EXTERNAL INCOMPRESSIBLE VISCOUS FLOW
– BOUNDARY LAYER CONCEPTS

FLUID MECHANICS
Fox & Mc.DONALD

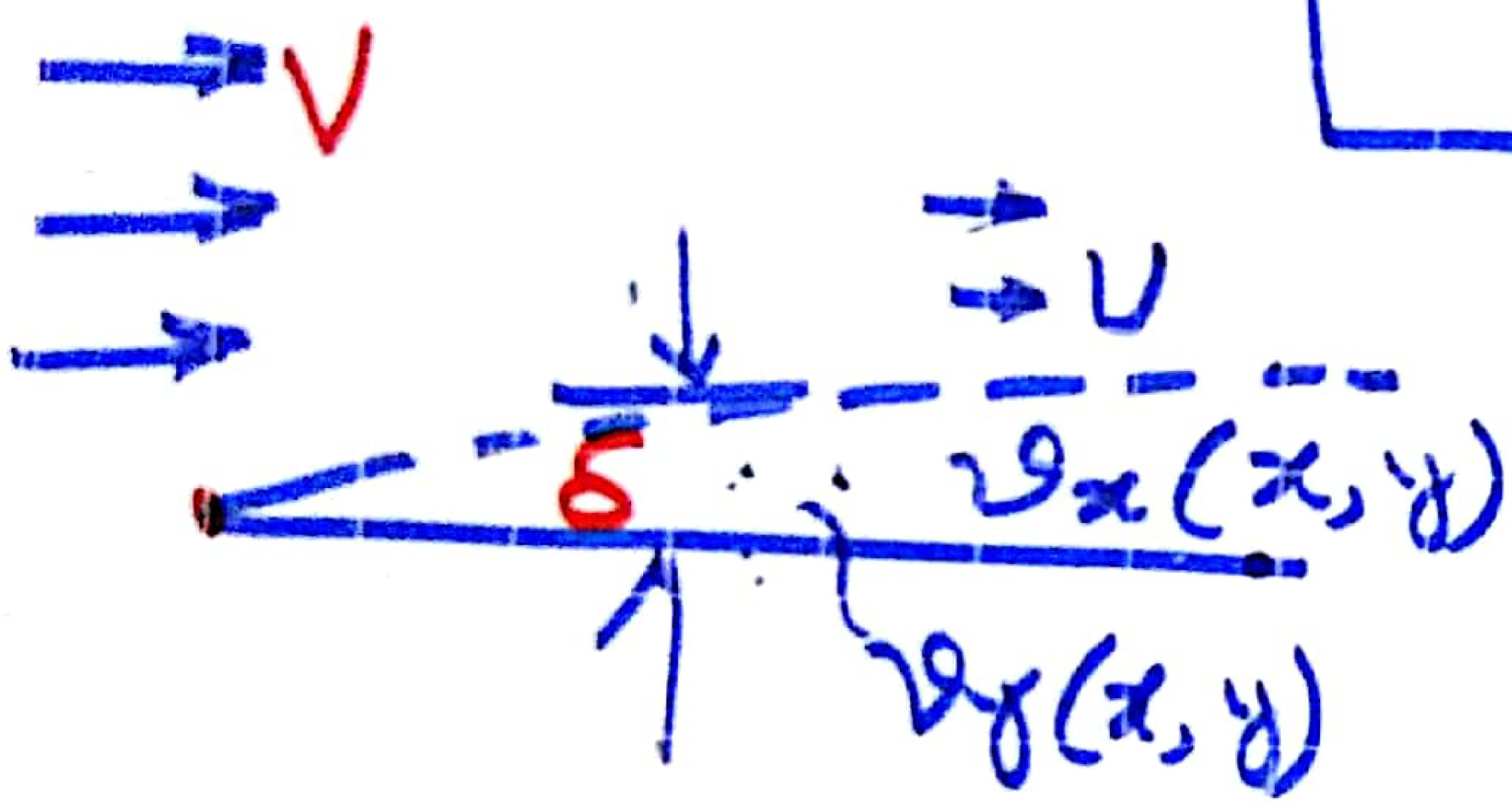
BOUNDARY LAYER APPROXIMATIONS

LAMINAR FLOW ON A FLAT PLATE

x COMP.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right]$$

$$\frac{\partial^2 v_x}{\partial x^2} \ll \frac{\partial^2 v_x}{\partial y^2}$$



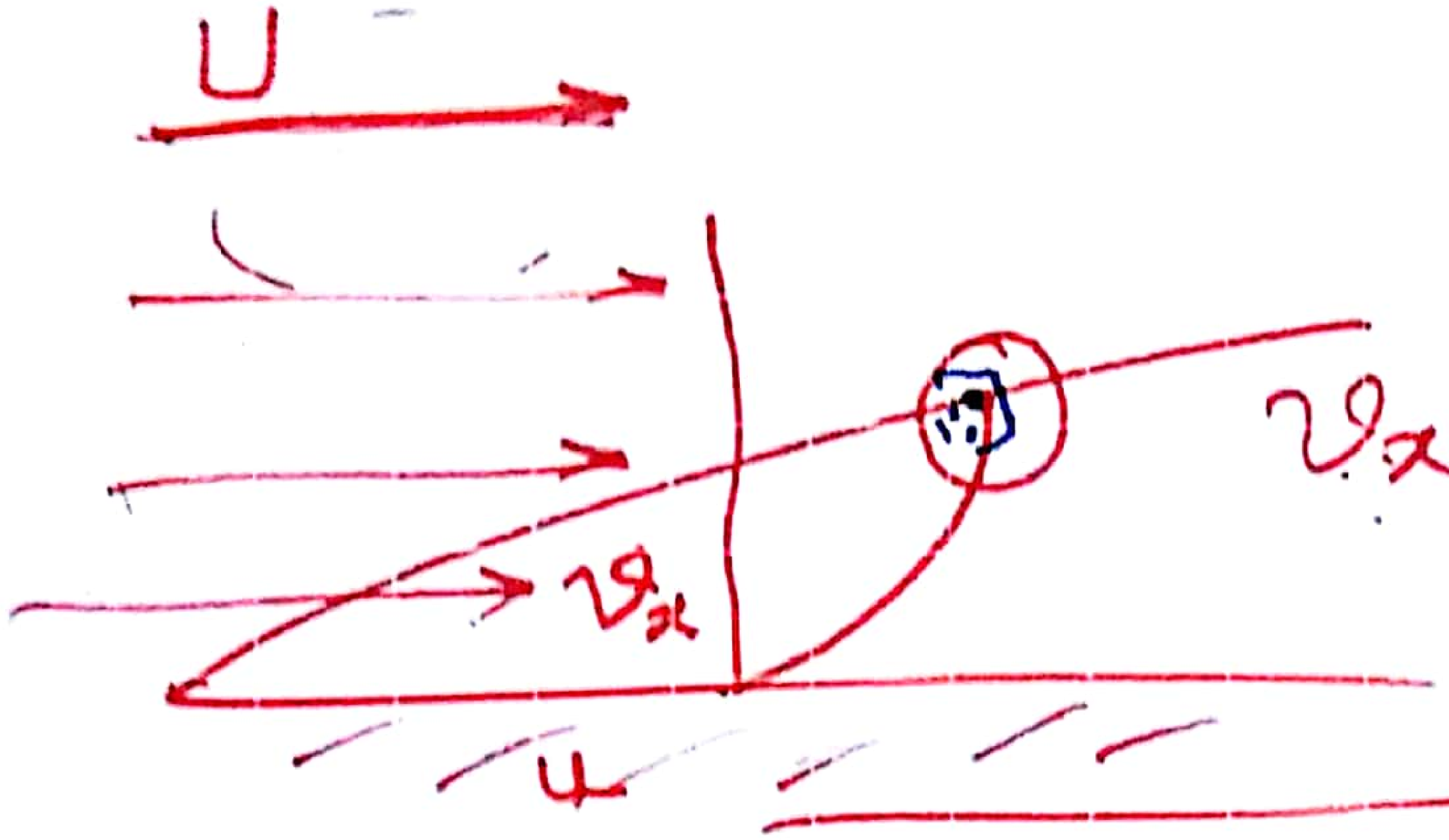
$$\boxed{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}}$$

$\delta(x)$ SMALL

$\frac{\partial v_x}{\partial x}$

$$\frac{\partial v_x}{\partial y} \sim \frac{0 - U}{\delta}$$

✓ $y = 0$	$v_x = 0$
✓ $y = \infty$	$v_x = U$
✓ $x = 0$	$v_x = (V = U) = \underline{U}$



y at which $v_x = 0.99\,U_\infty$ IS CALLED THE B.L. THICKNESS

DISTURBANCE THICKNESS

$v_x = f(x, y)$ $\qquad \dfrac{\partial v_x}{\partial y} \approx 0 \qquad v_x \neq f^n(y)$

DISPLACEMENT THICKNESS, δ^*

VISCOUS FLOW: AMOUNT OF REDUCTION IN MASS FLOW RATE IN A B.L. $= \displaystyle\boxed{\int_0^\infty \rho\,(U-u)\,dy}$

INVISCID FLOW

$$\rho\, U \delta^* = \int_0^\infty \rho\,(U-u)\,dy$$

$$\boxed{\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U}\right) dy}$$

MOMENTUM THICKNESS (Θ)

$$\boxed{\int_0^{\infty} \underbrace{\rho u (\;\; dy}_{\text{MASS}} \underbrace{(U_\infty - u)}_{\text{REDUCTION IN VEL}}} =$$

REDUCTION IN M^2 FOR THE FLUID THAT ACTUALLY FLOWS IN THE B.L.

$$\underbrace{\rho U \Theta}_{m} \times \underbrace{U}_{\text{VEL.}} = \underline{\rho U^2 \Theta}$$

REDUCTION IN M^2 IN INVISCID FLOW

$$\rho U_\infty^2 \Theta = \int_0^{\infty} \rho u (U_\infty - U)\, dy$$

$$\underline{\underline{\Theta}} = \int_0^{\infty} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \approx \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

BLASIUS

$$\frac{v_x}{U} = g(\eta) \qquad \eta \sim \frac{y}{\delta(x)}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \quad \text{—} \; 2$$

$$\cancel{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0} \quad \text{—} \; 1$$

$$\boxed{v_x = \frac{\partial \psi}{\partial y}}, \quad v_y = -\frac{\partial \psi}{\partial x}.$$

ψ = STREAM FUNCTION

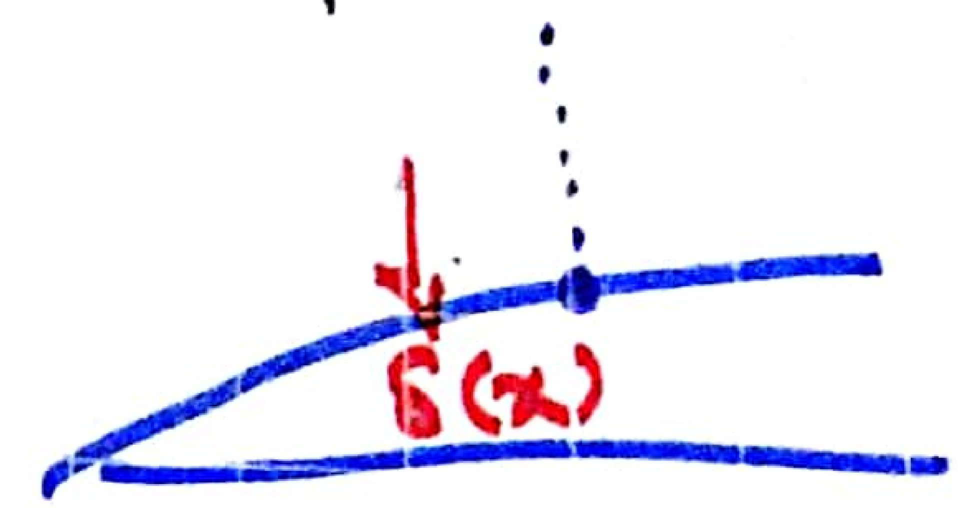
$$\frac{\partial \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial y \partial x} = 0$$

ψ EXACT DIFFERENTIAL

$$v_x \frac{\partial v_x}{\partial x} + v_y \cancel{\frac{\partial v_x}{\partial y}}^{\approx 0} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

NEAR THE B.L.

$$v_x \sim U, \quad y \sim \delta, \quad \frac{\partial v_x}{\partial y} \approx 0$$



$$U \frac{U}{x} \sim \nu \frac{U}{\delta^2}$$

$\frac{y}{\sqrt{x}}$

$$\delta^2 \sim \frac{\nu x}{U}$$

$$\boxed{\delta \sim \sqrt{\frac{\nu x}{U}}}$$

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{U}{\nu x}}$$

$f \quad \eta$

BLASIUS 2

$$\longrightarrow v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

i) INTRODUCING A STREAM FN ψ.

ii) BY INVOKING THE METHOD OF COMBINATION OF VARIABLES

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

DIMENSIONLESS STR. FN

$$f(\eta) = \frac{\psi}{\sqrt{\nu x U}}$$

$f \quad \eta$ $\rightarrow$ ODE

v_x , v_y

— x , y

$f(\eta) \longrightarrow \eta \longrightarrow$ ODE $\longleftrightarrow$ PDE

Fox & McDONALD
FLUID MECH.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

$$f = \frac{\psi}{\sqrt{\nu x U}}$$

$$\psi = \sqrt{\nu x U}\, f(\eta)$$

$$v_x = \frac{\partial \psi}{\partial y}$$

$$= \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \sqrt{\nu x U}\,\frac{df}{d\eta}\cdot\sqrt{\frac{U}{\nu x}}$$

$$= U\frac{df}{d\eta}$$

$$\boxed{v_x = U\frac{df}{d\eta}}$$

$$v_y = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[f\cdot\sqrt{\nu x U}\right].$$

$$= -\left[\sqrt{\nu x U}\cdot\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\, f\right]$$

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$$v_y = -\left[\sqrt{\nu x U}\cdot\frac{df}{d\eta}\cdot\frac{\partial\eta}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\, f\right].$$

$$= -[\ ]$$

$$\boxed{v_y = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta\frac{df}{d\eta} - f\right].}$$

$$\boxed{v_x = U\frac{df}{d\eta}}$$

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = \nu\frac{\partial^2 v_x}{\partial y^2}$$

$$\boxed{2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0}$$

ODE

$$2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0.$$

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$$\eta = y\sqrt{\frac{U}{\nu x}}$$

$$v_x = U\frac{df}{d\eta}.$$

$$v_y = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta\frac{df}{d\eta} - f\right].$$

BC.

$\eta = 0 \qquad f = \frac{df}{d\eta} = 0$

$\eta = \infty \qquad f' = 1$

NUM. SOL^N

HOWARTH

η	f	f'	f''
0	0	0	0.332
5.0	3.28329	0.99155	0.01591
8.0	6.27923	~1.0	0.00001
8.4	6.67923	~1.0	0.000001

$v_x \sim U.$

$f' \sim 1$

$\eta = 5\ [f' = 1]$

$\eta - f$

BLASIUS 6

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$$\eta = y\sqrt{\frac{U}{\nu x}}$$

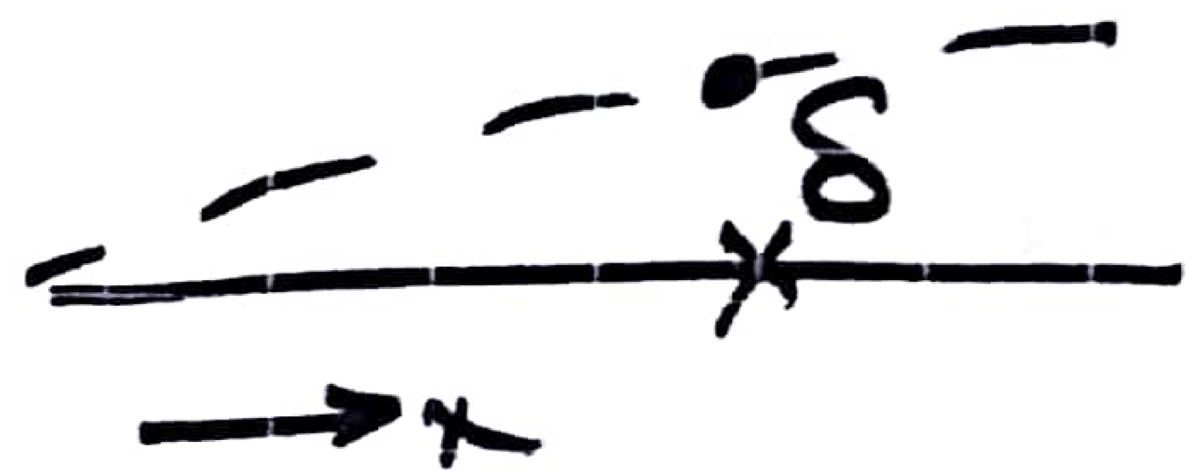
$\eta = 5.0$ EDGE OF THE B. L. $f' = 0.991$

$$\frac{v_x}{U} = 0.99$$

$$\boxed{5.0 = \delta\sqrt{\frac{U}{\nu x}}}$$

$$\delta = \frac{5.0}{\sqrt{U/\nu x}} = \frac{5.0\,x}{\sqrt{Re_x}}$$

$$\delta = \frac{5.0\,x}{\sqrt{Re_x}}$$



$$\tau_w = \mu \frac{\partial v_x}{\partial y}\bigg|_{y=0}$$

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

$$\frac{\psi}{\sqrt{\nu x U}} = f,$$

$$= \mu \frac{\partial}{\partial y} \frac{\partial \psi}{\partial y}\bigg|_{y=0}.$$

$$= \mu \frac{\partial}{\partial y} U \frac{df}{d\eta}\bigg|_{\eta=0}. \quad = \mu U \frac{d^2 f}{d\eta^2} \cdot \frac{\partial \eta}{\partial y}\bigg|_{\eta=0}$$

0.332

$$\Rightarrow \boxed{\tau_w = \mu U \sqrt{\frac{U}{\nu x}} \frac{d^2 f}{d\eta^2}\bigg|_{\eta=0}.}$$

$\eta = 0 \quad f'' = 0.332$

$$\tau_w = 0.332\, U \sqrt{\rho \mu U / x} = \frac{0.332 \rho U^2}{\sqrt{Re_x}}$$

BLASIUS

$$\boxed{\tau_w = \frac{0.332\,\rho U^2}{\sqrt{Re_x}}}$$

• SHEAR STRESS COEFF. C_f

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}$$

$$\delta = \frac{5.0x}{\sqrt{Re_x}}$$