1.1 Introduction to Integral Transform

In this lesson we will discuss the idea of integral transform, in general, and Laplace transform in particular. Integral transforms turn out to be a very efficient method to solve certain ordinary and partial differential equations. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The idea of solving differential equations is given in Figure 33.1.

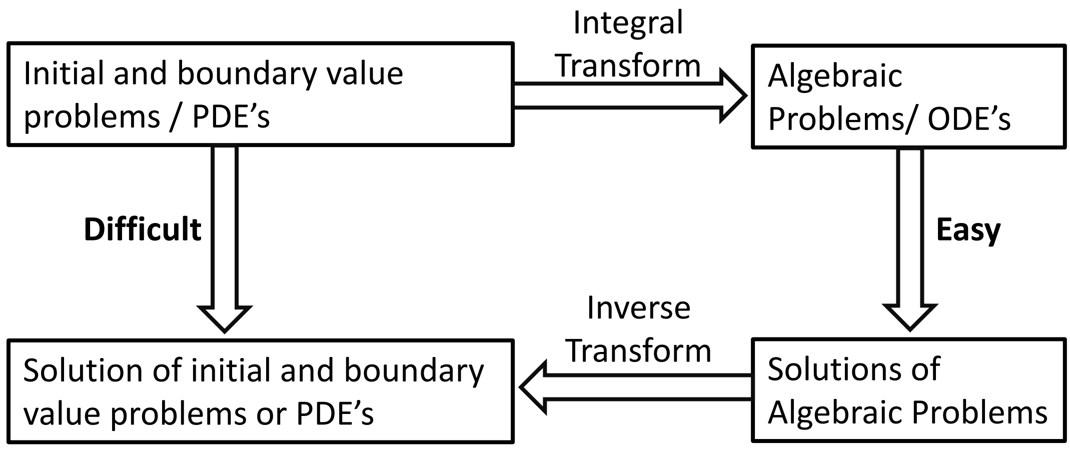


Figure 1.1: Idea of Solving Differential/Integral Equations

1.2 Concept of Transformations

An integral of the form

$$\int_a^b K(s,t)f(t)\,\mathrm{d}t$$

is called integral transform of $f(t)$. The function $K(s,t)$ is called kernel of the transform. The parameter s belongs to some domain on the real line or in the complex plane. Choosing different kernels and different values of a and b, we get different integral transforms. Examples include Laplace, Fourier, Hankel and Mellin transforms. For $K(s,t) = e^{-st},\ a = 0,\ b = \infty$, the improper integral

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

is called Laplace transform of $f(t)$. If we set $K(s,t) = e^{-ist},\ a = -\infty,\ b = \infty$, then

$$\int_{-\infty}^{\infty} e^{ist} f(t)\,\mathrm{d}t$$

where $i = \sqrt{-1}$ is called the Fourier transform of $f(t)$. A common property of integral transforms is linearity, i.e.,

$$\text{I.T.}\,[\alpha\, f(t) + \beta\, g(t)] = \int_a^b K(s,t)\,[\alpha\, f(t) + \beta\, g(t)]\ \mathrm{d}t = \alpha\,\text{I.T.}(f(t)) + \beta\,\text{I.T.}\ (g(t))$$

The symbol I.T. stands for integral transforms.

1.3 Laplace Transform

The Laplace transform of a function f is defined as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$$

provided the improper integral converges for some s.

Remark 1: *The integral $\int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$ is said to be convergent (absolutely convergent) if*

$$\lim_{R\to\infty} \int_0^R e^{-st} f(t)\,\mathrm{d}t \quad \left(\lim_{R\to\infty} \int_0^R \mid e^{-st} f(t) \mid\ \mathrm{d}t\right)$$

exists as a finite number.

1.4 Laplace Transform of Some Elementary Functions

We now give Laplace transform of some elementary functions. Laplace transform of these elementary functions together with properties of Laplace transform will be used to evaluate Laplace transform of more complicated functions.

1.5 Example Problems

1.5.1 Problem 1

Evaluate Laplace transform of $f(t) = 1,\ t \geq 0$.

Solution: Using definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-st}}{-s}\Big|_0^\infty$$

Assuming that s is real and positive, therefore

$$L[f(t)] = \frac{1}{s},\ \text{since } \lim_{R\to\infty} e^{-sR} = 0$$

What will happen if we take s to be a complex number, i.e., $s = x + iy$. Since $e^{-iyR} = \cos yR - i\sin yR$, and therefore $\mid e^{-iyR} \mid = 1$, then, we find

$$\lim_{R\to\infty} \mid e^{-xR} \mid\mid e^{-iyR} \mid = 0 \ \text{ for Re}(s) = x > 0$$

Thus, we have

$$L[f(t)] = L[1] = \frac{1}{s},\ \text{Re}(s) > 0.$$

1.5.2 Problem 2

Find the Laplace transform of the functions $e^{at},\ e^{iat},\ e^{-iat}$.

Solution: Using the definition of Laplace transform

$$\begin{aligned} L[e^{at}] = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t &= \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty \\ &= \frac{1}{s-a},\ \text{provided Re}(s) > a \text{ (or } s > a) \end{aligned}$$

Similarly, we can evaluate

$$\begin{aligned} L[e^{iat}] = \int_0^\infty e^{-(s-ia)t}\,\mathrm{d}t &= \frac{e^{-(s-ia)t}}{-(s-ia)}\Big|_0^\infty \\ &= \frac{1}{s-ia},\ \text{provided Re}(s) > 0. \end{aligned}$$

Here we have used the fact that, for $s = x + iy$, we have

$$\lim_{R\to\infty}\left|\frac{e^{-(s-ia)R}}{-(s-ia)}\right| = -\frac{1}{s-ia}\lim_{R\to\infty}\left|e^{-xR}e^{-i(y-a)R}\right| = 0$$

Similarly, we get

$$L[e^{-iat}] = \frac{1}{s+ia}.$$

1.5.3 Problem 3

Find the Laplace transform of the unit step function (commonly known as the Heaviside function). This function is given as

$$u(t-a) = \begin{cases} 0 & \textit{if } t < a, \\ 1 & \textit{if } t \geq a. \end{cases}$$

Solution: Let us find the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^\infty = \frac{e^{-as}}{s},$$

where of course $s > 0$ and $a \geq 0$.

1.5.4 Problem 4

Find the Laplace transform of $t^n,\ n = 1, 2, 3, ...$

Solution: Using definition of Laplace transform we get

$$\begin{aligned} L[t^n] = \int_0^\infty e^{-st}t^n\,\mathrm{d}t &= \left[t^n\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}nt^{n-1}\,\mathrm{d}t \\ &= 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,\mathrm{d}t = \frac{n}{s}L[t^{n-1}] \end{aligned}$$

Putting $n = 1$:

$$L[t] = \frac{1}{s}L[1] = \frac{1}{s^2} = \frac{1!}{s^2}$$

Putting $n = 2$:

$$L[t^2] = \frac{2}{s^3} = \frac{2!}{s^3}$$

If we assume $L[t^n] = \frac{n!}{s^{n+1}}$, then

$$L[t^{n+1}] = \frac{n+1}{s} L[t^n] = \frac{(n+1)!}{s^{n+2}} \quad \Rightarrow \quad L[t^n] = \frac{n!}{s^{n+1}}, \;\; \text{Re}(s) > 0.$$

One can also extend this result for non-integer values of n.

1.5.5 Problem 5

Find $L[t^\gamma]$ for non-integer values of γ.

Solution: Using the definition of Laplace transform we get

$$L[t^\gamma] = \int_0^\infty e^{-st} t^\gamma \,\mathrm{d}t, \;\; (\gamma > -1)$$

Note that the above integral is convergent only for $\gamma > -1$. We substitute $u = st \;\Rightarrow \mathrm{d}u = s\mathrm{d}t$ where $s > 0$. Thus we get

$$L[t^\gamma] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\gamma \frac{1}{s} \,\mathrm{d}u = \frac{1}{s^{\gamma+1}} \int_0^\infty e^{-u} u^\gamma \,\mathrm{d}u$$

We know

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u} \,\mathrm{d}u \;\; (p > 0)$$

Then,

$$L[t^\gamma] = \frac{\Gamma(\gamma+1)}{s^{\gamma+1}}, \;\; \gamma > -1, s > 0$$

Note that for $\gamma = 1, 2, 3, ...$, the above formula reduces to the formula we got in previous example for integer values, i.e., $L[t^\gamma] = \dfrac{\gamma!}{s^{\gamma+1}}$.

1.5.6 Problem 6

Let $f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$. Find $L[f(t)]$.

Solution: Applying the definition of Laplace transform we obtain

$$L[f(t)] = L\left[\sum_{k=0}^{n} a_k t^k\right]$$

Using the linearity of the transform we get

$$L[f(t)] = \sum_{k=0}^{n} L[t^k] = \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}.$$

Remark 2: *For an infinite series $\sum_{n=0}^{\infty} a_n t^n$, it is not possible, in general, to obtain Laplace transform of the series by taking the transform term by term.*

2.1 Basic Properties (Linearity)

Now we compute the Laplace transform of some elementary functions, before discussing the restriction that have to be imposed on $f(t)$ so that it has a Laplace transform. With the help of Laplace transform of elementary function we can get Laplace transform of complicated function using properties of the transform that will be discussed later. Another important aspect of the finding Laplace transform of elementary function relies on using them for getting inverse Laplace transform.

2.2 Example Problems

2.2.1 Problem 1

Find Laplace transform of (i) $\cosh \omega t$*, (ii)* $\cos \omega t$*, (iii)* $\sinh \omega t$ *(iv)* $\sin \omega t$.

Solution: (i) Using the definition of Laplace transform we get

$$L[\cosh \omega t] = L\left[\frac{e^{\omega t} + e^{-\omega t}}{2}\right]$$

Using linearity of the transform we obtain

$$L[\cosh \omega t] = \frac{1}{2}\Big(L\left[e^{\omega t}\right] + L\left[e^{-\omega t}\right]\Big)$$

Applying the Laplace transform of exponential function we obtain

$$L[\cosh \omega t] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s+\omega}\right] = \frac{s}{s^2-\omega^2}$$

(ii) Following similar steps we obtain

$$L[\cos \omega t] = L\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

Using linearity, we obtain

$$L[\cos \omega t] = \frac{1}{2}L\left[e^{i\omega t}\right] + \frac{1}{2}L\left[e^{-i\omega t}\right]$$

We know the Laplace transform of exponential functions which can be used now to get

$$L[\cos \omega t] = \frac{1}{2}\left\{\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right\} = \frac{1}{2}\frac{2s}{s^2+\omega^2}$$

Thus we have

$$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

Similarly we get the last two cases (iii) and (iv) as

$$L[\sinh\omega t] = \frac{\omega}{s^2-\omega^2} \quad \text{and} \quad L[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

2.2.2 Problem 2

Find the Laplace transform of $(3+e^{6t})^2$.

Solution: We determine the Laplace transform as follows

$$L(3+e^{6t})^2 = L(3+e^{6t})(3+e^{6t}) = L(9+6e^{6t}+e^{12t})$$

Using linearity we get

$$\begin{aligned} L(3+e^{6t})^2 &= L(9) + L(6e^{6t}) + L(e^{12t}) \\ &= 9L(1) + 6L(e^{6t}) + L(e^{12t}) \end{aligned}$$

Using the Laplace transform of elementary functions appearing above we obtain

$$L(3+e^{6t})^2 = \frac{9}{s} + \frac{6}{s-6} + \frac{1}{s-12}$$

2.2.3 Problem 3

Find the Laplace transform of $\sin^3 2t$.

Solution: We know that

$$\sin 3t = 3\sin t - 4\sin^3 t$$

This implies that we can write

$$\sin^3 2t = \frac{1}{4}\left(3\sin 2t - \sin 6t\right)$$

Applying Laplace transform and using its linearity property we get

$$L[\sin^3 2t] = \frac{1}{4}\left(3L[\sin 2t] - L[\sin 6t]\right)$$

Using the Laplace transforms of $\sin at$ we obtain

$$L[\sin^3 2t] = \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\frac{6}{s^2+36}$$

Thus we get

$$L[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$$

2.2.4 Problem 4

Find Laplace transform of the function $f(t) = 2^t$.

Solution: First we rewrite the given function as

$$f(t) = 2^t = e^{\ln 2^t} = e^{t\ln 2}$$

Now $f(t)$ is function of the form e^{at} and therefore

$$L[f(t)] = \frac{1}{s-\ln 2}, \quad \text{for } s > \ln 2$$

2.2.5 Problem 5

Find (a) $L[t^3 - 4t + 5 + 3\sin 2t]$ *and (b)* $L[H(t-a) - H(t-b)]$.

Solution: (a) Using linearity of the transform we get

$$L[t^3 - 4t + 5 + 3\sin 2t] = L[t^3] - 4L[t] + L[5] + 3L[\sin 2t]$$

Using Laplace transform evaluated in previous previous examples, we have

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{6}{s^4} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{(s^2+4)}$$

On simplification we find

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{(5s^5 + 2s^4 + 20s^3 10s^2 + 24)}{[s^4(s^2+4)]}$$

(b) Using Linearity property we get

$$L[H(t-a) - H(t-b)] = L[H(t-a)] - L[H(t-b)]$$

Applying the definition of Laplace transform we obtain

$$\begin{aligned} L[H(t-a) - H(t-b)] &= \int_0^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_0^\infty H(t-b)e^{-st}\,\mathrm{d}t \\ &= \int_a^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_b^\infty H(t-b)e^{-st}\,\mathrm{d}t \end{aligned}$$

Integration gives

$$L[H(t-a) - H(t-b)] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

This implies

$$L[H(t-a) - H(t-b)] = \frac{e^{-as} - e^{-bs}}{s}$$

2.2.6 Problem 6

Find Laplace transform of the following function

$$f(t) = \begin{cases} t/c, & \text{if } 0 < t < c\,; \\ 1, & \text{if } t > c. \end{cases}$$

Here c is some constant.

Solution: Using the definition of Laplace transform we have

$$L[f(t)] = \int_0^c e^{-st}\left(\frac{t}{c}\right)\,\mathrm{d}t + \int_c^\infty e^{-st}\,\mathrm{d}t$$

Integrating by parts we find

$$L[f(t)] = \left[\frac{t}{c}\left(-\frac{e^{-st}}{s}\right) - \frac{1}{c}\left(-\frac{e^{-st}}{s^2}\right)\right]_0^c + \left[-\frac{e^{-st}}{s}\right]_c^\infty$$

On simplifications we obtain

$$L[f(t)] = \frac{1 - e^{sc}}{cs^2}$$

2.2.7 Problem 7

Find Laplace transform of the function $f(t)$ given by

$$f(t) = \begin{cases} 0, & \text{if } 0 < t < 1\,; \\ t, & \text{if } 1 < t < 2; \\ 0, & \text{if } t > 2. \end{cases}$$

Solution: By the definition of Laplace transform we have

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_1^2 e^{-st} t\,\mathrm{d}t$$

Integrating by parts we obtain

$$\begin{aligned} L[f(t)] &= \left[t\left(-\frac{e^{-st}}{s}\right)\right]_1^2 + \int_1^2 \frac{e^{-st}}{s}\,\mathrm{d}t \\ &= -\frac{2e^{-2s} - e^{-s}}{s} - \frac{e^{-2s} - e^{-s}}{s^2} \end{aligned}$$

2.2.8 Problem 8

Find Laplace transform of $\sin\sqrt{t}$.

Solution: We have

$$\sin\sqrt{t} = t^{1/2} - \frac{1}{3!}t^{3/2} + \frac{1}{5!}t^{5/2} - \frac{1}{7!}t^{7/2} + ...$$

Then, taking the Laplace transform of each term in the series we get

$$\begin{aligned} L[\sin\sqrt{t}] &= L[t^{1/2}] - \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] + \dots \\ &= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{s^{9/2}} + \dots \end{aligned}$$

Further simplifications leads to

$$\begin{aligned} L[\sin\sqrt{t}] &= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{3!}\frac{3}{2}\frac{1}{s} + \frac{1}{5!}\frac{5}{2}\frac{3}{2}\frac{1}{s^2} - \frac{1}{7!}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{s^3} + \dots\right] \\ &= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{2^2 s} + \frac{1}{2!}\frac{1}{(2^2 s)^2} - \frac{1}{3!}\frac{1}{(2^2 s)^3} + \dots\right] \end{aligned}$$

Thus, we have

$$L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}.$$

3.1 Existence of Laplace Transform

In this lesson we shall discuss existence theorem on Laplace transform. Since every Laplace integral is not convergent, it is very important to know for which functions Laplace transform exists.

Consider the function $f(t) = e^{t^2}$ and try to evaluate its Laplace integral. In this case we realize that

$$\lim_{R\to\infty}\int_0^R e^{t^2-st}\,\mathrm{d}t = \infty,\ \text{for any choice of } s$$

Naturally question arises in mind that for which class of functions, the Laplace integral converges? So before answering this question we go through some definition.

3.2 Piecewise Continuity

A function f is called piecewise continuous on $[a,b]$ if there are finite number of points $a < t_1 < t_2 < \ldots < t_n < b$ such that f is continuous on each open subinterval $(a,t_1),(t_1,t_2),\ldots,(t_n,b)$ and all the following limits exists

$$\lim_{t\to a+} f(t),\ \lim_{t\to b-} f(t),\ \lim_{t\to t_j+} f(t),\ \text{and}\ \lim_{t\to t_j-} f(t),\ \forall j.$$

Note: A function f is said to be piecewise continuous on $[0,\infty)$ if it is piecewise continuous on every finite interval $[0,b],\ b\in R_+$.

3.2.1 Example 1

The function defined by

$$f(t) = \begin{cases} t^2, & 0\le t\le 1; \\ 3-t, & 1<t\le 2; \\ t+1, & 2<t\le 3; \end{cases}$$

is piecewise continuous on $[0,3]$.

3.2.2 Example 2

The function defined by

$$f(t) = \begin{cases} \frac{1}{2-t}, & 0 \leq t < 2; \\ t+1, & 2 \leq t \leq 3; \end{cases}$$

is not piecewise continuous on $[0, 3]$.

3.3 Example Problems

3.3.1 Problem 1

Discuss the piecewise continuity of

$$f(t) = \frac{1}{t-1}$$

.

Solution: $f(t)$ is not piecewise continuous in any interval containing 1 since

$$\lim_{t \to 1\pm} f(t)$$

do not exists.

3.3.2 Problem 2

Check whether the function

$$f(t) = \begin{cases} \frac{1-e^{-t}}{t}, & t \neq 0; \\ 0, & \textit{otherwise} \end{cases}$$

is piecewise continuous or not.

Solution: The given function is continuous everywhere other than at 0. So we need to check limits at this point. Since both the left and right limits

$$\lim_{t \to 0-} f(t) = 1 \quad \text{and} \quad \lim_{t \to 0+} f(t) = 1$$

exists, the given function is piecewise continuous.

3.4 Functions of Exponential Orders

A function f is said to be of exponential order α if there exist constant M and α such that for some $t_0 \geq 0$

$$|f(t)| \leq Me^{\alpha t} \text{ for all } t \geq t_0$$

Equivalently, a function $f(t)$ is said to be of exponential order α if

$$\lim_{t\to\infty} e^{-\alpha t}|f(t)| = \text{ a finite quantity}$$

Geometrically, it means that the graph of the function f on the interval (t_0, ∞) does not grow faster than the graph of exponential function $Me^{\alpha t}$

3.5 Example Problems

3.5.1 Problem 1

Show that the function $f(t) = t^n$ has exponential order α for any value of $\alpha > 0$ and any natural number n.

Solution: We check the limit

$$\lim_{t\to\infty} e^{-\alpha t}t^n$$

Repeated application of L'hospital rule gives

$$\lim_{t\to\infty} e^{-\alpha t}t^n = \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} = 0$$

Hence the function is of exponential order.

3.5.2 Problem 2

Show that the function $f(t) = e^{t^2}$ is not of exponential order.

Solution: For given function we have

$$\lim_{t\to\infty} e^{-\alpha t}e^{t^2} = \lim_{t\to\infty} e^{t(t-\alpha)} = \infty$$

for all values of α. Hence the given function is not of exponential order.

3.5.3 Theorem (Sufficient Conditions for Laplace Transform)

If f is piecewise continuous on $[0, \infty)$ and of exponential order α then the Laplace transform exists for $Re(s) > \alpha$. Moreover, under these conditions Laplace integral converges absolutely.

Proof: Since f is of exponential order α, then

$$|f(t)| \leq M_1 e^{\alpha t}, \;\; t \geq t_0 \tag{3.1}$$

Also, f is piecewise continuous on $[0, \infty)$ then

$$|f(t)| \leq M_2, \;\; 0 \leq t \leq t_0 \tag{3.2}$$

From equation (3.1) and (3.2) we have

$$|f(t)| \leq M e^{\alpha t}, \;\; t \geq 0$$

Then

$$\int_0^R |e^{-st} f(t)| dt \leq \int_0^R |e^{-(x+iy)t} M e^{\alpha t}| dt$$

Here we have assumed s to be a complex number so that $s = x + iy$. Noting that $|e^{-iy}| = 1$ we find

$$\int_0^R |e^{-st} f(t)| dt \leq M \int_0^R e^{-(x-\alpha)t} dt$$

On integration we obtain

$$\int_0^R |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha} - \frac{M}{x-\alpha} e^{-(x-\alpha)R}$$

Letting $R \to \infty$ and noting $Re(s) = x > \alpha$, we get

$$\int_0^\infty |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha}$$

Hence the Laplace integral converges absolutely and thus converges. This implies the existence of Laplace transform. For piecewise continuous functions of exponential order, the Laplace transform always exists. Note that it is a sufficient condition, that means if a function is not of exponential order or piecewise continuous then the Laplace transform may or may not exist. ■

Remark 1: *We have observed in the proof of existence theorem that*

$$\left|\int_0^\infty e^{-st} f(t)dt\right| \leq \int_0^\infty |e^{-st} f(t)|dt \leq \frac{M}{Re(s)-\alpha} \quad \textit{for } Re(s) > \alpha$$

We now deduce two important conclusions with this observation:

- $L[f(t)] = \displaystyle\int_0^\infty e^{-st} f(t)dt = F(s) \to 0$ *as* $Re(s) \to \infty$

- *if* $L[f(t)] \not\to 0$ *as* $s \to \infty$ *(or* $Re(s) \to \infty$*) then* $f(t)$ *cannot be piecewise continuous function of exponential order. For example functions such as* $F_1(s) = 1$ *and* $F_2(s) = s/(s+1)$ *are not Laplace transforms of piecewise continuous functions of exponential order, since* $F_1(s) \not\to 0$ *and* $F_2(s) \not\to 0$ *as* $s \to \infty$.

Remark 2: *It should be noted that the conditions stated in existence theorem are sufficient rather than necessary conditions. If these conditions are satisfied then the Laplace transform must exist. If these conditions are not satisfied then Laplace transform may or may not exist. We can observe this fact in the following examples:*

- *Consider, for example,*

  $$f(t) = 2te^{t^2}\cos(e^{t^2})$$

  Note that $f(t)$ *is continuous on* $[0, \infty)$ *but not of exponential order, however the Laplace transform of* $f(t)$ *exists, since*

  $$L[f(t)] = \int_0^\infty e^{-st} 2te^{t^2}\cos(e^{t^2})dt$$

  Integration by parts leads to

  $$L[f(t)] = e^{-st}\sin(e^{t^2})\Big|_0^\infty + s\int_0^\infty e^{-st}\sin(e^{t^2})dt$$

  Using the definition of Laplace transform we obtain

  $$L[f(t)] = -\sin(1) + sL[\sin(e^{t^2})]$$

  Note that $L[\sin(e^{t^2})]$ *exists because the function* $\sin(e^{t^2})$ *satisfies both the conditions of existence theorem. This example shows that Laplace transform of a function which is not of exponential order exists.*

- *Consider another example of the function*

$$f(t) = \frac{1}{\sqrt{t}},$$

which is not piecewise continuous since $f(t) \to \infty$ *as* $t \to 0$*. But we know that*

$$L[f(t)] = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

This example shows that Laplace transform of a function which is not piecewise continuous exists. These two examples clearly shows that the conditions given in existence theorem are sufficient but not necessary.

4.1 Properties of Laplace Transform

In this lesson we discuss some properties of Laplace transform. There are several useful properties of Laplace transform which can extend its applicability. In this lesson we mainly present shifting and translation properties.

4.2 First Shifting Property

If $L[f(t)] = F(s)$ then $L\left[e^{at}f(t)\right] = F(s-a)$, where a is any real or complex constant.

Proof: By the definition of Laplace transform we find

$$\begin{aligned} L\left[e^{at}f(t)\right] &= \int_0^\infty e^{at}f(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t \end{aligned}$$

Again by the definition of Laplace transform we get

$$L\left[e^{at}f(t)\right] = F(s-a).$$

4.3 Example Problems

4.3.1 Problem 1

Find the Laplace transform of $e^{-t}\sin^2 t$.

Solution: First we get the Laplace transform of $\sin^2 t$ as

$$\begin{aligned} L\left[\sin^2 t\right] &= L\left[\frac{1-\cos 2t}{2}\right] \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)} = F(s). \end{aligned}$$

Now using the first shifting property we obtain

$$L\left[e^{-t}\sin^2 t\right] = F(s+1) = \frac{2}{(s+1)(s^2+2s+5)}$$

4.3.2 Problem 2

Find $L[e^{-2t}\sin 6t]$.

Solution: Setting $f(t) = \sin 6t$ we find

$$L[f(t)] = F(s) = \frac{6}{s^2+36}$$

Now using the first shifting property we get

$$L[e^{-2t}\sin 6t] = \frac{6}{(s+2)^2+36}$$

4.3.3 Problem 3

Evaluate $L[e^{2t}(t+3)^2]$.

Solution: By the definition and linearity of Laplace transform we have

$$\begin{aligned} L[(t+3)^2] =& L[t^2+6t+9] = L[t^2] + 6L[t] + 9L[1] \\ =& \frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s} \end{aligned}$$

Further simplifications lead to

$$L[(t+3)^2] = \frac{2+6s+9s^2}{s^3} = F(s)$$

Using the first shifting property we get

$$\begin{aligned} L[e^{2t}(t+3)^2] = F(s-2) =& \frac{2+6(s-2)+9(s-2)^2}{(s-2)^3} \\ =& \frac{9s^2-30s+26}{(s-2)^3} \end{aligned}$$

4.3.4 Problem 4

Using shifting property evaluate $L[\sinh 2t\cos 2t]$ *and* $L[\sinh 2t\sin 2t]$

Solution: We know that

$$L[\sinh 2t] = \frac{2}{s^2-4} = F(s)$$

Using shifting property we can get

$$L[e^{2it}\sinh 2t] = F(s-2i)$$

This implies

$$L[e^{2it}\sinh 2t] = \frac{2}{(s-2i)^2-4} = \frac{2}{(s^2-8)-4is}$$

Multiplying numerator and denominator by $(s^2-8)+4is$, we find

$$L[e^{2it}\sinh 2t] = \frac{2(s^2-8)+8is}{(s^2-8)^2+16s^2} = \frac{2(s^2-8)+8is}{(s^4+64)}$$

Replacing e^{2it} by $\cos 2t + i\sin 2t$ and using linearity of the transform we obtain

$$L[\cos 2t\sinh 2t] + iL[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} + i\frac{8s}{(s^4+64)}$$

Equating real and imaginary parts we have

$$L[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} \quad \text{and} \quad L[\cos 2t\sinh 2t] = \frac{8s}{(s^4+64)}$$

4.4 Second Shifting Property

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a) & \text{when } t > a \\ 0 & \text{when } 0 < t < a \end{cases}$

then

$$L[g(t)] = e^{-as}F(s).$$

Proof: By the definition of Laplace transform we have

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-st}g(t)\,\mathrm{d}t \\ &= \int_a^\infty e^{-st}f(t-a)\,\mathrm{d}t \end{aligned}$$

Substituting $t-a=u$ so that $\mathrm{d}t = \mathrm{d}u$, we find

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-s(u+a)}f(u)\,\mathrm{d}u \\ &= e^{-sa}\int_0^\infty e^{-su}f(u)\,\mathrm{d}u \end{aligned}$$

Again using the definition of Laplace transform we get

$$L[g(t)] = e^{-as}F(s).$$

Alternative form: It is sometimes useful to present this property in the following compact form.

If $L[f(t)] = F(s)$ then

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

where

$$H(t) = \begin{cases} 1 & \text{when } t > 0 \\ 0 & \text{when } t < 0 \end{cases}$$

Note that $f(t-a)H(t-a)$ is same as the function $g(t)$ given above.

4.5 Example Problems

4.5.1 Problem 1

Find $L[g(t)]$ *where* $g(t) = \begin{cases} 0 & \textit{when } 0 \leq t < 1 \\ (t-1)^2 & \textit{when } t \geq 1 \end{cases}$

Solution: On comparison with the function $g(t)$ given in second shifting theorem we get

$$f(f) = t^2 \quad \Rightarrow \quad L[f(t)] = \frac{2}{s^3}$$

Using the second shifting property we find

$$L[g(t)] = e^{-s}\left(\frac{2}{s^3}\right).$$

4.5.2 Problem 2

Find the Laplace transform of the function $g(t)$*, where*

$$g(t) = \begin{cases} \cos(t-\pi/3), & t > \pi/3; \\ 0, & 0 < t < \pi/3. \end{cases}$$

Solution: Comparing with the notations used in the second shifting theorem we have $f(t) = \cos t$. Thus, we find

$$L[f(t)] = F(s) = \frac{s}{s^2+1}.$$

Hence by the second shifting theorem we obtain

$$L[g(t)] = e^{-\frac{\pi}{3}s}F(s) = e^{-\frac{\pi}{3}s}\frac{s}{s^2+1}.$$

4.6 Change of Scale Property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: By definition, we have

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,\mathrm{d}t.$$

Substituting $at = u$ so that $a\,\mathrm{d}t = \,\mathrm{d}u$ we find

$$L[f(at)] = \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\frac{1}{a}\,\mathrm{d}u.$$

Using definition of the Laplace transform we get

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).$$

4.6.1 Example

If

$$L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$$

then find $L[f(2t)]$.

Solution: Direct application of the second shifting theorem we obtain

$$L[f(2t)] = \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

On simplifications, we get

$$L[f(2t)] = \frac{1}{4}\frac{s^2-2s+4}{(s+1)^2(s-2)}.$$

5.1 Laplace Transform of Derivatives

Before we state the derivative theorem, it should be noted that this results is the key aspect for its application of solving differential equations.

5.1.1 Derivative Theorem

Suppose f is continuous on $[0, \infty)$ and is of exponential order α and that $f^{'}$ is piecewise continuous on $[0, \infty)$. Then

$$L[f^{'}(t)] = sL[f(t)] - f(0), \quad \textit{Re(s)} > \alpha.$$

Proof: By the definition of Laplace transform, we have

$$L[f^{'}(t)] = \int_0^\infty f^{'}(t)e^{-st}\,\mathrm{d}t$$

Integrating by parts, we get

$$L[f^{'}(t)] = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)e^{-st}(-s)\,\mathrm{d}t$$

Using the definition of Laplace transform we obtain

$$L[f^{'}(t)] = -f(0) + sL[f(t)], \quad \text{Re(s)} > \alpha.$$

This completes the proof. ∎

Remark 1: *Suppose $f(t)$ is not continuous at $t = 0$, then the results of the above theorem takes the following form*

$$L[f^{'}(t)] = -f(0+0) + sL[f(t)]$$

Remark 2: *An interesting feature of the derivative theorem is that $L[f^{'}(t)]$ exists without the requirement of $f^{'}$ to be of exponential order. Recall the existence of Laplace transform of $f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$ which is obvious now by the derivative theorem because*

$$f(t) = \left(\sin\left(e^{t^2}\right)\right)^{'}.$$

Remark 3: *The derivative theorem can be generalized as*

$$\begin{aligned} L[f^{''}(t)] &= -f^{'}(0) + sL[f^{'}(t)] \\ &= -f^{'}(0) + s\left\{-f(0) + sL[f(t)]\right\} = s^2 L[f(t)] - sf(0) - f^{'}(0). \end{aligned}$$

In general, for nth derivative we have

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{n-1}(0).$$

5.2 Example Problems

5.2.1 Problem 1

Determine $L[\sin^2 \omega t]$.

Solution: Let us assume that

$$f(t) = \sin^2 \omega t$$

Now we compute the derivative of f as

$$f^{'}(t) = 2 \sin \omega t \cos \omega t \omega = \omega \sin 2\omega t.$$

Using the derivative theorem we have

$$L[f^{'}(t)] = -f(0) + sL[f(t)]$$

Substituting the function $f(t)$ and its derivative we find

$$L[\omega \sin 2\omega t] = sL[\sin^2 \omega t] - 0$$

Therefore, we have

$$L[\sin^2 \omega t] = \frac{\omega}{s}\left(\frac{2\omega}{s^2 + 4\omega^2}\right)$$

5.2.2 Problem 2

Using derivative theorem, find $L[t^n]$.

Solution: Let

$$f(t) = t^n.$$

Then

$$f^{'}(t) = nt^{n-1}, \quad f^{''}(t) = n(n-1)t^{n-2}, \ldots, \quad f^n(t) = n!.$$

From derivative theorem we have

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ldots - f^{n-1}(0).$$

Therefore, we find

$$L[n!] = s^n L[t^n] \Rightarrow L[t^n] = \frac{n!}{s^{n+1}}.$$

5.2.3 Problem 3

Using derivative theorem, find $L[\sin kt]$.

Solution: Let $f(t) = \sin kt$ and therefore we have

$$f'(t) = k\cos kt \quad \text{and} \quad f''(t) = -k^2 \sin kt$$

Substituting in the derivative theorem

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

yields

$$L[-k^2 \sin kt] = s^2 L[\sin kt] - 0 - k$$

On simplifications we get

$$L[\sin kt] = \frac{k}{s^2 + k^2}$$

5.2.4 Problem 4

Using $L[t^2] = 2/s^3$ *and derivative theorem, find* $L[t^5]$.

Solution: Let $f(t) = t^5$ so that $f'(t) = 5t^4, \quad f''(t) = 20t^3 \quad f'''(t) = 60t^2$. The derivative theorem for third derivative reads as

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

This implies

$$L[60t^2] = s^3 L[f(t)] \quad \Rightarrow \quad L[f(t)] = \frac{120}{s^6}.$$

5.2.5 Problem 5

Using the Laplace transform of $L\left[\sin\sqrt{t}\right]$ *and applying the derivative theorem, find the Laplace transform of the function*

$$\frac{\cos\sqrt{t}}{\sqrt{t}}$$

Solution: We know that

$$L\left[\sin\sqrt{t}\right] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Let $f(t) = \sin\sqrt{t}$, then we have

$$f(0) = 0 \;\text{ and }\; f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Substitution of $f(t)$ in the derivative theorem

$$L[f'(t)] = sL[f(t)] - f(0)$$

yields

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s\frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Thus, we get

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

5.3 Laplace Transform of Integrals

5.3.1 Theorem

Suppose $f(t)$ *is piecewise continuous on* $[0,\infty)$ *and the function*

$$g(t) = \int_0^t f(u)\,\mathrm{d}u$$

is of exponential order. Then

$$L[g(t)] = \frac{1}{s}F(s).$$

Proof: Clearly $g(0) = 0$ and $g^{'}(t) = f(t)$. Note that $g(t)$ is piecewise continuous and is of exponential order as well as $g^{'}(t) = f(t)$ is piecewise continuous. Then, we get using the derivative theorem

$$L[g^{'}(t)] = sL[g(t)] - g(0)$$

Since $g(0) = 0$ we obtain the desired result as

$$L[g(t)] = \frac{1}{s}L[f(t)]$$

This completes the proof. ■

5.4 Example Problems

5.4.1 Problem 1

Given that

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s.$$

Find the Laplace transform of the integral

$$\int_0^t \frac{\sin u}{u}\,\mathrm{d}u.$$

Solution: Direct application of the above result gives

$$\begin{aligned} L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] &= \frac{1}{s}L\left[\frac{\sin t}{t}\right] \\ &= \frac{1}{s}\int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}s\right] \end{aligned}$$

Thus, we have

$$L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] = \frac{1}{s}\cot^{-1}s$$

5.4.2 Problem 2

Find Laplace transform of the following integral

$$\int_0^t u^n e^{-au} du$$

Solution: With the application of the first shifting theorem we know that

$$L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$$

It follows from the above result on Laplace transform of integrals

$$L\left[\int_0^t u^n e^{-au} du\right] = \frac{1}{s} L[t^n e^{-at}] = \frac{n!}{s(s+a)^{n+1}}.$$

5.5 Multiplication by t^n

5.5.1 Theorem

If $F(s)$ is the Laplace transform of $f(t)$, i.e., $L[f(t)] = F(s)$ then,

$$L[tf(t)] = -\frac{d}{ds} F(s)$$

and in general the following result holds

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s).$$

Proof: By definition we know

$$F(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

Using Leibnitz rule for differentiation under integral sign we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty (-t) e^{-st} f(t)\, \mathrm{d}t$$

Thus we get

$$\frac{dF(s)}{ds} = -L[tf(t)]$$

Repeated differentiation under integral sign gives the general rule. ■

Applicability of the above result will now be demonstrated by some examples.

5.6 Example Problems

5.6.1 Problem 1

Find Laplace transform of the function $t^2 \cos at$.

Solution: We know from Laplace transform of elementary functions that

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Direct application of the above rule gives

$$L\left[t^2 \cos at\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left(\frac{s^2+a^2-2s^2}{\left(s^2+a^2\right)^2}\right) = \frac{d}{ds}\left(\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right)$$

On simplifications we find

$$L\left[t^2 \cos at\right] = \frac{2s\left(s^2-3a^2\right)}{\left(s^2+a^2\right)^3}$$

5.6.2 Problem 2

Evaluate (i) $L[te^{-t}]$ *(ii)* $L[t^2e^{-t}]$ *(iii)* $L[t^ke^{-t}]$

Solution: (i) We know that

$$L[e^{-t}] = \frac{1}{s+1}$$

Using the above mentioned rule we find

$$L[te^{-t}] = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

(ii) Applying the same idea once again, we obtain

$$L[t^2e^{-t}] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

(iii) Similarly, we can further generalize this result as

$$L[t^ke^{-t}] = \frac{k!}{(s+1)^{k+1}}$$

5.6.3 Problem 3

Find the Laplace transform of $f(t) = (t^2 - 3t + 2)\sin t$

Solution: Using linearity of the Laplace transform we have

$$L[f(t)] = L[t^2 \sin t] - 3L[t \sin t] + 2L[\sin t] \tag{5.1}$$

Since we know

$$L[\sin t] = \frac{1}{1+s^2}$$

then

$$L[t \sin t] = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

and

$$L[t^2 \sin t] = -\frac{d}{ds}\frac{2s}{(1+s^2)^2} = \frac{2(1+s^2)^2 - 8s^2(1+s^2)}{(1+s^2)^4} = \frac{6s^2-2}{(1+s^2)^3}$$

Substituting the above values in the equation (5.1), we find

$$L[f(t)] = \frac{6s^2-2}{(1+s^2)^3} - \frac{6s}{(1+s^2)^2} + \frac{2}{1+s^2}$$

Further simplifications lead to

$$L[f(t)] = \frac{6s^2 - 2 - 6s(1+s^2) + 2(1+s^2)^2}{(1+s^2)^3}$$

Finally, we obtain

$$L[f(t)] = \frac{(2s^4 - 6s^3 + 10s^2 - 6s)}{(s^6 + 3s^4 + 3s^2 + 1)}$$

5.7 Division by t

5.7.1 Theorem

If f is piecewise continuous on $[0, \infty)$ and is of exponential order α such that

$$\lim_{t \to 0+} \frac{f(t)}{t}$$

exists, then,

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(u)\, \mathrm{d}u, \quad [s > \alpha]$$

Proof: This can easily be proved by letting $g(t) = \dfrac{f(t)}{t}$ so that $f(t) = tg(t)$.

Hence,

$$F(s) = L[f(t) = L[tg(t)] = \frac{d}{ds}L[g(t)]$$

Integrating with respect to s we get,

$$-L[g(t)]\Big|_s^\infty = \int_s^\infty F(s)\,\mathrm{d}s.$$

Since $g(t)$ is piecewise continuous and of exponential order, it follows that $\lim_{s\to\infty} L[g(t)] \to 0$. Thus we have

$$L[g(t)] = \int_s^\infty F(s)\,\mathrm{d}s.$$

This completes the proof. ■

Remark: *It should be noted that the condition* $\lim_{t\to 0+}[f(t)/t]$ *is very important because without this condition the function* $g(t)$ *may not be piecewise continuous on* $[0,\infty)$. *Thus without this condition we can not use the fact* $\lim_{s\to\infty} L[g(t)] \to 0$.

5.7.2 Corollary

If $L[f(t)] = F(s)$ *then* $\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty f(s)\,ds$*, provided that the integrals converge.*

Proof: We know that

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

Using the definition of Laplace transform we get

$$\int_0^\infty e^{-st}\frac{f(t)}{t}\,\mathrm{d}t = \int_s^\infty F(u)\,\mathrm{d}u$$

Taking limit $s \to 0$ in above two integrals we obtain

$$\int_0^\infty \frac{f(t)}{t}\,\mathrm{d}t = \int_0^\infty F(u)\,\mathrm{d}u$$

This completes the proof. ■

5.8 Example Problems

5.8.1 Problem 1

Find the Laplace transform of the function

$$f(t) = \frac{\sin at}{t}$$

Solution: We know,

$$L[\sin at] = \frac{a}{s^2+a^2} \quad \text{and} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

On integrating we get,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,\mathrm{d}s = \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

Thus we have

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

5.8.2 Problem 2

Find the Laplace transform of the function

$$f(t) = \frac{2\sin t \sinh t}{t}$$

Solution: Using Division by t property of the Laplace transform we get

$$L[f(t)] = \int_s^\infty L\left[\sin t\left(e^t - e^{-t}\right)\right]\,\mathrm{d}s \tag{5.2}$$

Now we evaluate $L\left[\sin t\left(e^t - e^{-t}\right)\right]$ using linearity of the Laplace transform as

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = L\left[e^t \sin t\right] - L\left[e^{-t}\sin t\right]$$

Applying the first shifting theorem we obtain

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = \frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}$$

Substituting this value in the equation (5.2) we find

$$L[f(t)] = \int_s^\infty \left[\frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}\right] \mathrm{d}s$$

On integrating, we have

$$\begin{aligned}L[f(t)] =& \tan^{-1}(s-1)\Big|_s^\infty - \tan^{-1}(s+1)\Big|_s^\infty \\ =& \frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\end{aligned}$$

On cancellation of $\pi/2$ we get

$$L[f(t)] = \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

This can be further simplified to obtain

$$L[f(t)] = \tan^{-1}\left(\frac{2}{s^2}\right)$$

In this lesson we evaluate Laplace transform of periodic functions. Periodic functions frequently occur in various engineering problems. We shall now show that with the help of a simple integral, we can evaluate Laplace transform of periodic functions. We shall further continue the discussion for stating initial and final value theorems of Laplace transforms and their applications with the help of simple examples.

6.1 Laplace Transform of a Periodic Function

Let f be a periodic function with period T so that $f(t) = f(t+T)$ then,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t.$$

Proof: By definition we have,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

We break the integral into two integrals as

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_T^\infty e^{-st} f(t)\,\mathrm{d}t$$

Substituting $t = \tau + T$ in the second integral

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_0^\infty e^{-(\tau+T)s} f(\tau+T)\,\mathrm{d}\tau$$

Noting $f(\tau+T) = f(\tau)$ we find

$$L[f(t)] = \int_0^T e^{-st} f(t)\,\mathrm{d}t + e^{-sT} L[f(t)],$$

On simplifications, we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t.$$

This completes the proof. ■

Remark 1: *Just to remind that if a function f is periodic with period $T > 0$ then $f(t) = f(t+T)$, $-\infty < t < \infty$. The smallest of T, for which the equality $f(t) = f(t+T)$ is true, is called fundamental period of $f(t)$. However, if T is the period of a function f then nT, n is any natural number, is also a period of f. Some familiar periodic functions are* $\sin x$, $\cos x$, $\tan x$ *etc.*

6.2 Example Problems

6.2.1 Problem 1

Find Laplace transform for

$$f(t) = \begin{cases} 1 & \textit{when } 0 < t \leq 1 \\ 0 & \textit{when } 1 < t < 2 \end{cases}$$

with $f(t+2) = f(t),\ t > 0.$

Solution: Using the above result on periodic function, we have,

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\,\mathrm{d}t$$

On integration we obtain

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left(\frac{1}{-s}\right) \left[e^{-s} - 1\right] = \frac{1}{s(1+e^{-s})}$$

6.2.2 Problem 2

Find Laplace transform for

$$f(t) = \begin{cases} \sin t & \textit{when } 0 < t < \pi \\ 0 & \textit{when } \pi < t < 2\pi \end{cases}$$

with $f(t+2\pi) = f(t),\ t > 0.$

Solution: Since $f(t)$ is periodic with period 2π we have

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}} \int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

We now evaluate the above integral as

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} f(t)\,\mathrm{d}t + \int_{\pi}^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

Substituting the given value of $f(t)$ we obtain

$$\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t = \int_0^{\pi} e^{-st} \sin t\,\mathrm{d}t + 0 = \frac{1+e^{-s\pi}}{1+s^2}$$

This implies

$$L[f(t)] = \frac{1}{1-e^{-2s\pi}} \frac{1+e^{-s\pi}}{1+s^2} = \frac{1}{(1+s^2)(1-e^{-s\pi})}$$

6.2.3 Problem 3

Find the Laplace transform of the square wave with period T:

$$f(t) = \begin{cases} h & \textit{when } 0 < t < T/2 \\ -h & \textit{when } T/2 < t < T \end{cases}$$

Solution: Using Laplace transform of periodic function we find

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t$$

Substituting $f(t)$ we obtain

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left(\int_0^{T/2} he^{-st}\,\mathrm{d}t - \int_{T/2}^{T} he^{-st}\,\mathrm{d}t \right)$$

Evaluating integrals we get

$$L[f(t)] = \frac{1}{(1-e^{-sT})} \frac{h}{s} \left(1 - 2e^{-sT/2} + e^{-sT} \right) = \frac{h(1-e^{-sT/2})}{s(1+e^{-sT/2})}$$

6.3 Limiting Theorems

These theorems allow the limiting behavior of the function to be directly calculated by taking a limit of the transformed function.

6.3.1 Theorem (Initial Value Theorem)

Suppose that f *is continuous on* $[0,\infty)$ *and of exponential order* α *and* f' *is piecewise continuous on* $[0,\infty)$ *and of exponential order. Let*

$$F(s) = L[f(t)],$$

then

$$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s), \quad \textit{[assuming } s \textit{ is real]}$$

Proof: By the derivative theorem,

$$L[f^{'}(t)] = sL[f(t)] - f(0+)$$

Note that $\lim_{s\to\infty} L[f^{'}(t)] = 0$, since $f^{'}$ is piecewise continuous on $[0, \infty)$ and of exponential order. Therefore we have

$$0 = \lim_{s\to\infty} sF(s) - f(0+)$$

Hence we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

This completes the proof. ■

6.3.2 Theorem (Final Value Theorem)

Suppose that f is continuous on $[0, \infty)$ and is of exponential order α and $f^{'}$ is piecewise continuous on $[0, \infty)$ and furthermore $\lim_{t\to\infty} f(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sL[f(t)] = \lim_{s\to 0} sF(s)$$

Proof: Note that f has exponential order $\alpha = 0$ since it is bounded, since $\lim_{t\to 0+} f(t)$ and $\lim_{t\to\infty} f(t)$ exist and $f(t)$ is continuous in $[0, \infty)$. By the derivative theorem, we have

$$L[f^{'}(t)] = sF(s) - f(0), \quad s > 0,$$

Taking limit as $s \to 0$, we obtain

$$\lim_{s\to 0} \int_0^\infty e^{-st} f^{'}(t)\, \mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

Taking the limit inside the integral

$$\int_0^\infty f^{'}(t)\, \mathrm{d}t = \lim_{s\to 0} sF(s) - f(0)$$

On integrating we obtain

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

Cancellation of $f(0)$ gives the desired results. ■

Remark 2: *In the final value theorem, existence of* $\lim_{t\to\infty} f(t)$ *is very important. Consider* $f(t) = \sin t$. *Then* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{1+s^2} = 0$. *But* $\lim_{t\to\infty} f(t)$ *does not exist. Thus we may say that if* $\lim_{s\to 0} sF(s) = L$ *exists then either* $\lim_{t\to\infty} f(t) = L$ *or this limit does not exist.*

6.3.3 Example

Without determining $f(t)$ *and assuming that* $f(t)$ *satisfies the hypothesis of the limiting theorems, compute*

$$\lim_{t\to 0+} f(t) \textit{ and } \lim_{t\to\infty} f(t) \quad \textit{if} \quad L[f(t)] = \frac{1}{s} + \tan^{-1}\left(\frac{a}{s}\right).$$

Solution: By initial value theorem, we get

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \left[1 + s\tan^{-1}\left(\frac{a}{s}\right)\right]$$

Application of L'hospital rule gives

$$\lim_{t\to 0+} f(t) = 1 + \lim_{s\to\infty} \frac{\frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right)}{-\frac{1}{s^2}} = 1 + a$$

Using the final value theorem we find

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \left[1 + s\tan^{-1}\frac{a}{s}\right] = 1.$$

Remark 3: *Final value theorem says* $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, *if* $\lim_{t\to\infty} f(t)$ *exists. If* $F(s)$ *is finite as* $s \to 0$ *then trivially* $\lim_{t\to\infty} f(t) = 0$. *However, there are several functions whose Laplace transform is not finite as* $s \to 0$, *for example,* $f(t) = 1$ *and its Laplace transform* $F(s)$ *is equal to* $\frac{1}{s}$, $s > 0$. *In this case we have* $\lim_{s\to 0} sF(s) = \lim_{s\to 0} 1 = 1 = \lim_{t\to\infty} f(t)$.

In this lesson we introduce the concept of inverse Laplace transform and discuss some of its important properties that will be helpful to evaluate inverse Transform of some complicated functions. As mention in the beginning of this module that the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the transformed domain we will like to get back to the time domain and therefore we need to introduce the concept of inverse Laplace transform. Further, we introduce the convolution property of the Laplace transform. We shall start with the definition of convolution followed by an important theorem on Laplace transform of convolution. Convolution theorem plays an important role for finding inverse Laplace transform of complicated functions and therefore very useful for solving differential equations.

7.1 Inverse Laplace Transform

If $F(s) = L[f(t)]$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$L^{-1}[F(s)] = f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself as it requires complex numbers and path integrals. The easiest way of computing the inverse is using table of Laplace transform. For example,

$$L[\sin wt] = \frac{w}{s^2 + w^2}$$

This implies

$$L^{-1}\left[\frac{w}{s^2 + w^2}\right] = \sin wt,\ t \geq 0$$

and similarly

$$L[\cos wt] = \frac{s}{s^2 + w^2} \quad \Rightarrow \quad L^{-1}\left[\frac{s}{s^2 + w^2}\right] = \cos wt,\ t \geq 0$$

7.2 Uniqueness of Inverse Laplace Transform

If we have a function $F(s)$, to be able to find $f(t)$ such that $L[f(t)] = F(s)$, we need to first know if such a function is unique.

Consider

$$g(t) = \begin{cases} 1 & \text{when } t = 1 \\ \sin(t) & \text{when otherwise} \end{cases}$$

$$L[g(t)] = \frac{1}{s^2+1} = L[\sin t]$$

Thus we have two different functions $g(t)$ and $\sin t$ whose Laplace transform are same. However note that the given two functions are different at a point of discontinuity. Thanks to the following theorem where we have uniqueness for continuous functions:

7.2.1 Theorem (Lerch's Theorem)

If f and g are continuous and are of exponential order, and if $F(s) = G(s)$ for all $s > s_0$ then $f(t) = g(t)$ for all $t > 0$.

Proof: If $F(s) = G(s)$ for all $s > s_0$ then,

$$\begin{aligned}
&\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty e^{-st} g(t)\mathrm{d}t, \;\; \forall s > s_0 \\
&\Rightarrow \int_0^\infty e^{-st}[f(t) - g(t)]\mathrm{d}t = 0, \;\; \forall s > s_0 \\
&\Rightarrow f(t) - g(t) \equiv 0, \;\; \forall t > 0. \\
&\Rightarrow f(t) = g(t), \;\; \forall t > 0.
\end{aligned}$$

This completes the proof. ■

Remark: *The uniqueness theorem holds for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function or the function $g(t)$ defined above. Since the Laplace integral however does not "see" values at the discontinuities. So in this case we can only conclude that $f(t) = g(t)$ outside of discontinuities.*

We now state some important properties of the inverse Laplace transform. Though, these properties are the same as we have listed for the Laplace transform, we repeat them without proof for the sake of completeness and apply them to evaluate inverse Laplace transform of some functions.

7.3 Linearity of Inverse Laplace Transform

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of the function $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[a_1F_1(s) + a_2F_2(s)] = a_1L^{-1}[F_1(s)] + L^{-1}[F_2(s)] = a_1f_1(t) + a_2f_2(t)$$

where a_1 and a_2 are constants.

7.4 Example Problems

7.4.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{6}{2s-3} + \frac{8-6s}{16s^2+9}$$

Solution: Using linearity of the inverse Laplace transform we have

$$f(t) = 6L^{-1}\left[\frac{1}{2s-3}\right] + 8L^{-1}\left[\frac{1}{16s^2+9}\right] - 6L^{-1}\left[\frac{s}{16s^2+9}\right]$$

Rewriting the above expression as

$$f(t) = 3L^{-1}\left[\frac{1}{s-(3/2)}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(9/16)}\right] - \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(9/16)}\right]$$

Using the result

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{as}$$

and taking the inverse transform we obtain

$$f(t) = 3e^{3t/2} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}.$$

7.4.2 Problem 2

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}$$

Solution: We use the method of partial fractions to write F in a form where we can use the table of Laplace transform. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1) + s(Bs+C) = s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$, and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity of the inverse Laplace transform we get

$$L^{-1}\left[\frac{s^2+s+1}{s^3+s}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] = 1 + \sin t.$$

7.5 First Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t)$, *then* $L^{-1}[F(s-a)] = e^{at} f(t)$

7.6 Example Problems

7.6.1 Problem 1

Evaluate $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Solution: Rewriting the given expression as

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1}{(s-(-1))^2}\right]$$

Applying the first shifting property of the inverse Laplace transform

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

Thus we obtain

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}.$$

7.6.2 Problem 2

Find $L^{-1}\left[\dfrac{1}{s^2+4s+8}\right]$.

Solution: First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$L^{-1}\left[\frac{1}{s^2+4}\right]=\frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right]=L^{-1}\left[\frac{1}{(s+2)^2+4}\right]=\frac{1}{2}\,e^{-2t}\sin(2t).$$

7.7 Second Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)]=f(t),$ *then* $L^{-1}\left[e^{-as}F(s)\right]=f(t-a)\,H(t-a)$

7.8 Example Problems

7.8.1 Problem 1

Find the inverse Laplace transform of

$$F(s)=\frac{e^{-s}}{s(s^2+1)}$$

Solution: First we compute the inverse Laplace transform

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=L^{-1}\left[\frac{1}{s}-\frac{s}{(s^2+1)}\right]$$

Using linearity of the inverse transform we get

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{(s^2+1)}\right]=1-\cos t$$

We now find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]=L^{-1}\left[e^{-s}L[1-\cos t]\right]$$

Using the second shifting theorem we obtain

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = \big[1-\cos(t-1)\big]\,H(t-1).$$

7.8.2 Problem 2

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+4} + \frac{e^{-2s}}{s^2+4} + \frac{e^{-3s}}{(s+2)^2}$$

Solution: First we find that

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

and using the first shifting property

$$L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

By linearity we have

$$f(t) = L^{-1}\left[\frac{e^{-s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-3s}}{(s+2)^2}\right]$$

Putting it all together and using the second shifting theorem we get

$$f(t) = \frac{1}{2}\sin 2(t-1)\,H(t-1) + \frac{1}{2}\sin 2(t-2)\,H(t-2) + e^{-2(t-3)}\,(t-3)\,H(t-3)$$

7.9 Convolution

The convolution of two given functions $f(t)$ and $g(t)$ is written as $f * g$ and is defined by the integral

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{7.1}$$

As you can see, the convolution of two functions of t is another function of t.

7.10 Example Problems

7.10.1 Problem 1

Find the convolution of $f(t) = e^t$ *and* $g(t) = t$ *for* $t \geq 0$.

Solution: By the definition we have

$$(f * g)(t) = \int_0^t e^{\tau}(t - \tau)\, d\tau$$

Integrating by parts, we obtain

$$(f * g)(t) = e^t - t - 1.$$

7.10.2 Problem 2

Find the convolution of $f(t) = \sin(\omega t)$ *and* $g(t) = \cos(\omega t)$ *for* $t \geq 0$.

Solution: By the definition of convolution we have

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\, \cos\big(\omega(t - \tau)\big)\, d\tau.$$

We apply the identity $\cos(\theta)\sin(\psi) = \frac{1}{2}\big(\sin(\theta + \psi) - \sin(\theta - \psi)\big)$ to get

$$(f * g)(t) = \int_0^t \frac{1}{2}\left(\sin(\omega t) - \sin(\omega t - 2\omega\tau)\right)\, d\tau$$

On integration we obtain

$$(f * g)(t) = \left[\frac{1}{2}\,\tau\sin(\omega t) + \frac{1}{4\omega}\,\cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} = \frac{1}{2}\,t\sin(\omega t).$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

7.11 Properties of Convolution

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions, then

(i) $f * g = g * f$, [symmetry]

(ii) $c(f * g) = cf * g = f * cg$, [c=constant]

(iii) $f * (g * h) = (f * g) * h$, [associative property]

(iv) $f * (g + h) = f * g + f * h$, [distributive property]

Proof: We give proof of (i) and all others can be done similarly. By the definition of convolution we have

$$f * g = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

Substituting $t - \tau = u \Rightarrow -d\tau = du$ we get

$$f * g = -\int_t^0 f(t-u)g(u)\mathrm{d}u = \int_0^t f(t-u)g(u)\mathrm{d}u = g * f$$

This completes the proof. ■

The most interesting property for us, and the main result of this lesson is the following theorem.

7.12 Convolution Theorem

If f and g are piecewise continuous on $[0, \infty)$ and of exponential order α, then

$$L[(f * g)(t)] = L[f(t)]L[g(t)].$$

Proof: From the definition,

$$L[(f * g)(t)] = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\mathrm{d}t, \text{ [Re(s)} > \alpha \text{]}$$

Changing the order of integration,

$$L[(f * g)(t)] = \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\mathrm{d}t\mathrm{d}\tau,$$

We now put $t - \tau = u \Rightarrow -d\tau = du$ and get,

$$\begin{aligned} L[(f * g)(t)] &= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^\infty e^{-s\tau)} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-su} g(u)\mathrm{d}u = L[f(t)]L[g(t)] \end{aligned}$$

This completes the proof. ■

8.1 Example Problems on Convolution

8.1.1 Problem 1

Find the inverse Laplace transform of the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\frac{1}{s^2}.$$

Solution: We recognize the two elementary entries

$$L^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \qquad \text{and} \qquad L^{-1}\left[\frac{1}{s^2}\right] = t.$$

Therefore,

$$L^{-1}\left[\frac{1}{s+1}\frac{1}{s^2}\right] = \int_0^t \tau e^{-(t-\tau)}\, d\tau$$

On integration by parts we obtain

$$L^{-1}\left[\frac{1}{s+1}\frac{1}{s^2}\right] = e^{-t} + t - 1.$$

8.1.2 Problem 2

Use the convolution theorem to evaluate

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right].$$

Solution: Note that

$$L[\sin t] = \frac{1}{s^2+1} \qquad \text{and} \qquad L[\cos t] = \frac{s}{s^2+1}$$

Using convolution theorem,

$$L[\sin t * \cos t] = L[\sin t]L[\cos t] = \frac{s}{(s^2+1)^2}.$$

Therefore, we have

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \int_0^t \sin\tau\cos(t-\tau)\mathrm{d}\tau.$$

Using the trigonometric equality $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ we get

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}\int_0^t [\sin t+\sin(2\tau-t)]\mathrm{d}\tau.$$

On integration we find

$$\begin{aligned} L^{-1}\left[\frac{s}{(s^2+1)^2}\right] &= \frac{1}{2}t\sin t+\frac{1}{2}\left[-\frac{\cos(2\tau-t)}{2}\right]_0^t \\ &= \frac{1}{2}t\sin t+\frac{1}{4}[\cos t-\cos t]. \end{aligned}$$

Finally we have the following result

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}t\sin t.$$

8.1.3 Problem 3

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right]$$

Solution: We know the following elementary transforms

$$L\left[\frac{1}{\sqrt{t}}\right] = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \Rightarrow L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{t\pi}}$$

and

$$L^{-1}\left[\frac{1}{s-1}\right] = e^t.$$

Then by the convolution theorem, we find

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{1}{\sqrt{t\pi}} * e^t = \int_0^t \frac{1}{\sqrt{\tau\pi}}e^{t-\tau}\mathrm{d}\tau.$$

Substitution $u=\sqrt{\tau} \Rightarrow du = \dfrac{1}{2\sqrt{\tau}}d\tau$ gives

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\mathrm{d}\tau = 2\frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\mathrm{d}u.$$

Thus, we have

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = e^t\mathrm{erf}(\sqrt{t}).$$

8.1.4 Problem 4

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right].$$

Solution: We know

$$L^{-1}\left[\frac{1}{s^3}\right]=\frac{t^2}{2}\ \text{ and }\ L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t.$$

By the convolution theorem we have

$$\begin{aligned} L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] &= \frac{1}{2}t^2 * \sin t = \frac{1}{2}\int_0^t \sin\tau\,(t-\tau)^2\,\mathrm{d}\tau \\ &= \frac{1}{2}\left[\left(-\cos\tau(t-\tau)^2\right)\Big|_0^t - 2\int_0^t (t-\tau)\cos\tau\mathrm{d}\tau\right] \\ &= \frac{1}{2}\left[t^2 - 2\left((t-\tau)\sin\tau\right)\Big|_0^t - 2\int_0^t \sin\tau\mathrm{d}\tau\right]. \end{aligned}$$

Finally we get the desired inverse Laplace transform as

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]=\frac{t^2}{2}+\cos t-1.$$

We shall continue discussing various properties of inverse Laplace transform. We mainly cover change of scale property, inverse Laplace transform of integrals and derivatives etc.

8.2 Change of Scale Property

If $L^{-1}[F(s)]=f(t)$ *then* $L^{-1}[F(as)]=\frac{1}{a}f\left(\frac{t}{a}\right)$

8.2.1 Example

If

$$L^{-1}\left[\frac{s}{s^2-16}\right]=\cosh 4t,$$

then find

$$L^{-1}\left[\frac{s}{2s^2-8}\right]$$

Solution: Given that

$$L^{-1}\left[\frac{s}{s^2-16}\right]=\cosh 4t$$

Replacing s by $2s$ and using scaling property we find

$$L^{-1}\left[\frac{2s}{4s^2-16}\right]=\frac{1}{2}\cosh 2t$$

Thus, we obtain

$$L^{-1}\left[\frac{s}{2s^2-8}\right]=\frac{1}{2}\cosh 2t$$

8.3 Inverse Laplace Transform of Derivatives (Derivative Theorem)

If $L^{-1}[F(s)]=f(t)$ *then* $L^{-1}\left[\frac{d^n}{ds^n}F(s)\right]=(-1)^n t^n f(t),\;\; n=1,2\ldots$

8.3.1 Example

Find the inverse Laplace transform of

$$(i)\;\frac{2as}{(s^2+a^2)^2}\qquad\qquad (ii)\;\frac{s^2-a^2}{(s^2+a^2)^2}$$

Solution: Note that

$$\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)=\frac{-2as}{(s^2+a^2)^2}\;\text{ and }\;\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=\frac{a^2-s^2}{(s^2+a^2)^2}$$

Direct application of the derivative theorem we obtain

$$(i)\qquad L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right]=(-1)\,t\,L^{-1}\left[-\frac{a}{s^2+a^2}\right]=t\sin at$$

and

$$(ii)\qquad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]=(-1)\,t\,L^{-1}\left[-\frac{s}{s^2+a^2}\right]=t\cos at$$

8.4 Inverse Laplace Transform of Integrals

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \dfrac{f(t)}{t}$

8.4.1 Example

Find the inverse Laplace transform $f(t)$ *of the function*

$$\int_s^\infty \frac{1}{s(s+1)}\,ds$$

Solution: By the method of partial fraction we obtain

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 1 - e^{-t}.$$

Using the inverse Laplace transform of integrals we get

$$L^{-1}\left[\int_s^\infty \frac{1}{s(s+1)}\right] = \frac{1-e^{-t}}{t}.$$

8.5 Multiplication by Powers of s

If $L^{-1}[F(s)] = f(t)$ *and* $f(0) = 0,$ *then* $L^{-1}\left[sF(s)\right] = f'(t)$

8.5.1 Example

Using $L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t$*, and with the application of above result compute* $L^{-1}\left[\dfrac{s}{s^2+1}\right]$.

Solution: Direct application of the above result leads to

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}\sin t = \cos t.$$

8.6 Division by Powers of s

If $L^{-1}[F(s)] = f(t)$, *then*

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du.$$

8.7 Example Problems

8.7.1 Problem 1

Compute

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

Solution: we could proceed by applying this integration rule.

$$L^{-1}\left[\frac{1}{s}\,\frac{1}{s^2+1}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+1}\right]\ \mathrm{d}u = \int_0^t \sin\tau\ \mathrm{d}u = 1-\cos t.$$

8.7.2 Problem 2

Find inverse Laplace transform of $\dfrac{1}{(s^2+1)^2}$

Solution: We know that

$$L^{-1}\left[\frac{s}{(1+s^2)^2}\right] = \frac{1}{2}t\sin t.$$

We now apply the above result as

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right] = L^{-1}\left[\frac{1}{s}\,\frac{s}{(1+s^2)^2}\right] = \frac{1}{2}\int_0^t t\sin t\,\mathrm{d}t.$$

Evaluating the above integral we get

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right] = \frac{1}{2}(-t\cos t + \sin t).$$

8.7.3 Problem 3

Find inverse Laplace transform of $\dfrac{s-1}{s^2(s^2+1)}$.

Solution: It is easy to compute

$$L^{-1}\left[\frac{s-1}{(s^2+1)}\right] = L^{-1}\left[\frac{s}{(s^2+1)}\right] - L^{-1}\left[\frac{1}{(s^2+1)}\right] = \cos t - \sin t.$$

Now repeated application of the above result we get

$$L^{-1}\left[\frac{s-1}{s(s^2+1)}\right] = \int_0^t (\cos t - \sin t)\,\mathrm{d}t = \sin t + \cos t - 1.$$

Finally, we obtain the desired transform as

$$L^{-1}\left[\frac{s-1}{s^2(s^2+1)}\right] = \int_0^t (\sin t + \cos t - 1)\,\mathrm{d}t = 1 - t + \sin t - \cos t.$$

9.1 Laplace Transform of Some Special Functions

In this lesson we discuss Laplace transform of some special functions like error functions, Dirac delta functions, etc. There functions appears in various applications of science and engineering to some of them we shall encounter while solving differential equations using Laplace transform.

9.2 Error Function

The error appears in probability, statistics and solutions of some partial differential equations. It is defined as

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, \mathrm{d}u$$

Its complement, known as complementary error function, is defined as

$$\text{erfc}(t) = 1 - \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, \mathrm{d}u$$

We find Laplace transform of different forms of error function in the following examples.

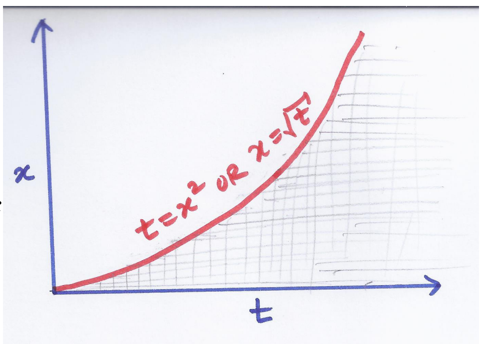
9.3 Example Problems

9.3.1 Problem 1

Find $L[erf(\sqrt{t})]$.

Solution: From definition of the error function and the Laplace transform we have,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{\sqrt{t}} e^{-st} e^{-x^2} \mathrm{d}x \mathrm{d}t$$



By changing the order of integration we get,

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty \int_{t=x^2}^\infty e^{-st} e^{-x^2} \mathrm{d}t \mathrm{d}x$$

Evaluating the inner integral we obtain

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \int_{x=0}^\infty e^{-x^2} \frac{e^{-sx^2}}{s} \mathrm{d}x = \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_{x=0}^\infty e^{-(1+s)x^2} \mathrm{d}x$$

Substituting $\sqrt{(1+s)}x = u \Rightarrow dx = \dfrac{1}{\sqrt{1+s}}du$

$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\frac{1}{s\sqrt{1+s}}\int_{x=0}^{\infty} e^{-u^2}\text{d}u = \frac{1}{s\sqrt{s+1}}$$

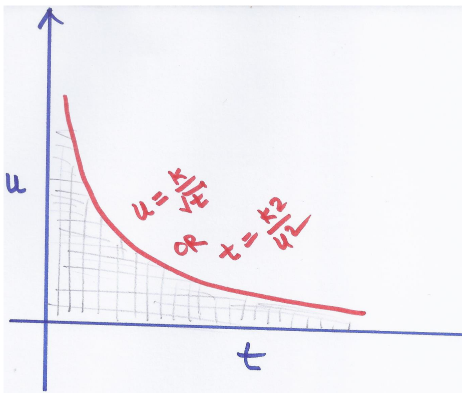
Note that we have used the value of Gaussian integral $\displaystyle\int_{x=0}^{\infty} e^{-u^2}\text{d}u = \frac{\sqrt{\pi}}{2}$.

9.3.2 Problem 2

Find $L\left[erf\left(\dfrac{k}{\sqrt{t}}\right)\right]$. *and show that* $L^{-1}\left[\dfrac{e^{-2k\sqrt{s}}}{s}\right] = erfc\left(\dfrac{k}{\sqrt{t}}\right)$

Solution: By the definition of Laplace transform we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \int_0^{\infty} e^{-st}\frac{2}{\sqrt{\pi}}\int_0^{\frac{k}{\sqrt{t}}} e^{-u^2}\text{d}u\,\text{d}t$$



Changing the order of integration we get

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\int_0^{\frac{k^2}{u^2}} e^{-st}e^{-u^2}\text{d}t\,\text{d}u$$

Evaluation of the inner integral leads to

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{1}{s}\int_0^{\infty} e^{-u^2}\left(1 - e^{-s\frac{k^2}{u^2}}\right)\text{d}u$$

Using the value of Gaussian integral we have

$$L\left[\text{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{1}{s}\left[\frac{\sqrt{\pi}}{2} - \int_0^{\infty}\left(e^{-u^2-s\frac{k^2}{u^2}}\right)\text{d}u\right] \tag{9.1}$$

Let us assume

$$I(s) = \int_0^{\infty} e^{-u^2-s\frac{k^2}{u^2}}\text{d}u$$

By differentiation under integral sign

$$\frac{dI}{ds} = \int_0^{\infty} e^{-u^2-s\frac{k^2}{u^2}}\left(-\frac{k^2}{u^2}\right)\text{d}u$$

Substitution $\frac{\sqrt{s}k}{u} = x \Rightarrow -\frac{\sqrt{s}k}{u^2}du = dx$ leads to

$$\frac{dI}{ds} = -\frac{k}{\sqrt{s}}\int_0^{\infty} e^{-x^2-s\frac{k^2}{x^2}}\text{d}x = -\frac{k}{\sqrt{s}}I$$

Solving the above differential equation we get

$$\ln I(s) = -2k\sqrt{s} + \ln c \quad \Rightarrow \quad I(s) = ce^{-2k\sqrt{s}}$$

Further note that

$$I(0) = \int_0^\infty e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad c = \frac{\sqrt{\pi}}{2}$$

Therefore, we get

$$I(s) = \frac{\sqrt{\pi}}{2} e^{-2k\sqrt{s}}$$

Substituting this value in the equation (9.1), we obtain

$$L\left[\operatorname{erf}\left(\frac{k}{\sqrt{t}}\right)\right] = \frac{2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-2k\sqrt{s}}\right] = \frac{1 - e^{-2k\sqrt{s}}}{s}$$

Taking inverse Laplace transform on both sides we get

$$\operatorname{erf}\left(\frac{k}{\sqrt{t}}\right) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right]$$

This leads to the desired result as

$$L^{-1}\left[\frac{e^{-2k\sqrt{s}}}{s}\right] = 1 - \operatorname{erf}\left(\frac{k}{\sqrt{t}}\right) = \operatorname{erf}_c\left(\frac{k}{\sqrt{t}}\right)$$

10.1 Dirac-Delta Function

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behaviour is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi_\epsilon^a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1/\epsilon & \text{if } a \le t < a+\epsilon, \\ 0 & \text{if } a+\epsilon \le t. \end{cases}$$

Let us take the Laplace transform of a square pulse,

$$L\left[\varphi_\epsilon^a(t)\right] = \int_0^\infty e^{-st}\varphi_\epsilon(t)\,\mathrm{d}t$$

Substituting the value of the function we obatin

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,\mathrm{d}t$$

On integration we get

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{e^{-sa}}{s\epsilon}\left[1 - e^{-s\epsilon}\right]$$

We generally want ϵ to be very small. That is, we wish to have the pulse be very short and very tall. By letting ϵ go to zero we arrive at the concept of the *Dirac delta function*, $\delta(t-a)$. Thus, the Dirac-Delta can be thought as the limiting case of $\varphi_\epsilon(t)$ as $\epsilon \to 0$

$$\delta(t-a) = \lim_{\epsilon\to 0}\varphi_\epsilon^a(t)$$

So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, the Dirac-delta function is defined as having the following properties:

(i) $\delta(t-a) = 0, \;\; \forall t, \; t \neq a$

(ii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)\;dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } a \text{, i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)f(t)\;dt = \begin{cases} f(a) & \text{if the interval } [c,d] \text{ contains } a \text{, i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)\, dt = e^{-as}$$

In particular,

$$L\left[\delta(t)\right] = 1.$$

Remark: *Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice*

$$\mathcal{L}\big[u(t-a)\big] = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{"}\ \frac{d}{dt}\big[u(t-a)\big] = \delta(t-a)\ \text{"}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

10.1.1 Example

Compute $L^{-1}\left[\frac{s+1}{s}\right]$.

Solution: We write,

$$L^{-1}\left[\frac{s+1}{s}\right] = L^{-1}\left[1+\frac{1}{s}\right] = L^{-1}\left[1\right] + L^{-1}\left[\frac{1}{s}\right] = \delta(t) + 1.$$

The resulting object is a generalized function which makes sense only when put under an integral.

10.2 Bessel's Functions

The Bessel's functions of order n (of first kind) is defined as

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{t}{2}\right)^{n+2r}.$$

This Bessel's function is a solution of the Bessel's equation of order n

$$y^{(n)} + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right) y = 0$$

The Bessel's functions of order 0 and 1 are given as

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots$$

and

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2 4} + \frac{t^5}{2^2 4^2 6} + \dots$$

Note that $J_0'(t) = -J_1(t)$.

10.2.1 Example

Find the Laplace transform of $J_0(t)$ and $J_1(t)$.

Solution: Taking Laplace transform of the $J_0(t)$ we have

$$L[J_0(t)] = L\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots\right]$$

Using linearity of the Laplace transform we get

$$L[J_0(t)] = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 4^2}\frac{4!}{s^5} - \frac{1}{2^2 4^2 6^2}\frac{6!}{s^7} + \dots$$

This can be rewritten as

$$L[J_0(t)] = \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{3}{4}\frac{1}{s^4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^6} + \dots\right]$$

With Binomial expansion we can write

$$L[J_0(t)] = \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2} = \frac{1}{\sqrt{1+s^2}}$$

Further note that $L[J_1(t)] = -L[J_0'(t)]$ and therefore using the derivative theorem we find

$$L[J_1(t)] = -sL[J_0(t)] + J_0(0)] = 1 - sL[J_0(t)], \text{ since } J_0(0) = 1$$

Hence, we obtain

$$L[J_1(t)] = 1 - \frac{s}{\sqrt{1+s^2}}$$

10.3 Laguerre Polynomials

Laguerre polynomials are defined as

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right),\ n = 0, 1, 2, \ldots$$

The Laguerre polynomials are solutions of Laguerre's differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0,\ n = 0, 1, 2, \ldots$$

10.3.1 Example

Show that $L[L_n(t)] = \dfrac{(s-1)^n}{s^{n+1}}$

Solution: By definition of the Laplace transform we have

$$\begin{aligned} L[L_n(t)] &= \int_0^\infty e^{-st}\frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \\ &= \frac{1}{n!}\int_0^\infty e^{-(s-1)t}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \end{aligned}$$

Integrating by parts, we find

$$L[L_n(t)] = \frac{1}{n!}\left[e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\Big|_0^\infty + (s-1)\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Noting that each term in $\frac{d^{n-1}}{dt^{n-1}}$ contains some integral power of t so that it vanishes as $t \to 0$ and $e^{-(s-1)t}$ vanishes for $t \to \infty$ provided $s > 1$. Thus, we have

$$L[L_n(t)] = \frac{s-1}{n!}\left[\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Repeated use of integration by parts leads to

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\left[\int_0^\infty e^{-(s-1)t}e^{-t}t^n\,\mathrm{d}t\right] = \frac{(s-1)^n}{n!}L[t^n]$$

Hence, we get

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\frac{n!}{s^{n+1}} = \frac{(s-1)^n}{s^{n+1}}$$

11.1 Miscellaneous Example Problems

11.1.1 Problem 1

Using the convolution theorem prove that

$$B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\quad [m,n>0].$$

Solution: Let $f(t)=t^{m-1},\ g(t)=t^{n-1}$, then

$$(f*g)(t)=\int_0^t \tau^{m-1}(t-\tau)^{n-1}\mathrm{d}\tau,$$

Substituting $\tau=ut$ so that $\mathrm{d}\tau=t\,\mathrm{d}u$ we obtain

$$(f*g)(t)=\int_0^1 t^{m-1}u^{m-1}t^{n-1}(1-u)^{n-1}t\mathrm{d}u$$

We simplify the above expression to get

$$(f*g)(t)=t^{m+n-1}\int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u=t^{m+n-1}B(m,n)$$

Taking Laplace transform and using convolution property, we find

$$L[t^{m+n-1}B(m,n)]=L[f(t)]*L[g(t)]=L[t^{m-1}]*L[t^{n-1}]=\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Taking inverse Laplace transform,

$$t^{m+n-1}B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

Hence, we get the desired result as

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

11.1.2 Problem 2

Show that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t=\frac{\pi}{2}.$$

Solution: We know

$$L[\sin t] = \frac{1}{s^2+1}$$

Therefore, we get

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \frac{\pi}{2} - \tan^{-1} s.$$

Taking limit as $s \to 0$ (see remarks below for details) we find

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}.$$

Remark 1: *Suppose that f is piecewise continuous on $[0, \infty)$ and $L[f(t)] = F(s)$ exists for all $s > 0$, and $\int_0^\infty f(t)\,\mathrm{d}t$ converges. Then $\lim_{s\to 0+} F(s) = \lim_{s\to 0+} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty f(t)\,\mathrm{d}t$.*

Remark 2: *If f is a piecewise continuous function ans $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s)$ converges uniformly for all $s \in E$, then $F(s)$ is a continuous function on E, that is, for $s \to s_0 \in E$,*

$$\lim_{s\to s_0} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s_0) = \int_0^\infty \lim_{s\to s_0} e^{-st} f(t)\,\mathrm{d}t.$$

Remark 3: *Recall that the integral $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is said to converge uniformly for s in some domain Ω if for any $\epsilon > 0$ there exists some number τ_0 such that if $\tau \geq \tau_0$ then*

$$\left|\int_\tau^\infty e^{-st} f(t)\,\mathrm{d}t\right| < \epsilon$$

for all s in Ω.

11.1.3 Problem 3

Using Laplace transform, evaluate the following integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2+a^2}\,dx$$

Solution: Let

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2 + a^2} \, \mathrm{d}x$$

Taking Laplace transform, we get

$$F(s) = \int_0^\infty \frac{x}{x^2 + a^2} \frac{x}{x^2 + s^2} \, \mathrm{d}x$$

Using the method of partial fractions we obtain

$$F(s) = \int_0^\infty \frac{1}{x^2 + s^2} \, \mathrm{d}x - \frac{a^2}{s^2 - a^2} \int_0^\infty \left(\frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} \right) \, \mathrm{d}x$$

Evaluating the above integrals we have

$$F(s) = \frac{1}{s} \tan^{-1} \left(\frac{x}{s} \right) \Big|_0^\infty - \frac{a^2}{s^2 - a^2} \left[\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) - \frac{1}{s} \tan^{-1} \left(\frac{x}{s} \right) \right]_0^\infty$$

On simplification we obtain

$$F(s) = \frac{1}{2} \frac{\pi}{s + a}$$

Taking inverse Laplace transform we find

$$f(t) = \frac{1}{2} \pi e^{-at}$$

Hence the value of the given integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2 + a^2} \, \mathrm{d}x = 2 \int_0^\infty \frac{x \sin xt}{x^2 + a^2} \, \mathrm{d}x = \pi e^{-at}.$$

11.1.4 Problem 4

Evaluate $\displaystyle\int_0^\infty \frac{\cos tx}{x^2 + 1} \, \mathrm{d}x, \quad t > 0.$

Solution: Let

$$f(t) = \int_0^\infty \frac{\cos tx}{x^2 + 1} \, \mathrm{d}x.$$

Taking Laplace transform on both sides,

$$\begin{aligned} L[f(t)] &= \int_0^\infty \frac{s}{(x^2 + 1)(s^2 + x^2)} \, \mathrm{d}x \\ &= \frac{s}{s^2 + 1} \int_0^\infty \left(\frac{1}{x^2 + 1} - \frac{1}{s^2 + x^2} \right) \, \mathrm{d}x \\ &= \frac{s}{s^2 - 1} \left[\tan^{-1} x - \frac{1}{s} \tan^{-1} \left(\frac{1}{s} \right) \right]_0^\infty \\ &= \frac{s}{s^2 - 1} \left(\frac{\pi}{2} - \frac{\pi}{2s} \right) = \frac{\pi}{2} \frac{1}{s + 1}. \end{aligned}$$

Taking inverse Laplace transform on both sides,

$$f(t) = \frac{\pi}{2} e^{-t}.$$

11.1.5 Problem 5

Evaluate $\int_0^\infty e^{-x^2} \mathrm{d}x.$

Solution: Let

$$g(t) = \int_0^\infty e^{-tx^2} \, \mathrm{d}x$$

Now taking Laplace on both sides,

$$L[g(t)] = \int_0^\infty \frac{1}{s + x^2} \, \mathrm{d}x = \frac{1}{\sqrt{s}} \arctan\left(\frac{x}{\sqrt{s}}\right) \Big|_0^\infty = \frac{1}{\sqrt{s}} \frac{\pi}{2}$$

Taking inverse Laplace transform we obtain

$$g(t) = \frac{\pi}{2} L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{\pi}{2} \frac{1}{\sqrt{\pi}\sqrt{t}}.$$

Hence for $t = 1$ we get

$$\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

In previous lessons we have evaluated Laplace transforms and inverse Laplace transform of various functions that will be used in this and following lessons to solve ordinary differential equations. In this lesson we mainly solve initial value problems.

12.1 Solving Differential/Integral Equations

We perform the following steps to obtain the solution of a differential equation.

(i) Take the Laplace transform on both sides of the given differential/integral equations.

(ii) Obtain the equation $L[y] = F(s)$ from the transformed equation.

(iii) Apply the inverse transform to get the solution as $y = L^{-1}[F(s)]$.

In the process we assume that the solution is continuous and is of exponential order so that Laplace transform exists. For linear differential equations with constant coefficients one can easily prove that under certain assumption that the solution is continuous and is of exponential order. But for the ordinary differential equations with variable coefficients we should be more careful. The whole procedure of solving differential equations will be clear with the following examples.

12.2 Example Problems

12.2.1 Problem 1

Solve the following initial value problem

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y^{'}(0) = 0.$$

Solution: Take the Laplace transform on both sides, we get

$$L[y^{''}] + L[y] = L[1]$$

Using derivative theorems we find

$$s^2 L[y] - sy(0) - y^{'}(0) + L[y] = L[1]$$

We plug in the initial conditions now to obtain

$$L[y]\left(1 + s^2\right) = \frac{1}{s} \Rightarrow L[y] = \frac{1}{s\left(1 + s^2\right)}$$

Using partial fractions we obtain

$$L[y] = \frac{1}{s} - \frac{s}{1+s^2}$$

Taking inverse Laplace transform we get

$$y(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s}{1+s^2}\right] = 1 - \cos t$$

12.2.2 Problem 2

Solve the initial value problem

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

Solution: We will take the Laplace transform on both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} L[x''(t) + x(t)] &= L[\cos(2t)], \\ s^2X(s) - sx(0) - x'(0) + X(s) &= \frac{s}{s^2+4}. \end{aligned}$$

Plugging the initial conditions, we obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

We now solve for $X(s)$ as

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}$$

We use partial fractions to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

12.2.3 Problem 3

Solve the following initial value problem

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}, \quad y(0) = 2,\, y^{'}(0) = 6.$$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y^{'}(0) - 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{2}{(s-3)^3}$$

Using initial values we obtain

$$s^2Y(s) - 2s - 6 - 6\left[sY(s) - 2\right] + 9Y(s) = \frac{2}{(s-3)^3}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2(s-3)}{(s-3)^2}$$

Taking inverse Laplace transform, we find

$$y(t) = \frac{2}{4!}t^4e^{3t} + 2e^{3t} = \frac{1}{12}t^4e^{3t} + 2e^{3t}.$$

12.2.4 Problem 4

Solve

$$y'' + y = CH(t-a), \quad y(0) = 0,\, y^{'}(0) = 1.$$

Solution: Taking Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y^{'}(0) + Y(s) = C\int_a^{\infty} e^{-st}\,\mathrm{d}t$$

We substitute the given initial values to obtain

$$(s^2+1)Y(s) = 1 + C\frac{e^{-as}}{s}$$

Solve for $Y(s)$ as

$$Y(s) = \frac{1}{s^2+1} + C\frac{e^{-as}}{s(s^2+1)}$$

Method of partial fractions leads to

$$y(t) = \sin t + CL^{-1}\left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-as}\right]$$

By inverse Laplace transform we obtain

$$y(t) = \sin t + CH(t-a)[1-\cos(t-a)].$$

12.2.5 Problem 5

Solve the following initial value problem

$$x''(t) + x(t) = H(t-1) - H(t-5), \quad x(0) = 0, \quad x'(0) = 0,$$

Solution: We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We can easily show that

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t.$$

In other words $L[1-\cos t] = \dfrac{1}{s(s^2+1)}$. So using the shifting theorem we find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-s}L[1-\cos t]\right] = \left[1-\cos(t-1)\right]H(t-1).$$

Similarly, we have

$$L^{-1}\left[\frac{e^{-5s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-5s}L[1-\cos t]\right] = \left[1-\cos(t-5)\right]H(t-5).$$

Hence, the solution is

$$x(t) = \left[1-\cos(t-1)\right]H(t-1) - \left[1-\cos(t-5)\right]H(t-5).$$

In this lesson we continue the application of Laplace transform for solving initial and boundary value problems. In this lesson we will also look for differential equations with variable coefficients and some boundary value problems.

13.0.1 Problem 1

Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-3)H(t-3), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution: Recall the second shifting theorem

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

We now apply the Laplace transform to the differential equation to get

$$s^2Y(s) - sy(0) - y^{'}(0) + 2\left[sY(s) - y(0)\right] + 2Y(s) = e^{-3s}$$

Plugging the initial values we find

$$\left[s^2 + 2s + 2\right]Y(s) = e^{-3s}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{1}{[(s+1)^2+1]}e^{-3s}$$

Taking inverse Laplace transform with the use of first and second shifting properties we obtain

$$y(t) = L^{-1}\left[\frac{1}{[(s+1)^2+1]}e^{-3s}\right] = H(t-3)e^{-(t-3)}\sin(t-3).$$

13.0.2 Problem 2

Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

Solution: We first apply the Laplace transform to the equation. Denoting the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual, we have

$$s^2X(s) + \omega_0^2X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2 + \omega_0^2}.$$

We know

$$L^{-1}\left[\frac{1}{s^2 + \omega_0^2}\right] = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore, using the convolution theorem, we find

$$x(t) = \int_0^t f(\tau)\frac{\sin\big(\omega_0(t-\tau)\big)}{\omega_0}\,d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0 t)}{\omega_0} f(t-\tau)\,d\tau.$$

13.0.3 Problem 3

Find the general solution of

$$y'' + y = e^{-t}.$$

Solution: Taking Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s+1}$$

Denoting $y(0)$ by y_0 and $y'(0)$ by y_1 we find

$$(s^2+1)Y(s) - sy_0 - y_1 = \frac{1}{s+1}$$

Now we solve for $Y(s)$ to obtain

$$Y(s) = \frac{1}{(s+1)(s^2+1)} + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Method of partial fractions leads to

$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s-1}{s^2+1}\right] + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Taking the inverse transform we get

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t + y_0\cos t + y_1\sin t$$

This can be rewritten as

$$y(t) = \frac{1}{2}e^{-t} + \left(y_0 - \frac{1}{2}\right)\cos t + \left(y_1 + \frac{1}{2}\right)\sin t$$

Note that y_0 and y_1 are arbitrary, so the general solution is given by

$$y(t) = \frac{1}{2}e^{-t} + C_0 \cos t + C_1 \sin t.$$

13.0.4 Problem 4

Solve the following boundary value problem

$$y'' + y = \cos t, \quad y(0) = 1, y\left(\frac{\pi}{2}\right) = 1.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2Y(s) - sy(0) - y^{'}(0) + Y(s) = \frac{s}{s^2+1}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{s}{(s^2+1)^2} + \frac{s}{s^2+1} + \frac{y^{'}(0)}{s^2+1}$$

Taking inverse Laplace transform on both sides we get,

$$y(t) = \frac{1}{2}t\sin t + \cos t + y^{'}(0)\sin t.$$

Given $y(\frac{\pi}{2}) = 1$, therefore

$$1 = \frac{1}{2}\frac{\pi}{2} + 0 + y^{'}(0) \Rightarrow y^{'}(0) = \left(1 - \frac{\pi}{4}\right).$$

Hence, we obtain the solution as

$$y(t) = \frac{1}{2}t\sin t + \cos t + \left(1 - \frac{\pi}{4}\right)\sin t.$$

13.0.5 Problem 5

Solve the following fourth order initial value problem

$$\frac{d^4y}{dx^4} = -\delta(x-1),$$

with the initial conditions

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

Solution: We apply the transform and get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) + C_1x + \frac{C_2}{6}x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6}2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3}C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6}3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

We now demonstrate the potential of Laplace transform for solving ordinary differential equations with variable coefficients.

13.0.6 Problem 6

Solve the initial value problem

$$y^{''} + ty^{'} - 2y = 4; \qquad y(0) = -1,\ y^{'}(0) = 0.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2Y(s) - sy(0) - y^{'}(0) + \left(-\frac{d}{ds}L[y^{'}]\right) - 2Y(s) = 4L[1]$$

Using the given initial values and applying derivative theorem once again, we get

$$s^2Y(s) + s - \frac{d}{ds}\left(sY(s) - y(0)\right) - 2Y(s) = \frac{4}{s}$$

On simplification we find the following differential equation

$$\frac{dY}{ds} + \left(\frac{3}{s} - s\right)Y(s) = -\frac{4}{s^2} + 1$$

Integrating factor of the above differential equation is given as

$$e^{\int\left(\frac{3}{s}-s\right)\mathrm{d}s} = s^3e^{-\frac{s^2}{2}}$$

Hence, the solution of the differential equation can be written as

$$Y(s)s^3e^{-\frac{s^2}{2}} = \int\left(-\frac{4}{s^2} + 1\right)s^3e^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

On integration we find

$$Y(s)s^3e^{-\frac{s^2}{2}} = 4e^{-\frac{s^2}{2}} - \left(s^2e^{-\frac{s^2}{2}}\right) + \int 2se^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

We can simplify the above expression to get

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \left(\frac{c}{s^3}\right)e^{\frac{s^2}{2}}$$

Since, $Y(s) \to 0$ as $s \to \infty, c$ must be zero. Putting $c = 0$ and taking inverse Laplace transform we get the desired solution as

$$y(t) = t^2 - 1$$

13.0.7 Problem 7

Solve the initial value problem

$$ty^{''}+y^{'}+ty=0;\ \ y(0)=1,\ y^{'}(0)=0$$

Solution: Taking Laplace transform on both sides we get,

$$-\frac{d}{ds}L[y^{''}]+L[y^{'}]+\left(-\frac{d}{ds}L[y]\right)=0$$

Application of derivative theorem leads to

$$-\frac{d}{ds}\left\{s^2Y(s)-sy(0)-y^{'}(0)\right\}+\{sY(s)-y(0)\}-\frac{d}{ds}Y(s)=0$$

Plugging initial values, we find

$$\left(s^2+1\right)Y^{'}(s)+sY(s)=0$$

On integration we get

$$Y(s)=\frac{c}{\sqrt{1+s^2}}$$

Taking inverse Laplace transform we find

$$y(t)=cJ_0(t)$$

Noting $y(0)=1,\ J_0(0)=1$, we find $c=1$. Thus, the required solution is

$$y(t)=J_0(t).$$

In this lesson we discuss application of Laplace transform for solving integral equations, integro-differential equations and simultaneous differential equations.

14.1 Integral Equation

An equation of the form

$$f(t) = g(t) + \int_0^t K(t,u)f(u)\,\mathrm{d}u,$$

or

$$g(t) = \int_0^t K(t,u)f(u)\,\mathrm{d}u$$

are known as the integral equations, where $f(t)$ is the unknown function. When the kernel $K(t,u)$ is of the particular form $K(t,u) = K(t-u)$ then the equations can be solved using Laplace transforms. We apply the Laplace transform to the first equation to obtain

$$F(s) = G(s) + K(s)F(s),$$

where $F(s)$, $G(s)$, and $K(s)$ are the Laplace transforms of $f(t)$, $g(t)$, and $K(t)$ respectively. Solving for $F(s)$, we find

$$F(s) = \frac{G(s)}{1-K(s)}.$$

To find $f(t)$ we now need to find the inverse Laplace transform of $F(s)$. Similar steps can be followed to solve the integral equation of second type mentioned above.

14.2 Example Problems

14.2.1 Problem 1

Solve the following integral equation

$$f(t) = e^{-t} + \int_0^t \sin(t-u)f(u)\,\mathrm{d}u.$$

Solution: Applying Laplace transform on both sides and using convolution theorem we get,

$$L[f(t)] = \frac{1}{s+1} + L[\sin t]L[f(t)]$$

On simplifications, we obtain

$$L[f(t)]\left[1-\frac{1}{s^2+1}\right]=\frac{1}{s+1}$$

This further implies

$$L[f(t)]=\frac{s^2+1}{s^2(s+1)}$$

Partial fractions leads to

$$L[f(t)]=\frac{2}{s+1}+\frac{1}{s^2}-\frac{1}{s}$$

Taking inverse Laplace transform we obtain the desired solution as

$$f(t)=2e^{-t}+t-1$$

14.2.2 Problem 2

Solve the differential equation

$$x(t)=e^{-t}+\int_0^t \sinh(t-\tau)x(\tau)\,d\tau.$$

Solution: We apply Laplace transform to obtain

$$X(s)=\frac{1}{s+1}+\frac{1}{s^2-1}X(s),$$

or

$$X(s)=\frac{\frac{1}{s+1}}{1-\frac{1}{s^2-1}}=\frac{s-1}{s^2-2}=\frac{s}{s^2-2}-\frac{1}{s^2-2}.$$

It is not difficult to take inverse Laplace transform to find

$$x(t)=\cosh(\sqrt{2}\,t)-\frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,t).$$

14.2.3 Problem 3

Solve the following integral equation for $x(t)$

$$t^2=\int_0^t e^{\tau}x(\tau)\,d\tau,$$

Solution: We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s}\,L[e^t x(t)] = \frac{1}{s}\,X(s-1),$$

where $X(s) = L[x(t)]$. Thus, we have

$$X(s-1) = \frac{2}{s^2} \qquad \text{or} \qquad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again to obtain

$$x(t) = 2e^{-t}t.$$

14.3 Integro-Differential Equations

In addition to the integral we have a differential term in the integro differential equations. The idea of solving ordinary differential equations and integral equations are now combined. We demonstrate the procedure with the help of the following example.

14.3.1 Example

Solve

$$\frac{dy}{dt} + 4y + 13\int_0^t y(u)\,\mathrm{d}u = 3e^{-2t}\sin 3t, \qquad y(0) = 3.$$

Solution: Taking Laplace transform and using its appropriate properties we obtain,

$$sY(s) - y(0) + 4Y(s) + 13\frac{Y(s)}{s} = 3\frac{3}{(s+2)^2+9}.$$

Collecting terms of $Y(s)$ we get

$$\frac{s^2+4s+13}{s}Y(s) = \frac{9}{(s+2)^2+9} + 3$$

On simplification we have

$$Y(s) = \frac{9s}{[(s+2)^2+9]^2} + \frac{3s}{(s+2)^2+9}$$

Taking inverse Laplace transform and using shifting theorem we get

$$y(t) = e^{-2t} L^{-1} \left[\frac{9(s-2)}{(s^2+9)^2} + \frac{3(s-2)}{s^2+9} \right].$$

We now break the functions into the known forms as

$$\begin{aligned} y(t) =& e^{-2t} L^{-1} \left[\frac{9s}{(s^2+9)^2} - \frac{18}{(s^2+9)^2} + \frac{3s}{s^2+9} + \frac{1}{s^2+9} - \frac{7}{s^2+9} \right] \\ =& e^{-2t} L^{-1} \left[\frac{9s}{(s^2+9)^2} + \frac{s^2-9}{(s^2+9)^2} + \frac{3s}{s^2+9} - \frac{7}{s^2+9} \right] \end{aligned}$$

Using the the following basic inverse transforms

$$L^{-1} \left[\frac{a}{s^2+a^2} \right] = \sin at, \quad L^{-1} \left[\frac{s}{s^2+a^2} \right] = \cos at$$

$$L^{-1} \left[\frac{2as}{(s^2+a^2)^2} \right] = t \sin at, \quad L^{-1} \left[\frac{s^2-a^2}{(s^2+a^2)^2} \right] = t \cos at.$$

We find the desired solution as

$$y(t) = e^{-2t} \left[\frac{3}{2} t \sin 3t + t \cos 3t + 3 \cos 3t - \frac{7}{3} \sin 3t \right]$$

14.4 Simultaneous Differential Equations

At the end we show with the help of an example the application of Laplace transform for solving simultaneous differential equations.

14.4.1 Example

Solve

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = y - 2x$$

subject to the initial conditions

$$x(0) = 8, \quad y(0) = 3.$$

Solution: Taking Laplace transform on both sides we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

and

$$sY(s) - y(0) = Y(s) - 2X(s)$$

Collecting terms of $X(s)$ and $Y(s)$ we have the following equations

$$(s-2)X(s) + 3Y(s) = 8 \tag{14.1}$$

$$2X(s) + (s-1)Y(s) = 3 \tag{14.2}$$

Eliminating $Y(s)$ we obtain

$$\left[(s-1)(s-2) - 6\right] X(s) = 8(s-1) - 9$$

On simplifications we receive

$$X(s) = \frac{8s - 17}{(s-4)(s+1)}$$

Partial fractions lead to

$$X(s) = \frac{5}{s+1} + \frac{3}{s-4},$$

Taking inverse Laplace transform both sides we get

$$x(t) = 5e^{-t} + 3e^{4t}$$

Now we solve the above equations (14.1) and for (14.2) $Y(s)$

$$\left[6 - (s-1)(s-2)\right] Y(s) = 16 - 3(s-2)$$

On simplifications we get

$$Y(s) = \frac{3s - 22}{s^2 - 3s - 4} = \frac{3s - 22}{(s-4)(s+1)}$$

Using the method of partial fractions we obtain

$$Y(s) = \frac{5}{s+1} - \frac{2}{s-4}$$

Taking inverse transform we get

$$y(t) = 5e^{-t} - 2e^{4t}.$$

In this last lesson of this module we demonstrate the potential of Laplace transform for solving partial differential equations. If one of the independent variables in partial differential equations ranges from 0 to ∞ then Laplace transform may be used to solve partial differential equations.

15.1 Solving Partial Differential Equations

Working steps are more or less similar to what we had for solving ordinary differential equations. We take the Laplace transform with respect to the variable that ranges from 0 to ∞. This will convert the partial differential equation into an ordinary differential equation. Then, the transformed ordinary differential equation must be solved considering the given conditions.

Denoting the Laplace transform of unknown variable $u(x,t)$ with respect to t by $U(x,s)$ and using the definition of Laplace transform we have

$$U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\,\mathrm{d}t$$

Then, for the first order derivatives, we have

$$(i)\;\; L\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,\mathrm{d}t = \frac{d}{dx}\int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t = \frac{dU}{dx}$$

$$\begin{aligned}(ii)\;\; L\left[\frac{\partial u}{\partial t}\right] &= \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,\mathrm{d}t = e^{-st}u\Big|_0^\infty - \int_0^\infty u(-s)e^{-st}\,\mathrm{d}t \\ &= -u(x,0) + s\int_0^\infty ue^{-st}\,\mathrm{d}t \\ \Rightarrow\;\; L\left[\frac{\partial u}{\partial t}\right] &= -u(x,0) + sU(x,s)\end{aligned}$$

$$(iii)\;\; L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2U}{dx^2}$$

$$(iv)\;\; L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

$$(v)\;\; L\left[\frac{\partial^2 u}{\partial x\partial t}\right] = s\frac{d}{dx}U(x,s) - \frac{d}{dx}u(x,0)$$

Remark: *In order to derive the above results, besides the assumptions of piecewise continuity and exponential order of $u(x,t)$ with respect to t, we have also used the following assumptions: (i) The differentiation under integral sign is valid and (ii) The limit of the Laplace transform is the Laplace transform of the limit, i.e.,* $\lim_{x\to x_0} L\,[u(x,t)] = L\,[\lim_{x\to x_0} u(x,t)]$.

15.2 Example Problems

15.2.1 Problem 1

Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad u(x,0) = x,\; u(0,t) = t$$

Solution: Taking Laplace transform

$$\frac{d}{dx}U(x,s) = sU(x,s) - u(x,0)$$

Using the initial values we get

$$\frac{d}{dx}U(x,s) - sU(x,s) = -x$$

The integrating factor is

$$I.F. = e^{-\int s\,\mathrm{d}x} = e^{-sx}$$

Hence, the solution can be written as

$$U(x,s)e^{-sx} = -\int xe^{-sx}\,\mathrm{d}x + c$$

On integration by parts we find

$$U(x,s)e^{-sx} = -x\frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{s}\,\mathrm{d}x + c$$

Simplify, the above expression we have

$$U(x,s) = \frac{x}{s} + \frac{1}{s^2} + ce^{sx}$$

Using given boundary condition we find

$$\frac{1}{s^2} = \frac{1}{s^2} + c \cdot 1 \quad \Rightarrow \quad c = 0$$

With this we obtain

$$U(x,s) = \frac{x}{s} + \frac{1}{s^2}$$

Taking inverse Laplace transform, we find the desired solution as

$$u(x,t) = x + t$$

15.2.2 Problem 2

Solve the following partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \quad x > 0,\, t > 0$$

with the following initial and boundary condition

$$u(x,0) = 0,\, x > 0 \text{ and } u(0,t) = 0,\, t > 0$$

Solution: Taking Laplace transform with respect to t we have

$$sU(x,s) - u(x,0) + x\frac{d}{dx}U(x,s) = \frac{x}{s}, \quad s > 0$$

Using the given initial value we find

$$\frac{d}{dx}U(x,s) + \frac{s}{x}U(x,s) = \frac{1}{s}$$

Its integrating factor is x^s and therefore the solution can be written as

$$U(x,s)x^s = \int \frac{1}{s}x^s \,\mathrm{d}x + c \quad \Rightarrow \quad U(x,s) = \frac{1}{s(s+1)}x + \frac{c}{x^s}$$

Boundary condition provides

$$u(0,t) = 0 \quad \Rightarrow \quad U(0,s) = 0, \quad \Rightarrow \quad c = 0$$

Thus we have

$$U(x,s) = \frac{x}{s(s+1)} = x\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t) = x\left[1 - e^{-t}\right]$$

15.2.3 Problem 3

Solve the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\, t > 0$$

with the initial and boundary conditions

$$u(x,0) = 1,\, u(0,t) = 0,\, \lim_{x\to\infty} u(x,t) = 1$$

Solution: Taking Laplace transform we find

$$sU(x,s) - u(x,0) = \frac{d^2}{dx^2}U(x,s)$$

Using the given initial condition we have

$$\frac{d^2}{dx^2}U(x,s) - sU(x,s) = -1$$

Its solution is given as

$$U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s}$$

The given boundary conditions give

$$\lim_{x\to\infty} U(x,s) = \frac{1}{s} \quad \Rightarrow \quad c_1 = 0$$

and

$$U(0,s) = 0 \quad \Rightarrow \quad c_1 + c_2 + \frac{1}{s} = 0 \quad \Rightarrow \quad c_2 = -\frac{1}{s}$$

Hence, we have

$$U(x,s) = -\frac{1}{s}e^{-\sqrt{s}x} + \frac{1}{s}$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t) = 1 - L^{-1}\left[\frac{1}{s}e^{-\sqrt{s}x}\right] = 1 - \left[1 - \text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] = \text{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

15.2.4 Problem 4

Solve the one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0,\ t > 0$$

with the initial conditions

$$y(x, 0) = 0,\ y_t(x, 0) = 0$$

and boundary conditions

$$y(0, t) = \sin \omega t,\ \lim_{x \to \infty} y(x, t) = 0$$

Solution: Taking Laplace transform we get

$$s^2 Y(x, s) - s y(x, 0) - y_t(x, 0) - a^2 \frac{d^2}{dx^2} Y(x, s) = 0$$

With the given initial condition we have the resulting differential equation

$$\frac{d^2 Y}{dx^2} - \frac{s^2}{a^2} = 0$$

Its general solution is given as

$$Y(x, s) = c_1 e^{\frac{s}{a} x} + c_2 e^{-\frac{s}{a} x}$$

The given boundary conditions provides

$$\lim_{x \to \infty} Y(x, s) = 0 \quad \Rightarrow \quad c_1 = 0,$$

and

$$Y(0, s) = \frac{\omega}{s^2 + \omega^2} \quad \Rightarrow \quad c_2 = \frac{\omega}{s^2 + \omega^2}$$

Thus we have

$$Y(x, s) = \frac{\omega}{s^2 + \omega^2} e^{-\frac{s}{a} x}$$

Taking inverse Laplace transform we obtain

$$y(x, t) = \sin \left[\omega \left(t - \frac{x}{a} \right) \right] H \left(t - \frac{x}{a} \right).$$