Lecture - 4.
25/07/17
Tuesday

Laplace Transform of error funcⁿ:-

errof function $(erf(x)) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha$.

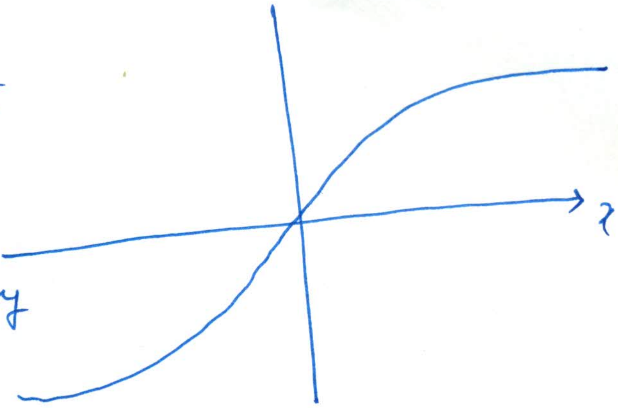
$erf(0) = 0, \quad erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\alpha^2} d\alpha = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$

$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \times \sqrt{\pi} = 1$.

$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-\alpha^2} d\alpha \quad ; \alpha = -u$

$= \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} \times -du = -erf(x)$

$\therefore$ erf is an odd funcⁿ.



graph of $erf(x)$.

$L[erf(x)]$ is determined in terms of complementary error funcⁿ $(erfc(x))$.

$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\alpha^2} d\alpha$.

$= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-\alpha^2} d\alpha - \int_0^x e^{-\alpha^2} da \right]$

$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\alpha^2} d\alpha - \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha = 1 - erf(x)$.

This portion is repeated in p.2

$$L\left[\operatorname{erf}(x)\right] =$$

$\operatorname{erfc}(x)$ = complementary error function.

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\alpha^2}\, d\alpha .$$

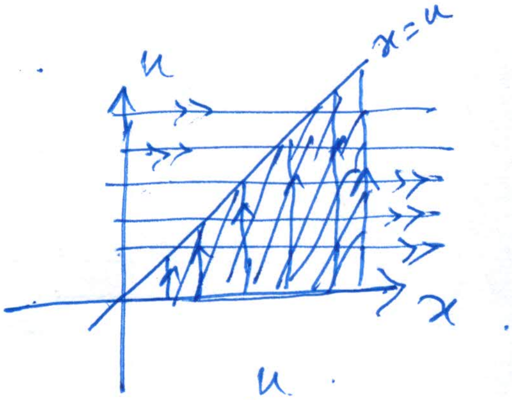
$$= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-\alpha^2}\, d\alpha - \int_0^x e^{-\alpha^2}\, d\alpha \right].$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \times \operatorname{erf}(x) \right].$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

$$L\left[\operatorname{erf}(x)\right] = \int_0^\infty \operatorname{erf}(x)\, e^{-sx}\, dx .$$

$$= \int_0^\infty \left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \right) e^{-sx}\, dx$$



$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\, du \left(\int_{x=u}^\infty e^{-sx}\, dx \right).$$

$$= \left. \frac{e^{-sx}}{s} \right|_{\infty}^{u}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \times \frac{e^{-su}}{s}\, du .$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{s} \int_0^\infty e^{-(u^2+su)} du.$$

$$= \frac{1}{s} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\left(u^2+2.u.\frac{s}{2}+\frac{s^2}{4}\right)} e^{\frac{s^2}{4}} du.$$

$$= \frac{e^{\frac{s^2}{4}}}{s} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\left(u+\frac{s}{2}\right)^2} du.$$

$u + \frac{s}{2} = t.$

$du = dt.$

$$= \frac{e^{\frac{s^2}{4}}}{s} \cdot \frac{2}{\sqrt{\pi}} \int_{\frac{s}{2}}^\infty e^{-t^2} dt.$$

$$= \frac{e^{\frac{s^2}{4}}}{s} \operatorname{erfc}\left(\frac{s}{2}\right)$$

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§ Find $L\left[\operatorname{erf}(\sqrt{x})\right]$

$$L\left[\operatorname{erf}(\sqrt{x})\right] = \int_0^\infty \operatorname{erf}\sqrt{x}\, e^{-sx} dx.$$

$$= \int_0^\infty \left(\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-u^2} du\right) e^{-sx} dx.$$

In the inner integral put $u^2 = t.$

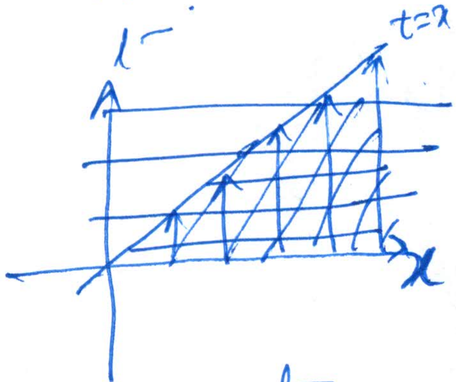
$2u\, du = dt.$

$du = \frac{dt}{2u} = \frac{dt}{2\sqrt{t}}.$

$$\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-u^2} du = \frac{\cancel{2}}{\sqrt{\pi}} \int_0^x e^{-t} \frac{dt}{\cancel{2}\sqrt{t}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^x \frac{e^{-t}\,dt}{\sqrt{t}} \quad \text{(inner integral)}.$$

$$L\left[\operatorname{erf}(\sqrt{x})\right] = \int_0^\infty \frac{1}{\sqrt{\pi}} \left(\int_0^x \frac{e^{-t}\,dt}{\sqrt{t}} \right) e^{-sx}\,dx$$



$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \left(\int_{x=t}^\infty e^{-sx}\,dx \right) dt.$$

$$\left. \frac{e^{-sx}}{s} \right|_\infty^t$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \, \frac{e^{-st}}{s}\,dt.$$

$$= \frac{1}{s\sqrt{\pi}} \int_0^\infty \frac{e^{-(s+1)t}}{\sqrt{t}}\,dt.$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,dx.$$

$$(s+1)t = z.$$

$$t = \frac{z}{s+1}; \quad dt = \frac{dz}{s+1}$$

$$= \frac{1}{s\sqrt{\pi}} \int_0^\infty \frac{e^{-z}}{\frac{\sqrt{z}}{\sqrt{s+1}}} \times \frac{dz}{s+1}.$$

$$= \frac{1}{s\sqrt{\pi}} \int_0^\infty e^{-z} z^{-\frac{1}{2}} \frac{dz}{\sqrt{s+1}}.$$

$$= \frac{1}{s\sqrt{s+1}} \cdot \frac{1}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-z} \cdot z^{-\frac{1}{2}}\,dz}_{=\Gamma(\frac{1}{2})} = \frac{1}{s\sqrt{s+1}} \, \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi}.$$

$$= \frac{1}{s\sqrt{s+1}}$$

Bessel Function.

$J_\nu(x) \rightarrow$ Bessel funct. of order ν.

Bessel funct. satisfies Bessel diff. equat.

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0.$$

It arises whenever one tries to solve Laplace equation / Helmholtz equation in cylindrical / spherical polar coordinates

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+\nu}}{r! \, \Gamma(r+\nu+1)}.$$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{(r!)^2 \cancel{\Gamma(r+1)}}$$

$$L[J_0(x)] = \int_0^\infty J_0(x) e^{-sx} dx.$$

$$= \int_0^\infty \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} e^{-sx} dx.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 \, 2^{2r}} \left(\int_0^\infty x^{2r} e^{-sx} dx \right)$$

$$\int_0^\infty x^n e^{-sx} dx = L[x^n] = \frac{n!}{s^{n+1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2\, 2^{2r}} \cdot \frac{(2r)!}{s^{2r+1}}.$$

$$= \frac{1}{s} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2\, 2^{2r}} \cdot \frac{(2r)!}{s^{2r}}$$

$$= \frac{1}{s}\left[1 - \frac{2!}{(1!)^2 . 2^2} \cdot \frac{1}{s^2} + \frac{4!}{(2!)^2\, 2^4\, s^4} - \frac{6!}{(3!)^2 . 2^6\, s^6} + \frac{8!}{(4!)^2\, 2^8\, s^8} - \cdots \right].$$

$$\frac{2!}{(1!)^2\, 2^2\, s^2} = \frac{2}{2.2.s^2} = \frac{1}{2s^2}.$$

$$\frac{4!}{(2!)^2\, 2^4\, s^4} = \frac{\cancel{4}^{2}.3.\cancel{2}.1}{\cancel{2}.\cancel{2}.2^{\cancel{4}3}\, s^4} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2s^4}.$$

$$\frac{6!}{(3!)^2\, 2^6\, s^6} = \frac{6.\cancel{5}.\cancel{4}^{2}.\cancel{3}.2.1}{\cancel{3}.\cancel{2}.\cancel{3}.\cancel{2}.2^6\, s^6} = \cancel{\frac{1.3.5}{3!}}$$

$$= \frac{6.5.4.3.2.1.}{3!\; 3.2.1.\; 2^6 . s^6} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{1}{3!\, s^6}.$$

$$= \frac{1}{s}\left[1 - \frac{1}{2} \cdot \frac{1}{s^2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right) \cdot \frac{1}{2!}\left(\frac{1}{s^2}\right)^2 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} - 2\right) \cdot \frac{1}{3!}\left(\frac{1}{s^2}\right)^3 - \cdots \right]$$

$$= \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s}\cdot\left(\frac{s^2+1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{s}\cdot\left(\frac{s^2}{s^2+1}\right)^{\frac{1}{2}}.$$

$$= \frac{1}{s}\cdot\frac{s}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}}.$$

$$L\left[J_0(x)\right] = \frac{1}{\sqrt{s^2+1}}.$$

<u>H.W.</u> Find $L\left[E_i(t)\right]$.

$$E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u}\,du.$$