1.1 Introduction to Fourier Series

Before we start discussion on Fourier transform it is very important to discuss Fourier series firstly because it gives a pathway to understand Fourier transform. Fourier series has a wide range of applications, viz. in analysis of current flow, sound waves, image analysis and many more. They are also used to solve differential equations. In a general sense, we use Fourier series to represent a periodic functions. Indeed, not only periodic functions but also to represent and approximate functions defined on a finite interval.

1.2 Periodic Functions

If a function f is periodic with period $T > 0$ then $f(t) = f(t+T)$, $-\infty < t < \infty$. The smallest of T, for which the equality $f(t) = f(t+T)$ is true, is called fundamental period of $f(t)$. However, if T is the period of a function f then nT, n is any natural number, is also a period of f. Some familiar periodic functions are $\sin x$, $\cos x$, $\tan x$ etc.

1.2.1 Properties of Periodic Functions

We consider two important properties of periodic function. These properties will be used to discuss the Fourier series.

1. It should be noted that the sum, difference, product and quotient of two functions is also a periodic function. Consider for example:

$$\begin{aligned} f(x) &= \underbrace{\sin x}_{} + \underbrace{\sin 2x}_{} + \underbrace{\cos 3x}_{} \\ \text{period:}\quad & 2\pi \quad \frac{2\pi}{2} = \pi \quad \frac{2\pi}{3} \end{aligned}$$

   Period of f = common period of $(\sin x, \sin 2x, \cos 3x)$ = 2π

   One can also confirms the period of the function $f(x)$ as

$$\begin{aligned} f(x+2\pi) &= \sin(x+2\pi) + \sin(2x+2\pi) + \cos(3x+2\pi) \\ &= \sin(x) + \sin(2x) + \cos(3x) = f(x) \end{aligned}$$

2. If a function is integrable on any interval of length T, then it is integrable on any other intervals of the same length and the value of the integral is the same, that is,

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_b^{b+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x \text{ for any value of } a \text{ and } b$$

This property has been depicted in Figure 1.2.1.

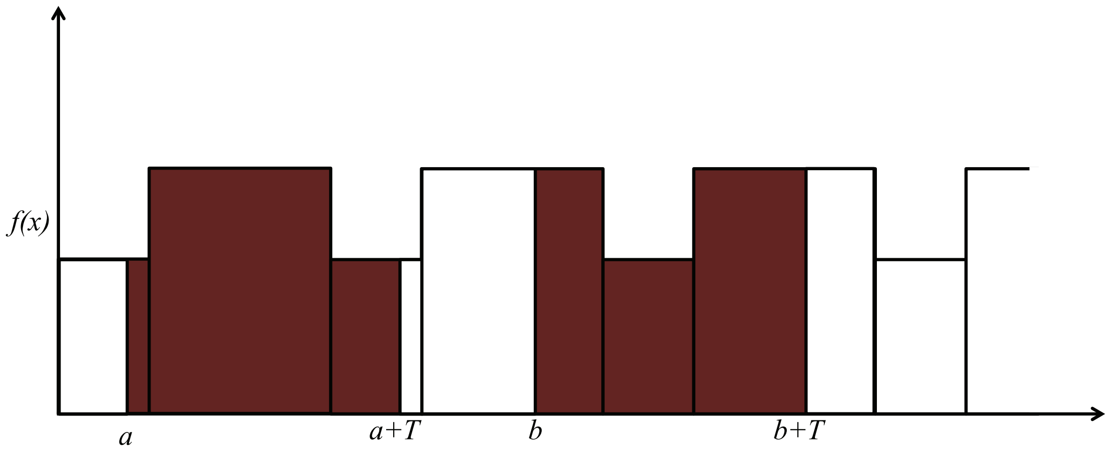

Figure 1.1: Area showing integral of a typical periodic function

1.3 Trigonometric Polynomials and Series

- Trigonometric polynomial of order n is defined as

$$S_n(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right)$$

Here a_n and b_n are some constants. Since the sum of the periodic functions again represents a periodic function. Therefore S_n will be a periodic function. What will be the period of the function S_n? The period can be identified simply by looking at the common period of the functions involved in the sum as

$$\begin{aligned} \text{Period of } S_n(x) &= \text{common period of} \left(\cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \ldots, \sin \frac{n\pi x}{l}, \cos \frac{n\pi x}{l} \right) \\ &= 2\pi/(\pi/l) = 2l. \end{aligned}$$

- The infinite trigonometric series

$$S(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right),$$

if it converges, also represents a function of period $2l$.

Now the question aries whether any function of period $T = 2l$ can be represented as the sum of a trigonometric series? The answer to this question is affirmative and it is possible for a very wide class of periodic functions. In the next lesson we will see how to obtain the constants a_n and b_n in order this trigonometric series to represent a given periodic function.

Remark 1: *Though sine and cosine functions are quite simple in nature but their sum function may be quite complex. One can see the plot of* $\sin x + \sin 2x + \cos 3x$ *in Figure 1.3. However, the function has a period* 2π *which is a common period of* $\sin x, \sin 2x, \cos 3x$.

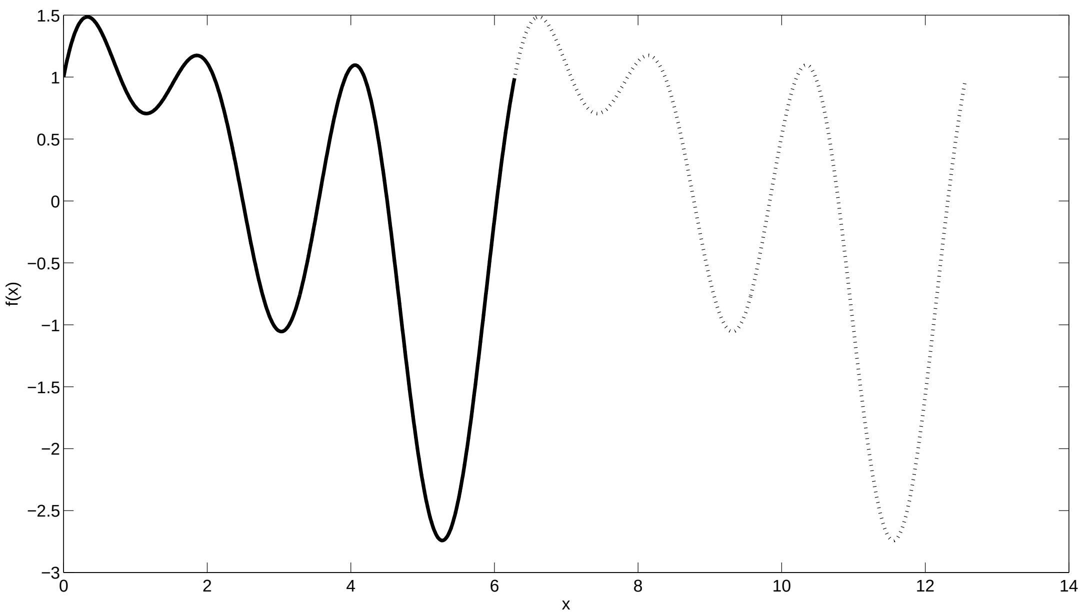

Figure 1.2: Plot of a trigonometric polynomial $f(x) = \sin x + \sin 2x + \cos 3x$

1.4 Orthogonality Property of Trigonometric System

We call two functions $\phi(x)$ and $\psi(x)$ to be orthogonal on the interval $[a, b]$ if

$$\int_a^b \phi(x)\psi(x)\, dx = 0$$

With this definition we can say that the basic trigonometric system viz.

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

is orthogonal on the interval $[-\pi, \pi]$ or $[0, 2\pi]$. In particular, we shall prove that any two distinct functions are orthogonal.

To show the orthogonality we take different possible combination as:

For any integer $n \neq 0$**:** We have the following integrals to show the orthogonality of the function 1 with any member of sine or cosine family

$$\int_{-\pi}^{\pi} 1 \cdot \cos(nx)\,\mathrm{d}x = \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi} = 0, \qquad \int_{-\pi}^{\pi} 1 \cdot \sin(nx)\,\mathrm{d}x = -\left.\frac{\cos(nx)}{n}\right|_{-\pi}^{\pi} = 0$$

We have also the following useful results

$$\int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2} = \pi, \qquad \int_{-\pi}^{\pi} \sin^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2} = \pi$$

For any integer m **and** n **(m** $\neq$ **n):** Now we show that any two different members of the same family (sine or cosine) are orthogonal. For the cosine family we have

$$\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x]\ \mathrm{d}x = 0$$

and for the sine family we have

$$\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x]\ \mathrm{d}x = 0$$

For any integer m **and** n**:** Here we show that any two members of the two different family (sine and cosine) are orthogonal

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = 0$$

Note that the integrand is an odd function and therefore the integral is zero.

The above result can be summarized in a more general setting in the following theorem.

1.4.1 Theorem

The trigonometric system

$$1,\ \cos\frac{\pi x}{l},\ \sin\frac{\pi x}{l},\ \cos\frac{2\pi x}{l},\ \sin\frac{2\pi x}{l},\ldots$$

is orthogonal on the interval $[-l,l]$ *or* $[a,a+2l]$*, where* a *is any real number.*

Proof: Note that the common period of the trigonometric system

$$1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l},\ldots$$

is $2l$. Similar to the evaluation of the integral appeared above to show orthogonality of the basic trigonometric system, we have the following results:

a) $$\int_{-l}^{l}\cos\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l}\cos\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m\neq n \\ l & \text{if } m=n\neq 0 \end{cases}$$

b) $$\int_{-l}^{l}\sin\frac{m\pi x}{l}\,\sin\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l}\sin\frac{m\pi x}{l}\,\sin\frac{n\pi x}{l}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m\neq n \\ l & \text{if } m=n\neq 0 \end{cases}$$

c) $$\int_{-l}^{l}\sin\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l}\sin\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0$$

This completes the proof of the above theorem. ■

To summarize, the value of the integral over length of period of integrand is equal to zero if the integrand is a product of two different members of trigonometric system. If the integrand is product of two same member from sine or cosine family then the value of the integral will be half of the interval length on which the integral is performed. These results will be used to establish Fourier series of a function of period $2l$ defined on the interval $[-l,l]$ or $[a,a+2l]$. It should be noted that for $l=\pi$ we obtain results for standard trigonometric system of common period 2π.