Saksham Sahai Srivastava
17CH30032

PAD - Assignment 1

1. Analysis of Vibrations

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = F_0 u(t)$$

Taking Laplace transform, we get:

$$m[s^2\bar{x}(s) - s\bar{x}(0) - x(0)] + cs\bar{x}(s) + k\bar{x}(s) = F_0 u(s)$$

$$\therefore \bar{x}(s)[ms^2 + cs + k] = F_0\bar{u}(s)$$

$$\frac{\bar{x}(s)}{\bar{u}(s)} = \frac{F_0}{ms^2 + cs + k}$$

$$\boxed{\frac{\bar{x}(s)}{\bar{u}(s)} = \frac{F_0/k}{\frac{m}{k}s^2 + \frac{c}{k}s + 1}}$$

Comparing with the standard form:

$$\frac{\bar{y}(s)}{\bar{x}(s)} = \frac{K_p}{\tau^2 s^2 + 2\xi\tau s + 1}$$

we get, $\tau^2 = m/k$

$$\tau = \sqrt{m/k}$$

$$2\xi\tau = c/k$$

$$\xi = \frac{2}{2k}\sqrt{\frac{k}{m}} = \frac{c}{2\sqrt{mk}}$$

$K_p = F_o / k$

Natural period of oscillation $= \tau = \sqrt{m/k}$

Damping coefficient $\xi = \dfrac{c}{2\sqrt{mk}}$

Steady state gain $= K_p = F_o/k$

Natural Period of Oscillation:

Period of Oscillation

$$T = \frac{1}{2\pi}\left(\frac{2\pi\tau}{\sqrt{1-\xi^2}}\right)$$

for natural oscillation $\xi = 0$

$$T_n = \tau$$

$$T_n = \sqrt{\frac{m}{n}}$$

Daming coefficient:

$$\xi = \frac{c}{2\sqrt{mk}}$$

Steady Gain:

$$K_p = \frac{F_o}{k}$$

Scanned with CamScanner

(b) Transfer Function:

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\xi \tau s + 1}$$

Now, substituting values,

$$G(s) = \frac{F_0/k}{\frac{m}{k}s^2 + \frac{c}{k}s + 1}$$

(c) Underdamped, Overdamped & Critically damped:

We have $G(s) = \dfrac{F_0/k}{\frac{m}{k}s^2 + \frac{c}{k}s + 1}$

Roots of characteristic equation

$$\frac{m}{k}s^2 + \frac{c}{k}s + 1 = 0$$

$$s = \frac{-c/k \pm \sqrt{(c/k)^2 - 4(m/k)}}{2}$$

For underdamped case, roots are complex conjugate

So, $\left(\frac{c}{k}\right)^2 - 4\left(\frac{m}{k}\right) < 0$

$$c^2 - 4mk < 0$$

$$\frac{c}{2\sqrt{mk}} < 1$$

$$\xi < 1$$

For critically damped,

$$\left(\frac{c}{k}\right)^2 - 4\left(\frac{m}{k}\right) = 0$$

$$c = 2\sqrt{mk}$$

$$\xi = 1$$

For overdamped case,

$$\left(\frac{c}{k}\right)^2 - \frac{4m}{k} > 0$$

$$c^2 > 4mk$$

$$\frac{c}{2\sqrt{mk}} > 1$$

$$\xi > 1$$

(d) Response for step input

$$u(s) = A/s$$

For underdamped

$$\bar{y}(s) = \frac{1}{s}\left[\frac{F_0/k}{\tau^2 s^2 + 2\xi\tau s + 1}\right]$$

$$\bar{y}(s) = \frac{AF_0}{k} \cdot \frac{1}{\tau^2}\left[\frac{1}{s} - \frac{s + 2\xi/\tau}{s^2 + \frac{2\xi s}{\tau} + \frac{1}{\tau^2}}\right]\tau^2$$

Taking inverse laplace

$$Y(t) = \frac{AF_0}{K}\left[1 - e^{-\xi t/\tau}\left[\sqrt{1-\xi^2}\cos\sqrt{\frac{1-\xi^2}{\tau^2}}\,t + \xi\sin\sqrt{\frac{1-\xi^2}{\tau^2}}\,t\right]\right]$$

$$p\cos A + q\sin A = r\sin(A+\phi)$$

$$r = \sqrt{p^2+q^2} \qquad \phi = \tan^{-1}\left(\frac{p}{q}\right)$$

$$Y(t) = \frac{AF_0}{K}\left[1 - \frac{e^{-\xi t/\tau}}{\sqrt{1-\xi^2}}\sin(\omega t+\phi)\right]$$

$$\omega = \frac{\sqrt{1-\xi^2}}{\tau}$$

Critically damped ($\xi = 1$)

$$y(s) = \frac{A}{s}\cdot\frac{F_0/K}{\tau^2 s^2 + 2\xi\tau s + 1}$$

$$y(s) = \frac{AF_0}{K}\cdot\frac{1}{\tau^2}\left[\frac{1}{s(s+1/\tau)^2}\right]$$

By partial fraction,

$$y(s) = \frac{AF_0}{K}\left[\frac{1}{s} - \frac{1}{s+1/\tau} - \frac{1/\tau}{(s+1/\tau)^2}\right]$$

By Inverse Laplace

$$Y(t) = \frac{AF_0}{K}\left[1 - e^{-t/\tau}\left(1+\frac{t}{\tau}\right)\right]$$

Overdamped Response ($\xi > 1$)

$$\bar{y}(s) = \frac{A}{s}\left(\frac{F_o}{R}\right)\left(\frac{1}{\tau^2}\right) \cdot \frac{1}{\left(s^2 + \frac{2\xi}{\tau}s + \frac{1}{\tau^2}\right)}$$

$$\bar{y}(s) = \left(\frac{AF_o}{R}\right)\left[\frac{1}{s} - \frac{s + \xi/\tau}{\left(s + \frac{\xi}{\tau}\right)^2 - \left(\frac{\xi^2 - 1}{\tau^2}\right)} - \frac{\xi/\tau \sqrt{\frac{\xi^2-1}{\tau^2}} \sqrt{\frac{\tau^2}{\xi^2-1}}}{\left(s + \xi/\tau\right)^2 - \left(\frac{\xi^2-1}{\tau^2}\right)^2}\right]$$

Taking Inverse Laplace

$$y(t) = \frac{AF_o}{R}\left[1 - e^{-\xi t/\tau} \cosh \sqrt{\frac{\xi^2-1}{\tau^2}}\, t + \frac{\xi}{\sqrt{\xi^2-1}} \sinh \sqrt{\frac{\xi^2-1}{\xi^2}}\, t\right]$$

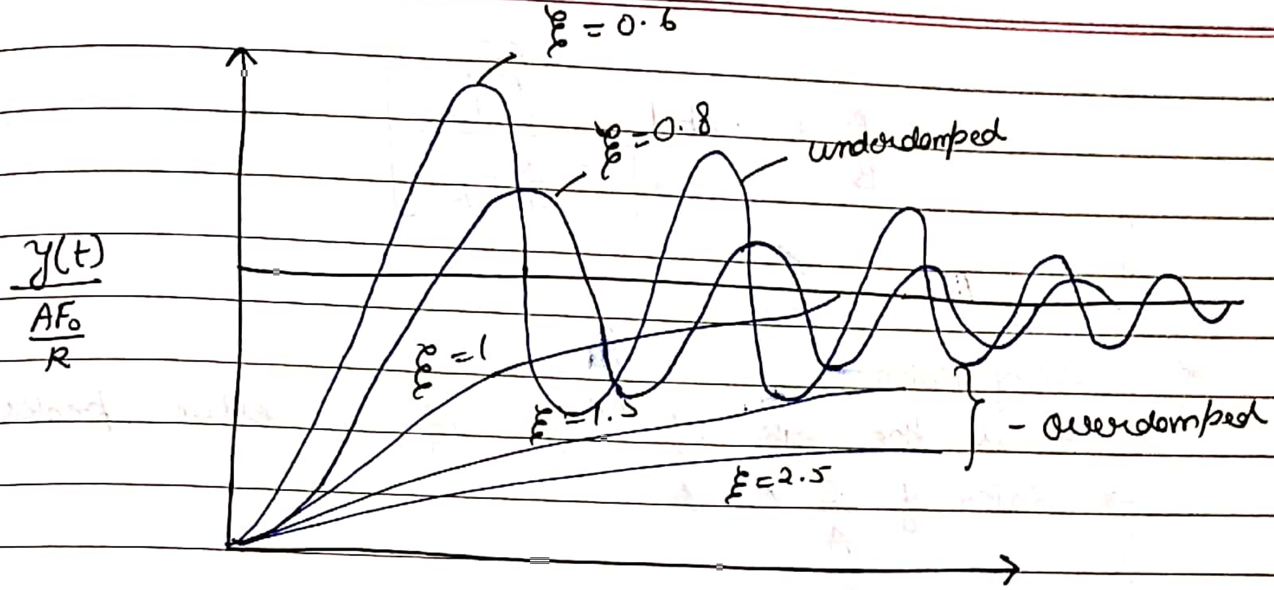
* For underdamped system, decreasing ξ makes the response more oscillatory.

* For Critically system, response reaches steady state more quickly without oscillations.

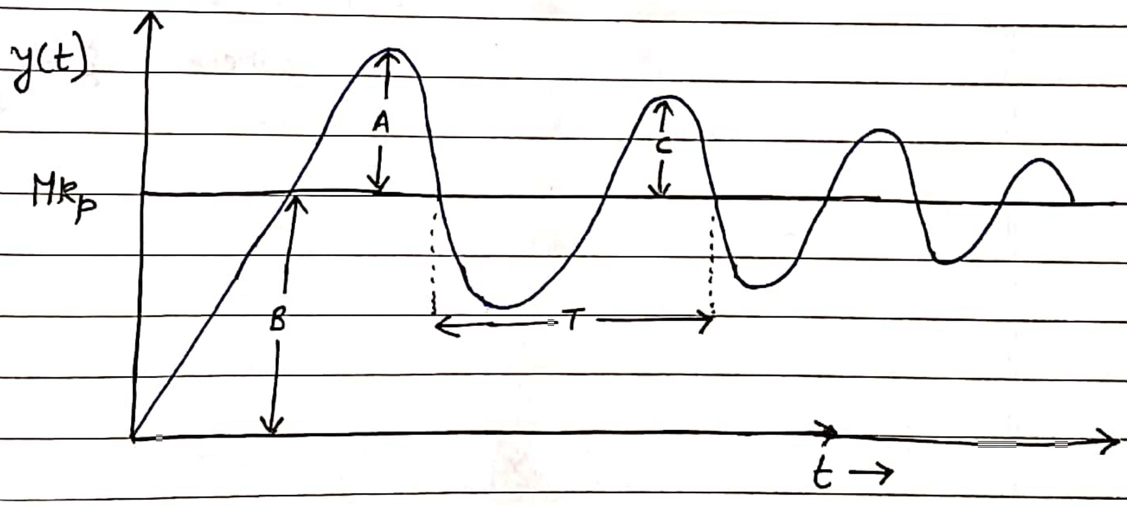
* For Overdamped system, as ξ increases, the response becomes more sluggish.

$$\boxed{\xi \propto \frac{1}{\text{speed of response}}}$$

Scanned with CamScanner



(e) Underdamped Response



(1) Overshoot :

→ measure of how much response exceeds the ultimate value following a step change

→ Ratio of A/B

$$\text{Overshoot} = \frac{A}{B} = \exp\left[\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right]$$

$$\xi = \frac{c}{2\sqrt{mk}}$$

$$\frac{A}{B} = \exp\left[\frac{\frac{-\pi c}{2\sqrt{mk}}}{\sqrt{1-\frac{c^2}{4mk}}}\right]$$

2. Decay Ratio:

It is the ratio of sizes of successive peaks

→ Ratio of $\frac{C}{A}$

$$\text{Decay Ratio} = \frac{C}{A} = \exp\left[\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right] = (\text{overshoot})^2$$

where $\xi = \frac{c}{2\sqrt{mk}}$

2. Response of a body towards changes in reservoir temperature

Given Equation

$$\frac{dT}{dt} = -h(T - T_\infty)$$

$T_\infty \rightarrow$ reservoir temp

$T_0 \rightarrow$ at $t=0$

$$\frac{dT}{dt} = -hT + hT_\infty$$

$$\frac{dT}{dt} + hT = hT_\infty$$

$$\frac{1}{h} \cdot \frac{dT}{dt} + T = T_\infty \quad \rightarrow (i)$$

For, 1^{st} order system

$$\tau \cdot \frac{dy}{dt} + y = X(t) \cdot K_p \quad \rightarrow (ii)$$

Comparing (i) & (ii)

$\boxed{\tau = 1/h}$ Time constant $\quad$ $\boxed{K_p = 1}$ Steady state gain

$$G(s) = \frac{1}{\left(\frac{1}{h}\right)s + 1}$$

(a) Step input of magnitude A

$$G(s) = \frac{T'(s)}{T'_\infty(s)} = \frac{1}{\frac{1}{h}s + 1}$$

$T'(s) = T(t) - T_s$

$T'_\infty(s) = T_\infty(t) - T_{\infty s}$

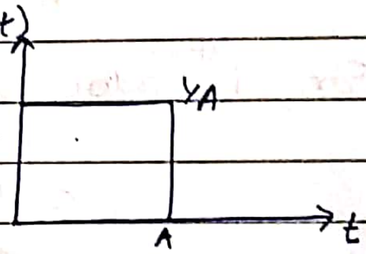
$$T_{\infty}'(s) = A/s \qquad \tau = 1/h$$

$$T'(s) = \frac{A}{s}\left(\frac{1}{\frac{1}{h}s+1}\right) \qquad k=1$$

$$T(t) = T_s + A(1-e^{-ht})$$

(b) Pulse function:

$$\delta = \begin{cases} 1/A & 0 \le t \le A \\ 0 & \text{otherwise} \end{cases}$$



$$T_{\infty}'(s) = \frac{1}{A}\frac{(1-e^{-As})}{s}$$

$$T'(s) = \frac{(1-e^{-As})}{As}\left(\frac{1}{\frac{1}{h}s+1}\right)$$

By partial fraction

$$T'(s) = \frac{1}{A}\left[\left(\frac{1}{s} - \frac{1}{s+h}\right) - \left(\frac{1}{s} - \frac{1}{s+h}\right)e^{-As}\right]$$

Inverse laplace

$$T = T_s + \frac{1}{A}\left[(1-e^{-ht}) - (1-e^{-h(t-A)})u(t-A)\right]$$

Impulse function:

Impulse function $= \delta(t)$

$$T_{\infty}'(s) = 1$$

$$T'(s) = \frac{1}{\frac{1}{h}s+1}$$

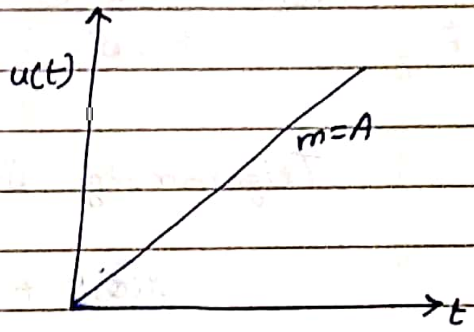
Taking inverse laplace

$$T'(t) = h e^{-ht} \qquad \tau = \frac{1}{h}$$

$$\boxed{T(t) = T_s + h e^{-ht}}$$

Ramp Function:

$$u(t) = At$$



$$T'_{\infty}(s) = A/s^2$$

$$T'(s) = \frac{A}{s^2}\left(\frac{1}{\frac{1}{h}s+1}\right) \qquad \tau = 1/h$$

$$T'(s) = \frac{A}{\tau}\left[\frac{\tau^2}{s+1/\tau} + \frac{-\tau^2}{s} + \frac{\tau}{s^2}\right]$$

By laplace inverse

$$T(t) - T_s = A\left(t - \tau\left(1-e^{-t/\tau}\right)\right) \quad \text{where } \tau = \frac{1}{h}$$

For sinusoidal response

$$T'_{\infty}(t) = A\sin\omega t$$

$$T'_{\infty}(s) = \frac{A\omega}{s^2+\omega^2}$$

$$T'(s) = \frac{A\omega}{s^2+\omega^2}\left(\frac{1/\tau}{s+1/\tau}\right)$$

By partial fraction

$$T'(s) = \frac{A\omega}{\tau}\left[\frac{\tau^2}{(1+\tau^2\omega^2)\left(s+\frac{1}{\tau}\right)} - \frac{\tau^2 s}{(1+\tau^2\omega^2)(s^2+\omega^2)} + \frac{\tau}{(1+\tau^2\omega^2)(s^2+\omega^2)}\right]$$

Taking laplace inverse.

$$T'(t) = \frac{A\omega\tau}{1+\tau^2\omega^2}e^{-t/\tau} - \frac{A\omega\tau\cos\omega t}{1+\tau^2\omega^2} + \frac{A\sin\omega t}{1+\tau^2\omega^2}$$

Trigonometry Identity

$$a\cos\theta + b\sin\theta = r\sin(\theta+\phi)$$

For this case, $r = \frac{A}{\sqrt{1+\tau^2\omega^2}}$ $\quad \phi = \tan^{-1}(\omega\tau)$

$$T(t) = T_s + \frac{A\omega\tau e^{-t/\tau}}{1+\tau^2\omega^2} + \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t+\phi)$$

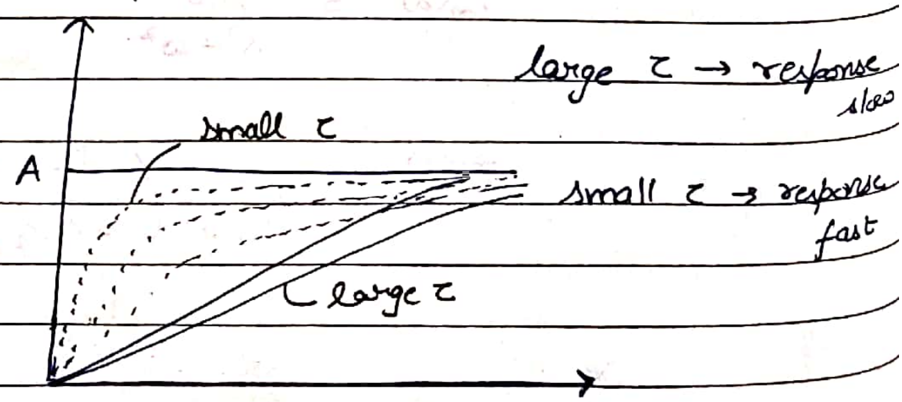
$$\tau = 1/h$$

Effect of Time constant

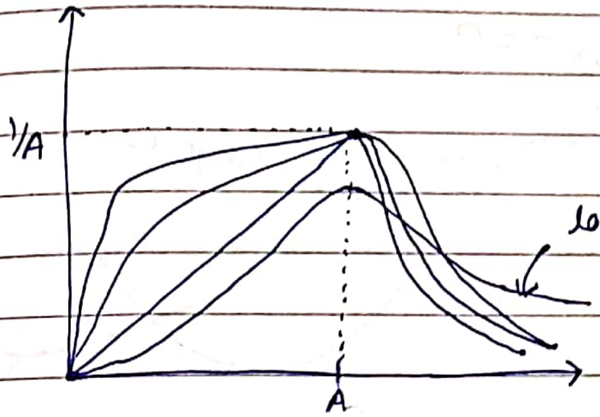
C → very small time constant

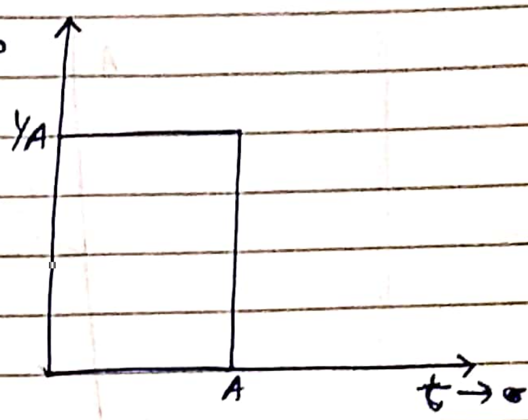
D → extremely large time constant

(i) For step input

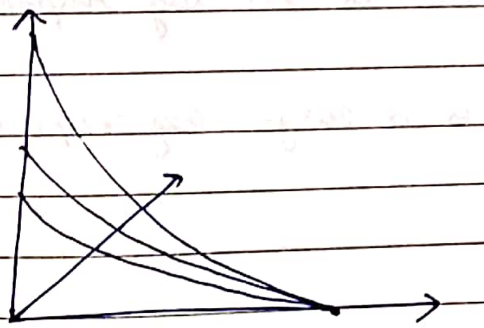


(ii) For Pulse



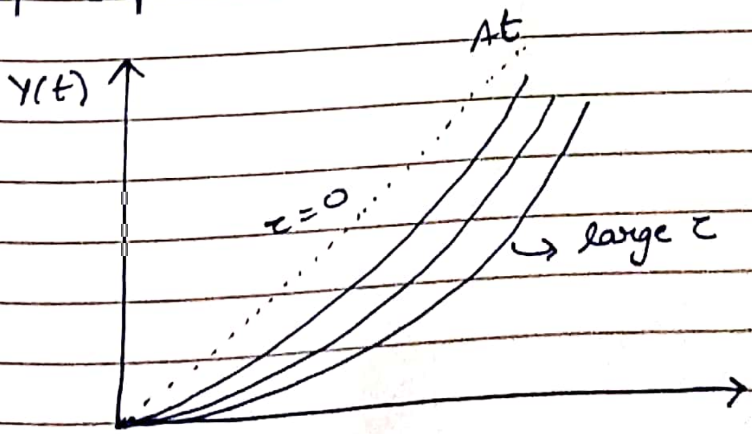


(iii) Unit Impulse

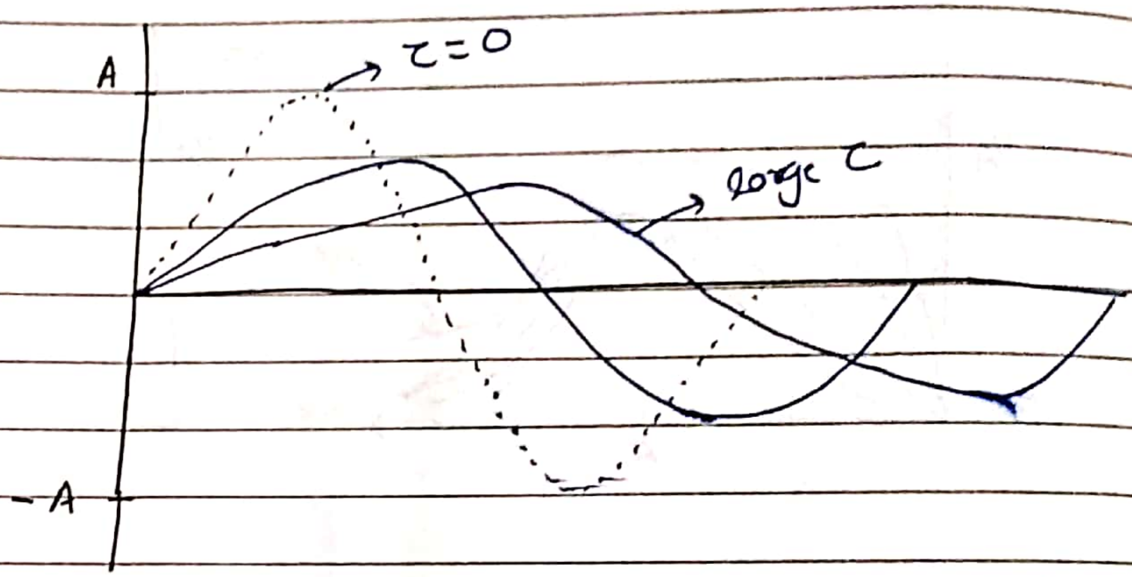


as $\tau \to 0$ $Y(t) \to \infty$

(iv) Ramp Input



(V) Sinosuidal Input



For very small 'τ' $\rightarrow$ There is no lag response

For large 'τ' $\rightarrow$ There is a large lag response.