① Linear Vector Spaces.

two generic operators

real / complex vectors

- An algebraic structure $(V, F, \otimes, \odot)$ equipped with a non-empty set V, a field F, a binary operator -

real / compl. scalars

an element not from the set.

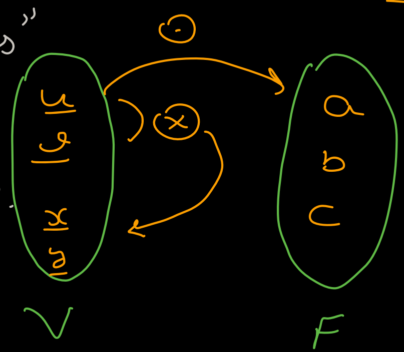
- $\otimes : V \times V \to V$, and an external mapping
- $\odot : F \times V \to V$ such that $\to$
  - (i) - (iv) $(V, \otimes)$ is an Abelian group $\forall\ \underline{u}, \underline{v} \in V$ & $a, b \in F$
  - (v) $a \odot (\underline{u} \otimes \underline{v}) = (a \odot \underline{u}) \otimes (a \odot \underline{v})$
  - (vi) $(\underline{u} \otimes \underline{v}) \odot b = (\underline{u} \odot b) \otimes (\underline{v} \odot b)$
  - (vii) $\forall\ a, b \in F$ and $\underline{u} \in V$

$$a \odot (b \odot u) = (a \odot b) \odot \underline{u}$$

  - (viii) $\forall\ \underline{u} \in V\ \exists\ e \in F$ s.t.

$$e \odot \underline{u} = \underline{u}$$

- elements of $V \to$ "vectors"
- elements of $F \to$ "scalars"



- $\underline{x} = \underline{u} \otimes \underline{v}$
- $\underline{y} = a \odot \underline{u}$

Listing all rules in simple terms.

vector add.

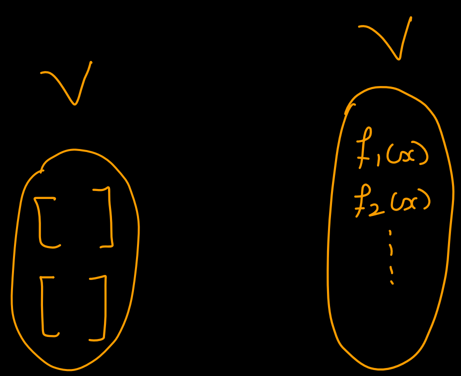
$\otimes \equiv +$

$\odot \equiv \times$

&

Scalar mult.

- $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
- $\underline{u} + 0 = \underline{u}$
- $\underline{u} + (-\underline{u}) = 0$
- $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
- $a \cdot (\underline{u} + \underline{v}) = a \cdot \underline{u} + a \cdot \underline{v}$
- $(\underline{u} + \underline{v}) \cdot b = b \cdot u + b \cdot v$
- $(a \times b) \times \underline{u} = a \times (b \times \underline{u})$
- $1 \times \underline{u} = \underline{u}$



- all the elements in V are vectors.

&

general defn. of vector.

② Space

- collection of vectors which have been formed following these rules

e.g. Euclidean space.

- vectors of the form (x, y, z).

③ if f is an operator, then for it to be linear $\Rightarrow$

① $f(\underline{u} + \underline{v}) = f(\underline{u}) + f(\underline{v})$

② $f(a \cdot \underline{u}) = a \cdot f(\underline{u})$.

$$\Rightarrow f(a \cdot \underline{u} + b \cdot \underline{v}) = a \cdot f(\underline{u}) + b \cdot f(\underline{v})$$

④ Q. Show that the set of all ordered n-tuples of a real field F is a vector space over the same field.

- Let $\underline{u}, \underline{v} \in V$ such that $\Rightarrow$

$$\underline{u} = (u_1, u_2, \ldots, u_n), \quad u_i \in \mathbb{R}$$

$$\underline{v} = (v_1, v_2, \ldots, v_n), \quad v_i \in \mathbb{R}$$

(T.B.D.)

- show each property one by one.
  - if one fails, you don't need to show all. you also don't need to test in seq.

⑤ $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ = vector (in this course we will use column matrices to represent vectors)

⑥ if an ODE is defined for vectors in a linear vector space, then its solutions also lie in the l.v.s.

(7) let C be a set of ordered pairs (a, b), with the following ops:

(i) $(a, b) \otimes (c, d) = (a+c, b+d)$

(ii) $k \odot (a, b) = (k \odot a, b)$, $k \in \mathbb{R}$.

show that C forms a linear vector space.

(5) $\rightarrow$ $a \odot (\underline{u} \otimes \underline{v}) = (a \odot \underline{u}) \otimes (a \odot \underline{v})$

LHS: $a \odot (\underline{u} \otimes \underline{v})$

$= k \odot ((a, b) \otimes (c, d))$

$= k \odot (a+c, b+d)$.

$= (k(a+c), b+d)$

RHS $= (k \odot (a, b)) \otimes (k \odot (c, d))$

$= (ka, b) \otimes (kc, d)$

$= (ka + kc, b+d)$

$= (k(a+c), b+d)$

$\therefore$ LHS = RHS.

||ly show other props.

8) Let S be a set of elements of the form $(x+2y, y, -x+3y) \in \mathbb{R}^3$

Show that S is a subspace of $\mathbb{R}^3$

- $(S, F, \otimes, \odot)$ is said to be a subspace of $(C, F, \otimes, \odot)$ iff $S \subset V$

subset.

$\Rightarrow f(a \cdot \underline{u} + b \cdot \underline{v}) = a \cdot f(\underline{u}) + b \cdot f(\underline{v})$

any op.

- The problem here is that given two elements $s_1, s_2 \in S$, $s_1 \otimes s_2$ or $s_1 \odot s_2$ may not $\in S$, which would make S a subset not a subspace.

let $\underline{u} = (x_1 + 2y_1, y_1, -x_1 + 3y_1)$

$\underline{v} = (x_2 + 2y_2, y_2, -x_2 + 3y_2)$

$x_i, y_i \in \mathbb{R}$

$\therefore \underline{u} + \underline{v} = (x_1 + x_2 + 2 \cdot (y_1 + y_2), y_1 + y_2, -(x_1 + x_2) + 3 \cdot (y_1 + y_2))$

$$= (x_3 + 2y_3 , y_3 , -x_3 + 3 \cdot y_3)$$

$$\in R.$$

$$a \cdot \bar{u} = (ax_1 + 2ay_1 , ay_1 , -ax_1 + 3ay_1)$$

$$= (x_4 + 2y_4 , y_4 , -x_4 + 3y_4)$$

$\therefore$ S is a subspace of $\mathbb{R}^3$.

(a) Subspace

A non-empty subset S of V $(S \subset V)$ is a subspace of V iff $\Rightarrow$

$$\alpha \cdot \underline{u} + \beta \cdot \underline{v} \in S$$

$$\forall \underline{u}, \underline{v} \in S \ \& \ \alpha, \beta \in F$$

Ideas

1. Checking if a given algebraic structure is a linear vector space
2. Checking if a subset of an l.v.s is a subspace