Lecture - 19
9/10/17

Examples on Fourier cosine & sine integrals.

Ex 1. Find the 1) Fourier cosine integral (F C I) (representation) and 2) F. S. I. (sine) of

$$f(x) = \sin x, \quad 0 \le x \le \pi$$
$$= 0, \quad x > \pi.$$

1) Define
$$F(x) = \begin{cases} f(x), & x \ge 0 \\ f(-x), & x < 0 \end{cases}$$

Then $F(x)$ is an even function defined in $[-\infty, \infty]$.

$$\therefore F(x) = \int_0^\infty A(\alpha) \cos \alpha x \, d\alpha, \quad -\infty < x < \infty.$$

where
$$A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{F(u)}_{\text{even}} \underbrace{\cos \alpha u}_{\text{even}} \, du.$$

$$\therefore \cancel{A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\alpha}} = \frac{2}{\pi} \int_0^\infty F(u) \cos \alpha u \, du.$$

$$= \frac{2}{\pi} \int_0^\infty f(u) \cos \alpha u \, du$$

$$= \frac{2}{\pi} \int_0^\pi \sin u \cos \alpha u \, du.$$

$$= \frac{1}{\pi} \int_0^{\pi} \left\{ \sin(1+\alpha)u + \sin(1-\alpha)u \right\} du.$$

$$= \frac{-1}{\pi} \left[\frac{\cos(1+\alpha)u}{1+\alpha} \Bigg|_0^{\pi} + \frac{\cos(1-\alpha)u}{1-\alpha} \Bigg|_0^{\pi} \right].$$

$$= -\frac{1}{\pi} \left[\frac{\cos(1+\alpha)\pi - 1}{1+\alpha} + \frac{\cos(1-\alpha)\pi - 1}{1-\alpha} \right].$$

$$\cancel{= -\frac{1}{\pi} \left[\frac{(-1)^{1+\alpha} - 1}{1+\alpha} + \frac{(-1)^{1-\alpha} - 1}{1-\alpha} \right]}.$$

$$A(\alpha) = \frac{2}{\pi(1-\alpha^2)} (1 + \cos\alpha\pi) \quad (\text{pls. check}).$$

$$F(x) = \int_0^{\infty} A(\alpha) \cos\alpha x \, d\alpha.$$

$$F(x) = \frac{2}{\pi} \int_0^{\infty} \frac{(1+\cos\alpha\pi)}{1-\alpha^2} \cos\alpha x \, d\alpha, \quad -\infty < x < \infty$$

Or, $$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{1+\cos\alpha\pi}{1-\alpha^2} \cos\alpha x \, d\alpha, \quad 0 \le x < \infty.$$

Or, $$\sin x = \frac{2}{\pi} \int_0^{\infty} \frac{1+\cos\alpha\pi}{1-\alpha^2} \cos\alpha x \, d\alpha, \quad 0 \le x \le \pi.$$

2) F. S. I. of $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

Define $G(x) = \begin{cases} f(x), & 0 \le x < \infty \\ -f(-x), & -\infty < x \le 0. \end{cases}$

$G(x)$ is an odd function.

$$G(x) = \int_0^\infty B(\alpha) \sin \alpha x \, d\alpha.$$

$$B(\alpha) = \frac{2}{\pi} \int_0^\infty G(u) \sin \alpha u \, du.$$

$$= \frac{2}{\pi} \int_0^\infty f(u) \sin \alpha u \, du.$$

$$= \frac{2}{\pi} \int_0^\pi \sin u \sin \alpha u \, du.$$

$$= \frac{2}{\pi} \frac{\sin \alpha \pi}{1-\alpha^2}$$

note: α may not be an integer

$$G(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha \pi}{1-\alpha^2} \sin \alpha x \, d\alpha, \quad -\infty < x < \infty.$$

or, $f(x) =$ " , $0 \le x < \infty$.

or, $\sin x = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha\pi \sin \alpha x}{1-\alpha^2} d\alpha, \quad 0 \le x \le \pi.$

Fourier transform of $f(x)$.

Complex of Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(u-x)}\, du\, d\alpha .$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\alpha \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\underbrace{e^{i\alpha(u-x)}}_{= e^{i\alpha u}\cdot e^{-i\alpha x}}\, du\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha x}\, d\alpha \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du\right) \longrightarrow (1).$$

Let us call $F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du . \longrightarrow (2).$

Then (1) can be written as,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha \longrightarrow (3).$$

$F(\alpha)$ defined in (2) is known as Fourier transform of $f(x)$ <u>defined in $(-\infty, \infty)$</u>.

So, $\mathcal{F}[f(x); \alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = F(\alpha)$

Inverse Fourier transform of $F(\alpha)$ is,

$$f(x) = \mathcal{F}^{-1}[F(\alpha); x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha .$$

Look at ~~(1)~~ expression before (1)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\alpha(u-x)}\, du \right).$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{i\alpha(u-x)}\, d\alpha\, du.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \int_{-\infty}^{\infty} e^{i\alpha(u-x)}\, d\alpha.$$

put $\alpha' = -\alpha$.

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \int_{\infty}^{-\infty} e^{-i\alpha'(u-x)} \times -d\alpha'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \int_{-\infty}^{\infty} e^{-i\alpha'(u-x)}\, d\alpha'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha(u-x)}\, d\alpha\, du.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du \right)$$

If $F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du, \; = \mathcal{F}[f(x); \alpha]$

then $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha = \mathcal{F}^{-1}[F(\alpha); x]$

We will follow:

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx.$$

& $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha.$$

<u>Q1.</u> Find the Fourier transform of ~~$f(x)$~~ = ~~1~~ $1/x$. Hence find the value of

$$f(x) = e^{-|x|} \qquad \int_0^{\infty} \frac{\cos \omega x}{1+x^2}\, dx.$$

$$f(x) = \begin{cases} e^{-x}, & x \geq 0. \\ e^{x}, & x < 0. \end{cases}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{+x} \cdot e^{i\omega x}\, dx + \int_{0}^{\infty} e^{-x}\, e^{i\omega x}\, dx \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1+i\omega)x}\, dx + \int_{0}^{\infty} e^{-(1-i\omega)x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left. \frac{e^{(1+i\omega)x}}{1+i\omega} \right|_{-\infty}^{0} + \left. \frac{e^{-(1-i\omega)x}}{1-i\omega} \right|_{\infty}^{0} \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1+i\omega} + \frac{1}{1-i\omega}\right] = \frac{2}{\sqrt{2\pi}\,(1+\omega^2)}$$

$$F(\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\omega^2}.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega.$$

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\omega^2}\, e^{-i\omega x}\, d\omega.$$

$$\int_0^{\infty} \frac{\cos \omega x}{1+x^2}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{\cos \omega x - i \sin \omega x}{1+\omega^2}\, d\omega.$$

Equating real part,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos \omega x}{1+\omega^2}\, d\omega = e^{-|x|}.$$

$$\text{or,} \quad \frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x}{1+\omega^2}\, d\omega = e^{-|x|}.$$

$$\text{or,} \quad \int_0^{\infty} \frac{\cos \omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-|x|}.$$

$$\text{or,} \quad \int_0^{\infty} \frac{\cos \omega x}{1+x^2}\, dx = \frac{\pi}{2} e^{-|\omega|} \quad //.$$