9th Nov

$$u(x_0,y_0) = \underbrace{\iint f\, g\, dx\,dy}_{I_1} - a\underbrace{\int \frac{\partial g}{\partial x}\Big|_{x=0} dy}_{I_2} + b\underbrace{\int_{y=0}^{1} \frac{\partial g}{\partial x}\Big|_{x=1} dy}_{I_3}$$

$$g = \sum_n \sum_m -4\,\frac{\sin(n\pi x_0)\,\sin(m\pi y_0)\,\sin(n\pi x)\,\sin(m\pi y)}{\pi^2(m^2+n^2)}$$

$$I_1 = f\int_0^1\int_0^1 g\, dx\, dy$$

$$= -4f \sum\sum \frac{\sin(n\pi x_0)\,\sin(m\pi y_0)}{\pi(m^2+n^2)} \underbrace{\int_0^1 \sin(n\pi x)\,dx}_{\frac{1-\cos n\pi}{n\pi}} \cdot \underbrace{\int_0^1 \sin m\pi y\, dy}_{\frac{1-\cos m\pi}{m\pi}}$$

$$= -4f \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(1-\cos n\pi)(1-\cos m\pi)}{mn\,\pi^4(m^2+n^2)} \sin(n\pi x_0)\,\sin(m\pi y_0)$$

$$I_2 = a\int_{y=0}^{1} \sum_n \sum_m \frac{-4\sin(n\pi x_0)\,\sin(m\pi y_0)}{\pi^2(m^2+n^2)} \cdot \sin(m\pi y)(n\pi)\,dy$$

$$= -4a\sum\sum (n)\frac{\sin(n\pi x_0)\,\sin(m\pi y_0)}{\pi(m^2+n^2)} \underbrace{\int_0^1 \sin(m\pi y)\,dy}_{\frac{1-\cos m\pi}{m\pi}}$$

$$= -4a\cdot\sum\sum \frac{n(1-\cos m\pi)}{m\,\pi^2(m^2+n^2)} \cdot \sin(m\pi y_0) \cdot \sin n\pi x_0$$

Similarly I_3 can be solved.

Soln of N.H Parabolic PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

@ $t = 0$ $u = a$.

$x = 0$ $u = b$.

$u = c$.

$= 1$

construct causal green's funcⁿ.

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} + \delta(x - x_0)\,\delta(t - t_0)$$

@ $t = 0$ $g = 0$

$\left.\begin{array}{l} x = 0 \\ = 1 \end{array}\right\} g = 0$.

Soln

eigenval. prob. in x-direcⁿ.

$$g(x,t/x_0,t_0) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

corresponding E-value prob:

Partial eigenfunctionⁿ expansion method.

operator for prob $\longrightarrow$ $L: \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

" " eigenvalϕ $\longrightarrow$ $-\frac{\partial^2}{\partial x^2}$

corresponding Evalue prob.

$L\phi + \lambda\phi = 0$.

$-\frac{d^2\phi}{dx^2} - \alpha^2\phi = 0$

$\frac{d^2\phi}{dx^2} + \alpha^2\phi_n = 0$, $\Rightarrow$ @ $\left.\begin{array}{l} x = 0 \\ = 1 \end{array}\right\} \phi_n = 0$

$\phi = C_n \sin n\pi x$. $\alpha_n = n\pi$, $n = 1, 2, \ldots \infty$

$$a_n = \frac{\langle g, \phi_n \rangle}{\langle \phi, \phi \rangle} = \langle g\,\phi_n \rangle$$

Make $\langle \phi_n, \phi_n \rangle = 1$

$$a_n^2 \int_0^1 \sin^2(n\pi x)\, dx = 1$$

$$\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} = \delta(x - x_0)\, \delta(t - t_0)$$

$$\int_0^1 \phi_n(x) \frac{\partial g}{\partial t}\, dx - \int_0^1 \phi_n \frac{\partial^2 g}{\partial x^2}\, dx = \delta(t - t_0)\, \phi_n(x_0)$$

(B.L eqn fluid mechanics)

$$\phi_n(x) = x^2 \qquad g_i = at$$

$$\int_0^1 \phi_n \frac{\partial g}{\partial t}\, dx = \frac{\partial}{\partial x} \int \phi_n g\, dx$$

We are operating $\int$ over x not t

$$\frac{d}{dt} \int_0^1 \phi_n . g\, dx - \cancel{\phi_n \frac{\partial g}{\partial x}\Big|_0^1} + \int_0^1 \frac{d\phi_n}{dx} \frac{\partial g}{\partial x}\, dx$$

$= 0$ i.e.

$$\frac{d a_n}{dt} + \frac{d\phi_n}{dx} g \Big|_0^1 - \int \frac{d^2 \phi_n}{dx^2} g\, dx = \delta(t - t_0)\, \phi_n(x_0)$$

$$\frac{d a_n}{dt} + \alpha_n^2 \int \phi_n g\, dx = \delta(t - t_0)\, \phi_n(x_0)$$

$$\frac{d a_n}{dt} + \alpha_n^2 a_n = \delta(t - t_0)\, \phi_n(x_0)$$

$$\frac{d a_n}{dt} + \alpha_n^2 a_n = \delta(t - t_0)\, \phi_n(x_0)$$

$$\frac{da_n}{dt} + \alpha_n^2 a_n = 0 \quad \forall\ t \neq t_0,$$

$a_n = A \exp(-\alpha_n^2 t) \quad t < t_0,$

$= B \exp(-\alpha_n^2 t) \quad \forall\ t > t_0$

@ $t=0$ $g=0$ $\Rightarrow a_n = 0,$

$\hookrightarrow A=0.$

$a_n = 0$ for $0 \le t \le t_0.$

$= B\exp[-\alpha_n^2 t]$ for $t > t_0$

$\boxed{\phi_n = \sqrt{2}\sin\pi x}$

Jump Dis cont.

$$\int_{t_0^-}^{t_0^+} \frac{da_n}{dt}\,dt + \alpha_n^2 \cancel{\int_{t_0}^{t_0} a_n\,dt} = \int_{t=t_0}^{t_0+\epsilon} \delta(t-t_0)\,\phi_n(x_0)\,dx$$

$a_n(t_0^+) - a_n(t_0^-) = \phi_n(x_0)$

$a_n(t_0^+) = \phi_n(x_0)$

$B\exp[-\alpha_n^2(t_0+\epsilon)] = \phi_n(x_0).$

$B = \phi_n(x_0)\exp(\alpha_n^2 t)$

$a_n = 0$ for $t < t_0$

$= \phi_n(x_0)\exp[-\alpha_n^2(t-t_0)]$ $t > t_0.$

$g(x,t/x,t_0) = \sum a_n(t)\,\phi_n(x)$

$g = 0$ for $t < t_0$

$H \longrightarrow$ Heavy side function

$H(t-t_0) = 0$ $\forall t < t_0.$

$= 1$ $t > t_0$

$$g(x,t/x_0,t_0) = H(t-t_0)\sum_{n=1}^{\infty} 2\sin(n\pi x)\exp\left[-n^2\pi^2(t-t_0)\right]$$

To find adj op:

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

$$\langle g^*, Lg\rangle$$

$$= \iint g^*\left(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2}\right) dx\,dt$$

$$= \int_x \left[[g^* g]_0^t - \int \frac{\partial g^*}{\partial t} g\,dt\right] dx - \int\left[\left[g^*\frac{\partial g}{\partial x}\right]_0^1 - \int_0 \frac{\partial g^*}{\partial x}\frac{\partial g}{\partial x} dx\right] dt$$

$$= \int_x \left[g^*(t) g(t)\right) - g^*(t-)g(t)$$

attention

today's form imp//

10th.

$$g(x,t/x_0,t_0) = H(t-t_0)\sum_{n=1}^{\infty} 2\sin n\pi x \sin n\pi x_0 \exp[-n^2\pi^2(t-t_0)]$$

$$g(x,t_1/x_0,t_0) = g^*(x_0,t_0/x,t)$$

$x,t \longrightarrow x_1,t_1$

$$g(x_1,t_1,x_0,t_0) = H(t_1-t_0)\sum_{n=1}^{\infty} 2\sin(n\pi x_1)\sin(n\pi x_0) \exp[-n^2\pi^2(t_1-t_0)]$$

$$g^*(x,t,x_1,t_1) = H(t_1-t_0)$$

$$g^*(x,t,x_1,t_1) = H(t_1-t)\sum_{n=1}^{\infty} 2\sin(n\pi x_1)\sin(n\pi x)\exp[-n^2\pi^2(t_1-t)]$$

$H(t_1-t) = 0$ for $t > t_1$

$= 1$ for $t_0 < t_1$

$$g^*(x,t/x_1,t_1) = 2\sum_{n=1}^{\infty} \sin n\pi x \sin(n\pi x_0)\exp[-n^2\pi^2(t_1-t)]$$
for $0 \le t \le t_1$

$= 0$ for $t > t_1$

① $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} = f.$

② $\dfrac{\partial g^*}{\partial t} - \dfrac{\partial^2 g^*}{\partial x^2} = \delta(x-x_1)\delta(t-t_1)$

$\langle g^*, 1\rangle = \langle u, 2\rangle$

$$\iint g^* \frac{\partial u}{\partial t}\,dt\,dx - \iint g^* \frac{\partial^2 u}{\partial x^2}\,dx\,dt = \iint f g^*\,dx\,dt - u(x_0,t_0)$$

$$+\iint u\frac{\partial g^*}{\partial t}\,dt\,dx + \iint u\frac{\partial^2 g^*}{\partial t^2}\,dx\,dt$$

Scanned with CamScanner

$$\iint_{x\,t}\left(g^{*}\frac{\partial u}{\partial t}+u\frac{\partial g^{*}}{\partial t}\right)dt\,dx-\int_{t}\left[g^{*}\frac{\partial u}{\partial x}\Big|_{0}^{1}-\int\frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x}\right]dt$$

$$+\int\left[u\frac{\partial g^{*}}{\partial x}\Big|_{0}^{1}-\int_{x}\frac{\partial u}{\partial x}\frac{\partial g^{*}}{\partial x}dx\right]dt$$

pic

$$=\iint_{x\,t}\frac{\partial}{\partial t}(ug^{*})\,dt\,dx-\int_{t}\left[c\frac{\partial g^{*}}{\partial x}\Big|_{x=1}-b\frac{\partial g^{*}}{\partial x}\Big|_{x=0}\right]dt$$

$$=\int_{x}\left[u\,g^{*}\right]_{0}^{t}dx-c\int_{t}\frac{\partial g^{*}}{\partial x}\Big|_{x=1}dt+b\int_{0}^{t}\frac{\partial g^{*}}{\partial x}\Big|_{x=0}dt$$

$$-\int_{x}\left[u(t)g^{*}(t)-a\,g^{*}(0)\right]dx-c\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=1}dt+b\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=0}dt$$

$$-a\int_{0}^{1}g^{*}(t=0)\,dx+c\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=1}dt-b\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=0}dt.$$

wh.

$$R.H.S=\iint f\,g^{*}\,dx\,dt-u(x_1,t_1)$$

$$u(x,t)=\underbrace{\int_{t=0}\int_{x=0}f g^{*}\,dx\,dt}_{I_1}+\overbrace{a\int g^{*}(t=\cdot)\,dx}^{I_2}$$

IH of $t=0$

$$-\underbrace{c\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=1}dt}_{I_3}-\underbrace{b\int_{0}^{t_1}\frac{\partial g^{*}}{\partial x}\Big|_{x=0}dt}_{I_4}.$$

$$I_1 = \int_0^1 2 \sum \sin(n\pi x) \sin(n\pi x_1) \exp(-n^2\pi^2 t_1)\, dx$$

$$= 2a \sum_{n=1}^{\infty} \left(\frac{1-\cos n\pi x}{n\pi}\right) \sin(n\pi x_1) \exp(-n^2\pi^2 t_1)$$

$$I_2 = -2c \int_0^t \sum_{n=1}^{\infty} n\pi \frac{\cos n\pi}{n\pi} \sin n (n\pi x_1) \exp(-n^2\pi^2 t_1)$$

$$= -2c \sum_{n=1}^{\infty} n\pi \cos n\pi \sin n\pi x_1 \; 1 - \exp(-n^2\pi^2 t_1)$$

$g = g^{*}$ (explain)