

## 5.1 Laplace Transform of Derivatives

Before we state the derivative theorem, it should be noted that this results is the key aspect for its application of solving differential equations.

### 5.1.1 Derivative Theorem

*Suppose  $f$  is continuous on  $[0, \infty)$  and is of exponential order  $\alpha$  and that  $f'$  is piecewise continuous on  $[0, \infty)$ . Then*

$$L[f'(t)] = sL[f(t)] - f(0), \quad \text{Re}(s) > \alpha.$$

**Proof:** By the definition of Laplace transform, we have

$$L[f'(t)] = \int_0^{\infty} f'(t)e^{-st} dt$$

Integrating by parts, we get

$$L[f'(t)] = f(t)e^{-st} \Big|_0^{\infty} - \int_0^{\infty} f(t)e^{-st}(-s) dt$$

Using the definition of Laplace transform we obtain

$$L[f'(t)] = -f(0) + sL[f(t)], \quad \text{Re}(s) > \alpha.$$

This completes the proof. ■

**Remark 1:** *Suppose  $f(t)$  is not continuous at  $t = 0$ , then the results of the above theorem takes the following form*

$$L[f'(t)] = -f(0+) + sL[f(t)]$$

**Remark 2:** *An interesting feature of the derivative theorem is that  $L[f'(t)]$  exists without the requirement of  $f'$  to be of exponential order. Recall the existence of Laplace transform of  $f(t) = 2te^{t^2} \cos(e^{t^2})$  which is obvious now by the derivative theorem because*

$$f(t) = \left( \sin(e^{t^2}) \right)'.$$

**Remark 3:** *The derivative theorem can be generalized as*

$$\begin{aligned}L[f''(t)] &= -f'(0) + sL[f'(t)] \\ &= -f'(0) + s\{-f(0) + sL[f(t)]\} = s^2L[f(t)] - sf(0) - f'(0).\end{aligned}$$

*In general, for  $n$ th derivative we have*

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0).$$

## 5.2 Example Problems

### 5.2.1 Problem 1

*Determine  $L[\sin^2 \omega t]$ .*

**Solution:** Let us assume that

$$f(t) = \sin^2 \omega t$$

Now we compute the derivative of  $f$  as

$$f'(t) = 2 \sin \omega t \cos \omega t \omega = \omega \sin 2\omega t.$$

Using the derivative theorem we have

$$L[f'(t)] = -f(0) + sL[f(t)]$$

Substituting the function  $f(t)$  and its derivative we find

$$L[\omega \sin 2\omega t] = sL[\sin^2 \omega t] - 0$$

Therefore, we have

$$L[\sin^2 \omega t] = \frac{\omega}{s} \left( \frac{2\omega}{s^2 + 4\omega^2} \right)$$

### 5.2.2 Problem 2

*Using derivative theorem, find  $L[t^n]$ .*

**Solution:** Let

$$f(t) = t^n.$$

Then

$$f'(t) = nt^{n-1}, \quad f''(t) = n(n-1)t^{n-2}, \dots, \quad f^n(t) = n!.$$

From derivative theorem we have

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0).$$

Therefore, we find

$$L[n!] = s^n L[t^n] \Rightarrow L[t^n] = \frac{n!}{s^{n+1}}.$$

### 5.2.3 Problem 3

*Using derivative theorem, find  $L[\sin kt]$ .*

**Solution:** Let  $f(t) = \sin kt$  and therefore we have

$$f'(t) = k \cos kt \quad \text{and} \quad f''(t) = -k^2 \sin kt$$

Substituting in the derivative theorem

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

yields

$$L[-k^2 \sin kt] = s^2 L[\sin kt] - 0 - k$$

On simplifications we get

$$L[\sin kt] = \frac{k}{s^2 + k^2}$$

### 5.2.4 Problem 4

*Using  $L[t^2] = 2/s^3$  and derivative theorem, find  $L[t^5]$ .*

**Solution:** Let  $f(t) = t^5$  so that  $f'(t) = 5t^4$ ,  $f''(t) = 20t^3$ ,  $f'''(t) = 60t^2$ . The derivative theorem for third derivative reads as

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

This implies

$$L[60t^2] = s^3 L[f(t)] \Rightarrow L[f(t)] = \frac{120}{s^6}.$$

### 5.2.5 Problem 5

Using the Laplace transform of  $L[\sin \sqrt{t}]$  and applying the derivative theorem, find the Laplace transform of the function

$$\frac{\cos \sqrt{t}}{\sqrt{t}}$$

**Solution:** We know that

$$L[\sin \sqrt{t}] = \frac{1}{2s} \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$$

Let  $f(t) = \sin \sqrt{t}$ , then we have

$$f(0) = 0 \quad \text{and} \quad f'(t) = \frac{\cos \sqrt{t}}{2\sqrt{t}}$$

Substitution of  $f(t)$  in the derivative theorem

$$L[f'(t)] = sL[f(t)] - f(0)$$

yields

$$L\left[\frac{\cos \sqrt{t}}{2\sqrt{t}}\right] = s \frac{1}{2s} \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$$

Thus, we get

$$L\left[\frac{\cos \sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$$

## 5.3 Laplace Transform of Integrals

### 5.3.1 Theorem

Suppose  $f(t)$  is piecewise continuous on  $[0, \infty)$  and the function

$$g(t) = \int_0^t f(u) \, du$$

is of exponential order. Then

$$L[g(t)] = \frac{1}{s} F(s).$$

**Proof:** Clearly  $g(0) = 0$  and  $g'(t) = f(t)$ . Note that  $g(t)$  is piecewise continuous and is of exponential order as well as  $g'(t) = f(t)$  is piecewise continuous. Then, we get using the derivative theorem

$$L[g'(t)] = sL[g(t)] - g(0)$$

Since  $g(0) = 0$  we obtain the desired result as

$$L[g(t)] = \frac{1}{s}L[f(t)]$$

This completes the proof. ■

## 5.4 Example Problems

### 5.4.1 Problem 1

Given that

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{1+s^2} ds.$$

Find the Laplace transform of the integral

$$\int_0^t \frac{\sin u}{u} du.$$

**Solution:** Direct application of the above result gives

$$\begin{aligned} L\left[\int_0^t \frac{\sin u}{u} du\right] &= \frac{1}{s}L\left[\frac{\sin t}{t}\right] \\ &= \frac{1}{s} \int_s^\infty \frac{1}{1+s^2} ds = \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s\right] \end{aligned}$$

Thus, we have

$$L\left[\int_0^t \frac{\sin u}{u} du\right] = \frac{1}{s} \cot^{-1} s$$

### 5.4.2 Problem 2

Find Laplace transform of the following integral

$$\int_0^t u^n e^{-au} du$$

**Solution:** With the application of the first shifting theorem we know that

$$L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$$

It follows from the above result on Laplace transform of integrals

$$L\left[\int_0^t u^n e^{-au} du\right] = \frac{1}{s} L[t^n e^{-at}] = \frac{n!}{s(s+a)^{n+1}}.$$

## 5.5 Multiplication by $t^n$

### 5.5.1 Theorem

If  $F(s)$  is the Laplace transform of  $f(t)$ , i.e.,  $L[f(t)] = F(s)$  then,

$$L[tf(t)] = -\frac{d}{ds}F(s)$$

and in general the following result holds

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s).$$

**Proof:** By definition we know

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Using Leibnitz rule for differentiation under integral sign we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty (-t) e^{-st} f(t) dt$$

Thus we get

$$\frac{dF(s)}{ds} = -L[tf(t)]$$

Repeated differentiation under integral sign gives the general rule. ■

Applicability of the above result will now be demonstrated by some examples.

## 5.6 Example Problems

### 5.6.1 Problem 1

Find Laplace transform of the function  $t^2 \cos at$ .

**Solution:** We know from Laplace transform of elementary functions that

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Direct application of the above rule gives

$$L[t^2 \cos at] = \frac{d^2}{ds^2} \left( \frac{s}{s^2 + a^2} \right) = \frac{d}{ds} \left( \frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right) = \frac{d}{ds} \left( \frac{a^2 - s^2}{(s^2 + a^2)^2} \right)$$

On simplifications we find

$$L[t^2 \cos at] = \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}$$

### 5.6.2 Problem 2

Evaluate (i)  $L[te^{-t}]$  (ii)  $L[t^2e^{-t}]$  (iii)  $L[t^ke^{-t}]$

**Solution:** (i) We know that

$$L[e^{-t}] = \frac{1}{s+1}$$

Using the above mentioned rule we find

$$L[te^{-t}] = -\frac{d}{ds} \frac{1}{s+1} = \frac{1}{(s+1)^2}$$

(ii) Applying the same idea once again, we obtain

$$L[t^2e^{-t}] = -\frac{d}{ds} \frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

(iii) Similarly, we can further generalize this result as

$$L[t^ke^{-t}] = \frac{k!}{(s+1)^{k+1}}$$

### 5.6.3 Problem 3

Find the Laplace transform of  $f(t) = (t^2 - 3t + 2) \sin t$

**Solution:** Using linearity of the Laplace transform we have

$$L[f(t)] = L[t^2 \sin t] - 3L[t \sin t] + 2L[\sin t] \quad (5.1)$$

Since we know

$$L[\sin t] = \frac{1}{1 + s^2}$$

then

$$L[t \sin t] = -\frac{d}{ds} \frac{1}{1 + s^2} = \frac{2s}{(1 + s^2)^2}$$

and

$$L[t^2 \sin t] = -\frac{d}{ds} \frac{2s}{(1 + s^2)^2} = \frac{2(1 + s^2)^2 - 8s^2(1 + s^2)}{(1 + s^2)^4} = \frac{6s^2 - 2}{(1 + s^2)^3}$$

Substituting the above values in the equation (5.1), we find

$$L[f(t)] = \frac{6s^2 - 2}{(1 + s^2)^3} - \frac{6s}{(1 + s^2)^2} + \frac{2}{1 + s^2}$$

Further simplifications lead to

$$L[f(t)] = \frac{6s^2 - 2 - 6s(1 + s^2) + 2(1 + s^2)^2}{(1 + s^2)^3}$$

Finally, we obtain

$$L[f(t)] = \frac{(2s^4 - 6s^3 + 10s^2 - 6s)}{(s^6 + 3s^4 + 3s^2 + 1)}$$

## 5.7 Division by $t$

### 5.7.1 Theorem

If  $f$  is piecewise continuous on  $[0, \infty)$  and is of exponential order  $\alpha$  such that

$$\lim_{t \rightarrow 0+} \frac{f(t)}{t}$$

exists, then,

$$L \left[ \frac{f(t)}{t} \right] = \int_s^\infty F(u) \, du, \quad [s > \alpha]$$



**Proof:** This can easily be proved by letting  $g(t) = \frac{f(t)}{t}$  so that  $f(t) = tg(t)$ .

Hence,

$$F(s) = L[f(t)] = L[tg(t)] = \frac{d}{ds}L[g(t)]$$

Integrating with respect to  $s$  we get,

$$-L[g(t)] \Big|_s^\infty = \int_s^\infty F(s) ds.$$

Since  $g(t)$  is piecewise continuous and of exponential order, it follows that  $\lim_{s \rightarrow \infty} L[g(t)] \rightarrow 0$ .

Thus we have

$$L[g(t)] = \int_s^\infty F(s) ds.$$

This completes the proof. ■

**Remark:** It should be noted that the condition  $\lim_{t \rightarrow 0^+} [f(t)/t]$  is very important because without this condition the function  $g(t)$  may not be piecewise continuous on  $[0, \infty)$ . Thus without this condition we can not use the fact  $\lim_{s \rightarrow \infty} L[g(t)] \rightarrow 0$ .

### 5.7.2 Corollary

If  $L[f(t)] = F(s)$  then  $\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty f(s) ds$ , provided that the integrals converge.

**Proof:** We know that

$$L \left[ \frac{f(t)}{t} \right] = \int_s^\infty F(u) du$$

Using the definition of Laplace transform we get

$$\int_0^\infty e^{-st} \frac{f(t)}{t} dt = \int_s^\infty F(u) du$$

Taking limit  $s \rightarrow 0$  in above two integrals we obtain

$$\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(u) du$$

This completes the proof. ■

## 5.8 Example Problems

### 5.8.1 Problem 1

Find the Laplace transform of the function

$$f(t) = \frac{\sin at}{t}$$

**Solution:** We know,

$$L[\sin at] = \frac{a}{s^2 + a^2} \quad \text{and} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u) \, du$$

On integrating we get,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2 + a^2} \, ds = \tan^{-1}\left(\frac{s}{a}\right) \Big|_s^\infty$$

Thus we have

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

### 5.8.2 Problem 2

Find the Laplace transform of the function

$$f(t) = \frac{2 \sin t \sinh t}{t}$$

**Solution:** Using Division by  $t$  property of the Laplace transform we get

$$L[f(t)] = \int_s^\infty L[\sin t (e^t - e^{-t})] \, ds \tag{5.2}$$

Now we evaluate  $L[\sin t (e^t - e^{-t})]$  using linearity of the Laplace transform as

$$L[\sin t (e^t - e^{-t})] = L[e^t \sin t] - L[e^{-t} \sin t]$$

Applying the first shifting theorem we obtain

$$L[\sin t (e^t - e^{-t})] = \frac{1}{1 + (s - 1)^2} - \frac{1}{1 + (s + 1)^2}$$

Substituting this value in the equation (5.2) we find

$$L[f(t)] = \int_s^\infty \left[ \frac{1}{1 + (s-1)^2} - \frac{1}{1 + (s+1)^2} \right] ds$$

On integrating, we have

$$\begin{aligned} L[f(t)] &= \tan^{-1}(s-1) \Big|_s^\infty - \tan^{-1}(s+1) \Big|_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \end{aligned}$$

On cancellation of  $\pi/2$  we get

$$L[f(t)] = \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

This can be further simplified to obtain

$$L[f(t)] = \tan^{-1} \left( \frac{2}{s^2} \right)$$