In this chapter, we start discussion on even and odd function. As mentioned earlier if the function is odd or even then the Fourier series takes a rather simple form of containing sine or cosine terms only. Then we discuss a very important topic of developing a desired Fourier series (sine or cosine) of a function defined on a finite interval by extending the given function as odd or even function.

7.1 Even and Odd Functions

A function is said to be an even about the point a if $f(a-x) = f(a+x)$ for all x and odd about the point a if $f(a-x) = -f(a+x)$ for all x. Further, note the following properties of even and odd functions:

a) The product of two even or two odd functions is again an even function.

b) The product of and even function and an odd function is an odd function.

Using these properties we have the following results for the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} \displaystyle\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, & \text{when } f \text{ is even function about } 0 \\ 0, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} 0, & \text{when } f \text{ is even function about } 0 \\ \displaystyle\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

From these observation we have the following results

7.1.1 Proposition

Assume that f is a piecewise continuous function on $[-\pi,\pi]$. Then

a) *If f is an even function then the Fourier series takes the simple form*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \qquad \textit{with} \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, n = 0, 1, 2, \ldots.$$

Such a series is called a cosine series.

b) *If f is an odd function then the Fourier series of f has the form*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \textit{with} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, n = 1, 2, \ldots.$$

Such a series is called a sine series.

7.2 Example Problems

7.2.1 Problem 1

Obtain the Fourier series to represent the function $f(x)$

$$f(x) = \begin{cases} x, & \textit{when } 0 \le x \le \pi \\ 2\pi - x, & \textit{when } \pi < x \le 2\pi \end{cases}$$

Solution: The given function is an even function about $x = \pi$ and therefore

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x = 0.$$

The coefficient a_0 will be calculated as

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\ \mathrm{d}x\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

The other coefficients a_n are given as

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\cos(nx)\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\cos(nx)\,\mathrm{d}x\right]$$

It can be further simplified as

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{when } n \text{ is even} \\ -\frac{4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \text{ where } 0 \le x \le 2\pi. \tag{7.1}$$

In this case as the function is continuous and f' is piecewise continuous, the series converges uniformly to $f(x)$ and we can write the equality (7.1).

7.2.2 Problem 2

Determine the Fourier Series of $f(x) = x^2$ on [-π, π] and hence find the value of the infinite series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: The function $f(x) = x^2$ is even on the interval [-π, π] and therefore b_n=0 for all n. The coefficient a_0 is given as

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} x^2\,\mathrm{d}x = \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

The other coefficients can be calculated by the general formula as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} 2x\sin(nx)\,\mathrm{d}x\right]$$

Again integrating by parts we obtain

$$a_n = \frac{4}{n\pi}\left[x\frac{\cos(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\cos(nx)}{n}\,\mathrm{d}x\right] = \frac{4}{n\pi}\left[\frac{\pi(-1)^n}{n} - 0\right] = \frac{4(-1)^n}{n^2}$$

Therefore the Fourier series is given as

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx) \quad \text{for} \quad x \in [-\pi, \pi]. \tag{7.2}$$

If we substitute $x = 0$ in the equation (7.2) we get

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

If we now substitute $x = \pi$ in the equation (7.2) we get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^2} \Rightarrow \frac{1}{4}\frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$