1 ODE Example

$$\frac{d^2u}{dx^2} = x \quad (1)$$

B.C. $\begin{cases} \text{at} \quad x = 0 \rightarrow u = 1 \quad (2) \\ \text{at} \quad x = 1 \rightarrow u = 2 \quad (3) \end{cases}$

3 sources of non-homogeniety

$$u = u_1 + u_2 + u_3$$

$\therefore$ u_1 : $\dfrac{d^2u_1}{dx^2} = x$ → define governing equation in terms of 1 N.H.

why?

$\left.\begin{array}{l} \text{at} \quad x = 0 \\ \text{at} \quad x = 1 \end{array}\right\} u_1 = 0$ → this is a design choice.

u_2 : $\dfrac{d^2u_2}{dx^2} = 0$

at $x = 0 \rightarrow u_2 = 1$

at $x = 1 \rightarrow u_2 = 0$

u_3 : $\dfrac{d^2u_3}{dx^2} = 0$

at $x = 0 \rightarrow u_3 = 0$

at $x = 1 \rightarrow u_3 = 2$

u_1 : $\frac{du_1}{dx} = \frac{x^2}{2} + c_1$.

$\therefore \quad u_1 = \frac{x^3}{6} + c_1 \cdot x + c_2$.

- use $u_1(x=0) = 0$

$\Rightarrow \quad c_2 = 0$.

- use $u_1(x=1) = 0$

$\Rightarrow \quad 0 = \frac{1}{6} + c_1 \cdot 1$

$\Rightarrow \quad c_1 = -1/6$.

$$\therefore \quad \boxed{u_1 = \frac{x^3}{6} - \frac{x}{6}}$$

$\therefore \quad u = u_1 + u_2 + u_3$.

solve for each & substitute.

In this case the prob was a simple ODE, but the same logic can be used for PDEs as well.

(2) Parabolic PDE - Principle of Linear Superposition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

at $t = 0$, $u = u_0$

at $x = 0$, $u = 0$

at $x = 1$, $u = 1$ 2 N.H.

$\therefore$ 2 N.H.

$\therefore$ $u = u_1 + u_2$

u_1 : $\dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2}$

$$\begin{cases} \text{at } t = 0 \rightarrow u = u_0 \\ \left.\begin{array}{l} \text{at } x = 0 \\ \text{at } x = 1 \end{array}\right\} u = 0 \end{cases}$$

when both B.C. are homo. & I.C. is N.H. $\Rightarrow$ "well posed / behaved" problem

steady state problem, solvable problem.

u_2 : $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$

"ill posed problem" & solved as two sub-problems

$$\left\{\begin{array}{l} \text{at } t = 0 \rightarrow u_2 = 0 \\ \text{at } x = 0 \rightarrow u_2 = 0 \\ \text{at } x = 1 \rightarrow u_2 = 1 \end{array}\right.$$

sub-problems – time dep. & time indep.

→ steady state → transient

$$u_2(x,t) = u_2^S(x,t) + u_2^t(x,t)$$

$$\therefore \frac{\partial}{\partial t}(u_2^S + u_2^t) = \frac{\partial^2}{\partial x^2}(u_2^S + u_2^t)$$

$$\therefore \frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^S}{\partial x^2} + \frac{\partial^2 u_2^t}{\partial x^2}$$

at $x = 1 \rightarrow u_2 = 1$

$$\therefore u_2^S + u_2^t = 1$$

&

s.s. problem doesn't have an I.C. anyway.

to make transient prob. well posed. ← $\underline{\underline{\text{SELECT}}}$

$$u_2^S(x=1) = 1$$

$$u_2^t(x=1) = 0$$

$$\therefore u_2^S : \frac{d^2 u_2^S}{dx^2} = 0$$

at $x = 0$, $u_2^s = 0$

at $x = 1$, $u_2^s = 1$.

$$\therefore \frac{d u_2^s}{dx} = c_1$$

$$\therefore u_2^s = c_1 x + c_2.$$

$$\therefore c_2 = 0 \ \& \ c_1 = 1$$

$$\therefore \boxed{u_2^s = x}$$

$$u_2^t : \frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^t}{\partial x^2}.$$

at $x = 0 \to u_2^t = 0$

at $x = 1 \to u_2^t = 0$

at $t = 0$

how is this a general solution?

$$- \quad u_2 = 0$$

$$\therefore u_2^s + u_2^t = 0$$

$$\therefore u_2^t = -u_2^s = -x.$$

ask.

(3) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

at $t = 0$, $u = \underline{\underline{u_{01}}}$ N.H 1

at $x = 0$, $\frac{\partial u}{\partial x} + \alpha u = \underline{\underline{u_{02}}}$ N.H 2

at $x = 1$, $\frac{\partial u}{\partial x} = \underline{\underline{u_{03}}}$ N.H 3

$\therefore$ 3 sources of N.H.

$$u = u_1 + u_2 + u_3$$

u_1 : $$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}$$

w.P. $\begin{cases} \text{at } t = 0 \rightarrow u_1 = u_{01} \\ \text{at } x = 0 \rightarrow \frac{\partial u_1}{\partial x} + \alpha u_1 = 0 \\ \text{at } x = 1 \rightarrow \frac{\partial u_1}{\partial x} = 0 \end{cases}$

u : $$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}$$

$$\text{I.P.}\begin{cases} \text{at} \quad t = 0 \rightarrow u_2 = 0 \\ \text{at} \quad x = 0 \rightarrow \dfrac{\partial u_2}{\partial x} + \alpha u_2 = u_{02} \\ \text{at} \quad x = 1 \rightarrow \dfrac{\partial u_2}{\partial x} = 0 \end{cases}$$

$$u : \quad \frac{\partial u_3}{\partial t} = \frac{\partial^2 u_3}{\partial x^2}$$

$$\text{I.P.}\begin{cases} \text{at} \quad t = 0 \rightarrow u_3 = 0 \\ \text{at} \quad x = 0 \rightarrow \dfrac{\partial u_3}{\partial x} + \alpha u_3 = u_{03} \\ \text{at} \quad x = 1 \rightarrow \dfrac{\partial u_3}{\partial x} = 0 \end{cases}$$

$$\rightarrow u_2 = u_2^s(x) + u_2^t(x, t).$$

$$u_2^s : \quad \frac{d^2 u_2^s}{dx^2} = 0. \qquad (\because \text{steady state})$$

$$\text{at } x = 0 \longrightarrow \frac{d u_2^s}{dx} + \alpha u_2^s = u_{02}$$

at $x = 1 \longrightarrow \dfrac{d u_2^s}{dx} = 0$

u_2^t : $\dfrac{\partial^2 u_2^t}{\partial x^2} = \dfrac{\partial u_2^t}{\partial t}$

at $x = 0 \longrightarrow \dfrac{\partial u_2^t}{\partial x} + \alpha u_2^t = 0$

at $x = 1 \longrightarrow \dfrac{\partial u_2^t}{\partial x} = 0$

at $t = 0$ $\quad u_2 = 0$

$\therefore \quad u_2^s + u_2^t = 0$