Lecture - 2
Tuesday
18/07/17
12:00-12:55

Ex 1 Find LT. of $f(t)$ when.

(a) $f(t) = (\sin t - \cos t)^2$.

$L[f(t)] = L[(\sin t - \cos t)^2]$.

$= L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$

$= L[1 - \sin 2t]$ $\quad L[\sin pt] = \frac{p}{s^2+p^2}$.

$= L[1] - L[\sin 2t]$. $\quad$ Here $p = 2$.

$= \frac{1}{s} - \frac{2}{s^2+4} = \frac{s^2-2s+4}{s(s^2+4)}$ // $L[1] = \frac{1}{s}$.

Ex-2 $f(t) = 0; \; 0 < t < 1$
$= t; \; 1 < t < 4$.
$= 0; \; t > 4$.

$f(t) = t; \; 1 < t < 4$.
$= 0$ otherwise.

$L[f(t)] = \int_0^\infty f(t) e^{-st} dt$.

$= \int_1^4 t e^{-st} dt = \left[t \cdot \frac{e^{-st}}{-s} \right]_1^4 + \int_1^4 \frac{e^{-st}}{s} dt$.

$\int_1^4 e^{-st} dt = \left. \frac{e^{-st}}{s} \right|_4^{t=1} = \frac{e^{-s} - e^{-4s}}{s}$.

Diff. both sides w.r.to s, get value of $\int_1^4 t e^{-st} dt$

or, you can integrate by parts.

$$L[f(t)] = e^{-s}\left\{\frac{1}{s}+\frac{1}{s^2}\right\} - e^{-4s}\left\{\frac{4}{s}+\frac{1}{s^2}\right\}.$$

Exercise.

1. Find $L[\cos^2 kt]$ $\rightarrow$ $L\left[\frac{1}{2}\times 2\cos^2 kt\right]$

$= L\left[\frac{1}{2}(1+\cos 2kt)\right].$

2. $L\left[(5e^{2t}-3)^2\right]$

$= L\left[25e^{4t}+9-30e^{2t}\right].$

Scaling & shifting theorems.

Scaling. Let $\bar{f}(s)$ be the LT of $f(t)$ w.r. to the transform variable s.

i.e $L[f(t); s] = \bar{f}(s)$

then $L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$; $a>0$.

Proof. $L[f(at)] = \int_0^\infty f(at)e^{-st}\,dt.$ $\quad at = z.$ $\quad t = \frac{z}{a}$ $\quad dt = \frac{1}{a}dz.$

$= \int_0^\infty f(z)e^{-s\cdot\frac{z}{a}}\cdot\frac{1}{a}\,dz.$

$= \frac{1}{a}\int_0^\infty f(z)e^{-s_1 z}\,dz$; $\quad s_1 = \frac{s}{a}.$

$= \frac{1}{a}L[f; s_1] = \frac{1}{a}L\left[f; \frac{s}{a}\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$

1st shifting theorem

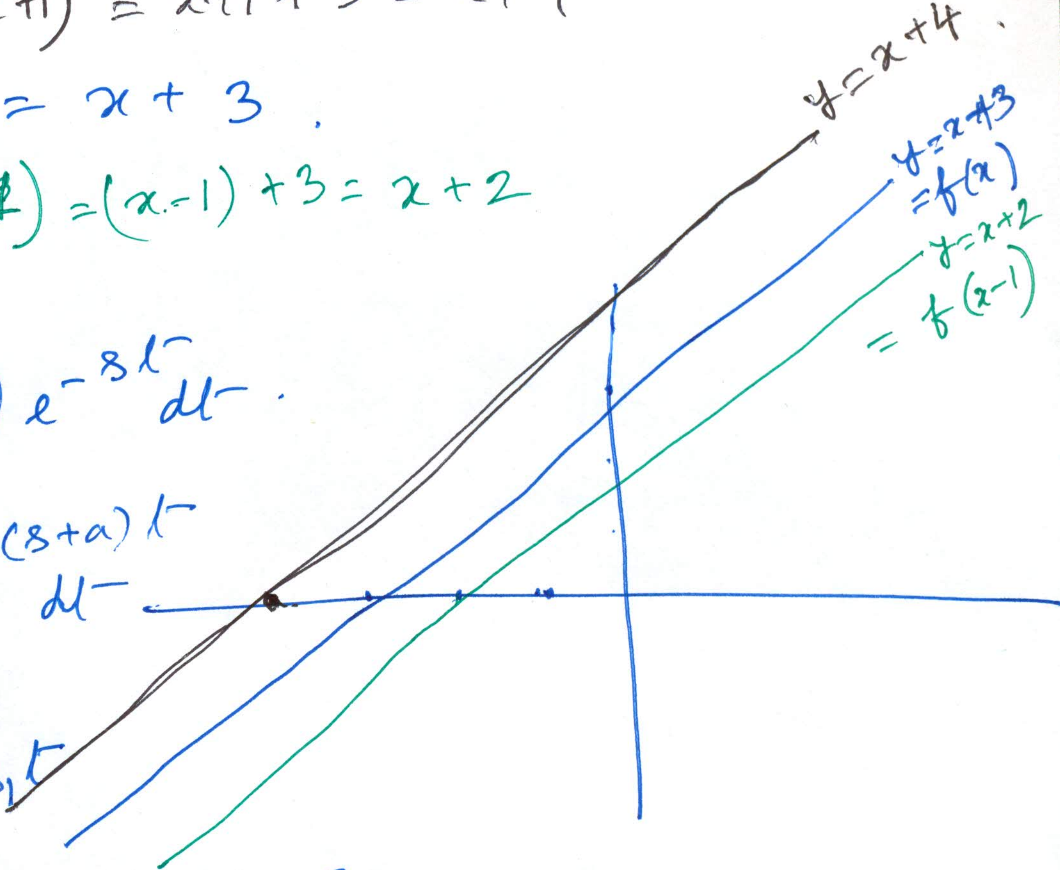
If $L[f(t)] = \bar{f}(s)$, then

$$L[e^{-at} f(t)] = \bar{f}(s+a)$$

$y = f(x+1) = x+1+3 = x+4$

$y = f(x) = x+3$.

$y = f(x-1) = (x-1)+3 = x+2$



$$= \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt.$$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}\, dt$$

$$= \int_0^\infty f(t)\, e^{-s_1 t} \quad ; \quad s_1 = s+a.$$

$$= \bar{f}(s_1) = \bar{f}(s+a).$$

Ex 1. If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$.

find $L\left\{\frac{\sin at}{t}\right\}$.

$f(t) = \frac{\sin t}{t}$, $f(at) = \frac{\sin at}{at}$.

$L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$; given $\bar{f}(s) = \tan^{-1}\frac{1}{s}$.

$\therefore L\left[\frac{\sin at}{at}\right] = \frac{1}{a}\cdot\tan^{-1}\frac{1}{\left(\frac{s}{a}\right)} = \frac{1}{a}\tan^{-1}\frac{a}{s}$.

or, $\frac{1}{a}L\left[\frac{\sin at}{t}\right] = \frac{1}{a}\tan^{-1}\frac{a}{s}$.

$\therefore L\left[\frac{\sin at}{t}\right] = \tan^{-1}\frac{a}{s}$.

If $L[f(t)] = \bar{f}(s)$

$L[e^{-at}f(t)] = \bar{f}(s+a)$

$L[e^{bt}f(t)] = \bar{f}(s-b)$

Ex-2. Find $L\left[t^{\frac{7}{2}}e^{-3t}\right]$

$L\left[t^{\frac{7}{2}}\right] = \frac{\Gamma\left(\frac{7}{2}+1\right)}{s^{\frac{7}{2}+1}}$.

$L[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$

$\underline{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$L\left[t^{\frac{7}{2}}\right] = \frac{\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}{s^{\frac{9}{2}}} = \frac{\frac{7}{2}\Gamma\left(\frac{5}{2}+1\right)}{s^{9/2}}.$$

$$= \frac{\frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}{s^{9/2}} = \frac{105}{16}\frac{\sqrt{\pi}}{s^{9/2}}.$$

$$L\left[e^{3t}\,t^{\frac{7}{2}}\right] = \frac{105}{16}\cdot\frac{\sqrt{\pi}}{(s-3)^{9/2}}$$ using 1st shifting theorem.

Exercise 1.

Find $L\left[\cosh at \cos bt\right]$.

Ex. 2.

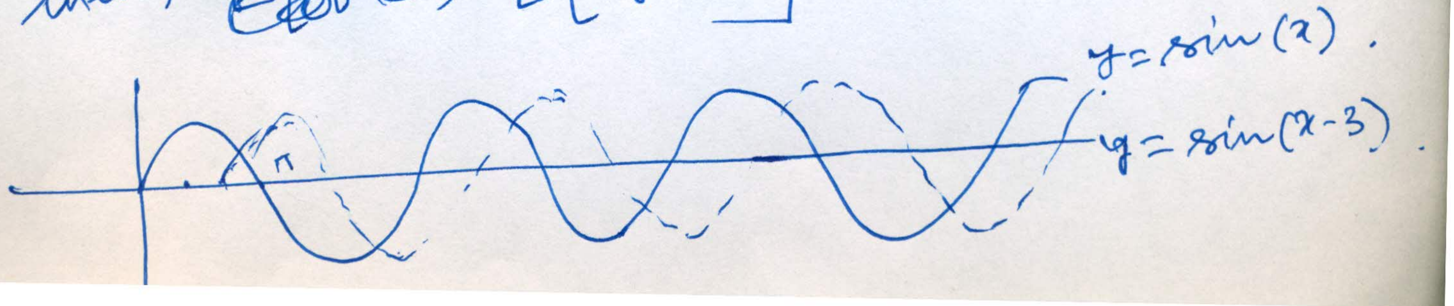
$L\left[\left\{3t^5 - 2t^4 + 4e^{-5t} - 3\sin 6t + 4\cos 4t\right\}e^{2t}\right]$

2nd shifting theorem.

If $L[f(t)] = \bar{f}(s)$

and if $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then, $L[g(t)] = e^{-as}\,\bar{f}(s)$.



$$L[g(t)] = \int_0^\infty g(t)\, e^{-st}\, dt.$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt.$$

$t - a = x.$

$t = (x + a)$

$dt = dx.$

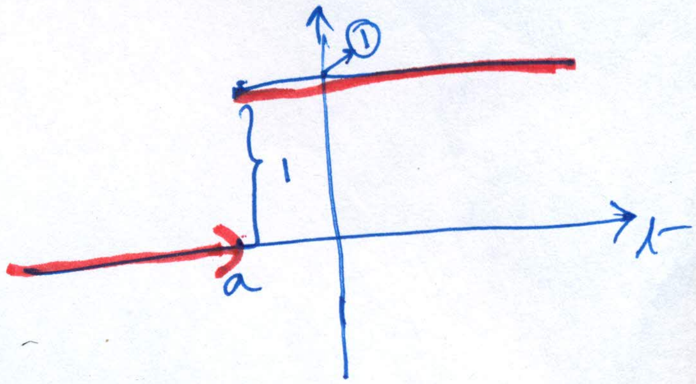
$$= \int_0^\infty f(x)\, e^{-s(x+a)}\, dx.$$

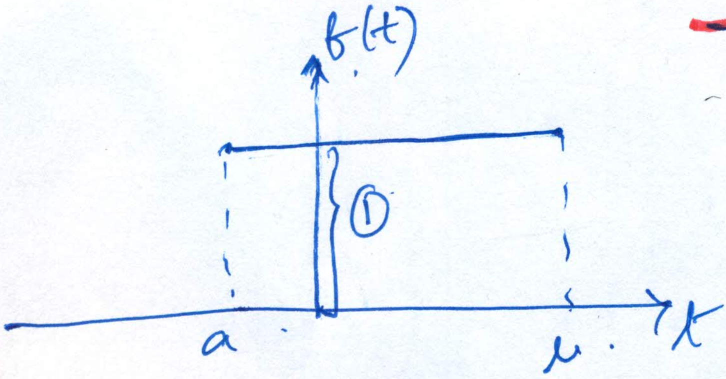
$$= \int_0^\infty f(x)\, e^{-sx} . e^{-sa}\, dx.$$

$$= e^{-sa} \int_0^\infty f(x)\, e^{-sx}\, dx = e^{-sa} \bar{f}(s) //.$$

Heaviside's unit step funct.

$$H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a. \end{cases}$$





$$f(t) = \begin{cases} 1, & a < t < b. \\ 0, & \text{otherwise}. \end{cases}$$

Rectangle function.

$$f(t) = H(t-a) - H(t-b)$$

$$f(t) = \begin{cases} 1, & 2 < t < 3. \\ 0, & \text{otherwise}. \end{cases}$$

$$f(t) = H(t-2) - H(t-3).$$

$t = 1.5$, R.H.S $= 0 - 0 = 0$.

$t = 2.5$, R.H.S $= 1 - 0 = 1$.

$t = 3.5$ R.H.S $= 1 - 1 = 0$.

$$f(t) = \begin{cases} f_1(t), & t < a. \\ f_2(t), & a < t < b \\ f_3(t), & t > b. \end{cases}$$

$$f(t) = \{1 - H(t-a)\} f_1(t) + f_2(t)\{H(t-a) - H(t-b)\} + f_3(t) H(t-b).$$

$$= \{1 - H(t-a)\} f_1(t) + \{H(t-a) - H(t-b)\} f_2(t) + H(t-b) f_3(t).$$

$$= f_1(t) + (f_2 - f_1) H(t-a) + \cancel{H(t-b)} - (f_3 - f_2) H(t-b)$$