1. Uniqueness of basis → basis is not unique
2. Equality of vector spaces.

} to be covered today.

   1. same field
   2. same ops.
   3. same basis

   } Required for equality

   $(V, F, +, *)$

   ⇒ give $F, +, *$ & the basis, we can generate V.

3. Recap

   1. Basis

$$\underline{v} = \sum_{i=1}^{N} c_i \cdot \underline{v}_i \quad , \quad \text{not all } c_i = 0$$

$$\underbrace{v_1, v_2, \cdots, v_N}_{\text{lin dep}} \quad v_{N+1}$$

⇒ basis

⇒ N = dime. of vector space.

   2. Span.

(4) Check if 4 given vectors span $\mathbb{R}^4$

$[\underline{\underline{A}} \mid \underline{b}] =$

(see prev. class).

→ Given that they are not lin dep, how many of them can be lin dep?

⑤ Equality of vector spaces

1. Same Field
2. Same ops.
3. Same basis.

but basis isn't unique!

∴ if they're diff, then check if each vector in one basis can be repr. as an L.C. of the other basis.

(then the two basis would be equivalent)

⑥ Solve the foll. set of eq^n s →

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 3x_2 + x_3 = 0$$

$$5x_1 + 6x_2 + 4x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 5 & 6 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \therefore \quad \begin{array}{l} x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}$$

$\Rightarrow$ eq^{n}s 1, 2 & 3 are lin. dep.

$\Rightarrow$ ∞ solutions.

DoF analysis

\# var $= 3$

\# eq $= 2$

$\Rightarrow$ DoF $= 1$.

$\therefore$ let $x_2 = \alpha$.

$\therefore$ $x_3 = +\alpha$

$x_1 = -2\alpha$.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ ∞ soln.

$\Rightarrow$ $[-2 \quad 1 \quad 1]^T$ is a basis of the lin. vector space of solns of $\underline{A} \cdot \underline{x} = \underline{0}$

& dime = $\underline{1}$.

& dim. of soln space.

called "Null space"

(7) $x + y = 0$

$x + 2y + z = 0$

$y + 2z + t = 0$

$z + 2t = 0.$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$\Rightarrow$ all 4 vectors are lindep.

$\therefore$ $[x \quad y \quad z \quad t]^T = [0 \quad 0 \quad 0 \quad 0]^T$

$\Rightarrow$ trivial soln

$\Rightarrow$ dim. $\underline{\underline{D.N.E}}$ ($\because$ 0 vector does not have a span).

(8) Find c s.t. $AX = B$ can be solved

$$x_1 + x_2 + 2\cdot x_3 = 2$$
$$2\cdot x_1 + 3\cdot x_2 - x_3 = 5$$
$$3\cdot x_1 + 4x_2 + x_3 = c$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right]$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c-6 \end{array}\right]$$

$1 - 6$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$$

Rank

$\therefore \rho(A) = 2$

for the system to be solvable,

we must have $\rho(A) = \rho(A|b)$

$\therefore \quad c - 7 = 0$

$\therefore \quad \boxed{c = 7}$

(9)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right]$$

Pivoting

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 0 & 0 & 3b_3 - b_4 - 3b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$3(b_3 - 2b_1)$

$- (b_4 - 3b_1)$

$3b_3 - b_4$

$- 3b_1$

$$\therefore \quad 3b_3 - b_4 - 3b_1 = 0$$

$$\& \quad b_2 - 2b_1 = 0$$

$\therefore$ DoF

\# var $= 4$

\# eq $= 2$

$\therefore$ 2 DoF.

let $b_1 = \alpha$

$$\therefore b_2 = 2\alpha.$$

let $b_3 = \beta$.

$$\therefore \quad 3\beta - b_4 - 3\alpha = 0$$

$$\therefore \quad b_4 = 3(\beta - \alpha)$$

$$\therefore \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \\ \beta \\ -3(\alpha - \beta) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore$ 2 basis vectors $\Rightarrow$ dim $= 2$.

$\therefore$ The lin. $\vec{v}$ space which solves $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a dim. of 2 with bases $\Rightarrow [1 \quad 2 \quad 0 \quad -3]^T$ & $[0 \quad 0 \quad 1 \quad 3]^T$.

This space does <u>not</u> solve the problem. It only allows $\underline{\underline{A}}\,\underline{x} = \underline{b}$ to have a solution.

"Range Space":

- If $\underline{\underline{A}}\,\underline{x} = \underline{0}$ has only a trivial soln, then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ will have a unique soln. (for any $\underline{b}$).
- if $\underline{\underline{A}}\,\underline{x} = \underline{0}$ has ∞ soln, $\underline{\underline{A}}\,\underline{x} = \underline{b}$ may or may not have a soln. If it does, then it'll have ∞ soln.

(10) Determine dim & basis of null & range spaces of A

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 4 & -1 & 9 \\ 2 & 1 & 1 \end{bmatrix}$$

$$b = [3 \quad 0 \quad 0 \quad 2]^T.$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$1 - 3(-2)$

$= 7$

$-12 + 3 \times (+6)$

6

$-3 - 1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ only trivial soln

$\Rightarrow$ Null space = [0 0 0 0]
only trivial, dim. DNE

$\Rightarrow$ $\underline{\underline{A}}\ \underline{x} = \underline{b}$ will have a unique solution.

Solving for range space

(for b always use b_1, b_2, b_3, b_4)

(see photos for soln).

$\Big[$

$\star$ o Dim(null) + Dim(range)
= Dim(original space)

$\rightarrow$ No midsem stuff in endsem.


Bc- No null space
No dimension!!!
Yeh kaunsa Janwar hai!!!
Yeh Behinchod bol rahe hain...