Ex

Express the following funct^{n} in terms of Heaviside's unit step funct^{n}:

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi. \end{cases}$$

Lecture-3
Monday
24/7/17

$$f(t) = \{1 - H(t-\pi)\}\sin t + \{H(t-\pi) - H(t-2\pi)\}\sin 2t + H(t-2\pi).\sin 3t.$$

$$= \sin t + H(t-\pi)\{\sin 2t - \sin t\} + H(t-2\pi)\{\sin 3t - \sin 2t\}$$

Laplace transform of $H(t-a)$

$H(t-a) = 1,\ t > a.$
$\qquad\quad\ \ 0,\ t < a.$

$$L\{H(t-a)\} = \int_0^\infty H(t-a)e^{-st}dt.$$

$$= \int_0^a \underbrace{H(t-a)}_{=0} e^{-st}dt + \int_a^\infty \underbrace{H(t-a)}_{=1} e^{-st}dt.$$

$$= \int_a^\infty e^{-st}dt. = \left.\frac{e^{-st}}{s}\right|_\infty^a = \frac{e^{-as}}{s}.$$

2nd shifting theorem. $\Rightarrow f(t-a)H(t-a)$

If $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a. \end{cases}$ $\quad \bar{f}(s) = L\{f(t)\}$

Then, $L\{g(t)\} = e^{-as}\bar{f}(s).$

or, $L\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s).$

1. Given $f(t) = e^{-t}, \quad 0 < t < 3$
$\qquad\qquad = 0, \quad t > 3.$

Write $f(t)$ in terms of Heaviside's unit-step funcn. and hence find the LT of $f(t)$.

$$f(t) = \{1 - H(t-3)\} e^{-t}$$

$$L\{f(t)\} = L\left[\{1 - H(t-3)\} e^{-t}\right].$$

$$= L\left[e^{-t} - H(t-3) e^{-t}\right]$$

$$= L[e^{-t}] - L\left[H(t-3) e^{-t}\right].$$

$$= \frac{1}{s+1} - P$$

$L\{e^{at}\}.$
$= \frac{1}{s-a}.$

1st shifting thm.
$L[e^{-at} f(t)]$
$= \bar{f}(s+a)$

Note. $L[H(t-3)] = \frac{e^{-3s}}{s}.$

1st shifting thm: $L[\underbrace{H(t-3)}_{f(t)} \underbrace{e^{-t}}_{e^{-at}}] = \bar{f}(s+a).$
$= \bar{f}(s+1) \because a=1$

$f(t) = H(t-3).$

$\therefore \bar{f}(s) = L\{H(t-3)\} = \frac{e^{-3s}}{s}.$

$\therefore$ by 1st shifting thm,

$$L\{H(t-3) e^{-t}\} = \frac{e^{-3(s+1)}}{s+1} = P.$$

$$\therefore L\{f(t)\} = \frac{1}{s+1} - \frac{e^{-3(s+1)}}{s+1}.$$

Application of 2nd shifting Thm.

$$f(t) = \{1 - H(t-3)\} e^{-t}.$$

$$L\{f(t)\} = \frac{1}{s+1} - L\{H(t-3) e^{-t}\}.$$

$$= \frac{1}{s+1} - L\{H(t-3) e^{-(t-3)} e^{-3}\}$$

$$= \frac{1}{s+1} - e^{-3} L\{\underline{H(t-3) e^{-(t-3)}}\}.$$

$$= \frac{1}{s+1} - e^{-3} e^{-as} \bar{f}(s)$$; a here $= 3$

$f(t) = e^{-t}$.

$$= \frac{1}{s+1} - \frac{e^{-3} \cdot e^{-3s}}{s+1}.$$

$$= \frac{1}{s+1} - \frac{e^{-3(s+1)}}{s+1}.$$

2. Find $L\left[g(t)\right]$, $g(t) = 0, \ 0<t<5$
$\qquad\qquad\qquad\qquad = t-3, \ t>5$

<u>Way 1</u>
Direct. definit.

$$= \int_0^\infty g(t)e^{-st}\,dt = \int_5^\infty (t-3)e^{-st}\,dt.$$

<u>Way 2</u>

$$g(t) = H(t-5)(t-3).$$

$$= H(t-5)(t-5+2)$$

$$= \cancel{H(t-5)}\ \underbrace{(t-5)H(t-5)}_{\text{2nd shifting thm.}} + \underbrace{2H(t-5)}_{2L(H(t-5))}.$$

$$= \frac{e^{-5s}}{s^2} + \frac{2e^{-5s}}{s}.$$

3. Find $L\{g(t)\}$ where, $g(t) = \begin{cases} 0, & 0<t<\frac{\pi}{2} \\ \sin t, & t>\frac{\pi}{2} \end{cases}$.

$$g(t) = \sin t\, H\left(t-\tfrac{\pi}{2}\right)$$
$$= \cos\left(t-\tfrac{\pi}{2}\right)H\left(t-\tfrac{\pi}{2}\right)$$

$$L\{g(t)\} = L\left\{\cos\left(t-\tfrac{\pi}{2}\right)H\left(t-\tfrac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s}L\{\cos t\}$$

$$= e^{-\frac{\pi}{2}s}\cdot\frac{s}{s^2+1}.$$

Laplace transform of periodic functions.

Thm. If $f(t)$ is a funct. of period T, then.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt .$$

Proof. $f(t)$ is a funct. of period T

$\Rightarrow f(t+nT) = f(t) \quad n = 1, 2, 3, \ldots$

i.e $f(t+T) = f(t)$, $f(t+2T) = f(t)$,

$f(t+3T) = f(t)$, $\ldots$

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt .$$

$$= \underbrace{\int_0^T f(t) e^{-st} dt}_{I_0} + \underbrace{\int_T^{2T} f(t) e^{-st} dt}_{I_1} + \underbrace{\int_{2T}^{3T} f(t) e^{-st} dt}_{I_2} .$$

$$+ \ldots + \underbrace{\int_{(n-1)T}^{nT} f(t) e^{-st} dt}_{I_{n-1}} + \ldots$$

$$I_1 = \int_T^{2T} f(t) e^{-st} dt, \qquad t - T = u$$

or, $t = u + T$

$dt = du$.

$$= \int_0^T f(u+T) e^{-s(u+T)} du$$

$$= e^{-sT} \int_0^T f(u+T) e^{-su} du = e^{-sT} \int_0^T f(u) e^{-su} du$$

$\because f$ is periodic

$$= e^{-sT} \int_0^T f(t) e^{-st} dt$$

$$I_{n-1} = \int_{(n-1)T}^{nT} f(t)\, e^{-st}\, dt .$$

Put $t - (n-1)T = u$.

$$= \int_0^T f\left\{u + (n-1)T\right\} e^{-s\{u+(n-1)T\}}\, du .$$

$$= e^{-s(n-1)T} \int_0^T f(\cancel{u}\,t)\, e^{-s\cancel{u}t}\, d\cancel{u}\,t .$$

$$\therefore L\left\{f(t)\right\} = I_0 + I_1 + I_2 + \cdots + I_{n-1} + \cdots$$

$$= \underline{\int_0^T f(t)\, e^{-st}\, dt} + e^{-sT} \int_0^T \underline{f(t)\, e^{-st}\, dt} + e^{-2sT} \int_0^T \underline{f(t)\, e^{-st}\, dt}$$

$$+ \cdots + e^{-s(n-1)T} \int_0^T \underline{f(t)\, e^{-st}\, dt} + \cdots .$$

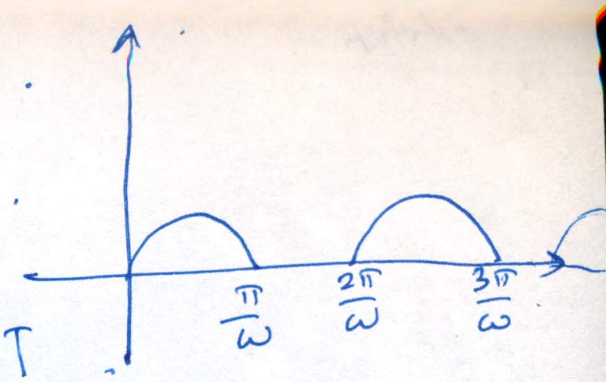
$$= \int_0^T f(t)\, e^{-st}\, dt \left(1 + e^{-sT} + e^{-2sT} + \cdots \right)$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt . \quad //$$

1. Find $L\{f(t)\}$ where

$$f(t) = \sin \omega t, \quad 0 < t < \frac{\pi}{\omega}$$
$$= 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

and periodic of period $\frac{2\pi}{\omega} = T$



$$L\{f(t)\} = \frac{1}{1 - e^{-s \times \frac{2\pi}{\omega}}} \int_0^{\frac{2\pi}{\omega}} f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\underbrace{\int_0^{\pi/\omega} f(t)}_{\sin \omega t} + \underbrace{\int_{\pi/\omega}^{\frac{2\pi}{\omega}} f(t)}_{0} \right] \cancel{f(t)}\, e^{-st}\, dt$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} \sin \omega t\, e^{-st}\, dt$$

$\sin \omega t = \operatorname{Im} e^{i\omega t}$

$$I = \int_0^{\pi/\omega} \sin \omega t\, e^{-st}\, dt$$

$$= \operatorname{Im} \int_0^{\pi/\omega} e^{i\omega t} e^{-st}\, dt$$

$$= \operatorname{Im} \int_0^{\pi/\omega} e^{-(s-i\omega)t}\, dt = \operatorname{Im} \left[\frac{e^{-(s-i\omega)t}}{s - i\omega} \right]_0^{\frac{\pi}{\omega}}$$

$$= \operatorname{Im} \left[\frac{1 - e^{-(s-i\omega)\frac{\pi}{\omega}}}{s - i\omega} \right]$$

$$I = \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{s\pi}{\omega}}} \;//.$$

Solve.

2. If $f(t) = \frac{k}{p} t ; \quad 0 < t < p.$

and $f(t+p) = f(t)$.

find $L\{f(t)\}$ $\boxed{= \frac{k}{ps^2} - \frac{k e^{-sp}}{s(1-e^{-ps})}}$.

3. If $f(t) = 1 , \quad 0 \le t < 2$ $\quad f(t+4) = f(t)$

$= -1 ; \quad 2 \le t < 4 ;$

find $L\{f(t)\} = \frac{1-e^{-2s}}{s(1+e^{-2s})}$.

$1-a^2 = (1-a)(1+a)$.

$1-e^{-4s} = 1-(e^{-2s})^2 = (1+e^{-2s})(1-e^{-2s})$

LT of some special functions.

① Sine Integral:

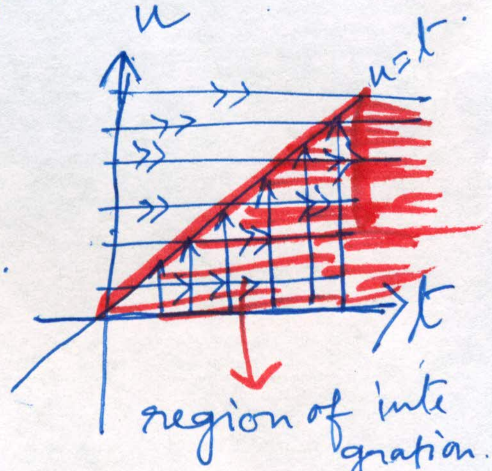
$$S_i(t) = \int_0^t \frac{\sin u}{u}\, du .$$

Find $L[S_i(t)]$.

$$L[S_i(t)] = \int_0^\infty S_i(t)\, e^{-st}\, dt .$$

$$= \int_0^\infty e^{-st}\, dt \left[\int_0^t \frac{\sin u}{u}\, du \right]$$

$$= \int_{u=0}^{\infty} \frac{\sin u}{u} \left(\int_{t=u}^{\infty} e^{-st}\, dt \right) du .$$



$$\int_{t=u}^{\infty} e^{-st}\, dt = \left. \frac{e^{-st}}{s} \right|_{\infty}^{u} = \frac{e^{-su}}{s} .$$

$$= \int_{u=0}^{\infty} \frac{\sin u}{u} \cdot \frac{e^{-su}}{s}\, du .$$

$$L[S_i(t)] = \frac{1}{s} \int_0^\infty \frac{\sin u\, e^{-su}}{u}\, du .$$

$$= \frac{1}{s} I(s)$$

$$I(s) = \int_0^\infty \frac{\sin u \, e^{-su}}{u} du. \Rightarrow I(\infty) = 0.$$

$$\frac{dI(s)}{ds} = \int_0^\infty \frac{-u \, e^{-su} \sin u.}{u} du. = -\int_0^\infty \sin u \, e^{-su} du.$$

$$= -\frac{1}{s^2+1}.$$

$$I(s) = -\tan^{-1} s. + C.$$

$$I(\infty) = -\tan^{-1} \infty + C. \quad \text{or,} \quad 0 = -\frac{\pi}{2} + C.$$

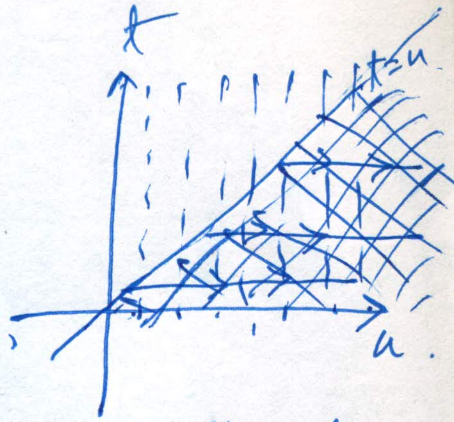
$$\therefore \quad C = \frac{\pi}{2}.$$

$$I(s) = \frac{\pi}{2} - \tan^{-1} s. = \cot^{-1} s = \tan^{-1} \frac{1}{s}.$$

$$L[si(t)] = \frac{1}{s} \tan^{-1} \frac{1}{s}..$$

2. Find $L[C_i(t)]$; $C_i(t) = \int_t^{\infty} \frac{\cos u}{u} du$.

$$L[C_i(t)] = \int_0^{\infty} C_i(t) e^{-st} dt.$$



$$= \int_0^{\infty} e^{-st} dt \left(\int_t^{\infty} \frac{\cos u}{u} du \right)$$

$$= \int_{u=0}^{\infty} \frac{\cos u}{u} du \left(\int_{t=0}^{u} e^{-st} dt \right)$$

$$\int_0^{u} e^{-st} dt.$$

$$= \int_{u=0}^{\infty} \frac{\cos u}{u} \cdot \frac{(1-e^{-su})}{s} du \quad = \left. \frac{e^{-st}}{s} \right|_u^0$$

$$L[C_i(t)] = \frac{1}{s} \int_0^{\infty} \frac{(1-e^{-su})}{u} \cos u \, du = \frac{1}{s} I(s).$$

$$I(s) = \int_0^{\infty} \frac{1-e^{-su}}{u} \cos u \, du; \quad I(0) = 0.$$

$$\frac{dI}{ds} = \int_0^{\infty} \cos u \, e^{-su} du = \frac{s}{s^2+1}$$

$$I(s) = \frac{1}{2} \ln |s^2+1|$$

$$I(s) = \frac{1}{2} \ln |s^2+1| + C.$$

$$0 = I(0) = \frac{1}{2} \ln 1 + C = C.$$

$$\therefore L[C_i(t)] = \frac{1}{2s} \ln |s^2+1| = \frac{1}{s} \ln \sqrt{s^2+1}.$$