Lecture - 28
31/10/17

Wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{(Hyperbolic type)}.$$

Problem.

Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}; \quad -\infty < x < \infty, \quad t > 0 \quad \to (1).$$

$u(x,t)$ = displacement from its horizontal equilibrium position

subject to the initial conditions.

$$u(x,0) = f(x), \quad -\infty < x < \infty \to (2)$$

$$u_t(x,0) = g(x), \quad -\infty < x < \infty \to (3).$$

$u(x,0) = f(x)$

• Cauchy problem of wave equation.

Solution. Apply complex F. T. w. r. to x on both sides of (1),

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{i\omega x}\,dx = \frac{1}{c^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial t^2} e^{i\omega x}\,dx.$$

$$\text{or, } (-i\omega)^2 U(\omega,t) = \frac{1}{c^2}\frac{d^2}{dt^2}U(\omega,t)$$

$$U(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\,dx$$

$$\text{So, } \frac{d^2}{dt^2}U(\omega,t) + c^2\omega^2 U(\omega,t) = 0.$$

$$U(\omega,t) = A_1 \cos c\omega t + A_2 \sin c\omega t.$$

Now, apply Fourier transform on (2) & (3),

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)\, e^{i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,dx = F(\omega), \text{ say}.$$

$$\text{or, } U(\omega,0) = F(\omega).$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,0)\, e^{i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\, e^{i\omega x}\, dx.$$

or. $\frac{d}{dt} U(\omega, 0) = G(\omega)$, say.

$$\begin{cases} U(\omega,t) = A_1(\omega) \cos c\omega t + A_2(\omega) \sin c\omega t \\ U(\omega,0) = F(\omega), \quad \frac{d}{dt} U(\omega,0) = G(\omega) \end{cases}$$

$$A_1 = F(\omega), \quad A_2 = \frac{G(\omega)}{c\omega}.$$

$$\therefore U(\omega,t) = F(\omega) \cos c\omega t + \frac{G(\omega)}{c\omega} \sin c\omega t.$$

$$u(x,t) = \mathcal{F}^{-1}\left[U(\omega,t)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\omega,t)\, e^{-i\omega x}\, d\omega.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left\{ F(\omega) \cos c\omega t + \frac{G(\omega)}{c\omega} \sin c\omega t \right\} e^{-i\omega x}\, d\omega.$$

$$(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) \cos c\omega t\, e^{-i\omega x}\, d\omega + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{G(\omega)}{c\omega} \sin c\omega t\, e^{-i\omega x}\, d\omega. \longrightarrow (*)$$

$$\cancel{\left(\frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)\, dx \right)}.$$

Now,
$$f(x) = \mathcal{F}^{-1}\left[F(\omega)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega .$$

$$f(x-ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega\cdot(x-ct)}\, d\omega .$$

$$f(x+ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega(x+ct)}\, d\omega .$$

$$\therefore\ f(x-ct) + f(x+ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\left(e^{i\omega ct} + e^{-i\omega ct}\right) d\omega .$$

$$\therefore\ \frac{f(x-ct) + f(x+ct)}{2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \cos\omega ct\, d\omega$$
$\longrightarrow (\#1)$ //

Also,
$$g(x) = \mathcal{F}^{-1}\left[G(\omega)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega x}\, d\omega .$$

$$\int_{x-ct}^{x+ct} g(u)\, du = \int_{x-ct}^{x+ct} du \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega u}\, d\omega\right).$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\, d\omega \int_{x-ct}^{x+ct} e^{-i\omega u}\, du .$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)\, d\omega \times \left[\frac{e^{-i\omega u}}{i\omega}\right]_{x+ct}^{x-ct} .$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{G(\omega)\, d\omega}{\omega} \times \frac{e^{-i\omega(x-ct)} - e^{-i\omega(x+ct)}}{i}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{G(\omega)}{\omega} e^{-i\omega x} \times 2\sin\omega ct\, d\omega .$$

$$\therefore \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G(\omega)}{c\omega}\,e^{-i\omega x}\sin c\omega t\,d\omega, \quad \longrightarrow (\#2)$$

By virtue of (#1) & (#2), eqn. (#) becomes

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.$$

→ D'Alembert's solution for the Cauchy problem of wave equation.

Ex 2. Solve: $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$; $0<x<l$, $t>0$. $\longrightarrow (1)$

$u(0,t) = u(l,t) = 0$.

$u(x,0) = x\sin\frac{\pi x}{l}$, $u_t(x,0) = 0$.

Apply Laplace transform on both sides of (1) w.r.to t.

$$\int_0^{\infty}\frac{\partial^2 u}{\partial t^2}e^{-st}\,dt = c^2\int_0^{\infty}\frac{\partial^2 u}{\partial x^2}e^{-st}\,dt.$$

or, $s^2\bar{u}(x,s) - s\,u(x,0) - u_t(x,0) = c^2\frac{d^2}{dx^2}\bar{u}(x,s)$.

or, $s^2\bar{u}(x,s) - s\,x\sin\frac{\pi x}{l} = c^2\frac{d^2}{dx^2}\bar{u}(x,s)$

$$\frac{d^2}{dx^2}\bar{u}(x,s) - \frac{s^2}{c^2}\bar{u}(x,s) = -\frac{s x}{c^2}\sin\frac{\pi x}{l}.$$

$$\therefore \left(D^2 - \frac{s^2}{c^2}\right)\bar{u}(x,s) = -\frac{s\lambda}{c^2}\sin\frac{\pi x}{l} \; ; \; D \equiv \frac{d}{dx}$$

$$C.F = A_1 e^{\frac{s}{c}x} + A_2 e^{-\frac{s}{c}x}.$$

$$P.I = \frac{1}{D^2 - \frac{s^2}{c^2}}\left(-\frac{s\lambda}{c^2}\sin\frac{\pi x}{l}\right) \qquad \Bigg| \qquad \frac{1}{F(D^2)}\sin ax = \frac{1}{F(-a^2)}\sin ax$$

$$= \frac{1}{-\frac{\pi^2}{l^2} - \frac{s^2}{c^2}} \times -\frac{s\lambda}{c^2}\sin\frac{\pi x}{l}.$$

$$= \frac{s\lambda l^2}{c^2\pi^2 + s^2 l^2}\sin\frac{\pi x}{l}.$$

$$\therefore \bar{u}(x,s) = C.F + P.I$$

$$= A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{s\lambda l^2 \sin}{c^2\pi^2 + s^2 l^2}\frac{\pi x}{l}.$$

$u(0,t) = 0, \quad u(l,t) = 0.$

Apply LT on the above conditions to get -

$\bar{u}(0,s) = 0, \; \bar{u}(l,s) = 0.$

$$\left.\begin{aligned} A_1 + A_2 &= 0. \\ A_1 e^{\frac{sl}{c}} + A_2 e^{-\frac{sl}{c}} &= 0 \end{aligned}\right\} \quad PX = 0. \quad X = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$|P| = e^{-\frac{sl}{c}} - e^{\frac{sl}{c}} \neq 0 \text{ identically}. \quad P = \begin{pmatrix} 1 & 1 \\ e^{\frac{sl}{c}} & e^{-\frac{sl}{c}} \end{pmatrix}$$

$\therefore A_1 = 0 = A_2$

$$\therefore \bar{u}(x,s) = \frac{s\lambda l^2}{c^2\pi^2 + s^2 l^2} \sin\frac{\pi x}{l}.$$

$$u(x,t) = L^{-1}_{(t)}\left[\frac{s\lambda l^2}{c^2\pi^2+s^2 l^2} \sin\frac{\pi x}{l}\right]$$

$$= \lambda l^2 \sin\frac{\pi x}{l} \, L^{-1}\left(\frac{s}{c^2\pi^2+s^2 l^2}\right)$$

$$= \cancel{\phi} \quad \lambda \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c}{l}t\right)$$

$$\frac{s}{s^2+a^2}$$