Linearization of systems.

Single variable

System: $\frac{dx}{dt} = f(x)$ — nonlinear function ... (1)

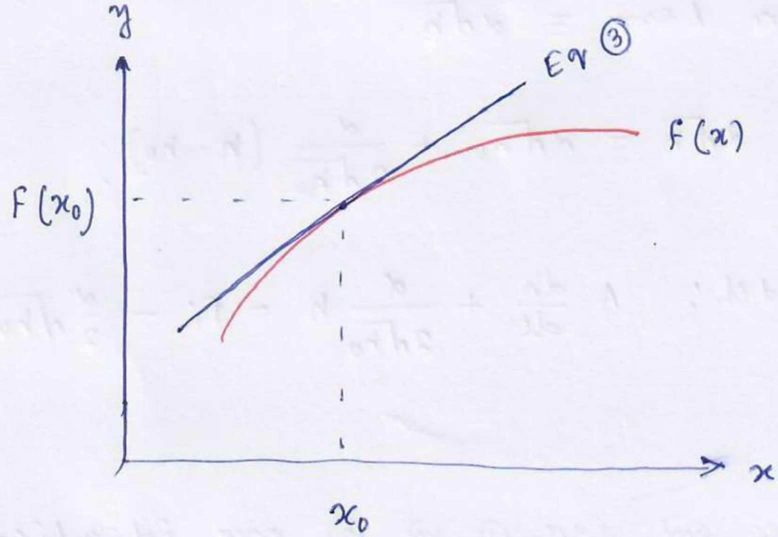
Taylor series around x_0:

$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} \frac{x-x_0}{1!} + \left(\frac{d^2f}{dx^2}\right)_{x=x_0} \frac{(x-x_0)^2}{2!} + \cdots + \cdots$$

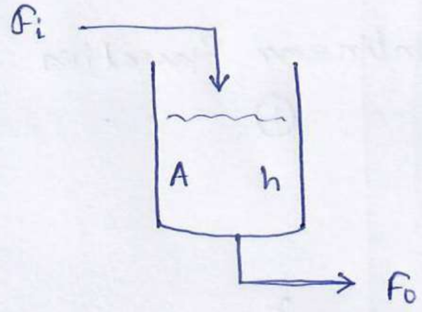
$$\cdots + \left(\frac{d^nf}{dx^n}\right)_{x=x_0} \frac{(x-x_0)^n}{n!} + \cdots + \cdots \quad (2)$$

Neglecting the 2nd- and higher-order terms:

$$f(x) \approx f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x-x_0) \quad (3)$$



Ex. single variable   Liquid tank system



Mass bal. $A \dfrac{dh}{dt} = F_i - F_o$

case 1. $F_o \propto h$

$F_o = \beta h$ (linear)

✓ Linear model : $A \dfrac{dh}{dt} + \beta h = F_i$

case 2. $F_o \propto \sqrt{h}$

$F_o = \alpha \sqrt{h}$ (nonlinear)

✓ Nonlinear model : $A \dfrac{dh}{dt} + \alpha \sqrt{h} = F_i$ ... ①

nonlinear term $\equiv \alpha \sqrt{h}$.

Taylor series: $\alpha \sqrt{h} = \alpha \sqrt{h_0} + \dfrac{\alpha}{2\sqrt{h_0}} (h - h_0)$.

✓ Linearized model: $A \dfrac{dh}{dt} + \dfrac{\alpha}{2\sqrt{h_0}} h = F_i - \dfrac{\alpha}{2}\sqrt{h_0}$ ... ②

Remarks:

1. At ss, $h = h_0$ and eqs. ① and ② are identical.
2. As $(h_0 - h)$ increases, the linearized approximation becomes progressively less accurate.

Deviation variables

system : $\frac{dx}{dt} = f(x)$

✓ At SS : $\frac{dx_s}{dt} = 0 = f(x_s)$ --- x_s = SS value of x ---- ①

✓ Linearized model : $\frac{dx}{dt} = f(x_s) + \left(\frac{df}{dx}\right)_{x_s} (x - x_s)$ --- ②

✓ ② − ① $\frac{d(x-x_s)}{dt} = \left(\frac{df}{dx}\right)_{x_s} (x - x_s)$ --- ③

✓ Deviation variable : $x' = x - x_s$ ≡ instantaneous − SS

✓ Eq ③ yields : $\boxed{\frac{dx'}{dt} = \left(\frac{df}{dx}\right)_{x_s} x'}$

Ex. Liquid tank system (revisited)

Linearized model : $A\frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_s}} h = F_i - \frac{\alpha}{2}\sqrt{h_s}$ -- ④

At SS : $A\frac{dh_s}{dt} + \frac{\alpha}{2\sqrt{h_s}} h_s = F_i - \frac{\alpha}{2}\sqrt{h_s}$. -- ⑤

④ − ⑤ $\boxed{A\frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i'}$

where : $h' = h - h_s$

$F_i' = F_i - F_{is}$

Linearization of systems (contd..)

Multivariable systems.

System: $$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2).$$

✓ Taylor series

$$f_1(x_1, x_2) = f_1(x_{1s}, x_{2s}) + \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s}) + \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{1s}, x_{2s})} (x_2 - x_{2s}) +$$

$$\left(\frac{\partial^2 f_1}{\partial x_1^2}\right)_{(x_{1s}, x_{2s})} \frac{(x_1 - x_{1s})^2}{2!} + \left(\frac{\partial^2 f_1}{\partial x_2^2}\right)_{(x_{1s}, x_{2s})} \frac{(x_2 - x_{2s})^2}{2!} +$$

$$\left(\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s})(x_2 - x_{2s})$$

✓ similar expression we can have for $f_2(x_1, x_2)$.

Neglecting 2nd- and higher-ord terms:

$$\frac{dx_1}{dt} = f_1(x_1, x_2) = f_1(x_{1s}, x_{2s}) + \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s}) + \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{1s}, x_{2s})} (x_2 - x_{2s}) \quad \text{---(1)}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) = f_2(x_{1s}, x_{2s}) + \left(\frac{\partial f_2}{\partial x_1}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s}) + \left(\frac{\partial f_2}{\partial x_2}\right)_{(x_{1s}, x_{2s})} (x_2 - x_{2s}) \quad \text{---(2)}$$

At steady state: $$\frac{dx_{1s}}{dt} = f_1(x_{1s}, x_{2s}) \quad \text{--} (3)$$

$$\frac{dx_{2s}}{dt} = f_2(x_{1s}, x_{2s}). \quad \text{--} (4)$$

✓ (1) − (3) + (2) − (4) :

$$\frac{d(x_1 - x_{1s})}{dt} = \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s}) + \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{1s}, x_{2s})} (x_2 - x_{2s})$$

$$\frac{d(x_2 - x_{2s})}{dt} = \left(\frac{\partial f_2}{\partial x_1}\right)_{(x_{1s}, x_{2s})} (x_1 - x_{1s}) + \left(\frac{\partial f_2}{\partial x_2}\right)_{(x_{1s}, x_{2s})} (x_2 - x_{2s})$$

✓ In tems of deviation variables:

$$\frac{dx_1'}{dt} = a_{11} x_1' + a_{12} x_2' \qquad x_1' = x_1 - x_{1s}$$

$$\frac{dx_2'}{dt} = a_{21} x_1' + a_{22} x_2' \qquad x_2' = x_2 - x_{2s}$$

$$a_{11} = \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{1s}, x_{2s})}$$

✓ It gives:

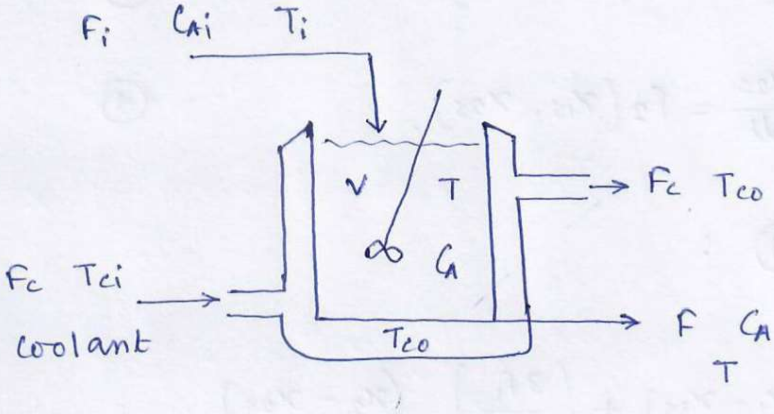
$$a_{21} = \left(\frac{\partial f_2}{\partial x_1}\right)_{(x_{1s}, x_{2s})}$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

$$a_{12} = \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{1s}, x_{2s})}$$

$$a_{22} = \left(\frac{\partial f_2}{\partial x_2}\right)_{(x_{1s}, x_{2s})}$$

Linearization of a CSTR (multivariable system)



Exo. reaction

$A \longrightarrow B$.

✓ Model.

$$\begin{cases} \dfrac{dV}{dt} = F_i - F & \text{-- Total} \\ \dfrac{dC_A}{dt} = \dfrac{F_i}{V}\left(C_{Ai} - C_A\right) - K_0 C_A e^{-E/RT} & \text{-- Comp A} \\ \dfrac{dT}{dt} = \dfrac{F_i}{V}\left(T_i - T\right) - \dfrac{Q}{V\rho C_p} + \dfrac{(-\Delta H) K_0 C_A e^{-E/RT}}{\rho C_p} & \text{-- Energy} \end{cases}$$

✓ If vol. (V) is assumed const (i.e., $F_i = F$)

step 1. Dynamic model gets the following form

$$\frac{dC_A}{dt} = \frac{1}{\tau}\left(C_{Ai} - C_A\right) - K_0 C_A e^{-E/RT}$$

$$\frac{dT}{dt} = \frac{1}{\tau}\left(T_i - T\right) - \frac{Q}{V\rho C_p} + S K_0 C_A e^{-E/RT}$$

where $\tau = V/F_i$ (residence time), $S = (-\Delta H)/\rho C_p$

step 2. Nonlinear term: $e^{-E/RT} \cdot C_A$. Applying Taylor series:

$$e^{-E/RT} \cdot C_A = e^{-E/RT_0} \cdot C_{A0} + \frac{E}{RT_0^2} e^{-E/RT_0} \cdot C_{A0}\left(T - T_0\right) + e^{-E/RT_0} \cdot \left(C_A - C_{A0}\right)$$

Step 3. Linearized model.

$$\frac{dC_A}{dt} = \frac{1}{\tau}\left(C_{Ai} - C_A\right) - K_0\left[e^{-E/RT_0} \cdot C_{A0} + \frac{E}{RT_0^2} e^{-E/RT_0} \cdot C_{A0}(T-T_0) + e^{-E/RT_0}(C_A - C_{A0})\right] \quad \text{--- (1)}$$

$$\frac{dT}{dt} = \frac{1}{\tau}\left(T_i - T\right) - \frac{Q}{V\rho C_p} + SK_0\left[\qquad \text{same} \qquad\right] \quad \text{--- (2)}$$

Step 4. Steady state model

$$\frac{dC_{A0}}{dt} = 0 = \frac{1}{\tau}\left(C_{Ai0} - C_{A0}\right) - K_0 e^{-E/RT_0} \cdot C_{A0} \quad \text{--- (3)}$$

$$\frac{dT_0}{dt} = 0 = \frac{1}{\tau}\left(T_{i0} - T_0\right) - \frac{Q_0}{V\rho C_p} + SK_0 e^{-E/RT_0} \cdot C_{A0} \quad \text{--- (4)}$$

Step 5. Model (linearized) in terms of deviation variables

Linearized model (1) − SS model (3) + (2) − (4)

$$\frac{dC_A'}{dt} = \frac{1}{\tau}\left(C_{Ai}' - C_A'\right) - \frac{K_0 E}{RT_0^2} e^{-E/RT_0} C_{A0} T' - K_0 e^{-E/RT_0} C_A'$$

$$\frac{dT'}{dt} = \frac{1}{\tau}\left(T_i' - T'\right) - \frac{Q'}{V\rho C_p} + SK_0\left[\frac{E}{RT_0^2} e^{-E/RT_0} \cdot C_{A0} T' + e^{-E/RT_0} \cdot C_A'\right]$$

where,

$C_A' = C_A - C_{A0}$ $\qquad$ $C_{Ai}' = C_{Ai} - C_{Ais}$ $\qquad$ $T' = T - T_0$

$T_i' = T_i - T_{i0}$ $\qquad$ $Q' = Q - Q_0$.

Transfer Functions.

$$\underset{\text{Input}}{f(t)} \longrightarrow \boxed{\text{Process}} \longrightarrow \underset{\text{Output}}{y(t)}$$

SISO process

$$\bar{f}(s) \longrightarrow \boxed{G(s)} \longrightarrow \bar{y}(s)$$

Block dig

$$TF = G(s) = \frac{\bar{y}(s)}{\bar{f}(s)}$$

SISO System

nth-order linear (or linearized) differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b\, f(t) \quad \cdots (1)$$

Both $f(t)$ and $y(t)$ are deviation variables, and so

$$y(0) = \left(\frac{dy}{dt}\right)_{t=0} = \left(\frac{d^2 y}{dt^2}\right)_{t=0} = \cdots = \left(\frac{d^{n-1} y}{dt^{n-1}}\right)_{t=0} = 0$$

$$L\left[\frac{d^n y(t)}{dt^n}\right] = s^n \bar{y}(s) \underbrace{- s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{n-1}(0)}_{=0} = s^n \bar{y}(s)$$

So Eq (1) yields:

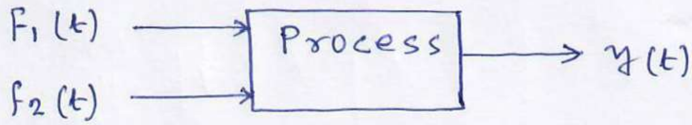
$$a_n s^n \bar{y}(s) + a_{n-1} s^{n-1} \bar{y}(s) + \cdots + a_1 s \bar{y}(s) + a_0 \bar{y}(s) = b \bar{f}(s)$$

Rearranging,

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{b}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = G(s)$$

$G(s)$ = tranfer function (TF).

MISO system

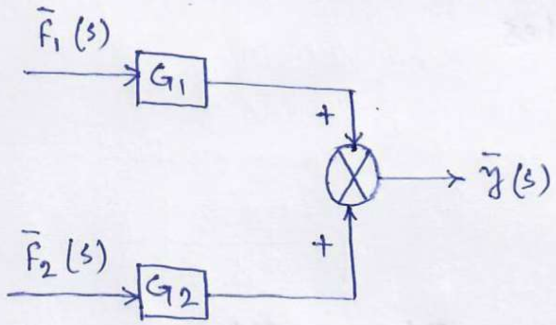


$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_1 f_1(t) + b_2 f_2(t)$$

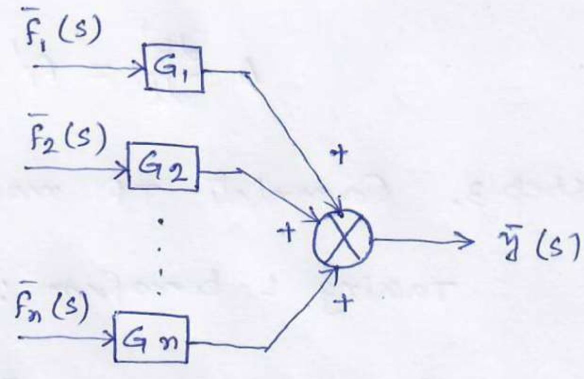
In Laplace domain:

$$\bar{y}(s) = \frac{b_1}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \bar{f}_1(s) + \frac{b_2}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \bar{f}_2(s)$$

$$= G_1(s)\,\bar{f}_1(s) + G_2(s)\,\bar{f}_2(s).$$



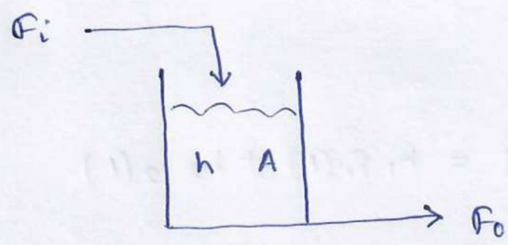
Block dig.



Block dig

Transfer function

Ex. Liquid tank system



$F \rightarrow$ volumetric flow rate

$h \rightarrow$ liq. height

Step 1. Develop the model

$$A \frac{dh}{dt} = F_i - F_o \quad \text{... linear}$$

Step 2. Model in deviation variables

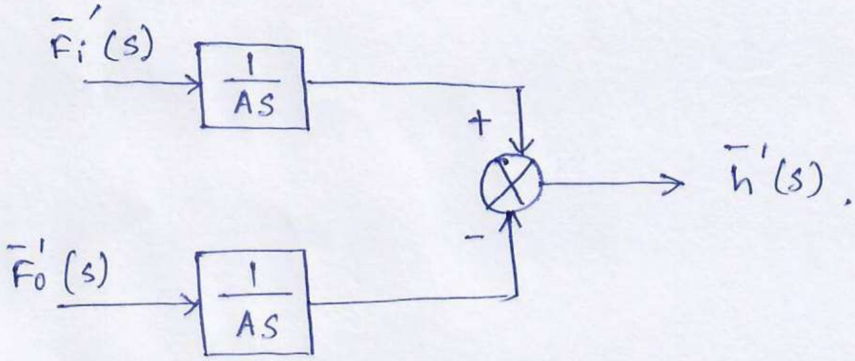
$$A \frac{dh'}{dt} = F_i' - F_o'$$

Step 3. Formulate TF model.

Taking L-transform : $A s \bar{h}'(s) = \bar{F}_i'(s) - \bar{F}_o'(s)$.

Rearranging : $\bar{h}'(s) = \frac{1}{As} \bar{F}_i'(s) - \frac{1}{As} \bar{F}_o'(s)$.

Step 4. Developing block diagram



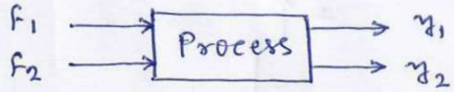
$F_i \rightarrow$ Input (+ve)

$F_o \rightarrow$ Output (−ve)

Transfer functions (contd--)

MIMO system

2x2 system



$$\left.\begin{aligned}\frac{dy_1}{dt} &= a_{11} y_1 + a_{12} y_2 + b_{11} f_1 + b_{12} f_2 \\ \frac{dy_2}{dt} &= a_{21} y_1 + a_{22} y_2 + b_{21} f_1 + b_{22} f_2\end{aligned}\right\} \text{Model}$$

$$\Rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Initial conditions: $y_1(0) = y_2(0) = 0$

Taking L-transform:

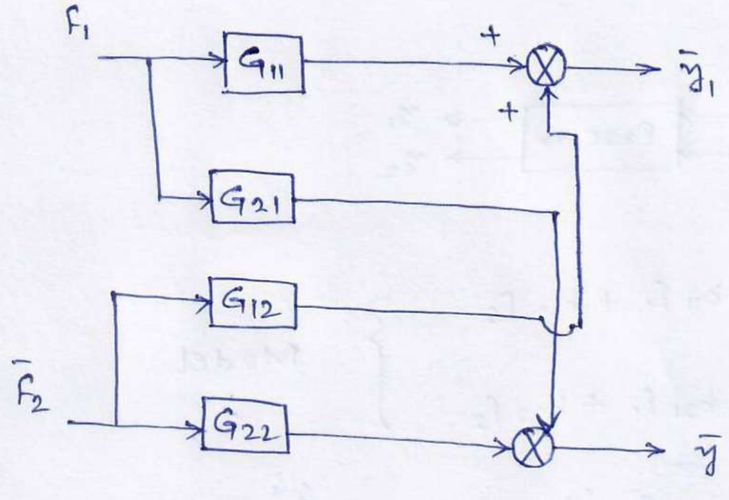
$$\bar{y}_1(s) = \overbrace{\frac{(s-a_{22})\, b_{11} + a_{12} b_{21}}{p(s)}}^{G_{11}} \bar{f}_1(s) + \overbrace{\frac{(s-a_{22})\, b_{12} + a_{12} b_{22}}{p(s)}}^{G_{12}} \bar{f}_2(s)$$

$$\bar{y}_2(s) = \underbrace{\frac{(s-a_{11})\, b_{21} + a_{21} b_{11}}{p(s)}}_{G_{21}} \bar{f}_1(s) + \underbrace{\frac{(s-a_{11})\, b_{22} + a_{21} b_{12}}{p(s)}}_{G_{22}} \bar{f}_2(s)$$

where $p(s) = s^2 - (a_{11} + a_{22})\, s - (a_{12} a_{21} - a_{11} a_{22})$.

It gives:

$$\begin{bmatrix} \bar{y}_1(s) \\ \bar{y}_2(s) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \end{bmatrix}$$



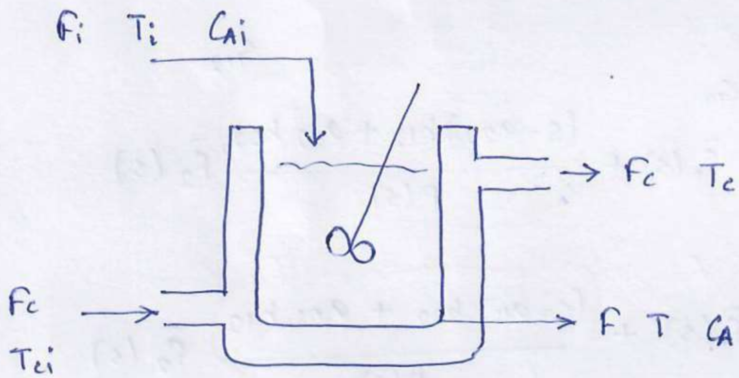
Block dig

G_{ij}

$i \to$ Output

$j \to$ Input.

Ex. Jacketed CSTR (Revisited).



- Exo. reaction $A \to B$
- $V =$ const.

Linearized model (developed)

$$\frac{dC_A'}{dt} + \left[\frac{1}{\tau} + K_0 e^{-E/RT_0}\right] \underset{\text{output}}{C_A'} + \left[\frac{K_0 E}{RT_0^2} e^{-E/RT_0} C_{A0}\right] \underset{\text{output}}{T'} = \frac{1}{\tau} \underset{\text{input}}{C_{Ai}'}$$

$$\frac{dT'}{dt} + \left[\frac{1}{\tau} - \frac{S K_0 E}{RT_0^2} e^{-E/RT_0} C_{A0} + \frac{UA_t}{V\rho C_p}\right] T' - \left[S K_0 e^{-E/RT_0}\right] C_A' = \frac{1}{\tau} \underset{\text{input}}{T_i'} + \frac{UA_t}{V\rho C_p} \underset{\text{input}}{T_c'}$$

where: $Q = UA_t(T - T_c)$.

✓ Considering :

$$a_{11} = \frac{1}{\tau} + K_0 e^{-E/RT_0} \qquad a_{12} = \frac{K_0 E}{R T_0^2} e^{-E/RT_0} \cdot C_{A0}$$

$$a_{21} = -S K_0 e^{-E/RT_0} \qquad a_{22} = \frac{1}{\tau} - \frac{S K_0 E}{R T_0^2} e^{-E/RT_0} \cdot C_{A0} + \frac{UA_t}{V\rho C_p}.$$

$$b_1 = \frac{1}{\tau} \qquad b_2 = \frac{UA_t}{V\rho C_p}.$$

✓ CSTR model yields: 2×3 system (nonsquare).

$$\frac{dC_A'}{dt} + a_{11} C_A' + a_{12} T' = b_1 C_{Ai}'$$

$$\frac{dT'}{dt} + a_{21} C_A' + a_{22} T' = b_1 T_i' + b_2 T_c'$$

Initial conditions: $C_A'(0) = T'(0) = 0.$

✓ Taking L-transform and rearranging,

$$\bar{C}_A'(s) = \frac{b_1(s + a_{22})}{P(s)} \bar{C}_{Ai}'(s) - \frac{a_{12} b_1}{P(s)} \bar{T}_i'(s) - \frac{a_{12} b_2}{P(s)} \bar{T}_c'(s)$$

$$\bar{T}'(s) = -\frac{a_{21} b_1}{P(s)} \bar{C}_{Ai}'(s) + \frac{b_1(s + a_{11})}{P(s)} \bar{T}_i'(s) + \frac{b_2(s + a_{11})}{P(s)} \bar{T}_c'(s)$$

where : $P(s) = s^2 + (a_{11} + a_{22}) s + (a_{11} a_{22} - a_{12} a_{21})$

✓ It yields:

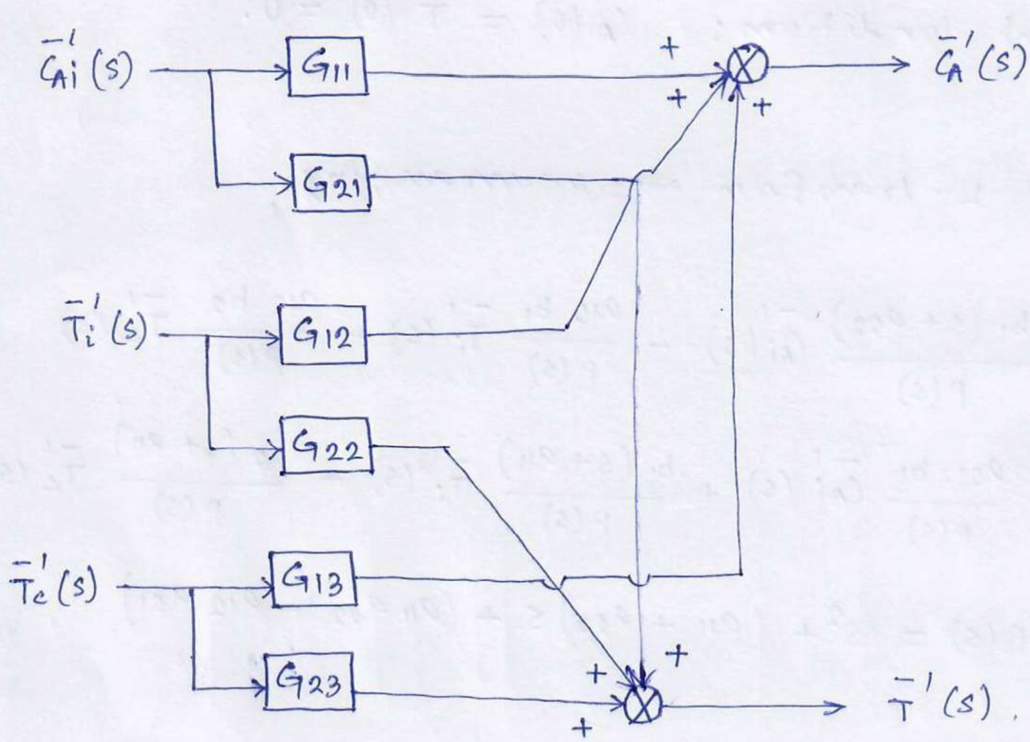
$$\bar{C}_A'(s) = G_{11}\,\bar{C}_{Ai}'(s) + G_{12}\,\bar{T}_i'(s) + G_{13}\,\bar{T}_c'(s)$$

$$\bar{T}'(s) = G_{21}\,\bar{C}_{Ai}'(s) + G_{22}\,\bar{T}_i'(s) + G_{23}\,\bar{T}_c'(s).$$

✓ In matrix form:

$$\begin{bmatrix} \bar{C}_A'(s) \\ \bar{T}'(s) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \end{bmatrix} \begin{bmatrix} \bar{C}_{Ai}'(s) \\ \bar{T}_i'(s) \\ \bar{T}_c'(s) \end{bmatrix}$$

✓ Block diagram



Poles and zeros of TF

$$\bar{F}(s) \longrightarrow \boxed{G(s)} \longrightarrow \bar{y}(s) \qquad G(s) = \frac{\bar{y}(s)}{\bar{F}(s)} = \frac{Q(s)}{P(s)}$$

$\equiv$ ratio of two polynomials

zeros

The roots of the polynomial $Q(s)$ are called "zeros of the TF or system". $Q(s) = 0$.

Poles

The roots of the polynomial $P(s)$ are called "poles of the TF or system". $P(s) = 0$.

Ex1.

$$G(s) = \frac{K}{s+a} = \frac{Q(s)}{P(s)}.$$

It has no zeros and one pole at $s = -a$.

Ex2.

$$G(s) = \frac{s-1}{s^2-3s+2}$$

zeros : 1

poles : 1, 2

At the zeros of a system, TF becomes 0.

At the poles - - - - , TF becomes ∞.

General form of TF

✓ An nth-order system can be described by a linear ODE

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m f}{dt^m} + \cdots + b_1 \frac{df}{dt} + b_0 f$$

✓ $$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \left(\frac{b_m}{a_n}\right) \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

z_i = zeros + p_i = poles.

✓

$n > m$	strictly proper	physically realizable
$n = m$	semiproper	physically realizable
$n < m$	Improper	

Ex.

$n > m$ $\quad G(s) = \dfrac{K}{\tau s + 1}$ lag system

$n = m$ $\quad G(s) = \dfrac{\tau_1 s + 1}{\tau_2 s + 1}$ lead-lag system

$n < m$ $\quad G(s) = \dfrac{\tau_1 s + 1}{\tau_2}$ lead system

Remarks

1. For a physically realizable system $n \geq m$.
2. A system is stable if all poles lie in the left half of s plane.
3. Locations of the zeros have no effect on the stability of the system. They certainly affect the dynamic response, not stability.

Response of a system: Qualitative analysis

✓ Dynamic response of y

$$\bar{y}(s) = G(s)\,\bar{f}(s) \quad \cdots (1)$$

where,

$$G(s) = \frac{Q(s)}{P(s)} \qquad \text{e.g.} \quad G(s) = \frac{\tau_1 s + 1}{\tau_2 s + 1}$$

$$\bar{f}(s) = \frac{r(s)}{q(s)} \qquad \text{e.g.} \quad L[At] = \frac{A}{s^2}$$

So,

$$\bar{y}(s) = \frac{Q(s)}{P(s)} \cdot \frac{r(s)}{q(s)}$$

✓ Let us use G(s) for qualitative stability analysis with the following general form:

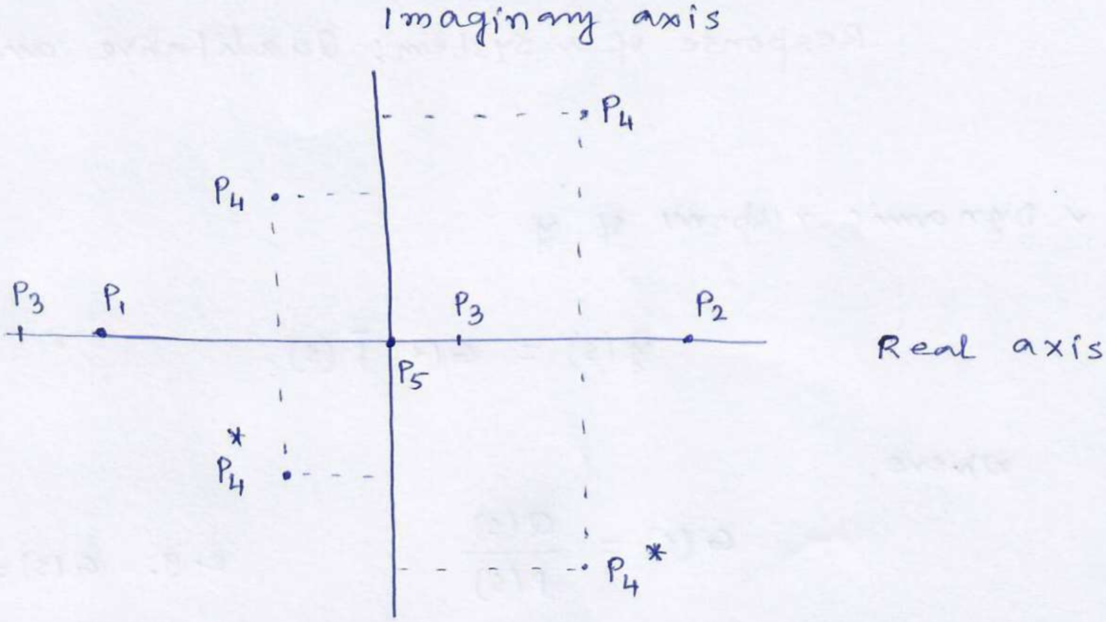
$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-P_1)(s-P_2)(s-P_3)^m(s-P_4)(s-P_4^*)(s-P_5)} \quad \cdots (2)$$

$P \rightarrow$ roots of $P(s)$ [i.e., poles of the system]

✓ Partial fractions expansion

$$G(s) = \frac{c_1}{s-P_1} + \frac{c_2}{s-P_2} + \left\{ \frac{c_{31}}{s-P_3} + \frac{c_{32}}{(s-P_3)^2} + \cdots + \frac{c_{3m}}{(s-P_3)^m} \right\} +$$

$$\frac{c_4}{s-P_4} + \frac{c_4^*}{s-P_4^*} + \frac{c_5}{s-P_5}$$



Location of poles in the complex plane / s plane ($s \equiv a + jb$).

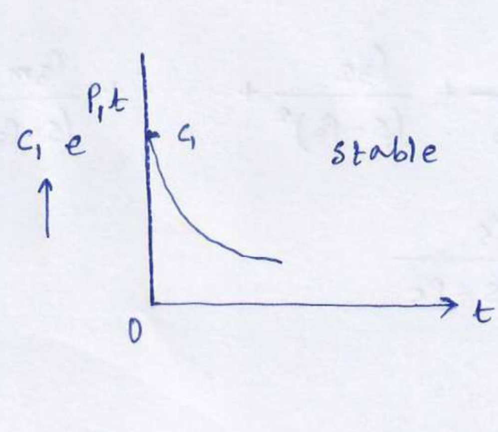
1. Real distinct poles.

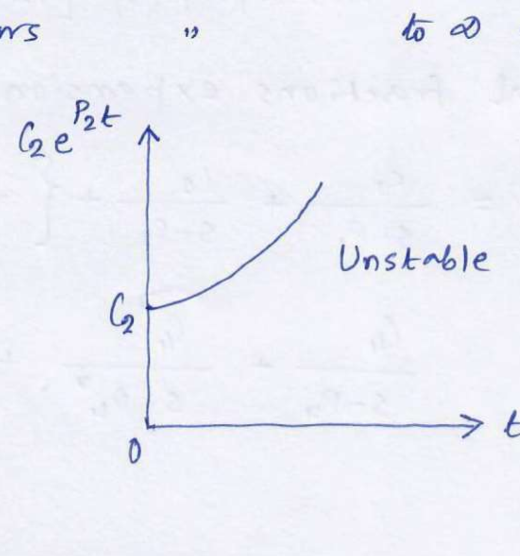
$$G(s) = \frac{c_1}{s - P_1} + \frac{c_2}{s - P_2} \quad \text{--- s domain}$$

$$L^{-1}\left[G(s)\right] = c_1 e^{P_1 t} + c_2 e^{P_2 t} \quad \text{--- L domain}$$

Since $P_1 < 0$, $c_1 e^{P_1 t}$ decays exponentially to 0 as $t \to \infty$

Since $P_2 > 0$, $c_2 e^{P_2 t}$ grows " to ∞ as $t \to \infty$.





two poles are -ve $\Longrightarrow$ stable

two poles are +ve $\Longrightarrow$ Unstable

One +ve one -ve $\Longrightarrow$ Unstable.

2. Multiple real poles.

$$G(s) = \frac{C_{31}}{s-P_3} + \frac{C_{32}}{(s-P_3)^2} + \cdots + \frac{C_{3m}}{(s-P_3)^m}$$

$$L^{-1}[G(s)] = \underbrace{\left[C_{31} + \frac{C_{32}}{1!}t + \frac{C_{33}}{2!}t^2 + \cdots + \frac{C_{3m}}{(m-1)!}t^{m-1}\right]}_{\text{grows toward } \infty \text{ with time}} \underbrace{e^{P_3 t}}_{\text{depends on } P_3}$$

If $P_3 > 0$, $e^{P_3 t} \to \infty$ as $t \to \infty$ $\quad \therefore L^{-1}[G(s)] \to \infty$ Unstable

$P_3 < 0$, $e^{P_3 t} \to 0$ as $t \to \infty$ $\quad L^{-1}[G(s)] \to 0$ stable

$P_3 = 0$, $e^{P_3 t} = 1$ for all times $\quad L^{-1}[G(s)] \to \infty$ Unstable.

3. Complex conjugate poles.

$$G(s) = \frac{C_4}{s-P_4} + \frac{C_4^*}{s-P_4^*}$$

Let

$P_4 = \alpha + j\beta$

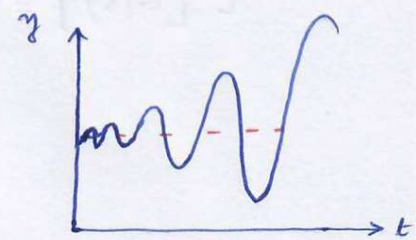
$P_4^* = \alpha - j\beta$

$$L^{-1}[G(s)] = L^{-1}\left[\frac{C_4}{s-(\alpha+j\beta)} + \frac{C_4^*}{s-(\alpha-j\beta)}\right] = [\quad]\, \underbrace{e^{\alpha t}}_{\text{depends on } \alpha}\, \underbrace{\sin(\beta t+\varphi)}_{\text{periodic oscillating function}}$$

Effect of α ($\equiv$ real part of complex poles)

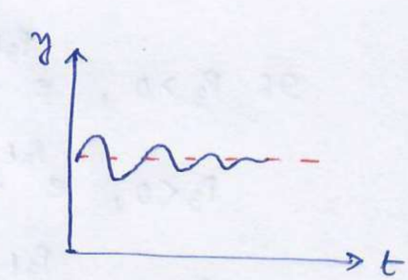
(i) If $\alpha > 0$, $e^{\alpha t} \to \infty$ as $t \to \infty$

$\therefore$ $e^{\alpha t}\sin(\beta t+\varphi)$ grows to ∞ in oscillatory manner. Unstable



(ii) If $\alpha < 0$, $e^{\alpha t} \to 0$ as $t \to \infty$

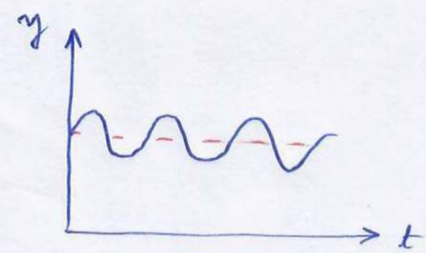
$\therefore$ $e^{\alpha t}\sin(\beta t+\varphi)$ decays to 0 in oscillatory manner. Stable



(iii) If $\alpha = 0$, $e^{\alpha t} = 1$ for all times

$\therefore$ $e^{\alpha t}\sin(\beta t+\varphi) = \sin(\beta t+\varphi)$

oscillates with const. amplitude



Marginally stable

4. Poles at the origin

$$G(s) = \frac{C_5}{s - P_5}$$

P_5 located at the origin of complex plane $P_5 = 0 + j \cdot 0$

$$= \frac{C_5}{s}$$

$$L^{-1}[G(s)] = C_5 = \text{Const.}$$

$y(t) = C_5 f(t)$