Lecture-22
Tuesday
17/10/17

Show that

Exercise : ~~The~~ the function $\cos\left[\frac{x^2}{2}-\frac{\pi}{8}\right]$ is self-reciprocal w.r.to F.T and F.C.T.

Hint

$$\mathcal{F}\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] = \mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos\omega x\,dx.$$

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$$= \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\cos\left\{\frac{x^2}{2}-\frac{\pi}{8}+\omega x\right\}+\cos\left\{\frac{x^2}{2}-\frac{\pi}{8}-\omega x\right\}\right]dx.$$

$\downarrow$

Put $x=-x$.

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\cos\left(\frac{x^2}{2}+\omega x-\frac{\pi}{8}\right)dx.$$

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$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\cos\left[\left(\frac{x}{\sqrt{2}}+\frac{\omega}{\sqrt{2}}\right)^2-\left(\frac{\pi}{8}+\frac{\omega^2}{2}\right)\right]dx.$$

Put $\frac{x}{\sqrt{2}}+\frac{\omega}{\sqrt{2}}=v$.

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$$= \frac{1}{\sqrt{\pi}}\left[\cos\left(\frac{\omega^2}{2}+\frac{\pi}{8}\right)\underbrace{\int_{-\infty}^{\infty}\cos v^2\,dv}_{=\sqrt{\frac{\pi}{2}}}+\sin\left(\frac{\omega^2}{2}+\frac{\pi}{8}\right)\underbrace{\int_{-\infty}^{\infty}\sin v^2\,dv}_{=\sqrt{\frac{\pi}{2}}}\right]$$

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$$= \cos\left(\frac{\omega^2}{2}-\frac{\pi}{8}\right).$$

Complete it.

Properties of F.T.

(1) Shifting Property.

If $F(\omega) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$.

then $\mathcal{F}[f(x-a)] = e^{i\omega a} F(\omega)$.

Pf. $\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{i\omega x} dx$.

Put $x - a = t$ ∴ $dx = dt$.
$x = t + a$

$$\therefore \mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega(t+a)} dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \underline{e^{i\omega a}} dt.$$

$$= e^{i\omega a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. = e^{i\omega a} F(\omega).$$

(2) Scaling property.

If $F(\omega) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$.

then $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$.

Proof- Case 1. $a > 0$.

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{i\omega x}\, dx.$$

Let $ax = t$. Then $dx = \frac{dt}{a}$.

$$\therefore \mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\frac{\omega}{a}t} \frac{dt}{a}.$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\cdot\left(\frac{\omega}{a}\right)t} dt.$$

$$= \frac{1}{a} \cdot F\left(\frac{\omega}{a}\right), \quad a > 0$$

Case 2. $a < 0$, $a = -b$; $b > 0$.

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{i\omega x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-bx)\, e^{i\omega x} dx.$$

Put $-bx = t \Rightarrow -b\,dx = dt$.

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{i\frac{\omega}{-b}t} \frac{dt}{-b}.$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{\omega}{a}\right)t}\right) \times \frac{1}{-a}.$$

$$= -\frac{1}{a} F\left(\frac{\omega}{a}\right), \quad a < 0.$$

Combining the two cases,

$$\mathcal{F}\left[f(ax)\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

3. Translation property.

$$\mathcal{F}\left[e^{iax} f(x)\right] = F(\omega + a).$$

$$\text{L.H.S} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) \cdot e^{i\omega x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(\omega+a)x}\, dx = F(\omega+a).$$

4. Conjugate property.

$$\mathcal{F}\left\{\overline{f(-x)}\right\} = \overline{\mathcal{F}\left\{f(x)\right\}}$$

$\overline{e^{i\omega x}} = e^{-i\omega x}$

$\bar{z}_1 \bar{z}_2 = \overline{z_1 z_2}$

$$\text{L.H.S} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{i\omega x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\; \overline{e^{-i\omega x}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)\, e^{-i\omega x}}\, dx.$$

Let $-x = y$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(y)\, e^{i\omega y}}\, dy$$

$$= \overline{\mathcal{F}\left[f(x)\right]}$$

5. Duality Property.

If $\mathcal{F}[f(x);\omega] = F(\omega)$,

then $\mathcal{F}[F(x);\omega] = f(-\omega)$.

If $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$.

then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \longrightarrow (1).$$

In (1), replace x by ω and ω by x. & get.

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-ix.\omega} dx.$$

Putting $-\omega$ in place of ω, get

$$f(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{i\omega x} dx = \mathcal{F}[F(x);\omega].$$

6. ~~Der~~ F.T. of derivatives.

If. $f(x) \in C'(-\infty,\infty)$, ~~then~~ and if $f(x) \to 0$. as $x \to \pm\infty$ (as $|x| \to \infty$), then.

$$\mathcal{F}[f'(x)] = -i\omega F(\omega).$$

$$\mathcal{F}\left[f'(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{i\omega x} \Big|_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \right]$$

$$= -i\omega \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = (-i\omega) F(\omega).$$

Extension. If $f(x) \in C^n(-\infty, \infty)$ & if $f(x), f'(x), \ldots, f^{(n-1)}(x) \to 0$ as $|x| \to \infty$, then.

$$F\left[f^{(n)}(x); \omega\right] = (-i\omega)^n F(\omega).$$

Fourier transform of a constant.

$$F(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c\, e^{i\omega x} dx \rightarrow \text{does not exist in ordinary sense,}$$

Sufficient condi^{n} for existence of F.T.

F.T. of $f(x)$ will exist if it satisfies.

1) it is absoolutely integrable in $(-\infty, \infty)$.

2) it satisfies Dirichlet's conditions.

Dirichlet's condition in $(-\infty, \infty)$.

1) $f(x)$ ~~satisfies~~ has finite no. of maxima & minima in $(-\infty, \infty)$

2) $f(x)$ has finite no. of finite discontinuities in $(-\infty, \infty)$.

F.T. of $\delta(t)$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt.$$

$$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt$$

$$= f(a)$$

$$= \frac{1}{\sqrt{2\pi}} \times e^{i\omega . 0} = \frac{1}{\sqrt{2\pi}} = F(\omega).$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega.$$

$$\text{or, } \delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega x} d\omega.$$

$$\text{or, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} d\omega = \delta(x) \cdot \times \sqrt{2\pi}$$

$$\downarrow$$

$$\omega = -y.$$

$$\text{So, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} dy = \delta(x) \sqrt{2\pi}.$$

replace x by ω.

$$\text{or, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c e^{i\omega y} dy = \sqrt{2\pi}\, c\, \delta(\omega).$$

$$\mathcal{F}[c] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c e^{i\omega x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c e^{i\omega y} dy.$$

So, F.T. of a const. funct. doesn't exist in ordinary sense. It exists in terms of generalized funct. //.