In this lesson we evaluate Laplace transform of periodic functions. Periodic functions frequently occur in various engineering problems. We shall now show that with the help of a simple integral, we can evaluate Laplace transform of periodic functions. We shall further continue the discussion for stating initial and final value theorems of Laplace transforms and their applications with the help of simple examples.

6.1 Laplace Transform of a Periodic Function

Let f be a periodic function with period T so that $f(t) = f(t+T)$ then,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t.$$

Proof: By definition we have,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

We break the integral into two integrals as

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_T^\infty e^{-st} f(t)\,\mathrm{d}t$$

Substituting $t = \tau + T$ in the second integral

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_0^\infty e^{-(\tau+T)s} f(\tau+T)\,\mathrm{d}\tau$$

Noting $f(\tau+T) = f(\tau)$ we find

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + e^{-sT} L[f(t)],$$

On simplifications, we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t.$$

This completes the proof. ■

Remark 1: *Just to remind that if a function f is periodic with period $T > 0$ then $f(t) = f(t+T)$, $-\infty < t < \infty$. The smallest of T, for which the equality $f(t) = f(t+T)$ is true, is called fundamental period of $f(t)$. However, if T is the period of a function f then nT, n is any natural number, is also a period of f. Some familiar periodic functions are* $\sin x$, $\cos x$, $\tan x$ *etc.*

6.2 Example Problems

6.2.1 Problem 1

Find Laplace transform for

$$f(t) = \begin{cases} 1 & \textit{when } 0 < t \leq 1 \\ 0 & \textit{when } 1 < t < 2 \end{cases}$$

with $f(t+2) = f(t),\ t > 0.$

Solution: Using the above result on periodic function, we have,

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\,\mathrm{d}t$$

On integration we obtain

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left(\frac{1}{-s}\right) \left[e^{-s} - 1\right] = \frac{1}{s(1+e^{-s})}$$

6.2.2 Problem 2

Find Laplace transform for

$$f(t) = \begin{cases} \sin t & \textit{when } 0 < t < \pi \\ 0 & \textit{when } \pi < t < 2\pi \end{cases}$$

with $f(t+2\pi) = f(t),\ t > 0.$

Solution: Since $f(t)$ is periodic with period 2π we have

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}} \int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

We now evaluate the above integral as

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} f(t)\,\mathrm{d}t + \int_{\pi}^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

Substituting the given value of $f(t)$ we obtain

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} \sin t\,\mathrm{d}t + 0 = \frac{1+e^{-s\pi}}{1+s^2}$$

This implies

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}} \frac{1+e^{-s\pi}}{1+s^2} = \frac{1}{(1+s^2)(1-e^{-s\pi})}$$

6.2.3 Problem 3

Find the Laplace transform of the square wave with period T:

$$f(t) = \begin{cases} h & \textit{when } 0 < t < T/2 \\ -h & \textit{when } T/2 < t < T \end{cases}$$

Solution: Using Laplace transform of periodic function we find

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t$$

Substituting $f(t)$ we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left(\int_0^{T/2} he^{-st}\,\mathrm{d}t - \int_{T/2}^{T} he^{-st}\,\mathrm{d}t \right)$$

Evaluating integrals we get

$$L[f(t)] = \frac{1}{(1-e^{-sT})} \frac{h}{s} \left(1 - 2e^{-sT/2} + e^{-sT} \right) = \frac{h(1-e^{-sT/2})}{s(1+e^{-sT/2})}$$

6.3 Limiting Theorems

These theorems allow the limiting behavior of the function to be directly calculated by taking a limit of the transformed function.

6.3.1 Theorem (Initial Value Theorem)

Suppose that f is continuous on $[0,\infty)$ and of exponential order α and f' is piecewise continuous on $[0,\infty)$ and of exponential order. Let

$$F(s) = L[f(t)],$$

then

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s), \quad \textit{[assuming } s \textit{ is real]}$$

Proof: By the derivative theorem,

$$L[f^{'}(t)] = sL[f(t)] - f(0+)$$

Note that $\lim_{s\to\infty} L[f^{'}(t)] = 0$, since $f^{'}$ is piecewise continuous on $[0, \infty)$ and of exponential order. Therefore we have

$$0 = \lim_{s\to\infty} sF(s) - f(0+)$$

Hence we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

This completes the proof. ■

6.3.2 Theorem (Final Value Theorem)

Suppose that f is continuous on $[0, \infty)$ and is of exponential order α and $f^{'}$ is piecewise continuous on $[0, \infty)$ and furthermore $\lim_{t\to\infty} f(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sL[f(t)] = \lim_{s\to 0} sF(s)$$

Proof: Note that f has exponential order $\alpha = 0$ since it is bounded, since $\lim_{t\to 0+} f(t)$ and $\lim_{t\to\infty} f(t)$ exist and $f(t)$ is continuous in $[0, \infty)$. By the derivative theorem, we have

$$L[f^{'}(t)] = sF(s) - f(0), \quad s > 0,$$

Taking limit as $s \to 0$, we obtain

$$\lim_{s\to 0} \int_0^\infty e^{-st} f^{'}(t)\, \mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

Taking the limit inside the integral

$$\int_0^\infty f^{'}(t)\, \mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

On integrating we obtain

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

Cancellation of $f(0)$ gives the desired results. ■

Remark 2: *In the final value theorem, existence of* $\lim_{t\to\infty} f(t)$ *is very important. Consider* $f(t) = \sin t$*. Then* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{1+s^2} = 0$*. But* $\lim_{t\to\infty} f(t)$ *does not exist. Thus we may say that if* $\lim_{s\to 0} sF(s) = L$ *exists then either* $\lim_{t\to\infty} f(t) = L$ *or this limit does not exist.*

6.3.3 Example

Without determining $f(t)$ *and assuming that* $f(t)$ *satisfies the hypothesis of the limiting theorems, compute*

$$\lim_{t\to 0+} f(t) \textit{ and } \lim_{t\to\infty} f(t) \quad \textit{if} \quad L[f(t)] = \frac{1}{s} + \tan^{-1}\left(\frac{a}{s}\right).$$

Solution: By initial value theorem, we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \left[1 + s\tan^{-1}\left(\frac{a}{s}\right)\right]$$

Application of L'hospital rule gives

$$\lim_{t\to 0+} f(t) = 1 + \lim_{s\to\infty} \frac{\frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right)}{-\frac{1}{s^2}} = 1 + a$$

Using the final value theorem we find

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \left[1 + s\tan^{-1}\frac{a}{s}\right] = 1.$$

Remark 3: *Final value theorem says* $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$ *if* $\lim_{t\to\infty} f(t)$ *exists. If* $F(s)$ *is finite as* $s \to 0$ *then trivially* $\lim_{t\to\infty} f(t) = 0$*. However, there are several functions whose Laplace transform is not finite as* $s \to 0$*, for example,* $f(t) = 1$ *and its Laplace transform* $F(s)$ *is equal to* $\frac{1}{s}, \;\; s > 0$*. In this case we have* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} 1 = 1 = \lim_{t\to\infty} f(t)$.