① Example 1

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

B.C. $\begin{cases} \left.\dfrac{dy}{dx} + \beta \cdot y \right|_{x=0} = 0 \\ y(x=1) = 0 \end{cases}$

Define $x^* = 1 - x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx^*} \cdot \frac{dx^*}{dx} = -\frac{dy}{dx^*}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{dy}{dx^*}\right)$$

$$= \frac{d}{dx^*}\left(-\frac{dy}{dx^*}\right) \cdot \frac{dx^*}{dx}$$

$$= -\frac{d^2y}{dx^{*2}} \cdot (-1)$$

$$= \frac{d^2y}{dx^*}$$

$\therefore$ new B.C. $\begin{cases} \left. \dfrac{d^2y}{dx^*} + \alpha^2 \cdot y \right|_{x^*=1} = 0 \\ y(x^* = 0) = 0 \end{cases}$

now this question is is the same as the prev. one.

$$\therefore \quad y = c_1 \cdot \sin(\alpha_n \cdot x^*)$$

$$= c_1 \cdot \sin(\alpha_n \cdot (1-x)).$$

(2) Special ODEs for spherical co-ords.

$$(1-x^2) \cdot y'' - 2 \cdot x \cdot y' + n \cdot (n+1) \cdot y = 0$$

$n \in \mathbb{Z}^{+}$

Legendre Equation

B.C. $\{$ at $x = \pm 1$, $y = 0$

$$y(x) = c_1 \cdot P_n(x) + c_2 \cdot Q_n(x).$$

$P_n(x)$: Legendre poly. of order n

$Q_n(x)$: Legendre func. of order n.

(3) Properties

- Inner product : $\langle u, v \rangle = \int u \cdot v \cdot dx$

- $P_n(1) = 1$

- $P_n(-1) = (-1)^n$

- $\int_{-1}^{1} P_n^2(x) = \frac{2}{2 \cdot n + 1}$

- $\int_{-1}^{1} P_n(x) \cdot P_m(x)\, dx = 0$

  ↳ orthogonal prop.

- $\int u(x) \cdot v(x) \cdot \omega(x)\, dx$

  if u, v are orthogonal w.r.t weight func. $\omega(x)$

- if $Q_n(x = \pm 1) \equiv$ undefined $| \to \infty$,

  then $c_2 = 0$

  $\therefore \quad y_n(x) = c_1 \cdot P_n(x)$.

④ Special ODEs for cylindrical co-ord system.

$$x^2 \cdot y'' + x \cdot y' + x^2 \cdot y = 0$$

Bessel Equation of order 0.

Solution : $y(x) = c_1 \cdot J_0(x) + c_2 \cdot Y_0(x)$

J_0: Bessel func of 1st kind of order 0

Y_0: Bessel func. of 2nd kind of order 0

$$x^2 \cdot y'' + x \cdot y' + (x^2 - \alpha^2) \cdot y = 0$$

$\alpha \in \mathbb{Z}$.

Solution : $y(x) = c_1 \cdot J_\alpha(x) + c_2 \cdot Y_\alpha(x)$

J_α: Bessel func of 1st kind of order α

Y_α: Bessel func. of 2nd kind of order α

(see photos)

⑤ Euler Equation

$$x^2 \cdot y'' + \alpha \cdot x \cdot y' + b \cdot y = 0$$

given : $y = x^r$

$$\Rightarrow x^2 \cdot r \cdot (r-1) \cdot x^{r-2} + \alpha \cdot x \cdot r \cdot x^{r-1} + b \cdot x^r = 0$$

$$\Rightarrow (r^2 - r) x^r + a \cdot r \cdot x^r + b \cdot x^r = 0$$

characteristic eqn :

$$r^2 - r + a \cdot r + b = 0$$

$$\therefore r^2 + (a-1) r + b = 0$$

solutions $\rightarrow \{r_1, r_2\}$

⑥ Operator (L)

+ adjoint of $L \rightarrow L^*$

① $A \cdot x = b$

$\therefore x = A^{-1} \cdot b$

discrete domain

② $Lu = f$

$u = L^{-1} f \equiv L^* f$

continuous domain

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Consider a general 2nd order ODE:

$$a_0(x)\cdot u'' + a_1(x)\cdot u' + a_2(x)\cdot u = 0$$

B.C. $\begin{cases} u(x=a) = 0 \\ u(x=b) = 0 \end{cases}$

$$\therefore L = a_0\cdot\frac{d^2()}{dx^2} + a_1\cdot\frac{d()}{dx} + a_2\cdot()$$

find L^*

$\langle Lu, v\rangle$

$$= \int_a^b v\cdot Lu\cdot dx$$

$$= \int_a^b v\,(a_0\cdot u'' + a_1\cdot u' + a_2)\,dx.$$

(integr. by parts)

$$= v\cdot a_0\cdot u'\Big|_a^b + \int (v\cdot a_0)'\cdot u'\cdot dx$$

$$+ v\cdot a_1\cdot u\cdot\Big|_a^b + \int (v\cdot a_1)'\cdot u\cdot dx$$

$$+ \int_a^b v\cdot a_2\,dx$$

$$= \left[v \cdot a_0 \cdot u' + v \cdot a_1 \cdot u \right]_a^b$$

$$- \int_a^b \left(a_0' \cdot v + a_0 \cdot v' \right) u' \cdot dx$$

$$- \int_a^b \left(a_1' \cdot v + a_1 \cdot v' \right) \cdot u \cdot dx$$

$$+ \int v \cdot a_2 \cdot u \cdot dx$$

$$= \left[v \cdot a_0 \cdot u' + v \cdot a_1 \cdot u \right]_a^b - \left(a_0' v + a_0 \cdot v' \right) \cdot u \Big|_a^b$$

$$+ \int \left(a_0'' \cdot v + a_0' \cdot v' + a_0' \cdot v' + a_0 \cdot v'' \right) \cdot u \cdot du$$

$$- \int \left(a_1' \cdot v + a_1 \cdot v \right) u \, dx \quad + \int v \cdot a_2 \cdot u \cdot dx$$

$$= J(u, v) + \int (L^* v) \cdot u \cdot dx$$

Bilinear Concommittant

$$a_0 \cdot \frac{d^2}{dx^2}() + (2a_0' - a_1) \cdot \frac{d}{dx}() + (a_0'' - a_1' + a_2) \cdot ()$$

⑧ $\langle Lu, v\rangle = J(u,v) + \langle L^* v, u\rangle$

if $L = L^* \Rightarrow$ self adjoint.

⑨ Example 2

$$\frac{d^2 u}{dx^2} + \lambda \cdot u = 0$$

$$\text{B.C.} \begin{cases} u(x=0) = 0 \\ u(x=1) = 0 \end{cases}$$

$$Lu + \lambda u = 0$$