Controller Design based on State Space Model

State Space Model

State Variable:

A minimum set of variables which mathematically defines or captures the state of a dynamical system.

The dynamic behaviour of any system can be mathematically represented by State Space model. It is of two types:

1. Nonlinear State Space Model

$$\dot{x} = f(x) + g(x)u \qquad (1)$$

$$y = h(x, u) \qquad (2)$$

Eqn (1) is called the nonlinear state equation and eqn (2) is called nonlinear output map.

Here, $x = [x_1 \;\; x_2 \;\; x_3 \;\; \dots\dots \;\; x_n]^T \quad u = [u_1 \;\; u_2 \;\; \dots\dots u_m]^T$

$y = \left[y_1 \;\; y_2 \;\; y_3 \;\; \dots\dots y_p\right]^T$

State Space Model

2. Linear State Space model

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

Where, X, U and Y are the state variable vector, input vector and output vector in deviation form.

Here, $X = [X_1 \;\; X_2 \;\; X_3 \;\;\; \dots \dots X_n]^T \quad U = [U_1 \;\; U_2 \;\;\; \dots \dots U_m]^T$
$Y = \left[Y_1 \;\; Y_2 \;\; Y_3 \;\;\; \dots \dots Y_p\right]^T$

The matrices A and B are properties of the system and are determined by the system structure and elements.

The output equation matrices C and D are determined by the particular choice of output variables.

Development State Space Model

1. Develop dynamic model from 1st principles i.e,
$$\dot{x} = f(x, u) \quad \ldots (1)$$

2. Define control objective $y = h(x, u) \ldots\ldots(2)$
3. Rearrange equation (1) and (2) to get control-affine nonlinear state space form
$$\dot{x} = f(x) + g(x)u$$
$$y = h(x, u)$$
4. Linearize equation (1) and (2) around nominal values of x_s and u_s and form linear state space equation after subtracting steady state equation $f(x_{s,}\ u_s) = 0$; $h(x_{s,}\ u_s) = 0$

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

Dynamic model:

$$\dot{x} = f(x, u) \quad \dots (1)$$
$$y = h(x, u) \dots .. (2)$$

Using Taylor series approximation around (x^s , u^s) and neglecting HOT

$$\dot{x}_1 = f_1(x_1^s, x_2^s, \dots \dots . x_n^s, u_1^s, u_2^s, \dots .. u_m^s) +$$

$$\frac{\partial f_1}{\partial x_1}(x_1 - x_1^s) + \frac{\partial f_1}{\partial x_2}(x_2 - x_2^s) + \dots \dots + \frac{\partial f_1}{\partial x_n}(x_n - x_n^s)$$

$$+\frac{\partial f_1}{\partial u_1}(u_1 - u_1^s) + \frac{\partial f_1}{\partial u_2}(u - u_2^s) + \dots \dots + \frac{\partial f_1}{\partial u_m}(u_m - u_m^s)$$

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$$\dot{x}_n = f_n(x_1^s, x_2^s, \dots \dots . x_n^s, u_1^s, u_2^s, \dots .. u_m^s) +$$

$$\frac{\partial f_n}{\partial x_1}(x_1 - x_1^s) + \frac{\partial f_n}{\partial x_2}(x_2 - x_2^s) + \dots \dots + \frac{\partial f_n}{\partial x_n}(x_n - x_n^s)$$

$$+\frac{\partial f_n}{\partial u_1}(u_1 - u_1^s) + \frac{\partial f_n}{\partial u_2}(u - u_2^s) + \dots \dots + \frac{\partial f_n}{\partial u_m}(u_m - u_m^s)$$

Subtracting the steady state equations, we can write:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ . \\ . \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ X_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ . \\ . \\ U_m \end{bmatrix}$$

i.e,

$$\dot{X} = AX + BU$$

Where,

A= $\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ $a_{ij} = \frac{\partial f_i}{\partial x_j}$ *and B =* $\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$ $b_{ij} = \frac{\partial f_i}{\partial u_j}$

Similarly for equation (2),

$$y_1 = h_1(x_1^s, x_2^s, \dots \dots . x_n^s, u_1^s, u_2^s, \dots . . u_m^s) +$$

$$\frac{\partial h_1}{\partial x_1}(x_1 - x_1^s) + \frac{\partial h_1}{\partial x_2}(x_2 - x_2^s) + \ldots\ldots + \frac{\partial h_1}{\partial x_n}(x_n - x_n^s)$$

$$+ \frac{\partial h_1}{\partial u_1}(u_1 - u_1^s) + \frac{\partial h_1}{\partial u_2}(u - u_2^s) + \ldots\ldots + \frac{\partial h_1}{\partial u_m}(u_m - u_m^s)$$

……

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$$y_p = h_p(x_1^s, x_2^s, \dots \dots . x_n^s, u_1^s, u_2^s, \dots . . u_m^s) +$$

$$\frac{\partial h_p}{\partial x_1}(x_1 - x_1^s) + \frac{\partial h_p}{\partial x_2}(x_2 - x_2^s) + \ldots\ldots + \frac{\partial h_p}{\partial x_n}(x_n - x_n^s)$$

$$+ \frac{\partial h_p}{\partial u_1}(u_1 - u_1^s) + \frac{\partial h_p}{\partial u_2}(u - u_2^s) + \ldots\ldots + \frac{\partial h_p}{\partial u_m}(u_m - u_m^s)$$

Subtracting the steady state equations, we can write:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ Y_p \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ X_n \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ . \\ . \\ U_m \end{bmatrix}$$

i.e,

$$Y = CX + DU$$

Where,

C= $\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}$ $c_{ij} = \frac{\partial h_i}{\partial x_j}$ *and D* = $\begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}$ $d_{ij} = \frac{\partial h_i}{\partial u_j}$

Linear State-space model to Transfer function model

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

Taking Laplace Transform,

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Or,

$$(sI - A)X(s) = BU(s); i.e, X(s) = (sI - A)^{-1}BU(s)$$

So,

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

Normally transfer function model is expressed in terms of process and disturbance transfer function. So, input variables U are partitioned to manipulated M and load/disturbance variable L i.e, U = [M L]

$$Y(s) = [C(sI - A)^{-1}B_M + D_M]M(s) + [C(sI - A)^{-1}B_L + D_L]L(s)$$

$$Y(s) = G_P(s)M(s) + G_L(s)L(s)$$

Example: Van De Vusse Reactor

Dynamic Model:

$$\frac{dC_A}{dt} = \frac{F}{V}\left(C_{Af} - C_A\right) - k_1 C_A \; - k_3 C_A^2 = f_1(C_A, C_B, {}^F\!/_V)$$

$$\frac{dC_B}{dt} = -\frac{F}{V} C_B + k_1 C_A \; - k_2 C_B = f_2(C_A, C_B, {}^F\!/_V)$$

Non-linear state space model:

States are : $x_1 = C_A$, $x_2 = C_B$ input : u = F/V and output $y = C_B$

State equation:

$$\frac{dx_1}{dt} = (-k_1 x_1 - k_3 x_1^2) + \left(x_{1f} - x_1\right)u$$

$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2 + (-x_2)u$$

i.e, $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{Bmatrix} (-k_1 x_1 - k_3 x_1^2) \\ k_1 x_1 - k_2 x_2 \end{Bmatrix} + \begin{Bmatrix} \left(x_{1f} - x_1\right) \\ (-x_2) \end{Bmatrix} \mathrm{u} = \mathrm{f(x) + g(x)u}$

Output map: $y = h(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example: Van De Vusse Reactor

Linear State Space Model:

$$f_1(C_A, C_B, {}^F\!/_V) = \frac{F}{V}\left(C_{Af} - C_A\right) - k_1 C_A - k_3 C_A^2$$

$$f_2(C_A, C_B, {}^F\!/_V) = -\frac{F}{V} C_B + k_1 C_A - k_2 C_B$$

Let, $X_1 = C_A - C_A^s$; $X_2 = C_B - C_B^s; U = \frac{F}{V} - \frac{F^s}{V}$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial C_A} & \frac{\partial f_1}{\partial C_B} \\ \frac{\partial f_2}{\partial C_A} & \frac{\partial f_2}{\partial C_B} \end{bmatrix} = \begin{bmatrix} -{}^{F^s}\!/_V - k_1 - 2k_3 C_A^s & 0 \\ k_1 & -{}^{F^s}\!/_V - k_2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial (\frac{F}{V})} \\ \frac{\partial f_2}{\partial (\frac{F}{V})} \end{bmatrix} = \begin{bmatrix} C_{Af} - C_A^s \\ -C_B^s \end{bmatrix}$$

$$C = [0 \quad 1] \quad D = 0$$

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Example: Van De Vusse Reactor

Transfer Domain Model -- Laplace Domain:

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) = [C(sI - A)^{-1}B]U(s)$$

$$sI - A = \begin{bmatrix} s + {F^s}/{V} + k_1 + 2k_3C_A^s & 0 \\ -k_1 & s + {F^s}/{V} + k_2 \end{bmatrix} = \begin{bmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{det}\begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix}$$

where, $det = (s - a_{11})(s - a_{22}) - a_{12}a_{21}$

$$C(sI - A)^{-1} = [0 \quad 1]\frac{1}{det}\begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} = \frac{1}{det}[a_{21} \quad s - a_{11}]$$

$$C(sI - A)^{-1}\mathrm{B} = \frac{1}{det}[a_{21} \quad s - a_{11}]\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{a_{21}b_1 + (s - a_{11})b_2}{(s - a_{11})(s - a_{22}) - a_{12}a_{21}}$$

Example: Van De Vusse Reactor

Transfer Domain Model -- Laplace Domain:

$$C(sI-A)^{-1}\mathrm{B} = \frac{a_{21}b_1 + (s-a_{11})b_2}{(s-a_{11})(s-a_{22}) - a_{12}a_{21}}$$

$$= \frac{k_1\left(C_{Af} - C_A^s\right) - C_B^s s - C_B^s\left({F^s}/{V} + k_1 + 2k_3C_A^s\right)}{(s + {F^s}/{V} + k_1 + 2k_3C_A^s)(s + {F^s}/{V} + k_2)}$$

$$= \frac{-C_B^s s + [k_1\left(C_{Af} - C_A^s\right) - C_B^s\left({F^s}/{V} + k_1 + 2k_3C_A^s\right)]}{(s + {F^s}/{V} + k_1 + 2k_3C_A^s)(s + {F^s}/{V} + k_2)}$$

State Space realization from Transfer Function

- Process of converting transfer function to state space form is not unique.
- Various realizations possible
- All realizations are equivalent
- One realization may have some advantages over others for a particular task

Possible realizations :

- ➢First Companion form (Controllable Canonical Form)
- ➢Jordan Canonical form
- ➢Alternate first companion form (Toeplitz form)
- ➢Second Companion form (Observable canonical form)

Controllable Canonical Form

Consider Laplace domain transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots\ldots\ldots + a_{n-1}s + a_n}$$

$$= \frac{y(s)}{z(s)}\frac{z(s)}{u(s)} = \left(b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1}s + b_n\right)\left(\frac{1}{s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n}\right)$$

$$Considering, \frac{z(s)}{u(s)} = \frac{1}{s^n + a_1 s^{n-1} + \ldots\ldots\ldots + a_{n-1}s + a_n}$$

$$i.e, \quad \frac{d^n z}{dt^n} + a_1 \frac{d^{n-1}z}{dt^{n-1}} + \ldots\ldots\ldots + a_{n-1}\frac{dz}{dt} + a_n z = u$$

Controllable Canonical Form

Let us choose the states x_1 to x_n as,

$$x_1 = z; x_2 = \frac{dz}{dt}; x_3 = \frac{d^2 z}{dt^2} ......; \quad x_n = \frac{d^{n-1} z}{dt^{n-1}}$$

Therefore, the state equations are:

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = x_3;$$
$$.$$
$$.$$
$$\dot{x}_{n-1} = x_n;$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - ...... - a_2 x_{n-1} - a_1 x_n + u;$$

Controllable Canonical Form

Now for the output map,

$$\frac{y(s)}{z(s)} = b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_{n-1} s + b_n$$

$$\begin{aligned}
i.e, y &= b_0 \frac{d^n z}{dt^n} + b_1 \frac{d^{n-1} z}{dt^{n-1}} + \ldots\ldots + b_{n-1} \frac{dz}{dt} + b_n z \\
&= b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \ldots\ldots + b_{n-1} x_2 + b_n x_1 \\
&= b_0 \left(-a_n x_1 - a_{n-1} x_2 - \ldots\ldots - a_2 x_{n-1} - a_1 x_n \right) \\
&\quad + b_1 x_n + b_2 x_{n-1} + \ldots\ldots + b_{n-1} x_2 + b_n x_1 + b_0 u \\
&= \left(b_n - a_n b_0 \right) x_1 + \left(b_{n-1} - a_{n-1} b_0 \right) x_2 + \ldots\ldots \\
&\quad + \left(b_2 - a_2 b_0 \right) x_{n-1} + \left(b_1 - a_1 b_0 \right) x_n + b_0 u
\end{aligned}$$

Controllable Canonical Form

So, in standard vector-matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & . & . & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + [b_0] u$$

Jordan Canonical Form

Consider Laplace domain transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \dots\dots + a_{n-1}s + a_n}$$

$$= b_0 + \frac{r_1}{(s-\lambda_1)} + \frac{r_2}{(s-\lambda_2)} + \dots\dots + \frac{r_n}{(s-\lambda_n)}$$

Therefore,

$$y(s) = b_0 u(s) + \frac{r_1 u(s)}{(s-\lambda_1)} + \frac{r_2 u(s)}{(s-\lambda_2)} + \dots\dots + \frac{r_n u(s)}{(s-\lambda_n)}$$

$$= b_0 u(s) + r_1 x_1(s) + r_2 x_2(s) + \dots\dots + r_n x_n(s)$$

Where, $x_1, x_2 \dots\dots x_n$ are considered as the states of the system

Jordan Canonical Form

Therefore, the state equations are

$$x_1(s) = \frac{u(s)}{(s-\lambda_1)} \qquad \dot{x}_1 = \lambda_1 x_1 + u$$

$$x_2(s) = \frac{u(s)}{(s-\lambda_2)} \qquad \dot{x}_2 = \lambda_2 x_2 + u$$

....

$$x_n(s) = \frac{u(s)}{(s-\lambda_n)} \qquad \dot{x}_n = \lambda_n x_n + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & & & \\ 0 & 0 & \lambda_3 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_4 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} u$$

and output map

$$y = r_1 x_1 + r_2 x_2 + \ldots\ldots + r_n x_n + b_0 u$$

What will happen in case of repeated roots?

Jordan Canonical Form (repeated roots)

For repeated roots, the partial fraction expression may be written as

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots\dots + a_{n-1} s + a_n}$$

$$= b_0 + \frac{r_{11}}{(s-\lambda_1)^3} + \frac{r_{12}}{(s-\lambda_1)^2} + \frac{r_{13}}{(s-\lambda_1)} + \frac{r_4}{(s-\lambda_2)} + \dots\dots + \frac{r_n}{(s-\lambda_n)}$$

Therefore,

$$y(s) = b_0 u(s) + \frac{r_{11}\, u(s)}{(s-\lambda_1)^3} + \frac{r_{12}\, u(s)}{(s-\lambda_1)^2} + \frac{r_{13}\, u(s)}{(s-\lambda_1)} + \frac{r_4\, u(s)}{(s-\lambda_2)} + \dots\dots + \frac{r_n\, u(s)}{(s-\lambda_n)}$$

$$= b_0 u(s) + r_{11} x_1(s) + r_{12} x_2(s) + r_{13} x_3(s) + r_4 x_4(s) + \dots\dots + r_n x_n(s)$$

Jordan Canonical Form (repeated roots)

Now the state equations are

$$x_1(s) = \frac{x_2(s)}{(s-\lambda_1)} \qquad \dot{x}_1 = \lambda_1 x_1 + x_2$$

$$x_2(s) = \frac{x_3(s)}{(s-\lambda_1)} \qquad \dot{x}_2 = \lambda_1 x_2 + x_3$$

$$x_3(s) = \frac{u(s)}{(s-\lambda_1)} \qquad \dot{x}_3 = \lambda_1 x_3 + u$$

$$x_4(s) = \frac{u(s)}{(s-\lambda_2)} \qquad \dot{x}_4 = \lambda_2 x_4 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & . & . & 0 \\ 0 & \lambda_1 & 1 & & & \\ 0 & 0 & \lambda_1 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_2 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} u$$

....

$$x_n(s) = \frac{u(s)}{(s-\lambda_n)} \qquad \dot{x}_n = \lambda_n x_n + u$$

Output map

$$y = r_{11}x_1 + r_{12}x_2 + r_{13}x_3 + r_4 x_4 + ... + r_n x_n + b_0 u$$

Alternate Canonical form (Toeplitz form)

Consider the transfer function as

$$g(s)=\frac{y(s)}{u(s)}=\frac{b_0 s^n+b_1 s^{n-1}+......+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+.......+a_{n-1}s+a_n}$$

$$i.e,\quad \frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+.......+a_{n-1}\frac{dy}{dt}+a_n y=b_0\frac{d^n u}{dt^n}+b_1\frac{d^{n-1}u}{dt^{n-1}}+.......+b_{n-1}\frac{du}{dt}+b_n u$$

Define State equations and output map as the following:

$$y=x_1+p_0 u$$

$$\dot{x}_1=x_2+p_1 u$$

$$\dot{x}_2=x_3+p_2 u$$

$$......$$

$$\dot{x}_{n-1}=x_n+p_{n-1}u$$

$$\dot{x}_n=-a_1 x_n-a_2 x_{n-1}-......-a_n x_1+p_n u$$

Alternate Canonical form (Toeplitz form)

From the above definition, we can write

$$y = x_1 + p_0 u$$

$$\dot{y} = \dot{x}_1 + p_0 \dot{u} = x_2 + p_1 u + p_0 \dot{u}$$

$$\ddot{y} = \dot{x}_2 + p_1 \dot{u} + p_0 \ddot{u} = x_3 + p_2 u + p_1 \dot{u} + p_0 \ddot{u}$$

....

$$\frac{d^{n-1}y}{dt^{n-1}} = x_n + p_{n-1}u + p_{n-2}\dot{u} + \ldots.. + p_1 \frac{d^{n-2}u}{dt^{n-2}} + p_0 \frac{d^{n-1}u}{dt^{n-1}}$$

$$\frac{d^n y}{dt^n} = \dot{x}_n + p_{n-1}\dot{u} + p_{n-2}\ddot{u} + \ldots.. + p_1 \frac{d^{n-1}u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n}$$

$$= -a_1 x_n - a_2 x_{n-1} - \ldots\ldots - a_n x_1 + p_n u + p_{n-1}\dot{u} + p_{n-2}\ddot{u} + \ldots.. + p_1 \frac{d^{n-1}u}{dt^{n-1}} + p_0 \frac{d^n u}{dt^n}$$

Alternate Canonical form (Toeplitz form)

Therefore,

$$\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+.......+a_{n-1}\frac{dy}{dt}+a_n y$$

$$=\left(-a_1x_n-a_2x_{n-1}-......-a_nx_1+p_nu+p_{n-1}\dot{u}+p_{n-2}\ddot{u}+.....+p_1\frac{d^{n-1}u}{dt^{n-1}}+p_0\frac{d^n u}{dt^n}\right)$$

$$+a_1\left(x_n+p_{n-1}u+p_{n-2}\dot{u}+.....+p_1\frac{d^{n-2}u}{dt^{n-2}}+p_0\frac{d^{n-1}u}{dt^{n-1}}\right)+........$$

$$+a_{n-1}\left(x_2+p_1u+p_0\dot{u}\right)+a_n(x_1+p_0u)$$

$$=(p_n+a_1p_{n-1}+......+a_{n-1}p_1+a_np_0)u+(p_{n-1}+a_1p_{n-2}+......+a_{n-2}p_1+a_{n-1}p_0)\dot{u}+.....$$

$$+(p_1+a_1p_0)\frac{d^{n-1}u}{dt^{n-1}}+p_0\frac{d^n u}{dt^n}$$

$$=b_0\frac{d^n u}{dt^n}+b_1\frac{d^{n-1}u}{dt^{n-1}}+.......+b_{n-1}\frac{du}{dt}+b_nu$$

Alternate Canonical form (Toeplitz form)

Equating the coefficients of $u, \dot{u}, \ldots, \dfrac{d^{n-1}u}{dt^{n-1}}, \dfrac{d^n u}{dt^n}$

$$b_0 = p_0;$$

$$b_1 = p_1 + a_1 p_0;$$

......

$$b_{n-1} = p_{n-1} + a_1 p_{n-2} + \ldots\ldots + a_{n-2} p_1 + a_{n-1} p_0$$

$$b_n = p_n + a_1 p_{n-1} + \ldots\ldots + a_{n-1} p_1 + a_n p_0$$

In vector-Matrix form,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & . & . & 0 & 0 \\ a_1 & 1 & 0 & . & . & 0 & 0 \\ a_2 & a_1 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n-1} & a_{n-2} & . & . & . & 1 & 0 \\ a_n & a_{n-1} & a_{n-2} & . & . & a_1 & 1 \end{bmatrix}}_{Toeplitz\ \ Matrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ . \\ . \\ p_{n-1} \\ p_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ b_{n-1} \\ b_n \end{bmatrix}$$

2^{nd} Companion form (Observer Canonical form)

Consider the transfer function

$$g(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1} s + b_n}{\left(s^n + a_1 s^{n-1} + \dots\dots + a_{n-1} s + a_n\right)}$$

$$i.e, \quad \left(s^n + a_1 s^{n-1} + \dots\dots + a_{n-1} s + a_n\right) y(s) = \left(b_0 s^n + b_1 s^{n-1} + \dots\dots + b_{n-1} s + b_n\right) u(s)$$

Rearranging the terms,

$$s^n \left[y(s) - b_0 u(s)\right] + s^{n-1}\left[a_1 y(s) - b_1 u(s)\right] + \dots\dots + \left[a_n y(s) - b_n u(s)\right] = 0$$

Simplify :

$$y(s) - b_0 u(s) = \frac{1}{s}\left[b_1 u(s) - a_1 y(s)\right] + \frac{1}{s^2}\left[b_2 u(s) - a_2 y(s)\right] + \dots\dots + \frac{1}{s^n}\left[b_n u(s) - a_n y(s)\right]$$

$$y(s) = b_0 u(s) + \frac{1}{s}\left[b_1 u(s) - a_1 y(s)\right] + \frac{1}{s^2}\left[b_2 u(s) - a_2 y(s)\right] + \dots\dots + \frac{1}{s^n}\left[b_n u(s) - a_n y(s)\right]$$

2^{nd} Companion form (Observer Canonical form)

Rearranging the terms,

$$y(s) = b_0 u(s) + \underbrace{\frac{1}{s}\left[\left[b_1 u(s) - a_1 y(s) \right] + \underbrace{\frac{1}{s}\left[\left[b_2 u(s) - a_2 y(s) \right] + \ldots.. + \underbrace{\frac{1}{s}\left[b_n u(s) - a_n y(s) \right]}_{x_n(s)} \right]}_{x_2(s)} \right]}_{x_1(s)}$$

The equations now can be written as

$$y = x_1 + b_0 u$$
$$\dot{x}_1 = x_2 - a_1 y + b_1 u = -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u$$
$$\dot{x}_2 = x_3 - a_2 y + b_2 u = -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$$

.....

.....

$$\dot{x}_{n-1} = x_n - a_{n-1} y + b_{n-1} u = -a_{n-1} x_1 + x_n + (b_{n-1} - a_{n-1} b_0) u$$
$$\dot{x}_n = -a_n y + b_n u = -a_n x_1 + (b_n - a_n b_0) u$$

2^{nd} Companion form (Observer Canonical form)

In vector Matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & . & . & 0 \\ -a_2 & 0 & 1 & & & \\ -a_3 & 0 & 0 & 1 & . & 0 \\ . & 0 & 0 & 0 & . & 0 \\ . & . & . & . & . & . \\ -a_{n-1} & . & . & . & . & 1 \\ -a_n & 0 & 0 & . & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ . \\ . \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

2^{nd} Companion form (Observer Canonical form)

On the other hand, if we formulate

$$y(s) = b_0 u(s) + \underbrace{\frac{1}{s}\left[\left[b_1 u(s) - a_1 y(s)\right] + \underbrace{\frac{1}{s}\left[\left[b_2 u(s) - a_2 y(s)\right] + \ldots\ldots + \underbrace{\frac{1}{s}\left[b_n u(s) - a_n y(s)\right]}_{x_1(s)} \right]}_{x_{n-1}(s)} \right]}_{x_n(s)}$$

Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . & . & -a_n \\ 1 & 0 & 0 & 0 & 0 & -a_{n-1} \\ 0 & 1 & 0 & 0 & . & -a_{n-2} \\ 0 & 0 & 1 & 0 & . & -a_{n-3} \\ . & . & . & . & . & . \\ 0 & . & . & . & . & -a_2 \\ 0 & 0 & 0 & . & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ b_{n-2} - a_{n-2} b_0 \\ . \\ . \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$