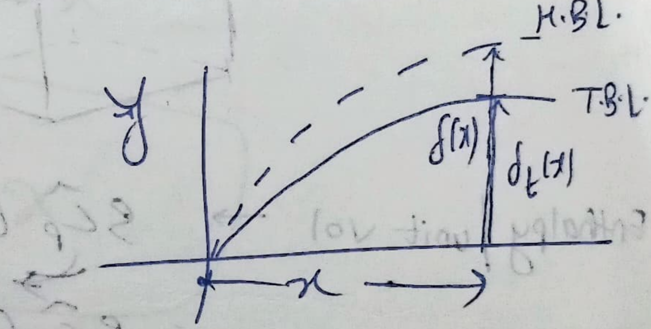
$$\frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad \Big\} \text{ Energy balance in T.B.L.} \quad \ldots \boxed{A}$$



$$\int_0^{\delta_t(x)} \left(\frac{\partial (Tu)}{\partial x} + \frac{\partial (Tv)}{\partial y} \right) dy = \alpha \int_0^{\delta_t(x)} \frac{\partial^2 T}{\partial y^2}\, dy$$

$$\int_0^{\delta_t(x)} \frac{\partial (Tu)}{\partial x}\, dy + (Tv)\Big|_0^{\delta_t(x)} = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$\int_0^{\delta_t(x)} \frac{\partial (Tu)}{\partial x}\, dy + (Tv)\Big|_{\delta_t(x)} - \underbrace{(Tv)\Big|_0}_{0} = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$\int_0^{\delta_t(x)} \frac{\partial (Tu)}{\partial x}\, dy + (T_\infty v)\Big|_{\delta_t(x)} = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0} \quad \ldots (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\int_0^{\delta_t(x)} \frac{\partial u}{\partial x}\,dy + \int_0^{\delta_t(x)} \frac{\partial v}{\partial y}\,dy = 0$$

$$v\Big|_{\delta(x)} = -\int_0^{\delta_t(x)} \frac{\partial u}{\partial x}\,dy$$

$$\therefore \cancel{T_\infty} v\Big|_{\delta_t(x)} = -\int_0^{\delta_t(x)} \cancel{T_\infty} \frac{\partial u}{\partial x}\,dy$$

Substituting in (1)

$$\therefore \int_0^{\delta_{tx}} \frac{\partial (Tu)}{\partial x}\,dy - \int_0^{\delta_t(x)} T_\infty \frac{\partial u}{\partial x}\,dy = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$= \frac{d}{dx} \int_0^{\delta_t(x)} u\,(T - T_\infty)\,dy = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$\frac{d}{dx} \int_0^{\delta_t(x)} u\,(T - T_\infty)\,dy = -\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$$

$$\hat{T} = \frac{T - T_\infty}{T_w - T_\infty} \;;\quad \hat{y}_t = \frac{y}{\delta_t(x)} \;;\quad \hat{u} = u/u_\infty$$

$$\hat{T}(T_w - T_\infty) + T_\infty = T \;;\quad y = \delta_t(x)\,\hat{y}_t \;;\quad u = u_\infty \hat{u}$$

$$T = \hat{T}(T_w - T_\infty) + T_\infty \;;\quad \hat{y}_t = \frac{y}{\delta_t(x)} \;;\quad \hat{u} = u/u_\infty$$

$$\delta T = \delta \hat{T}(T_w - T_\infty) \;;\quad \delta \hat{y}_t = \frac{\delta y}{\delta_t(x)} \;;\quad \partial \hat{u} = \frac{\partial u}{u_\infty}$$

$$\therefore \frac{d}{dx} \int_0^1 \hat{u}\, u_\infty (\hat{T})(T_w - T_\infty)\,\delta_t(x)\,d\hat{y}_t = -\alpha \frac{\partial \hat{T}}{\partial \hat{y}_t}\Big|_{y=0} \frac{(T_w - T_\infty)}{\delta_t(x)}$$

$$\boxed{\frac{d}{dx}\left[\delta_t(x) \int_0^1 \hat{u}\,\hat{T}\,d\hat{y}_t\right] = \frac{-\alpha}{u_\infty \delta_t(x)} \frac{\partial \hat{T}}{\partial \hat{y}_t}\Big|_{y=0}}$$

$$\hat{u} = \frac{3}{2}\hat{y} - \frac{\hat{y}^3}{2}$$

$$\hat{T} = a + b\hat{y_t} + c\hat{y_t}^2 + d\hat{y_t}^3$$

conditions:

a) $\hat{y_t} = 0 \; ; \; \hat{T} = 1$

b) $\hat{y_t} = 1 \; ; \; \hat{T} = 0$

c) $\hat{y_t} = 1 \; ; \; \frac{\partial \hat{T}}{\partial \hat{y}} = 0$

d) $\hat{y_t} = 0 \; ; \; \frac{\partial^2 \hat{T}}{\partial \hat{y}^2} = 0$ [from eq.^{n} $\boxed{A}$]

$$\hat{T} = 1 - \frac{3}{2}\hat{y_t} + \frac{1}{2}\hat{y_t}^3$$

$$\frac{\hat{y}}{\hat{y_t}} = \psi(\nu, \alpha)$$

$$\frac{d}{dx}\Big[\delta_t(x)\underbrace{\Big(\frac{3}{20}\psi - \frac{3}{280}\psi^3\Big)}_{\varepsilon_1}\Big] = \frac{3}{2}\frac{\alpha}{u_\infty \delta_t(x)}$$

$$\frac{d}{dx}\big(\delta_t(x)\varepsilon_1\big) = \frac{3}{2}\frac{\alpha}{u_\infty \delta_t(x)}$$

$$\varepsilon_1 \frac{\delta_t(x)^2}{\cancel{2}} = \frac{3}{\cancel{2}}\frac{\alpha x}{u_\infty}$$

$$(\delta_t(x))^2 = \frac{3}{\varepsilon_1}\frac{\alpha x}{u_\infty}$$

$$\boxed{\delta_t(x) = \sqrt{\frac{3}{\varepsilon_1}\frac{\alpha x}{u_\infty}}} \; ; \; \frac{\hat{y}}{\hat{y_t}} = \psi(\nu, \alpha) \; ; \; \psi = \frac{\delta_t(x)}{\delta(x)}$$

$$\delta(x) = 4.64\sqrt{\frac{\nu x}{u_\infty}}$$

$$\therefore \psi(\nu,\alpha) = \frac{\delta_t(x)}{\delta(x)} = \frac{\sqrt{3\alpha}}{\sqrt{\xi_1 u_\infty}} \times \frac{\sqrt{u_0}}{4.64\sqrt{\nu}}$$

$$\psi(\nu,\alpha) = \frac{1}{4.64}\sqrt{\frac{3\alpha}{\xi_1\nu}}$$

$$= \frac{\sqrt{3}}{4.64} \cdot \sqrt{\frac{\alpha}{\nu\xi_1}}$$

$$\psi = \frac{\sqrt{3}}{4.64}\sqrt{\frac{\alpha}{\nu}} \times \frac{1}{\sqrt{\left(\frac{3}{20}\psi - \frac{3}{280}\psi^3\right)}}$$

as $\psi < 1$ (~~assuming thickness of HDBL > TBL~~) we can neglect ψ^3 term.

$$\psi = \frac{\sqrt{3}}{4.64}\sqrt{\frac{\alpha}{\nu}} \times \frac{1}{\sqrt{\frac{3\psi}{20}}}$$

$$\psi^{3/2} = \frac{1}{4.64} \times \sqrt{\frac{\alpha}{\nu}} \times \sqrt{20}$$

$$= \frac{1 \times \sqrt{20}}{4.64\sqrt{Pr}}$$

$$\psi^{3/2} = \frac{\sqrt{20}}{4.64\sqrt{Pr}}$$

$$\psi^3 = \frac{20}{(4.64)^2(Pr)}$$

$$\psi = \left(\frac{(20)}{(4.64)^2(Pr)}\right)^{1/3}$$

$$\psi = \frac{0.976}{(Pr)^{1/3}}$$

$$\psi = (0.976) \times (Pr)^{-1/3}$$

$$\boxed{\psi \approx (Pr)^{-1/3}}$$

$$-\quad \psi(\nu, \alpha) = \frac{\delta_t(x)}{\delta(x)}$$

$$\therefore \delta_t(x) = 4.64 (Re_x)^{-1/2} \cdot x \times \cancel{\delta(x)} \psi(\nu, \alpha)$$

$$\delta_t(x) = 4.64 (Re_x)^{-1/2} x \, (Pr)^{-1/3}$$

$$\boxed{\therefore \delta_t(x) = (4.64) (Re(x))^{-1/2} (Pr)^{-1/3} x}$$

$$\boxed{\delta_t(x) = \frac{4.64 x}{\sqrt[3]{Pr} \sqrt{Re}}}$$