① Equality

$A = B$ iff $x \in A \Rightarrow x \in B$ AND
$x \in B \Rightarrow x \in A$

② Alphabets, operators.

(subset, operator) is well defined for a given set if

③ Problems

① If G is a group s.t.

$(a \cdot b)^2 = a^2 \cdot b^2 \quad \forall a, b \in G,$

group op. NOT multiplication

then P.T. G is Abelian.

- we know that G is a group $\therefore$ we only need to demonstrate commutativity.

- Let us define the group's operator to be $\otimes$ (some random op.)

$\therefore (a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b).$

- we need to prove that →

$$a \otimes b = b \otimes a.$$

- we know that G is a group ∴ associative.

- let $a \otimes b = c$. $c \in G$ by definition of groups.

- $(ab)^2 = (a \otimes b) \otimes (a \otimes b)$

$$= c \otimes (a \otimes b)$$

$$= (c \otimes a) \otimes b$$

- Assume G is Abelian

∴ $c \otimes a = a \otimes c$

∴ $(ab)^2 = (a \otimes c) \otimes b$

$$= a \otimes (c \otimes b)$$

- Let $\overset{\to c}{(a \otimes b)} \otimes b = d$.

∴ $a \otimes \underbrace{(b \otimes b)}_{} = d$

∴ $a \otimes f = d$ ↙say.

$(ab)^2 = a \otimes ((a \otimes b) \otimes b)$

$$= a \otimes (a \otimes (b \otimes b))$$

$$= a \otimes (a \otimes f)$$

$$= (a \otimes a) \otimes f$$

$$= (a \otimes a) \otimes (b \otimes b)$$

$$= a^2 b^2$$

Now work similarly - starting from $a^2 b^2$ show that it's equal to $(ab)^2$.

required for the proof.

$$a^2 b^2 = (a \otimes a) \otimes (b \otimes b)$$

TO DO.

Method 2

- Given $(a\,b)^2 = a^2 b^2$

$\therefore \underbrace{(a \otimes b)}_{c} \otimes (a \otimes b) = \underbrace{(a \otimes a)}_{d} \otimes (b \otimes b)$

$\therefore c \otimes (a \otimes b) = d \otimes (b \otimes b)$

$\therefore (c \otimes a) \otimes b = (d \otimes b) \otimes b.$

- $\because$ G is a group, we know that an identity element & inverse element exists

$\therefore \forall\, b \;\exists\, b^{-1} \in G$ s.t. $b \otimes b^{-1} = e$, $e \in G$

(let $(c \otimes a) \times b = f$)

$\therefore ((c \otimes a) \otimes (b \otimes b^{-1})) = (d \otimes b) \otimes (b \otimes b^{-1})$

$\therefore ((c \otimes a) \otimes e = (d \otimes b) \otimes e$

$\therefore c \otimes a = d \otimes b$

$\therefore (a \otimes b) \otimes a = (a \otimes a) \otimes b$

$\therefore a \otimes (b \otimes a) = a \otimes (a \otimes b)$

$\therefore (a^{-1} \otimes a) \otimes (b \otimes a) = (a^{-1} \otimes a) \otimes (a \times b)$

$\therefore \quad e \otimes (b \otimes a) = e \otimes (a \otimes b)$

$\therefore \quad b \otimes a = a \otimes b$

Q.E.D.

② Verify that a set of $n \times n$ non-singular matrices with real elements forms a group under matrix addition.

- We need to show that →

associativity ① $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R$

identity element must exist ② $\forall a \in R, \ \exists \ b$ s.t. $a + e = e + a = a$

inverse element must exist ③ $\forall a \in R, \ \exists \ b$ s.t. $a + b = b + a = e, \ e \in R$

- Problem → $0_{n \times n}$ is singular.

$\therefore$ Does not exist in the set.

$\therefore$ This is <u>not</u> a group.

<u>Formal Proof</u>

- Let $(G, +)$ be the group.

- If $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}} \in G$ s.t →

our notation for matrices.

$$\underline{\underline{A}} = (a_{ij})_{n\times n}, \quad a_{ij} \in \mathbb{R} \ \forall i,j.$$

$$\underline{\underline{B}} = (b_{ij})_{n\times n}, \quad b_{ij} \in \mathbb{R} \ \forall i,j.$$

$$\underline{\underline{C}} = (c_{ij})_{n\times n}, \quad c_{ij} \in \mathbb{R} \ \forall i,j.$$

$$-(\underline{\underline{A}} + \underline{\underline{B}}) + \underline{\underline{C}}$$

(see photos).

③ Verify if the group $(\mathbb{R}^n[x], +, \cdot)$, i.e real polynomials in x with degree $= n$ form a ring.

- we need to show that →

associativity ① $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R$

identity element must exist ② $\forall a \in R, \exists b$ s.t. $a + e = e + a = a$

inverse element must exist. ③ $\forall a \in R, \exists b$ s.t. $a + b = b + a = e$, $e \in R$.

commutativity of + ④ $a + b = b + a, \forall a, b \in R$.

assoc. of $\times$ ⑤ $(a \times b) \times c = a \times (b \times c) \quad \forall a, b, c \in R$

$\times$ distributes over + ⑥ $\left.\begin{array}{l} a \times (b + c) = (a \times b) + (a \times c) \\ (a + b) \times c = (a \times c) + (b \times c) \end{array}\right\} \forall a, b, c \in R$

identity for $\times$ ⑦ $\forall a \in R \; \exists e$ s.t. $a \times e = e \times a = a, \; e \in R$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots\cdots + b_n x^n.$$
$$h(x) = c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_n x^n$$

associativity of $\times$ will fail to hold.

④ Let G be a set with elements of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ s.t. $a, b, c, d \in \{0, 1\}$ & $a \cdot d - c \cdot b \neq 0$.

Verify if G is a group under mat mul. Also find the order of G.

- several issues. check each condition.