Fourier Series (F.S)

Lecture - 15
5/9/17

Ex-1. Find F.S. of $f(x)$ where $f(x)$ is given by

$$f(x) = \begin{cases} x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

$x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$a = -\frac{\pi}{2}$, $b = \frac{3\pi}{2}$

$b - a = \frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$

~~$f(x)$~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(x)\,dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} \underbrace{x}_{\text{odd}}\,dx = 0.$$

$$a_n = \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \underbrace{\underbrace{x}_{\text{odd}} \underbrace{\cos nx}_{\text{even}}}_{\text{odd}}\,dx = 0.$$

$$b_n = \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(x) \sin nx\,dx = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \underbrace{x \sin nx}_{\text{even}}\,dx.$$

$$= \frac{2}{\pi} \int_0^{\pi/2} x \sin nx\,dx.$$

$$= \frac{2}{\pi} \left[x \frac{\cos nx}{n} \Big|_{\frac{\pi}{2}}^{0} + \int_0^{\pi/2} 1.\frac{\cos nx}{n}\,dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{2} \frac{\cos \frac{n\pi}{2}}{n} + \frac{\sin nx}{n^2} \Big|_0^{\pi/2} \right] = \frac{2}{\pi} \cdot \frac{\sin \frac{n\pi}{2}}{n^2} - \frac{\cos \frac{n\pi}{2}}{n}.$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \frac{\sin \frac{n\pi}{2}}{n^2} - \frac{\cos \frac{n\pi}{2}}{n} \right) \sin nx .$$

2. Find F.S. for $f(x)$ in $[0, 2]$ where

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 1-x, & 1 < x < 2 \end{cases}, \quad a=0, \quad l=2 .$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x .$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\, dx = \int_0^1 x\, dx + \int_1^2 (1-x)\, dx .$$

$$x-1=z.$$

$$= \int_0^1 x\, dx - \int_0^1 z\, dz = 0 .$$

$$a_n = \begin{cases} -\frac{4}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even} . \end{cases}$$

$$b_n = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} . \end{cases}$$

$$f(x) = -\frac{4}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right]$$
$$+ \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right] .$$

3. Find FS of $f(x)$ where

$f(x) = x\sin x, \quad 0 < x < 2\pi$,

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx .$$

$$= -2 .$$

$$a_n = \frac{\cancel{2}}{\cancel{2}\pi\times 2}\int_0^{2\pi} x\, 2\sin x\cos nx\,dx .$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{\sin(1+n)x + \sin(1-n)x\right\}dx .$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin(n-1)x\,dx .$$

$$= \frac{2}{n^2-1} .$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} x\sin x\sin nx\,dx .$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{\cos(1-n)x - \cos(1+n)x\right\}dx .$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\,dx .$$

$$= 0 .$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= -1 + \sum_{n=1}^{\infty} \frac{2}{n^2-1} \cos nx$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx = -\frac{1}{2}.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin x \, dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \, 2\sin^2 x \, dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(1-\cos 2x)\, dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \, dx - \underbrace{\frac{1}{2\pi}\int_0^{2\pi} x \cos 2x \, dx}_{=0 \text{ ? check.}}$$

$$= \frac{1}{2\pi} \times \frac{4\pi^2}{2} = \pi.$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx.$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx.$$