① Discussion of test problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- $u(x=0) = 0$
- $u(x=1) = 0$
- $u(y=0) = 0$
- $u(y=1) = 1$

Solution

- Take $u(x,y) = X(x) \cdot Y(y)$

$$\therefore \frac{1}{X} \cdot \frac{d^2 X}{dx^2} + \frac{1}{Y} \cdot \frac{d^2 Y}{dy^2} = 0$$

$$\therefore \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = -\frac{1}{Y} \cdot \frac{d^2 Y}{dy^2} = -\alpha^2$$

- $$\frac{d^2 X}{dx^2} + \alpha^2 X = 0$$

  $X(x=0) = 0$

  $X(x=1) = 0$

$$\therefore \quad \alpha_n = n\pi \, , \quad n = 1, 2, \ldots$$

$$X_n = c_1 \cdot \sin(n\pi x)$$

- $$\frac{d^2 Y}{dy^2} - \alpha^2 \cdot Y = 0$$

$$\therefore \quad Y_n = c_2 \cdot e^{-\alpha_n \cdot y} + c_3 \cdot e^{\alpha_n \cdot y}$$

BC : $Y(y=0) = 0$

$$\therefore \quad 0 = c_2 + c_3$$

$$\therefore \quad Y_n = c_2' \cdot \sinh(\alpha_n \cdot y)$$

- $$u = \sum_{n=1}^{\infty} u_n$$

$$= \sum_{n=1}^{\infty} c_n \cdot \sin(n\pi x) \cdot \sinh(n\pi y)$$

B.C. : $u(y=1) = 1$

$$\therefore \quad 1 = \sum c_n \cdot \sin(n\pi x) \cdot \sinh(n\pi)$$

$$\therefore \quad c_n = \frac{2 \cdot (1 - \cos(n\pi))}{n \cdot \pi \cdot \sinh(n\pi)}$$

work out simplification

$$\therefore \quad u = 2 \cdot \sum_{n=1}^{\infty} \frac{(1-\cos(n\pi))}{n\pi \cdot \sinh(n\pi)} \cdot \sin(n\pi x)\sinh(n\pi y)$$

2

(see photos)

③ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x=0) = u_{01}$$

$$u(x=1) = 0$$

$$u(y=0) = u_{02}$$

$$\left.\frac{\partial u}{\partial y} + \beta \cdot u \right|_{y=1} = 0$$

- given an arbitrary B.C., convert the problem to a standard version & then solve.

TODO.

(4) $\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$u(x=0) = 0$

$u(x=1) = u_0$

$u(y=0) = 0$

$u(y=1) = 0$

$u(z=0) = 0$

$u(z=1) = 0$

∘ $u = X(x) \cdot Y(y) \cdot Z(z)$

$$\therefore \frac{1}{X} \cdot \frac{d^2X}{dx^2} + \frac{1}{Y} \cdot \frac{d^2Y}{dy^2} + \frac{1}{Z} \cdot \frac{d^2Z}{dz^2} = 0$$

$$\therefore \frac{1}{Y} \cdot \frac{d^2Y}{dy^2} = -\frac{1}{X} \cdot \frac{d^2X}{dx^2} - \frac{1}{Z} \cdot \frac{d^2Z}{dz^2} = -\alpha^2$$

∘ $$\frac{d^2Y}{dy^2} + \alpha^2 \cdot Y = 0.$$

$$Y(y=0) = 0$$

$$Y(y=1) = 0$$

$$\therefore \quad \alpha_n = n\pi \quad , \quad n = 1, 2, \cdots \infty$$

$$Y_n = c_1 \cdot \sin(n\pi) \cdot$$

- $$-\frac{1}{X} \cdot \frac{d^2X}{dx^2} - \frac{1}{Z} \cdot \frac{dZ^2}{dz^2} = -\alpha^2$$

$$\therefore \quad \frac{1}{Z} \cdot \frac{dZ^2}{dz^2} = \frac{-1}{X} \cdot \frac{d^2X}{dx^2} + \alpha^2 = -\beta^2$$

- $$\frac{dZ^2}{dz^2} + \beta^2 \cdot Z = 0$$

$$Z(z=0) = 0$$

$$Z(z=1) = 0$$

$$\therefore \quad \beta_m = m\pi \quad , \quad m = 1, 2, \cdots \infty$$

$$\therefore \quad Z_m = c_2 \cdot \sin(m \cdot \pi \cdot z)$$

- $$\frac{1}{X} \cdot \frac{d^2X}{dx^2} = \alpha_n^2 + \beta_m^2$$

$$\therefore \quad \frac{d^2 X_{mn}}{dx^2} - \gamma_{mn}^2 \cdot X = 0$$

$$= \alpha_n^2 + \beta_m^2$$

$$= \pi^2 (m^2 + n^2)$$

$$\therefore \quad X_{mn} = c_3 \cdot e^{\gamma_{mn} \cdot x} + c_4 \cdot e^{-\gamma_{mn} \cdot x}$$

$$X_{mn}(x = 0) = 0$$

$$\Rightarrow \quad 0 = c_3 + c_4$$

$$\therefore \quad X_{mn} = c_3' \cdot \sinh(\gamma_{mn} \cdot x)$$

- $$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \cdot \sinh(\gamma_{mn} \cdot x) \cdot \sin(n\pi y) \cdot \sin(m\pi z)$$

$$u(x = 1) = u_0$$

$$\therefore \quad u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \cdot \sinh(\gamma_{mn}) \cdot \sin(n\pi y) \cdot \sin(m\pi z)$$

$$\Rightarrow \iint_{y\,z} u_0 \cdot \sin(n\pi y) \cdot \sin(m\pi z)\, dy\, dz = c_{mn} \cdot \sinh(\gamma_{mn}) \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\therefore \quad c_{mn} = \frac{4 \cdot u_0 (1 - \cos(n\pi))(1 - \cos(m\pi))}{\sinh(\gamma_{mn}) \cdot m \cdot n \cdot \pi^2}$$

how does this simplify? work it out.

5 Hyperbolic PDE.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

- $u(t=0) = u_{01}$
- $\frac{\partial u}{\partial t}(t=0) = u_{02}$

- $u(x=0) = 0$
- $u(x=1) = 0$

- $u = u_1 + u_2$

- $\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_1}{\partial x^2}$

- $u_1(t=0) = u_{01}$
- $\frac{\partial u_1}{\partial t}(t=0) = 0$

- $u_1(x=0) = 0$
- $u_1(x=1) = 0$

- $$\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial^2 u_2}{\partial x^2}$$

  - $u_2(t=0) = 0$
  - $\frac{\partial u_2}{\partial t}(t=0) = u_{02}$
  - $u_2(x=0) = 0$
  - $u_2(x=1) = 0$.

- Take $u = X(x) \cdot T(t)$

$$\therefore \frac{1}{T} \cdot \frac{d^2 T}{dt^2} = \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = -\alpha^2$$

$$\therefore \frac{d^2 X}{dx^2} + \alpha^2 \cdot X = 0$$

$$\therefore X = c_1 \cdot \sin(n\pi x)$$

$$\alpha_n = n\pi, \quad n = 1, 2, 3 \ldots$$

$$\& \quad \frac{d^2 T_n}{dt^2} + \alpha_n^2 \cdot T_n = 0$$

$$\therefore \quad T_n = c_3 \cdot \sin(n\pi t) + c_4 \cdot \cos(n\pi t)$$

$$\therefore \quad u = \sum_{n=1}^{\infty} \sin(n\pi x)\left[c_3' \sin(n\pi t) + c_4' \cos(n\pi t)\right]$$

B.C: $u(t=0) = u_{01}$

$$\therefore \quad u_{01} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot c_4'$$

$$\therefore \quad \frac{2u_{01}(1-\cos(n\pi x))}{n\pi} = c_4'$$

B.C: $\dfrac{\partial u}{\partial t}(t=0) = u_{02}$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin(n\pi x)\left[n \cdot \pi \cdot c_3' \cos(n\pi t) - n \cdot \pi \cdot c_4' \cdot \sin(n\pi t)\right]$$

$$\therefore \quad u_{02} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot n \cdot \pi \cdot c_3'$$

$$\therefore \quad \frac{u_{02}(1-\cos(n\pi))}{n\pi} = \frac{n \cdot \pi \cdot c_3'}{2}$$

$$\therefore \quad c_3' = \frac{2u_{02} \cdot (1-\cos(n\pi))}{n^2\pi^2}$$

$$\therefore \quad u = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \frac{2 \cdot (1-\cos(n\pi))}{n\pi} \left[\frac{u_{02} \cdot \sin(n\pi t)}{n\pi} + u_{0i} \cos(n\pi t) \right]$$

Alternate Solution

- Take $u = u_1 + u_2$

$\underline{\underline{u_1}}$	$\underline{\underline{u_2}}$
$u_1 : \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_1}{\partial x^2}$	$u_2 : \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial^2 u_2}{\partial x^2}$
$u_1(t=0) = u_{10}$	$u_2(t=0) = 0$
$\frac{\partial u_1}{\partial t}(t=0) = 0$	$\frac{\partial u_2}{\partial t}(t=0) = u_{20}$
$u_1(x=0) = 0$	$u_2(x=0) = 0$
$u_1(x=1) = 0$	$u_2(x=1) = 0$

- Take $u_1 = X(x) \cdot T(t)$

$$\therefore \frac{1}{T} \cdot \frac{\partial^2 T}{\partial t^2} = \frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = -\alpha^2$$

- $$\frac{\partial^2 X}{\partial x^2} + \alpha^2 \cdot X = 0$$

$$\therefore \quad X_n = C_1 \cdot \sin(n\pi x)$$

$$\alpha_n = n\pi$$

$$n = 1, 2, \ldots$$

- $$\frac{\partial^2 T}{\partial t^2} + T \cdot \alpha^2 = 0$$

$$T_n = c_2 \cdot \sin(n\pi t) + c_3 \cdot \cos(n\pi t)$$

- $$u_1 = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \left(c_2' \sin(n\pi t) + c_3' \cos(n\pi t)\right)$$

$$\therefore u_{10} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot c_3'$$

$$\therefore \frac{c_3'}{2} = \frac{(1-\cos(n\pi)) \cdot u_{10}}{n\pi}$$

- $$u_2 = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \left(c_4' \sin(n\pi t) + c_5' \cdot \cos(n\pi t)\right)$$

$$\therefore 0 = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot c_5'$$

$$\therefore c_5' = 0.$$

$$\frac{\partial u_2}{\partial t} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \left(c_4' \cdot n\pi \cdot \cos(n\pi t)\right)$$

$$\therefore u_{20} = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot c_4' \cdot n\pi$$

$$\therefore \frac{u_{20}(1-\cos(n\pi))}{n\pi} = \frac{c_4' \cdot n\pi}{2}$$

$$\therefore c_4' = \frac{2u_{20}(1-\cos(n\pi))}{(n\pi)^2}$$