① 4D Parabolic PDE.

(see photos)

(2) Solution of elliptical PDEs

$$\nabla^2 u = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

B.C.

$$\begin{cases} u(x=0) = u_0 \\ u(x=1) = 0 \\ u(y=0) = 0 \\ u(y=1) = 0 \end{cases}$$

- Eigenvalue problem exists in y

- Separate variables

$$u = X(x) \cdot Y(y)$$

$$\therefore Y \cdot \frac{d^2X}{dx^2} + X \cdot \frac{d^2Y}{dy^2} = 0$$

$$\therefore \frac{1}{X} \cdot \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} = 0$$

$$\therefore -\frac{1}{X} \cdot \frac{d^2X}{dx^2} = +\frac{1}{Y} \cdot \frac{d^2Y}{dy^2} = -\alpha_n^2$$

$$\frac{Y}{} \quad \frac{d^2Y}{dy^2} + \alpha_n^2 \cdot Y = 0$$

$$Y(y=0) = 0$$

$$Y(y=1) = 0 .$$

$$\Rightarrow \quad \alpha_n = n\pi$$

$$n = 1, 2, 3, 4 \ldots\ldots$$

$$Y_n = C_1 \cdot \sin(n\pi y) .$$

$$-\frac{1}{X} \cdot \frac{d^2X}{dx^2} = -\alpha^2$$

$$\therefore \quad \frac{d^2X}{dx^2} - \alpha^2 X = 0$$

$$\therefore \quad X_n = C_2 \cdot e^{\alpha_n \cdot x} + C_3 \cdot e^{-\alpha_n \cdot x}$$

$$X_n(x=1) = 0$$

$$\Rightarrow \quad 0 = C_2 e^{\alpha_n} + C_3 \cdot e^{-\alpha_n}$$

$$\Rightarrow \quad C_2 = -C_3 \cdot e^{-2\alpha_n}$$

$$\Rightarrow \quad C_3 = C_2 \cdot e^{2\alpha_n}$$

$$\therefore \quad X_n = C_2\left(e^{\alpha_n \cdot x} + e^{2\alpha_n} \cdot e^{-\alpha_n \cdot x}\right)$$

$$\therefore \quad X_n = C_2 \cdot e^{\alpha_n}\left(e^{-\alpha_n} \cdot e^{\alpha_n \cdot x} - e^{\alpha_n} \cdot e^{-\alpha_n x}\right)$$

$$= C_2 \cdot e^{\alpha_n}\left(e^{-(1-x)\cdot \alpha_n} - e^{(1-x)\alpha_n}\right)$$

$$= -2 \cdot C_2 \cdot e^{\alpha_n} \cdot \sinh\left(\alpha_n(1-x)\right)$$

$$\therefore \quad X_n = C_2' \cdot e^{\alpha_n} \cdot \sinh\left(\alpha_n(1-x)\right)$$

- $$u(x,y) = \sum_{n=1}^{\infty} X_n \cdot Y_n$$

$$= \sum_{n=1}^{\infty} C_n \cdot e^{\alpha_n} \cdot \sinh\left(\alpha_n(1-x)\right) \cdot \sin(\alpha_n y)$$

- B.C. $u(0,y) = u_0$

$$\therefore \quad u_0 = \sum_{n=1}^{\infty} C_n \cdot e^{\alpha_n} \cdot \sinh(\alpha_n) \cdot \sin(\alpha_n y)$$

- multiply both sides by $\sin(n\pi y)$ & integrate w.r.t y.

$$\therefore \quad u_0 \int_0^1 \sin(n\pi y)\, dy = C_n \cdot e^{\alpha_n} \cdot \sinh(\alpha_n)$$

③ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x=0) = 0$

$u(x=1) = 0$

$u(y=0) = 0$

$u(y=1) = 1.$