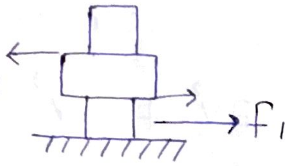
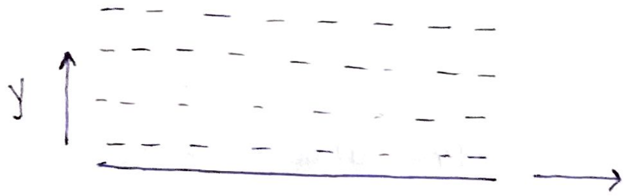
04/01/19 Monish Koushal

Unsteady / Transient transport of M^2/ Heat / Mass –

- Momentum



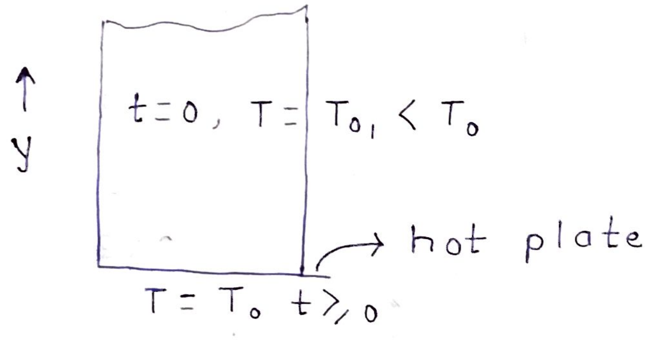


$t \geq 0, u = u_0$

Semi infinite plate

Here, momentum is taking some time to get transported from bottom layer to other layers above – hence "unsteady" state.

- Heat



- Mass

| → A (sensor)
· → B (Blood)

A is getting consumed / reacting with B. First, V decreases at its surface, then a conc^n gradient occurs in vertical direction, then it develops laterally. Therefore, unsteady state.

- Thermodynamics tells about equilibrium/ steady state.
- Heat Transfer deals with kinetics.

Cauchy Momentum Balance:

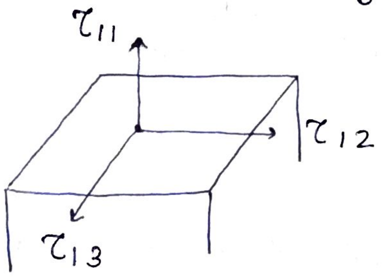
$$\underbrace{\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla v\right)}_{\text{acceleration}} = \underbrace{-\nabla p}_{\text{Pressure forces}} + \underbrace{\nabla\cdot\bar{\bar{\tau}}}_{\text{Stress}} + \underbrace{\rho\vec{g}}_{\text{Body force}}$$

A

* N.S is applicable for Newtonian, incompressible fluids.

Tensor - has no physical meaning
is just a matrix of 9 component

$$\underset{1\times 3}{\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}} = 1\times 3$$



Operators, tensors are introduced to simplify mathematics

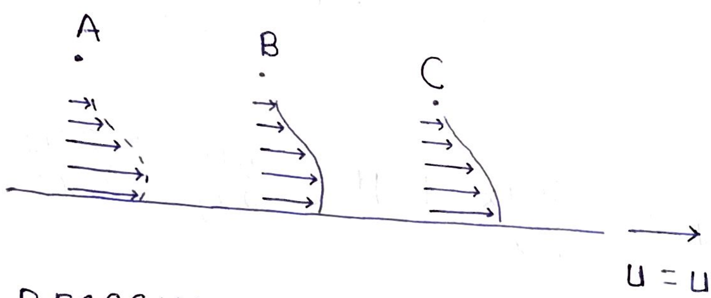
$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

In (A), we have one having 3 component.

$\nabla v \rightarrow$ tensor (3 X 3 combinations) = 9

- (A) combines all components in 1 eq^n.
- Momentum –

A, B and C are hydrodynamically equal. The conditions / environment surrounding them are same. So, why will the velocities be different ?



No pressure gradient. Flow is due to movement of bottom plate

$$\frac{\partial u}{\partial t} + \underbrace{\cancel{u \frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v}}_{0} \frac{\partial u}{\partial y} + \underbrace{\cancel{w}}_{0} \frac{\partial u}{\partial z} = -\frac{1}{\rho} \underbrace{\cancel{\frac{\partial p}{\partial x}}}_{0} + \nu \Delta^2 u$$

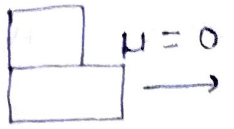
$$\rightarrow \quad \frac{\partial u}{\partial x} = 0 \; , \quad v = 0 \; , \quad w = 0$$

$$\therefore \quad \frac{\partial u}{\partial t} = \underbrace{\frac{\mu}{\rho}}_{\nu} \frac{\partial^2 u}{\partial y^2}$$

ν : momentum diffusivity

More momentum gets transferred with a greater (μ).

No slip doesn't tell about static / dynamic but deals with relative velocity.



Slipping occurs

There will be no momentum transport if $\mu = 0$

I.C : $u(y, t=0) = 0$

B.C : $u(y=0, \forall t) = u_0$

$u(y \to \infty, \forall t) = 0$

Scaling Analysis :

To see the relative importance of terms.

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + \vec{U} . \nabla \vec{V} \right) = \underbrace{- \nabla P + \nabla . \bar{\bar{\tau}}} + \rho \vec{g}$$

$$\tau_{ij} = - P \delta_{ij} + \lambda \frac{\partial u_K}{\partial x_K} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad ; \text{ Identity Matrix}$$

$$\frac{\partial u_K}{\partial x_K} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

P here is the thermodynamic pressure

$$P_{mech} = \frac{(\tau_{11} + \tau_{22} + \tau_{33})}{3}$$

For NV eq^n to work,

$P_{ther} = P_{mech}$.

Stokesian Fluid :-

If the relaxation time, i.e., time required by P_{ther} to become equal to P_{mech}, is very small, fluid is said to be stokesian.

$$-P_{mech} = -P_{therm} + \left(\lambda + \frac{2}{3}\mu\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

For NS eq^n to work,

$$P_{mech} = P_{therm}$$

if $\left(\lambda + \frac{2}{3}\mu\right) = 0$, fluids follow Stokes hypothesis.

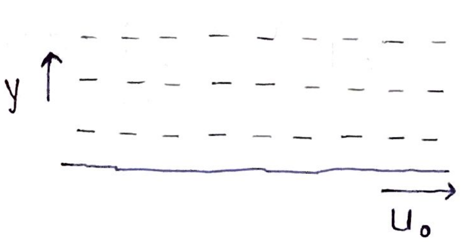
For incompressible fluid,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Hence, NS eq^n will be applicable.

→ No hydrodynamic B.L. Throughout viscous forces important.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$



I.C. : $t = 0 \quad u = 0 \ \forall y$

B.C. : $y = 0 \quad u = u_\infty \ \forall t$

$y \to \infty \quad u = 0 \ \forall t$

Scaling Analysis :-

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u}\right) = -\nabla P + \mu \nabla^2 \bar{V} + \rho \vec{g}$$

$u_0 = 10^{-3}$ m/s

$y = L = 10^{-4}$ m

$\rho = 10^3$ kg/m^3

$\mu = 10^{-3}$ Pa.s

Order of magnitude analysis :-

$$\sim \frac{U_o}{t_{ref}} \sim \frac{U_o^2}{L} \sim \frac{\rho U_o^2}{L} + \sim \frac{\mu U_o}{L^2}$$

If $\frac{\partial \bar{u}}{\partial t}$ and $\bar{u}.\nabla\bar{u}$ has to survive, order of the two terms should be equal

$$\frac{U_o}{t_{ref}} = \frac{U_o^2}{L} \Rightarrow t_{ref} = \frac{L}{U_o}$$

$$u^* = \frac{u}{u_o}$$

$$\rho\left(\frac{U_o^2}{L}\frac{\partial \bar{U}^*}{\partial t^*} + \frac{U_o^2}{L}\bar{u}^*.\nabla\bar{u}^*\right) = -\frac{\rho U_o^2}{L}\nabla P + \frac{\mu U_o}{L^2}\nabla^2\bar{U}^*$$

$$\Rightarrow \left(\frac{\partial \bar{u}^*}{\partial t^*} + \bar{u}^*.\nabla\bar{u}^*\right) = -\nabla P^* + \frac{1}{Re}\nabla^2\bar{u}^*$$

where, $Re = \frac{\rho U_o L}{\mu}$

$$Re = \frac{\rho U_o^2}{\mu . \frac{U_o}{L}} = \frac{\text{Inertial forces}}{\text{Viscous forces}}$$

For higher velocity, inertial forces dominate viscous forces, second term on R.H.S is zero, therefore potential flow regime. For lower velocity, L.H.S. = 0, therefore creeping flow regime.

For potential flow, bernoulli's eq^n applicable.

$$Re = \frac{10^3 \times 10^{-3} \times 10^{-4}}{10^{-3}} = 0.1$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\sim \frac{U_o}{t_{ref}} \sim \nu \frac{U_o}{\delta^2}$$; δ varies with time $\delta = \delta(t)$

$t_{ref} = t$ { t : running time }

$$\frac{\cancel{U_o}}{t} = \nu \frac{\cancel{U_o}}{\delta^2} \Rightarrow \delta \sim \sqrt{\nu t}$$

δ is momentum diffusion length scale.

Hydrodynamic B.L. : viscous forces important

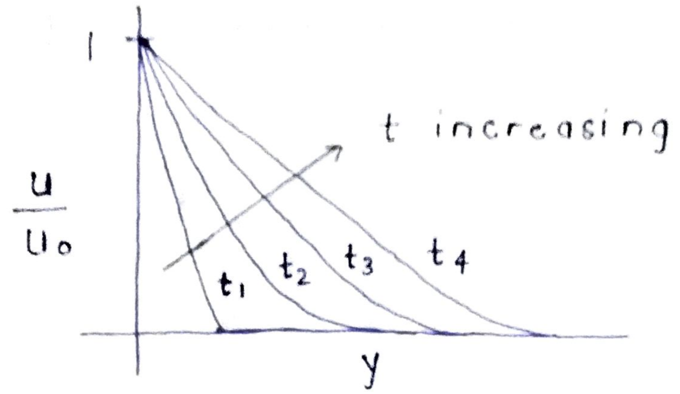
$$\delta = \sqrt{2^2 \nu t} = \sqrt{4 \nu t}$$

Error function :-

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

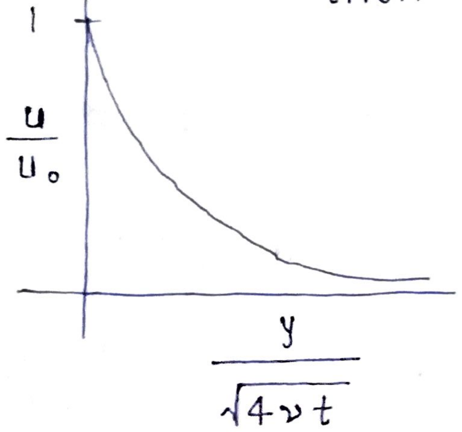
Gamma function :-

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$



Self - Similarity :

if the distance from the plate is non-dimensionalised , we get a single curve then self - similar



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\bar{u} = \frac{u}{u_0} \qquad \eta = \frac{y}{\delta}$$

$$\frac{\partial \bar{u}}{\partial t} = \nu \frac{\partial^2 \bar{u}}{\partial y^2}$$

B.C. :

$y = 0 \quad \bar{u} = 1$

$y = \infty \quad \bar{u} = 0$

I.C. : $t = 0 \quad \bar{u} = 0 \quad \forall y$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

Similarity transformation variable

$$\bar{u} = u/u_0 = f(\eta)$$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -\frac{1}{2}\left(\frac{\partial f}{\partial \eta}\right) \frac{y}{\sqrt{4\nu t} \cdot t}$$

$$= -\frac{\eta}{2t} \frac{\partial f}{\partial \eta}$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial f}{\partial \eta} \cdot \frac{1}{\sqrt{4\nu t}}$$

$$\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial \left(\frac{\partial \bar{u}}{\partial y}\right)}{\partial y} = \frac{\partial \left(\frac{\partial f}{\partial \eta} \frac{1}{\sqrt{4\nu t}}\right)}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$= \frac{1}{\sqrt{4\nu t}} \left(\frac{\partial^2 f}{\partial \eta^2}\right) \frac{1}{\sqrt{4\nu t}} = \frac{1}{4\nu t} \frac{\partial^2 f}{\partial \eta^2}$$

$$-\frac{\eta}{\cancel{2t}} \frac{\partial f}{\partial \eta} = \frac{\cancel{\nu}}{2\cancel{4}\cancel{\nu}\cancel{t}} \frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{\partial^2 f}{\partial \eta^2} + 2\eta \frac{\partial f}{\partial \eta} = 0$$

$$\eta = 0 \longrightarrow f = 1$$

$$(y = 0,\ t \to \infty)$$

$$\eta = \infty \longrightarrow f = 0$$

Let $\frac{\partial f}{\partial \eta} = \psi$.

$$\Rightarrow \frac{\partial \psi}{\partial \eta} + 2\eta \psi = 0$$

$$\frac{\partial \psi}{\psi} = -2\eta \, d\eta$$

$$\ln \psi = -\eta^2 + c$$

$$\Rightarrow \psi = c\, e^{-\eta^2}$$

$$\frac{\partial f}{\partial \eta} = c e^{-\eta^2}$$

$$f_{\eta=\eta} - f_{\eta=0} = \int_0^{\eta} c e^{-\eta^2} d\eta$$

$$f = 1 + c \int_0^{\eta} e^{-\eta^2} d\eta$$

$$n \to \infty \qquad f = 0$$

$$0 = 1 + c \int_0^{\infty} e^{-\eta^2} d\eta$$

$$\eta^2 = z \Rightarrow \eta = \sqrt{z} \Rightarrow d\eta = \frac{1}{2\sqrt{z}} dz$$

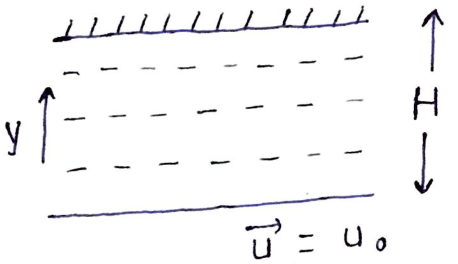
$$0 = 1 + \frac{c}{2} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$0 = 1 + \frac{c}{2} \Gamma(1/2) = 1 + \frac{c}{2}\sqrt{\pi}$$

$$\Rightarrow c = -\frac{2}{\sqrt{\pi}}$$

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta = 1 - \mathrm{erf}(\eta)$$

$$\frac{u}{u_0} = 1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right).$$

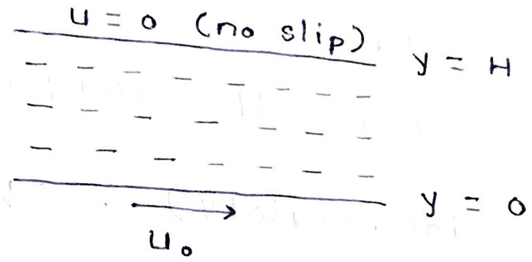


$$u = f(y, t)$$

$$\left.\begin{array}{ll} y = 0, & u = u_0 \\ y = H, & u = 0 \end{array}\right\} \text{B.C.}$$

$$t = 0 \qquad u = 0 \ \forall y \ \} \text{ I.C.}$$

6·02·2019



Unsteady state confined flows

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad u = u(y)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} \frac{u_0}{t_c} = \nu \frac{u_0}{H^2} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

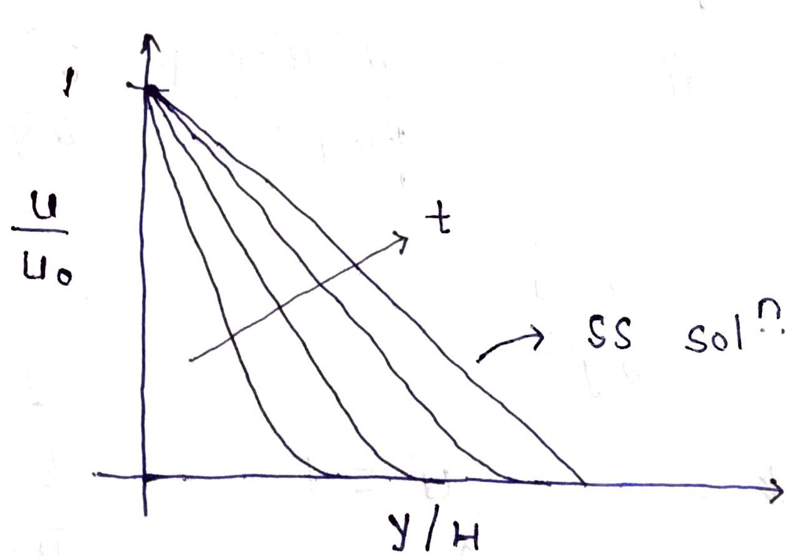
$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c \nu}{H^2} \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right) \Rightarrow$$ All terms have order of mag. equal to 1.

$$\frac{t_c \nu}{H^2} \sim O(1) \Rightarrow t_c = \frac{H^2}{\nu}$$

where t_c is M^2 diffusion time scale

$t << t_c$, it reduces to infinite domain problem and error function sol^n will work.

$t >> t_c$, steady state $u = ay + b$



This profile is not self similar like the one in previous problem.

$$t_c = \frac{(100 \times 10^{-6})^2}{10^{-3}/10^3} = 0.01\,s$$

(for water where $H = 100\,\mu m$)

$\frac{t_c \nu}{H^2} = 1$ (deliberately) , eq^n becomes

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

BC : $\bar{y} = 0 \quad \bar{u} = 1$

$\bar{y} = 1 \quad \bar{u} = 0$

I.C. : $\bar{t} = 0 \quad \forall \bar{y}$

$\bar{u} = 0$

$$\bar{u} = \underbrace{\bar{u}_{ss}}_{ss\ sol^n} + \hat{u} \longrightarrow \text{unsteady part}$$

SS solution –

$$0 = \frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2}$$

$\Rightarrow \bar{u}_{ss} = a\bar{y} + b$ $\qquad \bar{y} = 0 \quad \bar{u}_{ss} = 1$

$\bar{y} = 1 \quad \bar{u}_{ss} = 0$

$\Rightarrow \bar{u}_{ss} = 1 - \bar{y}$

Unsteady part –

$$\frac{\partial (\bar{u}_{ss} + \hat{u})}{\partial \bar{t}} = \underbrace{\cancel{\frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2}}}_{0} + \frac{\partial^2 \hat{u}}{\partial \bar{y}^2}$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial \bar{t}} = \frac{\partial^2 \hat{u}}{\partial \bar{y}^2}$$

B.C. $\bar{y} = 0 \quad \bar{u} = 1$

$\bar{u}_{ss} + \hat{u} = 1$

$\Rightarrow \hat{u} = 0$

I.C. :-

$\bar{t} = 0 \quad \bar{u} = 0$

$\bar{u}_{ss} + \hat{u} = 0$

$\hat{u} = -\bar{u}_{ss}$

$\hat{u} = \bar{y} - 1$

$\bar{y} = 1 \qquad \hat{u} = 0$

Homogeneous B.C.. So separation of variables work.

Separation of variables -

$\hat{u} = f(\bar{y})\, g(\bar{t}) = fg$

$fg' = gf''$

$$\frac{g'}{g} = \frac{f''}{f} = a$$

$$\frac{g'}{g} = a \Rightarrow \frac{dg}{g} = a\,d\bar{t} \Rightarrow g = ce^{a\bar{t}}$$

a should be negative. Let $a = -\lambda^2$

$g = ce^{-\lambda^2 \bar{t}}$

$$\frac{f''}{f} = -\lambda^2 \Rightarrow f'' + \lambda^2 f = 0$$

$f = a \cos \lambda \bar{y} + b \sin \lambda \bar{y}$

$\Rightarrow \forall\, \bar{t} \qquad \hat{u} = 0 \Rightarrow fg = 0 \Rightarrow f = 0$
$(\bar{y} = 0)$

when $\bar{y} = 1$, $\hat{u} = 0 \Rightarrow fg = 0 \Rightarrow f = 0$

From 1^{st} condition $a = 0$

From 2^{nd} condition

$f = b \sin \lambda \bar{y} \Rightarrow 0 = b \sin \lambda$

$b \neq 0$, $\therefore \sin \lambda = 0 \Rightarrow \lambda_n = n\pi$

$\Rightarrow \hat{u} = \sum b_n \sin(\lambda_n \bar{y}) c e^{-\lambda_n^2 \bar{t}}$

$\bar{u} = (1-\bar{y}) + \sum b_n (\sin(\lambda_n \bar{y})) c e^{-\lambda_n^2 t}$

$\hookrightarrow$ general sol^n

$\bar{u} = (1-\bar{y}) + \sum A_n \sin(\lambda_n \bar{y})\, e^{-\lambda_n^2 t}$

where $\{ A_n = c b_n \}$

At $t = 0$, $\bar{u} = 0 \quad \forall \bar{y}$

$\therefore \bar{y} - 1 = \sum A_n \sin \lambda_n \bar{y} \quad - (*)$

Let $g_n = \sin \lambda_n \bar{y}$

We have, $f'' + \lambda^2 f = 0$

$\Rightarrow \dfrac{d^2 f_n}{d\bar{y}^2} + \lambda_n^2 f_n = 0 \quad \{ f_n = A_n \sin \lambda_n$

for some m,

$$\int_0^1 f_m \frac{d^2 f_n}{d\bar{y}^2} d\bar{y} + \int_0^1 \lambda_n^2 f_n f_m \, d\bar{y} = 0$$

$$\underbrace{f_m \cancel{\frac{df_n}{d\bar{y}}} \Big|_0^1}_{0} - \int_0^1 \frac{df_m}{d\bar{y}} \frac{df_n}{d\bar{y}} d\bar{y} + \lambda_n^2 \int_0^1 \frac{f_m f}{d\bar{y}} = 0 \quad - (b)$$

Swapping m and n

$$-\int_0^1 \frac{df_n}{d\bar{y}} \frac{df_m}{d\bar{y}} d\bar{y} + \lambda_m^2 \int_0^1 f_m f_n \, d\bar{y} = 0 \quad - (c)$$

$$(\lambda_m^2 - \lambda_n^2) \int_0^1 f_m f_n \, d\bar{y} = 0$$

$$\Rightarrow m \neq n \qquad \int_0^1 f_m f_n \, d\bar{y} = 0$$

From ($*$)

$$(\bar{y} - 1) = \Sigma A_n (\sin \lambda_n \bar{y}) = \Sigma A_n q_n$$

Multiplying by q_m

$$\Rightarrow \int_0^1 (\bar{y} - 1) q_m \, d\bar{y} = \int_0^1 \Sigma A_n q_n q_m \, d\bar{y}$$

R.H.S. $= 0$ for $n \neq m$.

For $n = m$

$$\int_0^1 (\bar{y} - 1) q_m \, d\bar{y} = \int_0^1 A_n q_n^2 \, d\bar{y}$$

$$\int_0^1 (\bar{y} - 1) \sin \lambda_n \bar{y} \, d\bar{y} = \int_0^1 A_n \sin^2 \lambda_n \bar{y} \, d\bar{y}$$

$$\int_0^1 \bar{y} \sin \lambda_n \bar{y} \, d\bar{y} - \int_0^1 \sin \lambda_n \bar{y} \, d\bar{y} = \text{R.H.S}$$

$$\left[-\bar{y} \frac{\cos \lambda_n \bar{y}}{\lambda_n} - \left\{ -\frac{\sin \lambda_n \bar{y}}{\lambda_n^2} \right\} \right]_0^1$$

$$+ \left. \frac{\cos \lambda_n \bar{y}}{\lambda_n} \right|_0^1 = \text{R.H.S}$$

$$-\frac{1 \cdot (-1)^n}{\lambda_n} - 0 - \{0 - 0\} + \frac{(-1)^n}{\lambda_n}$$

$$-\frac{1}{\lambda_n} = \text{R.H.S}$$

$$R.H.S. = \frac{A_n}{2}$$

$$\Rightarrow -\frac{1}{\lambda_n} = \frac{A_n}{2}$$

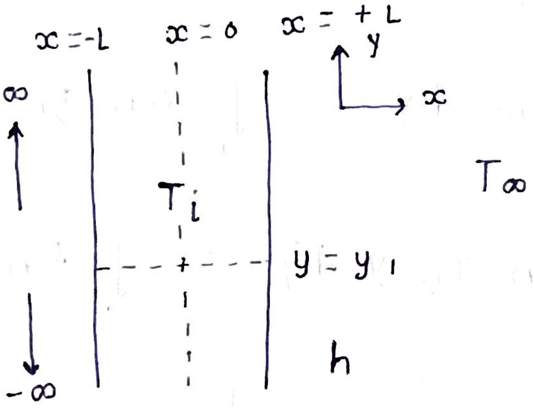
$$\Rightarrow A_n = -\frac{2}{\lambda_n} = -\frac{2}{n\pi}$$

$$\bar{u} = (1-\bar{y}) + \Sigma \left(A_n \sin \lambda_n \bar{y} \right) e^{-\lambda_n^2 \bar{t}}$$

$$\bar{u} = (1-\bar{y}) - \frac{2}{\pi} \Sigma \frac{\sin \lambda_n \bar{y}}{n} e^{-\lambda_n^2 \bar{t}}$$

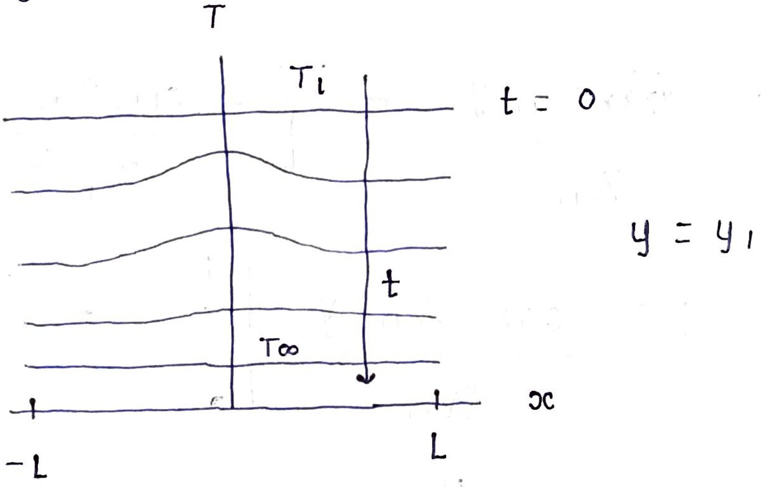
Unsteady Heat Transfer :-

1-D heat conduction



$T_i > T_\infty$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = \frac{k}{\rho C_p}$$



τ = thermal diffusion time scale

$$\tau = \frac{L^2}{\alpha}$$

$x = 0 \qquad \frac{\partial T}{\partial x} = 0$

B.C.

$x = \pm L \qquad -k \frac{\partial T}{\partial x}\Big|_{x=L} = h\,(T_{x=L} - T_\infty)$

I.C. $\quad t = 0, \quad T = T_i \quad \forall\, x$

Non-dimensionalising the eq^n

$\theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad \bar{x} = \frac{x}{L} \qquad \bar{t} = \frac{t}{t_c}$

$$\frac{(\cancel{T_i - T_\infty})}{t_c}\frac{\partial \theta}{\partial \bar{t}} = \frac{\alpha\,(\cancel{T_i - T_\infty})}{L^2}\frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\frac{1}{t_c}\frac{\partial \theta}{\partial \bar{t}} = \frac{\alpha}{L^2}\frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\left(\frac{\partial \theta}{\partial \bar{t}}\right) = \left(\frac{\alpha t_c}{L^2}\right)\left(\frac{\partial^2 \theta}{\partial \bar{x}^2}\right)$$

$\frac{\alpha t_c}{L^2} \sim 1 \quad \Rightarrow t_c = \frac{L^2}{\alpha}$

$\boxed{F_o = \frac{t\alpha}{L^2}}$

$= \frac{t}{t_c}$

$$\frac{\partial \theta}{\partial \bar{t}} = \frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\frac{\partial \theta}{\partial F_o} = \frac{\partial^2 \theta}{\partial \bar{x}^2}$$

B.C. :-

$\bar{x} = 0 \qquad \frac{\partial \theta}{\partial \bar{x}} = 0$

$\bar{x} = 1, \quad -k \frac{(\cancel{T_i - T_\infty})}{L}\frac{\partial \theta}{\partial \bar{x}}\Big|_{\bar{x}=1} = h\,(\cancel{T_i - T_\infty}) \times \theta|_{\bar{x}=1}$

$\Rightarrow \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = -\frac{hL}{k} \left.\theta\right|_{x=1}$

I.C :-

$\Rightarrow \theta'(1) = -Bi\, \theta(1)$

$Fo = 0,\ \theta = 1$

$T = f(\bar{x})\, g(F_o) = \theta$

$f g' = g f''$

$\frac{g'}{g} = \frac{f''}{f} = -\lambda^2$

$\frac{g'}{g} = -\lambda^2 \Rightarrow g = c_1 e^{-\lambda^2 F_o}$

$f'' + \lambda^2 f = 0 \Rightarrow f = a\cos(\lambda\bar{x}) + b\sin(\lambda\bar{x})$

$\left.\frac{\partial f}{\partial x}\right|_{x=0} = 0$

$f'(1) = -Bi\,(f(1))$

$\Rightarrow f' = -a\lambda \sin(\lambda\bar{x})$

$f'_{x=0} = b\lambda = 0$

$\Rightarrow b = 0$

$f'(1) = -a\lambda \sin\lambda = -Bi\, a \cos\lambda$

$\Rightarrow \lambda \tan\lambda = Bi$ $\rightarrow$ not a finite no. for every value of λ.

For some λ_n,

$\lambda_n \tan\lambda_n = Bi$

$\theta = \sum A_n \cos(\lambda_n \bar{x})\, e^{-\lambda_n^2 \bar{t}}$

Applying the initial condition,

$1 = \sum A_n \cos\lambda_n \bar{x}$

$\int_0^1 (\cos\lambda_m \bar{x})\, d\bar{x} = \sum A_n \int_0^1 (\cos\lambda_n \bar{x})(\cos\lambda_m \bar{x})\, d\bar{x}$

$$\frac{\sin \lambda_m \bar{x}}{\lambda_m}\Big|_0^1 = \frac{\sum A_n}{2}\int_0^1 \cos(\lambda_m+\lambda_n)\bar{x}\,d\bar{x}$$

$$+ \frac{\sum A_n}{2}\int_0^1 \cos(\lambda_m-\lambda_n)\bar{x}\,d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2}\left\{\frac{\sin(\lambda_m+\lambda_n)\bar{x}}{\lambda_m+\lambda_n} + \frac{\sin(\lambda_m-\lambda_n)\bar{x}}{\lambda_m-\lambda_n}\right\}_0^1$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2}\left\{\frac{\sin(\lambda_m+\lambda_n)}{(\lambda_m+\lambda_n)} + \frac{\sin(\lambda_m-\lambda_n)}{(\lambda_m-\lambda_n)}\right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2}\cdot\frac{1}{\lambda_m^2-\lambda_n^2}$$

$$\left\{(\lambda_m-\lambda_n)\sin(\lambda_m+\lambda_n) + (\lambda_m+\lambda_n)\sin(\lambda_m-\lambda_n)\right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2}\cdot\frac{1}{\lambda_m^2-\lambda_n^2}$$

$$\left\{\lambda_m\{\sin(\lambda_m+\lambda_n)+\sin(\lambda_m-\lambda_n)\} - \lambda_n\{\sin(\lambda_m+\lambda_n)-\sin(\lambda_m-\lambda_n)\}\right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \sum A_n \frac{(\lambda_m \sin\lambda_m \cos\lambda_n - \lambda_n \cos\lambda_m \sin\lambda_n)}{\lambda_m^2-\lambda_n^2} \quad (*)$$

$$\lambda_n \tan \lambda_n = \lambda_m \tan \lambda_m = Bi$$

$$\Rightarrow \lambda_n \sin \lambda_n \cos \lambda_m = \lambda_m \sin \lambda_m \cos \lambda_n$$

R.H.S. of (*) is zero for $m \neq n$

for $m = n$

$$\int_0^1 (\cos \lambda_m \bar{x})\, d\bar{x} = A_m \int_0^1 \cos^2 \lambda_m \bar{x}\, d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \int_0^1 (\cos 2\lambda_m \bar{x} + 1)\, d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \left[\frac{\sin 2\lambda_m \bar{x}}{2\lambda_m} + \bar{x} \right]_0^1$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \left[\frac{\sin 2\lambda_m}{2\lambda_m} + 1 \right]$$

$$\Rightarrow A_m = \frac{2 \sin \lambda_m}{\cancel{\lambda_m}} \left(\frac{2 \cancel{\lambda_m}}{2\lambda_m + \sin 2\lambda_m} \right)$$

$$\Rightarrow A_m = \frac{4 \sin \lambda_m}{2\lambda_m + \sin 2\lambda_m}$$

$$\theta = \sum A_m \cos (\lambda_m \bar{x})\, e^{-\lambda_m^2 F_o}$$

where $$A_m = \frac{4 \sin \lambda_m}{2\lambda_m + \sin 2\lambda_m}$$

Heisler Chart :

$\theta_0 = \theta_0 (F_o)$ $\{ \theta_0 = \text{Mid line temp.} \}$

For some $F_o = a$,

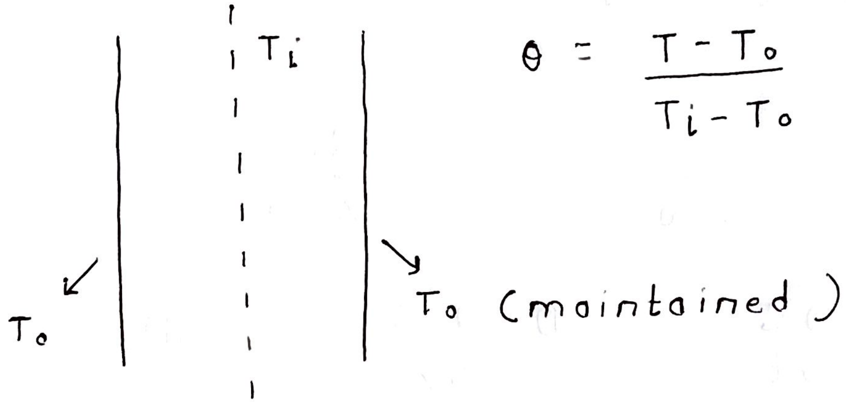
$$\theta_0 = \sum A_n e^{-\lambda_n^2 a} \quad - (1)$$

$$\theta_{F_o = a} (\bar{x}) = \sum A_n (\cos \lambda_n \bar{x}) e^{-\lambda_n^2 a} \quad - (2)$$

eq^n (2) / (1)

$$\frac{\theta_{(F_o = a)} (\bar{x})}{\theta_0 (F_o = a)} = \frac{\sum A_n (\cos \lambda_n \bar{x}) e^{-\lambda_n^2 a}}{\sum A_n e^{-\lambda_n^2 a}}$$

Q.



$$\theta = \frac{T - T_0}{T_i - T_0}$$

I.C. : $t = 0$ $T = T_i$ $\forall x$

B.C. : $x = 0$ $\frac{\partial T}{\partial x} = 0$ $\forall t$

$x = L$ $T = T_0$ $\forall t$

Significance of scaling analysis ?

Based on scaling analysis, we can design our equipments.

{Book : Gupta and Gupta}

5/03/2019

Diffusion from instantaneous point source

m_A

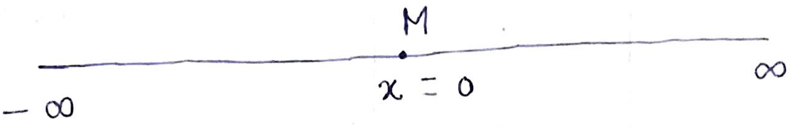
$r = 0, t = 0$

$$\rho_A = \frac{m_A}{(4\pi D_{AB} t)^{3/2}} \exp\left(-\frac{r^2}{4 D_{AB} t}\right)$$

(1) Check the validity of Fick's Law

(2) $\int \rho_A \, dV = m_A$ (for any t)

(3) $t \to 0$



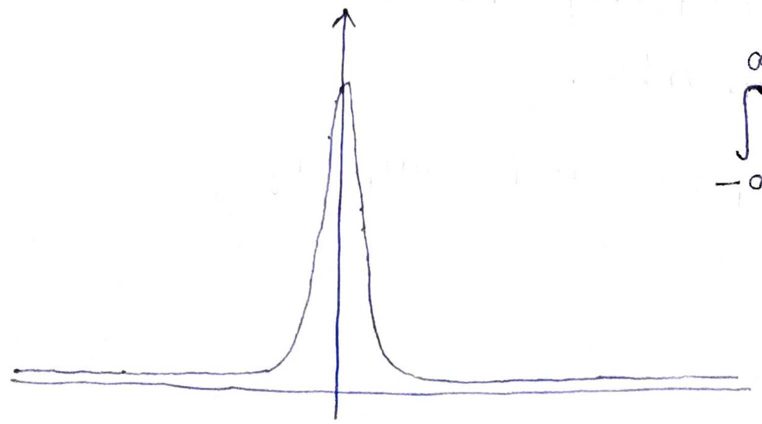
$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \quad - (1)$$

where,

$C = \frac{M}{L}$ (linear density)

B.C. : $x = \pm\infty$, $C \to 0$

I.C. : $t = 0$, $C = \frac{M}{L}\delta(x)$



$$\int_{-\infty}^{\infty} \delta(x) = 1$$

$$\hat{c}(\alpha, t) = \int_{-\infty}^{\infty} c(x,t)\, e^{-i\alpha x}\, dx$$

(Fourier Transform)

$$c_t = \frac{\partial c}{\partial t}$$

$$\int c_t\, e^{-i\alpha x}\, dx = \frac{\partial \hat{c}(\alpha, t)}{\partial t}$$

$$\int c_x\, e^{-i\alpha x}\, dx = i\alpha\, \hat{c}(\alpha, t)$$

$$\int c_{xx}\, e^{-i\alpha x}\, dx = -\alpha^2\, \hat{c}(\alpha, t)$$

Applying F.T. on eq^n 1

$$\frac{d\hat{c}}{dt} + D\alpha^2 \hat{c} = 0$$

$$\hat{c}(\alpha, t) = F(\alpha) \exp(-D\alpha^2 t)$$

Put $t = 0$,

$$F(\alpha) = \hat{c}(\alpha, 0) = \int \frac{M}{L}\, \delta(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{M}{L}$$

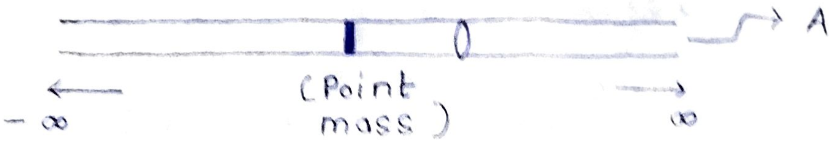
$$\hat{c}(\alpha, t) = \frac{M}{L} \exp(-D\alpha^2 t)$$

$$F(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha, t)\, e^{i\alpha x}\, d\alpha$$

$$c(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M}{L} \exp(-D\alpha^2 t)\, e^{i\alpha x}\, d\alpha$$

Profile is symmetric.

$$c(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M}{L} \left(\exp\left(-D\alpha^2 t\right)\right) \cos\alpha x$$



$$\Delta V = \lim_{\Delta x \to 0} A\, \Delta x$$

$$\Rightarrow \quad \Delta V \longrightarrow 0$$

$$c\,(t=0,\, x=0) = \lim_{\Delta V \to 0} \frac{M}{(A\,\Delta x)}$$

$$c\,(t=0) = \frac{M}{A}\,\delta(x) \leftarrow \text{I.C.}$$

B.C. : $\quad c\,(t,\, x = \pm\infty) = 0$

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

$$\frac{\partial \hat{c}}{\partial t} + D\alpha^2\,\hat{c} = 0$$

$$\hat{c}\,(\alpha, t) = F(\alpha)\,\exp\left(-D\alpha^2 t\right)$$

$$\hat{c}\,(\alpha, 0) = F(\alpha)$$

$$\hat{c}\,(\alpha, 0) = \int_{-\infty}^{\infty} \frac{M}{A}\,\delta(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{M}{A} = F(\alpha)$$

$$\hat{c}\,(\alpha, t) = \frac{M}{A}\,\exp\left(-D\alpha^2 t\right)$$

$$c(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M}{A}\,\exp\left(-D\alpha^2 t\right) e^{i\alpha x}\, d\alpha$$

$$\alpha = \frac{y}{\sqrt{Dt}} \Rightarrow d\alpha = \frac{dy}{\sqrt{Dt}}$$

$$c(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M}{A} \frac{\left(\exp(-y^2)\right)\left(\cos\left(\frac{yx}{\sqrt{Dt}}\right)\right)dy}{\sqrt{Dt}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M}{A\sqrt{Dt}} \exp(-y^2) \cos(\eta y)\, dy$$

where $\eta = x / \sqrt{Dt}$

$$I(\eta) = \int_0^{\infty} e^{-y^2} \cos(\eta y)\, dy \quad - \quad (*)$$

$$\frac{dI}{d\eta} = -\frac{1}{2} \int_0^{\infty} 2y\, e^{-y^2} \sin(\eta y)\, dy$$

$$\frac{dI}{d\eta} = \frac{1}{2} \int_0^{\infty} \sin(\eta y)\, d\left(e^{-y^2}\right)$$

$$= e^{-y^2} \sin(\eta y) \Big|_0^{\infty} - \frac{1}{2} \int_0^{\infty} \eta\, e^{-y^2} \cos(\eta y)\, dy$$

$$= -\frac{\eta}{2} \int_0^{\infty} e^{-y^2} \cos \eta y\, dy$$

$$= -\frac{\eta}{2} I$$

$$\frac{dI}{d\eta} + \frac{\eta I}{2} = 0$$

$$\Rightarrow I(\eta) = I_0 \exp\left(-\frac{\eta^2}{4}\right)$$

Put $\eta = 0$ in (*)

$$I_0 = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2} \exp\left(-\frac{\eta^2}{4}\right)$$

$$c(x,t) = \frac{M}{2A\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

$$c(x,0) = \frac{M}{A}\,\delta(x)$$

$$M = \int_{\forall} c(x,t)\, d\forall$$

$$= \int_{-\infty}^{\infty} \frac{M}{A}\,\delta(x)\, A\, dx$$

$$= M$$

$$\pi_1 = \frac{c}{M/(A\sqrt{Dt})}$$

$$\pi_2 = \frac{x}{\sqrt{Dt}}$$

$$\pi_1 = f(\pi_2) \Rightarrow c = \frac{M}{A\sqrt{Dt}}\, f\left(\frac{x}{\sqrt{Dt}}\right)$$

$$\eta = \frac{x}{\sqrt{Dt}} \Rightarrow \frac{d\eta}{dt} = -\frac{1}{2}\,\frac{x}{\sqrt{D}\sqrt{t}\,t} = -\frac{\eta}{2t}$$

$$\frac{d\eta}{dx} = \frac{1}{\sqrt{Dt}}$$

3 fundamental variables : M, L, T

c is a dependent variable.

$$c$$

$$\left.\begin{matrix} \left(\frac{M}{A}\right) \\ D \\ x \\ t \end{matrix}\right\} \text{Independent Variables}$$

Choose any fundamental quantity. Lets say T.

$$T = (c)^x \left(\frac{M}{A}\right)^y (D)^z$$

$$T = M^x L^{-3x} M^y L^{-2y} L^{2z} T^{-z}$$

$$z = -1 \qquad x + y = 0 \qquad 3x + 2y = 2z = -2$$

$$x = -2 \qquad y = 2$$

$$\pi_1 = \frac{t}{(c^{-2})\left(\frac{M}{A}\right)^2 D^{-1}} = \frac{c^2}{\left(\frac{M}{A}\right)^2 \frac{1}{Dt}}$$

$$\pi_1 = \frac{c}{M / A\sqrt{Dt}} = f(\pi_2)$$

$$\pi_2 = \frac{x}{\sqrt{Dt}}$$

$$\Rightarrow c = \frac{M}{A\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)$$

$$\frac{x}{\sqrt{Dt}} = \eta$$

$$\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t} , \quad \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{Dt}}$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left(\frac{M}{A\sqrt{Dt}} f(\eta)\right)$$

$$= \frac{M}{A\sqrt{Dt}} \frac{\partial f}{\partial t} - \frac{1}{2t} f \cdot \left(\frac{M}{A}\right) \frac{1}{\sqrt{Dt}}$$

$$= \frac{M}{A\sqrt{Dt}} \left(\frac{\partial f}{\partial \eta}\right)\left(\frac{-\eta}{2t}\right) + \frac{M}{A\sqrt{Dt}} \left(\frac{-f}{2t}\right)$$

$$\frac{\partial c}{\partial t} = -\frac{M}{2At\sqrt{Dt}} \left(\eta \frac{\partial f}{\partial \eta} + f\right)$$

$$\frac{\partial c}{\partial x} = \frac{M}{A\sqrt{Dt}} \cdot \frac{\partial f}{\partial \eta} \cdot \frac{1}{\sqrt{Dt}}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{M}{A\sqrt{Dt}} \cdot \frac{1}{Dt} \frac{d^2 f}{d\eta^2}$$

$$\Rightarrow \frac{d^2 f}{d\eta^2} + \frac{1}{2}\left(f + \eta \frac{df}{d\eta}\right) = 0$$

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2} \frac{d}{d\eta}(f\eta) = 0$$

$$\frac{d}{d\eta}\left(\frac{df}{d\eta} + \frac{f\eta}{2}\right) = 0$$

$$\frac{df}{d\eta} + \frac{f\eta}{2} = c_0$$

$$M = \int_{-\infty}^{\infty} C A \, dx$$

$$M = \int_{-\infty}^{\infty} \frac{M}{\cancel{A}\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right) \cancel{A} \, dx$$

$$1 = \int_{-\infty}^{\infty} f(\eta) \, d\eta = 1$$

We have,

$$c(\mp\infty, t) = 0 \Rightarrow f(\mp\infty) = 0$$

$$c(x, 0) = \frac{M}{A} \delta(x)$$

$$f\left(\frac{x}{\sqrt{Dt}}\right) = \delta(x)\sqrt{Dt}$$

Let $C_0 = 0$

$$\frac{df}{d\eta} = -\frac{f\eta}{2}$$

$$\Rightarrow \ln f = -\frac{\eta^2}{4} + c_1$$

$$\Rightarrow f = c_1 \exp\left(-\frac{\eta^2}{4}\right)$$

$$\int_{-\infty}^{\infty} f \, d\eta = \int_{-\infty}^{\infty} c_1 \exp\left(-\frac{\eta^2}{4}\right) d\eta$$

$$1 = 2\int_{0}^{\infty} c_1 \exp\left(-\frac{\eta^2}{4}\right) d\eta$$

Let $\Rightarrow \frac{\eta}{2} = \xi$

$$4 \int_{0}^{\infty} c_1 \exp(-\xi^2) \, d\xi = 1$$

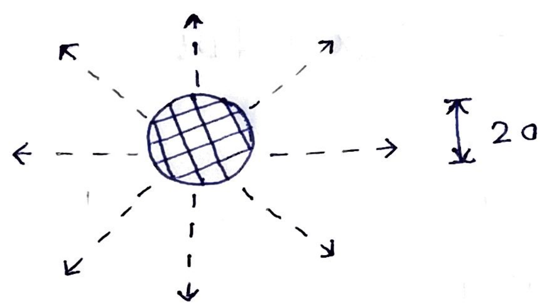
$$4 c_1 \frac{\sqrt{\pi}}{2} = 1$$

$$c_1 = \frac{1}{2\sqrt{\pi}}$$

$$c(x,t) = \frac{M}{A\sqrt{Dt}} \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4Dt}\right)$$

$$c(x,t) = \frac{M}{A\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

* Book : Conduction of heat in solids by Carslaw and Jaeger



$$t = 0, \quad C_0 = \frac{M}{\frac{4}{3}\pi a^3}$$

$$\frac{\partial c}{\partial t} = \frac{D}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right)$$

$$C = \frac{C_0}{2r(\pi D t)^{1/2}} \int_0^{\infty} r' \left\{ \exp\left(-\frac{(r-r')^2}{4Dt}\right) - \exp\left(-\frac{(r+r')^2}{4Dt}\right) \right. dr'$$

for $a \rightarrow 0$

$$c = \frac{M}{\frac{4}{3}\pi a^3} \quad \frac{1}{8} \quad \frac{e^{-r^2/4Dt}}{(\pi k t)^{1/2}}$$

Scaling Analysis :

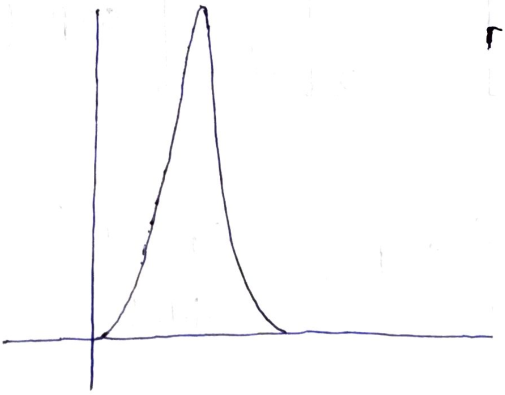
$$c = C_0 \frac{e^{-r^2/4Dt}}{8(\pi D t)^{1/2}}$$

* 3 D diffusion, no θ and ϕ.

$r \rightarrow 0$ to a. beyond this $c = 0$

$$c = C_0 = \frac{M}{\frac{4}{3}\pi a^3}, \quad 0 < r \leq a$$

$$c = 0, \quad r \geq a$$



$r \rightarrow$ space

no connection with time

Substitute $c = \frac{u}{r}$

Solve R.H.S :-

$$\frac{\partial (u/r)}{\partial t} = \frac{D}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial (u/r)}{\partial r}\right)$$

$$\Rightarrow \frac{\partial (u/r)}{\partial t} = \frac{D}{r^{\cancel{2}}} \cdot 2\cancel{r} \frac{\partial (u/r)}{\partial r} + \frac{D}{\cancel{r^2}} \cdot \cancel{r^2} \frac{\partial^2 (u/r)}{\partial r^2}$$

$$\frac{1}{\cancel{r}}\frac{\partial u}{\partial t} = \frac{D}{\cancel{r}}\frac{\partial^2 u}{\partial r^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial r^2} \Rightarrow u = f(r,t)$$

r' can be any value

$$u = t^{-1/2} \exp\left(\frac{-(r-r')^2}{4Dt}\right)$$ particular solution

$$T = f = \sum a_n \cos \lambda_n \bar{x} \exp(-t^2 \lambda_n)$$

$$f = a \cos \lambda \bar{x}$$

$$* \quad \frac{\partial u}{\partial t} = \left[\exp\left(\frac{-(r-r')^2}{4Dt}\right)\right]\left[\frac{(r-r')^2}{4D}\frac{1}{t^{3/2}}\right] - \frac{1}{2t^{3/2}}\left[\exp\left(\frac{-(r-r')^2}{4Dt}\right)\right]$$

$$\frac{\partial u}{\partial r} = t^{-1/2}\left\{\exp\left(\frac{-(r-r')^2}{4Dt}\right)\right\}\left(\frac{-1}{4Dt}\right)(2(r-r'))$$

$$\frac{\partial u}{\partial r} = -\frac{(r-r')^1}{2Dt^{3/2}}\exp\left[\frac{-(r-r')^2}{4Dt}\right]$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{-1}{2Dt^{3/2}}\left\{-\frac{(r-r')^2}{2Dt}\exp\left[\frac{-(r-r')^2}{4Dt}\right] + \exp\left(\frac{-(r-r')^2}{4Dt}\right)\right\}$$

$$= \frac{(r-r')^2}{4D^2 t^{5/2}} \exp\left[-\frac{(r-r')^2}{4Dt}\right] -$$

$$\frac{1}{2Dt^{3/2}} \exp\left[-\frac{(r-r')^2}{4Dt}\right]$$

$$* \quad f = \sum_{n=0}^{\infty} a_n \cos \lambda_n \bar{x}$$

$$\lambda_n = \left(n \pm \frac{1}{2}\right)\pi$$

a_n is dependent on λ_n.

For all $\lambda_n \rightarrow$ we get solution

$$u = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(r') \exp\left(-\frac{(r-r')^2}{4Dt}\right) dr'$$

$\rightarrow$ r' is the variable. choice dependent.

$t \rightarrow 0$, $u = f(r)$ (known) $\longrightarrow$

So f appears in integral.

$u = cr$

at $t = 0$, $c = c_0$

$\Rightarrow u = c_0 r$

$r' = r + 2\sqrt{Dt}\, \xi$

$dr' = 2\sqrt{Dt}\, d\xi$

[at some particular instant, t]

27.03.19.

$$\Rightarrow u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(r + 2\sqrt{Dt}\,\xi)\, e^{-\xi^2}\, d\xi$$

$t \rightarrow 0$

$$\Rightarrow u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(r)\, e^{-\xi^2}\, d\xi$$

$$= f(r) * \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi$$

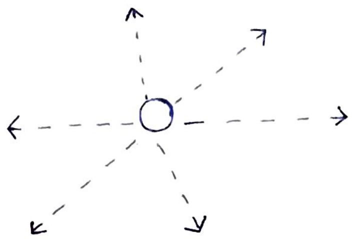
$$= f(r) \cdot \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\xi^2}\, d\xi$$

$$= f(r)\, \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$$u = f(r) \cdot 1$$

$$\Rightarrow u = f(r)$$

27.03.19



$$u = cr$$

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial r^2}$$

$$t = 0, \quad u = C_0 r, \quad 0 < r < a \qquad C_0 = \frac{M}{\frac{4}{3}\pi a^3}$$

$$= 0 \qquad a < r$$

$$u = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(r') \left(\exp\left(-\frac{(r-r')^2}{4Dt} \right)\right) d$$

where I.C. $t = 0 \quad u = f(r)$

or, $$u = \frac{1}{2\sqrt{\pi D t}} \int_{0}^{\infty} f(r') \left\{ \exp\left(-\frac{(r-r')^2}{4Dt} \right) - \exp\left(-\frac{(r+r')^2}{4Dt} \right) \right\} dr'$$

$$u = \frac{1}{2\sqrt{\pi D t}} \left(\exp\left(\frac{-r^2}{4Dt} \right)\right)$$

$$\int_0^a r' \exp\left(\frac{-r'^2}{4Dt} \right) \left\{ \exp\left(\frac{rr'}{2Dt} \right) - \exp\left(\frac{-rr'}{2Dt} \right) \right\} dr'$$

$$c = \frac{C_0}{2(\sqrt{\pi D t})^3}\left(\exp\left(\frac{-r^2}{4Dt}\right)\right)\left\{1 + \left(\frac{r^2}{Dt} - 6\right)\frac{a^2}{40Dt}\right\} \times \frac{1}{3}$$

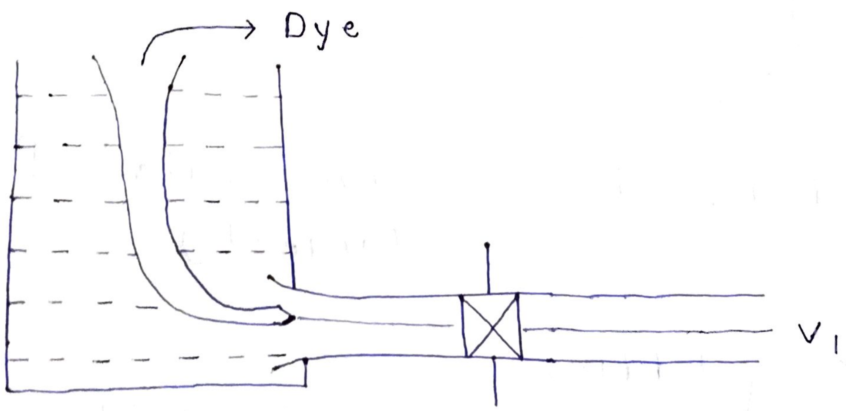
for $a \longrightarrow 0$

$$C = \frac{C_0}{2(\sqrt{\pi D t})^3}\exp\left(\frac{-r^2}{4Dt}\right) \times \frac{1}{3}$$

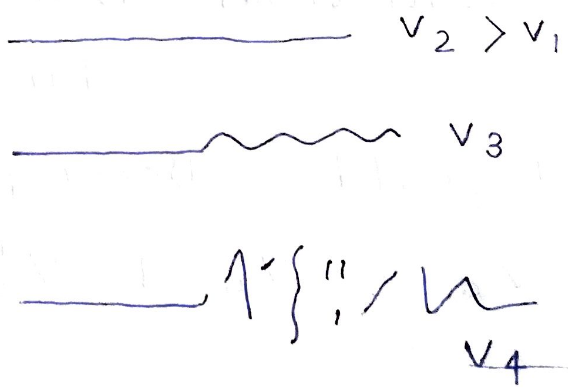
$$\Rightarrow C = \frac{M}{8(\pi D t)^{3/2}}\exp\left(\frac{-r^2}{4Dt}\right)$$

Turbulence :

Reynold's Experiment :



At low velocity, regular and orderly flow.

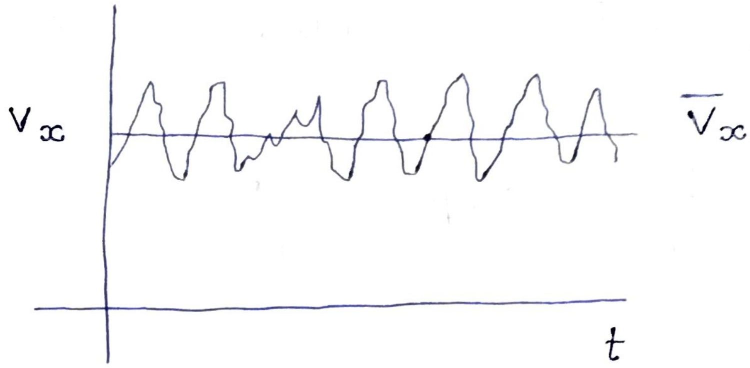


Turbulent : erratic fluctuations in velocity. Irregular flow

$$V_x(s,t) = \underset{\substack{\downarrow \\ \text{time independent}}}{\overline{V}_x(s)} + V_x'(s,t)$$

Such processes are known as stationary processes.

$\overline{V}_x(s)$: averaged over time



$$\underset{T \to \infty}{Lt} \frac{\int_{t-T}^{t+T} V_x' \, dt'}{2T} = \overline{V_x'} = 0$$

provided, $T >>$ time scale of fluctuation (largest)

Eddies
↳ lumps of fluid rotating

Statistical averaging ⟶ to get something deterministic / predictable

Reynold's Decomposition :

$$V_x = \overline{V}_x + V_x'(x, y, z, t)$$

$$V_x = \bar{V}_x + V'_x$$

$$\frac{\int_{t-T}^{t+T} V_x \, dt'}{2T} = \frac{\int_{t-T}^{t+T} \bar{V}_x \, dt'}{2T} + \frac{\int_{t-T}^{t+T} V'_x \, dt'}{2T}$$

$$\bar{V}_x = \bar{V}_x + \bar{V}'_x$$

$$\Rightarrow \bar{V}'_x = 0$$

$$\overline{\bar{V}_x \frac{\partial V_y'}{\partial x}} = \left(\int_{t-T}^{t+T} \bar{V}_x \frac{\partial V_y}{\partial x} \, dt' \right) / 2T$$

$$= \bar{V}_x \frac{\int_{t-T}^{t+T} \frac{\partial V'_y}{\partial x} \, dt'}{2T}$$

$$= \bar{V}_x \frac{\partial}{\partial x} \frac{\int_{t-T}^{t+T} V'_y \, dt'}{2T}$$

$$= \bar{V}_x \, . \, 0 = 0$$

$$Re = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{(\rho l^3) \frac{\bar{U}^2}{L}}{\mu \frac{\bar{U}}{L} . l^2}$$

$$= \frac{\rho \bar{U} l}{\mu}$$

Maximum length scale of an eddy ~ Radius of the pipe

For each eddy, we can have a local Re.

$l_e = f(Re, l_c)$

↳ length scale of eddy

rate of extraction of energy from bigger eddy $\sim \dfrac{\bar{U}^2}{t}$; $t = \dfrac{L}{\bar{U}}$

{ L is or the order of length scale of system. Hence $\bar{U}_{eddy} \sim \bar{U}$ }

$$\dot{E} = \frac{\bar{U}^3}{L}$$

rate of dissipitation by smallest eddy

$$\sim \dot{\epsilon} = \left(\mu \frac{\partial V_x}{\partial y}\right)\left(\frac{\partial V_x}{\partial y}\right)\rho^{-1}$$

$V_x \sim u_e$ (order of velocity of smallest eddy)

$y \sim l_e$

$$\Rightarrow \dot{\epsilon} \sim \nu \frac{u_e^2}{l_e^2}$$

For smallest eddy $Re \sim 1$

$$\Rightarrow \frac{u_e l_e}{\nu} \sim 1 \quad \Rightarrow \quad u_e \sim \nu / l_e$$

$$\therefore \dot{\epsilon} \sim \frac{\nu^3}{l_e^4}$$

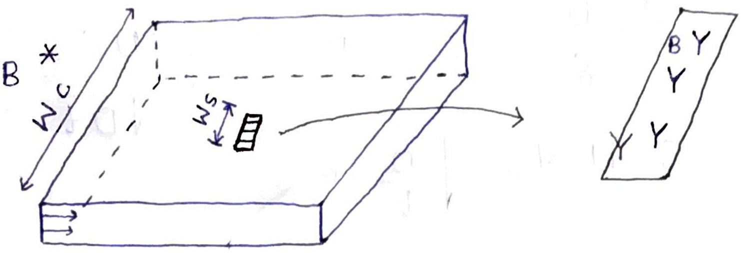
$\dot{E} \sim \dot{\epsilon}$ (breaking of eddy occurs till this point)

$$\frac{\bar{U}^3}{L} \sim \frac{\nu^3}{l_e^4}$$

$$l_e^4 \sim \frac{\nu^3 L}{\bar{U}^3}$$

$$l_e^{4} \sim \left(\frac{L}{\frac{\bar{U} L}{\nu}}\right)^{3} L \Rightarrow l_e = L\left(\frac{1}{Re}\right)^{3/4}$$

$A + B \longrightarrow AB^{*}$

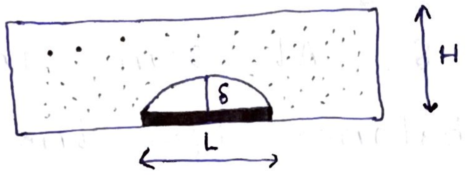


A

$C_o \longrightarrow$ bulk flow

Flux

$$j_D \sim D \frac{C_o}{\delta}$$



$$\delta \sim \sqrt{Dt}$$

$$\tau_D = \frac{\delta^2}{D} \quad \text{(Diffusional time scale)}$$

D : Depletion length

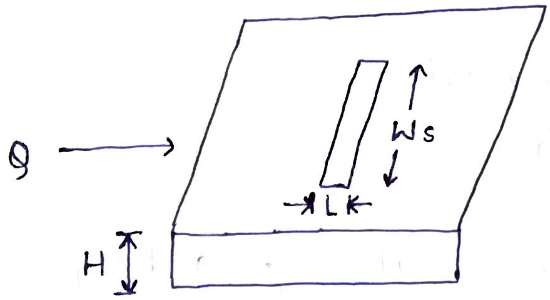
$$t \ll \frac{L^2}{D}, \quad \delta \ll H$$

$$t \ll \frac{H^2}{D}$$

$$J_D = j_D L W_s$$

A : Target molecule

B : Receptor



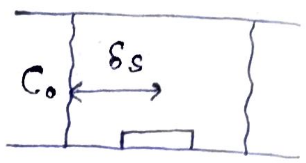
Reaction is instantaneous (transport ltd.)

C_o : bulk concentration of A

$$J_D = \frac{D C_o L W_s}{\delta} = \frac{D C_o L W_s}{\sqrt{Dt}} = C_o L W_s \sqrt{\frac{D}{t}}$$

Early time, channel dimensions has no effect on J_D.

$$t \gg \frac{H^2}{D}$$



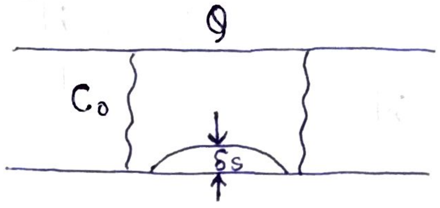
$$j_D = \frac{C_o D}{\delta} \; ; \quad J_D = \left(\frac{D C_o}{\sqrt{Dt}}\right) H W_c$$

$$J_D \sim C_o H W_c \sqrt{\frac{D}{t}} \qquad (t \gg H^2/D)$$

- Flow is present :

Diffusion increases the δ. Convection decreases the δ. At some instant, there is a balance b/w the two and δ will not change

$$J_{Diffusion} \sim \frac{D C_o H W_c}{\delta_s} \qquad J_{convection} = C_o Q$$



$$J_{diffusion} \approx J_{convection}$$

$$\frac{D \cancel{C_o} H W_c}{\delta_s} \approx \cancel{C_o} Q$$

$$\delta_s = \frac{D H W_c}{Q} \sim \frac{H}{\left(\frac{Q}{W_c D}\right)} \rightarrow Pe_H$$

$$Pe_H = \frac{\text{Diffusion time scale}}{\text{Convective time scale}} = \frac{H^2/D}{H/u}$$

$$Pe_H = \frac{H^2/D}{H/(Q/HW_c)} = \frac{Q}{D W_c}$$

$$\delta_s = \frac{H}{Pe_H} \qquad (\text{Valid for : } t \gg t_{con}$$