Dynamics of First-Order systems.

Definition : A first-order system is one whose output $y(t)$ is modeled by a first-order differential equation

✓ Linear (or linearized) system

$$a_1 \frac{dy}{dt} + a_0 y = b f(t)$$

$y \rightarrow$ output

$f \rightarrow$ Input/forcing function

$\hookrightarrow$ both deviation variables

case 1. $a_0 \neq 0$

$$\frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} f(t)$$

$$\tau_p \frac{dy}{dt} + y = K_p f(t)$$

where,

$\frac{a_1}{a_0} = \tau_p =$ time constant of the process (Unit $\equiv$ time)

$\frac{b}{a_0} = K_p =$ steady-state/static gain $\left(= \frac{\Delta y}{\Delta f} \right)_{ss}$

Taking L-transform and rearranging,

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1} \quad \text{--- TF}$$

(1st-order lag / linear lag)

case 2. $a_0 = 0$

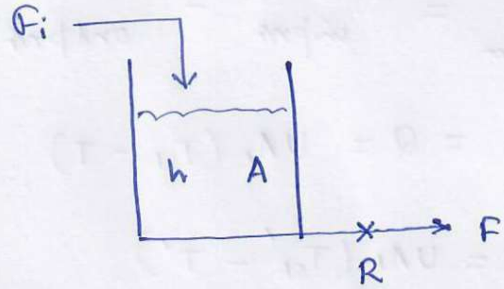
$$a_1 \frac{dy}{dt} = b\, f(t)$$

$$\frac{dy}{dt} = \frac{b}{a_0} f(t) = K_p' f(t)$$

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p'}{s} \quad \text{--- TF}$$

(purely capacitive / pure integrator)

Ex.1. First-order system (mass storage).



$$F = \frac{h}{R} = \frac{\text{driving force}}{\text{resistance}}$$

Model : $A \dfrac{dh}{dt} = F_i - F = F_i - \dfrac{h}{R}$

$$AR \frac{dh}{dt} + h = R\, F_i$$

$$\tau_p \frac{dh'}{dt} + h' = K_p F_i' \quad \text{--- in terms of deviation variables}$$

where,

$\tau_p = AR$ unit ?

$K_p = R$ unit ?

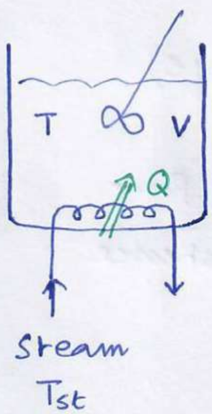
$$G(s) = \frac{\bar{h}'(s)}{\bar{F}_i'(s)} = \frac{K_p}{\tau_p s+1}.$$

time const $\tau_p = AR$ = storage capacitance × resistance to flow

$\uparrow A$ capacity to store mass is measured by A

$\uparrow R$

Ex 2. First-order system (energy storage).



✓ Energy bal.

Rate of accumulation = Rate of input − Rate of output

$$V\rho C_p \frac{dT}{dt} = Q = UA_t(T_{st} - T).$$

$$V\rho C_p \frac{dT'}{dt} = UA_t(T_{st}' - T').$$

✓ $$G(s) = \frac{\bar{T}'(s)}{\bar{T}_{st}'(s)} = \frac{UA_t}{V\rho C_p s + UA_t} = \frac{1}{\frac{V\rho C_p}{UA_t}s+1} \equiv \frac{K_p}{\tau_p s+1}$$

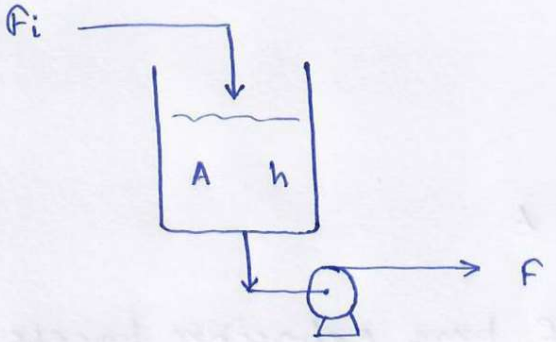
K_p = static gain = 1

$\tau_p = \frac{V\rho C_p}{UA_t}$ = (storage capacitance) × resistance to heat flow

$\uparrow V\rho C_p$ capacity to store thermal energy is measured by $V\rho C_p$.

$\uparrow \frac{1}{UA_t}$

Ex 3. Pure Integrator



Const. displacement pump

Mass bal.

$$A \frac{dh}{dt} = F_i - F$$

It yields:

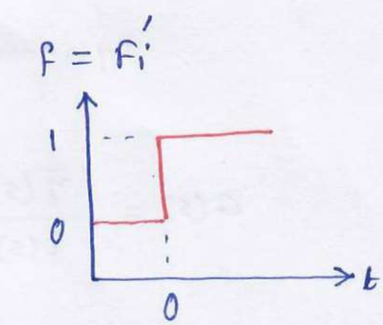
$$A \frac{dh'}{dt} = F_i' \qquad \text{Since } F = \text{Const}$$

$$G(s) = \frac{\bar{h}'(s)}{\bar{F}_i'(s)} = \frac{1}{As} = \frac{1/A}{s} = \frac{K_p'}{s} \qquad \frac{1}{s} \equiv \text{integrator}$$

✓ Dynamic response

$$\bar{h}'(s) = \frac{K_p'}{s} \cdot \bar{F}_i'(s)$$

Introduce a unit step change in the input / forcing function F_i'



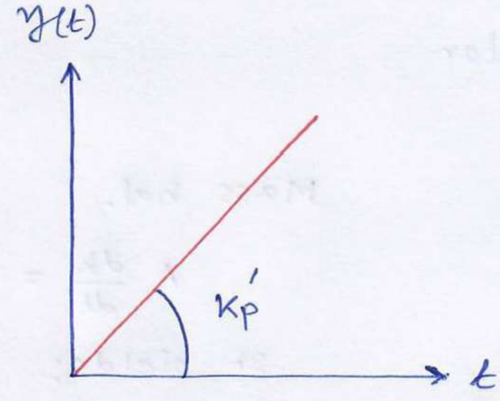
$$\bar{y} = \frac{K_p'}{s} \cdot \frac{1}{s}.$$

$$\bar{y} = \bar{h}'$$

$$\boxed{\begin{matrix} f(t) = 1 \\ \bar{f}(s) = \frac{1}{s} \end{matrix}}$$ -- time -- Laplace

Taking inverse of L-transform:

$$y(t) = K_p' \cancel{t} = K_p' t$$



Unbounded response of pure capacitive process

- It is called "pure integrator" because it behaves as if there were an integrator bet^{n} input and output.
- It is non-self-regulating system. As $F_i \uparrow$, h consistently $\uparrow$ without attaining any ss.

Dynamic response of first-order lag system.

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s+1} \quad \cdots \text{ derived before}$$

Consider a unit step change in f,

$$f(t) = 1 \quad \& \quad \bar{f}(s) = \frac{1}{s}.$$

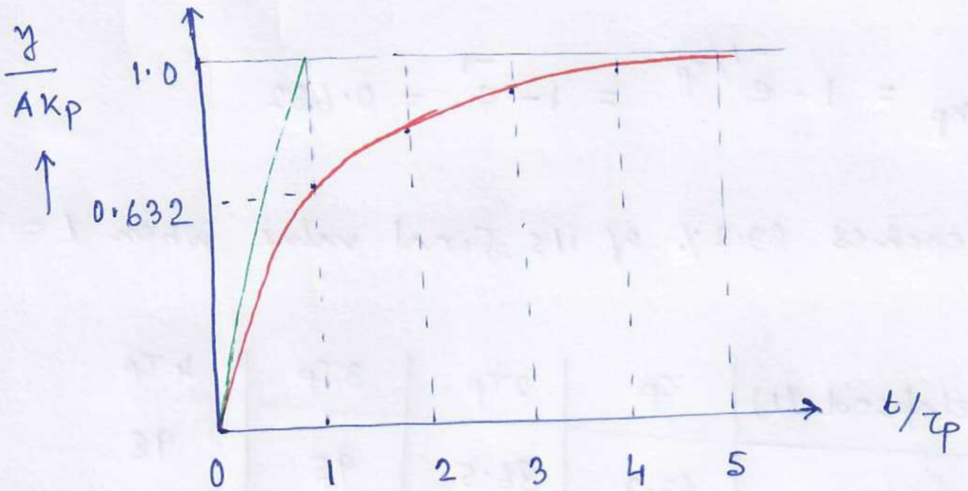
$$\bar{y}(s) = \frac{K_p}{s(\tau_p s+1)} = \frac{K_p}{s} - \frac{K_p \tau_p}{\tau_p s+1}$$

Inverting,

$$y(t) = K_p\left(1 - e^{-t/\tau_p}\right)$$

$$y(t) = K_p\left(1 - e^{-t/\tau_p}\right) \quad \cdots \quad \text{Unit step change}$$

$$y(t) = AK_p\left(1 - e^{-t/\tau_p}\right) \quad \cdots \quad \text{step change with magnitude } A$$

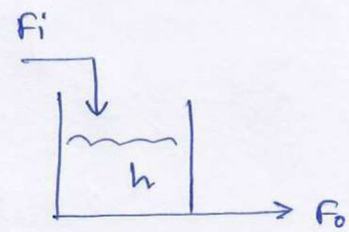


Dimensionless response y/AK_p vs. t/τ_p

Features.

1. First-order lag system is self-regulating (or) reaches a new ss.

If $F_i \uparrow$ $h \uparrow$ hydrostatic $p \uparrow$ $F_o \uparrow$



2. slope of response at $t=0$ is 1

$$\left.\frac{d[y/AK_p]}{d(t/\tau_p)}\right|_{t=0} = \left. e^{-t/\tau_p}\right|_{t=0} = 1$$

It implies that 'if the initial rate of change of $y(t)$ were to be maintained, the response would reach its final value in one time constant.'

$$t/\tau_p = 1$$

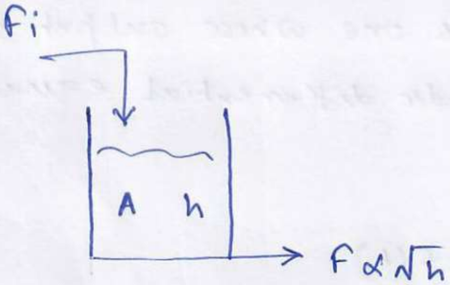
3. $$y/AK_p = 1 - e^{-t/\tau_p} = 1 - e^{-1} = 0.632$$

$y(t)$ reaches 63.2% of its final value when $t = \tau_p$.

Time elapsed (t)	τ_p	$2\tau_p$	$3\tau_p$	$4\tau_p$
% y/AK_p	63.2	86.5	95	98

4. As $t \to \infty$, $y/AK_p = 1$ = ultimate value of the response

Variable time const (τ_p) and gain (K_p)



Model : $A \frac{dh}{dt} = F_i - F$

$$A \frac{dh}{dt} + \alpha \sqrt{h} = F_i \quad \text{--- nonlinear model}$$

$$A \frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i' \quad \text{--- linearized model.}$$

$$\tau_p \frac{dh'}{dt} + h' = K_p F_i'$$

where, $\tau_p = \dfrac{2A\sqrt{h_s}}{\alpha}$, $K_p = \dfrac{2\sqrt{h_s}}{\alpha}$

value of h_s can vary by varying F_{is}. so, τ_p and K_p also vary accordingly.