Fourier Transform

Lecture-21
Monday
16/10/17

① $f(x)$ is even

Then Fourier transform (FT) of $f(x)$

= Fourier Cosine transform (FCT) of $f(x)$.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{even}} \underbrace{\cos\omega x}_{\text{even}}\, dx}_{\text{even}} + i \underbrace{\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{even}} \underbrace{\sin\omega x}_{\text{odd}}\, dx}_{\text{odd}} \right]$$

$= 0$.

$$= \frac{1}{\sqrt{2\pi}} \times 2 \int_{0}^{\infty} f(x) \cos\omega x\, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos\omega x\, dx = F_c(\omega).$$

② $f(x)$ is odd.

Then, FT. of $f(x) = i \times$ Fourier sine transform (FST) of $f(x)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos\omega x}_{\text{even}}\, dx}_{\text{odd}} + i \underbrace{\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin\omega x}_{\text{odd}}\, dx}_{\text{even}} \right]$$

$= 0$.

$$= \frac{1}{\sqrt{2\pi}} \times i\, 2 \int_{0}^{\infty} f(x) \sin\omega x\, dx .$$

$$= i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin\omega x\, dx = i\, F_s(\omega)$$

(1) Find the Fourier transform of $e^{-a^2x^2}$ ~~$(a>0)$~~.

Sol. Note. $f(x) = e^{-a^2x^2}$ is an even funct.

$\therefore$ F.T. of $f(x)$ = FCT of $f(x)$.

$$\text{F.C.T. of } f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos\omega x \, dx .$$

Let

$$I(\omega) = \int_0^\infty e^{-a^2x^2} \cos\omega x \, dx .$$

$$\frac{dI}{d\omega} = -\int_0^\infty x e^{-a^2x^2} \sin\omega x \, dx \longrightarrow (\star)$$

$$= \frac{1}{2a^2}\left[e^{-a^2x^2} \sin\omega x\right]_0^\infty - \int_0^\infty \frac{1}{2a^2} e^{-a^2x^2} \times \omega \cos\omega x \, dx$$

$$= -\frac{\omega}{2a^2} \int_0^\infty e^{-a^2x^2} \cos\omega x \, dx = -\frac{\omega}{2a^2} I(\omega).$$

$$\frac{d}{dx}\left(e^{-a^2x^2}\right) = -2a^2x \, e^{-a^2x^2}$$

$$\int -x e^{-a^2x^2} dx = \frac{1}{2a^2} e^{-a^2x^2}$$

$$\therefore \frac{dI}{d\omega} = -\frac{\omega}{2a^2} I(\omega).$$

$$\text{or, } \frac{dI}{I} = -\frac{1}{2a^2} \omega \, d\omega .$$

Integrating, $\log I = -\frac{\omega^2}{4a^2} + \log c$.

$$\therefore I(\omega) = c e^{-\frac{\omega^2}{4a^2}} .$$

$$I(0) = \int_0^\infty e^{-a^2x^2} \cos 0.x \, dx = \int_0^\infty e^{-a^2x^2} dx.$$

Put $ax = v$, $dx = \frac{dv}{a}$.

$$I(0) = \int_0^\infty e^{-v^2} \cdot \frac{dv}{a}.$$

$$= \frac{1}{a} \int_0^\infty e^{-v^2} dv$$

$v^2 = z$.

$2v\,dv = dz$.

$\therefore dv = \frac{dz}{2v} = \frac{dz}{2\sqrt{z}}$.

$$= \frac{1}{2a} \int_0^\infty e^{-z} \frac{dz}{\sqrt{z}}.$$

$$= \frac{1}{2a} \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right).$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

$$= \frac{1}{2a} \sqrt{\pi}.$$

We've $I(\omega) = c\, e^{-\frac{\omega^2}{4a^2}}$.

$I(0) = c \cdot 1$ But $I(0) = \frac{1}{a} \cdot \frac{\sqrt{\pi}}{2}$.

$\therefore c = \frac{1}{a} \frac{\sqrt{\pi}}{2}$.

$$\therefore I(\omega) = \frac{1}{a} \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{\omega^2}{4a^2}}. \longrightarrow (**)$$

$\therefore$ F.C.T. of $e^{-a^2x^2} = \sqrt{\frac{2}{\pi}} \cdot I(\omega)$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{a} \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{\omega^2}{4a^2}}.$$

F.C.T. of $e^{-a^2x^2} = \frac{1}{\sqrt{2}\,a} e^{-\frac{\omega^2}{4a^2}}$.

or, F.T. of $e^{-a^2x^2} = \frac{1}{\sqrt{2}\,a} e^{-\frac{\omega^2}{4a^2}}$

Thus

$$F\left(e^{-a^2x^2}\right) = F_c\left(e^{-a^2x^2}\right) = \frac{1}{\sqrt{2}\,a} e^{-\frac{\omega^2}{4a^2}}.$$

Put $a = \frac{1}{\sqrt{2}}$.

$$F\left(e^{-\frac{x^2}{2}}\right) = F_c\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}.$$

So, the form of the funct. $e^{-\frac{x^2}{2}}$ remains unchanged, when F.T. or F.C.T is applied on it. Such a function is called self-reciprocal w.r.to. Fourier transform. (or Fourier cosine transform)

Q. Show that $xe^{-\frac{x^2}{2}}$ is self-reciprocal w.r.to Fourier sine transform.

$$F_S\left[xe^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-\frac{x^2}{2}} \sin \omega x \, dx.$$

In the previous problem in equation (*), let us put $a = \frac{1}{\sqrt{2}}$.

Then $\frac{dI}{dw}(\omega) = -\int_0^\infty x e^{-\frac{x^2}{2}} \sin \omega x \, dx.$

$$\therefore F_S\left[xe^{-\frac{x^2}{2}}\right] = -\sqrt{\frac{2}{\pi}} \cdot \frac{dI}{dw} \longrightarrow (\#)$$

Now, in previous problem in eqn.(**), put $a = \frac{1}{\sqrt{2}}$.

$$I(\omega) = \frac{\sqrt{\pi}}{\sqrt{2}} e^{-\frac{\omega^2}{2}} \Rightarrow \frac{dI}{dw} = -\omega\sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\omega^2}{2}}$$

Putting the value of $\frac{dI}{dw}$ into (#), get -

$$F_S\left[x e^{-\frac{x^2}{2}}\right] = -\sqrt{\frac{2}{\pi}} \times -\omega \sqrt{\frac{\pi}{2}}\, e^{-\frac{\omega^2}{2}}$$

$$= \omega e^{-\frac{\omega^2}{2}}$$

This shows that $x e^{-\frac{x^2}{2}}$ is self-reciprocal w.r.to. Fourier sine transform.

Ex. Find Fourier transform of $\cos(a^2x^2)$.

Sol. To find $F\left(\cos(a^2x^2)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(a^2x^2) e^{i\omega x}\, dx$.

$F\left(\cos(a^2x^2)\right) = F_c\left(\cos(a^2x^2)\right)$, since $\cos(a^2x^2)$ is an even function.

$$F_c\left(\cos(a^2x^2)\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos a^2x^2 \cos \omega x\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^{\infty} 2\cos a^2x^2 \cos \omega x\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \cos(a^2x^2 + \omega x)\, dx + \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^{\infty} \cos(a^2x^2 - \omega x)\, dx}_{= I_2}.$$

For the 2nd integral put $x = -y$.
$dx = -dy$.

Then $I_2 = \frac{1}{\sqrt{2\pi}} \int_0^{-\infty} \cos(a^2y^2 + \omega y) \times -dy$

$$\therefore I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \cos(a^2x^2 + \omega x)\, dx .$$

$$\therefore F_c\left(\cos(a^2x^2)\right) = \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \cos(a^2x^2 + \omega x)\, dx + \int_{-\infty}^{0} \cos(a^2x^2 + \omega x)\, dx \right]$$

$$= \frac{1}{\sqrt{\pi \times 2}} \int_{-\infty}^{\infty} \cos(a^2x^2 + \omega x)\, dx .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos\left\{ (ax)^2 + 2 . ax . \frac{\omega}{2a} + \frac{\omega^2}{4a^2} - \frac{\omega^2}{4a^2} \right\} dx .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos\Big\{ \underbrace{\left(ax + \frac{\omega}{2a}\right)^2}_{A} - \underbrace{\frac{\omega^2}{4a^2}}_{B} \Big\} dx .$$

$$= \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\omega^2}{4a^2}\right) \int_{-\infty}^{\infty} \cos\left(ax + \frac{\omega}{2a}\right)^2 dx + \frac{1}{\sqrt{2\pi}} \sin\left(\frac{\omega^2}{4a^2}\right) \int_{-\infty}^{\infty} \sin\left(ax + \frac{\omega}{2a}\right)^2 dx$$

$$\left[\int_{-\infty}^{\infty} \cos v^2\, dv = \int_{-\infty}^{\infty} \sin v^2\, dv = \sqrt{\frac{\pi}{2}} \right]$$

Put $ax + \frac{\omega}{2a} = v .$

$\Rightarrow a\,dx = dv .$

$$= \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\omega^2}{4a^2}\right) \int_{-\infty}^{\infty} \cos(v^2) \frac{dv}{a} + \frac{1}{\sqrt{2\pi}} \sin\left(\frac{\omega^2}{4a^2}\right) \frac{1}{a} \int_{-\infty}^{\infty} \sin v^2\, dv$$

$\therefore F_c\left(\cos(a^2x^2)\right) = \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\omega^2}{4a^2}\right) \cdot \frac{\sqrt{\pi}}{a} \times \frac{1}{\sqrt{2}}$

$+ \frac{1}{\sqrt{2\pi}} \sin\left(\frac{\omega^2}{4a^2}\right) \cdot \frac{\sqrt{\pi}}{a} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{\cancel{\sqrt{2}}\,2a} \cos\left(\frac{\omega^2}{4a^2}\right) + \frac{1}{\sqrt{2}\,a} \sin\left(\frac{\omega^2}{4a^2}\right)$ //.

Find F_c of $\cos\left(\frac{x^2}{2}\right)$.

$F_c\left(\cos\left(\frac{x^2}{2}\right)\right) = \left\{\cos\left(\frac{\omega^2}{2}\right) + \sin\left(\frac{\omega^2}{2}\right)\right\} \frac{1}{\sqrt{2}}$.

Q. $F\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}} \rightarrow$ Using this result find the inverse Fourier transform of $e^{-t\omega^2}$ $(t > 0)$.

Sol. Given, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{i\omega x} dx = e^{-\frac{\omega^2}{2}}$.

To find,

$I(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\omega^2} e^{-i\omega x} d\omega$.

Put $\sqrt{t}\,\omega = \frac{1}{\sqrt{2}} v$. such that $t\omega^2 = \frac{v^2}{2}$.

or, $\sqrt{t}\, d\omega = \frac{1}{\sqrt{2}} dv$.

$$\therefore I(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{-i.x.\frac{v}{\sqrt{2t}}} \frac{dv}{\sqrt{2t}}.$$

$$= \frac{1}{\sqrt{2t}} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{-i\left(\frac{x}{\sqrt{2t}}\right)v} dv.$$

Put $v = -y$.

$$= \frac{1}{\sqrt{2t}} \times \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-\frac{y^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)y} \times -dy.$$

$$= \frac{1}{\sqrt{2t}} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)y} dy.$$

Compare this with $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{i(\omega)x} dx = e^{-\frac{\omega^2}{2}}.$

$$\therefore I(x,t) = \frac{1}{\sqrt{2t}} \times e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2t}}\right)^2}$$

$$= \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$$

//.

Test - 3

Q.1 Find the Fourier sine transform of $\dfrac{e^{-\alpha x}}{x}$; $\alpha > 0$. Hint. Leibnitz rule.

Q2. Show that $x^{-\frac{1}{2}}$ is self-reciprocal w.r.to Fourier cosine as well as Fourier sine transform.

Hint. previous day's note.

Solution of Q.1 = $\sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\frac{\omega}{\alpha}$.

Hint: $F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-\alpha x}}{x} \sin \omega x \, dx$. $\rightarrow$ (1)

$$\frac{dF_s}{d\omega} = \sqrt{\frac{2}{\pi}} \underbrace{\int_0^\infty e^{-\alpha x} \cos \omega x \, dx}_{\text{L.T. of } \cos \omega x = \frac{\alpha}{\alpha^2 + \omega^2}}$$

$$\therefore F_s = \sqrt{\frac{2}{\pi}} \alpha \int \frac{d\omega}{\alpha^2 + \omega^2} + C$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\alpha} \tan^{-1}\frac{\omega}{\alpha} + C$$

From (1), $F_s(0) = 0$. $\therefore F_s(0) = 0 = 0 + C$.

$$\therefore C = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{\omega}{\alpha}.$$