State Feedback Controller

- System State equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$
- State feedback Control: $u = -\mathbf{K}\mathbf{x}$
- Closed loop system : $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$

Equate closed loop System characteristic equation with desired characteristic to get K

$$\left| sI - A + BK \right| = (s - \mu_1)\cdots(s - \mu_n)$$

For n > 3, Use Bass-Gura method or Ackerman's Formula

State Observer Design

Design Philosophy

- In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback.
- In practice, all state variables are not available for feedback. Possible reasons are
  - Non-Availability of sensors
  - Expensive sensors
  - Available sensors are not acceptable (due to high noise, high power consumption etc.)
- A state observer estimates the state variables based on the measurements of the output over a period of time.
- The system should be “observable”.

Observer Types

- State Observers are of two types :
- Full Order Observer
  - If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a *full-order state observer*.
- Reduced Order Observer
  - Sometimes some of the state variables are measured and therefore need not be observed. If the state observer estimates only the unmeasured states then the observer is called reduced order observer.

Full Order Observer

Consider a Process:

$$\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$$

$$y = \boldsymbol{CX}$$

Let the observed state is $\overline{\boldsymbol{X}}$

Observer Model:

$$\dot{\overline{\boldsymbol{X}}} = \overline{\boldsymbol{A}}\overline{\boldsymbol{X}} + \overline{\boldsymbol{B}}\boldsymbol{U} + \boldsymbol{K_e}y$$

So Error in observation

$$\boldsymbol{E} = \boldsymbol{X} - \overline{\boldsymbol{X}}$$

Error Dynamics will be

$$\dot{\boldsymbol{E}} = \dot{\boldsymbol{X}} - \dot{\overline{\boldsymbol{X}}}$$

Full Order Observer

Error Dynamics:

$$\dot{\boldsymbol{E}} = \dot{\boldsymbol{X}} - \dot{\overline{\boldsymbol{X}}} = (\boldsymbol{AX} + \boldsymbol{BU}) - (\overline{\boldsymbol{A}}\overline{\boldsymbol{X}} + \overline{\boldsymbol{B}}\boldsymbol{U} + \boldsymbol{K_e}y)$$

Add and Substract $\overline{\boldsymbol{A}}\boldsymbol{X}$ and substitute $\mathrm{y} = \boldsymbol{CX}$

$$\dot{\boldsymbol{E}} = \boldsymbol{AX} - \overline{\boldsymbol{A}}\boldsymbol{X} + \overline{\boldsymbol{A}}\boldsymbol{X} - \overline{\boldsymbol{A}}\overline{\boldsymbol{X}} + (\boldsymbol{B} - \overline{\boldsymbol{B}})\boldsymbol{U} - \boldsymbol{K_e CX}$$

$$= (\boldsymbol{A} - \overline{\boldsymbol{A}} - \boldsymbol{K_e C})\boldsymbol{X} + \overline{\boldsymbol{A}}(\boldsymbol{X} - \overline{\boldsymbol{X}}) + (\boldsymbol{B} - \overline{\boldsymbol{B}})\boldsymbol{U}$$

$$= \overline{\boldsymbol{A}}\boldsymbol{E} + (\boldsymbol{A} - \overline{\boldsymbol{A}} - \boldsymbol{K_e C})\boldsymbol{X} + (\boldsymbol{B} - \overline{\boldsymbol{B}})\boldsymbol{U}$$

To make error **E** zero, Strategy should be

1. Make error independent of **X** : $\boldsymbol{A} - \overline{\boldsymbol{A}} - \boldsymbol{K_e C} = \boldsymbol{0}$
2. Eliminate input, U from error dynamic equation :

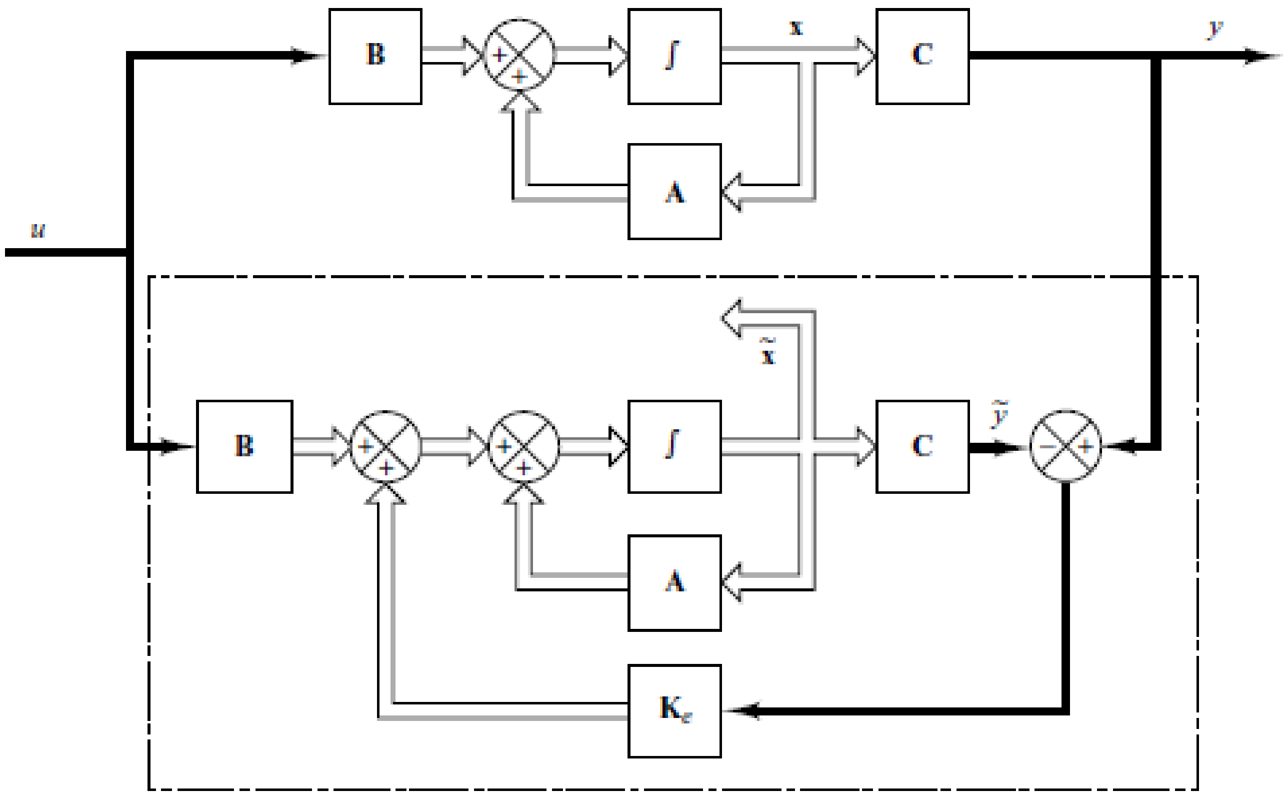
$$\boldsymbol{B} - \overline{\boldsymbol{B}} = \boldsymbol{0}$$

Full Order Observer

This leads to $\overline{A} = A - K_e C$ and $B = \overline{B}$

So, Error Dynamics : $\dot{E} = \overline{A}E = (A - K_e C)E$ and

Observer Dynamics: $\dot{\overline{X}} = A\overline{X} + BU + K_e(y - C\overline{X})$



Full-order state observer

Design of Observer : Full Order

- Objective: Find K_e such that
  - Estimation error is minimized.
  - Error dynamics is asymptotically stable and has sufficient speed of response.

Error Dynamics: $\dot{\boldsymbol{E}} = \overline{\boldsymbol{A}}\boldsymbol{E} = (\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})\boldsymbol{E}$

- So, $\lambda(\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})$ will determine the stability and speed of response of the error dynamics.
- If the $\lambda(\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.
- If the plant is completely observable, then it can be proved that it is possible to choose matrix $\mathbf{K}_e$ such that $\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C}$ has arbitrarily desired eigenvalues.

Design of Observer : Full Order

Controller Design	Observer Design
Dynamics:	Dynamics:
$\dot{\boldsymbol{X}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{X}$	$\dot{\boldsymbol{E}} = (\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})\boldsymbol{E}$
Aim: **X**(t) → 0 as t → ∞	Aim: **E**(t) → 0 as t → ∞
Place $\lambda(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ to desired location to get **K**	Place $\lambda(\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})$ to desired location to get $\boldsymbol{K_e}$

Note:

$$\boldsymbol{\lambda}(\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C}) = \boldsymbol{\lambda}[(\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})^T]$$
$$= \boldsymbol{\lambda}(\boldsymbol{A}^T - \boldsymbol{C}^T\boldsymbol{K_e}^T)$$

Observer Design – Dual system

Consider the dual system with input v and output y*

$$\dot{Z} = A^T Z + C^T v$$
$$y^* = B^T Z$$

Since original system is observable, the dual system is controllable.

State feedback controller design for dual system using pole placement technique yields

$$|sI - (A^T - C^T K_0)| = (s - \mu_1)(s - \mu_2) \ldots .(s - \mu_n)$$

Where $\mu_1, \mu_2, \ldots ., \mu_n$ are desired observer poles.

Now, equating observer characteristic equation with the r.h.s we get K_0. So Observer gain is $K_e = K_0^T$

Observer Design: Method 1

- For Systems of low order (n ≤ 3)
- Check Observability : Rank of N = order of system n
  - $N = [C^T \quad A^T C^T \quad \ldots.. \quad (A^T)^{n-1} C^T]$

- Define Observer gain $K_e = [K_1 \ K_2 \ K_3 \ \ldots. K_n]^T$
- Equate the observer characteristic equation with desired characteristic polynomial
  $|sI - (A - CK_e)| = (s - \mu_1)(s - \mu_2) \ldots. (s - \mu_n)$

- Solve for the gain elements by equating the like powers of polynomial in s.

Observer Design : Method 2

- Step 1: Find a_i from

$$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Step 2: Find α_i from

$$(s-\mu_1)(s-\mu_2)\ldots\ldots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$$

Step 3:

$$K_e = (WN^T)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ . \\ . \\ \alpha_1 - a_1 \end{bmatrix} \text{ where, } W = \begin{bmatrix} a_{n-1} & \cdots & a_1 & 1 \\ \vdots & \ddots & \cdots & \vdots \\ a_1 & & \cdots & 0 \\ 1 & \cdots & \cdots & 0 \end{bmatrix}$$

Observer Design : Method 3

- **Ackerman's Formula**
- $$\boldsymbol{K_e} = \phi(A)\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \vdots \\ \boldsymbol{CA^{n-2}} \\ \boldsymbol{CA^{n-1}} \end{bmatrix}^{\boldsymbol{-1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

Where,

$$\phi(\boldsymbol{A}) = \boldsymbol{A^n} + \alpha_1 \boldsymbol{A^{n-1}} + \;\dots\dots\dots\dots + \alpha_{n-1}\boldsymbol{A} + \alpha_n \boldsymbol{I}$$

Example: Biochemical Reactor

- Dynamic Model:

- $\frac{dC_b}{dt} = (\mu - F_d)C_b$ $\quad$ C_b biomass concentration

- $\frac{dC_s}{dt} = F_d\left(C_{sf} - C_s\right) - \frac{\mu C_b}{P}$

- $\mu = \frac{\mu_m C_s}{k_m + C_s}$ $\quad$ C_s Substrate Concentration

Data: μ_m=0.53 /hr ; k_m=0.12 g/l ; P= 0.4; F_d^s=0.3 /hr

$C_b^s = 1.530163$; $C_s^s = 0.174593$; $C_{sf} = 4.0$ g/l.

Ref: Process Control by B.W. Bequette

Biochemical Reactor

- Linearized Model:
- $x_1 = C_b - C_b^s;\;\; x_2 = C_s - C_s^s; u = F_d - F_d^s$
- $r = \left.\frac{\partial \mu}{\partial C_s}\right|_s = \frac{\mu_m k_m}{(k_m + C_s^s)^2}$
- $A = \begin{bmatrix} \mu^s - F_d^s & C_b^s\, r \\ -\frac{\mu^s}{P} & -F_d^s - \frac{r\, C_b^s}{P} \end{bmatrix} = \begin{bmatrix} 0 & 0.9056 \\ -0.75 & -2.564 \end{bmatrix}$
- $B = \begin{bmatrix} -C_s^b \\ C_{sf} - C_s^s \end{bmatrix} = \begin{bmatrix} -1.5301 \\ 3.8255 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ D=[0];

Observer Design

- Check Observabiity
  - $N = [C^T \ \ A^T C^T] = \begin{bmatrix} 0 & -0.75 \\ 1 & -2.564 \end{bmatrix}$ Rank(N) = 2
  - So system is observable.
- Characteristic Equation:
- $|sI \ - A| = \begin{vmatrix} s & 0.0679 \\ 0.75 & s + 0.1302 \end{vmatrix} = s^2 + 2.564s + 0.6792$

  $= s^2 + a_1 s + a_2 = 0$ So, a_1 =2.564 and a_2 = 0.6792

- Desired Characteristic Equation: (desired poles at -$\mu_{1,2}$)
- $(s + \mu_1)(s + \mu_2) = s^2 + (\mu_1 + \mu_2)s + \mu_1\mu_2$

  $= s^2 + \alpha_1 s + \alpha_2 = 0 \quad \alpha_1 = \mu_1 + \mu_2; \alpha_2 = \mu_1\mu_2$

Observer Design

- Observer Gain (Bass-Gura Method)

- $W = \begin{bmatrix} 2.564 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad N^T = \begin{bmatrix} 0 & 1 \\ -0.75 & -0.2.564 \end{bmatrix}$

- $K_e = (WN^T)^{-1} \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} -1.33 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1\mu_2 - 0.6792 \\ (\mu_1 + \mu_2) - 2.564 \end{bmatrix}$

- $= \begin{bmatrix} -1.33(\mu_1\mu_2 - 0.6792) \\ (\mu_1 + \mu_2) - 2.564 \end{bmatrix}$

Observer Design

- Observer Gain (Ackerman's Method)
- $\phi(A) = A^2 + \alpha_1 A + \alpha_2 I$
- $= \begin{bmatrix} 0 & 0.9056 \\ -0.75 & -2.564 \end{bmatrix}^2 + (\mu_1 + \mu_2)\begin{bmatrix} 0 & 0.9056 \\ -0.75 & -2.564 \end{bmatrix} + \mu_1\mu_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $= \begin{bmatrix} -0.6792 + \mu_1\mu_2 & 2.322 - 0.9056(\mu_1 + \mu_2) \\ 1.923 - 0.75(\mu_1 + \mu_2) & 5.8949 - 2.564(\mu_1 + \mu_2) + \mu_1\mu_2 \end{bmatrix}$

$$K_e = \phi(A)\begin{bmatrix} C \\ CA \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \phi(A)\begin{bmatrix} 0 & 1 \\ -0.75 & -2.564 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \phi(A)\begin{bmatrix} -1.333 \\ 0 \end{bmatrix}$$

So, $K_e = \begin{bmatrix} -1.33(\mu_1\mu_2 - 0.6792) \\ (\mu_1 + \mu_2) - 2.564 \end{bmatrix} = \begin{bmatrix} -5.0684 \\ 1.436 \end{bmatrix}$ for $\mu_{1,2} = -2 \pm 0.7j$

Separation Principle

System Dynamics: $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$

$$y = \boldsymbol{CX}$$

- State feedback control based on observed state:

$$\boldsymbol{U} = -\boldsymbol{K}\overline{\boldsymbol{X}}$$

State equation:

$$\dot{\boldsymbol{X}} = \boldsymbol{AX} - \boldsymbol{BK}\overline{\boldsymbol{X}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{X} + \boldsymbol{BK}(\boldsymbol{X} - \overline{\boldsymbol{X}})$$

Error : $\boldsymbol{E} = \boldsymbol{X} - \overline{\boldsymbol{X}}$ So, $\dot{\boldsymbol{X}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{X} + \boldsymbol{BKE}$

Observer Error Eqn. : $\dot{\boldsymbol{E}} = (\boldsymbol{A} - \boldsymbol{K_e}\boldsymbol{C})\boldsymbol{E}$

Separation Principle

- Combined Observer-Controller closed loop equation:

$$\begin{bmatrix} \dot{\boldsymbol{X}} \\ \dot{\boldsymbol{E}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{K} \\ 0 & \boldsymbol{A} - \boldsymbol{K}_{\boldsymbol{e}}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{E} \end{bmatrix}$$

Characteristic Equation:

$$\begin{vmatrix} s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{K} \\ 0 & s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{K}_{\boldsymbol{e}}\boldsymbol{C} \end{vmatrix} = 0$$

$$|s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}||s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{K}_{\boldsymbol{e}}\boldsymbol{C}| = 0$$

Poles due to Controller

Poles due to Observer

Hence Observer design and Pole placement are independent of each other!

This is known as "Separation Theorem".

Transfer Function of Observer Controller System

- Full order observer

$$\dot{\overline{\boldsymbol{X}}} = \boldsymbol{A}\overline{\boldsymbol{X}} + \boldsymbol{B}\boldsymbol{U} + \boldsymbol{K_e}(y - \boldsymbol{C}\overline{\boldsymbol{X}})$$

- State Feedback Controller: $\boldsymbol{U} = -\boldsymbol{K}\overline{\boldsymbol{X}}$

- Full order observer and Controller

$$\dot{\overline{\boldsymbol{X}}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} - \boldsymbol{K_e}\boldsymbol{C})\overline{\boldsymbol{X}} + \boldsymbol{K_e}y$$

- Taking Laplace Transform

$$s\overline{\boldsymbol{X}}(s) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} - \boldsymbol{K_e}\boldsymbol{C})\overline{\boldsymbol{X}}(s) + \boldsymbol{K_e}y(s)$$

$$\overline{\boldsymbol{X}}(s) = [s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} + \boldsymbol{K_e}\boldsymbol{C}]^{-1}\boldsymbol{K_e}y(s)$$

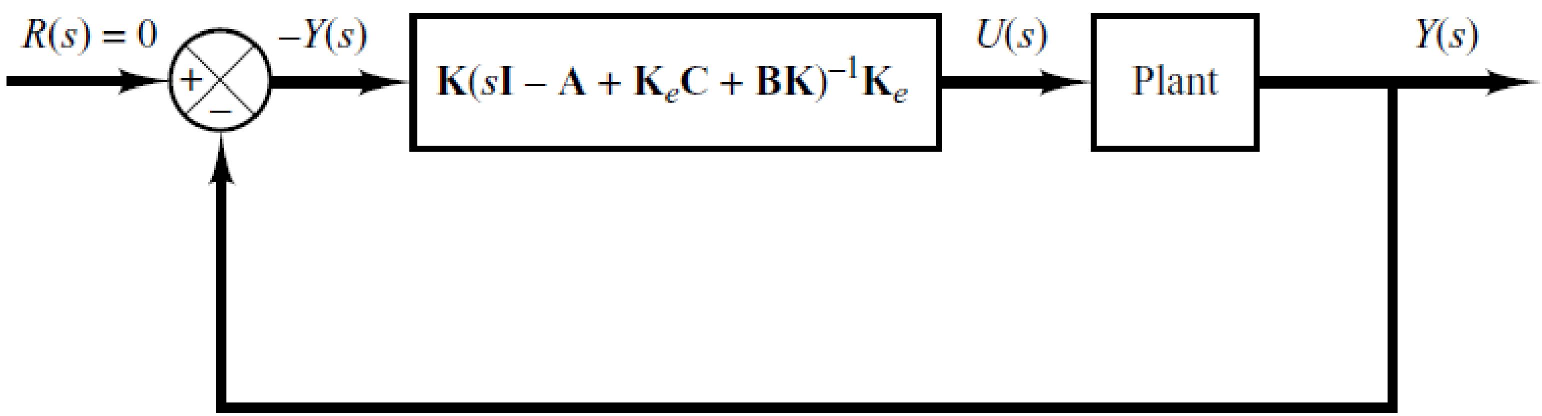
- Also, L.T. on Controller eqn. gives $\boldsymbol{U}(s) = -\boldsymbol{K}\overline{\boldsymbol{X}}(s)$

Transfer Function of Observer Controller System

- So, $U(s) = -\boldsymbol{K}[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} + \boldsymbol{K_e}\boldsymbol{C}]^{-1}\boldsymbol{K_e}y(s)$

- Observer-state feedback controller

$$\frac{U(s)}{-y(s)} = \boldsymbol{K}[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} + \boldsymbol{K_e}\boldsymbol{C}]^{-1}\boldsymbol{K_e}$$



Reduced Order Observer

Reduced Order Observer

- Some of the state variables are accurately measured.
- Suppose $\boldsymbol{X}$ is an *n - vector* and output *y* is an *m* **-** *vector* that can be measured .
- We need to estimate only (*n*-*m*) state variables.
- The reduced-order observer becomes (*n*-*m*)th order observer.

Reduced Order Observer

- System : $\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}u$ order = n
  $y = \boldsymbol{C}\boldsymbol{X}$ order = 1
- So, (n-1) state variables to be estimated.
- Let $X = \begin{bmatrix} X_a \\ \boldsymbol{X_b} \end{bmatrix}$ where X_a is the measured state and $\boldsymbol{X}_b$ is unmeasured states.
- State equation can be written as,

$$\begin{bmatrix} \dot{X}_a \\ \dot{\boldsymbol{X}}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{aa} & \boldsymbol{A}_{ab} \\ \boldsymbol{A}_{ba} & \boldsymbol{A}_{bb} \end{bmatrix} \begin{bmatrix} X_a \\ \boldsymbol{X}_b \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_a \\ \boldsymbol{B}_b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_a \\ \boldsymbol{X}_b \end{bmatrix}$$

X_a is scalar,

$\mathbf{X}_b$ is nx1 vector

Reduced Order Observer

- The equation for the measured portion of the state,

$$\dot{X}_a = \boldsymbol{A}_{aa}X_a + \boldsymbol{A}_{ab}\boldsymbol{X}_b + \boldsymbol{B}_a u$$

$$\dot{X}_a - \boldsymbol{A}_{aa}X_a - \boldsymbol{B}_a u = \boldsymbol{A}_{ab}\boldsymbol{X}_b$$

- The equation for the unmeasured portion of the state

$$\dot{X}_b = \boldsymbol{A}_{ba}X_a + \boldsymbol{A}_{bb}\boldsymbol{X}_b + \boldsymbol{B}_b u$$

Here, $A_{ba}X_a$ and $B_b u$ are known quantities.

Reduced Order Observer

- State/Output equation for Full order observer

$$\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{U}$$

$$y = \boldsymbol{C}\boldsymbol{X}$$

$$\dot{\overline{\boldsymbol{X}}} = \boldsymbol{A}\overline{\boldsymbol{X}} + \boldsymbol{B}\boldsymbol{U} + \boldsymbol{K_e}(y - \boldsymbol{C}\overline{\boldsymbol{X}})$$

- State/Output equation for Reduced order observer

$$\dot{X}_b = \boldsymbol{A}_{bb}\boldsymbol{X}_b + \boldsymbol{A}_{ba}X_a + \boldsymbol{B}_b u$$

$$\dot{X}_a - \boldsymbol{A}_{aa}X_a - \boldsymbol{B}_a u = \boldsymbol{A}_{ab}\boldsymbol{X}_b$$

What will be reduced order observer equation?

Comparison: Full Order with Reduced order

Full – Order State Observer	Reduced Order State observer
$\tilde{X}$	$\tilde{X}_b$
A	A_{bb}
Bu	$A_{ba}x_a + B_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
C	A_{ab}
$K_e(n \times 1\ matrix)$	$K_e[(n-1) \times 1\ matrix]$

Reduced Order Observer Design

- Full order Observer equation :

$$\dot{\tilde{X}} = (A - K_e C)\tilde{X} + Bu + K_e y$$

- Making substitutions from the table,

$$\dot{\tilde{X}}_b = (A_{bb} - K_e A_{ab})\tilde{X}_b + A_{ba} x_a + B_b u + K_e(\dot{x}_a - A_{aa} x_a - B_a u)$$

i.e.

$$\begin{aligned}
\dot{\tilde{X}}_b - K_e \dot{x}_a &= (A_{bb} - K_e A_{ab})\tilde{X}_b + (A_{ba} - K_e A_{aa})y + (B_b - K_e B_a)u \\
&= (A_{bb} - K_e A_{ab})(\tilde{X}_b - K_e y) \\
&\quad + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u
\end{aligned}$$

Reduced Order Observer Design

- Define

$$X_b - K_e y = \left(X_b - K_e x_a\right) \triangleq \eta$$

$$\tilde{X}_b - K_e y = \left(\tilde{X}_b - K_e x_a\right) \triangleq \tilde{\eta}$$

- Then $\dot{\tilde{\eta}} = (A_{bb} - K_e A_{ab})\tilde{\eta} +$

$$\left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right] y + (B_b - K_e B_a)u$$

This is reduced order observer.

Reduced Order Observer Design

We have:

$$\dot{X}_b = A_{bb}X_b + \left(A_{ba}x_a + B_b u\right)$$

$$\dot{\tilde{X}}_b = \left(A_{bb} - K_e A_{ab}\right)\tilde{X}_b + \left(A_{ba}x_a + B_b u\right) + K_e A_{ab} X_b$$

Subtracting:

$$\dot{X}_b - \dot{\tilde{X}}_b = \left(A_{bb}X_b - K_e A_{ab} X_b\right) - \left(A_{bb} - K_e A_{ab}\right)\tilde{X}_b$$

$$= \left(A_{bb} - K_e A_{ab}\right)\underbrace{\left(X_b - \tilde{X}_b\right)}_{E}$$

$i.e.\quad \dot{E} = \left(A_{bb} - K_e A_{ab}\right)E$

where $\; E \triangleq \left(X_b - \tilde{X}_b\right) = \left(\eta - \tilde{\eta}\right)$

Characteristic Equation

Necessary Condition

The error dynamics can be chosen provided the rank of matrix

$$\begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ . \\ . \\ A_{ab}A_{bb}^{n\text{-}2} \end{bmatrix}$$

is $(n-1)$. This is complete observability condition

Characteristic Equation:

$$\left|sI - A_{bb} + K_e A_{ab}\right| = (s-\mu_1)(s-\mu_2).......(s-\mu_{n\text{-}1})$$

$$= s^{n-1} + \hat{\alpha}_1 s^{n-2} + ... + \hat{\alpha}_{n\text{-}2} s + \hat{\alpha}_{n\text{-}1} = 0$$

where $\mu_1, \mu_2, ...\mu_{n\text{-}1}$ are desired eigenvalues of error dynamics

RO Observer Gain Computation

$$K_e = \hat{Q}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ . \\ . \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} = (\hat{W}\hat{N}^T)^{-1}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ . \\ . \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix}$$

where

$$\hat{N} = \left[\, A_{ab}^T \mid A_{bb}^T A_{ab}^T \mid \ldots.. \mid (A_{bb}^T)^{n-2} A_{ab}^T \,\right]: \quad (n-1)\times(n-1) \text{ matrix.}$$

$$\hat{W} = \begin{bmatrix} \hat{a}_{n-2} & \hat{a}_{n-3} & \ldots\ldots & \hat{a}_1 & 1 \\ \hat{a}_{n-3} & \hat{a}_{n-4} & \ldots\ldots & 1 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ \hat{a}_1 & 1 & & 0 & 0 \\ 1 & 0 & . \; . \; . & 0 & 0 \end{bmatrix}: \quad (n-1)\times(n-1) \text{ matrix .}$$

RO Observer Gain Computation

- $\hat{a}_1, \hat{a}_2, \ldots\ldots \hat{a}_{n\text{-}2}$ are coefficients in the characteristic equation $\left|sI - A_{bb}\right| = s^{n\text{-}1} + \hat{a}_1 s^{n\text{-}2} + \ldots. + \hat{a}_{n\text{-}2} s + \hat{a}_{n\text{-}1} = 0.$
- **Ackermann's formula :** $K_e = \phi(A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ . \\ . \\ . \\ A_{ab}A_{bb}^{n-3} \\ A_{ab}A_{bb}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$

where $\phi(A_{bb}) = A_{bb}^{n\text{-}1} + \hat{\alpha}_1 A_{bb}^{n\text{-}2} + \ldots.. + \hat{\alpha}_{n-2} A_{bb} + \hat{\alpha}_{n-1} I$

Separation Principle

- The system characteristic equation can be derived as

$$|sI - A + BK||sI - A_{bb} + K_e A_{ab}| = 0$$

Poles due to pole placement

Poles due to reduced order Observer

- Therefore the pole-placement design and the design of the reduced order observer are independent of each other.