Lecture - 23
23/10/17

Properties of Fourier Transform.

1. If $F(\omega) = \mathcal{F}\{f(x); \omega\}$, then

$$\lim_{|\omega| \to \infty} |F(\omega)| = 0.$$

Riemann-Lebesgue Lemma.

Proof. $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx. \longrightarrow (1).$

Note that,

$$e^{i\omega x} = - e^{i\omega x} \cdot e^{i\pi} \qquad \left(e^{i\pi} = \cos\pi + i\sin\pi = -1 \right)$$

$$= - e^{i\omega\left(x + \frac{\pi}{\omega}\right)}$$

So, $F(\omega) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega\left(x + \frac{\pi}{\omega}\right)} dx$

Let, $y = x + \frac{\pi}{\omega}$; $dx = dy$

Then, $F(\omega) = - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(y - \frac{\pi}{\omega}\right) e^{i\omega y} dy.$

$$= - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\omega}\right) e^{i\omega x} dx \longrightarrow (2).$$

So, (1) + (2) gives,

$$2F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ f(x) - f\left(x - \frac{\pi}{\omega}\right) \right\} e^{i\omega x} dx.$$

$$\therefore \left| F(\omega) \right| = \frac{1}{2\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} \left\{ f(x) - f\left(x - \frac{\pi}{\omega}\right) \right\} e^{i\omega x} \, dx \right|$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{\omega}\right) \right| \left| e^{i\omega x} \right| dx$$

$$\left| \int_a^b g(x)\, dx \right| \leq \int_a^b |g(x)|\, dx.$$

or,

$$|F(\omega)| \leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{\omega}\right) \right| dx.$$

$$\lim_{|\omega| \to \infty} |F(\omega)| \leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{|\omega| \to \infty} \left| f(x) - f\left(x - \frac{\pi}{\omega}\right) \right| dx.$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x) - f(x)| \, dx = 0.$$

$$\therefore \lim_{|\omega| \to \infty} |F(\omega)| = 0.$$

So, $\displaystyle\int_0^{\infty} \frac{\cos \omega x}{1+x^2} dx = C_0 \sinh \omega$. X

2. __Thm.__ If $f(x)$ is piecewise continuous, $f'(x)$ exists and is also continuous in $(-\infty, \infty)$ $[f(x) \in C'(-\infty, \infty)]$ & if $f(x)$ is absolutely integrable, then $F(\omega)$ is 1) continuous 2) bounded

Proof. 1) $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$.

$F(\omega + h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\omega+h)x} dx$.

$\therefore \left| F(\omega+h) - F(\omega) \right| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{i\omega x} \left\{ e^{ihx} - 1 \right\} dx \right|$.

$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \left| e^{i\omega x} \right| \left| e^{ihx} - 1 \right| dx$.

or,
$\left| F(\omega+h) - F(\omega) \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \left| e^{ihx} - 1 \right| dx$.

$\therefore \lim_{h\to 0} \left| F(\omega+h) - F(\omega) \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} |f(x)| \left| e^{ihx} - 1 \right| dx$.

$= 0$.

So, $\lim_{h\to 0} F(\omega+h) = F(\omega)$. //.

2) $f(x)$ is absolutely integrable in $(-\infty, \infty)$

$\Rightarrow \int_{-\infty}^{\infty} |f(x)| dx \leq M$ for some +ve no. M.

$\left| F(\omega) \right| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \right|$

$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \left| e^{i\omega x} \right| dx$.

or, $\left| F(\omega) \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx \leq \frac{M}{\sqrt{2\pi}} = M_0$.

$\therefore F(\omega)$ is bounded.

Q. Find the F.T. of $\frac{1}{1+x^2}$.

Note. $\frac{1}{1+x^2}$ is an even funct.

$\therefore$ F.T. of $\frac{1}{1+x^2}$ = F.C.T. of $\frac{1}{1+x^2}$.

$$\mathcal{F}\left(\frac{1}{1+x^2}\right) = \mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos \omega x}{1+x^2}\, dx.$$

$$I(\omega) = \int_0^\infty \frac{\cos \omega x}{1+x^2}\, dx. \qquad I(0) = \int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}.$$

$$I'(\omega) = -\int_0^\infty \frac{x \sin \omega x}{1+x^2}\, dx.$$

$$= -\int_0^\infty \frac{x^2+1-1}{x} \cdot \frac{\sin \omega x}{1+x^2}\, dx.$$

$$= -\left[\int_0^\infty \frac{\sin \omega x}{x}\, dx - \int_0^\infty \frac{\sin \omega x}{x(1+x^2)}\, dx\right].$$

$I'(\omega) = 0$? X $\qquad I'(0) = -\frac{\pi}{2}$ ✓

$$\int_0^\infty \frac{x \sin \omega x}{1+x^2}\, dx = \int_0^\infty \frac{x\left(\omega x - \frac{\omega^3 x^3}{3!} + \cdots\right)}{1+x^2}\, dx.$$

$$= \int_0^\infty \frac{\cancel{x^2}\left(\omega - \frac{\omega^3 x^2}{3!} + \cdots\right)}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)}\, dx.$$

Look at $\int_{-\infty}^{\infty} \frac{\omega}{1+\frac{1}{x^2}}\, dx$

As $x \to \infty$, the integral becomes infinite - ∞

Since the integral $\int_0^\infty \frac{x \sin \omega x}{1+x^2} dx$ is divergent

$$\lim_{\omega \to 0} \int_0^\infty \frac{x \sin \omega x}{1+x^2} dx \neq \int_0^\infty \lim_{\omega \to 0} \frac{x \sin \omega x}{1+x^2} dx.$$

$$\int_0^\infty \frac{\sin \omega x}{x(1+x^2)} dx. \qquad f(x) = \frac{\sin \omega x}{x(1+x^2)}$$

$$|f(x)| = \frac{|\sin \omega x|}{x(1+x^2)} \leq \frac{1}{x(1+x^2)} \leq \frac{1}{x^3}.$$

$\because \int_a^\infty \frac{dx}{x^3}$ is convergent.

$\therefore \int_a^\infty \frac{\sin \omega x}{x(1+x^2)} dx$ is convergent.

We can also show that $\int_0^a \frac{\sin \omega x}{x(1+x^2)} dx$ is convergt.

$$\therefore \lim_{\omega \to 0} \int_0^\infty \frac{\sin \omega x}{x(1+x^2)} dx = \int_0^\infty \lim_{\omega \to 0} \frac{\sin \omega x}{x(1+x^2)} dx.$$

$$= 0.$$

$$I'(0) = -\frac{\pi}{2}.$$

$$I'(\omega) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin \omega x}{x(1+x^2)} dx.$$

$$I''(\omega) = \int_0^\infty \frac{\cos \omega x}{1+x^2} dx = I(\omega).$$

$$\frac{d^2 I}{d\omega^2} - I(\omega) = 0 \implies I(\omega) = c_1 e^{-\omega} + c_2 e^{\omega}.$$

$$I(0) = \frac{\pi}{2}, \quad I'(0) = -\frac{\pi}{2}.$$

$c_1 + c_2 = \frac{\pi}{2}$

$-c_1 + c_2 = -\frac{\pi}{2}$. $\quad c_2 = 0$. $\quad c_1 = \frac{\pi}{2}$.

$\therefore I(\omega) = \frac{\pi}{2} e^{-\omega}$.

$\mathcal{F}\left(\frac{1}{1+x^2}\right) = \mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} e^{-\omega}$.

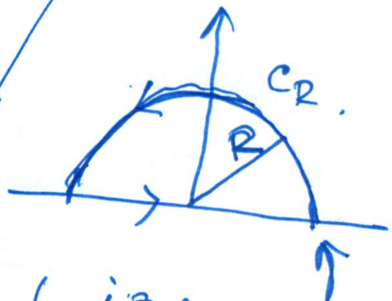
or, $F(\omega) = \sqrt{\frac{\pi}{2}}\, e^{-\omega}$.

Then $|F(\omega)| \to 0$ as $\omega \to \infty$.

$\int_0^\infty \frac{\cos \omega x}{1+x^2}\, dx$. Cauchy's residue thm.

$f(x) = \frac{\cos \omega x}{1+x^2}$, Jordan's lemma.

$\oint \frac{e^{i\omega z}}{1+z^2}\, dz$; z is complex



$f(z) = \frac{1}{1+z^2}$ has poles at $z = \pm i$.

$= 2\pi i \times$ residue at $z = i$

$\lim_{R\to\infty} \int_{C_R} e^{iz} f(z)\, dz = 0$.

$= 2\pi i \times \lim_{z\to i} (z-i) f(z)$

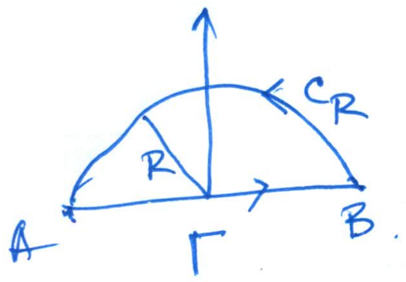
or $\lim_{R\to\infty} \int_{C_R} e^{-iz} f(z)\, dz = 0$.

$= 2\pi i \times \lim_{z\to i} \cancel{(z-i)} \frac{e^{i\omega z}}{\cancel{(z-i)}(z+i)}$

or $\lim_{R\to\infty} \int_{C_R} e^{-z} f(z)\, dz = 0$.

$= 2\pi i \times \frac{e^{-\omega}}{2i} = \pi e^{-\omega}$.

or $\lim_{R\to\infty} \int_{C_R} e^{z} f(z)\, dz = 0$



$$\oint_{\Gamma} \frac{e^{i\omega z}}{1+z^2}\,dz = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+x^2}\,dx + \int_{C_R} \frac{e^{i\omega z}}{1+z^2}\,dz.$$

On AB, $z = x + i.0$.

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{i\omega z}}{1+z^2}\,dz = 0, \text{ by Jordan's lemma}$$

(Schaum series Complex variables).

$$\therefore \oint_{\Gamma} \frac{e^{i\omega z}}{1+z^2}\,dz = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+x^2}\,dx.$$

or, $$\pi e^{-\omega} = \int_{-\infty}^{\infty} \frac{\cos\omega x + i\sin\omega x}{1+x^2}\,dx.$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos\omega x}{1+x^2}\,dx = \pi e^{-\omega}.$$

$$\int_{0}^{\infty} \frac{\cos\omega x}{1+x^2}\,dx = \frac{\pi}{2} e^{-\omega}.$$

$$\int_{-\infty}^{\infty} \frac{\sin\omega x}{1+x^2}\,dx = 0.$$

$\Rightarrow 0 = 0$ (identity).

Convolution theorem

L.T., $(f*g)(x) = \int_0^x f(u)\, g(x-u)\, du$,

For Fourier transform, we define convolution of $f(x)$ & $g(x)$ as,

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

Thm. $\mathcal{F}\left[(f*g)(x)\right] = F(\omega)\, G(\omega)$.

Proof. $\mathcal{F}\left[(f*g)(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{i\omega x}\, dx$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right) e^{i\omega x}\, dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \left(\int_{-\infty}^{\infty} g(x-u)\, e^{i\omega x}\, dx \right)$$

In the inner integral subst. $x-u=y$.

Then, $\int_{-\infty}^{\infty} g(x-u)\, e^{i\omega x}\, dx = \int_{-\infty}^{\infty} g(y)\, e^{i\omega(y+u)}\, dy$.

$$= \int_{-\infty}^{\infty} g(y)\, e^{i\omega y}\, \underline{\underline{e^{i\omega u}}}\, dy = e^{i\omega u} \int g(y)\, e^{i\omega y}\, dy.$$

$\therefore \mathcal{F}\left[(f*g)(x)\right] = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\omega u}\, du \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{i\omega y}\, dy \right)$

$\therefore \mathcal{F}[(f*g)(x)] = F(\omega)\, G(\omega)$.

Properties of convolution.

1. $(f*g)(x) = (g*f)(x)$.

So, $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$.

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)\, f(x-u)\, du.$$

2. $(f*g)*h = f*(g*h)$ (associativity)

3. $f*(g+h) = (f*g) + (f*h)$ (distributive property).

H.W: Prove the above properties.

Examples on convolution

1. Solve the integral equation:

$$f(x) + 4\int_{-\infty}^{\infty} e^{-2|x-t|} f(t)\,dt = e^{-2|x|} \rightarrow (1).$$

Let

$$g(x) = e^{-2|x|}, \quad g(x-t) = e^{-2|x-t|}.$$

(1) is,

$$\frac{f(x)}{\sqrt{2\pi}} + \frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t) f(t)\,dt = \frac{e^{-2|x|}}{\sqrt{2\pi}} = \frac{g(x)}{\sqrt{2\pi}}.$$

or, $$\frac{1}{\sqrt{2\pi}} f(x) + 4\,(f * g)(x) = \frac{1}{\sqrt{2\pi}} g(x).$$

Multiply both sides by $e^{i\omega x}$ and integrate w.r.t. x between $-\infty$ and ∞.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx + 4\int_{-\infty}^{\infty} (f*g)(x) e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x) e^{+i\omega x} dx.$$

or, $$F(\omega) + 4\sqrt{2\pi}\, \mathcal{F}\left((f*g)(x)\right) = G(\omega).$$

or, $$F(\omega) + 4\sqrt{2\pi}\, F(\omega) G(\omega) = G(\omega)$$

$$\therefore \quad F(\omega)\left\{1 + 4\sqrt{2\pi}\, G(\omega)\right\} = G(\omega).$$

$$\therefore \quad F(\omega) = \frac{G(\omega)}{1 + 4\sqrt{2\pi}\, G(\omega)}.$$

$$G(\omega) = \mathcal{F}\left[e^{-2|x|}; \omega\right]$$

$$\underset{(a>0)}{\mathcal{F}\left[e^{-a|x|}; \omega\right]} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{i\omega x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ax+i\omega x}\, dx + \int_{0}^{\infty} e^{-ax+i\omega x}\, dx\right] \qquad \begin{aligned}|x| &= x, x>0.\\ &= -x, x<0.\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(a+i\omega)x}\, dx + \int_{0}^{\infty} e^{-(a-i\omega)x}\, dx\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left.\frac{e^{(a+i\omega)x}}{a+i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a-i\omega)x}}{a-i\omega}\right|_{\infty}^{0}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\omega} + \frac{1}{a-i\omega}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+\omega^2}\right] = \frac{2a}{\sqrt{2\pi}\,(a^2+\omega^2)} = \mathcal{F}\left[e^{-a|x|}\right]$$

$$G(\omega) = \mathcal{F}\left[e^{-2|x|}\right] = \frac{4}{\sqrt{2\pi}\,(\omega^2+4)}.$$

$$\therefore F(\omega) = \frac{(4/\sqrt{2\pi}) \times \frac{1}{\omega^2+4}}{1 + 4\cancel{\sqrt{2\pi}} \times \frac{4}{\cancel{\sqrt{2\pi}}} \times \frac{1}{\omega^2+4}}.$$

$$\therefore F(\omega) = \frac{4}{\sqrt{2\pi}} \times \frac{1}{\omega^2 + 4 + 16} = \frac{4}{\sqrt{2\pi}} \times \frac{1}{\omega^2+20}.$$

$$\therefore f(x) = \mathcal{F}^{-1}\left[\frac{4}{\sqrt{2\pi}} \cdot \frac{1}{\omega^2+20}\right] \qquad \mathcal{F}\left[e^{-2\sqrt{5}|x|}\right].$$

$$= \mathcal{F}^{-1}\left[\frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{2\pi}} \frac{2\times 2\sqrt{5}}{\omega^2+(2\sqrt{5})^2}\right)\right]$$

$$= \frac{1}{\sqrt{5}} \mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} \cdot \frac{2\times 2\sqrt{5}}{\omega^2+(2\sqrt{5})^2}\right].$$

$$f(x) = \frac{1}{\sqrt{5}} e^{-2\sqrt{5}|x|}.$$

2. Find F.T. of $f(t) = \begin{cases} e^{-at}, & a>0,\ t\geq 0 \\ 0, & t<0. \end{cases}$

Hence find $F^{-1}\left[\frac{1}{(1-i\omega)^2}\right]$.

Sol. $\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt.$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-at} e^{i\omega t}\, dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a-i\omega)t}\, dt = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a-i\omega}.$$

$$F^{-1}\left[\frac{1}{(1-i\omega)^2}\right]$$

$$= (\sqrt{2\pi})^2 F^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{1}{1-i\omega}\cdot\frac{1}{\sqrt{2\pi}}\frac{1}{1-i\omega}\right]$$

$$= 2\pi \; F^{-1}\left[G(\omega)\cdot G(\omega)\right]$$

where $g(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$

$$\therefore 2\pi \; F^{-1}\left[G(\omega)\cdot G(\omega)\right]$$

$$= 2\pi \, (g * g)(t).$$

$$= 2\pi \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, g(t-u)\, du.$$

$g(u) = e^{-u}, \; u \geq 0.$ $\qquad g(t-u) = e^{-(t-u)}, \; t-u \geq 0 \Rightarrow u \leq t$

$\quad\;\; = 0, \; u < 0.$ $\qquad\qquad\qquad\quad = 0, \; t-u < 0.$

$$= 2\pi \times \frac{1}{\sqrt{2\pi}} \int_{0}^{t} e^{-u} e^{-(t-u)}\, du.$$

$$= \sqrt{2\pi}\, t\, e^{-t}$$

Ex. Using convolution theorem for Fourier transform, solve

$$\frac{d^2y}{dx^2} - y = -H(1-|x|), \quad -\infty < x < \infty \quad \longrightarrow (1).$$

$y(x), y'(x) \to 0$ as $|x| \to \infty$.

Take F.T. on both sides of (1).

$$\mathcal{F}\left[\frac{d^2y}{dx^2}\right] - \mathcal{F}[y] = -\mathcal{F}\left[H(1-|x|)\right].$$

Let $\mathcal{F}[y] = Y(\omega)$.

$$\therefore (-i\omega)^2 Y(\omega) - Y(\omega) = -\mathcal{F}\left[H(1-|x|)\right].$$

$\mathcal{F}\left[f^{(n)}(x)\right] = (-i\omega)^n F(\omega)$ provided, $f(x), f'(x), \ldots, f^{(n-1)}(x) \to 0$ as $|x| \to \infty$.

$$\mathcal{F}\left[H(1-|x|)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(1-|x|)\, e^{i\omega x}\, dx.$$

$H(y) = 1, \; y > 0.$
$\quad\;\; = 0, \; y < 0.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega x}\, dx.$$

$H(1-|x|) = 1, \; 1-|x| > 0$
$\Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1.$

$$= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{i\omega x}}{i\omega}\right|_{-1}^{1} = \frac{e^{i\omega} - e^{-i\omega}}{i \cdot \omega} \times \frac{1}{\sqrt{2\pi}}$$

$$= 2\,\frac{\sin \omega}{\omega} \times \frac{1}{\sqrt{2\pi}}$$

$$\therefore (-\omega^2 - 1) Y(\omega) = -2 \frac{\sin \omega}{\omega}.$$

$$\therefore Y(\omega) = \frac{2}{\sqrt{2\pi}} \frac{\sin \omega}{\omega} \times \frac{1}{1+\omega^2}.$$

$$\mathcal{F}\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+\omega^2}$$

$$= G_1(\omega) \times \underbrace{c}_{\text{some const.}} G_2(\omega)$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin \omega}{\omega} \left(\times \frac{\sqrt{2\pi}}{2}\right) \frac{1}{\sqrt{2\pi}} \frac{2}{\omega^2+1}$$

$$\therefore Y(\omega) = \sqrt{\frac{\pi}{2}} \left[\underbrace{\left(\frac{2}{\sqrt{2\pi}} \frac{\sin \omega}{\omega}\right)}_{G_1(\omega)} \times \underbrace{\left(\frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\omega^2+1}\right)}_{G_2(\omega)} \right].$$

$$g_1(x) = H(1-|x|), \quad g_2(x) = e^{-|x|}.$$

$$\therefore \mathcal{F}^{-1}[Y(\omega)] = \sqrt{\frac{\pi}{2}}\, \mathcal{F}^{-1}[G_1(\omega) \cdot G_2(\omega)].$$

$$\text{or, } y(x) = \sqrt{\frac{\pi}{2}} (g_1 * g_2)(x).$$

$$= \sqrt{\frac{\pi}{2}} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|u|} H[1-|x-u|]\, du.$$

$$H[1-|x-u|] = 1,\ 1-|x-u| > 0 \Rightarrow |x-u| < 1.$$
$$= 0,\ 1-|x-u| < 0 \Rightarrow |x-u| > 1.$$

$$-1 < x-u < 1 \Rightarrow -1 < u-x < 1 \Rightarrow x-1 < u < x+1.$$

$$\therefore\ y(x) = \frac{1}{2}\int_{x-1}^{x+1} e^{-|u|}\, du.$$

<u>Case 1</u> $\underline{x < -1}$ $\quad x+1 < 0.$

$x-1 < -2.$

Here. $|u| = -u.\ \therefore e^{-|u|} = e^{u}.$

$$\therefore y(x) = \frac{1}{2}\int_{x-1}^{x+1} e^{u}\, du = \frac{1}{2}\left[e^{u}\right]_{x-1}^{x+1} = \frac{1}{2}\left[e^{x+1} - e^{x-1}\right]$$

$$y(x) = e^{x}\sinh(1).$$

<u>Case 2.</u> $-1 < x < 1.$ $\quad x-1 < 0.\quad x+1 > 0.$

$$y(x) = \frac{1}{2}\int_{x-1}^{x+1} e^{-|u|}\, du = \frac{1}{2}\int_{x-1}^{0} e^{-|u|}\, du + \frac{1}{2}\int_{0}^{x+1} e^{-|u|}\, du.$$

$$= \frac{1}{2}\int_{x-1}^{0} e^{u}\, du + \frac{1}{2}\int_{0}^{x+1} e^{-u}\, du = 1 - e^{x}\cosh 1.$$

<u>Case 3.</u> $x > 1.$ $\quad x-1 > 0,\quad x+1 > 2.$

$$\therefore y(x) = \frac{1}{2}\int_{x-1}^{x+1} e^{-|u|}\, du = \frac{1}{2}\int_{x-1}^{x+1} e^{-u}\, du = \frac{1}{2}\left[e^{-u}\right]_{x+1}^{x-1}$$

$$= \frac{1}{2} e^{-x}\left(e^{1} - e^{-1}\right) = e^{-x}\sinh(1)$$

$$y(x) = \begin{cases} e^{x}\sinh 1, & x < -1 \\ 1 - e^{x}\cosh 1, & -1 < x < 1 \\ e^{-x}\sinh 1, & x > 1. \end{cases}$$

TC Classes

Class test - 13/11/17 8:30 - 9:30 a.m.

23/10/17 — 8:00 - 9:55 (NR421)

30/10/17 — "

6/11/17 — "

23/10/17 — 5:00 - 5:55 p.m. (V4)

30/10/17 — "

6/11/17 — "

24/10/17 — 12:00 - 12:55 (NR 421)

31/10/17 — "

7/11/17 — "

14/11/17 — " (Discussion).