Inner product spaces

Prove $\mathbb{R}^n$ is an inner product space

$\mathbb{R}^n$ over $\mathbb{R}$ (vector space)

standard inner product $\sum a_i b_i$

$\langle \underline{u}, \underline{v} \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

where $\underline{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\underline{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, $a_i, b_i \in \mathbb{R}$

① linearity :- $\langle \underline{u}+\underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$

Let $\underline{w} = [c_1 \;\; c_2 \; \dots \; c_n]^T$, $c_i \in \mathbb{R}$

$\underline{u}+\underline{v} = [a_1+b_1 \quad a_2+b_2 \quad a_3+b_3 \; \dots \; a_n+b_n]^T$

$$\langle \underline{u}+\underline{v}, \underline{w} \rangle = (a_1+b_1)c_1 + (a_2+b_2)c_2 + \dots + (a_n+b_n)c_n$$

$$= a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2 + \dots + a_nc_n + b_nc_n$$

$$= (a_1c_1 + a_2c_2 + \dots + a_nc_n) + (b_1c_1 + b_2c_2 + \dots b_nc_n)$$

$$= \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$$

$\alpha \underline{u} = [\alpha a_1 \quad \alpha a_2 \; \dots \; \alpha a_n]^T$

$$\langle \alpha\underline{u}, \underline{u} \rangle = \alpha a_1 b_1 + \alpha a_2 b_2 + \alpha a_3 b_3 + \dots + \alpha a_n b_n$$

$$= \alpha(a_1b_1 + a_2b_2 + \dots + a_nb_n)$$

$$= \alpha \langle \underline{u}, \underline{v} \rangle$$

② Symmetry :

$\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

$\langle \underline{u}, \underline{v} \rangle = a_1b_1 + a_2b_2 + \dots + a_nb_n$

since $a_i, b_i \in \mathbb{R}$

$= b_1a_1 + b_2a_2 + \dots + b_na_n$

$= \langle \underline{v}, \underline{u} \rangle$

③ Positive definiteness

$\langle \underline{u}, \underline{u} \rangle \geq 0$

0 iff $\underline{u} = \underline{0}$

$\langle \underline{u}, \underline{u} \rangle = a_1^2 + a_2^2 + \dots + a_n^2$

$\forall a_i \in \mathbb{R}$

$a_i^2 > 0 \Rightarrow \sum a_i^2 > 0$

$\therefore \langle \underline{u}, \underline{u} \rangle > 0$.

- If nothing is mentioned in the question, consider standard inner product case ✓ i.e $\sum a_i b_i$

Ex :- $\underline{u} = \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 3-i \\ 4+i \end{bmatrix}$

$\langle \underline{u}, \underline{v} \rangle = (1+i)(3-i) + (2-i)(4+i) = 13 + 10i$

we cannot compare complex no's

$\therefore$

$\Rightarrow \langle \underline{u}, \underline{u} \rangle = (1+i)^2 + (2-i)^2 =$ complex no: $\rightarrow$ don't know whether it is +ve or -ve

multiply with its complex conjugate

$\langle \underline{u}, \overline{\underline{u}} \rangle = (1+i)(1-i) + (2-i)(2+i) = 7 > 0$

$\Rightarrow$ standard inner product :- $\langle \underline{u}, \underline{v} \rangle = a_1\overline{b_1} + a_2\overline{b_2} + \dots + a_n\overline{b_n}$

$\langle \underline{u}, \underline{v} \rangle = a_1\overline{b_1} + a_2\overline{b_2} + \dots + a_n\overline{b_n}$

$= \overline{b_1}a_1 + \overline{b_2}a_2 + \dots + \overline{b_n}a_n$

symmetry for complex no:

$= \overline{\langle \underline{v}, \underline{u} \rangle} = \overline{b_1}a_1 + \overline{b_2}a_2 + \dots + \overline{b_n}a_n$

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$$\langle \alpha \underline{u}, \underline{v} \rangle = \alpha a_1 \bar{b_1} + \alpha a_2 \bar{b_2} + \dots + \alpha a_n \bar{b_n}$$

$$= \alpha \langle \underline{u}, \underline{v} \rangle \quad \text{--- (2)}$$

$$\Rightarrow \boxed{\langle \underline{u}, \alpha \underline{v} \rangle = \overline{\langle \alpha \underline{v}, \underline{u} \rangle}} \rightarrow \text{Symmetry}$$

$$= \bar{\alpha} \overline{\langle \underline{v}, \underline{u} \rangle}$$

$$\boxed{\langle \underline{u}, \alpha \underline{v} \rangle = \bar{\alpha} \langle \underline{u}, \underline{v} \rangle} \rightarrow \text{(1)}$$

Q) Find $\langle \underline{u}, \underline{v} + \underline{w} \rangle$ for real and complex numbers

$\underline{u} = [a_1 \;\; a_2 \; \dots \; a_n]^T \quad \underline{v} = [b_1 \;\; b_2 \; \dots \; b_n]^T$

$\underline{w} = [c_1 \;\; c_2 \; \dots \; c_n]^T$

$\langle \underline{u}, \underline{v} + \underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ → real numbers

$= \langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle$ → symmetry

$= \langle \underline{v} + \underline{w}, \underline{u} \rangle$

→ $\langle \underline{u}, \underline{v} + \underline{w} \rangle = \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$ → complex no.

$\overline{\langle \underline{v}, \underline{u} \rangle} + \overline{\langle \underline{w}, \underline{u} \rangle}$

$= \overline{\langle \underline{v} + \underline{w}, \underline{u} \rangle}$

$\langle \underline{u}, \underline{v} + \underline{w} \rangle = \overline{\langle \underline{v} + \underline{w}, \underline{u} \rangle}$

$= \overline{\langle \underline{v}, \underline{u} \rangle} + \overline{\langle \underline{w}, \underline{u} \rangle}$

$= \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

2) Consider a vector space of all continuous functions defined on the interval $[a, b]$

$$< f(x), g(x) > = \int_a^b f(x)\, \overline{g(x)}\, dx \rightarrow$$ holds true always (standard inner product)

Sol) ① $< f+g, h > = \int_a^b (f+g)\bar{h}\, dx$

(I)
$$= \int_a^b f\bar{h}\, dx + \int_a^b g\bar{h}\, dx$$

$$= <f, h> + <g, h>$$

② $$<\alpha f , g> = \int_a^b \alpha f \bar{g}\, dx = \alpha \int_a^b f\bar{g}\, dx = \alpha <f, g>$$

need not prove both

$$<f, \alpha g> = \int_a^b f\, \bar{\alpha}\, \bar{g}\, dx = \bar{\alpha} \int_a^b f\, \bar{g}\, dx = \bar{\alpha} <f, g>$$

II) symmetry

③ $$<f, g> = \int_a^b f\bar{g}\, dx$$

$$< g, f > = \int_a^b \bar{f} g\, dx$$

$$\overline{<g, f>} = \overline{\int_a^b \bar{f} g\, dx} = \int_a^b f\, \bar{g}\, dx$$

(III) positive definiteness

$$\Leftrightarrow \quad <f, f> = \int_a^b \underbrace{f \cdot \bar{f}}\, dx > 0 \checkmark$$

analyse it numerically

$\sum f \bar{f} \checkmark$

→ $\langle \underline{u}, \alpha \underline{v} \rangle = \overline{\langle \alpha \underline{v}, \underline{u} \rangle} = \bar{\alpha}\, \overline{\langle \underline{v}, \underline{u} \rangle}$

$= \bar{\alpha} \langle \underline{u}, \underline{v} \rangle$

→ const, variable, parameter

$y = mx + c$ $x, y \to$ variable

parameter

$y = 2x + 3$

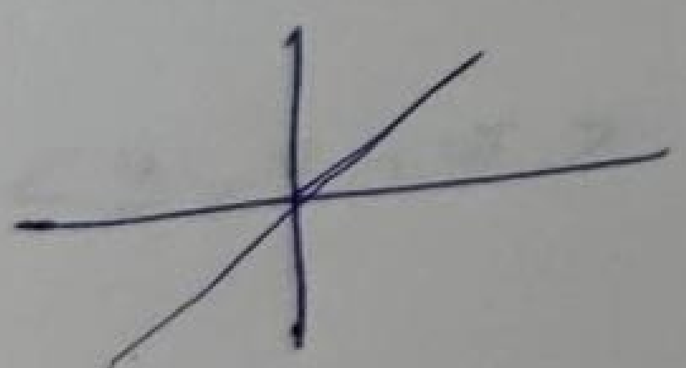
const

Q) if $f(x) = x$, $g(x) = e^{-ix}$ (if limits are not given find the domain)

$\langle f(x), g(x) \rangle = \overline{\langle g(x), f(x) \rangle}$

$\langle f, g \rangle = \int_{-\infty}^{\infty} x e^{ix} dx = \left[x \frac{e^{ix}}{i} \right]_{-\infty}^{\infty} - \int \frac{e^{ix}}{i} dx$

$= \left[x \frac{e^{ix}}{i} + e^{ix} \right]_{-\infty}^{+\infty}$



$=$

$\overline{\langle g, f \rangle} = \overline{\int_{-\infty}^{\infty} e^{-ix} x \, dx} = \overline{\left[x \frac{e^{-ix}}{-i} \right] - \int \frac{e^{-ix}}{-i} dx}$

$= \overline{\left(\frac{x e^{-ix}}{-i} \right]_{-\infty}^{\infty} + \left(\frac{e^{-ix}}{i} \right]_{-\infty}^{\infty}}$

Q) find the value of k so that the following is an inner product.

$$< \underline{u}, \underline{v} > = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + k x_2 y_2$$

Sol) $\underline{u} = [x_1 \; x_2]^T \quad \underline{v} = [y_1 \; y_2]^T \quad \underline{w} = [x_3 \; y_3]^T$

$$\cancel{< \underline{u} + \underline{v}, \underline{w} > = (x_1+x_2)(y_1+y_2) - 3(x_1+x_2) y_3 - 3 x_3 (y_1+y_2) + k (x_3) y_3.}$$

$$\cancel{= x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 - 3x_1 y_3 - 3x_2 y_3 - 3x_3 y_1 - 3x_3 y_2 + k x_3 y_3.}$$

will not give anything (①, ②)

⇒ +ve definiteness :-

$$< \underline{u}, \underline{u} > > 0$$

$$\cancel{x_1^2 - 3x_1} \quad \cancel{x_1 y_1 - 3x_1 y_1 - 3x_1 y_1 + k x_1 y_1}$$

$$x_1^2 - 6x_1 x_2 + x_2^2 k > 0$$

$$\left(\frac{x_1}{x_2}\right)^2 - 6\left(\frac{x_1}{x_2}\right) + k > 0$$

$$\frac{6 \pm \sqrt{36 - 4k}}{2}$$

$$y^2 - 6y + k > 0$$

$$2y - 6 = 0$$

$$y = 3$$

min > 0

$$9 - 6(3) + k > 0$$

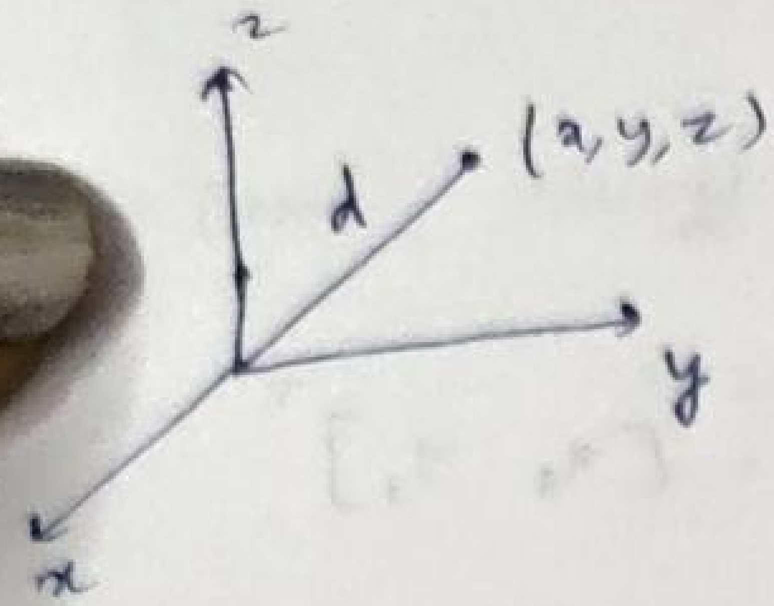
$$\boxed{k > 9}$$

$$36 - 4k > 0$$

$$\boxed{k > +9}$$

Discriminant

$< \underline{u}, \underline{u} > = a_1^2 + a_2^2 + \dots + a_n^2 \rightarrow$ magnitude



$d = \sqrt{x^2+y^2+z^2}$

$\sqrt{< \underline{u}, \underline{u} >}$ ✓ → vectors

$\Rightarrow < f, g > = \int_a^b f(x)\, \overline{g(x)}\, dx$

$\sqrt{< f(x), f(x) >}$ → magnitude of a function
↓ norm

→ direction → ??
↘ angle between (2)

→ $< \underline{u}, \underline{v} > = 0 \rightarrow 90^\circ$ → like dot product
↘ They are orthogonal

→ norm = unity, 90° → orthonormal

Q) Verify whether the following functions lie in $L^2(0, \infty)$

↓
space of square integrable functions

(i) $f(x) = \frac{1}{1+x}$

(ii) $f(x) = e^x$.

→ Functions which satisfy following criteria are said to be space of square integrable functions

$\int_a^b |f(x)|^2 dx$ is finite : $\int_a^b |f(x)|^2 dx < \infty$

$$\int_0^\infty \left(\frac{1}{1+x}\right)^2 dx = \left[-\frac{1}{1+x}\right]_0^\infty = 1 \rightarrow \text{belongs}$$

$$\int_0^\infty e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^\infty = \infty. \rightarrow \text{does not belong.}$$

$\Rightarrow \int_a^b (f(x))^2 dx < \infty.$

— Complex domain :- $\int_a^b f(x) \cdot \overline{f(x)}\, dx$

$\rightarrow \langle f(x), f(x) \rangle = \| f(x) \|$ square of norm of f(x).

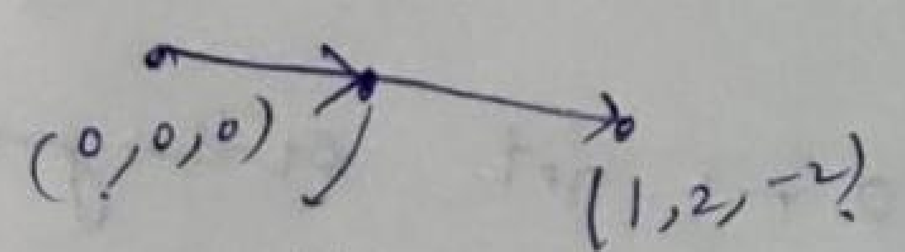
function we are trying to analyze has finite norm.

$\Rightarrow (1, 2, -2)$

$\hat{i} + 2\hat{j} + (-2\hat{k})$

$[1 \;\; 2 \;\; -2]^T$

$\| v \| = 3 \rightarrow \sqrt{1+4+4}$



unit vectors $\left(\frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}\right)$

if a vector has to be basis then the norm should be finite. Same thing is being done with functions (To check whether a function is eligible to be a basis) be one of the functions in the [generalization]

$\Rightarrow$ Norm $= 0 \rightarrow$ cannot be used as a one of the elements of basis.

$\rightarrow$ Norm $= 0 \rightarrow$ it is square integrable but it is normalizable

Normalize $\rightarrow$ divide by norm and division by 0 is not possible

$\rightarrow$ Non-orthogonal vectors $\rightarrow$ ~~Is~~ it possible to get a set of vectors orthogonal vectors from them?

Q) Consider the following vectors in $\mathbb{C}^2(\mathbb{C})$

$\underline{u_1} = [1\ \ 0]^T$

$\underline{u_2} = [i\ \ 1]^T$

(a) Determine whether they can form a basis.

sol) $\cancel{\begin{bmatrix} a_1+ib_1 \\ a_2+ib_2 \end{bmatrix}} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & i & a \\ 0 & 1 & b \end{array}\right]$$

$\lambda_2 = b$

$\lambda_1 + \lambda_2 i = a$

$\lambda_1 = a - bi$

$\cancel{\lambda_1 = a - b}$

Yes it forms basis.

(b) Do they form an orthogonal basis. → No

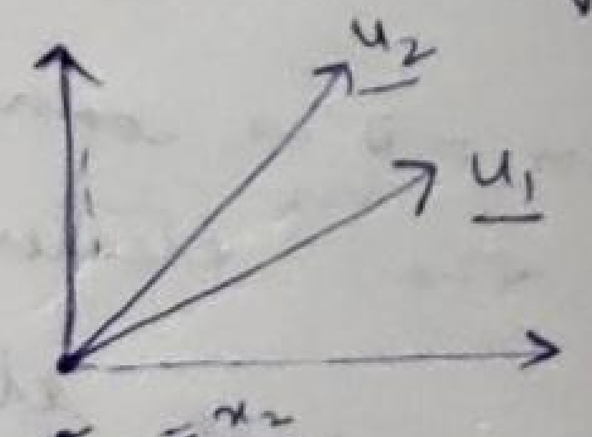
$\langle u_1, u_2 \rangle = 0.$

$\sum_{i=1}^{2} a_i \overline{b_i} = -i \neq 0$ → not orthogonal

(c) Determine the set of orthonormal basis from the given basis.

every element has norm unity

*** Gram-schmidt orthogonalization

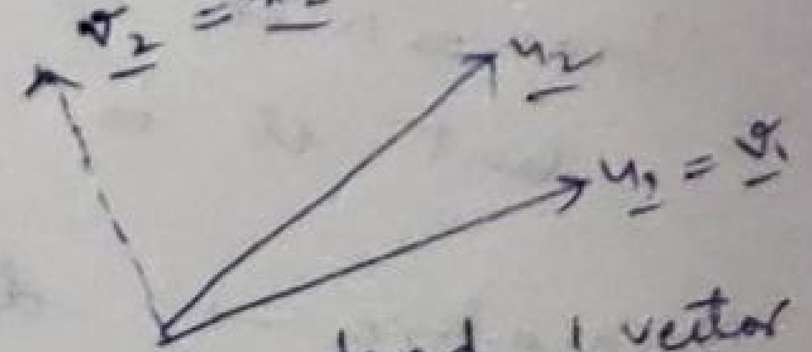


$\underline{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\cancel{\underline{u_2}}$

$\Downarrow$

$\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \underline{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → get norm and divide $\underline{v_1}$ by the norm

$\dfrac{\hat{i} + 0\hat{j}}{\sqrt{1^2+0^2}}$



find 1 vector which is orthogonal to $\underline{u_1}$

keep 1 vector same in the 2nd system.

$\underline{v_2} = \underline{u_2} - \alpha \underline{x_1}$ → linear combination: write $\underline{v_2}$ in terms of other 2 vectors.

second vector in orthonormal domain

$$\underline{v_2} = \underline{u_2} - \alpha \underline{x_1} = \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} i-\alpha \\ 1 \end{bmatrix}$$

$$\langle \underline{v_2}, \underline{x_1} \rangle = 0$$

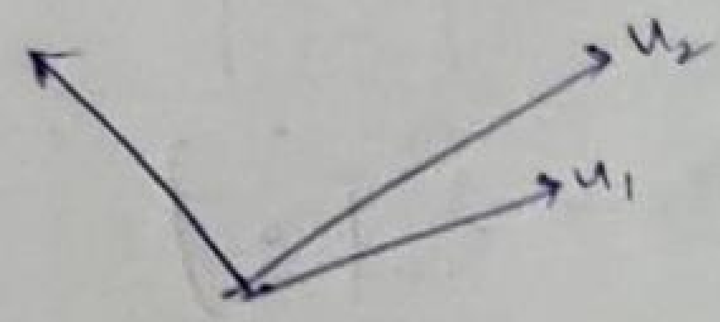
$$1(i-\alpha) = 0$$

$$\boxed{\alpha = i}$$

$$\boxed{\underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \Rightarrow \underline{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

method-2

$$\underline{u_2} = \underline{v_1} = \begin{bmatrix} i \\ 1 \end{bmatrix} \Rightarrow \underline{x_1} = \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



$$\underline{v_2} = \underline{u_1} - \alpha_1 \underline{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \alpha_1 \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 - \alpha_1 i/\sqrt{2} \\ -\alpha/\sqrt{2} \end{bmatrix}$$

$$\langle \underline{v_2}, \underline{x_1} \rangle = 0$$

$$\left(1 - \frac{\alpha_1 i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{-\alpha_1}{\sqrt{2}}\right) = 0.$$

$$\left(\frac{-i}{\sqrt{2}} - \frac{\alpha_1}{2}\right) - \frac{\alpha_1}{2} = 0. \qquad \boxed{\frac{-i}{\sqrt{2}} = \alpha_1}$$

Q) Determine the set of orthonormal basis from the following

$\underline{u_1} = [1 \; 0 \; 1]^T$

$\underline{u_2} = [1 \; 0 \; -1]^T$

$\underline{u_3} = [0 \; 3 \; 4]^T$

$$\underline{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{x_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{u_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{\alpha}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \langle \underline{v_2}, \underline{x_1} \rangle = 0$$

$$\underline{v_2} = \begin{bmatrix} 1 - \frac{\alpha}{\sqrt{2}} \\ 0 \\ -1 - \frac{\alpha}{\sqrt{2}} \end{bmatrix}.$$

$$\langle \underline{v_2}, \underline{x_1} \rangle = 0.$$

$$\left(1 - \frac{\alpha}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-1 - \frac{\alpha}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0.$$

$$1 - \frac{\alpha}{\sqrt{2}} - 1 - \frac{\alpha}{\sqrt{2}} = 0 \quad \boxed{\alpha = 0}$$

$\downarrow$

$\underline{u_1}, \underline{u_2}$ are already orthogonal

$$\underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\rightarrow \underline{u_3} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \underline{v_3} = \underline{u_3} - \alpha_1 \underline{x_1} - \alpha_2 \underline{x_2}$$

$$\underline{v_3} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \frac{\alpha_1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\alpha_2}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{v_3} = \begin{bmatrix} -\frac{\alpha_1}{\sqrt{2}} - \frac{\alpha_2}{\sqrt{2}} \\ 3 \\ 4 - \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}} \end{bmatrix}.$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\frac{2\sqrt{2}}{\sqrt{2}}$

$-2 + \frac{2\sqrt{2}}{\sqrt{2}}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\langle \underline{v_3}, \underline{x_1} \rangle = 0 \qquad \langle \underline{v_3}, \underline{x_2} \rangle = 0$$

$-\frac{2\sqrt{2}}{\sqrt{2}} -$

$$\frac{1}{\sqrt{2}}\left[-\frac{\alpha_1}{\sqrt{2}} - \frac{\alpha_2}{\sqrt{2}}\right] + \frac{1}{\sqrt{2}}\left[4 - \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}}\right) = 0$$

$$-\frac{2\alpha_1}{\sqrt{2}} + 4 = 0.$$

$$\alpha_1 = 2\sqrt{2}$$

$$\left(-\frac{\alpha_1}{\sqrt{2}} - \frac{\alpha_2}{\sqrt{2}}\right)\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{2}}}\left[4 - \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}}\right) = 0.$$

$$-\frac{2\alpha_2}{\sqrt{2}} - 4 = 0 \qquad \alpha_2 = -2\sqrt{2}$$

$$\underline{v_3} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \underline{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

⇒ check whether the vectors are orthogonal before solving →

Q) Let $C[-\pi, \pi]$ be a space of all continuous functions with general inner product defined. Verify whether the following are orthogonal.

use this as standard results

(i) $\sin t, \cos t$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \frac{\sin 2t}{2}\, dt = \int \left[\frac{\sin 2t}{2}\right]_{-\pi}^{\pi} dt$$

$$= \left[-\frac{\cos 2t}{4}\right]_{-\pi}^{\pi}$$

$$= 0$$

orthogonal

$$\rightarrow \frac{\sin t}{\sqrt{\int_{-\pi}^{\pi} \sin^2 t\, dt}} = \frac{\sin t}{\sqrt{\pi}} \Big\} \text{ normalized function.}$$

normalization

$$\|f(t)\| = \sqrt{\int_{-\pi}^{\pi} \sin^2 t\, dt}.$$

$\frac{1-\cos 2t}{2}$

$\frac{1}{2}t - \frac{\sin 2t}{4}$

$\frac{1}{2}\pi + \frac{\pi}{2}$ ⓟ

$$\frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos t \rightarrow \text{normalized function}$$

(ii) $\sin n\pi x$, $\sin(m\pi x)$. $x \in [-1, 1]$

$$\langle f, g \rangle = \int_{-1}^{1} (\sin n\pi x) \sin m\pi x \, dx.$$

$$= \int_{-1}^{1} \left(\frac{\cos(n\pi x - m\pi x) - \cos(n\pi x + m\pi x)}{2} \right) dx$$

$$= \frac{1}{2} \left[\frac{\sin(n\pi x - m\pi x)}{(n-m)\pi} - \frac{\sin(n\pi x + m\pi x)}{(n+m)\pi} \right]_{-1}^{1}$$

$$\cancel{= \frac{1}{2} \left[\sin((n-m)\pi^2) + \sin((n+m)\pi^2) \right.}$$

$$\cancel{= \frac{1}{2} \left[\frac{\sin((n-m)\pi^2)}{(n-m)\pi} + \frac{\sin((n+m)\pi^2)}{(n+m)\pi} \right]}$$

$n, m \in$ Integer $$= \frac{1}{2} \left[\frac{\sin(n-m)\pi}{(n-m)\pi} - \frac{\sin((n+m)\pi)}{(n+m)\pi} \right]_{-1}^{1}$$

$n \neq m \Rightarrow \langle f, g \rangle = 0 \rightarrow$ orthogonal

$\rightarrow n = m \rightarrow \langle f, g \rangle = 1 \rightarrow$ not orthogonal

$$\langle f, g \rangle = \int_{-1}^{1} \sin n\pi x \sin m\pi x \, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$1 \rightarrow (\|f\|)^2$

$\boxed{n, m = +ve}$

$\langle f, g \rangle \neq 0 \rightarrow$ if $n \neq -m$

use as standard formula

(iii) $\sin n\pi x, \cos m\pi x$ orthogonal

$$\langle f, g\rangle = \int_{-1}^{1} \sin n\pi x \cos m\pi x \, dx$$

$$= \int_{-1}^{1} \frac{\sin(n\pi x + m\pi x) + \sin(n\pi x - m\pi x)}{2} \, dx$$

*

(iv) $\cos m\pi x, \cos n\pi x$ ✓ orthogonal.

2) If $f(x)$ is expressed as an infinite series of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ Then determine the coeff of expansion.

$$\int_{-1}^{1} f(x) \sin m\pi x \, dx = \sum_{n=1}^{\infty} \int_{-1}^{1} a_n \sin n\pi x \sin m\pi x \, dx.$$

$$\int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_{-1}^{1} a_n \sin^2 n\pi x \, dx.$$

$$\boxed{\int_{-1}^{1} f(x) \sin n\pi x \, dx = a_n}$$

$\rightarrow f(x) = a_1 \sin \pi x + a_2 \sin 2\pi x + \ldots$

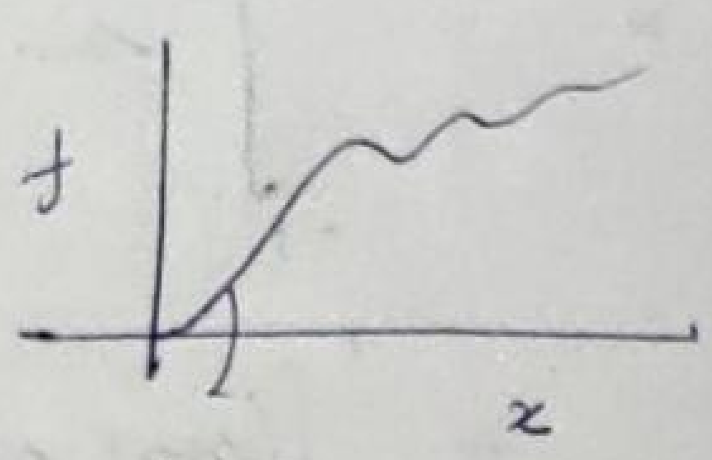
$$\rightarrow \langle f, \sin m\pi x\rangle = \int_{-1}^{1} a_m \sin^2 m\pi x \, dx.$$

$$a_m = \langle f(x), \sin m\pi x\rangle$$

$$a_m = \frac{\int_{-1}^{1} f(x) \sin m\pi x \, dx}{\int_{-1}^{1} \sin^2 m\pi x \, dx}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin n\pi x + \sum_{n=1}^{\infty} b_n \cos n\pi x \rightarrow \text{fourier series}$$

→ why is fourier series used?
(or) any other expansion



you have to write the function in terms of cos, sin

a_n = weight functions

~~⇒ Adj~~

Adjoint operator

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = 14$$

$$\langle \underline{u}, \underline{\underline{A}}\,\underline{v} \rangle = -2$$

→ $$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= a_{11}u_1v_1 + a_{12}u_2v_1 + a_{21}u_1v_1 + a_{22}u_2v_2$$

$$= u_1(a_{11}v_1 + a_{21}v_2) + (u_2(a_{12}v_1 + a_{22}v_2))$$

$$< \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11}v_1 + a_{21}v_2 \\ a_{12}v_1 + a_{22}v_2 \end{bmatrix} > = < \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} >$$

$$= < \underline{u}, \underline{\underline{A}}^* \underline{v} >.$$

$$< \underline{\underline{A}}\,\underline{u}, \underline{v} > = < \underline{u}, \underline{\underline{A}}^* \underline{v} >$$

$< u, \underline{A}^* v > = 14$

$\underline{\underline{A}}^*$ is adjoint operator of $\underline{\underline{A}}$

Every matrix can be an operator

$\rightarrow$ $\circledast : \hat{L} \times \underset{\text{vector}}{v}^{2\times 1} \longrightarrow v^{2\times 1}$ $\rightarrow$ generate new vectors within the same vector space

(operator maps the vector)

$\circledast : \hat{L} \times v^{m\times n} \longrightarrow v^{n\times p}$ $\quad$ $2\times 3 \quad 3\times 1$, 2×1 $\rightarrow$ original vector space

Q) $\underline{\underline{A}} = \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix}$

$\underline{u} = [u_1 \quad u_2]^T$

$\underline{v} = [v_1 \quad v_2]^T$

$$< \underline{\underline{A}}\,\underline{u}, \underline{v} > = < \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} >$$

$$= < \begin{bmatrix} u_1 - 2iu_2 \\ 3u_1 + iu_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} >$$

$$= (u_1 - 2iu_2)v_1 + (3u_1 + iu_2)v_2$$

$$= u_1(v_1 + 3v_2) + u_2(-2iv_1 + iv_2)$$

$$= \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 + 3v_2 \\ -2iv_1 + iv_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -2i & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

complex number ↓ Inner product is different take complex conjugate

$$\underline{\underline{A}}^* = \begin{bmatrix} 1 & 3 \\ +2i & -i \end{bmatrix}$$

$\cancel{\underline{\underline{A}}^* = \underline{\underline{A}}^T}$

$\underline{\underline{A}}^* = \bar{A}^T$ → need not derive this

Q) $$\underline{\underline{A}} = \begin{bmatrix} 3 & 2+i & 2 \\ 2-i & 2 & -i \\ 2 & i & 1 \end{bmatrix}$$

$$\underline{\underline{\bar{A}}} = \begin{bmatrix} 3 & 2-i & 2 \\ 2+i & 2 & i \\ 2 & -i & 1 \end{bmatrix}$$

$$\underline{\underline{A}}^* = \underline{\underline{\bar{A}}}^T = \begin{bmatrix} 3 & 2+i & 2 \\ 2-i & 2 & -i \\ 2 & i & 1 \end{bmatrix}$$

Hermitian matrix. (Hermitian conjugate)

$\underline{\underline{A}} = \underline{\underline{\bar{A}}}^T = \underline{\underline{A}}^*$

$\langle \underline{\underline{A}}\, \underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}^* \underline{v} \rangle$

if $\underline{\underline{A}}^* = \underline{\underline{A}}$

$\langle \underline{\underline{A}}\, \underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}\, \underline{v} \rangle$ → self adjoint operator

d) The eigen values of a self adjoint operator are real

$$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}\,\underline{v} \rangle$$

let $\underline{u}$ be an eigen vector of $\underline{\underline{A}}$ with the corresponding eigen value as λ

find eigen vector

$$\boxed{\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}} \rightarrow \text{scaling}$$

$$\langle \underline{\underline{A}}\,\underline{u}, \underline{u} \rangle = \langle \lambda \underline{u}, \underline{u} \rangle = \lambda \langle \underline{u}, \underline{u} \rangle \rightarrow ①$$

$$\langle \underline{u}, \underline{\underline{A}}\,\underline{u} \rangle = \langle \underline{u}, \lambda \underline{u} \rangle = \bar{\lambda} \langle \underline{u}, \underline{u} \rangle \rightarrow ②$$

① = ② → for self adjoint

$$\lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \in \mathbb{R}$$

→ The eigen vectors of a self adjoint operator corresponding to distinct eigen values are always orthogonal.

$$\underline{\underline{A}}\,\underline{u} = \lambda_u \underline{u}$$

$$\underline{\underline{A}}\,\underline{v} = \lambda_v \underline{v}$$

$$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \langle \lambda_u \underline{u}, \underline{v} \rangle = \lambda_u \langle \underline{u}, \underline{v} \rangle \rightarrow ①$$

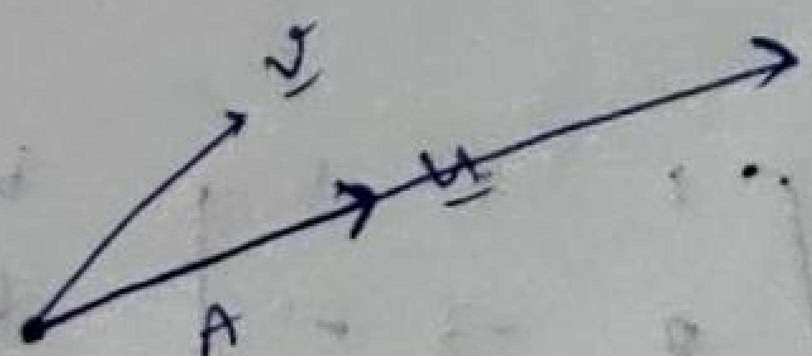
$$\langle \underline{u}, \underline{\underline{A}}\,\underline{v} \rangle = \lambda_v \langle \underline{u}, \underline{v} \rangle \rightarrow ②$$

① = ② iff $\langle \underline{u}, \underline{v} \rangle = 0$

$$\rightarrow \boxed{\langle \underline{u}, \underline{v} \rangle = 0}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 9 \\ 369 \end{bmatrix} \quad \underline{\underline{A}}\,\underline{u} = \underline{v}$$



eigen vector ↓ special vector such that direction is same norm changes

if $\underline{u}$ is eigen vector → $\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}$

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~~→ For~~

→ ✓

Q) $\frac{d^2 f}{dx^2} + \alpha f = 0$

$\hat{L} = \frac{d^2}{dx^2} + \alpha$ → it is a self adjoint operator

$\langle \hat{L} f, g \rangle = \langle f, \hat{L} g \rangle$

$\int_0^1 \left(\frac{d^2 f}{dx^2} + \alpha f \right) g \, dx$

$= \cancel{g \frac{df}{dx}\Big|_0^1} - \underbrace{f \frac{dg}{dx}\Big|_0^1}_{} + \ldots$

~~$g \frac{df}{dx} + \alpha f$~~

0 if $f'(0) = f'(1) = 0$ → force $g' = 0$

$g'(0) = g'(1) = 0$

→ neumann

Q) Identify the solvability condition & determine the range space of the following equation

$x_1 + x_2 + x_3 = b_1$

$2x_1 - x_2 + x_3 = b_2$

$x_1 - 2x_2 = b_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & -1 & 1 & b_2 \\ 1 & -2 & 0 & b_3 \end{array}\right]$$

$-1 \ -2$

$1 \ -2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -3 & -1 & b_2 - 2b_1 \\ 0 & -3 & -1 & b_3 - b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - (b_2 - 2b_1) \end{array}\right]$$

$$b_3 - b_1 - b_2 + 2b_1 = 0$$

$$b_3 - b_2 + b_1 = 0$$

$$\underline{b_3 = b_2 - b_1}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 - b_1 \end{bmatrix} \Rightarrow \boxed{DOF = 2}$$

$$b_1 = \alpha \quad b_2 = \beta$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

dimension = 2

basis

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Fredholm's alternative method (range space)

$\underline{\underline{A}}\, \underline{x} = \underline{b}$ is solvable if $\langle \underline{b}, \underline{y} \rangle = 0 \ \forall\ \underline{y} : \underline{\underline{A}}^{*} \underline{y} = 0$

null space of adjoint operator

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\underline{\underline{A}}^{*} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\underline{\underline{A}}^{*} \underline{y} = 0 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-\cancel{2x_2} + 3x_3 = 0$

$-3y_2 - 3y_3 = 0$

$y_1 + 2y_2 + y_3 = 0$

$\boxed{y_3 = -y_2}$

$y_1 + 2y_2 - y_2 = 0$

$\boxed{y_1 = -y_2}$ $\quad y_2 = \alpha$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ -\alpha \end{bmatrix}$$

DOF = 1

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\langle \underline{b}, \underline{y} \rangle = \left\langle \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \alpha \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\rangle = 0$$

$-b_1 + b_2 - b_3 = 0$

$b_3 = b_2 - b_1$ — $\boxed{\text{DOF} = 2}$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

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