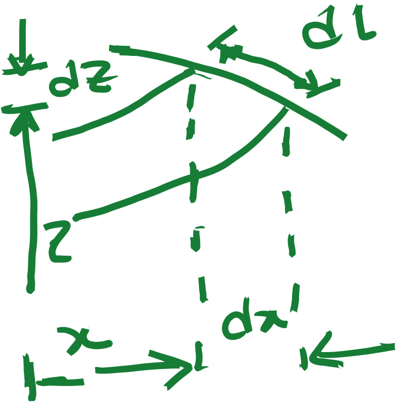
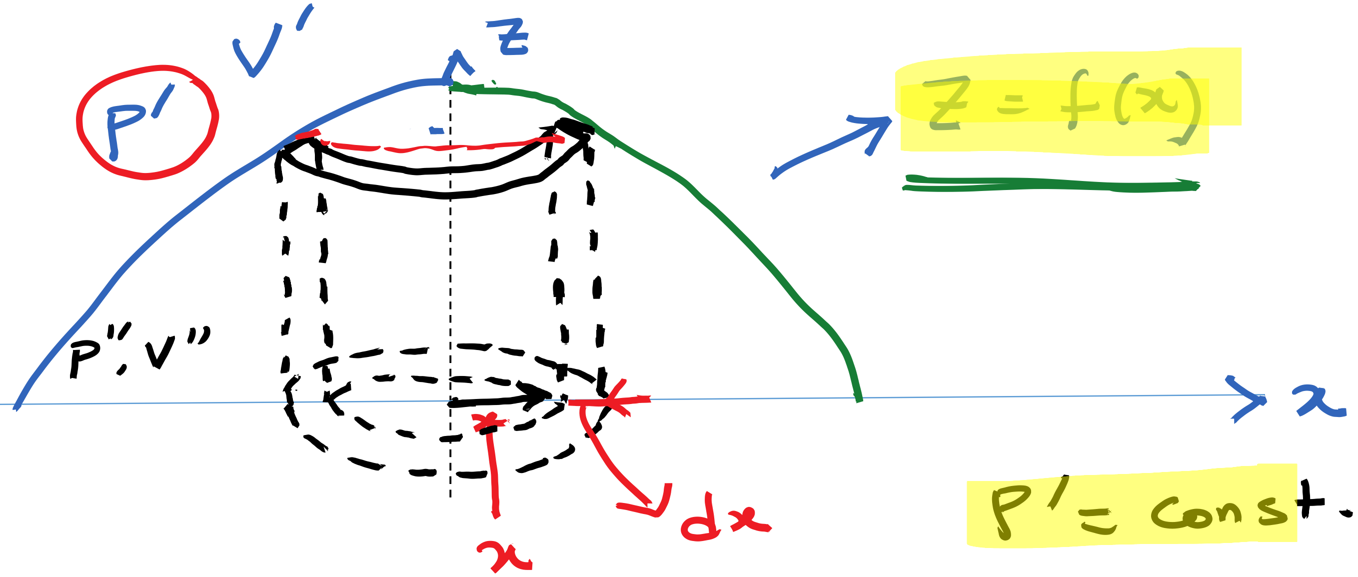
Generalized Expression of Young Laplace Equation for an axi-symmetric surface

17.03.2022
Lecture : 27





$Z = f(x)$

$V'' = \text{constant}$

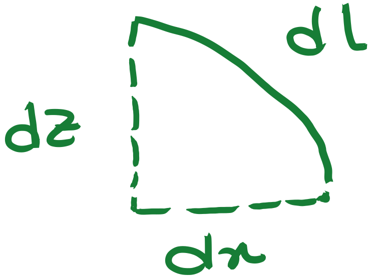
$P'' \to Pr$
$V'' \to Vol$] Inside

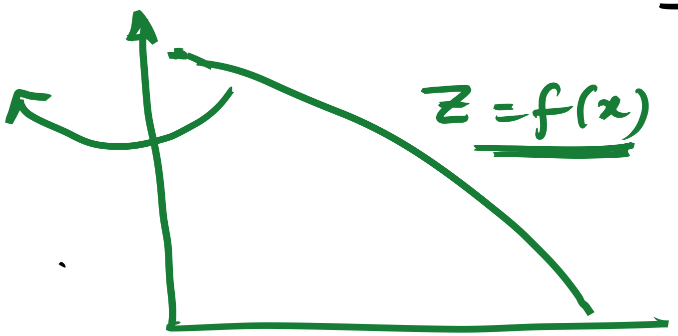
P'
V'] → out side

$V = V' + V''$

$P' = \text{const.}$

$V = \text{const}$
$T = \text{const}$
$V'' = \text{const}$





Helmholtz Free Energy:

$$F = U - TS + \sum_j \mu_j A_j = U - TS + \underline{\mu} A$$

μ = Here Surfece Tension

$\underline{\mu}$ → Surfece Tension

$$H = U + PV$$
$$G = H - TS$$
$$\Rightarrow \quad H = G + TS$$
$$U + PV = G + TS$$
$$\Rightarrow \quad U = G + TS - PV$$

$$\Rightarrow \quad F = G + \cancel{T}S - PV - T\cancel{S} + \mu A$$
$$= G - PV + \mu A$$

$$G = \sum \mu_i x_i = \underline{\text{Const}}$$

$\uparrow$ Chemical Potential

Since for the System Composition does not change, So $\underline{G = \text{Const}}$

$$F = G - PV + \gamma A \qquad V = V' + V''$$

$$= G - p'V' - p''V'' + \gamma A.$$

$$F = G - p''V'' - p'(V - V'') + \gamma A.$$

$$= \gamma A - p''V'' + p'V'' - \underbrace{p'V}_{c} + \underbrace{G}_{c}$$

Since
$p' = \text{const}$
$V = \text{const}$

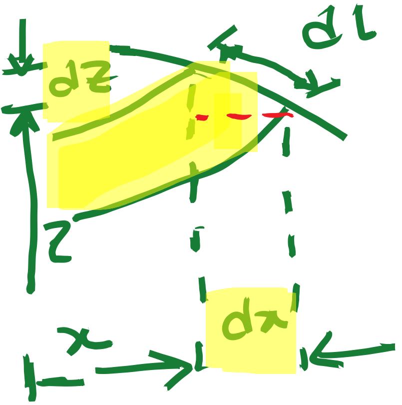
$$F = \gamma \underline{\underline{A}} + \underline{\underline{V''}} \underbrace{(p' - p'')}_{\Delta P} + C$$

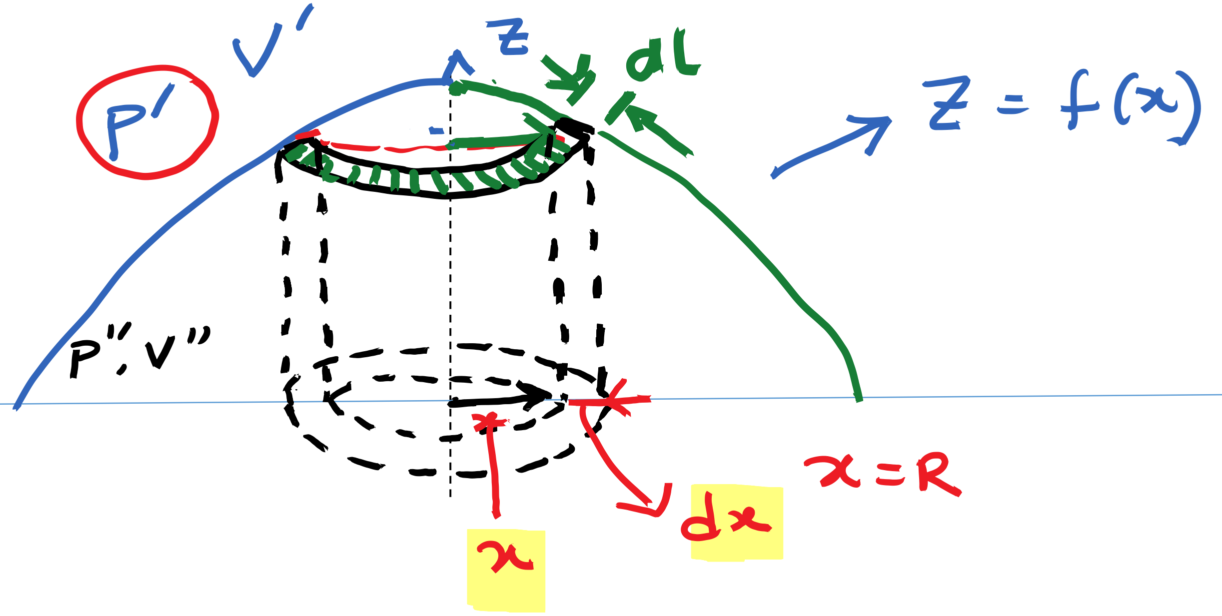
$\Delta P \rightarrow$ Across the Interface

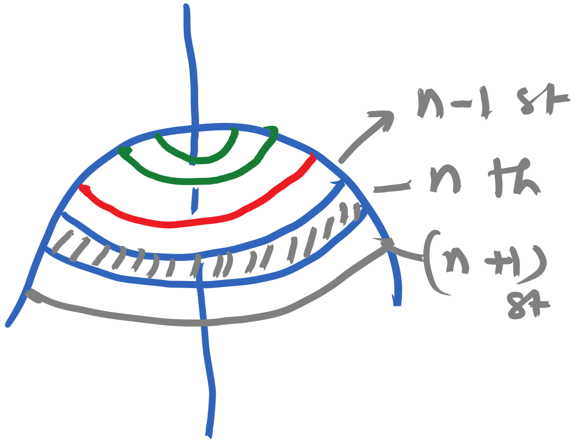
$$F = \gamma A + V''(\Delta P) + C$$

We now plug in Expression of A and V here x varies from $x = 0$ to $x = R$

$$Z_x = \frac{dZ}{dx}$$

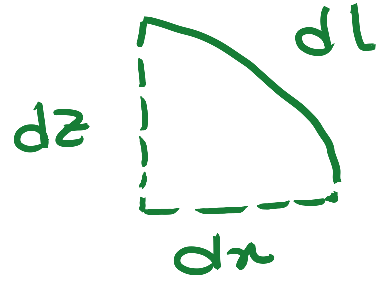






Σ Add the area of all the ring elements

→ We get total area:-



$$dA = 2\pi x \, dl$$

$$dl^2 = dZ^2 + dx^2$$

$$dl^2 = dx^2\left[1 + \left(\frac{dZ}{dx}\right)^2\right]$$

$$dl = dx\left[1 + Z_x^2\right]^{1/2}$$

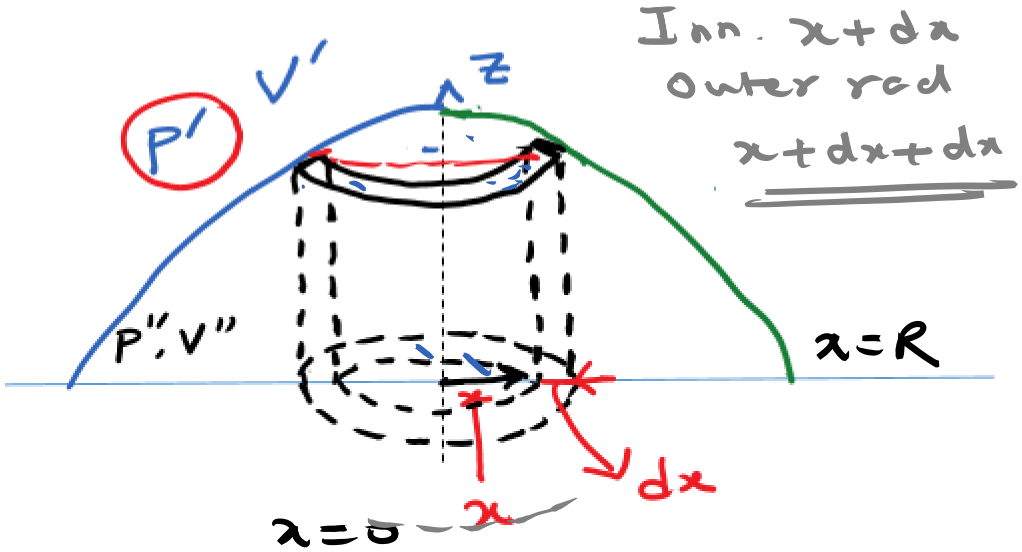
dl → Is curved / but smaller / so assumed be st. line

$$dA = 2\pi x \cdot (1+Z_x^2)^{1/2} dx$$

$$A = \int_0^R 2\pi x (1+Z_x^2)^{1/2} dx$$ ✓

$$dV'' = 2\pi x\, dx \cdot Z$$

$$V'' = \int_0^R 2\pi x Z\, dx$$ ✓



> Volume element. is like a cup.

Each volume element is a cup like str.

① $F = \gamma A + V''(\Delta P) + C$

Plugging in Expression of A and V.

$$F = \gamma \int_0^R 2\pi x \left(1 + Z_x^2\right)^{1/2} dx + (\Delta P) \int_0^R 2\pi x Z \cdot dx + C$$

② We would like to evaluate F as a function of shape, so that we can find out the equilibrium shape → which corresponds to minimize F_0 ⇒ FIND out the shape for which F is minimum.

$Z = f(x)$

Necessary Cond.

$\frac{dZ}{dx} = 0$,

Sufficiency Cond

$\frac{d^2Z}{dx^2} = +ve$

The general form of the eqn. which needs to be minimized

$$F = \int_0^R f\left(x, Z, \frac{dZ}{dx}\right) dx$$

$$F = \int_0^R f(x, z, z_x)\,dx$$

Problem related to integration of a differential.

Functional * NEW

Calculus of Variations

Necessary condition for Optimization of a Functional is NEW

$$\boxed{\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z_x}\right) = 0}$$

$$F = \gamma \int_0^R 2\pi x\,(1+z_x^2)^{1/2}\,dx + (\Delta P)\int_0^R 2\pi x z \cdot dx + C$$

Same expression without the integral

For our eqn.: $\boxed{f = \gamma \cdot 2\pi x\,(1+z_x^2)^{1/2} + \Delta P \cdot 2\pi x z} + C$

$$f = \gamma \cdot 2\pi x \left(1 + Z_x^2\right)^{1/2} + \Delta P \cdot 2\pi x \cdot Z$$