1. Solve the following ODEs

a) $(1-x^2)u'(x) - xu(x) = 0$;

b) $u'(x) = \frac{1+u(x)}{1+x}$;

c) $x\cos(xu)u' + (u\cos(xu) + 2x) = 0$.

Separable ODEs are a special case of first order ODEs that can be written as

$$u'(x) = f(x)g(u),$$

so the variables x and u on the right-hand side can be multiplicatively separated. Division with $g(u)$, integrating the equation in x and using the transformation theorem, we obtain

$$\int \frac{1}{g(u)}du = \int \frac{1}{g(u(x))}u'(x)dx = \int f(x)\,dx.$$

a) $(1-x^2)u'(x) - xu(x) = 0$

$$u'(x) = \frac{xu(x)}{(1-x^2)}$$

separable ODE: $f(x) = \frac{x}{1-x^2}$, $g(u) = u$

$$\int \frac{1}{u}du = \int \frac{x}{1-x^2}dx$$

$\left[z = 1-x^2,\ \frac{dz}{dx} = -2x \rightarrow dx = -\frac{dz}{2x} \right]$

$$\ln(u) + C = \int \frac{\cancel{x}}{z}(-1)\frac{dz}{2\cancel{x}}$$

$$= -\frac{1}{2}\ln(1-x^2) \quad \Big|\, e$$

$$e^{\ln u + C} = e^{-\frac{1}{2}\ln(1-x^2)}$$

$$ue^{C} = (1-x^2)^{-\frac{1}{2}} \qquad [e^{-C} = C']$$

$$\boxed{u = (1-x^2)^{-\frac{1}{2}}C'}$$

b) $u'(x) = \frac{1+u(x)}{1+x}$

Separable ODEs are a special case of first order ODEs that can be written as

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so the variables x and u on the right-hand side can be multiplicatively separated. Division with $g(u)$, integrating the equation in x and using the transformation theorem, we obtain

$$\int \frac{1}{g(u)} du = \int \frac{1}{g(u(x))} u'(x) dx = \int f(x)\, dx.$$

$f(x) = \frac{1}{1+x}$, $g(u) = 1+u$, seperable ODE

$$\int \frac{1}{1+u} du = \int \frac{1}{1+x} dx$$

$$\ln(1+u) = \ln(1+x) + C \quad | e$$

$$e^{\ln(1+u)} = e^{\ln(1+x)+C}$$

$$1+u = (1+x)e^{C} \quad [e^{C} = C']$$

$$\boxed{u = (1+x)C' - 1}$$

c) $x\cos(xu)u' + u\cos(xu) + 2x = 0$

$$u' = \frac{-(u\cos(xu) + 2x)}{x\cos(xu)} = \frac{-A(x,u)}{B(x,u)}, \text{ exact ODE?}$$

exact, if there is a function $H(x,u)$ such that $\frac{\partial H}{\partial x} = A$ and $\frac{\partial H}{\partial u} = B$. In other words, this means that the vectorfield $\begin{pmatrix} A \\ B \end{pmatrix}$ is a gradient field and H is its scalar potential. We recall that one can check whether $\begin{pmatrix} A \\ B \end{pmatrix}$ is a gradient field and correspondingly whether the above ODE is exact, if

$$\frac{\partial A}{\partial u} = \frac{\partial B}{\partial x}.$$

$$\Psi = \begin{pmatrix} u\cos(xu) + 2x \\ x\cos(xu) \end{pmatrix}$$

Solving the ODE is then done by computing the scalar potential $H(x,u)$, since

we could first check if $\frac{\partial A}{\partial u} = \frac{\partial B}{\partial x}$, however let us check instead immediatly if a scalar potential $H(u,x)$ exists:

$$\int u\cos(xu) + 2x \, dx = \sin(xu) + x^2 + C(u)$$

$$\int x\cos(xu)\, du = \sin(xu) + C(x)$$

$\to C(u) = \text{const}, \; C(x) = x^2$ $\to$ exact ODE ! $\frac{\partial A}{\partial u} = \frac{\partial B}{\partial x}$ ✓

$$\to H(x,u) = \sin(xu) + x^2 = c \quad | -x^2 \to | \arcsin$$

$$xu = \arcsin(c - x^2)$$

$$\boxed{u = \frac{1}{x}\arcsin(c - x^2)}$$

2. Find the solutions for the ODEs

a) $u' = \frac{x^2-u}{x}$;

b) $(x^2-u^2)u' - 2xu = 0$,

by computing an integrating factor.

exact, if there is a function $H(x,u)$ such that $\frac{\partial H}{\partial x} = A$ and $\frac{\partial H}{\partial u} = B$. In other words, this means that the vectorfield $\begin{pmatrix} A \\ B \end{pmatrix}$ is a gradient field and H is its scalar potential. We recall that one can check whether $\begin{pmatrix} A \\ B \end{pmatrix}$ is a gradient field and correspondingly whether the above ODE is exact, if

$$\frac{\partial A}{\partial u} = \frac{\partial B}{\partial x}.$$

Solving the ODE is then done by computing the scalar potential $H(x,u)$, since

a) $u' = \frac{x^2-u}{x} = -\frac{A(x,u)}{B(x,u)}$, exact ODE?

$\psi = \begin{pmatrix} x^2-u \\ -x \end{pmatrix}$ $\quad \int x^2-u \, dx = \frac{x^3}{3} - ux + C(u)$

$\int -x \, du = -xu + C(x)$

$\to C(u) = \text{const}, \quad C(x) = \frac{x^3}{3}$

$H(x,u) = \frac{x^3}{3} - ux = c \to$ exact ODE! $\frac{\partial A}{\partial u} = \frac{\partial B}{\partial x}$ ✓

$$\boxed{u = \frac{x^2}{3} - \frac{c}{x}}$$

b) $(x^2-u^2)u' - 2xu = 0$

$u' = \frac{2xu}{x^2-u^2} = -\frac{A(x,u)}{B(x,u)}$, exact ODE?

$A = -2xu, \quad B = x^2-u^2 \to \frac{\partial A}{\partial u} \overset{?}{=} \frac{\partial B}{\partial x} \to -2x \neq 2x$

As we know from the previous chapter, not all vector fields are gradient fields and consequently, we have that not all ODEs are exact. An inexact first order ODE is characterized by

$$\frac{\partial A}{\partial u} \neq \frac{\partial B}{\partial x}.$$

However, in some cases, it is possible to still solve such equations by introducing so called **integrating factors**. The idea is to multiply the ODE with a function $\mu(x,u)$, i.e., one tries to solve the equation $\mu(x,u)A(x,u)dx + \mu(x,u)B(x,u)du = 0$, where μ is such that

$$\boxed{\frac{\partial(\mu A)}{\partial u} = \frac{\partial(\mu B)}{\partial x}.}$$

Thus, the new ODE is exact. In general, if μ is an arbitrary function of both variables, there is no way to compute it. If, however, μ does only depend on one variable, i.e., $\mu = \mu(x)$ or $\mu = \mu(u)$ (or other cases like $\mu = \mu(x+y)$ or $\mu = \mu(xy)$) one has a chance.
For example, if $\mu = \mu(x)$ the condition for an exact ODE reduces to

$$\mu\frac{\partial A}{\partial u} = \mu\frac{\partial B}{\partial x} + B\frac{\partial \mu}{\partial x},$$

which is a separable ODE in μ that can be solved as explained above and we arrive at the integrating factor

$$\mu = \exp\left(\int f(x)dx\right) \quad \text{with} \quad f = \frac{1}{B}\left(\frac{\partial A}{\partial u} - \frac{\partial B}{\partial x}\right).$$

Similarly, if $\mu = \mu(u)$, we have

$$\mu = \exp\left(\int g(u)du\right) \quad \text{with} \quad g = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial u}\right).$$

Ansatz $\mu = \mu(u)$

$\to \mu \frac{\partial A}{\partial u} + A \frac{\partial \mu}{\partial u} = \mu \frac{\partial B}{\partial x}$

$\to \mu(u) = \exp\left(\int g(u) du\right)$

$= \exp\left(\int \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial u}\right) du\right)$

$= \exp\left(\int \frac{1}{-2xu}(2x - -2x) du\right)$

$= \exp\left(\int \frac{-\overset{2}{\cancel{4}}\cancel{x}}{\cancel{2}\cancel{x}u} du\right)$

$$= \exp\left(-2\int \frac{1}{u}\, du\right) = \exp(-2 \ln u)$$

$$\mu(u) = u^{-2}$$

$$\rightarrow \frac{\partial(\mu A)}{\partial u} = \frac{\partial(\mu B)}{\partial x} \qquad \mu A = -2\frac{x}{u}, \ \mu B = \frac{x^2}{u^2} - 1$$

$$2\frac{x}{u^2} = 2\frac{x}{u^2} \quad \checkmark$$

$$\psi = \begin{pmatrix} -2\frac{x}{u} \\ \frac{x^2}{u^2} - 1 \end{pmatrix} \qquad \int -2\frac{x}{u}\, dx = -\frac{x^2}{u} + C(u)$$

$$\int x^2 u^{-2} - 1\, du = \frac{x^2 u^{-1}}{-1} - u + C(x)$$

$$= -\frac{x^2}{u} - u + C(x)$$

$$\rightarrow C(u) = -u, \ C(x) = \text{const}$$

$$\rightarrow H(x,u) = -\frac{x^2}{u} - u = c \qquad \Big| \cdot u \ \Big| + u^2 \ \Big| + x^2$$

$$u^2 + cu + x^2 = 0$$

$$\boxed{u_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - x^2}}$$

3. Find the solutions to the following linear ODEs with initial conditions $u(0) = 0$, $u'(0) = 1$

a) $u'' + u' - 6u = 0$;

b) $u'' - 4u' + 5u = 0$.

a) $u'' + u' - 6u = 0$

Ansatz $u = ce^{\lambda x} \to \lambda^2 ce^{\lambda x} + \lambda ce^{\lambda x} - 6ce^{\lambda x} = 0$

$\lambda^2 + \lambda - 6 = 0$ ... char. polynomial

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6}$$

$$= -\frac{1}{2} \pm \sqrt{\frac{25}{4}} = -\frac{1}{2} \pm \frac{5}{2}$$

$\lambda_1 = 2$, $\lambda_2 = -3$

$$\boxed{u = c_1 e^{2x} + c_2 e^{-3x}}$$

$u(0) = 0$, $u'(0) = 1$

$\hookrightarrow c_1 + c_2 = 0$

$\to c_1 = -c_2$

$\hookrightarrow 2c_1 - 3c_2 = 1$

$2c_1 + 3c_1 = 1 \to 5c_1 = 1$

$\leftrightarrow c_1 = \frac{1}{5}$, $c_2 = -\frac{1}{5}$

$$\boxed{u = \frac{1}{5}\left(e^{2x} - e^{-3x}\right)}$$

b) $u'' - 4u' + 5u = 0$

Ansatz $u = ce^{\lambda x} \to \lambda^2 ce^{\lambda x} - 4\lambda ce^{\lambda x} + 5ce^{\lambda x} = 0$

$\lambda^2 - 4\lambda + 5 = 0$ ... char. polynomial

$\lambda_{1,2} = 2 \pm \sqrt{4-5} = 2 \pm \sqrt{-1} = [\sqrt{-1} = i] = 2 \pm i$

$[e^{ix} = \cos x + i\sin x]$

$\lambda_1 = 2 + i \to u_1 = c_1 e^{(2+i)x} = c_1 e^{2x} e^{ix} = c_1 e^{2x}(\cos(x) + i\sin(x))$

$\lambda_2 = 2 - i \to u_2 = c_2 e^{(2-i)x} = c_2 e^{2x} e^{-ix} = c_2(\cos(x) - i\sin(x))$

$$\boxed{u = e^{2x}\Big(c_1(\cos x + i\sin x) + c_2(\cos x - i\sin(x))\Big) = e^{2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right)}$$

↑ possibly should have stayed in this form, needlessly complicated

$u(0) = 0, \quad u'(0) = 1$

(1) $1 \cdot (c_1(1+0) + c_2(1-0)) = 0$

$u' = 2e^{2x}(c_1(\cos x + i\sin x) + c_2(\cos x - i\sin(x)))$

$+ e^{2x}\Big(c_1(-\sin x + i\cos(x)) + c_2(-\sin(x) - i\cos(x))\Big)$

(2) $2 \cdot 1(c_1(1+0) + c_2(1-0)) + 1(c_1(0+i) + c_2(0-i)) = 1$

(1): $c_1 + c_2 = 0 \to c_1 = -c_2$

(2): $2(c_1 + c_2) + c_1 i - c_2 i = 1$

$= 2(c_1 - c_1) + c_1 i + c_1 i = 1 \to 2c_1 i = 1$

$c_1 = \frac{1}{2i} = -\frac{i}{2}$

$c_2 = \frac{i}{2}$

$$u = e^{2x}\left(-\frac{i}{2}\left(\cos x + i\sin(x)\right) + \frac{i}{2}\left(\cos x - i\sin x\right)\right)$$

$$u = e^{2x}\left(-\frac{i}{2}e^{ix} + \frac{i}{2}e^{-ix}\right)$$

$$= e^{2x}\left(\frac{-ie^{ix} + ie^{-ix}}{2}\,\frac{i}{i}\right)$$

$$= e^{2x}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) \quad \left[\sin x = \frac{e^{ix} - e^{-ix}}{2i}\right]$$

$$\boxed{u = e^{2x}\left(\sin(x)\right)}$$