9.1 Laplace Transform of Some Special Functions

In this lesson we discuss Laplace transform of some special functions like error functions, Dirac delta functions, etc. There functions appears in various applications of science and engineering to some of them we shall encounter while solving differential equations using Laplace transform.

9.2 Error Function

The error appears in probability, statistics and solutions of some partial differential equations. It is defined as

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\,\mathrm{d}u$$

Its complement, known as complementary error function, is defined as

$$\mathrm{erfc}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\,\mathrm{d}u$$

We find Laplace transform of different forms of error function in the following examples.

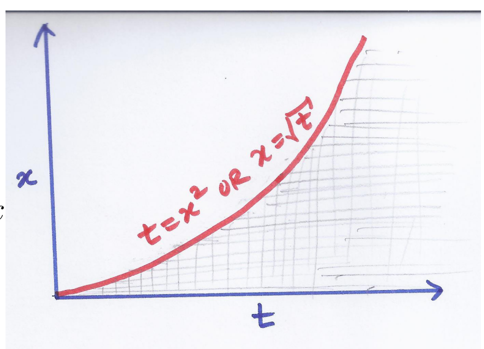
9.3 Example Problems

9.3.1 Problem 1

Find $L[erf(\sqrt{t})]$.

Solution: From definition of the error function and the Laplace transform we have,

$$L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{\sqrt{t}} e^{-st} e^{-x^2}\,\mathrm{d}x\mathrm{d}t$$



By changing the order of integration we get,

$$L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty \int_{t=x^2}^\infty e^{-st} e^{-x^2}\,\mathrm{d}t\mathrm{d}x$$

Evaluating the inner integral we obtain

$$L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty e^{-x^2} \frac{e^{-sx^2}}{s}\mathrm{d}x = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_{x=0}^\infty e^{-(1+s)x^2}\mathrm{d}x$$

Substituting $\sqrt{(1+s)}x = u \Rightarrow dx = \dfrac{1}{\sqrt{1+s}}du$

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \frac{1}{s\sqrt{1+s}} \int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{1}{s\sqrt{s+1}}$$

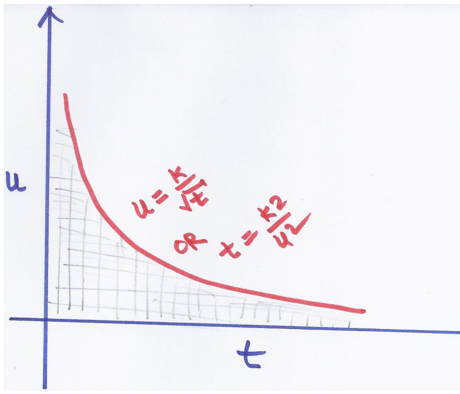
Note that we have used the value of Gaussian integral $\displaystyle\int_{x=0}^{\infty} e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{2}$.

9.3.2 Problem 2

Find $L\left[erf\left(\dfrac{k}{\sqrt{t}}\right)\right]$. *and show that* $L^{-1}\left[\dfrac{e^{-2k\sqrt{s}}}{s}\right] = erfc\left(\dfrac{k}{\sqrt{t}}\right)$

Solution: By the definition of Laplace transform we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \int_0^{\infty} e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{\sqrt{t}}} e^{-u^2} \mathrm{d}u \, \mathrm{d}t$$



Changing the order of integration we get

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\frac{k^2}{u^2}} e^{-st} e^{-u^2} \mathrm{d}t \, \mathrm{d}u$$

Evaluation of the inner integral leads to

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-u^2} \left(1 - e^{-s\frac{k^2}{u^2}}\right) \mathrm{d}u$$

Using the value of Gaussian integral we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}} \frac{1}{s} \left[\frac{\sqrt{\pi}}{2} - \int_0^{\infty} \left(e^{-u^2 - s\frac{k^2}{u^2}}\right) \mathrm{d}u\right] \tag{9.1}$$

Let us assume

$$I(s) = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \mathrm{d}u$$

By differentiation under integral sign

$$\frac{dI}{ds} = \int_0^{\infty} e^{-u^2 - s\frac{k^2}{u^2}} \left(-\frac{k^2}{u^2}\right) \mathrm{d}u$$

Substitution $\frac{\sqrt{s}k}{u} = x \Rightarrow -\frac{\sqrt{s}k}{u^2}du = dx$ leads to

$$\frac{dI}{ds} = -\frac{k}{\sqrt{s}} \int_0^{\infty} e^{-x^2 - s\frac{k^2}{x^2}} \mathrm{d}x = -\frac{k}{\sqrt{s}} I$$

Solving the above differential equation we get

$$\ln I(s) = -2k\sqrt{s} + \ln c \quad \Rightarrow \quad I(s) = ce^{-2k\sqrt{s}}$$

Further note that

$$I(0) = \int_0^\infty e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad c = \frac{\sqrt{\pi}}{2}$$

Therefore, we get

$$I(s) = \frac{\sqrt{\pi}}{2} e^{-2k\sqrt{s}}$$

Substituting this value in the equation (9.1), we obtain

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}\right] = \frac{1 - e^{-2k\sqrt{s}}}{s}$$

Taking inverse Laplace transform on both sides we get

$$\text{erf}\left(\frac{k}{\sqrt{t}}\right) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right]$$

This leads to the desired result as

$$L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - \text{erf}\left(\frac{k}{\sqrt{t}}\right) = \text{erf}_c\left(\frac{k}{\sqrt{t}}\right)$$