Ex. Solve

$$u_{xx} - u_t = 0 \xrightarrow{} (1), \quad 0<x<1, \quad t>0.$$

Boundary conditions.

subject to, $u(0,t) = 1$, $u(1,t) = 1$ $\forall t>0$.

$u(x,0) = 1 + \sin \pi x$; $0<x<1$.

$\rightarrow$ Initial condition ($t=0$).

~~Applying~~ Q. Can we apply fourier sine transform w.r.to t, because condition for $u(x,t)$ at $t=0$ is given? Ans - No.

Reason. On (1) you apply F.S.T, w.r.to t.

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \sin \omega t \, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin \omega t \, dt.$$

or, $$\frac{d^2}{dx^2} U_s(x,\omega) = \sqrt{\frac{2}{\pi}} \left[\Big(u(x,t) \sin \omega t \Big)_0^\infty - \omega \int_0^\infty u(x,t) \cos \omega t \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(0-0) - \omega \int_0^\infty u(x,t) \cos \omega t \, dt \right].$$

$$\therefore \frac{d^2}{dx^2} U_s(x,\omega) = -\omega \, U_c(x,\omega)$$

$\therefore$ if the order of the partial derivative w.r.to t (or x) is odd $\left(\text{here } 1, \frac{\partial u}{\partial t}\right)$, you can't apply fourier ~~cos~~ sine (or cosine) transform w.r.to t (or x).

Applying Laplace transform on (1) w.r.to t

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st}\,dt = \int_0^\infty \frac{\partial u(x,t)}{\partial t} e^{-st}\,dt.$$

Let $L[u(x,t);s] = \bar{u}(x,s)$.

Then, $\frac{d^2}{dx^2}\bar{u}(x,s) = s\bar{u}(x,s) - u(x,0)$

Or, $\frac{d^2}{dx^2}\bar{u}(x,s) - s\bar{u}(x,s) = -(1+\sin \pi x)$

$L[f'(t)] = \int_0^\infty f'(t) e^{-st}\,dt = s\bar{f}(s) - f(0)$

$(D^2 - s)\bar{u}(x,s) = \ldots$

auxiliary eq: $m^2 - s = 0$. $\therefore m = \pm\sqrt{s}$.

Complementar function (C.F.)

$$= c_1(s) e^{-\sqrt{s}x} + c_2(s) e^{\sqrt{s}x}.$$

Particular integral (P.I.)

$$= \left(\frac{1}{D^2 - s}\right)\left(-(1+\sin \pi x)\right)$$

$D \equiv \frac{d}{dx}$.

$$= -\frac{1}{D^2 - s}(1) - \frac{1}{D^2 - s}\sin \pi x.$$

$\frac{1}{F(D^2)}\sin xa = \frac{1}{F(-a^2)}\sin xa$

$$= -\frac{1}{-s\left(1 - \frac{D^2}{s}\right)}(1) -$$

$$= \frac{1}{s}\left(1 - \frac{D^2}{s}\right)^{-1}(1).$$

$$= \frac{1}{s}\left[1 + \frac{D^2}{s} + \frac{D^4}{s^2} + \cdots\right](1) - \frac{1}{-\pi^2 - s}\sin \pi x.$$

$$= \frac{1}{s} + \frac{1}{s + \pi^2}\sin \pi x.$$

$$\bar{u}(x,s) = \text{C.F.} + \text{P.I.} = c_1(s) e^{-\sqrt{s}x} + c_2(s) e^{\sqrt{s}x} + \frac{1}{s} + \frac{1}{s+\pi^2}\sin \pi x \rightarrow (2)$$

To get $c_1(s)$, $c_2(s)$, apply L.T. w.r.to t on, $u(0,t)=1$, $u(1,t)=1$.

This gives, $\bar{u}(0,s) = L[1] = \frac{1}{s}$ & $\bar{u}(1,s) = \frac{1}{s}$.

From (2),
$$c_1(s) + c_2(s) + \frac{1}{s} + 0 = \frac{1}{s}$$
$$c_1(s)e^{-\sqrt{s}} + c_2(s)e^{\sqrt{s}} + \frac{1}{s} + 0 = \frac{1}{s}$$

$$\therefore \left.\begin{aligned} c_1(s) + c_2(s) &= 0. \\ c_1(s)e^{-\sqrt{s}} + c_2(s)e^{\sqrt{s}} &= 0. \end{aligned}\right\} \quad AX = 0, \quad X = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ e^{-\sqrt{s}} & e^{\sqrt{s}} \end{pmatrix}$$

Now,
$$\det A = \begin{vmatrix} 1 & 1 \\ e^{-\sqrt{s}} & e^{\sqrt{s}} \end{vmatrix} = e^{\sqrt{s}} - e^{-\sqrt{s}} = 2\sinh\sqrt{s} \neq 0.$$
identically

$$\therefore c_1(s) = 0 = c_2(s).$$

$$\therefore \bar{u}(x,s) = \frac{1}{s} + \frac{1}{s+\pi^2}\sin\pi x.$$

Taking Laplace inversion,
$$u(x,t) = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+\pi^2}\sin\pi x\right).$$
$$= 1 + \sin\pi x\, e^{-\pi^2 t} \quad //.$$

Ex. Solve,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0,\ t>0 \longrightarrow (1)$$

if $u_x(0,t)=0$, $\quad u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x>1 \end{cases}$

and $u(x,t)$ is bounded when $x>0, t>0$.

Sol. Apply F.C.T. on both sides of (1),

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos\omega x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos\omega x \, dx.$$

$$\text{or, } \frac{d}{dt} U_c(\omega,t) = \sqrt{\frac{2}{\pi}} \left[\left(\frac{\partial u}{\partial x} \cos\omega x \right)_0^\infty + \omega \int_0^\infty \frac{\partial u}{\partial x} \sin\omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\underbrace{\frac{\partial u}{\partial x}(0,t)}_{=0} + \omega \left\{ \left(u(x,t) \sin\omega x \right)_0^\infty - \omega \int_0^\infty u(x,t) \cos\omega x \, dx \right\} \right].$$

$$= -\omega^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos\omega x \, dx.$$

$$\therefore \frac{d}{dt} U_c(\omega,t) = -\omega^2 U_c(\omega,t) \quad ; \quad U_c(\omega,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u \cos\omega x \, dx.$$

$$\int \frac{dU_c(\omega,t)}{U_c(\omega,t)} = -\int \omega^2 dt + C_0$$

$$\Rightarrow \ln U_c(\omega,t) = -\omega^2 t + \ln C$$

$$\therefore U_c(\omega,t) = C(\omega) e^{-\omega^2 t}$$

Put $t=0$: $\quad U_c(\omega,0) = C(\omega)$.

The initial condition is

$$u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} = f(x).$$

Taking F.C.T. on both sides,

$$U_c(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos \omega x \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin \omega x}{\omega} \Big|_0^1 - \int_0^1 \frac{\sin \omega x}{\omega} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} + \left[\frac{\cos \omega x}{\omega^2} \right]_0^1 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} + \frac{\cos \omega - 1}{\omega^2} \right].$$

$$c(\omega) = U_c(\omega, 0).$$

So, $U_c(\omega, t) = c(\omega) e^{-\omega^2 t}$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} + \frac{\cos \omega - 1}{\omega^2} \right] e^{-\omega^2 t}.$$

$$\therefore u(x,t) = \mathcal{F}_c^{-1}\left[U_c(\omega, t)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty U_c(\omega, t) \cos \omega x \, d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \left\{ \frac{\sin \omega}{\omega} + \frac{\cos \omega - 1}{\omega^2} \right\} e^{-\omega^2 t} \cos \omega x \, d\omega.$$

Solve.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0, \ t>0.$$

subject to, 1) $u(0,t) = 0$, $t>0$.

2) $u(x,0) = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1 \end{cases} = g(x)$

1. D.E. for $U_s(\omega, t)$:

$$\frac{dU_s}{dt} + \omega^2 U_s = 0.$$

2. Solutions $U_s(\omega, t) = c(\omega) e^{-\omega^2 t}$.

3. $c(\omega) = U_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin \omega x \, dx$,

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1-\cos \omega}{\omega}.$$

$$\therefore U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \cdot \frac{1-\cos \omega}{\omega} e^{-\omega^2 t}.$$

$$\therefore U(x,t) = \frac{2}{\pi} \int_0^\infty \frac{1-\cos \omega}{\omega} e^{-\omega^2 t} \sin \omega x \, d\omega.$$

<u>Exercise.</u>

1. Solve $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$; $x>0$, $t>0$.

$u(0,t) = 1$, $t>0$, $u(x,0) = e^{-x}$, $x>0$.