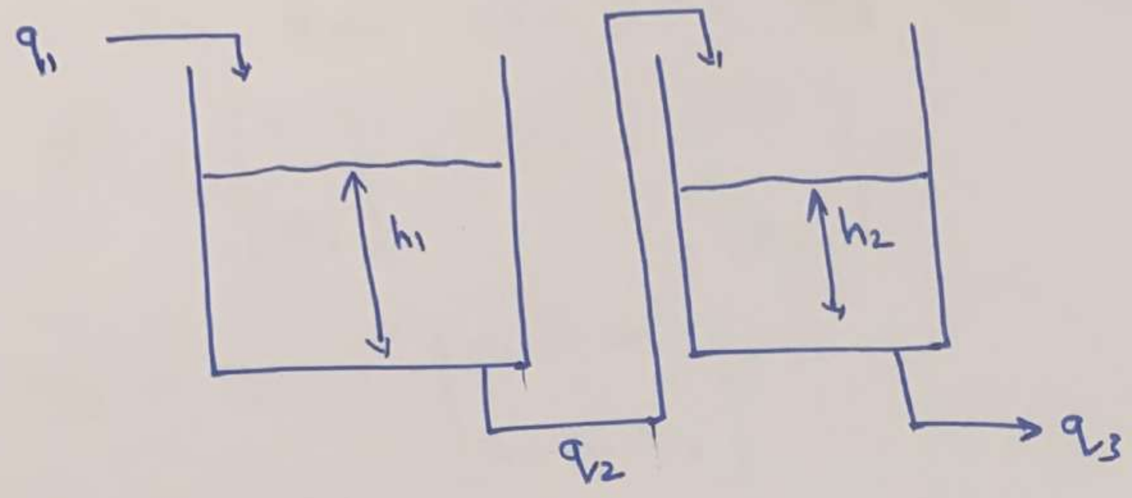
Q01: Given, a two-tank system with the following details:



$$\frac{dh_1}{dt} = q_1 - q_2 \qquad -(1)$$

$$\frac{dh_2}{dt} = q_2 - q_3 \qquad -(2)$$

$$q_1 = 0$$

$$q_2 = 0.94\, h_1$$

$$q_3 = 0.53 h_2$$

$$\Rightarrow \frac{dh_1}{dt} = -0.94 h_1$$

$$\frac{dh_2}{dt} = 0.94 h_1 - 0.53 h_2$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \qquad -(3)$$

$- \boxed{\text{1 mark till here}}$

(a) To verify the linearity of the system described by Eqⁿ (3).

Let the two solutions of Eqⁿ (3) be

$$\underline{x}_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underline{x}_1 + \underline{x}_2 = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

$$\alpha \underline{x}_1 = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}$$

If the system is linear then

(i) $\underline{x}_1 + \underline{x}_2$ is also a solution of Eqⁿ (3)

(ii) $\alpha \underline{x}_1$ is also a solution of Eqⁿ (3)

Consider the RHS of Eqⁿ (3)

$$\begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix} \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.94a_1 - 0.94b_1 \\ +0.94a_1 + 0.94b_1 - 0.53a_2 - 0.53b_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.94a_1 - 0.94b_1 \\ (0.94a_1 - 0.53a_2) + (0.94b_1 - 0.53b_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.94a_1 \\ 0.94a_1 - 0.53a_2 \end{bmatrix} + \begin{bmatrix} -0.94b_1 \\ 0.94b_1 - 0.53b_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{d}{dt}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{d}{dt}\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

$\boxed{\text{2 marks till here}}$

Hence, condition (i) is satisfied.

$$\begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix} = \alpha \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= \alpha \frac{d}{dt}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= \frac{d}{dt}\begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}$$

$\Rightarrow$ Condition (ii) is also satisfied.

Therefore, the system described by Eq^n (3) is a linear system.

$\boxed{\text{3 marks in total}}$

(b) Phase portrait of the system:

$$\underline{\underline{A}} = \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}$$

$\Rightarrow \quad \lambda_1 = -0.94$

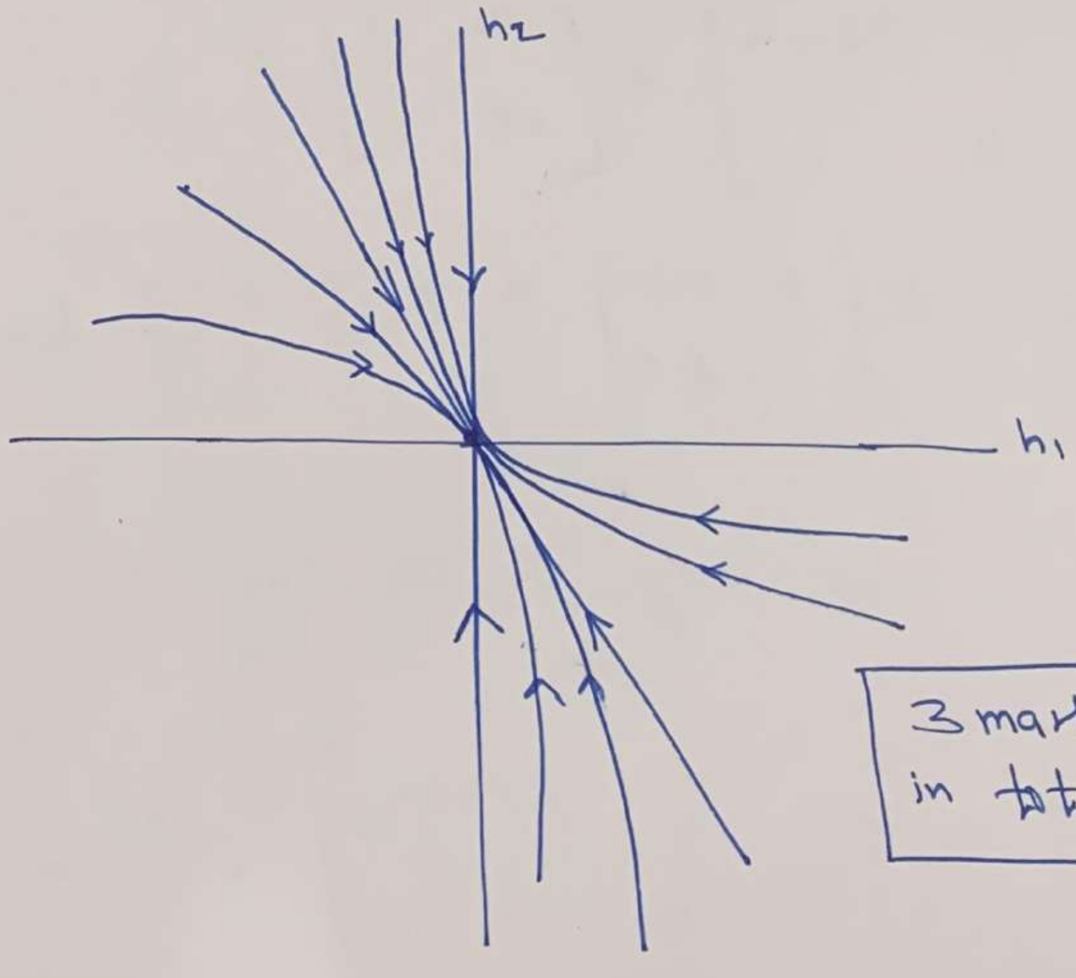
$\lambda_2 = -0.53$

$\underline{v}_1 = \begin{bmatrix} -0.44 \\ 1 \end{bmatrix}$

1 mark till here

$\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The phase portrait will be a sink solution since $\lambda_1 < 0$ and $\lambda_2 < 0$. with $|\lambda_1| > |\lambda_2|$ in the present case, we get the following:



3 marks in total

(c) Solution by similarity transformation:

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -0.94 & 0 \\ 0.94 & -0.53 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

Let the above equation be represented using the following notation:

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad -(4)$$

$$\Rightarrow \frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = (\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})(\underline{\underline{P}}^{-1}\underline{x})$$

$$\Rightarrow \frac{d}{dt}\underline{y} = \underline{\underline{\Lambda}}\,\underline{y} \qquad -(5)$$

where $\underline{\underline{P}} = [\underline{v_1} \mid \underline{v_2}] = \begin{bmatrix} -0.44 & 0 \\ 1 & 1 \end{bmatrix}$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -0.94 & 0 \\ 0 & -0.53 \end{bmatrix}$$

and $\underline{y} = \underline{\underline{P}}^{-1}\underline{x} = \begin{bmatrix} -2.3 & 0 \\ 2.3 & 1 \end{bmatrix}\underline{x}$

$\boxed{\text{2 marks till here}}$

Solution to Eq^n (5) is given as

$$\underline{y} = e^{\underline{\underline{\Lambda}}t}\,\underline{c}$$

But $e^{\underline{\underline{\Lambda}}t} = \begin{bmatrix} e^{-0.94t} & 0 \\ 0 & e^{-0.53t} \end{bmatrix}$

$$\Rightarrow \underline{y} = \begin{bmatrix} e^{-0.94t} & 0 \\ 0 & e^{-0.53t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \underline{y} = \begin{bmatrix} c_1 e^{-0.94t} \\ c_2 e^{-0.53t} \end{bmatrix} \quad - (6)$$

$\boxed{\text{3 marks till here}}$

Since $\underline{y} = \underline{\underline{P}}^{-1} \underline{x}$, $\underline{x} = \underline{\underline{P}}\, \underline{y}$

$$\Rightarrow \underline{x} = \begin{bmatrix} -0.44 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-0.94t} \\ c_2 e^{-0.53t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -0.44 c_1 e^{-0.94t} \\ c_1 e^{-0.94t} + c_2 e^{-0.53t} \end{bmatrix}$$

$$\Rightarrow h_1 = d e^{-0.94t} \quad ; \; d = -0.44 c_1$$

$$h_2 = c_1 e^{-0.94t} + c_2 e^{-0.53t}$$

$\boxed{\text{Total 5 marks}}$

Q02 : Given,

$$\frac{dx}{dt} = x(1-x) + \frac{x}{1+x} \qquad - (1)$$

Equilibrium solutions to Eq^n (1) can be determined by setting $\frac{dx}{dt} = 0$

$$\Rightarrow x(1-x) + \frac{x}{1+x} = 0$$

$$\Rightarrow (1+x)(x-x^2) + x = 0$$

$$\Rightarrow x - x^2 + x^2 - x^3 + x = 0$$

$$\Rightarrow 2x - x^3 = 0$$

$$\Rightarrow x_e = \pm\sqrt{2} \text{ or } x_e = 0 \qquad - \boxed{\text{1 mark till here}}$$

Checking for the stability of equilibrium sol's

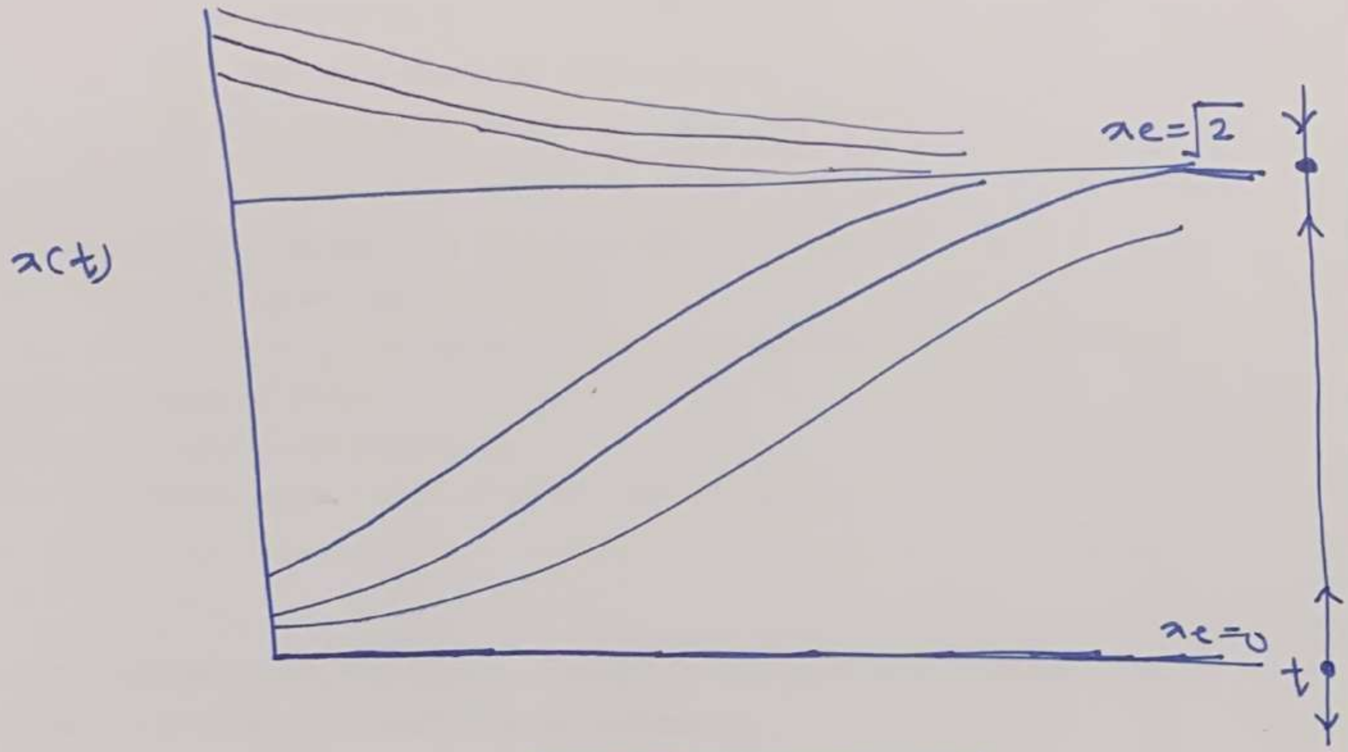
$$f(x) = x - x^2 + \frac{x}{1+x}$$

$$\Rightarrow \frac{df}{dx} = 1 - 2x + \frac{1}{1+x} - \frac{x}{(1+x)^2}$$

$$\Rightarrow \left.\frac{df}{dx}\right|_{x_e=0} = 2 > 0 \text{ - unstable}$$

$$\left.\frac{df}{dx}\right|_{x_e=\sqrt{2}} = -1.66 < 0 \text{ - stable} \qquad - \boxed{\text{3marks till here}}$$

Phase portrait of the physically realisable region, $x>0$, $t>0$



4 marks in total