^{Newton}(x) = x - \left(f'(x)\right)^{-1}f(x)$$

$$\|x_n - x^*\| \sim \|x_{n+1} - x_n\| = \|f'(x)^{-1}f(x)\|$$ from Taylor expansion of $f(x_n) = f(x_n) - f(x^*) \approx f'(x^*)(x_n - x^*)$

9.3) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \varepsilon\begin{pmatrix} (x_1 - x_2)^2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0.01(x_1 - x_2)^2 \\ 3 \end{pmatrix}$$

$$F(x) = \begin{pmatrix} 2x_1 + x_2 - \varepsilon(x_1 - x_2)^2 \\ x_1 + 2x_2 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$F'(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2 - 2\varepsilon(x_1 - x_2) & 1 + 2\varepsilon(x_1 - x_2) \\ 1 & 2 \end{pmatrix}$$

$$x_{n+1} = x_n - \underbrace{F'(x_n)}_{= A - \varepsilon f'(x_n)}{}^{-1} F(x_n)$$

$$x_{n+1} = x_n - A^{-1}F(x_n)$$

$$F'(x_n)x_{n+1} = F'(x_n)x_n - F(x_n)$$

b) +epsilon*f(x_n) term is a "step" like f'(x)^-1 * f(x)

9.4) $F(x) = Ax - b - \varepsilon f(x)$

$$F'(x) = A - \varepsilon f'(x)$$

$$x_{n+1}^{Newton} = x_n - F'(x_n)^{-1}F(x_n)$$

$$\|F(x_n)\| \leq C\|e_n\| \qquad e_n = x_* - x_n$$

show: $\|x_{n+1} - x_n\| \approx \|x_* - x_n\|$ for $x_{n+1}: Ax_{n+1} = b + \varepsilon f(x_n)$

$\|F(x_n)\| = C\|e_n\|$ $\quad e_n \quad x_* \quad x_n$

show: $\|x_{n+1} - x_n\| \approx \|x_* - x_n\|$ for x_{n+1}: $A x_{n+1} = b + \varepsilon f(x_n)$

$$\left(b + \varepsilon f(x)\right)^{-1} = b^{-1} + O(\varepsilon) \approx b^{-1}$$

$$x_{n+1}^N - x_{n+1} = x_n - \left(A^{-1} + \Theta(\varepsilon)\right)\left(A x_n - b - \varepsilon f(x)\right) - A^{-1} b - \varepsilon A^{-1} f(x_n)$$

$$= x_n - x_n - \Theta(\varepsilon) F(x_n) = -\Theta(\varepsilon) F(x)$$

$$\|x_{n+1}^N - x_{n+1}\| = \Theta(\varepsilon)\,\|F(x_n)\| \leq \Theta(\varepsilon)\, C\, \|e_n\|$$

not really true but somehow still holds

$$\|x_{n+1} - x_n\| + \underbrace{\|x_{n+1}^N - x_n\|}_{\sim \|e_n\|} \leq \Theta(\varepsilon)\, \tilde{C}\, \|e_n\|$$

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - x_{n+1}^N\| + \|x_{n+1}^N - x_n\|$$

$$\|F(x_n) - \underbrace{F(x^*)}_{=0}\|$$