10.1 Dirac-Delta Function

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behaviour is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi_\epsilon^a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1/\epsilon & \text{if } a \le t < a+\epsilon, \\ 0 & \text{if } a+\epsilon \le t. \end{cases}$$

Let us take the Laplace transform of a square pulse,

$$L\left[\varphi_\epsilon^a(t)\right] = \int_0^\infty e^{-st}\varphi_\epsilon(t)\,\mathrm{d}t$$

Substituting the value of the function we obatin

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,\mathrm{d}t$$

On integration we get

$$L\left[\varphi_\epsilon^a(t)\right] = \frac{e^{-sa}}{s\epsilon}\left[1 - e^{-s\epsilon}\right]$$

We generally want ϵ to be very small. That is, we wish to have the pulse be very short and very tall. By letting ϵ go to zero we arrive at the concept of the *Dirac delta function*, $\delta(t-a)$. Thus, the Dirac-Delta can be thought as the limiting case of $\varphi_\epsilon(t)$ as $\epsilon \to 0$

$$\delta(t-a) = \lim_{\epsilon\to 0}\varphi_\epsilon^a(t)$$

So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, the Dirac-delta function is defined as having the following properties:

(i) $\delta(t-a) = 0, \;\; \forall t, \; t \neq a$

(ii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)\,dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } a, \text{ i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) for any interval $[c, d]$

$$\int_c^d \delta(t-a)f(t)\,dt = \begin{cases} f(a) & \text{if the interval } [c,d] \text{ contains } a, \text{ i.e. } c \le a \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)\, dt = e^{-as}$$

In particular,

$$L\left[\delta(t)\right] = 1.$$

Remark: *Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice*

$$\mathcal{L}\big[u(t-a)\big] = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{,,}\ \frac{d}{dt}\big[u(t-a)\big] = \delta(t-a)\ \text{''}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

10.1.1 Example

Compute $L^{-1}\left[\frac{s+1}{s}\right]$.

Solution: We write,

$$L^{-1}\left[\frac{s+1}{s}\right] = L^{-1}\left[1+\frac{1}{s}\right] = L^{-1}\left[1\right] + L^{-1}\left[\frac{1}{s}\right] = \delta(t) + 1.$$

The resulting object is a generalized function which makes sense only when put under an integral.

10.2 Bessel's Functions

The Bessel's functions of order n (of first kind) is defined as

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{t}{2}\right)^{n+2r}.$$

This Bessel's function is a solution of the Bessel's equation of order n

$$y^{(n)} + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right) y = 0$$

The Bessel's functions of order 0 and 1 are given as

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots$$

and

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2 4} + \frac{t^5}{2^2 4^2 6} + \dots$$

Note that $J_0'(t) = -J_1(t)$.

10.2.1 Example

Find the Laplace transform of $J_0(t)$ and $J_1(t)$.

Solution: Taking Laplace transform of the $J_0(t)$ we have

$$L[J_0(t)] = L\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \dots\right]$$

Using linearity of the Laplace transform we get

$$L[J_0(t)] = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 4^2}\frac{4!}{s^5} - \frac{1}{2^2 4^2 6^2}\frac{6!}{s^7} + \dots$$

This can be rewritten as

$$L[J_0(t)] = \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{3}{4}\frac{1}{s^4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{s^6} + \dots\right]$$

With Binomial expansion we can write

$$L[J_0(t)] = \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2} = \frac{1}{\sqrt{1+s^2}}$$

Further note that $L[J_1(t)] = -L[J_0'(t)]$ and therefore using the derivative theorem we find

$$L[J_1(t)] = -sL[J_0(t)] + J_0(0)] = 1 - sL[J_0(t)], \text{ since } J_0(0) = 1$$

Hence, we obtain

$$L[J_1(t)] = 1 - \frac{s}{\sqrt{1+s^2}}$$

10.3 Laguerre Polynomials

Laguerre polynomials are defined as

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right),\ n = 0, 1, 2, \ldots$$

The Laguerre polynomials are solutions of Laguerre's differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0,\ n = 0, 1, 2, \ldots$$

10.3.1 Example

Show that $L[L_n(t)] = \dfrac{(s-1)^n}{s^{n+1}}$

Solution: By definition of the Laplace transform we have

$$\begin{aligned} L[L_n(t)] &= \int_0^\infty e^{-st}\frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \\ &= \frac{1}{n!}\int_0^\infty e^{-(s-1)t}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)\,\mathrm{d}t \end{aligned}$$

Integrating by parts, we find

$$L[L_n(t)] = \frac{1}{n!}\left[e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\Big|_0^\infty + (s-1)\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Noting that each term in $\frac{d^{n-1}}{dt^{n-1}}$ contains some integral power of t so that it vanishes as $t \to 0$ and $e^{-(s-1)t}$ vanishes for $t \to \infty$ provided $s > 1$. Thus, we have

$$L[L_n(t)] = \frac{s-1}{n!}\left[\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\,\mathrm{d}t\right]$$

Repeated use of integration by parts leads to

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\left[\int_0^\infty e^{-(s-1)t}e^{-t}t^n\,\mathrm{d}t\right] = \frac{(s-1)^n}{n!}L[t^n]$$

Hence, we get

$$L[L_n(t)] = \frac{(s-1)^n}{n!}\frac{n!}{s^{n+1}} = \frac{(s-1)^n}{s^{n+1}}$$