Controller Design based on State Space Model

Controllability

- A system is said to be *controllable* at time t_0 if it is possible by means of an ***unconstrained control vector*** to transfer the system from any initial state X_0 to any other state *in a **finite interval of time***
- Controllability depends upon the system matrix *A* and the control influence matrix *B*

Condition for Controllability: (single input case)

System: $$\dot{X} = AX + Bu$$

Solution: $$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

Assuming $X(t_1) = 0,$

$$0 = e^{At_1} X(0) + \int_0^{t_1} e^{A(t_1-\tau)} B\, u(\tau) d\tau$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau$$

Condition for Controllability: (single input case)

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)\, A^k \quad (\text{Sylvester's formula})$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$$

$$= -\sum_{k=0}^{n-1} A^k B \beta_k \qquad \text{where} \quad \beta_k \triangleq \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$$

$$= -\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$$

This system should have a non-trivial solution for $\begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$

Condition for Controllability (multiple input case)

The result obtained for single input case can be easily extended for m-dimensional control input U.

System: $\dot{X} = AX + BU$

$$X(0) = -\int_0^{t_1} e^{-A\tau} BU(\tau)d\tau$$

$$= [B \;\; AB \;\; \dots \;\; A^{n-1}B][\boldsymbol{\beta}_0 \;\; \boldsymbol{\beta}_1 \;\; \dots \;\; \boldsymbol{\beta}_{n-1}]^T$$

Condition for Controllability (multiple input case)

$$X(0) = [B \;\; AB \;\; \dots \;\; A^{n-1}B][\boldsymbol{\beta_0} \;\; \boldsymbol{\beta_1} \;\; \dots \;\; \boldsymbol{\beta_{n-1}}]^T$$

For completely state controllability, this equation must be satisfied. This requires the rank of

$$C_B = [B \;\; AB \;\; \dots \;\; A^{n-1}B]$$

Should be n.

Controllability

Result: If the rank of $C_B \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is n, then the system is controllable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$C_B = \begin{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$$

$rank\left(C_B\right) = 2 \quad \therefore$ The system is controllable.

Output Controllability

Result:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

$$X \in \mathbb{R}^n, \quad U \in \mathbb{R}^m, \quad Y \in \mathbb{R}^p$$

If the rank of $C_B \triangleq \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B & D \end{bmatrix}$ is p,

then the system is output controllable.

Note: The presence of DU term in the output equation always helps to establish output controllability.

Observability

- A system is said to be *observable* at time t_0 if, with the system in state $X(t_0)$,it is possible to determine this state from the observation of the output over a finite interval of time
- Observability depends upon the system matrix A and the output matrix C

Condition for Observability

System: $\dot{X} = AX$

$Y = CX$

Output: $Y(t) = Ce^{At}X(0)$

Sylvester Formula: $e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)A^k$

$So,$ $Y(t) = \sum_{k=0}^{n-1} \alpha_k(t)CA^kX(0)$

Condition for Observability

Expanding,

$$Y(t) = \alpha_0(t)CX(0) + \alpha_1(t)CAX(0) + \cdots + \alpha_{n-1}(t)CA^{n-1}X(0)$$

If the system is completely observable, then given Y(t) over a time interval $0 \leq t \leq t_1$, X(0) should be determined from above equation. This requires the following [np x n] matrix,

$$O_B = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C^T & A^T C^T & \ldots & (A^T)^{n-1} C^T \end{bmatrix}$$

must have rank n

Observability

Result: If the rank of $O_B \triangleq \begin{bmatrix} C^T & A^T C^T & \cdots & \left(A^T\right)^{n-1} C^T \end{bmatrix}$ is n,

then the system is observable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$O_B = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$rank\left(O_B\right) = 1 \neq 2 \quad \therefore$ The system is NOT observable.

Discuss State controllability of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$

Discuss State controllability of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$+ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$u

So, A= $\begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}$; B= $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$;

M = $M = [B \quad AB]$= $\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$. So rank(M) = 1

The system is not controllable. Why?

Discuss State controllability of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$

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The system is not controllable. Why?

$$\frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

So, Pole-Zero cancellation occurs.

Discuss controllability & observability of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0.8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Discuss controllability & observability of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0.8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, A= $\begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}$; B= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

$M = [B \quad AB]$= $\begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$. So rank(M) = 2 The system is controllable.

$N = [C^T \quad A^T C^T]$= $\begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$. So rank(N) = 1 The system is not observable

If we write the system in observable canonical form then,

The system is observable but not controllable. Why?

Discuss controllability & observability of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0.8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, A= $\begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}$; B= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

$M = [B \quad AB]$= $\begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$. So rank(M) = 2 The system is controllable.

$N = [C^T \quad A^T C^T]$= $\begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$. So rank(N) = 1 The system is not observable

If we write the system in observable canonical form then,

The system is observable but not controllable. Why?

$$\frac{X_1(s)}{U(s)} = \frac{s + 0.8}{(s + 0.8)(s + 0.5)}$$

So, Pole-Zero cancellation occurs.

Controllability and Observability in Transfer Function Domain

- The system is both controllable and observable if there is no Pole-Zero cancellation.

- **Note:** The cancelled pole-zero pair suppresses part of the information about the system

Principle of Duality

System S_1: $\dot{X} = AX + BU$ $\quad C_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

$Y_1 = CX$ $\quad O_B = \begin{bmatrix} C^T & A^TC^T & A^{T^2}C^T & \cdots & A^{T^{n-1}}C^T \end{bmatrix}$

System S_2: $\dot{Z} = A^TZ + C^TV$ $\quad C_B = \begin{bmatrix} C^T & A^TC^T & A^{T^2}C^T & \cdots & A^{T^{n-1}}C^T \end{bmatrix}$

$Y_2 = B^TZ$ $\quad O_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

The principle of duality states that the system S_1 is controllable if and only if system S_2 is observable; and vice-versa!

Hence, the problem of observer design for a system is actually a problem of control design for its dual system.

Stabilizability and Detectability

- Stabilizable system: Uncontrollable system in which uncontrollable part is stable

- Detectable system: Unobservable system in which the unobservable subsystem is stable

State Space Controller Design

System : $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$ $\quad \boldsymbol{X} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{U} \in \mathbb{R}^{\boldsymbol{m}}$

$\boldsymbol{Y} = \boldsymbol{CX} + \boldsymbol{DU}$ $\quad \boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{p}}$

Controllability:

State: Rank of $C_B = [B \; AB \; \ldots A^{n-1}B]$ is n

Output: Rank of $C_B = [CB \; CAB \; \ldots CA^{n-1}B \; D]$ is p

Observability:

Rank of $O_B = \left[C^T \quad A^T C^T \quad \ldots \ldots A^{T(n-1)} C^T\right]$ is n

State Feedback Controller

System :

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$\mathbf{x}$ = state vector (n-vector)

y = output signal (scalar)

u = control signal (scalar)

$\mathbf{A}$ = $n \times n$ constant matrix

$\mathbf{B}$ = $n \times 1$ constant matrix

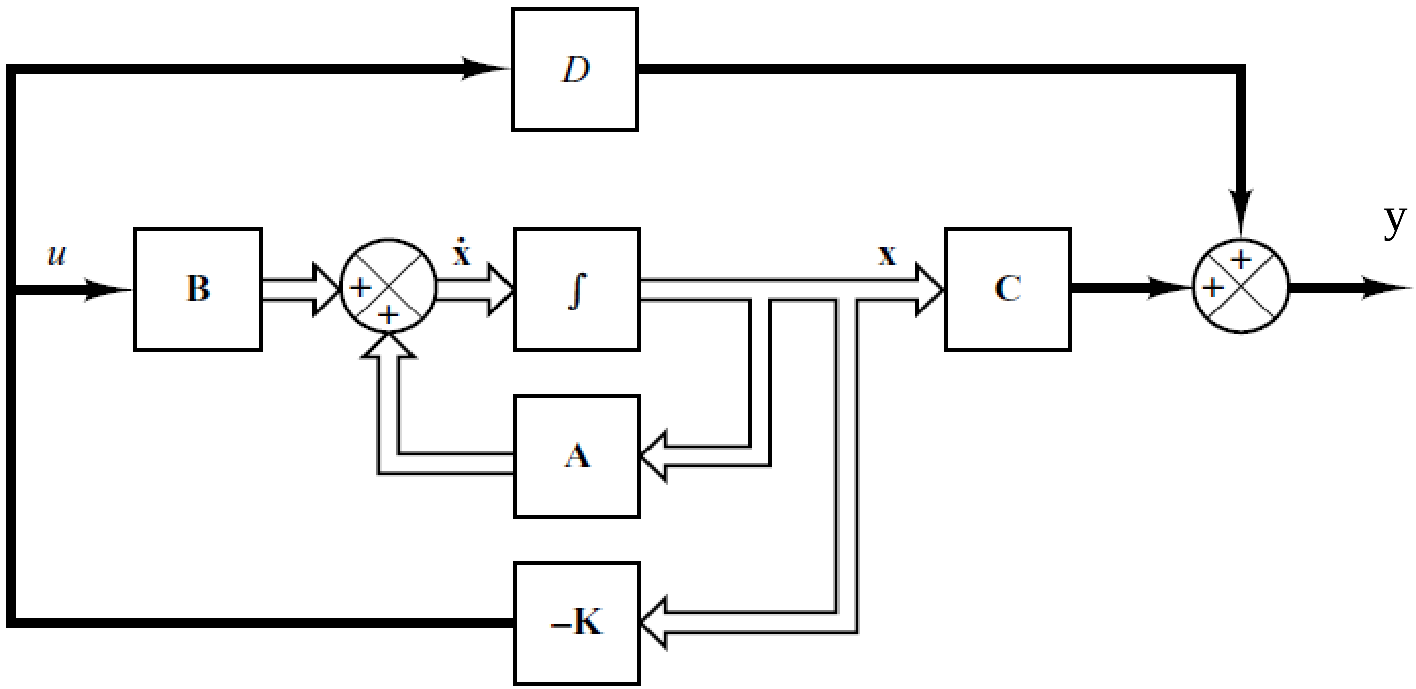
$\mathbf{C}$ = $1 \times n$ constant matrix

D = constant (scalar)

Control signal : $u = -\mathbf{K}\mathbf{x}$ where, **K** = 1 x n matrix

This means u is determined from instantaneous state.

Block Diagram



- This scheme is called state feedback and **K** matrix is called state feedback gain matrix.
- Non-zero output will be returned to zero reference input because of the state feedback scheme

Closed loop system

- System State equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$
- State feedback Control: $u = -\mathbf{K}\mathbf{x}$
- Closed loop system : $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$
- Solution is : $\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(0)$

  Where **X**(0) is the initial state
- eigenvalues of **A-BK** are called regulator poles
- Placement of regulator poles in left half of s-plane ensure **x**(t) approaches **0** as t tending to ∞
- Problem of placing regulator poles at desired location is called Pole Placement Problem

Necessary & sufficient condition

Arbitrary pole placement of a given system is possible if and only if the system is completely state controllable

Proof:

Suppose the system is not fully state controllable i.e,

rank $[\mathbf{P}]$ = rank $[\mathbf{B} \mid \mathbf{AB} \mid \ldots \mid \mathbf{A}^{n-1}\mathbf{B}] = q < n$

Let us *write*

$$\mathbf{P} = [\, \mathbf{f}_1 \mid \mathbf{f}_2 \mid \ldots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \mathbf{v}_{q+2} \mid \ldots \mid \mathbf{v}_n \,] \text{ of rank n.}$$

Where $\mathbf{f}_i$ (i = 1,q) are q linearly independent column vectors and $\mathbf{v}_j$ (j = q+1, n) are n-q additional chosen vectors to make rank $[\mathbf{P}]$ = n.

Necessary & sufficient condition

- By using **P** as transformation matrix for A and B,
- $\mathbf{A\,P = P\,\hat{A}}$ or $\left[\mathbf{Af}_1 \;\vdots\; \mathbf{Af}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{Af}_q \;\vdots\; \mathbf{Av}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{Av}_n\right]$

$$= \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]\hat{\mathbf{A}}$$

- Using Cauly-Hamilton's theorem to express $\mathbf{Af_1, Af_2, .. Af_q}$

$$\mathbf{Af}_1 = a_{11}\mathbf{f}_1 + a_{21}\mathbf{f}_2 + \cdots + a_{q1}\mathbf{f}_q$$

$$\mathbf{Af}_2 = a_{12}\mathbf{f}_1 + a_{22}\mathbf{f}_2 + \cdots + a_{q2}\mathbf{f}_q$$

.

.

.

$$\mathbf{Af}_q = a_{1q}\mathbf{f}_1 + a_{2q}\mathbf{f}_2 + \cdots + a_{qq}\mathbf{f}_q$$

Necessary & sufficient condition

- Hence,

$$\left[\mathbf{Af}_1 \;\vdots\; \mathbf{Af}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{Af}_q \;\vdots\; \mathbf{Av}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{Av}_n\right]$$

$$= \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]
\left[\begin{array}{ccc|ccc}
a_{11} & \cdots & a_{1q} & a_{1q+1} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2q} & a_{2q+1} & \cdots & a_{2n} \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
a_{q1} & \cdots & a_{qq} & a_{qq+1} & \cdots & a_{qn} \\
\hline
0 & \cdots & 0 & a_{q+1q+1} & \cdots & a_{q+1n} \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
0 & \cdots & 0 & a_{nq+1} & \cdots & a_{nn}
\end{array}\right]$$

Necessary & sufficient condition

- Hence,

$$\left[\mathbf{Af}_1 \mid \mathbf{Af}_2 \mid \cdots \mid \mathbf{Af}_q \mid \mathbf{Av}_{q+1} \mid \cdots \mid \mathbf{Av}_n\right]$$

$$= \left[\mathbf{f}_1 \mid \mathbf{f}_2 \mid \cdots \mid \mathbf{f}_q \mid \mathbf{v}_{q+1} \mid \cdots \mid \mathbf{v}_n\right]\left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right]$$

- Thus,

$$\mathbf{AP} = \mathbf{P}\left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right]$$

$$\mathbf{P}^{-1}\mathbf{AP} = \hat{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right]$$

Necessary & sufficient condition

- Similarly, $\mathbf{B} = \mathbf{P}\widehat{\mathbf{B}}$

$$\mathbf{B} = \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]\hat{\mathbf{B}}$$

- So,

$$b_{11}\mathbf{f}_1 + b_{21}\mathbf{f}_2 + \cdots + b_{q1}\mathbf{f}_q = \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]\begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{q1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Where **B** can be written in terms of independent q column vectors

$$\mathbf{B} = b_{11}\mathbf{f}_1 + b_{21}\mathbf{f}_2 + \cdots + b_{q1}\mathbf{f}_q$$

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{bmatrix}$$

Necessary & sufficient condition

- Now define, $\hat{\mathbf{K}} = \mathbf{KP} = [\mathbf{k}_1 \mid \mathbf{k}_2]$
- Then we have,

$$
\begin{aligned}
|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= |\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\mathbf{P}| \\
&= |s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP} + \mathbf{P}^{-1}\mathbf{BKP}| \\
&= |s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}| \\
&= \left| s\mathbf{I} - \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right] + \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array}\right] [\mathbf{k}_1 \mid \mathbf{k}_2] \right| \\
&= \begin{vmatrix} s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1 & -\mathbf{A}_{12} + \mathbf{B}_{11}\mathbf{k}_2 \\ \mathbf{0} & s\mathbf{I}_{n-q} - \mathbf{A}_{22} \end{vmatrix} \\
&= |s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1| \cdot |s\mathbf{I}_{n-q} - \mathbf{A}_{22}| = 0
\end{aligned}
$$

- Notice that the eigenvalues of $\mathbf{A}_{22}$ do not depend on $\mathbf{K}$. Thus if the system is not completely state controllable, then there are eigenvalues of $\mathbf{A}$ that cannot be arbitrarily placed.