1. Write down the 1D finite difference scheme to the Poisson equation

$$\begin{aligned} -u'' &= 1 + x \qquad && \text{in } (0,1) \\ u(0) &= u(1) = 0 \end{aligned}$$

on an equidistant grid of mesh-width $h{=}1/N$. Solve this ODE exactly, draw the solution and the finite difference approximation for $N = 4$ grid points. What is the error in the nodal values, i.e., $|u(x_j) - u_j|$ for $j \in \{0, \dots, N\}$?

$$-u'' = 1 + x \quad \Big| \int$$

$$-u' = x + \frac{x^2}{2} + C_1 \quad \Big| \int$$

$$\boxed{-u = \frac{x^2}{2} + \frac{x^3}{6} + C_1 x + C_2}$$

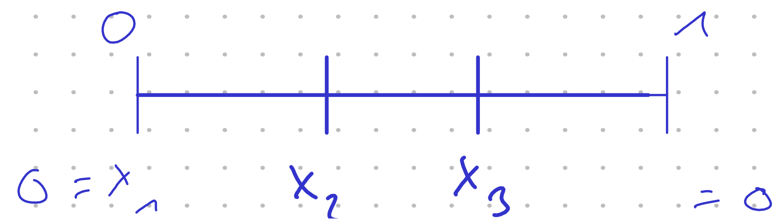
$$u = -\frac{x^3}{6} - \frac{x^2}{2} - C_1 x - C_2$$

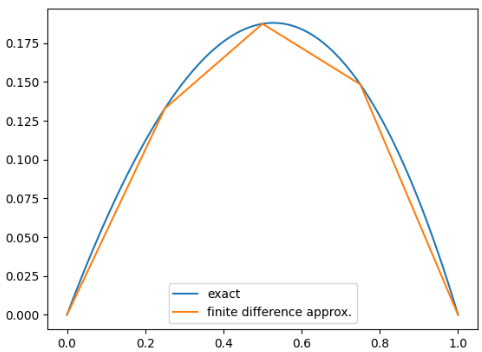
$$u(0) = 0 = -C_2 \longrightarrow \boxed{C_2 = 0}$$

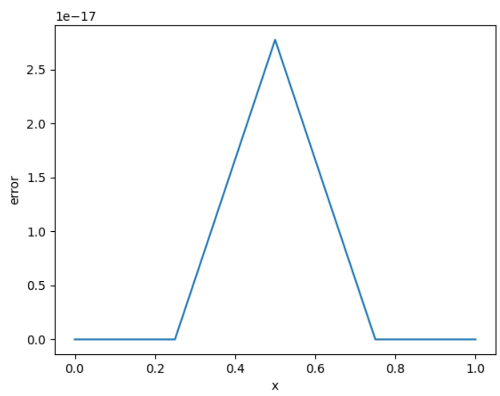
$$u(1) = 0 = -\frac{1}{6} - \frac{1}{2} - C_1$$

$$\boxed{\frac{\cancel{4}^{\,2}}{\cancel{6}_{\,3}} = C_1}$$

$$\boxed{-u = \frac{x^2}{2} + \frac{x^3}{6} - \frac{2}{3}x}$$







using results from next page →

FD form

$$-\frac{1}{h^2}\left(v_{i-1} - 2v_i + v_{i+1}\right) = 1 + x_i = 1 + \frac{i-1}{N},\ i = 2,\dots,N-1$$

$$v_1 = v_N = 0 \quad , h = \frac{1}{N} = \frac{1}{4}$$

$$\rightarrow -16\left(v_{i-1} - 2v_i + v_{i+1}\right) = 1 + \frac{i}{N}$$

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = -\frac{1}{16}\begin{pmatrix} 0 \\ 5/4 \\ 6/4 \\ 7/4 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix} = -\frac{1}{16}\begin{pmatrix} 5/4 \\ 6/4 \\ 7/4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 6 & 5/4 \\ 1 & -2 & 1 & 6/4 \\ 0 & 1 & -2 & 7/4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 6/4 \\ -2 & 1 & 6 & 5/4 \\ 0 & 1 & -2 & 7/4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 6/4 \\ 0 & -3 & 2 & 17/4 \\ 0 & 1 & -2 & 7/4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 6/4 \\ 0 & 1 & -2 & 7/4 \\ 0 & -3 & 2 & 17/4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 20/4 \\ 0 & 1 & -2 & 7/4 \\ 0 & 0 & -4 & 38/4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 40/8 \\ 0 & 1 & -2 & 14/8 \\ 0 & 0 & 1 & -19/8 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{17}{8} \\ 0 & 1 & 0 & -\frac{14}{8} \\ 0 & 0 & 1 & -\frac{19}{8} \end{array}\right) \rightarrow \begin{array}{l} v_2 = \frac{17}{128} \\ v_3 = \frac{19}{128} \\ v_4 = \frac{19}{128} \end{array} \qquad \frac{16 \cdot 8}{128} \qquad \frac{19 \cdot 3}{57}$$

node error is zero!

2. (Consistency error of 1D-FD) Assume that $u \in C^4([0,1])$ (i.e. 4-times continuously differentiable). Show that for h sufficiently small and a constant $C > 0$, there holds

$$\left| \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right) - u''(x) \right| \le Ch^2.$$

What is the error if we use the one-sided approximation twice, i.e.

$$u''(x) \approx \frac{u'(x+h) - u'(x)}{h} \approx \frac{\frac{u(x+2h)-u(x+h)}{h} - \frac{u(x+h)-u(x)}{h}}{h} = \frac{u(x+2h) - 2u(x+h) - u(x)}{h^2}?$$

hint: Use Taylor expansion.

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)h^2}{2} + \frac{u'''}{6}(x)h^3 + \frac{u''''}{24}(x)h^4$$

$$u(x-h) = u(x) - u'(x)h + \frac{u''}{2}(x)h^2 - \frac{u'''}{6}(x)h^3 + \frac{u''''}{24}(x)h^4$$

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + \frac{u''''}{12}(x)h^4$$

$$\frac{u(x+h) - 2u(x) + u(x-h) - u''(x)}{h^2} = \frac{u''''}{12}(x)h^2$$

$$\frac{u(x+h) - 2u(x) + u(x-h) - u''(x)}{h^2} \le Ch^2$$

$$u(x+2h) = u(x) + u'(x)2h + \frac{u''(x)}{2}4h^2 + \frac{u'''}{6}(x)8h^3$$

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)h^2}{2} + \frac{u'''}{6}(x)h^3$$

$$u(x+2h) - 2u(x+h) = u(x) + u''(x)h^2 + u'''(x)h^3$$

$\cdot 4$

2. (Consistency error of 1D-FD) Assume that $u \in C^4([0,1])$ (i.e. 4-times continuously differentiable). Show that for h sufficiently small and a constant $C > 0$, there holds

$$(1) \qquad \left| \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right) - u''(x) \right| \leq Ch^2.$$

What is the error if we use the one-sided approximation twice, i.e.

$$u''(x) \approx \frac{u'(x+h) - u'(x)}{h} \approx \frac{\frac{u(x+2h)-u(x+h)}{h} - \frac{u(x+h)-u(x)}{h}}{h} = \frac{u(x+2h) - 2u(x+h) - u(x)}{h^2}?$$

hint: Use Taylor expansion.

ξ: a point where the taylor expansion is exact

Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f''''(\xi)}{24}(x-a)^4, \ \xi \in [x,a]$$

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + \frac{u'''(x)}{6}h^3 + \frac{u''''(\xi_1)}{24}h^4 \qquad (2)$$

$$u(x-h) = u(x) - u'(x)h + \frac{u''(x)}{2}h^2 - \frac{u'''(x)}{6}h^3 + \frac{u''''(\xi_2)}{24}h^4 \qquad (3)$$

insert into (1)

$$\left| \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right) - u''(x) \right| = \quad \text{add (2) and (3)}$$

$$= \left| \frac{1}{h^2}\left(\cancel{2u(x)} + \cancel{u''(x)h^2} + \frac{u''''(\xi_1)}{24}h^4 + \frac{u''''(\xi_2)}{24}h^4 - \cancel{2u(x)}\right) - \cancel{u''(x)} \right|$$

$$= \left| \underbrace{\frac{h^2}{24}\left(u''''(\xi_1) + u''''(\xi_2)\right)}_{\text{since } u \in C^4 \text{ this is a constant } C^*} \right|$$

$$\boxed{\left| \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right) - u''(x) \right| = \left| h^2 \frac{C^*}{24} \right| \leq |hC|}$$

$$u(x+2h) = u(x) + u'(x)2h + u''(x)2h^2 + u'''(x)\frac{4}{3}h^3 + u''''(\xi_0)\frac{2}{3}h^4$$

$$\left|\frac{1}{h^2}\Big(u(x+2h) - 2u(x+h) + u(x)\Big) - u''(x)\right| =$$

$$u(x+2h) = u(x) + u'(x)2h + u''(x)2h^2 + u'''(x)\frac{4}{3}h^3 + u''''(\xi_0)\frac{2}{3}h^4$$

$$2u(x+h) = 2u(x) + 2u'(x)h + u''(x)h^2 + \frac{u'''(x)}{3}h^3 + \frac{u''''(\xi_1)}{12}h^4$$

$$-u(x) + u'''(x)h^3 + u''''(\xi_0)\frac{2}{3}h^4 - \frac{u''''(\xi_1)}{12}h^4 + u''(x)$$

$$= \left|\frac{1}{h^2}\Big(\cancel{-u(x)} + \cancel{u''(x)h^2} + u'''(x)h^3 + u''''(\xi_0)\frac{2}{3}h^4 - \frac{u''''(\xi_1)}{12}h^4 + \cancel{u(x)}\Big) - \cancel{u''(x)}\right| =$$

$$= \left|\frac{1}{h^2}\Big(u'''(x)h^3 + u''''(\xi_0)\frac{2}{3}h^4 - \frac{u''''(\xi_1)}{12}h^4\Big)\right| =$$

$$= \left|u'''(x)h + u''''(\xi_0)\frac{2}{3}h^2 - \frac{u''''(\xi_1)}{12}h^2\right| \leq C(h + h^2)$$

3. Show that the space $H^1(0,1)$ is a vector space. (1) Moreover, show that

$$(2)\quad \|u\|^2_{H^1(0,1)} := \|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)}$$

is a norm on $H^1(0,1)$ and

$$(3)\quad (u,v)_{H^1} := (u',v')_{L^2} + (u,v)_{L^2} = \int_0^1 u'v' + uv\,dx$$

is an inner product on $H^1(0,1)$.

$$\{v \in L^2(\Omega), v' \in L^2(\Omega)\} =: H^1(\Omega)$$

Note: differential omitted in the following

(1) i) $\int_0^1 v+u = \underbrace{\int_0^1 v}_{\in L^2} + \underbrace{\int_0^1 u}_{\in L^2} < \infty$

$$\sqrt{\int |(v+u)'|^2} = \|(v+u)'\|_{L^2} \le \underbrace{\|v'\|_{L_2}}_{\in L^2} + \underbrace{\|u'\|_{L^2}}_{\in L^2} < \infty$$

ii) $\int_0^1 0 < \infty$, $\int_0^1 v+0 < \infty$

$\int_0^1 0' < \infty$, $\int_0^1 (v+0)' = \int v' < \infty$

iii) $\int_0^1 av = a\underbrace{\int_0^1 v}_{\in L^2} < \infty$

$\int_0^1 (av)' = \int_0^1 av' = a\underbrace{\int_0^1 v'}_{\in L^2} < \infty$

(2) $\|u\|^2_{H^1(0,1)} := \|u\|^2_{L^2(0,1)} + \|u'\|^2_{L^2(0,1)}$

(7) is a norm on $H^1(0,1)$ and

Short: $\|\cdot\|_{L^2}$ is a norm, so $\|u\|^2_{H^1}$ is still a norm

Long:

i) $\|\alpha u\|^2_{H^1} \overset{?}{=} |\alpha|^2 \|u\|^2_{H^1}$

$\|\alpha u\|^2_{H^1} = \|\alpha u\|^2_{L^2} + \|\alpha u'\|^2_{L^2}$ | $\|\cdot\|^2_{L^2}$ is a norm $\to \|\alpha u\|_{L^2} = |\alpha| \|u\|_{L^2}$

$= |\alpha|^2 \|u\|^2_{L^2} + |\alpha|^2 \|u'\|^2_{L^2}$

$$\boxed{\|\alpha u\|^2_{H^1} = |\alpha|^2 (\|u\|^2_{L^2} + \|u'\|^2_{L^2})}$$

(also $\|\alpha u\|_{H^1} = |\alpha| (\|u\|_{H^1})$)

ii.) $\|u+v\|^2_{H^1} \overset{?}{\leq} \|u\|^2_{H^1} + \|v\|^2_{H^1}$

$\|u+v\|^2_{H^1} = \|u+v\|^2_{L^2} + \underbrace{\|(u+v)'\|^2_{L^2}}_{= \|u'+v'\|^2_{L^2}}$ | triangle inequality for L^2: $\|u+v\|^2_{L^2} \leq \|u\|^2_{L^2} + \|v\|^2_{L^2}$

$= \int_0^1 u^2 + 2uv + v^2 \, dx + \int_0^1 (u')^2 + 2u'v' + (v')^2$ Cauchy-Schwarz $\|uv\| \leq \|u\|\|v\|$

$= \|u\|^2_{H^1} + 2\|uv\|^2_{H^1} + \|v\|^2_{H^1} \leq \|u\|_{H^1} + 2\|u\|_{H^1}\|v\|_{H^1} + \|v\|^2_{H^1}$

$= (\|u\|_{H^1} + \|v\|_{H^1})^2 \to \|u+v\|_{H^1} \leq \|u\|_{H^1} + \|v\|_{H^1}$

$$\boxed{\|u+v\|^2_{H^1} \leq \|u\|^2_{H^1} + \|v\|^2_{H^1}}$$

iii.) $\|u\|_{H^1} \geq 0$ and $\|u\| \iff u = 0$

$\|u\|^2_{H^1} = \underbrace{\|u\|^2_{L^2}}_{\geq 0 \text{ (norm)}} + \underbrace{\|u'\|^2_{L^2}}_{\geq 0 \text{ (norm)}}$

$$\boxed{\|u\|^2_{H^1} \geq 0}$$

(3) $(u,v)_{H^1} := (u',v')_{L^2} + (u,v)_{L^2} = \int_0^1 u'v' + uv\,dx$

is an inner product on $H^1(0,1)$.

(3)

i) $(u,v)_{H^1} \overset{?}{=} \overline{(v,u)}$ complex conjugate symmetry

$$\boxed{(u,v)_{H^1} = \int_0^1 u'v' + uv\,dx = \int_0^1 v'u' + vu\,dx = (v,u)_{H_1}}$$

ii) $(\alpha u + \beta v, w)_{H_1} \overset{?}{=} \alpha (u,w)_{H_1} + \beta(\beta u, w)_{H_1}$ linearity

$$(\alpha u + \beta v, w)_{H^1} = \int_0^1 (\alpha u + \beta v)' w' + (\alpha u + \beta v) w\,dx$$

$$= \int_0^1 (\alpha u' + \beta v') w' + \alpha u w + \beta v w\,dx$$

$$= \int_0^1 \alpha u' w + \alpha u w\,dx + \int_0^1 \beta v' w' + \beta v w\,dx$$

$$= \alpha \int_0^1 u' w + uw\,dx + \beta \int_0^1 v' w' + vw\,dx$$

$$\boxed{(\alpha u + \beta v, w)_{H^1} = \alpha (u,w)_{H^1} + \alpha (v,w)_{H^1}}$$

iii) $(u,u)_{H^1} \geq 0$ positive definiteness

$$\boxed{(u,u)_{H^1} = \int_0^1 \underbrace{(u')^2}_{\text{positive}} + \underbrace{u^2}_{\text{functions}} \geq 0}$$

(0 if $u = 0$)

4. Show that a weak solution to the 1D Poisson equation

$$\boxed{a(u,v) := \int_\Omega u'v'dx = \int_\Omega fv dx =: l(v) \qquad \forall v \in V.}$$

$$-u'' = f \quad \text{in } (0,1)$$
$$u = 0 \quad \text{on } \{0,1\}$$

is also a classical (strong) solution, if additionally $u \in C^2([0,1])$.

Weak solution:

$$a(u,v) = \boxed{\int_0^1 u'v'\,dx = \int_0^1 fv\,dx} = l(v), \quad v \in H_1(0,1)$$

$$(u'v)' = \boxed{u''v + u'v'}$$

$$-u''v = -(u'v)' + u'v' \to \int -u''v = -u'v| + \int u'v'$$

$u \in C^2$, so u'' exists

$$-\int_0^1 \underbrace{u''}_{f'}\underbrace{v}_{g}\,dx = \underbrace{\cancel{-u'v|_0^1}}_{\text{end of } v \in H_0^1} + \int_0^1 \underbrace{u'}_{f}\underbrace{v'}_{g'}\,dx$$

$$\to -\int_0^1 u''v\,dx = \int_0^1 u'v'\,dx = \int_0^1 fv\,dx$$

$$\int_0^1 u''v\,dx + \int_0^1 fv\,dx = 0$$

$$\int_0^1 (u''+f)v\,dx = 0 \leftarrow \text{must hold for all } v \in H_1(0,1)$$

hence $(u''+f)$ must be 0

$$\boxed{\to -u'' = f}$$

5. For each of the following classes of functions, find a PDE that is satisfied by u for *all* choices of functions f, g (i.e. f, g should not appear in the PDE)

a) $u(x, y) = f(x) + g(y)$,

b) $u(x, y) = f(x + y)$,

c) $u(x, y) = f(x^2 - y^2)$.

a) $\frac{f(x)}{\partial y} = 0$ and $\frac{g(y)}{\partial x} = 0$, hence

$$\boxed{\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0}$$

$(x+y)^2$

b) $\frac{\partial f(x+y)}{\partial x} = \left| g(x,y) = x+y \right| = \frac{\partial f(g(x,y))}{\partial g} \frac{\partial g}{\partial x}$

$= \frac{\partial f(g(x,y))}{\partial g} \cdot 1$

and analoguous: $\frac{\partial f(x+y)}{\partial y} = \frac{\partial f(g(x+y))}{\partial g} \cdot 1$

$$\rightarrow \boxed{\frac{\partial u(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x}}$$

c) $\frac{\partial f(x^2-y^2)}{\partial x} = \left| g(x,y) = x^2 - y^2 \right| = \frac{\partial f(g(x,y))}{\partial g} \cdot \frac{\partial g}{\partial x}$

$= \frac{\partial f(g(x,y))}{\partial g} \cdot 2x$

and analogous: $$\frac{\partial f(x^2-y^2)}{\partial y} = \frac{\partial f(g(x,y))}{\partial g}(-2y)$$

$$\boxed{\longrightarrow \frac{\partial u(x,y)}{\partial x}\cdot(-\cancel{2}y) = \frac{\partial u(x,y)}{\partial y}(\cancel{2}x)}$$