Lecture on

Lecture 12-13

Application of Laplace Transforms

A. Solution of an integral equation (IE)

An IE is of the form

$$u(x) = f(x) + \int_a^b K(x,t)\, u(t)\, dt.$$

$f(x)$, $K(x,t)$ are known functions, $u(t) \to$ unknown.

$f(x)$ $\downarrow$ forcing funcⁿ. ; $K(x,t)$ $\to$ kernel of IE.

If $K(x,t)$ is of the form

$K(x,t) = K_1(x-t)$ and limits of integratⁿ are 0 and t,

i.e if the IE is of the form

$$u(x) = f(x) + \int_0^t K_1(x-t)\, u(t)\, dt$$

then the IE is called an IE of convolution type

Laplace transform is applied to solve linear IE's of convolution type.

1. Solve the IE

$$\phi(t) = 1 + 2\int_0^t \phi(t-u)\cos u\, du \quad \text{using LT.} \longrightarrow (1)$$

Sol. Taking LT on both sides of (1),

$$L[\phi(t)] = L[1] + 2L[(\phi \star \cos)(t)]$$

$$\text{or, } \bar{\phi}(s) = \frac{1}{s} + 2\bar{\phi}(s)L(\cos(t)) \quad \left[\because L(f \star g) = \bar{f}(s).\bar{g}(s)\right]$$

$$\text{or, } \bar{\phi}(s) = \frac{1}{s} + \frac{2\bar{\phi}(s).s}{s^2+1}$$

$$\text{or, } \left(1 - \frac{2s}{s^2+1}\right)\bar{\phi}(s) = \frac{1}{s} \Rightarrow \bar{\phi}(s) = \frac{s^2+1}{s(s-1)^2}$$

$$= \frac{(s-1)^2+2s}{s(s-1)^2}$$

$$\therefore \bar{\phi}(s) = \frac{1}{s} + \frac{2}{(s-1)^2}$$

$$\therefore \phi(t) = 1 + 2e^t L^{-1}\left(\frac{1}{s^2}\right) = 1 + 2te^t.$$

2. Solve: $\phi(t) = te^t - 2e^t\int_0^t e^{-u}\phi(u)\,du. \longrightarrow (1).$

Take LT on both sides of (1):

$$L[\phi(t)] = L(te^t) - 2L\left(\int_0^t e^{t-u}\phi(u)\,du\right)$$

$$\text{or, } \bar{\phi}(s) = -\frac{d}{ds}L(e^t) - 2L(e^t \star \phi)$$

$$\text{or, } \bar{\phi}(s) = -\frac{d}{ds}\cdot\frac{1}{s-1} - 2\cdot\frac{\bar{\phi}(s)}{s-1}.$$

$$\text{or, } \left(1 + \frac{2}{s-1}\right)\bar{\phi}(s) = \frac{1}{(s-1)^2} \Rightarrow \bar{\phi}(s) = \frac{(s-1)}{(s-1)^2 \times (s+1)}$$

$$\therefore \bar{\phi}(s) = \frac{1}{s^2-1} \Rightarrow \phi(t) = \sinh t.$$

B. Evaluation of definite integrals

1. Evaluate $\displaystyle\int_0^\infty \frac{\sin tx}{x(1+x^2)}\,dx$ using LT.

$\to = I(t)$, say.

Sol. $$\bar{I}(s) = \int_0^\infty e^{-st}\,dt \left(\int_0^\infty \frac{\sin tx}{x(1+x^2)}\,dx \right)$$

$$= \int_0^\infty \frac{dx}{x(1+x^2)} \int_0^\infty e^{-st} \underbrace{\sin xt\,dt}_{\text{LT of } \sin xt}$$

$$\bar{I} = \int_0^\infty \frac{dx}{\cancel{x}(1+x^2)} \cdot \frac{\cancel{x}}{x^2+s^2} = \int_0^\infty \frac{1}{s^2-1}\left(\frac{1}{x^2+1} - \frac{1}{x^2+s^2}\right)dx.$$

$$= \frac{1}{s^2-1}\int_0^\infty \frac{dx}{x^2+1} - \frac{1}{s^2-1}\int_0^\infty \frac{dx}{x^2+s^2}$$

$$= \frac{1}{s^2-1}\left[\tan^{-1} x \Big|_0^\infty - \frac{1}{s}\tan^{-1}\frac{x}{s}\Big|_0^\infty \right] = \frac{1}{s^2-1}\left(\frac{\pi}{2} - \frac{1}{s}\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}\cdot\frac{1}{s^2-1}\cdot\frac{s-1}{s} = \frac{\pi}{2}\cdot\frac{1}{s(s+1)} = \frac{\pi}{2}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\therefore\ I(t) = \frac{\pi}{2}(1-e^{-t})$$

2. Evaluate $\displaystyle I(t) = \int_0^\infty e^{-tx^2}\,dx.$

$$\bar{I}(s) = \int_0^\infty e^{-st}\,dt \int_0^\infty e^{-tx^2}\,dx = \int_0^\infty dx \int_0^\infty e^{-(s+x^2)t}\,dt = \int_0^\infty \frac{dx}{x^2+s}$$

$$= \frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\Big|_0^\infty = \frac{\pi}{2\sqrt{s}}.$$

$$\therefore\ I(t) = \frac{\pi}{2}\cdot\frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{1}{2}\cdot\sqrt{\frac{\pi}{t}}.$$

C. Solving integro-differential equation

- In an IE the unknown appears under integral sign.
- In an integro-differential equation unknown appears under integral as well as differentiation sign.

Ex-1 Solve:

$$u(t) = a \sin t + 2\int_0^t u'(\tau) \sin(t-\tau)\, d\tau \; ; \; u(0)=0 \quad \rightarrow (1)$$

[Note, like in the case of an ODE of order n, one must specify n conditions to get a particular solution.

In an integro-diff. equat^n. also, one must specify n no. of conditions if the unknown appears under n^{th} order differentiation]

Taking LT on both sides of (1), get

$$\bar{u}(s) = \frac{a}{s^2+1} + 2L\left[u'(t) * \sin t\right]$$

$$= \frac{a}{s^2+1} + 2L(u'(t))\, L(\sin t). \qquad \because L(f*g) = \bar{f}(s)\cdot\bar{g}(s)$$

$$= \frac{a}{s^2+1} + 2\left\{s\bar{u}(s) - \underbrace{u(0)}_{=0}\right\}\cdot\frac{1}{s^2+1}$$

$$\therefore \left(1 - \frac{2s}{s^2+1}\right)\bar{u}(s) = \frac{a}{s^2+1} \Rightarrow \bar{u}(s) = \frac{a}{(s-1)^2}$$

$$\therefore u(t) = a t e^t.$$

D. Solution of differential equations.

D1. Solving Initial Value Problem (IVP)

Note. LT is widely used to solve differential equations where the independent variable is time. The problems where conditions are prescribed at time $t=0$ (initial value) are known as IVP. Also, if for the range of the independent variable (x) is $[a, b]$, say and if the condition is given at one end point (mainly at left end), that is also treated as IVP.

Ex1 Solve. $y''' - 3y'' + 3y' - y = t^2 e^t$; $\rightarrow (1)$.

$y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

$\boxed{\text{Note: Here coefficients of } y, y', y'', y''' \text{ are constants}}$.

Sol. Take LT on both sides of (1) & get

$$L[y'''] - 3L[y''] + 3L[y'] - L[y] = L[t^2 e^t]$$

or, $$s^3 \bar{y}(s) - s^2 \underbrace{y(0)}_{=1} - s \underbrace{y'(0)}_{=0} - \underbrace{y''(0)}_{(=-2)}$$
$$-3\left\{s^2 \bar{y}(s) - s \underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\right\} + 3\left\{s\bar{y}(s) - \underbrace{y(0)}_{=1}\right\}$$
$$-\bar{y}(s) = \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

or, $$(s^3 - 3s^2 + 3s - 1)\bar{y}(s) - s^2 + 2 + 3s - 3 = \frac{2}{(s-1)^3}.$$

$$\therefore (s-1)^3 \bar{y}(s) = s^2 - 3s + 1 + \frac{2}{(s-1)^3}.$$

$$\therefore \bar{y}(s) = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}.$$

$$= \frac{(s-1)^2 - s + 1 - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{2}{(s-1)^6} - \frac{1}{(s-1)^3}$$

$$\therefore y(t) = e^t \left[1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right] //.$$

Ex-2. Solve: $t\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 4t\,y(t) = 0.$ $y(0) = 3,\ y'(0) = 0.$

$\longrightarrow (1)$

Note: the coefficients of $\dfrac{d^2y}{dt^2}$, $y(t)$ are variable.

Ans. Take LT on both sides of (1).

$$L\left[t\frac{d^2y}{dt^2}\right] + L\left[\frac{dy}{dt}\right] + 4L\left[t\,y(t)\right] = 0$$

$$\text{or, } -\frac{d}{ds}\left[s^2\bar{y}(s) - s\,y(0) - y'(0)\right] + s\bar{y}(s) - y(0) - 4\frac{d}{ds}\bar{y}(s) = 0$$

$$\text{or, } -2s\bar{y}(s) - s^2\bar{y}'(s) + \cancel{y(0)} + s\bar{y}(s) - \cancel{y(0)} - 4\frac{d}{ds}\bar{y}(s) = 0$$

$$\text{or, } -(s^2+4)\bar{y}'(s) - s\bar{y}(s) = 0.$$

$$\text{or, } \frac{\bar{y}'(s)}{\bar{y}(s)} + \frac{s}{s^2+4} = 0.$$

Integrating, $\log \bar{y}(s) + \frac{1}{2}\log|s^2+4| = \log c$

$$\therefore \bar{y}(s) = \frac{c}{\sqrt{s^2+4}} \Rightarrow y(t) = c\,J_0(2t)$$

$y(0) = 3 = c\,J_0(0) = c \quad \because J_0(0) = 1 \quad \therefore y(t) = 3J_0(2t).$

D2. Solving boundary value problems. (BVP)

A BVP is a problem of solving a DE, when the conditions are prescribed at two end (boundary) points.

Ex. Solve: $\frac{d^2y}{dt^2} + 9y = 18t \; ; \; 0 \le t \le \frac{\pi}{2}$ $\longrightarrow (1)$

with the boundary conditions $y(0)=0, \; y(\frac{\pi}{2})=0$.

Sol. Take LT on both sides of (1) & get

$$s^2\bar{y}(s) - s\underbrace{y(0)}_{=0\,(\text{given})} - y'(0) + 9\bar{y}(s) = \frac{18}{s^2}$$

or, $(s^2+9)\bar{y}(s) - A = \frac{18}{s^2}$; Here $A = y'(0)$, not given.

$$\therefore \bar{y}(s) = \frac{A}{s^2+9} + \frac{18}{s^2(s^2+9)} = \frac{A}{s^2+9} + \frac{2}{s^2} - \frac{2}{s^2+9}$$

$$\therefore y(t) = L^{-1}\left(\frac{A-2}{s^2+9}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) = \frac{A-2}{3}\sin 3t + 2t$$

A can be determined by using the condition $y(\frac{\pi}{2})=0$, which is ~~still~~ so far unused.

$$\therefore \; 0 = y\left(\frac{\pi}{2}\right) = \frac{A-2}{3}\sin\frac{3\pi}{2} + \pi = -\frac{A-2}{3} + \pi$$

$$\therefore \; A - 2 = 3\pi \Rightarrow A = 3\pi + 2.$$

$$\therefore \; y(t) = \pi \sin 3t + 2t.$$

D3. Solving system of diff. equations.

Solve : $\dfrac{d^2y}{dt^2} - 3y - 4x = 0 \longrightarrow (1)$

$\dfrac{d^2x}{dt^2} + y + x = 0. \longrightarrow (2).$

Initial conditions:

$y(0) = 0 = x(0)$

$y'(0) = 2,\ x'(0) = 0.$

Sol. Take LT of (1) & (2) w.r.t. the variable t.

$\therefore$ This gives, $s^2\bar{y}(s) - s\underset{=0}{y(0)} - \underset{=2}{y'(0)} - 3\bar{y}(s) - 4\bar{x}(s) = 0$

& $s^2\bar{x}(s) - s\underset{=0}{x(0)} - \underset{=0}{x'(0)} + \bar{y}(s) + \bar{x}(s) = 0.$

$\therefore (s^2-3)\bar{y}(s) - 4\bar{x}(s) = 2 \longrightarrow (3)$

$\bar{y}(s) + (s^2+1)\bar{x}(s) = 0. \longrightarrow (4)$

Mult. (4) by (s^2-3) & subtracting from (3) get

$$(s^2-3)\cancel{\bar{y}}(s) - 4\bar{x}(s) = 2$$

$$\underline{-\ (s^2-3)\cancel{\bar{y}}(s) \underset{-}{+} (s^2+1)(s^2-3)\bar{x}(s) \underset{-}{=} 0.}$$

$$-(s^4 - 2s^2 - 3 + 4)\bar{x}(s) = 2$$

$\Rightarrow (s^4 - 2s^2 + 1)\bar{x}(s) = -2 \quad \Rightarrow \bar{x}(s) = -\dfrac{2}{(s^2-1)^2}$

$\therefore \bar{x}(s) = +\dfrac{\cancel{2}}{\cancel{2}} \cdot \dfrac{(s-1)-(s+1)}{(s^2-1)^2} = \dfrac{1}{(s-1)(s+1)^2} - \dfrac{1}{(s+1)(s-1)^2}$

$= \dfrac{1}{2} \cdot \dfrac{(s+1)-(s-1)}{(s-1)(s+1)^2} - \dfrac{1}{2} \cdot \dfrac{(s+1)-(s-1)}{(s+1)(s-1)^2}$

$$\therefore \bar{x}(s) = \frac{1}{2}\cdot\frac{1}{(s-1)(s+1)} - \frac{1}{2}\cdot\frac{1}{(s+1)^2} - \frac{1}{2}\cdot\frac{1}{(s-1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)(s-1)}$$

$$= \frac{1}{2}\cdot\frac{(s+1)-(s-1)}{(s-1)\ (s+1)} - \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{2}\frac{1}{(s-1)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{(s+1)^2} - \frac{1}{2}\cdot\frac{1}{(s-1)^2}.$$

$$\therefore x(t) = \frac{1}{2}e^{t} - \frac{1}{2}te^{t} - \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t}$$

$$= \frac{1}{2}(1-t)e^{t} - \frac{1}{2}(1+t)e^{-t} = \sinh t - t\cosh t.$$

<u>Alt.</u> $\bar{x}(s) = -\frac{2}{(s^2-1)^2} = -2\cdot\frac{1}{s^2-1}\cdot\frac{1}{s^2-1}.$

Now, you may apply convolution thm to get $x(t)$.

To determine $y(t)$, look at (3).

$$(s^2-3)\bar{y}(s) - 4\bar{x}(s) = 2.$$

$$\therefore \bar{y}(s) = \frac{1}{s^2-3}\left[2 + 4\times -\frac{2}{(s^2-1)^2}\right] = \frac{2}{s^2-3}\cdot\frac{s^4-2s^2-3}{(s^2-1)^2}$$

$$= \frac{2\cancel{(s^2-3)}(s^2+1)}{(s^2-1)^2\cancel{(s^2-3)}} = \frac{(s+1)^2+(s-1)^2}{(s+1)^2(s-1)^2}$$

$$= \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2}$$

$$\therefore y(t) = te^{-t} + te^{t} = 2t\cosh t. \ //.$$

Exercise Problems

A1. Solve the IE:

$$\phi(t) = 1 - \sinh t + \int_0^t (1+u)\,\phi(t-u)\,du$$

Ans: $\phi(t) = \cosh t$.

A2. Solve the IE:

$$\phi(t) = 2t - 4\int_0^t \phi(u)\,(t-u)\,du\,;$$ Ans: $\phi(t) = \sin 2t$.

B1. Evaluate $\displaystyle\int_0^\infty \frac{\sin tx}{x}\,dx$ using LT. Ans. $\frac{\pi}{2}$

B2. " $\displaystyle\int_0^\infty \frac{\cos tx}{x^2+a^2}\,dx$ " LT Ans. $\frac{\pi}{a} e^{-at}$.

C1 Solve $\displaystyle\frac{dy}{dt} + 5\int_0^t y(u)\cos 2(t-u)\,du = 10\,;\; y(0) = 2$

Ans. $y(t) = \frac{1}{9}\left(8 + 40t + 10\cos 3t + \frac{50}{3}\sin 3t\right)$.

C2. Solve $y(t) = 2\sin t + 2\int_0^t y'(u)\sin(t-u)\,du\,;\; y(0)=0$

Ans. $y(t) = 2te^t$

D11. Solve $\displaystyle\frac{d^2y}{dt^2} + n^2 y = a\sin(nt+2)\,;\; y(0)=0,\; y'(0)=0$

Ans. $\frac{a}{2n^2}\left[\cos 2 \cdot \sin nt - nt\cos(nt+2)\right]$.

D12. Solve: $\displaystyle\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^t\,;$ $y(0)=1$, $y'(0)=0$, $y''(0)=-2$.

Ans. $e^t\left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)$.

Solve:

D13. $t\dfrac{d^2y}{dt^2} + (1-2t)\dfrac{dy}{dt} - 2y = 0$; $y(0)=1$, $y'(0)=2$

Ans. $y(t) = e^{2t}$.

D14. Solve: $t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + ty = 0$; $y(0)=1$, $y(\pi)=0$.

Ans. $y(t) = \dfrac{\sin t}{t}$.

D21 Solve the BVP: $\dfrac{d^2x}{dt^2} + 4x(t) = -8t^2$; $x(0) = 3$, $x\left(\frac{\pi}{4}\right) = 0$

Ans. $x(t) = 2\cos 2t + \left(\frac{\pi^2}{8} - 1\right)\sin 2t + 1 - 2t^2$.

D22. Solve the BVP: $y^{(iv)} - 16y = 30\sin t$ | $y^{(iv)} \equiv \dfrac{d^4y}{dt^4}$.

$y''(0) = 0$, $y'''(0) = -18$

$y''(\pi) = 0$, $y'''(\pi) = -18$

Ans: $y(t) = -\frac{1}{8}e^{2t} + \frac{1}{8}e^{-2t} + 7\sin t - \frac{9}{2}\sin 2t$

D31. Solve:

$$\left.\begin{aligned} x(t) - \frac{d^2y}{dt^2} + y(t) &= -e^{-t} - 1 \\ \frac{dx}{dt} + \frac{dy}{dt} - y(t) &= -3e^{-t} + t \end{aligned}\right\} \quad x(0) = 0,\ y(0) = 1,\ y'(0) = -2$$

Ans: $x(t) = t + e^{-t} - 1$; $y(t) = e^{-t} - t$.

D32. Solve.

$$\left.\begin{aligned} 3\frac{dx}{dt} + \frac{dy}{dt} + 2x &= 1 \\ \frac{dx}{dt} + 4\frac{dy}{dt} + 3y &= 0 \end{aligned}\right\} \quad x(0) = 0 = y(0)$$

Ans. $x(t) = \frac{1}{10}\left(5 - 3e^{-\frac{6}{11}t} - 2e^{-t}\right)$

$y(t) = \frac{1}{5}\left(e^{-t} - e^{-\frac{6}{11}t}\right)$.