4.1 Properties of Laplace Transform

In this lesson we discuss some properties of Laplace transform. There are several useful properties of Laplace transform which can extend its applicability. In this lesson we mainly present shifting and translation properties.

4.2 First Shifting Property

If $L[f(t)] = F(s)$ then $L\left[e^{at}f(t)\right] = F(s-a)$, where a is any real or complex constant.

Proof: By the definition of Laplace transform we find

$$\begin{aligned} L\left[e^{at}f(t)\right] &= \int_0^\infty e^{at}f(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t \end{aligned}$$

Again by the definition of Laplace transform we get

$$L\left[e^{at}f(t)\right] = F(s-a).$$

4.3 Example Problems

4.3.1 Problem 1

Find the Laplace transform of $e^{-t}\sin^2 t$.

Solution: First we get the Laplace transform of $\sin^2 t$ as

$$\begin{aligned} L\left[\sin^2 t\right] &= L\left[\frac{1-\cos 2t}{2}\right] \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)} = F(s). \end{aligned}$$

Now using the first shifting property we obtain

$$L\left[e^{-t}\sin^2 t\right] = F(s+1) = \frac{2}{(s+1)(s^2+2s+5)}$$

4.3.2 Problem 2

Find $L[e^{-2t}\sin 6t]$.

Solution: Setting $f(t) = \sin 6t$ we find

$$L[f(t)] = F(s) = \frac{6}{s^2+36}$$

Now using the first shifting property we get

$$L[e^{-2t}\sin 6t] = \frac{6}{(s+2)^2+36}$$

4.3.3 Problem 3

Evaluate $L[e^{2t}(t+3)^2]$.

Solution: By the definition and linearity of Laplace transform we have

$$\begin{aligned} L[(t+3)^2] =& L[t^2+6t+9] = L[t^2] + 6L[t] + 9L[1] \\ =& \frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s} \end{aligned}$$

Further simplifications lead to

$$L[(t+3)^2] = \frac{2+6s+9s^2}{s^3} = F(s)$$

Using the first shifting property we get

$$\begin{aligned} L[e^{2t}(t+3)^2] = F(s-2) =& \frac{2+6(s-2)+9(s-2)^2}{(s-2)^3} \\ =& \frac{9s^2-30s+26}{(s-2)^3} \end{aligned}$$

4.3.4 Problem 4

Using shifting property evaluate $L[\sinh 2t\cos 2t]$ *and* $L[\sinh 2t\sin 2t]$

Solution: We know that

$$L[\sinh 2t] = \frac{2}{s^2-4} = F(s)$$

Using shifting property we can get

$$L[e^{2it}\sinh 2t] = F(s-2i)$$

This implies

$$L[e^{2it}\sinh 2t] = \frac{2}{(s-2i)^2-4} = \frac{2}{(s^2-8)-4is}$$

Multiplying numerator and denominator by $(s^2-8)+4is$, we find

$$L[e^{2it}\sinh 2t] = \frac{2(s^2-8)+8is}{(s^2-8)^2+16s^2} = \frac{2(s^2-8)+8is}{(s^4+64)}$$

Replacing e^{2it} by $\cos 2t + i\sin 2t$ and using linearity of the transform we obtain

$$L[\cos 2t\sinh 2t] + iL[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} + i\frac{8s}{(s^4+64)}$$

Equating real and imaginary parts we have

$$L[\cos 2t\sinh 2t] = \frac{2(s^2-8)}{(s^4+64)} \quad \text{and} \quad L[\cos 2t\sinh 2t] = \frac{8s}{(s^4+64)}$$

4.4 Second Shifting Property

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a) & \text{when } t > a \\ 0 & \text{when } 0 < t < a \end{cases}$

then

$$L[g(t)] = e^{-as}F(s).$$

Proof: By the definition of Laplace transform we have

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-st}g(t)\,\mathrm{d}t \\ &= \int_a^\infty e^{-st}f(t-a)\,\mathrm{d}t \end{aligned}$$

Substituting $t-a=u$ so that $\mathrm{d}t = \mathrm{d}u$, we find

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-s(u+a)}f(u)\,\mathrm{d}u \\ &= e^{-sa}\int_0^\infty e^{-su}f(u)\,\mathrm{d}u \end{aligned}$$

Again using the definition of Laplace transform we get

$$L[g(t)] = e^{-as}F(s).$$

Alternative form: It is sometimes useful to present this property in the following compact form.

If $L[f(t)] = F(s)$ then

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

where

$$H(t) = \begin{cases} 1 & \text{when } t > 0 \\ 0 & \text{when } t < 0 \end{cases}$$

Note that $f(t-a)H(t-a)$ is same as the function $g(t)$ given above.

4.5 Example Problems

4.5.1 Problem 1

Find $L[g(t)]$ *where* $g(t) = \begin{cases} 0 & \textit{when } 0 \le t < 1 \\ (t-1)^2 & \textit{when } t \ge 1 \end{cases}$

Solution: On comparison with the function $g(t)$ given in second shifting theorem we get

$$f(f) = t^2 \quad \Rightarrow \quad L[f(t)] = \frac{2}{s^3}$$

Using the second shifting property we find

$$L[g(t)] = e^{-s}\left(\frac{2}{s^3}\right).$$

4.5.2 Problem 2

Find the Laplace transform of the function $g(t)$*, where*

$$g(t) = \begin{cases} \cos(t-\pi/3), & t > \pi/3; \\ 0, & 0 < t < \pi/3. \end{cases}$$

Solution: Comparing with the notations used in the second shifting theorem we have $f(t) = \cos t$. Thus, we find

$$L[f(t)] = F(s) = \frac{s}{s^2+1}.$$

Hence by the second shifting theorem we obtain

$$L[g(t)] = e^{-\frac{\pi}{3}s}F(s) = e^{-\frac{\pi}{3}s}\frac{s}{s^2+1}.$$

4.6 Change of Scale Property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: By definition, we have

$$L[f(at)] = \int_0^\infty e^{-st}f(at)\,\mathrm{d}t.$$

Substituting $at = u$ so that $a\,\mathrm{d}t = \mathrm{d}u$ we find

$$L[f(at)] = \int_0^\infty e^{-\left(\frac{s}{a}\right)u}f(u)\frac{1}{a}\,\mathrm{d}u.$$

Using definition of the Laplace transform we get

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).$$

4.6.1 Example

If

$$L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$$

then find $L[f(2t)]$.

Solution: Direct application of the second shifting theorem we obtain

$$L[f(2t)] = \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2-\frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

On simplifications, we get

$$L[f(2t)] = \frac{1}{4}\frac{s^2-2s+4}{(s+1)^2(s-2)}.$$