Lecture -17

12/9/17

Complex Fourier series.

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \longrightarrow (1)$$

$$f(x) = \sum \underline{\underline{C_n}}\, e^{inx}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}\right) + \sum_{n=1}^{\infty} \frac{b_n}{2}\frac{\left(e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{2}}\right)}{i}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\frac{a_n - ib_n}{2}}_{A_n} e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} \underbrace{\frac{a_n + ib_n}{2}}_{B_n} e^{-\frac{in\pi x}{l}}.$$

$$A_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - \frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right\}dx.$$

$$A_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx \longrightarrow (1)$$

$$\frac{a_0}{2} = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx. = A_0, \text{ by } (1).$$

$$B_n = \frac{a_n + i b_n}{2} = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + \frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$B_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{in\pi x}{l}}\,dx = A_{-n} \quad \text{by (1)}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} B_n e^{-\frac{in\pi x}{l}}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} A_{-n} e^{-\frac{in\pi x}{l}}$$

$n = -m$

$$= A_0 + \sum_{n=1}^{\infty} A_n e^{\frac{in\pi x}{l}} + \sum_{m=-1}^{-\infty} A_m e^{\frac{im\pi x}{l}}$$

$$= A_0 e^{\frac{i\cdot 0\pi x}{l}} + \sum_{n=1}^{\infty} A_n e^{\frac{in\pi x}{l}} + \sum_{n=-1}^{-\infty} A_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{\frac{in\pi x}{l}}$$

$$A_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

Ex. Express $f(x) = x$; $-2 \le x \le 2$ in terms of a complex Fourier series.

$$f(x) = x = \sum_{n=-\infty}^{\infty} A_n e^{i\frac{n\pi x}{2}}.$$

$$A_n = \frac{1}{4}\int_{-2}^{2} x e^{-\frac{in\pi x}{2}} dx.$$

$$= \frac{1}{4}\int_{-2}^{2} \underbrace{x \cos\frac{n\pi x}{2}}_{odd} dx - \frac{i}{4}\int_{-2}^{2} \underbrace{x \sin\frac{n\pi x}{2}}_{even} dx.$$

$$= 0 - \frac{i}{4}\times 2\int_{0}^{2} x \sin\frac{n\pi x}{2} dx$$

$$= \frac{i}{2}\left[x \cos\frac{n\pi x}{2} \cdot \frac{2}{n\pi}\Big|_0^2 - \int_0^2 \cos\frac{n\pi x}{2} \times \frac{2}{n\pi} dx \right].$$

$$= \frac{i}{2}\left[\frac{4}{n\pi}\cos n\pi - \underbrace{\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^2}_{=0} \right]$$

$$= \frac{2i}{\pi}\cdot\frac{(-1)^n}{n}.$$

$$\therefore \quad f(x) = x = \frac{2i}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{i\frac{n\pi x}{2}}.$$

$$A_0 = \frac{1}{4}\int_{-2}^{2} x \cdot e^{-\frac{i \cdot 0 \cdot \pi x}{2}} dx = \frac{1}{4}\int_{-2}^{2} x\, dx = 0.$$