Review
of
Feedback Control System

Transfer Function for a PID Controller

Ideal PID : $G_c(s) = \frac{c(s)}{e(s)} = K_c\left[1 + \frac{1}{\tau_I s} + \tau_D s\right]$

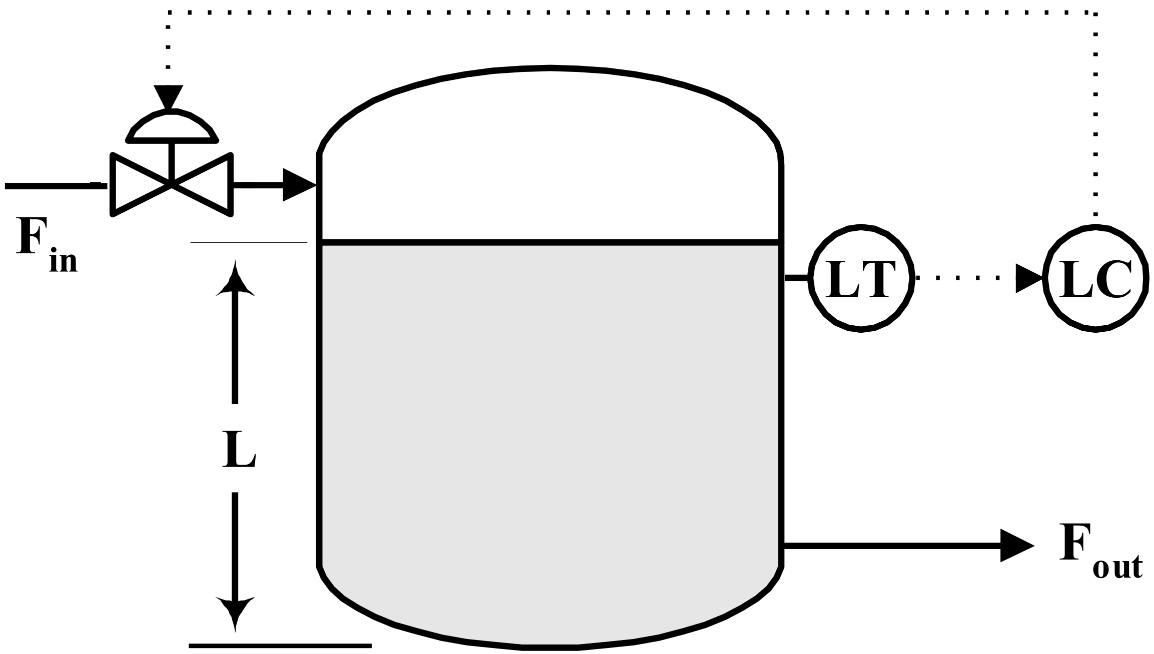
Real PID : $G_c(s) = \frac{c(s)}{e(s)} = K_c\left(1 + \frac{1}{\tau_I s}\right)\left(\frac{\tau_D s}{\beta\tau_D s+1}\right)$

Derivative Kick:
To avoid sudden jump of output due to setpoint change, sensor output is used in place of error term, i.e,

$$c(t) = c_0 + K_c\left[e(t) + \frac{1}{\tau_I}\int_0^t e(\zeta)d\zeta - \tau_D\frac{dy_s(t)}{dt}\right]$$

Direct / Reverse acting controller

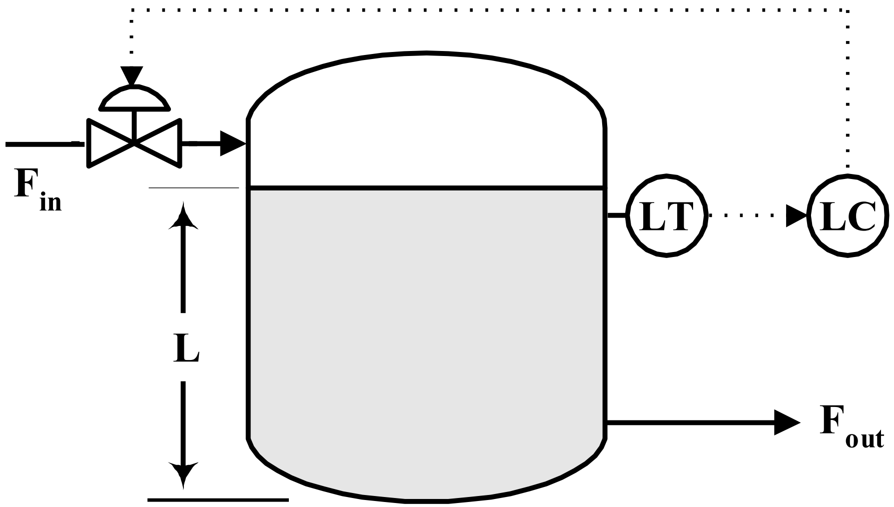
- What will be the sign of the controller gain?



Guidelines for Selecting Direct and Reverse Acting PID's

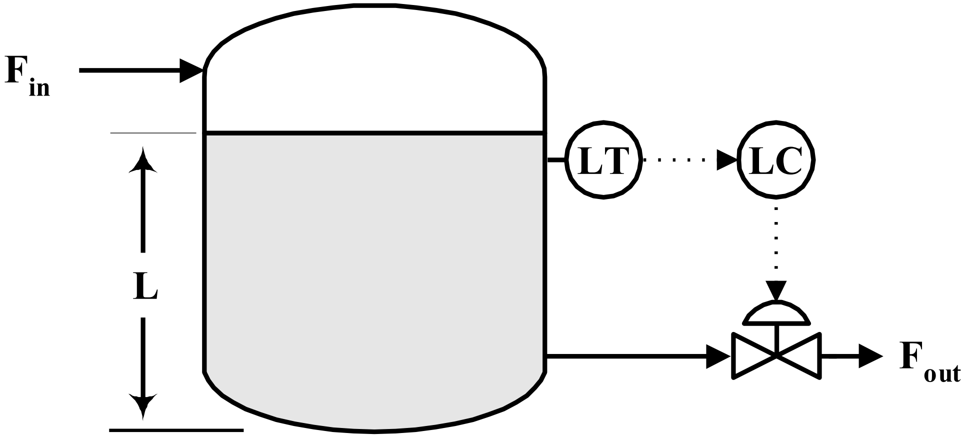
- Consider a direct acting final control element to be positive and reverse to be negative.
- If the sign of the product of the final control element and the process gain is positive, use the reverse acting PID algorithm.
- If the sign of the product is negative, use the direct acting PID algorithm
- If control signal goes to a control valve with a valve positioner, the actuator is considered direct acting.

Level Control Example



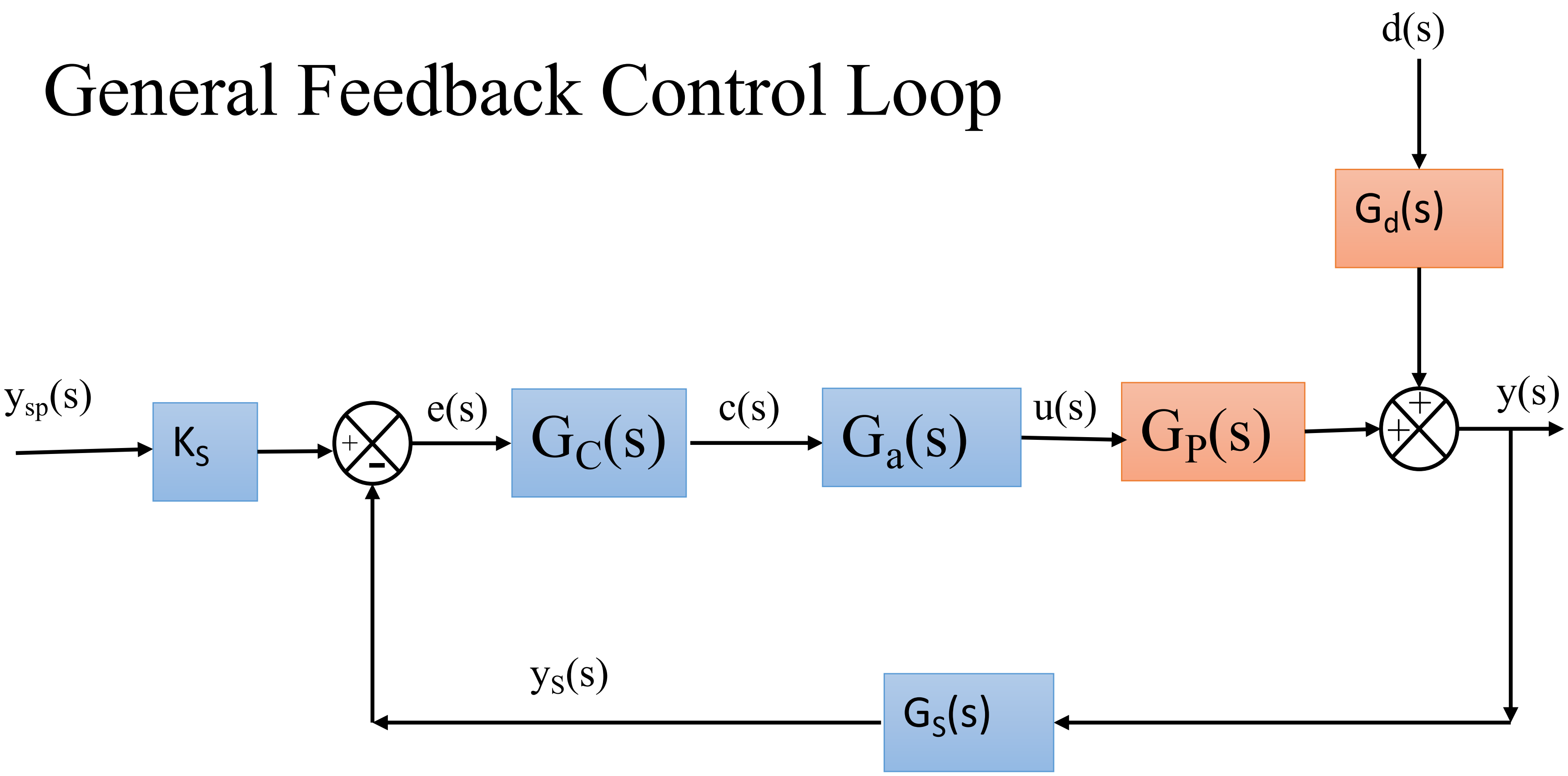
- Process gain is positive because when flow in is increased, the level increases.
- If the final control element is direct acting, use reverse acting PID.
- For reverse acting final control element, use direct acting PID.

Level Control Example



- Process gain is negative because when flow out is increased, the level decreases.
- If the final control element is direct acting, use direct acting PID.
- For reverse acting final control element, use reverse acting PID.

General Feedback Control Loop



$$\frac{y(s)}{y_{sp}(s)} = \frac{K_S G_p(s)\ G_a(s)\ G_c(s)}{G_p(s)\ G_a(s)\ G_c(s)\ G_s(s) + 1} \cong \frac{G_c(s)\,G(s)}{G_c(s)\ G(s) + 1}$$

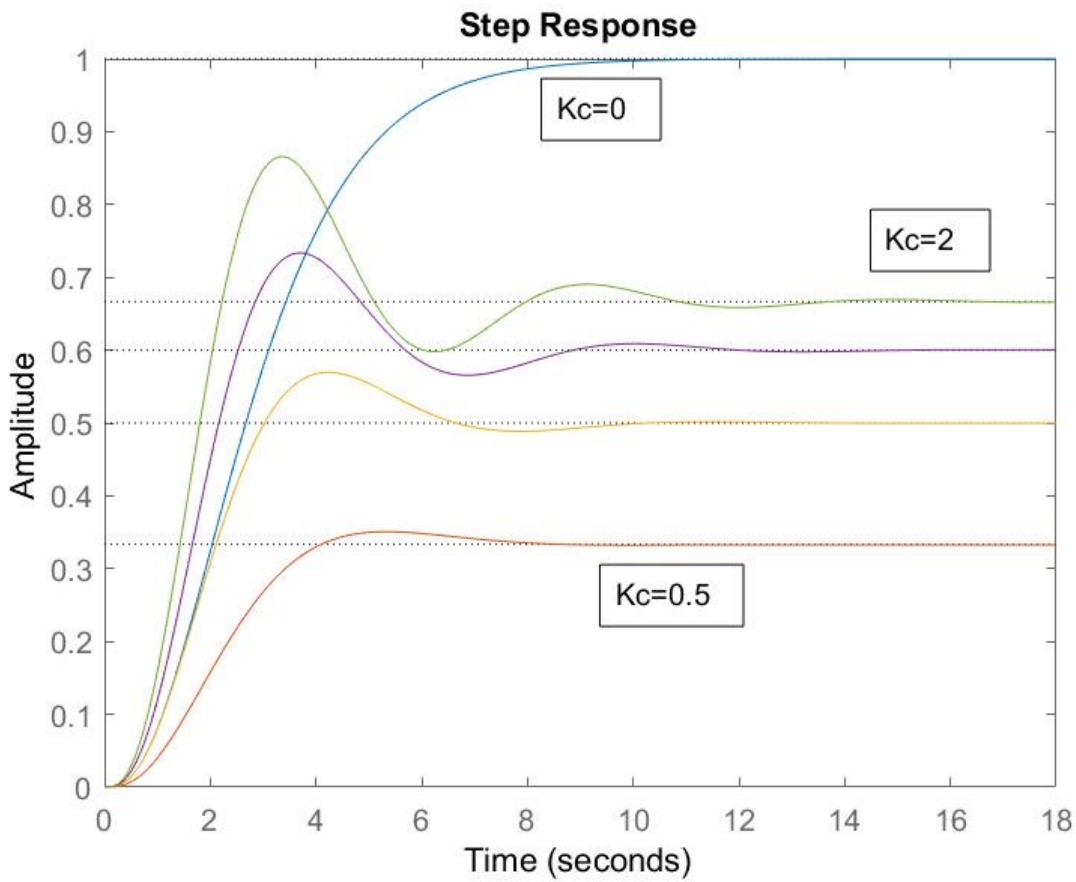
Characteristic Polynomial : $1 + G_{OL} = 0$ or $1 + G_c(s)G(s) = 0$

Controller actions on feedback dynamics

Process G(s) : $\frac{1}{(s+1)^3}$ Controller : Proportional, K_c;

Matlab:

```
s=tf('s'); g=1/(s+1)^3;
step(g); hold on
for kc=[0.5:0.5:2],
gcl=feedback(kc*g,1);
step(gcl);
end
```



Proportional Control

Important points:

- proportional feedback does not change the order of the system
  - started with a first order process
  - closed-loop process also first order
  - order of characteristic polynomial is invariant under proportional feedback

- speed of response of closed-loop process is directly affected by controller gain
  - increasing controller gain reduces the closed-loop time constant

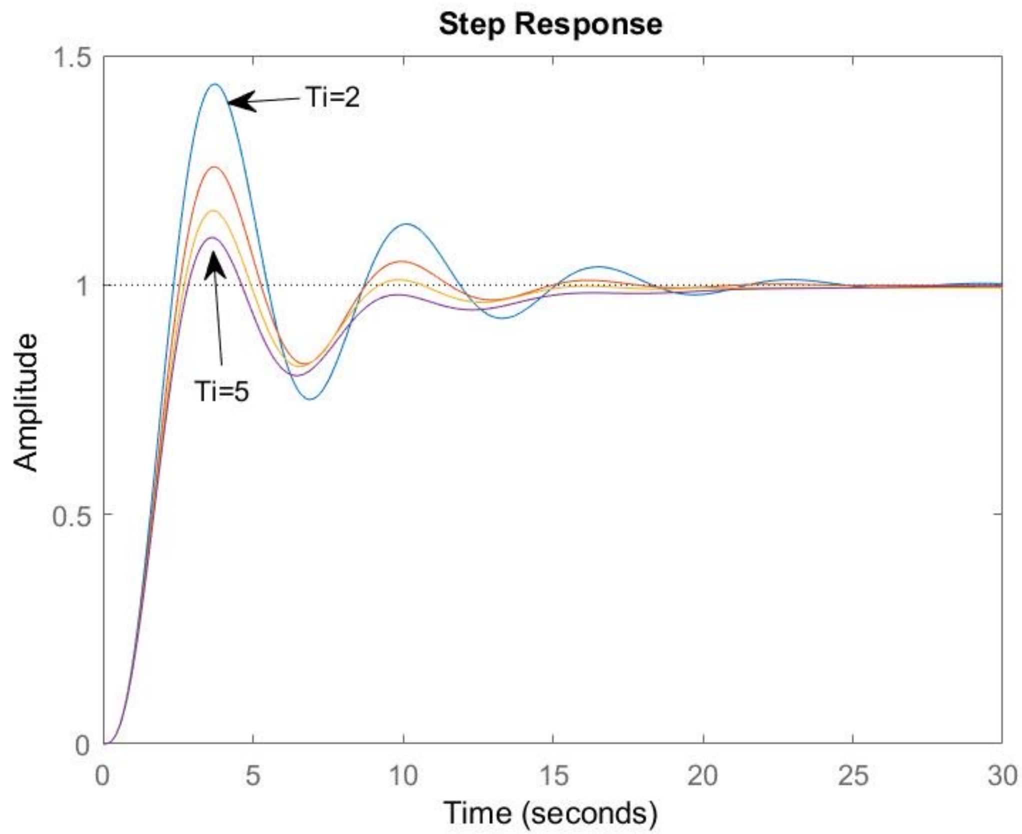
- In general, proportional feedback
  - reduces (does not eliminate) offset
  - speeds up response
  - for oscillatory processes, makes closed-loop process more oscillatory

Controller actions on feedback dynamics

Process G(s) : $\frac{1}{(s+1)^3}$ Controller : PI (I varying) $K_c(1+\frac{1}{\tau s})$;

Matlab:

```
figure; hold on; kc=2;
for Ti=[2:1:5],
gc=tf(kc*[1,1/Ti],[1,0]);
gcl=feedback(gc*g,1);
step(gcl);
end
```



Proporional - Integral Control

Important points:

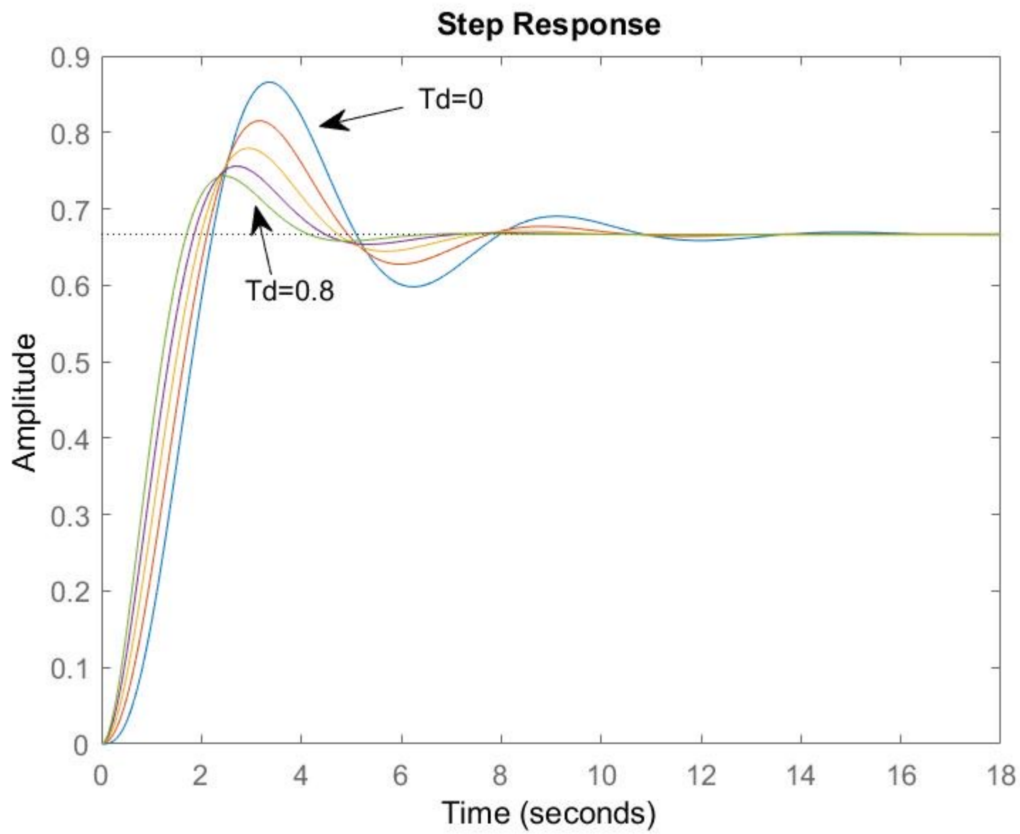
- integral action increases order of the system in closed-loop
- integral action eliminates offset
- integral action
  - should be small compared to proportional action
  - tuned to slowly eliminate offset
  - can increase or cause oscillation
  - can be de-stabilizing
- PI controller has two tuning parameters that can independently affect
  - speed of response
  - Nature of response (oscillation)
- PI is the most widely used controller in industry
  - optimal structure for first order processes

Controller actions on feedback dynamics

Process G(s) : $\frac{1}{(s+1)^3}$ Controller : PD, $K_c(1+\tau_D s)$;

Matlab:

```
figure; hold on; kc=2;
for Td=[0:0.2:0.8],
gc=tf(kc*[Td,1],[1]);
gcl=feedback(gc*g,1);
step(gcl); end
```

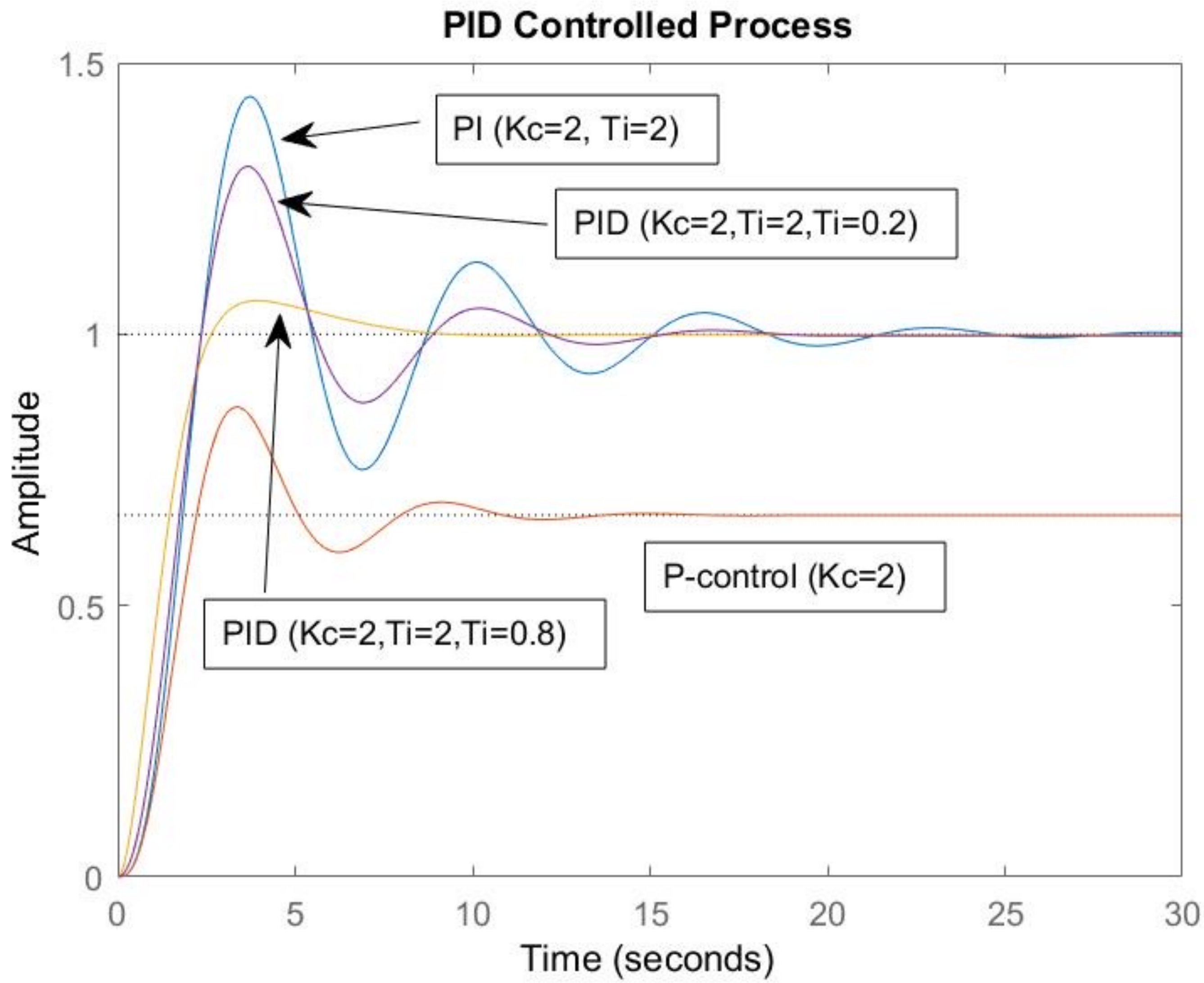


Proportional Derivative

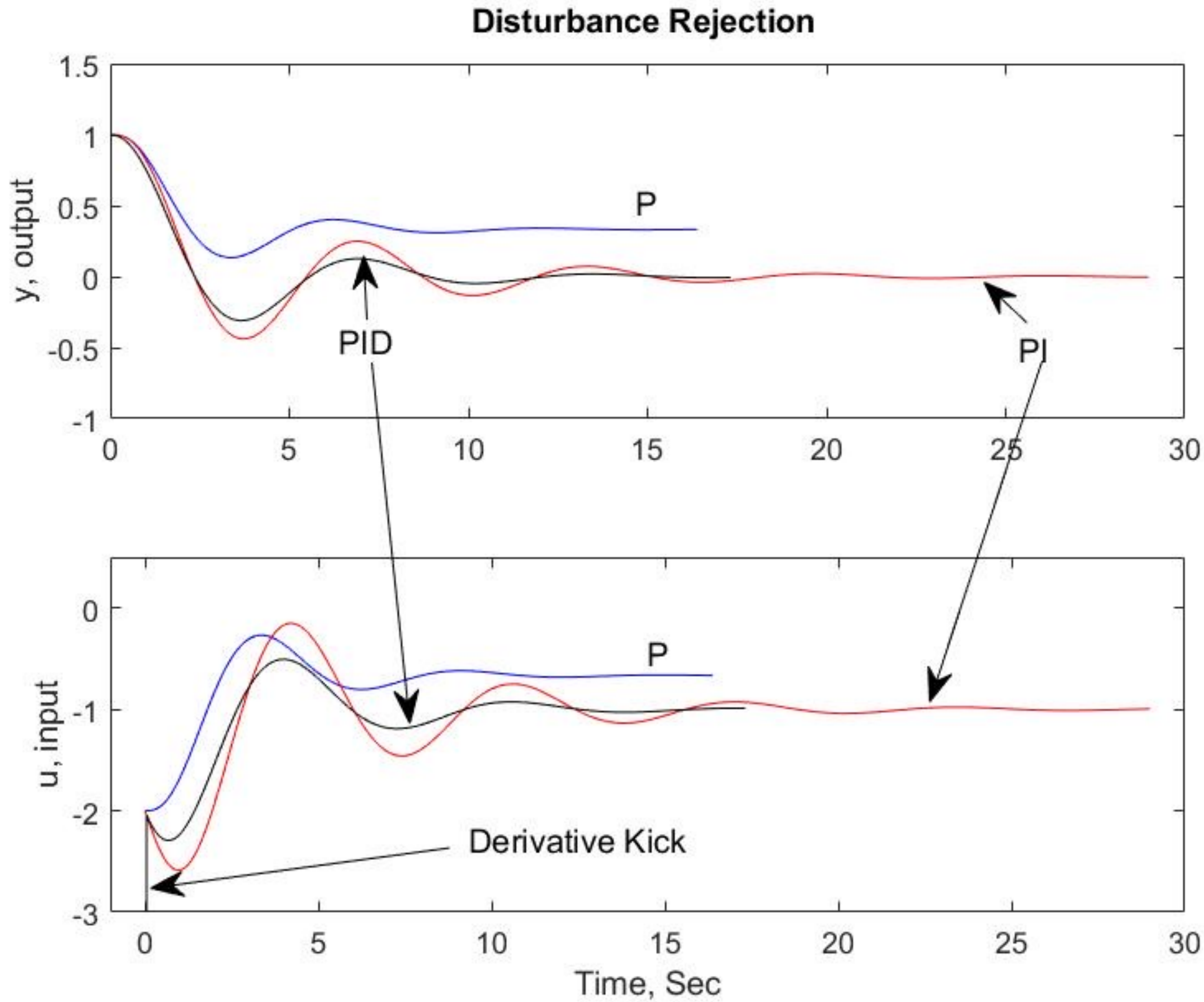
Important Points:

- derivative action does not increase the order of the system
- Used to compensate for trends in output
  - measure of speed of error signal change
  - provides predictive or anticipatory action
- adding derivative action affects the period of oscillation of the process
  - good for disturbance rejection
  - poor for tracking
- derivative action
  - should be small compared to integral action
  - has a stabilizing influence
  - difficult to use for noisy signals
  - usually modified in practical implementation

PID control (Setpoint Tracking or Servo Control)



Disturbance Rejection: Input dynamics



Closed-loop Stability

Every control problem involves a consideration of closed-loop stability

General concepts:

BIBO Stability:

" An (unconstrained) linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise it is unstable."

Comments:

- Stability is much easier to prove than instability
- This is just one type of stability

Closed-loop Stability

General Stability criterion:

"A closed-loop feedback control system is stable if and only if all roots of the characteristic polynomial ($1+G_{OL}=0$) are negative or have negative real parts. Otherwise, the system is unstable."

- Unstable region is the right half plane of the complex plane.
- Valid for any *linear systems*.
- Underlying system is *almost always nonlinear* so stability holds only locally. Moving away from the point of *linearization may cause instability*

Stability Analysis Methods

Problem reduces to finding roots of a polynomial

Traditional:

1. Routh array:
   - Test for positivity of roots of a polynomial
2. Direct substitution
   - Complex axis separates stable and unstable regions
   - Find controller gain that yields purely complex roots
3. Root locus diagram
   - Vary location of poles as controller gain is varied
   - Of limited use

- Bode stability criteria
- Niquist Stability criteria

Closed-loop stability

Routh array for a polynomial equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

is

$$\begin{array}{c|cccc}
1 & a_n & a_{n-2} & a_{n-4} & \cdots \\
2 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
3 & b_1 & b_2 & b_3 & \cdots \\
4 & c_1 & c_2 & \cdots & \\
\vdots & \vdots & & & \\
n+1 & z_1 & & &
\end{array}$$

where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_{n-3}a_n}{a_{n-1}}, b_2 = \frac{a_{n-1}a_{n-4} - a_{n-5}a_n}{a_{n-1}}, \ldots$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}, c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}, \ldots$$

Elements of left column must be positive to have roots with negative real parts

Example: Routh Array

Characteristic polynomial

$$2.36s^5 + 1.49s^4 - 0.58s^3 + 1.21s^2 + 0.42s + 0.78 = 0$$

Polynomial Coefficients

$$a_5 = 2.36, a_4 = 1.49, a_3 = -0.58, a_2 = 1.21, a_1 = 0.42, a_0 = 0.78$$

Routh Array

$$\begin{array}{c|ccc}
1 & a_5(2.36) & a_3(-0.58) & a_1(0.42) \\
2 & a_4(1.49) & a_2(1.21) & a_0(0.78) \\
3 & b_1(-2.50) & b_2(-0.82) & b_3(0) \\
4 & c_1(0.72) & c_2(0.78) & \\
5 & d_1(1.89) & d_2(0) & \\
6 & e_1(0.78) & &
\end{array}$$

- Closed-loop system is unstable

Direct Substitution

- Technique to find gain value that de-stabilizes the system.
- Observation:

  Process becomes unstable when poles appear on right half plane

  ⇒ Find value of K_c that yields purely complex poles

- Strategy:
  - Start with characteristic polynomial

$$1 + K_c G_a(s) G_p(s) G_s(s) = 1 + K_c \frac{r(s)}{q(s)}$$

  - Write characteristic equation:

$$q(s) + K_c r(s) = 0$$

  - Substitute for complex pole ($s = j\omega$)

$$q(j\omega) + K_c r(j\omega) = 0$$

  - Solve for K_c and ω

Example: Direct Substitution

Characteristic equation

$$1 + K_c \frac{s+1}{s^3 + 0.5s^2 - 0.5s - 0.75} = 0$$

$$s^3 + 0.5s^2 - 0.5s - 0.75 + K_c s + K_c = 0$$

$$s^3 + 0.5s^2 + (K_c - 0.5)s + (K_c - 0.75) = 0$$

Substitution for $s=j\omega$

$$(j\omega)^3 + 0.5(j\omega)^2 + (K_c - 0.5)\,j\omega + (K_c - 0.75) = 0$$

$$-j\omega^3 - 0.5\omega^2 + (K_c - 0.5)\,j\omega + (K_c - 0.75) = 0$$

Real Part: $-0.5\omega^2 + K_c - 0.75 = 0$ *Complex Part:* $(K_c - 0.5)\omega - \omega^3 = 0$

$$\therefore K_c = 0.5\omega^2 + 0.75 \Rightarrow (0.5\omega^2 + 0.75 - 0.5)\omega - \omega^3 = 0$$

$$\Rightarrow -0.5\omega^2 + 0.25 = 0$$

$$\Rightarrow \omega = \pm\sqrt{2}/2, \quad K_c = 1$$

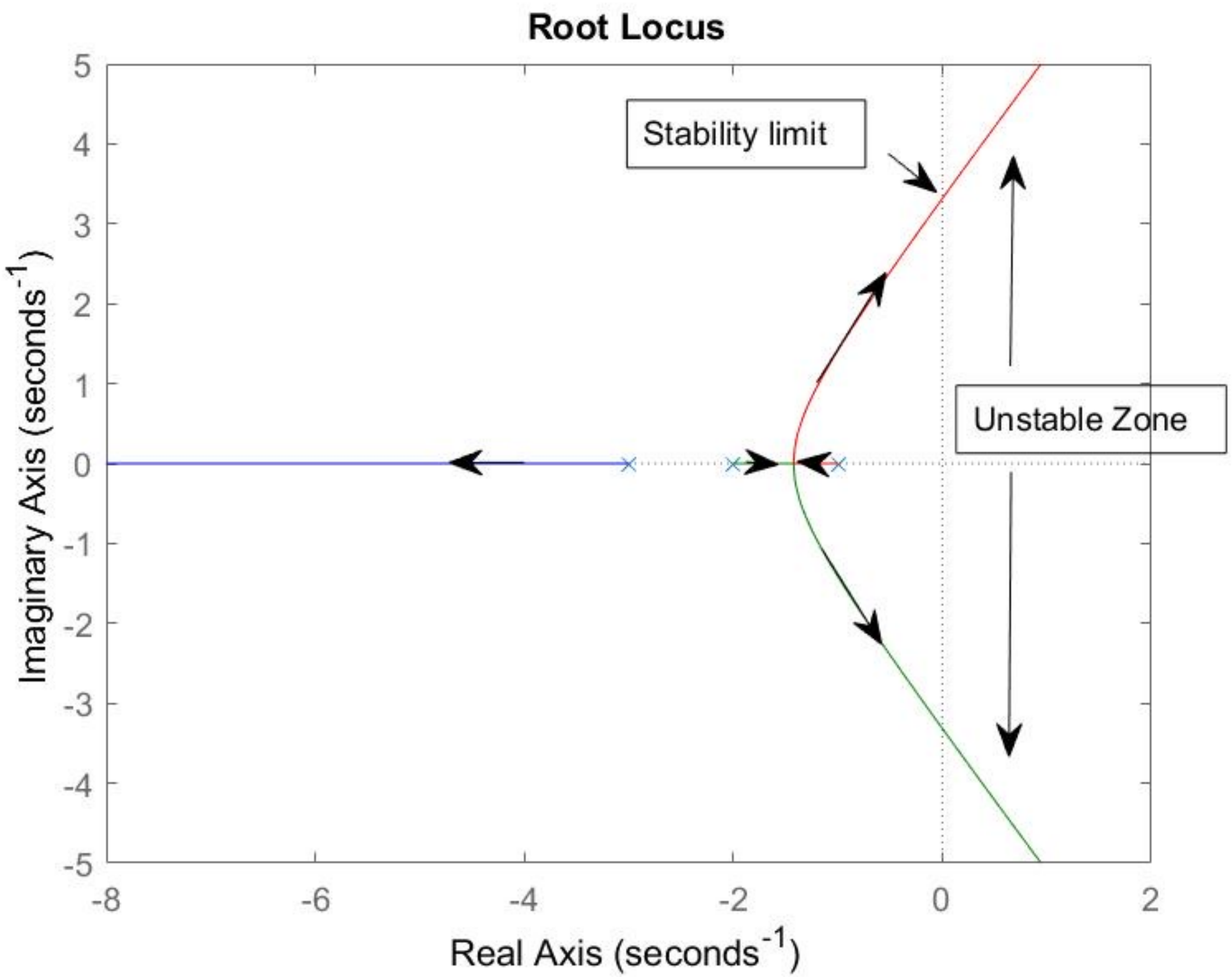
- *System is unstable if* $K_c > 1$

Root Locus Diagram

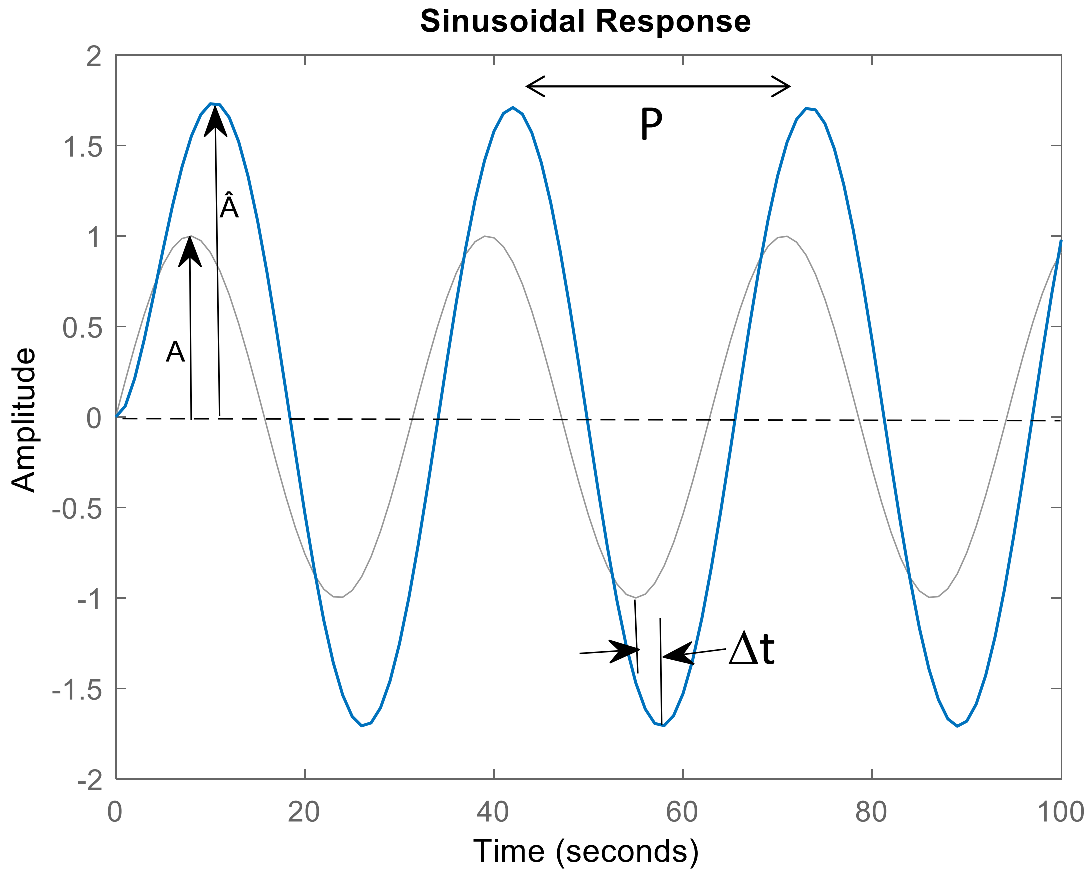
- Old method that consists in plotting roots of characteristic polynomial (closed loop poles) as controller gain is changed.

Matlab

```
s=tf('s');
G1=1/(s+1);
G2=1/(s+2);
G3=1/(s+3);
G=G1*G2*G3;
rlocus(G);
rlocfind(G);
```



Frequency Response



Amplitude Ratio (AR) = Â/A
Phase lag (ϕ) = -Δt/P*180

The process acts as a "filter" or "amplifier" that attenuates or amplifies the input at frequency ω.

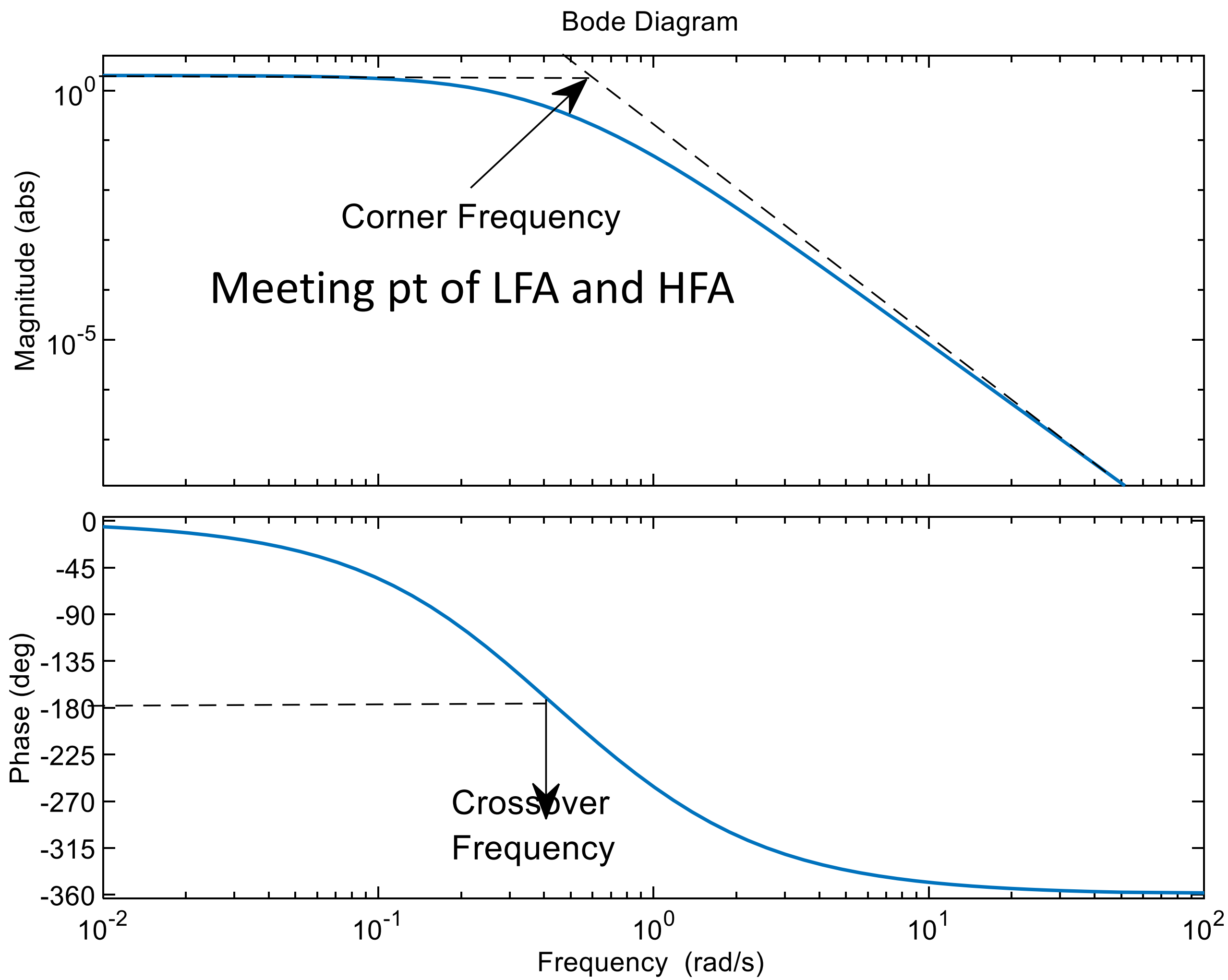
The frequency response of a process G(s) at a specific frequency ω is a complex number and can be obtained by substituting $s = j\omega$

Frequency Response

- $G(j\omega) = Re(\omega) + j\,Im(\omega) = \rho(\omega)e^{j\phi(\omega)}$
- $\rho(\omega) = AR = \sqrt{Re^2(\omega) + Im^2(\omega)} = |G(j\omega)|$
- $\phi = phase\ difference = arc\tan\left(\frac{Im(\omega)}{Re(\omega)}\right) = \angle G(j\omega)$

- Two ways of representing the frequency response
  - Bode Plot
    - Log-Log plot of AR vs ω and semi-log plot ϕ vs ω
  - Nyquist Plot
    - Plot of Re(ω) vs Im(ω) for various values of ω.
    - Frequency does not appear explicitly in this plot.

Bode Plot

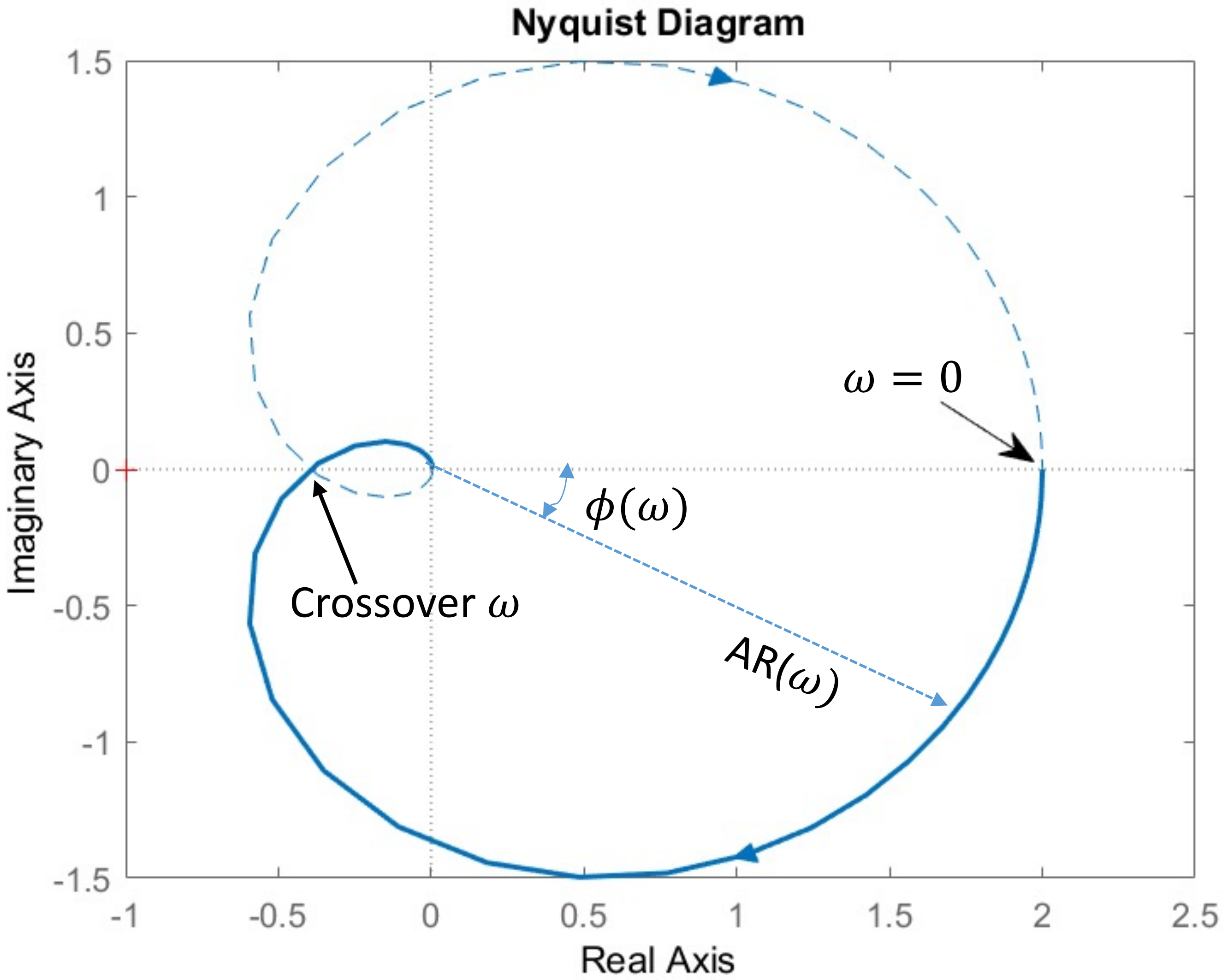


Steady state gain:

$$K = \rho(0)e^{j\phi(0)}$$

Bandwidth, ω_b, defines the highest frequency at which the process displays no attenuation. It is calculated based on the frequency when amplitude reaches 0.707 K.

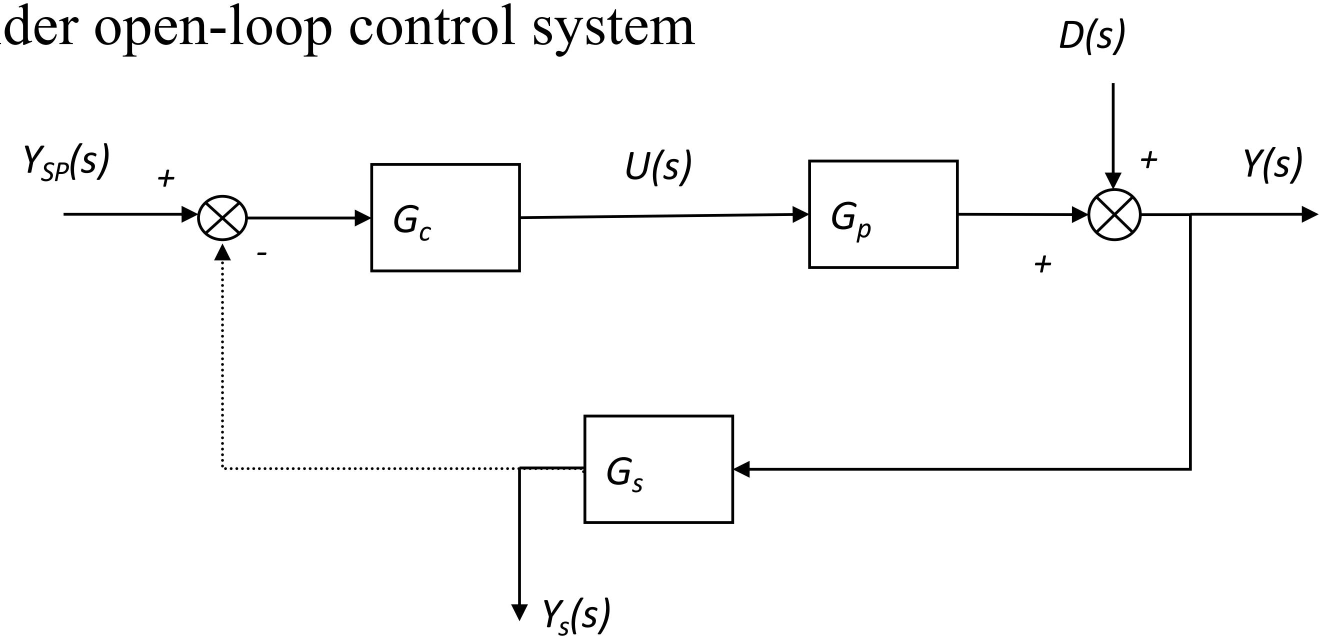
Roll-off is slope of decaying AR at high frequencies.



Nyquist Diagram
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
$\omega = 0$
$\phi(\omega)$
Crossover ω
$AR(\omega)$

Bode Stability Criterion

Consider open-loop control system



Open-loop Response to $Y_{SP}(s)$

1. Introduce sinusoidal input in setpoint ($D(s)$=0) and observe sinusoidal output
2. Fix gain such $AR=1$ and input frequency such that ϕ=-180
3. At same time, connect close the loop and set $Y_{SP}(s)=0$

Q: What happens if AR <1, 1 and >1?

Bode Stability Criterion

A closed-loop system is unstable if the frequency of the response of the open-loop G_{OL} has an amplitude ratio greater than one at the critical frequency. Otherwise it is stable.

Strategy:

1. Solve for ω in $\arg(G_{OL}(j\omega)) = -\pi$

2. Calculate AR $AR = G_{OL}(j\omega)$

Bode Stability Criterion

To check for stability:

1. Compute open-loop transfer function
2. Solve for ω in ϕ=-π
3. Evaluate AR at ω
4. If AR>1 then process is unstable

Find ultimate gain:

1. Compute open-loop transfer function without controller gain
2. Solve for ω in ϕ=-π
3. Evaluate AR at ω
4. Let $K_{cu} = \frac{1}{AR}$

Bode Criterion

Consider the transfer function and controller

$$G(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)} \qquad G_c(s) = 0.4\left(1 + \frac{1}{0.1s}\right)$$

- Open-loop transfer function

$$G_{OL}(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)} 0.4\left(1 + \frac{1}{0.1s}\right)$$

- Amplitude ratio and phase shift

$$\text{AR} = \frac{5}{\sqrt{1+\omega^2}} \frac{1}{\sqrt{1+0.25\omega^2}} 0.4\sqrt{1 + \frac{1}{0.01\omega^2}}$$

$$\phi = -0.1\omega - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\frac{1}{0.1\omega}\right)$$

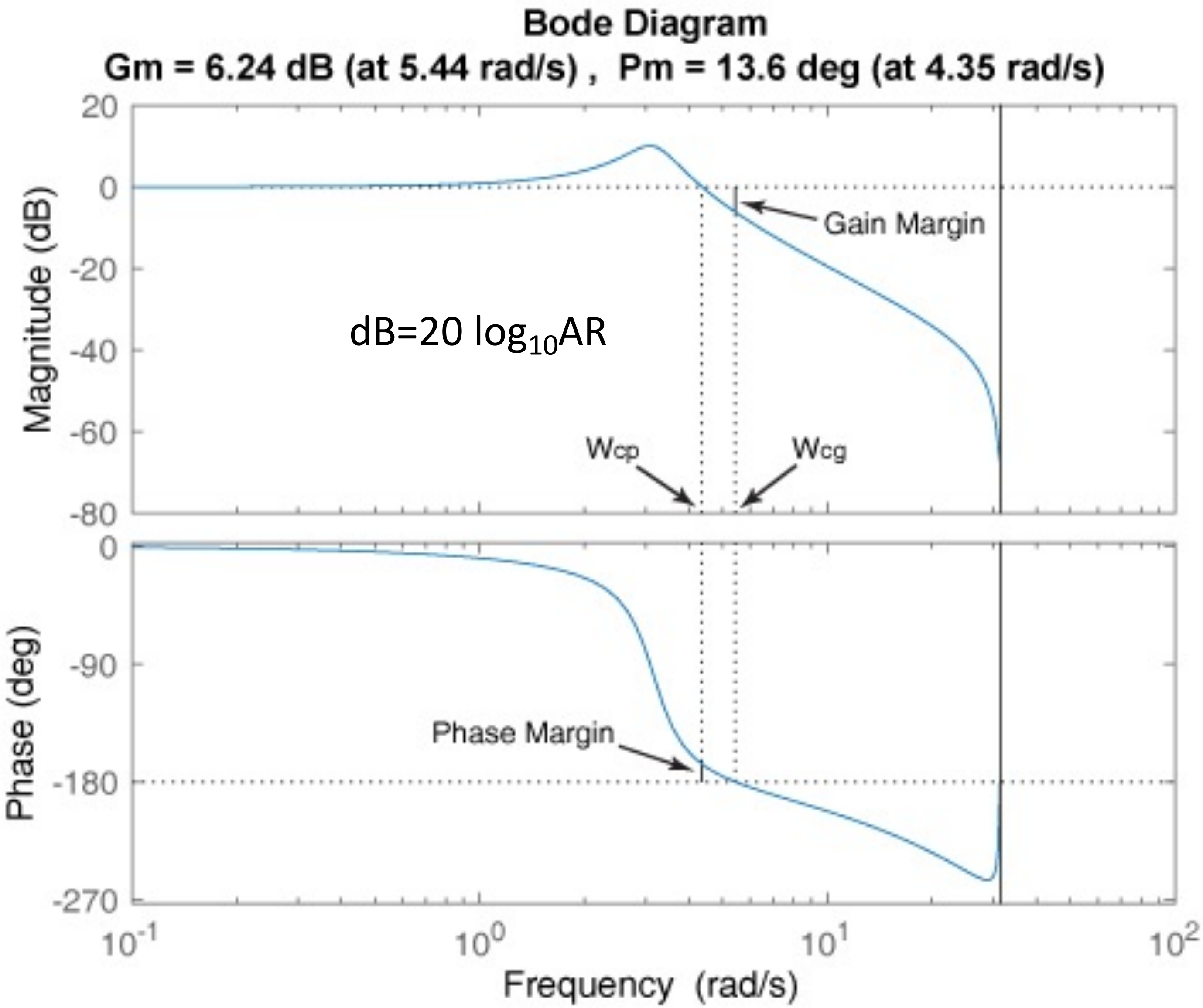
- At $\phi=-\pi$, $\omega=1.4128$, AR=6.746

Bode Stability

- Phase Crossover frequency (ω_{gc}) is the frequency at which phase lag is 180^0.
- Gain Crossover frequency (ω_{pc}) is the frequency at which AR = 1
- Gain Margin (GM) is the amount of gain required for the AR to reach 1 at phase crossover frequency.

  $GM=1/AR_C$, where AR_C is the Amplitude ratio at gain crossover frequency.
- Phase Margin (PM) is the amount of phase difference required for phase lag to reach 180^0 at gain crossover frequency. $PM=180+\phi_{pc}$ where ϕ_{pc} is phase lag at phase crossover frequency.

Gain Margin and Phase Margin



Wcg is the frequency where the gain margin is measured, which is a -180° phase crossing frequency.

Wcp is the frequency where the phase margin is measured, which is a 0-dB gain crossing frequency.

Nyquist Stability Criterion

If N is the number of times that the Nyquist plot encircles the point (-1,0) in the complex plane in the clockwise direction, and P is the number of open-loop poles of G_{OL} that lie in the right-half plane, then Z=N+P is the number of unstable roots of the closed-loop characteristic equation.

Strategy

1. Substitute $s=j\omega$ in $G_{OL}(s)$
2. Plot $G_{OL}(j\omega)$ in the complex plane
3. Count encirclements of (-1,0) in the clockwise direction

Nyquist Criterion

Consider the transfer function $$G(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)}$$

and the P controller $G_c(s) = 3.2$