① Special ODE

① $\frac{d^2y}{dx^2} + \lambda y = 0$

λ → scalar

B.C. : $y(x=0) = 0$

$y(x=1) = 1$

} Standard eigenvalue problem

$Ly + \lambda y = 0$

$\therefore Ly = -\lambda y$ → eigenvalue problem in cont. domain

→ disc. case → $A \cdot x = \lambda \cdot x$

case 1 : $\lambda = 0$

$\therefore \frac{d^2y}{dx^2} = 0$

$\Rightarrow y = 0$ (after checking B.C.)

→ trivial solution

case 2 : $\lambda < 0$

$\therefore$ let $\lambda = -\alpha^2$

Revise how to solve these from Math 2

$\therefore \dfrac{d^2y}{dx^2} = -\lambda y = \alpha^2 \cdot y$

$\therefore \quad y = c_1 \cdot e^{\alpha \cdot x} + c_2 \cdot e^{-\alpha \cdot x}$

$y(x=0) = 0 \Rightarrow c_1 = -c_2.$

$y(x=1) = 0 \Rightarrow 0 = c_1 \cdot e^{\alpha} - c_1 e^{-\alpha}$

$\Rightarrow c_1 \underbrace{(e^{\alpha} - e^{-\alpha})}_{+ve} = 0$

$\therefore c_1 = 0.$

$\therefore \quad y = 0$

& trivial solution.

Case 3 : $\lambda > 0$

let $\lambda = \alpha^2$

$\therefore \dfrac{d^2y}{dx^2} + \alpha^2 y = 0$

$\therefore y = c_1 \cdot \sin(\alpha \cdot x) + c_2 \cdot \cos(\alpha \cdot x)$

$y(x=0) = 0$

$\therefore 0 = c_1 \cdot 0 + c_2 \cdot 1$

$\Rightarrow c_2 = 0$

$y(x=1) = 0$

$\therefore \quad 0 = c_1 \cdot \sin \alpha$

assume $c_1 \neq 0$ ($\because$ we don't want another trivial soln)

$\therefore \quad \sin \alpha = 0$

$\therefore \quad \alpha_n = n\pi, \quad n \in 1, 2, 3 \ldots$

not 0 $\therefore$ no trivial soln.

$\therefore \quad y_n = c_1 \cdot \sin(n\pi x)$

(2) $\dfrac{d^2y}{dx^2} + \lambda y = 0$

scalar

B.C. : $\dfrac{dy}{dx}(x=0) = 0$

$y(x=1) = 1$

<u>case 1</u> : $\lambda = 0$

$\therefore \quad \dfrac{d^2y}{dx^2} = 0$

$\therefore \quad y = c_1 \cdot x + c_2$

$0 = c_1 + c_2$

$\therefore \quad c_2 = -c_1$

$\dfrac{dy}{dx} = c_1 = 0.$

$\therefore \quad y = 0 \rightsquigarrow$ Trivial Soln.

Case 2 : $\lambda = -\alpha^2 < 0$

$\therefore \quad y = c_1 \cdot e^{\alpha \cdot x} + c_2 \cdot e^{-\alpha x}$

$$\frac{dy}{dx} = c_1 \cdot \alpha \cdot e^{\alpha \cdot x} - c_2 \cdot \alpha \cdot e^{-\alpha \cdot x}$$

$0 = c_1 \cdot \alpha - c_2 \cdot \alpha$

$\therefore \quad c_1 = c_2$ (since $-\alpha^2 = \lambda \neq 0$)

$0 = c_1 \cdot e^{\alpha} + c_1 \cdot e^{-\alpha}$

$\therefore \quad c_1 \cdot (e^{\alpha} + e^{-\alpha}) = 0$

$\therefore \quad c_1 = 0$

$\therefore \quad y = 0$

$\rightsquigarrow$ trivial soln.

Case 3 : $\lambda = \alpha^2$

$$\frac{d^2y}{dx^2} + \alpha^2 \cdot y = 0$$

$\therefore \quad y = c_1 \cdot \sin(\alpha x) + c_2 \cdot \cos(\alpha x)$

$$\frac{dy}{dx} = \alpha \cdot c_1 \cdot \cos(\alpha x) - c_2 \cdot \alpha \sin(\alpha x)$$

$$\therefore \quad 0 = \alpha \cdot c_1 \cdot 1 - 0$$

$$\therefore \quad c_1 = 0$$

$$0 = c_2 \cdot \cos(\alpha)$$

$$\therefore \quad \alpha = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$\therefore \quad y_n = c_n \cdot \cos\left((2n-1)\frac{\pi}{2} \cdot x\right).$$

(3) $$\frac{d^2y}{dx^2} + \underset{\text{scalar}}{\lambda} y = 0$$

B.C. : $y(x=0) = 0$

β → scalar. (positive) ∵ it's a physical const.

$$\left.\frac{dy}{dx} + \beta \cdot y\right|_{x=1} = 0$$

<u>case 1</u> : $\lambda = 0$

$$\therefore \quad y = c_1 \cdot x + c_2$$

$$c_2 = 0$$

$$c_1 + \beta \cdot (c_1 \cdot 1) = 0$$

$$\therefore \quad c_1 = 0$$

$$\therefore \quad y = 0$$

trivial.

<u>case 2</u> : $\lambda = -\alpha^2$.

$$\frac{d^2y}{dx^2} = \alpha^2 \cdot y$$

$$\therefore \quad y = c_1 \cdot e^{\alpha x} + c_2 \cdot e^{-\alpha \cdot x}$$

$$0 = c_1 \cdot 1 + c_2 \cdot 1$$

$$\therefore \quad c_2 = -c_1 .$$

$$\frac{dy}{dx} = \alpha \cdot c_1 \cdot (e^{\alpha x} + e^{-\alpha \cdot x})$$

$$\therefore \quad \alpha \cdot c_1 \cdot (e^{\alpha} + e^{-\alpha}) + \beta \cdot c_1 \cdot (e^{\alpha} - e^{-\alpha}) = 0$$

$$\therefore \quad c_1 \cdot \big(\underbrace{\alpha (e^{\alpha} + e^{-\alpha}) + \beta (e^{\alpha} - e^{-\alpha})} \big) = 0$$

for +ve β, this term is $\neq 0$

$$\therefore \quad c_1 = 0$$

$$\therefore \quad y = 0$$

$\hookrightarrow$ Trivial soln

Case 2 : $\lambda = \alpha^2$.

$$\frac{d^2y}{dx^2} + \alpha^2 \cdot y = 0$$

$$y = c_1 \cdot \sin(\alpha x) + c_2 \cdot \cos(\alpha x)$$

$$0 = c_2 \cdot 1$$

$$\therefore c_2 = 0$$

$$\frac{dy}{dx} = c_1 \cdot \alpha \cdot \cos(\alpha x)$$

$$\therefore c_1 \cdot \alpha \cdot \cos(\alpha) + \beta \cdot (c_1 \cdot \sin(\alpha)) = 0$$

$$\therefore c_1 \cdot (\alpha \cdot \cos(\alpha) + \beta \cdot \sin(\alpha)) = 0.$$

$$\therefore c_1(\alpha + \beta \cdot \tan(\alpha)) = 0$$

$$\therefore \alpha_n + \beta \cdot \tan(\alpha_n) = 0.$$

Solve using NR method.

"Transcendental Equation".

Fancy term for a non-polynomial eq^n.

(2)

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

B.C : homogenious

→ we're so interested in this because it's common in the soln of P.D.Es in cartesian coordinates

"Standard E.V. Problem".

(3)

B.C.	Eigen-value	Eigen-function
Dirichlet	$n\pi$	$\sin(n\pi x)$
Neumann + Dirichlet	$(2n-1)\frac{\pi}{2}$	$\cos\left((2n-1)\frac{\pi}{2}\cdot x\right)$
Dir. + Robin	$\alpha_n + \beta\cdot\tan(\alpha_n)$	$\sin(\alpha x)$

solve for α_n