8.1 Half Range Series

Suppose that $f(x)$ is a function defined on $(0, l]$. Suppose we want to express $f(x)$ in the cosine or sine series. This can be done by extending $f(x)$ to be an even or an odd function on $[-l, l]$. Note that there exists an infinite number of ways to express the function in the interval $[-l, 0]$. Among all possible extension of f there are two, even and odd extensions, that lead to simple and useful series:

a) If we want to express $f(x)$ in cosine series then we extend $f(x)$ as an even function in the interval $[-l, l]$.

b) On the other hand, if we want to express $f(x)$ in sine series then we extend $f(x)$ as an odd function in $[-l, l]$.

We summarize the above discussion in the following proposition

8.1.1 Proposition

Let f be a piecewise continuous function defined on $[0, l]$. The series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \textit{with} \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

is called half range cosine series of f. Similarly, the series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \textit{with} \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

is called half range sine series of f.

Remark: *Note that we can develop a Fourier series of a function f defined in $[0, l]$ and it will, in general, contain all sine and cosine terms. This series, if converges, will represent a l-periodic function. The idea of half range Fourier series is entirely different where we extend the function f as per our desire to have sine or cosine series. The half range series of the function f will represent a $2l$-periodic function.*

8.2 Example Problems

8.2.1 Problem 1

Obtain the half range sine series for e^x in $0 < x < 1$.

Solution: Since we are developing sine series of f we need to compute b_n as

$$\begin{aligned}
b_n =& \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = 2\int_0^1 e^x \sin n\pi x = 2\left[e^x \sin n\pi x\big|_0^1 - n\pi\int_0^1 e^x\cos n\pi x\, dx\right]\mathrm{d}x \\
=& 2\left[-n\pi\{e^x\cos n\pi x\big|_0^1 + n\pi\int_0^1 e^x \sin n\pi x\,\mathrm{d}x\}\right] = -2n\pi(e(-1)^n - 1) - n^2\pi^2 b_n
\end{aligned}$$

Taking second term on the right side to the left side and after simplification we get

$$b_n = \frac{2n\pi\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}$$

Therefore, the sine series of f is given as

$$e^x = 2\pi\sum_{n=1}^{\infty}\frac{n\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}\sin n\pi x \quad \text{for} \quad 0 < x < 1$$

8.2.2 Problem 2

Let $f(x) = \sin\frac{\pi x}{l}$ on $(0, l)$. Find Fourier cosine series in the range $0 < x < l$.

Solution: Sine we want to find cosine series of the function f we compute the coefficients a_n as

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\int_0^l\left[\sin\frac{(n+1)\pi x}{l} + \sin\frac{(1-n)\pi x}{l}\right]\mathrm{d}x$$

For $n \neq 1$ we can can compute the integrals to get

$$a_n = \frac{1}{l}\left[-\frac{\cos\frac{(n+1)\pi x}{l}}{\frac{(n+1)\pi}{l}} + \frac{\cos\frac{(1-n)\pi x}{l}}{\frac{(n-1)\pi}{l}}\right]_0^l = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right]$$

It can be further simplified as

$$a_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{4}{\pi(n+1)(n-1)}, & \text{when } n \text{ is even}\end{cases}$$

The coefficient a_1 needs to calculated separately as

$$a_1 = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\left[\cos\frac{2\pi x}{l}\,\frac{l}{2\pi}\right]_0^l = \frac{1}{2\pi}(1-1) = 0$$

The Fourier cosine series of f is given as

$$\sin\frac{\pi x}{l} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \dots\right]$$

8.2.3 Problem 3

Expand $f(x) = x,\ \ 0 < x < 2$ in a (i) sine series and (ii) cosine series.

Solution: (i) To get sine series we calculate b_n as

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

Integrating by parts we obtain

$$b_n = \left[x\cos\frac{n\pi x}{2}\left(-\frac{2}{n\pi}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,\mathrm{d}x = -\frac{4}{n\pi}\cos n\pi.$$

Then for $0 < x < 2$ we have the Fourier sine series

$$x = -\frac{4}{\pi}\sum\nolimits_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin\frac{n\pi x}{2} = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \dots\right).$$

(ii) Now we express $f(x) = x$ in cosine series. We need to calculate a_n for $n \neq 0$ as

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x = \left[x\sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\right]_0^2 - \int_0^2 \sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\,\mathrm{d}x$$

After simplifications we obtain

$$a_n = \frac{2}{n\pi}\left(\frac{2}{n\pi}\right)\left[\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left(\cos n\pi - 1\right) = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

The coefficient a_0 is given as

$$a_0 = \int_0^2 x\,\mathrm{d}x = 2$$

Then the Fourier sine series of $f(x) = x$ for $0 < x < 2$ is given as

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right).$$

It is interesting to note that the given function $f(x) = x, \ \ 0 < x < 2$ is represented by two entirely different series. One contains only sine terms while the other contains only cosine terms.

Note that we have used series equal to the given function because the series converges for each $x \in (0, 2)$ to the function value. It should also be pointed out that one can deduce sum of several series by putting different values of $x \in (0, 2)$ in the above sine and cosine series.