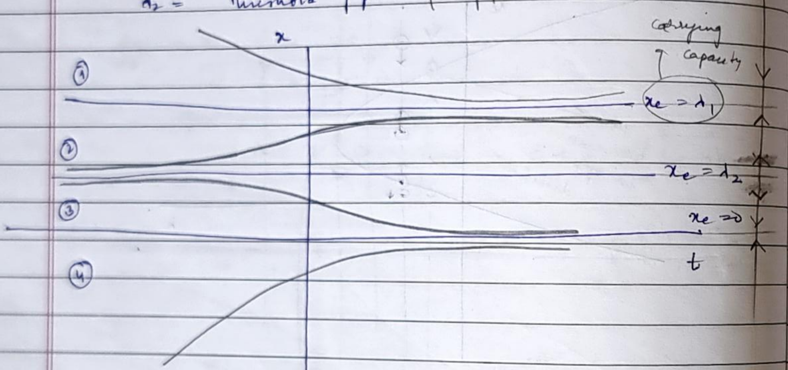
2) $\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$

$0 < \lambda_2 < \lambda_1$

λ_1 = carrying capacity

λ_2 = threshold population (min population)



① →

asymptotes → $\lambda_1, \lambda_2, 0$

(Extinction when the initial population is less than the threshold population)

Point of intersection of stable & unstable curves is bifurcation pt

→ $\frac{dx}{dt} = ax - ax^2$

$\frac{dx}{dt} = a - x^2$

$\frac{dx}{dt} = ax - x^2$

$\frac{dx}{dt} = ax - x^3$

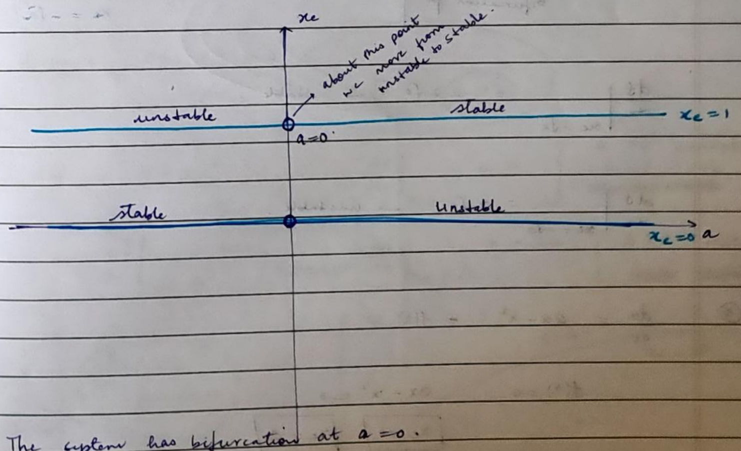
① $\frac{dx}{dt} = ax(1-x) = f(x)$

• Bifurcation diagram

nature of x_e will depend on a → stable or unstable.

$\frac{df}{dx} = a - 2ax$

$f(x) = 0 \rightarrow x_e = 0, \ x_e = 1$



The system has bifurcation at a = 0.

$\frac{df}{dx}$ → has no meaning until and unless calculated at x_e.

$\frac{df}{dx} < 0$ → stable

$\frac{df}{dx} > 0$ → unstable

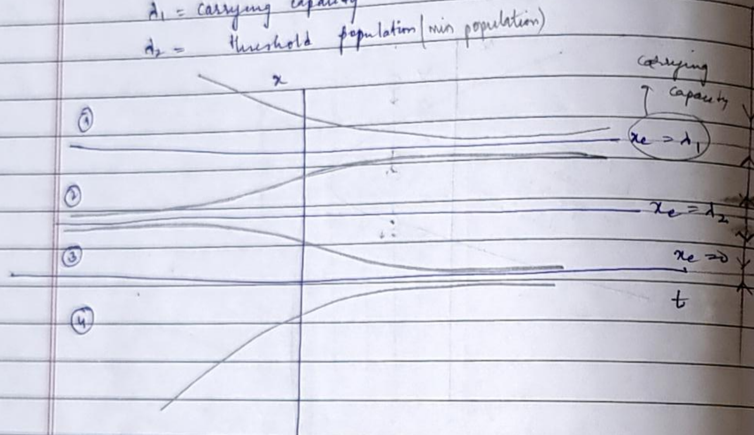
$\left.\frac{df}{dx}\right|_{x_e=0} = a \quad \begin{cases} a>0 \rightarrow \text{unstable} \\ a<0 \rightarrow \text{stable} \end{cases}$

$\left.\frac{df}{dx}\right|_{x_e=1} = -a \quad \begin{cases} a<0 \rightarrow \text{unstable} \\ a>0 \ \text{stable.} \end{cases}$

2) $\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$ $0 < \lambda_2 < \lambda_1$

λ_1 = carrying capacity

λ_2 = threshold population / min population)



① → asymptotes → $\lambda_1, \lambda_2, 0$

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① $\frac{dx}{dt} = ax(1-x) = f(x)$

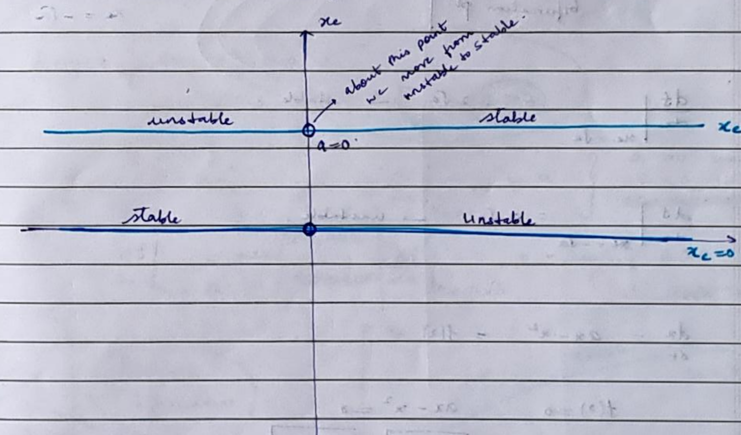
stable or unstable

nature of x_e will depend on a

$\frac{df}{dx} = a - 2ax$

• Bifurcation diagram

$f(x) = 0 \rightarrow x_e = 0, \; x_e = 1$



The system has bifurcation at $a = 0$.

$\frac{df}{dx}$ → has no meaning until and unless calculated at x_e.

$\left.\frac{df}{dx}\right|_{x_e=0} = a \begin{cases} a>0 \rightarrow \text{unstable} \\ a<0 \rightarrow \text{stable} \end{cases}$

$\frac{df}{dx} < 0 \rightarrow$ stable

$\frac{df}{dx} > 0 \rightarrow$ unstable

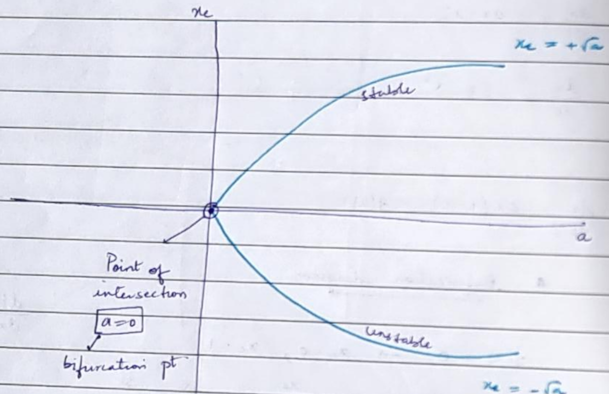
$\left.\frac{df}{dx}\right|_{x_e=1} = -a \begin{cases} a<0 \rightarrow \text{unstable} \\ a>0 \;\; \text{stable.} \end{cases}$

② $\frac{dx}{dt} = a - x^2 = f(x) = 0$ $\quad x^2 = a$

$\boxed{x_e = \pm\sqrt{a}}$ → equilibrium soln.

↳ $a > 0$

$\frac{df}{dx} = -2x$



$\left.\frac{df}{dx}\right|_{x_e = \sqrt{a}} = -2\sqrt{a}$ → stable

$\left.\frac{df}{dx}\right|_{x_e = -\sqrt{a}} = 2\sqrt{a}$ → unstable.

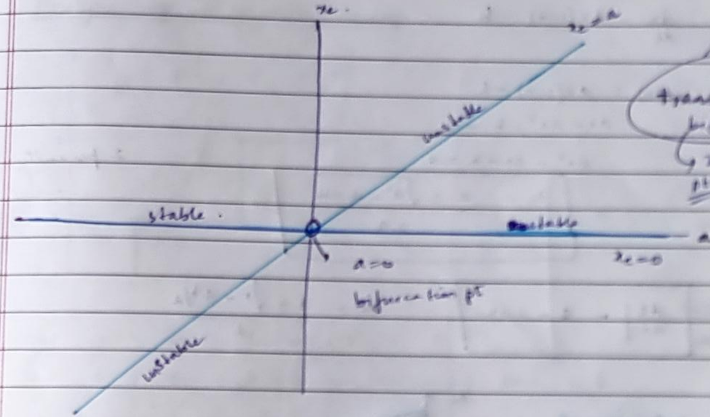
③ $\frac{dx}{dt} = ax - x^2 = f(x)$

$f(x) = 0 \qquad ax - x^2 = 0$

$\boxed{x_e = 0} \quad \boxed{x_e = a}$

$\frac{df}{dx} = a - 2x$ $\qquad \left.\frac{df}{dx}\right|_{x_e=0} = a \quad \begin{cases} a>0 \text{ unstable} \\ a<0 \text{ stable} \end{cases}$

$\left.\frac{df}{dx}\right|_{x_e=a} = -a \quad \begin{cases} a>0 \text{ stable} \\ a<0 \text{ unstable} \end{cases}$



pt of intersection

transcritical bifurcation

↳ $x_e = 0, a = 0$

pt

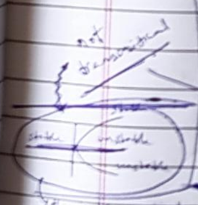
④ $\frac{dx}{dt} = ax - x^3 = f(x)$

$f(x) = 0 \qquad ax - x^3 = 0$

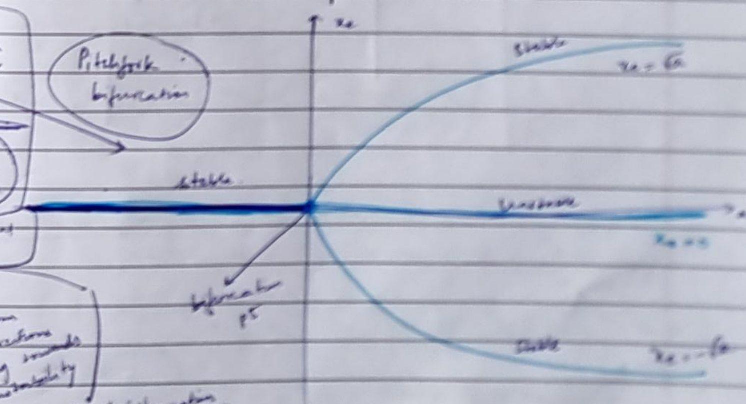
$x_e = 0, \quad x_e = \pm\sqrt{a} \quad a > 0$

$\frac{df}{dx} = a - 3x^2$ $\qquad \left.\frac{df}{dx}\right|_{x_e=0} = a \quad \begin{cases} a>0 \text{ unstable} \\ a<0 \text{ stable} \end{cases}$

$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = -2a \quad \{ a>0 \text{ stable}$ $\qquad \left.\frac{df}{dx}\right|_{x_e=\sqrt{a}} = -2a \quad \} \ a>0 \text{ stable}$



Pitchfork bifurcation



like all stable ones are coming from multiple directions & moving towards unstability

∴ bifurcation exists

$1 - \frac{4h}{a} > 0$ $\quad \frac{h}{a} < \frac{1}{4}$ $\quad h < \frac{a}{4}$

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8) $\frac{dx}{dt} = ax(1-x) - h$

$f(x) = ax(1-x) - h$

$ax - ax^2 - h = 0$

$ax^2 - ax + h = 0$

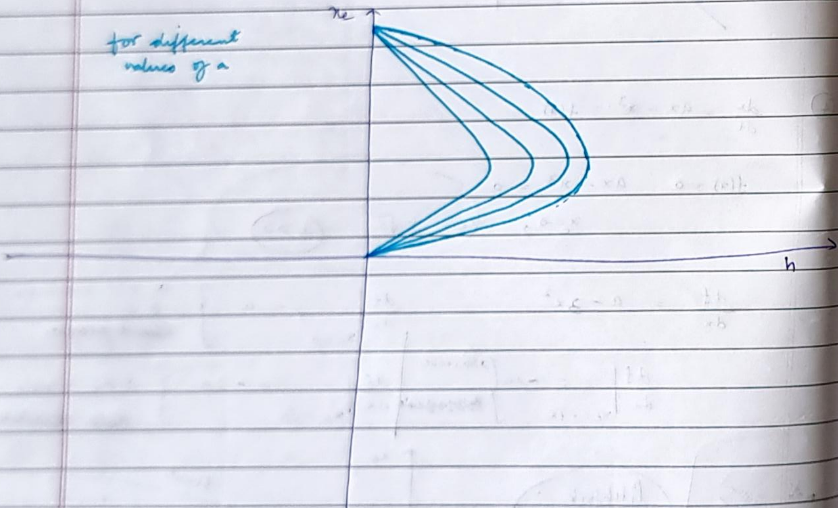
$$\boxed{x_e = \frac{a \pm \sqrt{a^2 - 4ah}}{2a}}$$

2 parameters

$$\boxed{\left.\frac{df}{dx}\right| = a - 2ax}$$

$$x_e = \frac{1 \pm \sqrt{1 - 4h/a}}{2}$$

for different values of a



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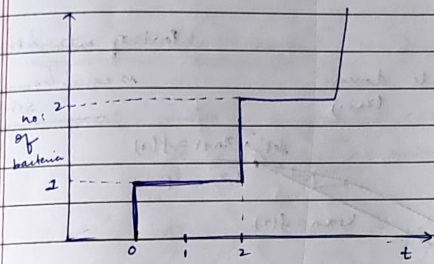
Analysis ~~of~~ in discrete domain



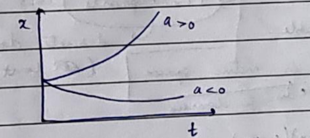
x, t both are discrete

at $t = 0$ no. of bacteria $= 1$

$t = 1$ the no. of bacteria will remain same $= 1$

→ The no. of bacteria is very large → taking continuous domain works

$$\frac{dx}{dt} = ax$$



$$\frac{x_{n+1} - x_n}{\Delta t} = a x_n$$

$$x_{n+1} = (1 + a\Delta t)\, x_n$$

⇒ Q) $\frac{dx}{dt} = ax(1-x)$

$$x_{n+1} = (1 + a\Delta t)\, x_n \left[1 - \left(\frac{a\Delta t}{1 + a\Delta t}\right) x_n\right]$$

⇒ Equilibrium soln :- $x_{n+1} = x_n$

$$x_n = (1 + a\Delta t)\, x_n \left[1 - \frac{a\Delta t\, x_n}{1 + a\Delta t}\right]$$

$$x_n \left[1 - (1 + a\Delta t)\left(1 - \frac{a\Delta t\, x_n}{1 + a\Delta t}\right)\right] = 0.$$

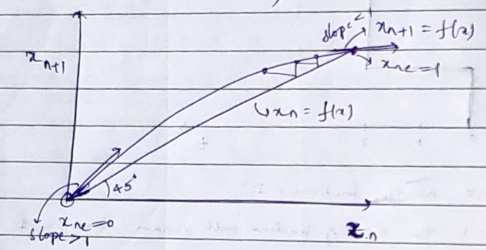
$x_{ne} = 0$ $\quad$ $x_{ne} = 1$

→ Graphical method to get equilibrium soln

Plot $f(x) = x$

Plot $g(x)$ = discrete domain (x_{n+1})

Point of intersection is equilibrium soln

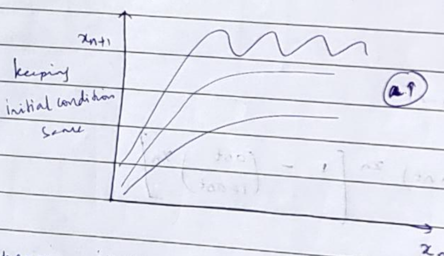


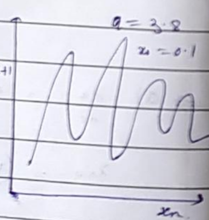
1. discretize the system
2. find equilibrium soln by plotting $f(x) = x$, $g(x) = x_{n+1}$ → Fixed points → discrete domain (continuous domain → equilibrium soln)
3. slope > 1 → unstable
   slope < 1 → stable

→ All the pts lie in metric spaces → discrete domain

Q) $x_{n+1} = a(x_n)(1 - x_n)$

as you change a it starts oscillating





→ change initial condition & check ✓ $\boxed{a = 3.8,\ x_0 = 0.11}$

→ future behavior has changed completely

→ chaotic system → future changes drastically depending on initial condition

---

metric spaces → have measurement of units defined.

Reactor stability Analysis

Transient operation of a jacketed CSTR

$$\frac{dc}{dt} = \frac{F}{V}(C_f - C) - r \quad \text{→ makes the system non-linear}$$

$$\frac{dT}{dt} = \frac{F}{V}(T_f - T) + \left(\frac{-\Delta H}{\rho C_p}\right) r - \frac{UA}{V\rho C_p}(T - T_j)$$

→ There can be multiple ss solutions. We need to identify which solution is stable. If the conditions change (stability), nature of solutions will change.

1. Find steady state solution. → Newton raphson → to find SS

Q)
$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$$
$$\frac{dx_2}{dt} = 2x_2 = f_2$$
} No bifurcation plot

SS: $x_1^2 - x_2^2 - 1 = 0$ $x_1^2 = 1$ $x_{1e} = \pm 1$

$2x_2 = 0$ $\boxed{x_{2e} = 0}$

$$\underbrace{\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{equilibrium soln}}$$

(dynamical variable as a variable)

→ (determine jacobian) $J = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$

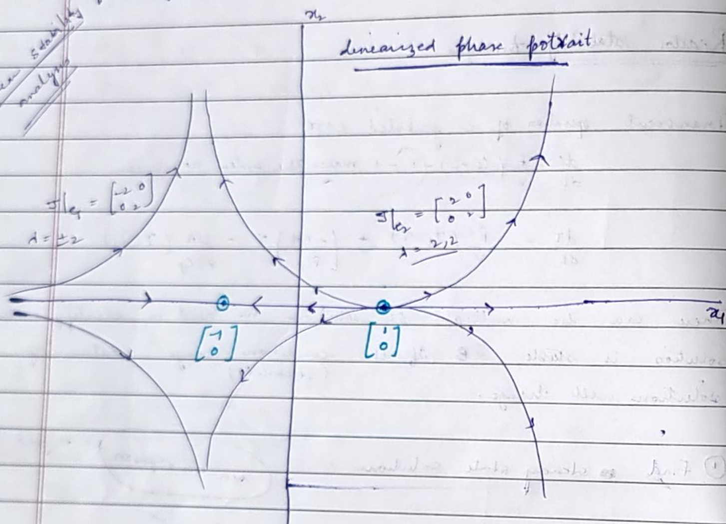
→ Determine the jacobian at equilibrium soln.

$$J|_{e_1} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad J|_{e_2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ → source soln}$$

$\lambda|_{e_1} = \pm 2$ → saddle. $\lambda|_{e_2} = 2, 2$

→ find eigen values.

linear stability analysis



$$y = f(x)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{bmatrix}$$

→ $\dot{y} = f(x)$

→ $y = y_e + \frac{dy}{dx}\Big|_e (x - x_e) + O\left(\frac{d^2y}{dx^2}\right)$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{e_1} + J\Big|_{e_1} \begin{bmatrix} x_1 - x_{1e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

$$\begin{bmatrix} f_1 - f_1|_{e_1} \\ f_2 - f_2|_{e_1} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - x_{1e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

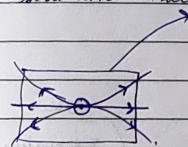
$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - x_{1e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Hartman-Grobman theorem

orbit structure → all phase lines.

topologically → qualitatively



hyperbolic equilibrium point:- pt which doesn't correspond to eigenvalue 0

(even if one of the eigen value is 0 → don't linearize) → linearized phase potrait will not work

$$\frac{F}{V} C_f - \frac{F}{V} C_s - k e^{-E/RT_s} C_s = 0$$

$$C_s = \frac{\left(\frac{F C_f}{V}\right)}{\left(\frac{F}{V}\right) + k e^{-E/RT_s}}$$

$$\frac{F T_f}{V} - \left(\frac{F}{V}\right) T_s + \left(\frac{-\Delta H}{\rho c_p}\right)\left(k e^{-E/RT_s}\right) C_s - \frac{UA}{V\rho c_p} T_s + \frac{UAT_j}{V\rho c_p} = 0$$

heat of reaction

$$\underbrace{\frac{\left[\frac{-(\Delta H)\, k\, (F/V)}{\rho c_p} C_f\right] e^{-E/RT_s}}{\left(\frac{F}{V}\right) + k e^{-E/RT_s}}}_{\text{nonlinear}} = \underbrace{\left(\frac{UA}{V\rho c_p} + \frac{F}{V}\right) T_s - \left(\frac{UAT_j}{V\rho c_p} + \frac{FT_f}{V}\right)}_{\text{linear}}$$

heat added to the system or taken out

convective heat transfer due to reactor fluid and jacket fluid classmate

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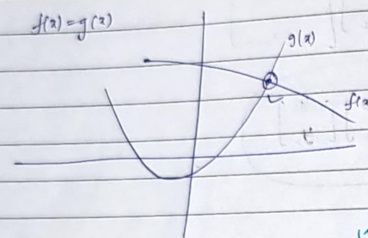
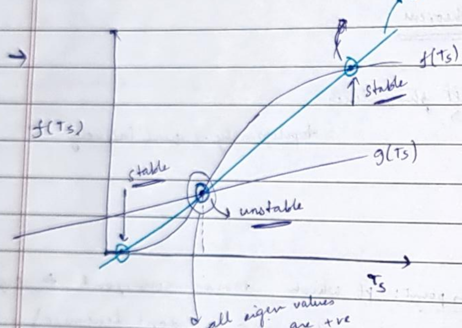


(plot to find the point of intersection)

$g(T_s)$ (3 ss solutions)

all solutions are physically realisable



stable

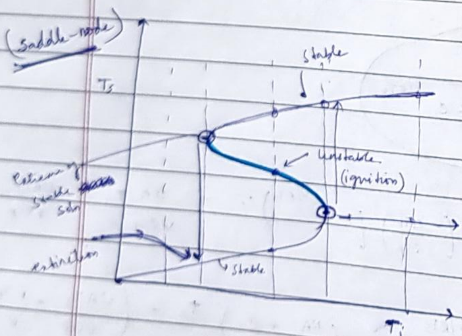
eigenvalues = −ve (or)

Re(eigenvalues) = −ve

might have complex eigen values.

will vary T_j to ~~[illegible]~~ maintain desired stable soln

(saddle-node)



(bifurcation diagram)

when there is a small change in temp, the ss temp shoots up (ignition)

(lorenz attractor)

classmate

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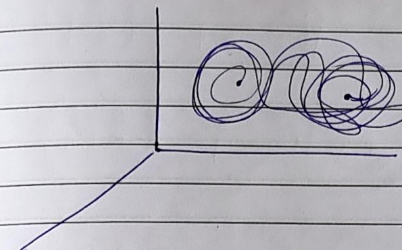

Chaotic

(lorenz attractor)