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Flow of a Viscous Liquid on a Rotating Disk*

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Equations describing the flow of a Newtonian liquid on a rotating disk have been solved so that characteristic curves and surface contours at successive times for any assumed initial fluid distribution may be constructed. It is shown that centrifugation of a fluid layer that is initially uniform does not disturb the uniformity as the height of the layer is reduced. It is also shown that initially irregular fluid distributions tend toward uniformity under centrifugation, and means of computing times required to produce uniform layers of given thickness at given angular velocity and fluid viscosity are demonstrated. Contour surfaces for a number of exemplary initial distributions (Gaussian, slowly falling, Gaussian plus uniform, sinusoidal) have been constructed. Edge effects on rotating planes with rising rims, and fluid flow on rotating nonplanar surfaces, are considered.

IT has been found possible in a number of applications to form uniform, solid films by applying the desired film material, in a fluid condition, to a rapidly spinning disk. A uniform film of fluid seems to form quickly under these conditions, and its corresponding solid film may be retained by evaporation of volatile constituents (or by cooling, if a molten material is involved). Preparation of thin, uniform films of paint and varnish[1] and of asphalt,[2] employing spinning disk techniques, have been reported. In one process employed in the production of color television screens, a solid film containing phosphor particles and photoresist polymer is prepared directly on the tube faceplate by applying a quantity of slurry at the center, distributing it by a combination of slow spinning and progressive inclination, removing excess slurry by rapid centrifugation, and, finally, drying while centrifugation continues. Three separate applications are required, since "dots" containing three kinds of phosphor are formed photographically from these solid films. Uniformity in the solid film can be achieved only if the rapid spinning operation can succeed in evening out the extreme thickness variations in the slurry covering introduced in the initial distribution stages. Uniformity in these films, in turn, is of great consequence, since residual thickness variations can result in nonuniformities in light intensity and chromaticity in the finished screen.

Because of the above practical considerations, it is of interest to analyze the problem of viscous flow on a rotating plane. At the outset, for simplicity, we shall assume that

1. the rotating plane is infinite in extent,
2. the plane is horizontal, so that there is no radial gravitational component,
3. the liquid layer is radially symmetric, and so thin that differences in gravitational potential normal to the surface of the disk have negligible effect in distributing the liquid[3] compared with the effect of centrifugal forces,
4. the viscosity is independent of the rate of shear, i.e., the liquid is Newtonian,
5. the liquid layer is everywhere so thin that shear resistance is appreciable only in horizontal planes,
6. the radial velocity is everywhere so small that Coriolis forces may be neglected.

We take cylindrical polar coordinates (r,θ,z) rotating with the spinning disk at angular velocity ω. The z dependence of the radial velocity v of the liquid at any point (r,θ,z) can be found by equating the viscous and centrifugal forces per unit volume;

$$-\eta\frac{\partial^2 v}{\partial z^2}=\rho\omega^2 r, \tag{1}$$

where η is the viscosity and ρ the density of the liquid.

Equation (1) may be integrated employing the boundary conditions that $v=0$ at the surface of the disk $(z=0)$ and $\partial v/\partial z=0$ at the free surface of the liquid $(z=h)$ where the shearing force must vanish. Thus

$$v=\frac{1}{\eta}\left(-\tfrac{1}{2}\rho\omega^2 rz^2+\rho\omega^2 rhz\right). \tag{2}$$

The radial flow q per unit length of circumference is

$$q=\int_0^h v\,dz=\frac{\rho\omega^2 rh^3}{3\eta}. \tag{3}$$

To obtain a differential equation for h we apply the equation of continuity,

$$r\frac{\partial h}{\partial t}=-\frac{\partial(rq)}{\partial r}. \tag{4}$$

* Work performed under contract to Radio Corporation of America, Tube Division, Lancaster, Pennsylvania.

[1] P. H. Walker and J. G. Thompson, Proc. Am. Soc. Testing Materials, **22**, Part II, 464 (1922).

[2] L. R. Kleinschmidt, ASTM Bull. **No. 193**, 53 (October, 1953).

[3] The interesting case of radial growth of liquid pools applied to stationary, horizontal plates has been studied by E. B. Bielak and E. W. J. Mardles, J. Colloid Sci. **9**, 233 (1954). See also J. R. Philip, Australian J. Phys. **9**, 570 (1956).

Thus, from Eq. (3)

$$\frac{\partial h}{\partial t}=-K\frac{1}{r}\frac{\partial}{\partial r}(r^2h^3), \tag{5}$$

where

$$K=\rho\omega^2/3\eta.$$

1. CASE OF INITIALLY UNIFORM DISTRIBUTION

Before seeking a general solution to Eq. (5), it will be instructive to consider the striking fact that this equation has a special solution which depends only on t. In this case

$$\frac{dh}{dt}=-2Kh^3, \tag{6}$$

whence

$$h=\frac{h_0}{(1+4Kh_0^2t)^{\frac{1}{2}}}, \tag{7}$$

where the constant h_0, *independent of* r, corresponds to the initial height of a fluid layer. Since the solution of Eq. (5) is uniquely determined when h is given at time $t=0$, it follows that Eq. (7) is the solution corresponding to an initially uniform distribution, $h=h_0$. Thus, if the initial distribution of fluid is everywhere uniform, it will remain so with time, as the thickness of fluid film is decreased by continuing application of centrifugal force. This conclusion immediately tells us that ultimate unformity in thin films is assured if an initial thick fluid distribution, before centrifugation, can somehow be made uniform. It does not tell us whether uniformity can be expected in the more practical case of an initial distribution that is irregular, however.

Equation (7) shows that the fluid layer decreases in thickness by a factor $1/\sqrt{2}$ in a time

$$\tau=1/4Kh_0^2, \tag{8}$$

which shows that a thick layer thins out much more rapidly than a thin one. This suggests, in turn, that a nonuniform layer should become increasingly more uniform as centrifugation continues. This conclusion is borne out by the general solution of Eq. (5), which we shall now derive.

2. GENERAL SOLUTION

The general solution can be obtained by considering instead of the first order partial differential Eq. (5) an equivalent set of two simultaneous first order ordinary differential equations. To do this we write Eq. (5) in the form

$$-2Kh^3=\frac{\partial h}{\partial t}+3Krh^2\frac{\partial h}{\partial r}. \tag{9}$$

Now imagine that the successive surface contours defined by Eq. (9) are given by the instantaneous positions of a set of points which move along a family of *characteristic curves*. Then the height h of any one of these moving points varies according to the total derivative expression

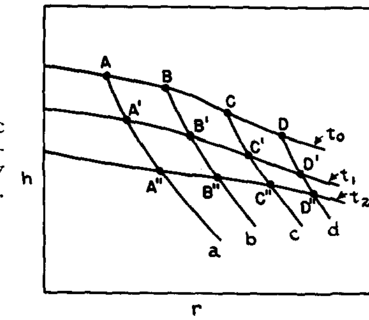

FIG. 1. Characteristic curves and surface contours for an arbitrary initial fluid distribution.

$$\frac{dh}{dt}=\frac{\partial h}{\partial t}+\frac{\partial h}{\partial r}\frac{dr}{dt}. \tag{10}$$

The differential equations for the characteristic curves can now be obtained by writing down the conditions for mutual consistency of Eqs. (9) and (10), namely

$$dh/dt=-2Kh^3, \tag{11}$$

and

$$dr/dt=3Krh^2. \tag{12}$$

The solution of the simultaneous Eqs. (11) and (12) is illustrated in Fig. 1. This figure schematically represents the fate of an arbitrary initial distribution at time $t=t_0$. The set of points A, B, C, D travel along characteristic curves a, b, c, d, and are found at positions A', etc., at time $t=t_1$, A'', etc., at time $t=t_2$. The loci of points A', etc., and A'', etc., represent the new surface contours at times t_1 and t_2. It is to be noted that the characteristic curves a, b, c, d are not the flow lines of particles on the surface of the liquid, but are simply a mathematical artifice for calculating the positions of successive surface contours.

Equation (11) integrates immediately to give

$$h=\frac{h_0}{(1+4Kh_0^2t)^{\frac{1}{2}}}. \tag{13}$$

On substituting this expression for h into Eq. (12) we obtain

$$\frac{dr}{dt}=\frac{3Kh_0^2r}{1+4Kh_0^2t}, \tag{14}$$

which has the integral

$$r=r_0(1+4Kh_0^2t)^{\frac{3}{4}}. \tag{15}$$

Equations (13) and (15) give the coordinates (r,h) after time t of a point on the surface in terms of its original coordinates (r_0,h_0). Thus, from any initial surface contour we can construct the new contour after any given time of centrifugation.

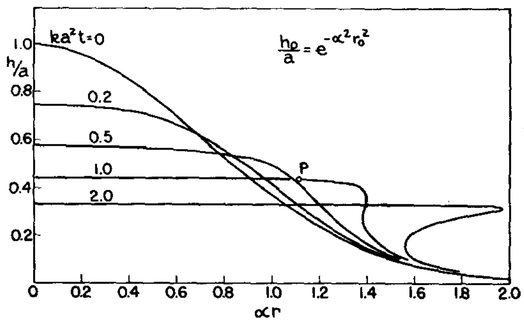

FIG. 2. Successive surface contours for an initially Gaussian fluid distribution.

3. GAUSSIAN INITIAL CONTOUR

Figure 2 shows several successive surface contours calculated from Eqs. (13) and (15) for the case of a Gaussian-shaped initial distribution of the form

$$h_0 = a \exp - \alpha^2 r_0^2. \tag{16}$$

A single set of curves can be made to include all possible choices of a and α by plotting the graphs in terms of the dimensionless quantities h/a, αr, and ka^2t, as has been done in this figure.

The expected flattening of the surface with increasing t is clearly shown. For $ka^2t=1$ the liquid is remarkably uniform in thickness. Also to be noted, for large values of r, is the formation of a steep wave front and the eventual folding over of the top of the wave. Although effects of this kind undoubtedly occur with the spinning disk, and also in the case of a viscous liquid draining down a vertical wall,† the differential Eq. (5) certainly does not represent them accurately since it was derived on the basis of approximately horizontal flow. Wave fronts as steep as that shown at $ka^2t=2.0$ are physically improbable because of obvious mechanical considerations. In fact, we cannot regard (5) or (9) as valid in the region of a vertical wave front. The condition for a vertical wave front is that $dr=0$, in Eq. (15), for some fixed value of t. Now

$$\frac{dr}{dr_0} = \frac{1+2Kh_0t[2h_0+3r_0(dh_0/dr_0)]}{(1+4Kh_0^2t)^{\frac{1}{2}}}. \tag{17}$$

This expression will become zero, for some value of t, if at any point on the initial contour,

$$\frac{dh_0}{dr_0} < -\frac{2h_0}{3r_0}. \tag{18}$$

† The problem of liquid draining down a vertical wall can also be solved in terms of the characteristic curves. In this case Eqs. (13) and (15) are replaced by

$$h = h_0$$
$$x = x_0 + g\rho h_0^2 t/\eta,$$

where x is a coordinate measured vertically downwards. Steep wavefronts will always occur if the initial contour has negative slope at any point. No smoothing action occurs in this case (recalling our assumption of Newtonian flow), because the characteristic curves all proceed parallel to the vertical wall.

This means that any curve that falls off more rapidly than the inverse two-thirds power of r_0 will develop vertical fronts. For the contour represented by Eq. (16), condition (18) is satisfied when $\alpha r_0 > 1/\sqrt{3}$.

4. SLOWLY-FALLING INITIAL CONTOUR

Figure 3 shows the case of an initial contour

$$h_0 = \frac{a}{(1+\alpha^2 r_0^2)^{\frac{1}{4}}}. \tag{19}$$

The surface flattens out smoothly without any sign of vertical fronts. This is in accord with condition (18) since the initial curve (19) falls off less rapidly than the inverse two-thirds power of r_0.

5. GAUSSIAN PLUS UNIFORM INITIAL CONTOUR

Figure 4 shows the behavior of a contour of the initial form

$$h_0/a = 1 + \exp - \alpha^2 r_0^2. \tag{20}$$

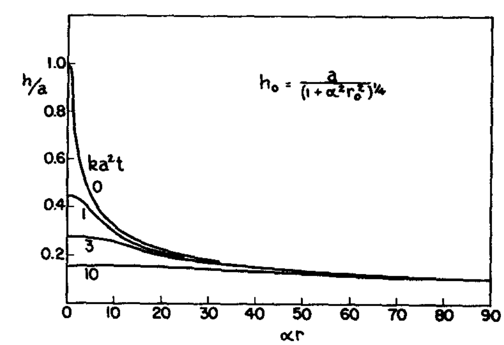

FIG. 3. Successive surface contours for a slowly falling initial surface.

In effect, this contour consists of a layer of fluid which is everywhere uniform, plus a second, superposed distribution which is Gaussian about the center of rotation. As shown, nonuniformity is reduced very effectively, and without establishment of wave fronts.

6. SINUSOIDAL INITIAL CONTOUR

Figure 5 shows the successive contours for the case of an initial surface of the form

$$h_0 = a(1+0.15 \cos \alpha r_0), \tag{21}$$

representing a level surface with a 15% ripple. Again we observe the smoothing effect produced by the flow and also the formation of vertical wave fronts. Condition (18) is satisfied for certain ranges of values of r_0.

7. LIMITS OF VALIDITY OF THE FLOW EQUATIONS

(a) Effects of Coriolis Force

The Coriolis acceleration perpendicular to the radius is given by

$$a_{\mathrm{cor}} = 2\omega v, \tag{22}$$

and the centrifugal acceleration by

$$a_{\text{cent}}=\omega^2 r. \tag{23}$$

Therefore, the condition for neglecting Coriolis acceleration (our simplifying assumption 6) is

$$2\omega v/\omega^2 r \ll 1, \tag{24}$$

or

$$v \ll \omega r/2. \tag{25}$$

Now from Eq. (2) we find the maximum (surface layer) value for v at a given radius r to be

$$v=\rho\omega^2 r h^2/2\eta. \tag{26}$$

Thus the condition becomes

$$\eta \gg \rho\omega h^2. \tag{27}$$

In practice this means that the theory is restricted to relatively thin layers of fluid of high viscosity for any fixed value of angular velocity. For example, the condition for validity for a fluid layer of density 1.0 g/cm^3, of thickness 1 mm, and rotating at 4π rad/sec, is that the fluid viscosity be much greater than 12 centipoise.

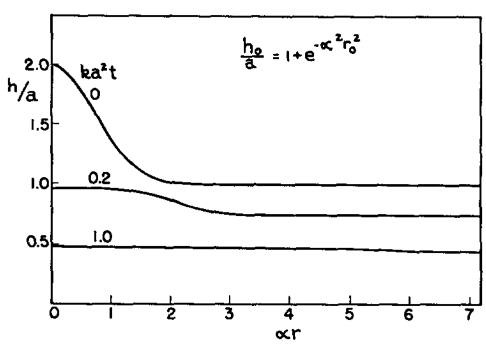

FIG. 4. Successive surface contours for initial Gaussian superposed upon uniform contour.

(b) Influence of Gravitational Force

The radial gravitational force per unit volume, when the rotational axis is vertical, is $-g\rho\partial h/\partial r$. In order that the effects of this force may be neglected, the necessary condition is

$$-g\rho(\partial h/\partial r) \ll \rho\omega^2 r. \tag{28}$$

This condition can be expressed in terms of the radius of curvature R of the liquid surface at the center of the tube face;

$$R=\left|\frac{1}{\partial^2 h/\partial r^2}\right| \gg \frac{g}{\omega^2}. \tag{29}$$

For example, at a rotational speed of 10 rad/sec ($\sim$100 rpm), the condition is that $R \gg 10$ cm, which is easily met in practical cases.

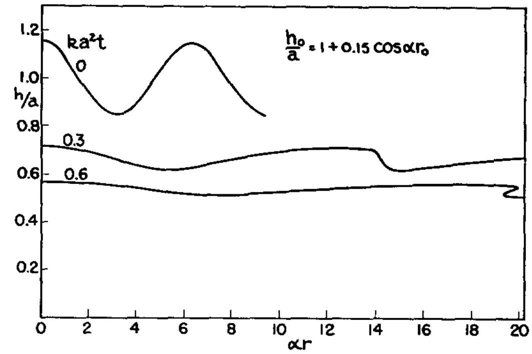

FIG. 5. Successive surface contours for initial surface containing a ripple.

(c) Flow on Finite Planes and on Curved Surfaces

Our simplifying assumption (1), that the rotating plane is infinite in extent imposes a restriction that is, of course, difficult to meet. For a finite disk the considerations of our preceding paragraphs are unaltered, with the possible exception of disruptive action of surface forces on the fluid film at its edges. Formation of a thin film by rotation is obviously most readily accomplished on a disk which permits excess fluid to be thrown off horizontally.[4] Where the rim rises above the plane of the disk, as is the case in the instance of color television phosphor screening, the situation is somewhat different. For relatively low values of η and high values of ω, liquid will be thrown upward at the rim as it is transported to it under centrifugation. If adjustment of these parameters is such that the liquid remains at the rim, there will be thickening of the film at its outer edge to a distance depending upon the volume of fluid in excess and the angular velocity of the disk. The latter dependence arises because the height to which fluid will rise in the rim is determined by the equilibrium which becomes established between gravitational and centrifugal force components along the slanted boundary of contained fluid. The extent of such edge effects can obviously be minimized by application of high angular velocity.

For a rotating disk that has a rising rim, our assumption (2), that the plane is horizontal, may also inject practical difficulty. In the color television screening process previously described, for example, a considerable quantity of excess fluid must be removed from the rim before the permanent film may be dried. This is accomplished by carrying out the final rapid centrifugation step with the axis of rotation inclined to an angle in excess of 90° with respect to the vertical. This does not require significant alteration of our flow equations as long as ω is large, so that centrifugal force per unit volume ($\rho\omega^2 r$) greatly exceeds gravitational force per unit volume (ρg, for 90° inclination of axis).

[4] The fate of fluid *after* leaving the rim of a rotating disk has been studied by Dixon, Russell, and Swallow, Brit. J. Appl. Phys. 3, 115 (1952).

It is of interest to consider the extent to which curvature of the rotating surface may affect the ultimate theoretical uniformity of fluid films which may form on them. The color television faceplates we have referred to, for example, are not planar, but spherical, with radius of curvature of the order of 65 cm. On a spherical surface, the expression for force per unit volume becomes

$$F=\rho\omega^2 r\cos\phi-\rho g\sin\phi, \tag{30}$$

where ϕ is the angle formed between the direction along the surface and the horizontal line, at any given point. Rigorous solution for the case of flow on a curved surface would require the introduction of the appropriate nonlinear terms into our Eqs. (1), (2), *et seq.* However, we can again see, by inspection of Eq. (30), that gravitational effects can be made inconsequential by appropriate adjustment of ω. If this is done, the principal source of nonuniformity in this case is related to the angle ϕ itself. If the maximum value of ϕ were about 20°, for example, the ratio of force per unit volume at the edge to that for a flat disk of similar radius would be cos 20°, or 0.94. This might suggest that maximum ultimate thickness variation would be about 6%, although the problem is actually much more complicated than this.

Experiments have been performed in these laboratories, employing spherical glass television faceplates having an angle ϕ of roughly 20°, in which phosphor suspension was added at the center of the previously wetted and slowly rotating faceplate, distributed by steadily increasing angular velocity, then thinned by rapid centrifugation. Excess suspension was removed by tilting the rotating faceplate, and the remanent film was then dried. Film thickness variations as low as 4% were observed by a light transmittance method of measurement.

8. DISCUSSION

The curves presented in Figs. 2–5 are perfectly general, and may be applied to any specific set of parameters of interest within the limitations discussed above. In addition, any other initial distribution of interest may be employed in our characteristic Eqs. (13) and (15) to obtain corresponding families of contour curves.

To illustrate the use of our reported theoretical results, let us consider a fluid of density 1 g—cm^{-3} and viscosity 1 poise, on a horizontal plane in an initially Gaussian distribution (Fig. 2). Let us further prescribe that a, the initial thickness at the center, has the value 0.1 cm, and that α, the reciprocal of the radius (r) at a position such that initial thickness is ae^{-1}, has the value 0.1 cm^{-1}. Using values of h/a and αr calculated for the curve for which $ka^2t=1.0$, we find that for this particular initial Gaussian distribution $h=0.0447$ cm at $r=0$, and $h=0.0431$ cm at $r=1.112/\alpha=11.12$ cm. The thickness variation between the center and point P is thus $\Delta h=0.0016$ cm, or about 3.6% of the value at the center. To find the time required to achieve this degree of uniformity, we must evaluate $k=\rho\omega^2/3\eta$. For $\omega=1$ rad/sec ($\sim$10 rpm), k has the value $\frac{1}{3}$ for a liquid having the values of ρ and η we have arbitrarily selected; if $\omega=4\pi$ rad/sec (120 rpm), $k=52.6$. For the corresponding times, then, we find 300 sec (at 1 rad/sec), and 1.9 sec (at 4π rad/sec).

Our last result indicates that removal of irregularities can be achieved very quickly, and that centrifugation could be effective in preparing uniform depositions of materials of very high viscosity, e.g., molten glass, whose flow properties are known to be Newtonian. It is interesting to compute the time required to obtain a very thin fluid film from a relatively thick layer of a highly viscous liquid. Let $\eta=100$ poise, $\rho=1.0$ g-cm^{-3}, and $a=1$ cm for an initially Gaussian distribution, and let us compute the time required to reduce this distribution to a film 100 μ (10^{-2} cm) in thickness. From Eq. (13),

$$1+4Kh_0^2t=(h_0/h)^2=10^4, \tag{31}$$

whence $t=10^4/4k$. Since in this example $k=0.528$ for a rotational speed of 4π rad/sec, we find t to be approximately 8 minutes.

Finally, we should note that the descriptions we have presented in this paper are essentially mathematical ones, and may differ from actual physical flow situations in a variety of respects, as can be seen particularly by reviewing the set of assumptions we have made. In particular, our assumption that the fluid exhibits Newtonian flow behavior will be readily violated in practical situations involving suspensions or some highly viscous fluids. Although the phosphor suspensions employed in the color television process we have alluded to exhibit very nearly ideal flow behavior when their water content is relatively high, it is at least likely that non-Newtonian characteristics exert a strong influence on the consequences of centrifugation when h has become very small, because of such phenomena as particle settling and moisture evaporation.