6.1 Different Notions of Convergence

6.1.1 Mean Square Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. Let f be defined on $[a, b]$. We say that the sequence $\{f_m\}_{m=1}^{\infty}$ converges in the mean square sense to f on $[a, b]$ if

$$\lim_{m \to \infty} \int_a^b |f(x) - f_m(x)|^2 \, \mathrm{d}x = 0$$

6.1.2 Pointwise Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a, b]$ if for each $x \in [a, b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. That is, for each $x \in [a, b]$ and $\varepsilon > 0$ there is a natural number $N(\varepsilon, x)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon, x)$$

6.1.3 Uniform Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on $[a, b]$ if for each $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon), \text{ and for all } x \in [a, b]$$

There is one more interesting fact about the uniform convergence. If $\{f_m\}_{m=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function to f on $[a, b]$, then f is continuous.

6.1.4 Example 1

Let $u_n = x^n$ *on* $[0, 1)$. *Clearly, the sequence* $\{u_n\}_{n=1}^{\infty}$ *converges pointwise to* 0, *that is, for fixed* $x \in [0, 1)$ *we have* $\lim\limits_{n\to\infty} u_n = 0$. *But it does not converge uniformly to* 0 *as we shall* *show that for given ε there does not exist a natural number N independent of x such that $|u_n - 0| < \varepsilon$. Suppose that the series converges uniformly, then for a given ε with*

$$|u_n - 0| < \varepsilon, \tag{6.1}$$

we seek for a natural number $N(\varepsilon)$ such that relation (6.1) holds for $n > N$. Note that relation (6.1) holds true if

$$x^n < \varepsilon \Longleftrightarrow n > \frac{\ln \varepsilon}{\ln x}$$

It should be evident now that for given x and ε one can define

$$N := \left[\frac{\ln \varepsilon}{\ln x}\right], \qquad \textit{where } [\,] \textit{ gives integer rounded towards infinity}$$

It once again confirms pointwise convergence. However if x is not fixed then $\ln \varepsilon / \ln x$ grows without bounds for $x \in [0, 1)$. Hence it is not possible to find N which depends only on ε and therefore the sequence u_n does not converge uniformly to 0.

6.1.5 Example 2

Let $u_n = \dfrac{x^n}{n}$ on $[0, 1)$. This sequence converges uniformly and of course pointwise to 0. For given $\varepsilon > 0$ take $n > N := \left[\frac{1}{\varepsilon}\right]$ then noting $\left[\frac{1}{\varepsilon}\right] > \frac{1}{\varepsilon}$ we have $|u_n - 0| < x^n/n < 1/n < \varepsilon$ for all $n > N$ Hence the sequence u_n converges uniformly.

Now we discuss these three types of convergence for the Fourier series of a function.

- Let f be a **piecewise continuous function** on $[-\pi, \pi]$ then the Fourier series of f convergence to f in the mean square sense. That is

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \left[\frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)\right]\right|^2 \mathrm{d}x = 0$$

- Let f be a **piecewise continuous function** on $[-\pi, \pi]$ and the appropriate **one sided derivatives** of f at each point in $[-\pi, \pi]$ exists then for each $x \in [-\pi, \pi]$ the Fourier series of f converges pointwise to the value $(f(x-) + f(x+))/2$.

- If f **is continuous** on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and f' **is piecewise continuous** on $[-\pi, \pi]$, then the Fourier series of f converges uniformly (and also absolutely) to f on $[-\pi, \pi]$.

6.2 Best Trigonometric Polynomial Approximation

An interesting property of the partial sums of a Fourier series is that among all trigonometric polynomials of degree N, the partial sum of Fourier Series yield the best approximation of f in the mean square sense. This result has been summarized in the following lemma.

6.2.1 Lemma

Let f be piecewise continuous function on $[-\pi, \pi]$ and let the mean square error is defined by the following function

$$E(c_0, \ldots, c_N, d_1, \ldots, d_N) = \int_{-\pi}^{\pi} \left| f - \left[\frac{c_0}{2} + \sum_{k=1}^{N} (c_k \cos kx + d_k \sin kx) \right] \right|^2 \mathrm{d}x$$

then $E(a_0, \ldots, a_N, b_1, \ldots, b_N) \leq E(c_0, \ldots, c_N, d_1, \ldots, d_N)$ for any real numbers $c_0, c_1, \ldots, c_N$ and $d_1, d_2, \ldots, d_N$. Note that a_k and b_k are the Fourier coefficients of f.

6.3 Example Problems

6.3.1 Problem 1

Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$.
Solution: At $x = 0$, the Fourier series will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{0 + (-\pi)}{2} = -\frac{\pi}{2}$$

Again, $x = \pm\pi$ are another points of discontinuity and the value of the series at these point will be

$$\frac{f(\pi-) + f((-\pi)+)}{2} = \frac{\pi + (-\pi)}{2} = 0;$$

At all other points the series will converge to functional value $f(x)$.

6.3.2 Problem 2

Let the Fourier series of the function $f(x) = x + x^2$, $-\pi < x < \pi$ *be given by*

$$x + x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx \right]$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$. *Applying the result on convergence of the Fourier series find the value of*

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \qquad \textit{and} \qquad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Solution: Clearly the required series may be obtained by substituting $x = \pm\pi$ and $x = 0$. At the points of discontinuity $x = \pm\pi$ the series converges to

$$\frac{f(\pi-) + f((-\pi)+)}{2} = \frac{(\pi + \pi^2) + (-\pi + \pi^2)}{2} = \pi^2;$$

Substituting $x = \pm\pi$ into the series we get

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{(2n)} \frac{4}{n^2} = \pi^2 \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

At the point $x = 0$ is a point of continuity and therefore the series will converge to 0. Substituting $x = 0$ into the series we obtain

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{(n)} \frac{4}{n^2} = 0 \Longrightarrow \sum_{n=1}^{\infty} (-1)^{(1+n)} \frac{1}{n^2} = \frac{\pi^2}{12}.$$