Lecture - 1
Monday
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Transform Calculus (MA20101)

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Books: 1. Integral Transform and their Applications
Debnath and Bhatta (CRC Press) – Available online
2. The Use of Integral Transforms
I. N. Sneddon - Available in library

$f(x)$ defined in $[a, b]$.

$$I[f(x); s] = \int_a^b f(x) \underline{K(x,s)}\, dx = F(s) \text{ or } \bar{f}(s)$$

$K(x,s) \to$ kernel of the transform.

$s \to$ transform variable / parameter.

$f(x) \to$ object function.

$F(s)/\bar{f}(s) \to$ (Integral) transform function.

$I \to$ Transform operator

Depending on $K(x,s)$ and the limits a & b, many integral transforms exist.

- Laplace transform.
- Fourier "
- Mellin "
- Z "
- Hankel " & so on.

Application of integral transforms

- solving differential equations.

Ordinary diff. equations.

$$\frac{d^2y}{dx^2}+y=\sin x.$$

$\rightarrow$ ODE ; $y=y(x)$

Partial differential equations.

$$z=z(x,y)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\underset{\text{(known)}}{f(x,y)}$$

- We can solve integral equat^n.

$$y(x)+\int_a^b y(t)\sin(x+t)\,dt=\cos x.$$

- We can evaluate some difficult integrals

$$T(c_1\underset{\sim}{v}_1+c_2\underset{\sim}{v}_2)=c_1T(\underset{\sim}{v}_1)+c_2T(\underset{\sim}{v}_2).$$

$$T\left(\sum_{i=1}^{2}c_i\underset{\sim}{v}_i\right)=\sum_{i=1}^{2}c_iT(\underset{\sim}{v}_i)$$

- Integral transforms are linear operators.

$$\mathcal{I}\left[\sum_{i=1}^{n}c_if_i(x);s\right]=\sum_{i=1}^{n}c_i\,\mathcal{I}\left[f_i(x);s\right].$$

$$\text{L.H.S}=\int_a^b\sum_{i=1}^{n}c_if_i(x)\,K(x,s)\,dx$$

$$=\int_a^b\left(c_1f_1(x)+c_2f_2(x)+\dots+c_nf_n(x)\right)K(x,s)\,dx.$$

$$=c_1\int_a^b f_1(x)K(x,s)\,dx+c_2\int_a^b f_2(x)K(x,s)\,dx+\dots+c_n\int_a^b f_n(x)K(x,s)\,dx.$$

$$=c_1\,\mathcal{I}[f_1(x);s]+c_2\,\mathcal{I}[f_2(x);s]+\dots+c_n\,\mathcal{I}[f_n(x);s]$$

$$= \sum_{i=1}^{n} c_i \, \mathcal{J}\left[f_i(x); s\right] = \text{R.H.S.}$$

Inverse transform

Suppose $\mathcal{J}\left[f(x); s\right] = F(s)$

Then in order to get $f(x)$ from $F(s)$, we introduce an inverse operator $\mathcal{J}^{-1}$ such that $\mathcal{J}^{-1}\{F(s)\} = f(x)$.

$\hookrightarrow$ inverse transform of $F(s)$.

If $\mathcal{J}\left[f(x); s\right] = F(s) \longrightarrow (1)$

then $\mathcal{J}^{-1}\left[F(s)\right] = f(x) \longrightarrow (2)$.

Operate $\mathcal{J}^{-1}$ on both sides of (1).

$$\mathcal{J}^{-1}\mathcal{J}\left[f(x); s\right] = \mathcal{J}^{-1}F(s) = f(x).$$

$$\Rightarrow \mathcal{J}^{-1}\mathcal{J} \equiv I \rightarrow \text{identity operator},$$

Operate $\mathcal{J}$ on both sides of (2).

$$\mathcal{J}\mathcal{J}^{-1}\left[F(s)\right] = \mathcal{J}\left[f(x)\right] = F(s).$$

$$\Rightarrow \mathcal{J}\mathcal{J}^{-1} = I \rightarrow \text{identity operator}.$$

i.e $\mathcal{J}\mathcal{J}^{-1} = \mathcal{J}^{-1}\mathcal{J} = I$.

Laplace Transform.

$f(t)$ is defined for $t \geq 0$.

$$L[f(t); s] = \bar{f}(s) = \int_0^\infty f(t) e^{-st} dt \qquad \text{Re}\, s > a$$

what is a?

$L \rightarrow$ Laplace transform operator.

$s \rightarrow$ transform variable (in general complex no.).

$$\mathcal{J}[f(t); s] = \int_a^b f(t) K(s,t) dt.$$

kernel $K(s,t) = e^{-st}$; $a = 0$, $b = \infty$.

Linearity property.

$$L\left[\sum_{i=1}^{n} c_i f_i(t); s\right] = \sum_{i=1}^{n} c_i L[f_i(t); s]$$

$c_i \rightarrow$ arbit. constants.

$$= \int_0^\infty \sum_{i=1}^{n} c_i f_i(t) e^{-st} dt = \sum_{i=1}^{n} c_i \int_0^\infty f_i(t) e^{-st} dt.$$

$$= \sum_{i=1}^{n} c_i \overline{f_i}(s) = \sum_{i=1}^{n} c_i L[f_i(t); s].$$

Existence of Laplace transform.

Note: suppose $s = 2$, $f(t) = e^{3t}$.

$$L[f(t); s] = \int_0^\infty e^{3t} e^{-2t} dt = \int_0^\infty e^{t} dt.$$

piecewise continuous function.

Definit^n 1. A funct^n $f(t)$ is said to be <u>piecewise continuous</u> on a closed interval $a \le t \le b$, if it is defined on that interval and is such that the interval can be subdivided into a finite no. of intervals, in each of which $f(t)$ is continuous & has finite right and left limits.

Definit^n 2. funct^n of exponential order 'a' (>0)

$f(t)$ is said to be of exponential order a (>0) ~~expli~~ as $t \to \infty$, if there exists a real +ve constant M, a no. a and a finite no. t_0 such that $|f(t)| < Me^{at}$ for all $t \ge t_0$.

or, $\lim_{t \to \infty} |f(t) e^{-at}| =$ finite quantity.

$$f(t) \sim O(e^{at})$$

Definit^n 3. $f(t)$ is said to be a funct^n of class A if.

1) it is piecewise continuous on every finite interval in the range $t \ge 0$.

2) it is of expondential order as $t \to \infty$.

Existence

Theorem. If $f(t)$ is a function of class A, then the Laplace transform of $f(t)$ exists for all s provided Re $s > a$ (where $f(t)$ is of exponential order a).

The above condition is sufficient but ~~nece~~ not necessary.

i.e $f(t)$ is of class A $\Rightarrow$ Laplace transform exists.

~~If you can check that $f(t)$ is not of class A, its LT will not exist.~~

Laplace transform exists ~~that~~ does not necessarily imply, that $f(t)$ is of class A.

Ex. $f(t) = t^n$; $-1 < n < 0$. | $f(t) = \frac{1}{t^{1/2}}$

$f(t)$ is not piecewise continuous in $t \geq 0$.

(Show that $f(t)$ is of exponential order. H.W.
Hint take any 'a' $\lim_{t \to \infty} t^n e^{at}$ = finite limit)

$f(t)$ is not a funct. of class A. //

$$f(t) = t^n, \quad -1 < n < 0.$$

$$L[f(t);s] = \int_0^\infty f(t)e^{-st}\,dt.$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}\,dx. \quad \alpha > 0.$$

$$= \int_0^\infty t^n e^{-st}\,dt.$$

Put $st = x$, $t = \frac{x}{s}$.

$$= \int_0^\infty \frac{x^n}{s^n} e^{-x}\frac{1}{s}\,dx.$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n\,dx = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^{(n+1)-1}\,dx.$$

$$= \frac{1}{s^{n+1}}\Gamma(n+1).$$

$n > -1$.

$\therefore n+1 > 0$.

Proof of existence theorem.

Suppose $f(t)$ is of class A.

i.e $f(t)$ is piecewise continuous in $t \geq 0$.

& $f(t)$ is of exponential order $a(>0)$ as $t \to \infty$.

$$\bar{f}(s) = L[f(t);s] = \int_0^\infty f(t)e^{-st}\,dt.$$

$$|\bar{f}(s)| = \left|\int_0^\infty f(t)e^{-st}\,dt\right| \leq \int_0^\infty |f(t)||e^{-st}|\,dt.$$

$\because f(t)$ is of exp. order 'a' as $t \to \infty$

then $\exists$, a +ve const. M, a no. t_0 : $|f(t)| < Me^{at}\ \forall\ t \geq t_0$

$\therefore \quad |\bar{f}(s)| \le \int_0^\infty |f(t)|\,|e^{-st}|\,dt.$

$< M \int_0^\infty e^{at} |e^{-st}|\,dt.$ $\quad s = s_0 + i s_1$

s_0, s_1 are real

$i = \sqrt{-1}.$

$|e^{-st}| = |e^{-(s_0+is_1)t}| = |e^{-s_0t - is_1t}|$

$= |e^{-s_0t}|\,|e^{-is_1t}| = e^{-s_0t}$ $\quad |e^{ix}| = 1$

$\therefore \quad |\bar{f}(s)| < M \int_0^\infty e^{at} \cdot e^{-s_0t}\,dt.$

$= M \int_0^\infty e^{\underline{-(s_0-a)t}}\,dt$

$= \dfrac{M}{s_0 - a}$ [Note integral exists only when $s_0 > a$. i.e $\mathrm{Re}\, s > a$]

$\int_0^\infty e^{t}\,dt$ does not exist.

$\int_0^\infty e^{-pt}\,dt$ exists if $p > 0$

$\Rightarrow \bar{f}(s)$ has always some finite value if $f(t)$ is p. c. in $t \ge 0$. and is of exp. order 'a' as $t \to \infty$.

Laplace Transform of elementary functions

1. $L[1] = \int_0^\infty e^{-st}\,dt = \left.\dfrac{e^{-st}}{s}\right|_\infty^0$

$\boxed{\bar{f}(s) = L[f(t)] = \int_0^\infty f(t)\,e^{-st}\,dt}$

$= \dfrac{1}{s}.$

Get L.T. of t^n n is a non-negative integer.

$L[t] = \int_0^\infty t\,e^{-st}\,dt$

$\int_0^\infty e^{-st}\,dt = \dfrac{1}{s}.$

Take derivative of both sides w.r.to $\underline{\underline{s}}$:

$\dfrac{d}{ds}\int_0^\infty e^{-st}\,dt = \dfrac{d}{ds}\left(\dfrac{1}{s}\right)$

or, $\int_0^\infty \dfrac{\partial}{\partial s}\left(e^{-st}\right)dt = -\dfrac{1}{s^2}$

or, $\int_0^\infty -t\,e^{-st}\,dt = -\dfrac{1}{s^2}.$

or, $L[t] = \int_0^\infty t\,e^{-st}\,dt = \dfrac{1}{s^2} = \dfrac{1!}{s^2} \longrightarrow (1).$

$L[t^2] = ?$ Diff. both sides of (1) w.r.to. s.

Obtain. $\int_0^\infty t^2 e^{-st}\,dt = \dfrac{2}{s^3} = \dfrac{2!}{s^3}. \longrightarrow (2).$

$L[t^3]$ by diff. (2). w.r.to s. $\int_0^\infty t^3 e^{-st}\,dt = \dfrac{6}{s^4} = \dfrac{3!}{s^4}$

$\therefore L[t^n] = \dfrac{n!}{s^{n+1}}$ (n is a non-negative integer)

~~where~~ $-1 < \alpha < 0$.

$L[t^\alpha] = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

2. L. T. of exponential funct. $e^{+\alpha t}$.

$L[e^{+\alpha t}] = \int_0^\infty e^{+\alpha t} e^{-st} dt$.

$= \int_0^\infty e^{-(s-\alpha)t} dt = \dfrac{1}{s-\alpha}$.

3. L. T. of $\cos pt$ or $\sin pt$.

$L[e^{ipt}] = \int_0^\infty e^{ipt} e^{-st} dt$.

$= \int_0^\infty e^{-(s-ip)t} dt = \dfrac{1}{s-ip}$.

$L[e^{ipt}] = \dfrac{s+ip}{s^2+p^2} = \dfrac{s}{s^2+p^2} + i\dfrac{p}{s^2+p^2}$.

$\downarrow$

$L[\cos pt + i \sin pt] =$ "

or, $L(\cos pt) + iL(\sin pt) = \dfrac{s}{s^2+p^2} + i\dfrac{p}{s^2+p^2}$.

$L(\cos pt) = \dfrac{s}{s^2+p^2}$, $L(\sin pt) = \dfrac{p}{s^2+p^2}$.

4. Find $L(\cosh pt)$ and $L(\sinh pt)$

$$\cosh pt = \frac{e^{pt} + e^{-pt}}{2}, \quad \sinh pt = \frac{e^{pt} - e^{-pt}}{2}.$$

$$\therefore L(\cosh pt) = \frac{1}{2} L[e^{pt}] + \frac{1}{2} L[e^{-pt}].$$

$$L(\sinh pt) = \frac{1}{2} L[e^{pt}] - \frac{1}{2} L[e^{-pt}].$$

$$L(\cosh pt) = \frac{s}{s^2 - p^2}. \quad \Bigg| \quad L(\sinh pt) = \frac{p}{s^2 - p^2}. \ //$$