<u>Assumptions of incompressible flow in natural convection</u>

The flow is assumed to be incompressible only if the total rate of the volumetric strain of the fluid element is zero.

Total rate of volumetric strain is defined as $\frac{D\forall}{Dt}$.

$\frac{D\forall}{Dt} = \vec{\nabla}.\vec{v}$ [Can be proved considering liner deformation of a fluid element]

Let us consider a fluid element (presented in Figure 1) and consider first the front plane only linearly deforms over time *Δt.*

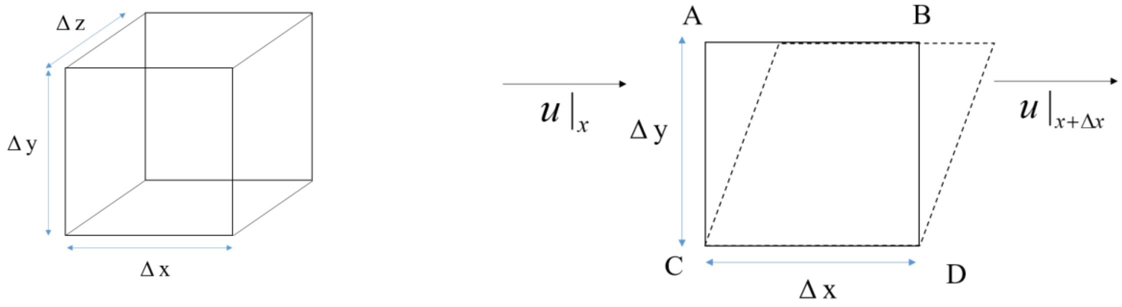

Figure 1. Linear deformation of a fluid element

Change in length of face AB = $u\,|_x \times \Delta t$

Change in length of face BD = $u\,|_{x+\Delta x} \times \Delta t$

Net change in length = $[u\,|_x + \frac{\partial u}{\partial x}\Delta x - u\,|_x] \times \Delta t$

$= \frac{\partial u}{\partial x}\Delta x \Delta t$.

Strain along length = Change in the length / length = $\frac{\partial u}{\partial x}\Delta x \Delta t \,/\, \Delta x = \frac{\partial u}{\partial x}\Delta t$.

So, rate of strain along $x = \frac{\partial u}{\partial x}$.

Similarly, rate of strain along y = $\frac{\partial v}{\partial y}$ and, rate of strain along z = $\frac{\partial w}{\partial z}$.

New length along x = Δx + net change in length along x = $\Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t = \Delta x\left(1+\frac{\partial u}{\partial x}\Delta t\right)$.

New length along $y = \Delta y\left(1+\frac{\partial v}{\partial y}\Delta t\right)$.

New length along $z = \Delta z(1+\frac{\partial w}{\partial z}\Delta t)$.

Total change in volume = New volume – Old volume

$$= \Delta x\Delta y\Delta z\left[\left(1+\frac{\partial u}{\partial x}\Delta t\right)\left(1+\frac{\partial v}{\partial y}\Delta t\right)\left(1+\frac{\partial w}{\partial z}\Delta t\right)\right]-\Delta x\Delta y\Delta z$$

$$= \Delta x\Delta y\Delta z\left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\Delta t\right] + \text{higher order terms}$$

$\therefore$ Total rate of change in volume = $\Delta x\Delta y\Delta z\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]$

$\therefore$ Total strain rate = $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \vec{\nabla}.\vec{v}$

This is equal to the total change in fluid volume per unit volume.

i.e., $\frac{1}{\forall}\frac{D\forall}{Dt} = \vec{\nabla}.\vec{v}$

So, when the fluid element in a flow field will have zero total volumetric strain, the flow will be designated as incompressible flow.

i.e., $\frac{1}{\forall}\frac{D\forall}{Dt} = \vec{\nabla}.\vec{v} = 0$.

From the conservation of mass, we can write:

$m = \rho \times \forall$

$\ln m = \ln \rho + \ln \forall$ (Taking log on both sides)

$\frac{1}{m}\frac{Dm}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\forall}\frac{D\forall}{Dt}$ (Taking total derivatives on both sides)

So $0 = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\forall}\frac{D\forall}{Dt}$ (Conservation of total mass)

The second term denotes the total rate of volumetric strain $(= \nabla.\vec{\mathrm{v}})$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}.\vec{v} = 0$$

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla}.\vec{v}) = 0$$

Now the condition for incompressible flow to occur

$$\frac{1}{\forall}\frac{D\forall}{Dt} = \vec{\nabla}.\vec{v} = 0$$

So $\frac{D\rho}{Dt} = 0$ That is the condition for incompressible flow that we get from the continuity equation.

$\frac{d\rho}{dt} + u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} = 0$, for an incompressible flow.

Keeping density constant is a special case of incompressible flow.

So, in case of natural convection the flow is assumed to be incompressible but the density may vary with space and time.