① Hyperbolic PDEs

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

- $u(t=0) = u_{10}$
- $u(x=0) = u_{30}$
- $\frac{\partial u}{\partial t}(t=0) = u_{20}$
- $u(x=1) = u_{40}$

- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$u(t=0) = u_{10}$ $\quad$ $u(x=0) = u_{30}$

$\frac{\partial u}{\partial t}(t=0) = 0$ $\quad$ $u(x=1) = 0$

Split $u = u_1 + u_2$

we know how to solve this problem:

u_1:

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_1}{\partial x^2}$$

$u_1(t=0) = u_{10}$ $\quad$ $u_1(x=0) = 0$

$\frac{\partial u_1}{\partial t}(t=0) = 0$ $\quad$ $u_1(x=1) = 0$

Ill-posed problem

$$u_2 : \quad \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial^2 u_2}{\partial x^2}$$

$$u_2(t=0) = 0 \qquad u_2(x=0) = u_{30}$$

$$\frac{\partial u_2}{\partial t}(t=0) = 0 \qquad u_2(x=1) = 0$$

$$\therefore \text{ take } \quad u_2(x,t) = u_2^t(x,t) + u_2^s(x)$$

$$\frac{\partial^2 u_2^t}{\partial t^2} = \frac{\partial^2 u_2^s}{\partial x^2} + \frac{\partial^2 u_2^t}{\partial x^2}$$

u_2^s

$$\frac{d^2 u_2^s}{dx^2} = 0$$

$$u_2(x=0) = u_{30}$$

$$\therefore u_2^s(x=0) = u_{30} \quad \text{(taking it like this)}$$

$$u_2^s(x=1) = 0$$

$$u_2^s = c_1 \cdot x + c_2$$

$$u_{30} = c_2$$

$$0 = c_1 + u_{30}$$

$$\therefore \boxed{u_2^s = u_{30}(1-x)}$$

u_2^t

$$\frac{\partial^2 u_2^t}{\partial t^2} = \frac{\partial^2 u_2^t}{\partial x^2}$$

$$u_2^t(x=0) = 0$$

$$u_2^t(x=1) = 0$$

$$u_2(t=0) = 0$$

$$\therefore u_2^t(t=0) = -u_{30}(1-x)$$

$$\frac{\partial u_2^t}{\partial t}(t=0) = 0$$

→ IMP.

$$u_2(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[c_5 \cdot \sin(n\pi t) + c_6 \cos(n\pi t)\right]$$

$$\frac{\partial u_2(x,t)}{\partial t}\Big|_{(t=0)} = 0$$

$$\frac{\partial u_2(x,t)}{\partial t} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot n \cdot \pi \left[c_5 \cos(n\pi t) - c_6 \sin(n\pi t)\right]$$

$$\therefore \quad 0 = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot n\pi \cdot c_5$$

$$\Rightarrow \quad c_5 = 0$$

$$-u_{30}(1-x) = \sum_{n=1}^{\infty} c_6 \sin(n\pi x)$$

$$\Rightarrow \quad -u_{30} \int_0^1 (1-x) \sin(n\pi x)\, dx = \frac{c_6}{2}$$

$$\Rightarrow u_{30} = -2u_{30} \cdot \left[\int_0^1 \sin(n\pi x)\, dx - \int_0^1 x \sin(n\pi x)\, dx \right]$$