In this lesson we introduce the concept of inverse Laplace transform and discuss some of its important properties that will be helpful to evaluate inverse Transform of some complicated functions. As mention in the beginning of this module that the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the transformed domain we will like to get back to the time domain and therefore we need to introduce the concept of inverse Laplace transform. Further, we introduce the convolution property of the Laplace transform. We shall start with the definition of convolution followed by an important theorem on Laplace transform of convolution. Convolution theorem plays an important role for finding inverse Laplace transform of complicated functions and therefore very useful for solving differential equations.

7.1 Inverse Laplace Transform

If $F(s) = L[f(t)]$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$L^{-1}[F(s)] = f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself as it requires complex numbers and path integrals. The easiest way of computing the inverse is using table of Laplace transform. For example,

$$L[\sin wt] = \frac{w}{s^2 + w^2}$$

This implies

$$L^{-1}\left[\frac{w}{s^2 + w^2}\right] = \sin wt,\ t \geq 0$$

and similarly

$$L[\cos wt] = \frac{s}{s^2 + w^2} \quad \Rightarrow \quad L^{-1}\left[\frac{s}{s^2 + w^2}\right] = \cos wt,\ t \geq 0$$

7.2 Uniqueness of Inverse Laplace Transform

If we have a function $F(s)$, to be able to find $f(t)$ such that $L[f(t)] = F(s)$, we need to first know if such a function is unique.

Consider

$$g(t) = \begin{cases} 1 & \text{when } t = 1 \\ \sin(t) & \text{when otherwise} \end{cases}$$

$$L[g(t)] = \frac{1}{s^2+1} = L[\sin t]$$

Thus we have two different functions $g(t)$ and $\sin t$ whose Laplace transform are same. However note that the given two functions are different at a point of discontinuity. Thanks to the following theorem where we have uniqueness for continuous functions:

7.2.1 Theorem (Lerch's Theorem)

If f and g are continuous and are of exponential order, and if $F(s) = G(s)$ for all $s > s_0$ then $f(t) = g(t)$ for all $t > 0$.

Proof: If $F(s) = G(s)$ for all $s > s_0$ then,

$$\begin{aligned}
&\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty e^{-st} g(t)\mathrm{d}t, \ \ \forall s > s_0 \\
&\Rightarrow \int_0^\infty e^{-st}[f(t) - g(t)]\mathrm{d}t = 0, \ \ \forall s > s_0 \\
&\Rightarrow f(t) - g(t) \equiv 0, \ \ \forall t > 0. \\
&\Rightarrow f(t) = g(t), \ \ \forall t > 0.
\end{aligned}$$

This completes the proof. ■

Remark: *The uniqueness theorem holds for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function or the function $g(t)$ defined above. Since the Laplace integral however does not "see" values at the discontinuities. So in this case we can only conclude that $f(t) = g(t)$ outside of discontinuities.*

We now state some important properties of the inverse Laplace transform. Though, these properties are the same as we have listed for the Laplace transform, we repeat them without proof for the sake of completeness and apply them to evaluate inverse Laplace transform of some functions.

7.3 Linearity of Inverse Laplace Transform

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of the function $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[a_1F_1(s) + a_2F_2(s)] = a_1L^{-1}[F_1(s)] + L^{-1}[F_2(s)] = a_1f_1(t) + a_2f_2(t)$$

where a_1 and a_2 are constants.

7.4 Example Problems

7.4.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{6}{2s-3} + \frac{8-6s}{16s^2+9}$$

Solution: Using linearity of the inverse Laplace transform we have

$$f(t) = 6L^{-1}\left[\frac{1}{2s-3}\right] + 8L^{-1}\left[\frac{1}{16s^2+9}\right] - 6L^{-1}\left[\frac{s}{16s^2+9}\right]$$

Rewriting the above expression as

$$f(t) = 3L^{-1}\left[\frac{1}{s-(3/2)}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(9/16)}\right] - \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(9/16)}\right]$$

Using the result

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{as}$$

and taking the inverse transform we obtain

$$f(t) = 3e^{3t/2} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}.$$

7.4.2 Problem 2

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}$$

Solution: We use the method of partial fractions to write F in a form where we can use the table of Laplace transform. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1) + s(Bs+C) = s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$, and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity of the inverse Laplace transform we get

$$L^{-1}\left[\frac{s^2+s+1}{s^3+s}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] = 1 + \sin t.$$

7.5 First Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t),$ *then* $L^{-1}[F(s-a)] = e^{at} f(t)$

7.6 Example Problems

7.6.1 Problem 1

Evaluate $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Solution: Rewriting the given expression as

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1}{(s-(-1))^2}\right]$$

Applying the first shifting property of the inverse Laplace transform

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

Thus we obtain

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}.$$

7.6.2 Problem 2

Find $L^{-1}\left[\dfrac{1}{s^2+4s+8}\right]$.

Solution: First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right] = L^{-1}\left[\frac{1}{(s+2)^2+4}\right] = \frac{1}{2}\,e^{-2t}\sin(2t).$$

7.7 Second Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t),$ *then* $L^{-1}\left[e^{-as}F(s)\right] = f(t-a)\,H(t-a)$

7.8 Example Problems

7.8.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{e^{-s}}{s(s^2+1)}$$

Solution: First we compute the inverse Laplace transform

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{s}{(s^2+1)}\right]$$

Using linearity of the inverse transform we get

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s}{(s^2+1)}\right] = 1-\cos t$$

We now find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-s}L[1-\cos t]\right]$$

Using the second shifting theorem we obtain

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = \big[1-\cos(t-1)\big]\,H(t-1).$$

7.8.2 Problem 2

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+4} + \frac{e^{-2s}}{s^2+4} + \frac{e^{-3s}}{(s+2)^2}$$

Solution: First we find that

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

and using the first shifting property

$$L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

By linearity we have

$$f(t) = L^{-1}\left[\frac{e^{-s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-3s}}{(s+2)^2}\right]$$

Putting it all together and using the second shifting theorem we get

$$f(t) = \frac{1}{2}\sin 2(t-1)\,H(t-1) + \frac{1}{2}\sin 2(t-2)\,H(t-2) + e^{-2(t-3)}\,(t-3)\,H(t-3)$$

7.9 Convolution

The convolution of two given functions $f(t)$ and $g(t)$ is written as $f * g$ and is defined by the integral

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{7.1}$$

As you can see, the convolution of two functions of t is another function of t.

7.10 Example Problems

7.10.1 Problem 1

Find the convolution of $f(t) = e^t$ *and* $g(t) = t$ *for* $t \geq 0$.

Solution: By the definition we have

$$(f * g)(t) = \int_0^t e^\tau (t - \tau)\, d\tau$$

Integrating by parts, we obtain

$$(f * g)(t) = e^t - t - 1.$$

7.10.2 Problem 2

Find the convolution of $f(t) = \sin(\omega t)$ *and* $g(t) = \cos(\omega t)$ *for* $t \geq 0$.

Solution: By the definition of convolution we have

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\, \cos\big(\omega(t - \tau)\big)\, d\tau.$$

We apply the identity $\cos(\theta)\sin(\psi) = \frac{1}{2}\big(\sin(\theta + \psi) - \sin(\theta - \psi)\big)$ to get

$$(f * g)(t) = \int_0^t \frac{1}{2}\, \big(\sin(\omega t) - \sin(\omega t - 2\omega\tau)\big)\, d\tau$$

On integration we obtain

$$(f * g)(t) = \left[\frac{1}{2}\,\tau\sin(\omega t) + \frac{1}{4\omega}\, \cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} = \frac{1}{2}\, t\sin(\omega t).$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

7.11 Properties of Convolution

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions, then

(i) $f * g = g * f, \quad$ [symmetry]

(ii) $c(f * g) = cf * g = f * cg, \quad$ [c=constant]

(iii) $f * (g * h) = (f * g) * h, \quad$ [associative property]

(iv) $f * (g + h) = f * g + f * h, \quad$ [distributive property]

Proof: We give proof of (i) and all others can be done similarly. By the definition of convolution we have

$$f * g = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

Substituting $t - \tau = u \Rightarrow -d\tau = du$ we get

$$f * g = -\int_t^0 f(t-u)g(u)\mathrm{d}u = \int_0^t f(t-u)g(u)\mathrm{d}u = g * f$$

This completes the proof. ■

The most interesting property for us, and the main result of this lesson is the following theorem.

7.12 Convolution Theorem

If f and g are piecewise continuous on $[0, \infty)$ and of exponential order α, then

$$L[(f * g)(t)] = L[f(t)]L[g(t)].$$

Proof: From the definition,

$$L[(f * g)(t)] = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\mathrm{d}t, \ \text{[Re(s)>} \alpha \text{]}$$

Changing the order of integration,

$$L[(f * g)(t)] = \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\mathrm{d}t\mathrm{d}\tau,$$

We now put $t - \tau = u \Rightarrow -d\tau = du$ and get,

$$\begin{aligned} L[(f * g)(t)] &= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^\infty e^{-s\tau)} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-su} g(u)\mathrm{d}u = L[f(t)]L[g(t)] \end{aligned}$$

This completes the proof. ■