1.1 Introduction to Integral Transform

In this lesson we will discuss the idea of integral transform, in general, and Laplace transform in particular. Integral transforms turn out to be a very efficient method to solve certain ordinary and partial differential equations. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The idea of solving differential equations is given in Figure 33.1.

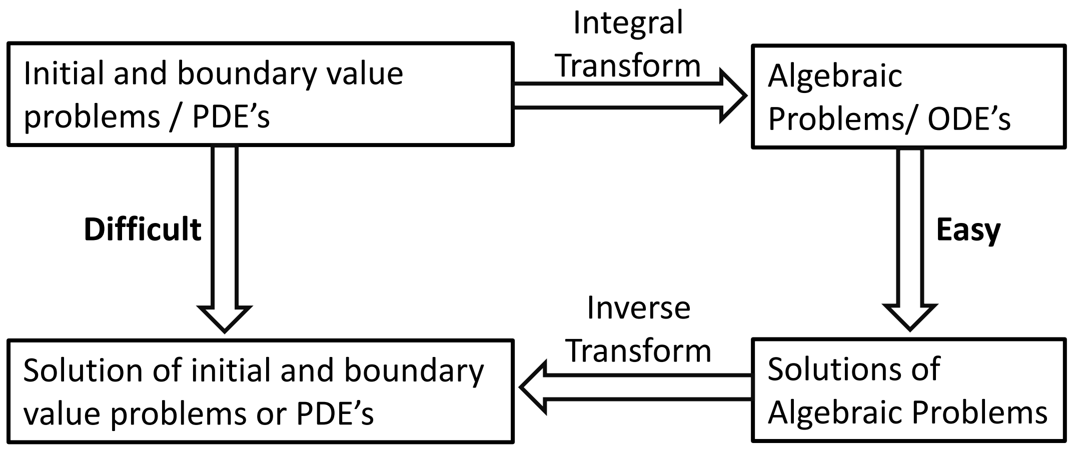

Figure 1.1: Idea of Solving Differential/Integral Equations

1.2 Concept of Transformations

An integral of the form

$$\int_a^b K(s,t) f(t)\,\mathrm{d}t$$

is called integral transform of $f(t)$. The function $K(s,t)$ is called kernel of the transform. The parameter s belongs to some domain on the real line or in the complex plane. Choosing different kernels and different values of a and b, we get different integral transforms. Examples include Laplace, Fourier, Hankel and Mellin transforms. For $K(s,t) = e^{-st},\ a = 0,\ b = \infty$, the improper integral

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

is called Laplace transform of $f(t)$. If we set $K(s,t) = e^{-ist},\ a = -\infty,\ b = \infty$, then

$$\int_{-\infty}^{\infty} e^{ist} f(t)\,\mathrm{d}t$$

where $i = \sqrt{-1}$ is called the Fourier transform of $f(t)$. A common property of integral transforms is linearity, i.e.,

$$\text{I.T.}\,[\alpha\, f(t) + \beta\, g(t)] = \int_a^b K(s,t)\,[\alpha\, f(t) + \beta\, g(t)]\ \mathrm{d}t = \alpha\,\text{I.T.}(f(t)) + \beta\,\text{I.T.}\ (g(t))$$

The symbol I.T. stands for integral transforms.

1.3 Laplace Transform

The Laplace transform of a function f is defined as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$$

provided the improper integral converges for some s.

Remark 1: *The integral $\int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$ is said to be convergent (absolutely convergent) if*

$$\lim_{R\to\infty} \int_0^R e^{-st} f(t)\,\mathrm{d}t \quad \left(\lim_{R\to\infty} \int_0^R \mid e^{-st} f(t) \mid \ \mathrm{d}t \right)$$

exists as a finite number.

1.4 Laplace Transform of Some Elementary Functions

We now give Laplace transform of some elementary functions. Laplace transform of these elementary functions together with properties of Laplace transform will be used to evaluate Laplace transform of more complicated functions.

1.5 Example Problems

1.5.1 Problem 1

Evaluate Laplace transform of $f(t) = 1,\ t \geq 0.$

Solution: Using definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-st}}{-s}\Big|_0^\infty$$

Assuming that s is real and positive, therefore

$$L[f(t)] = \frac{1}{s},\ \text{since}\ \lim_{R\to\infty} e^{-sR} = 0$$

What will happen if we take s to be a complex number, i.e., $s = x + iy.$ Since $e^{-iyR} = \cos yR - i\sin yR$, and therefore $\mid e^{-iyR} \mid = 1$, then, we find

$$\lim_{R\to\infty} \mid e^{-xR} \mid\mid e^{-iyR} \mid = 0 \ \text{ for Re}(s) = x > 0$$

Thus, we have

$$L[f(t)] = L[1] = \frac{1}{s},\ \text{Re}(s) > 0.$$

1.5.2 Problem 2

Find the Laplace transform of the functions $e^{at},\ e^{iat},\ e^{-iat}.$

Solution: Using the definition of Laplace transform

$$\begin{aligned} L[e^{at}] = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t &= \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty \\ &= \frac{1}{s-a},\ \text{provided Re}(s) > a\ (\text{or}\ s > a) \end{aligned}$$

Similarly, we can evaluate

$$\begin{aligned} L[e^{iat}] = \int_0^\infty e^{-(s-ia)t}\,\mathrm{d}t &= \frac{e^{-(s-ia)t}}{-(s-ia)}\Big|_0^\infty \\ &= \frac{1}{s-ia},\ \text{provided Re}(s) > 0. \end{aligned}$$

Here we have used the fact that, for $s = x + iy$, we have

$$\lim_{R\to\infty}\left|\frac{e^{-(s-ia)R}}{-(s-ia)}\right| = -\frac{1}{s-ia}\lim_{R\to\infty}\left|e^{-xR}e^{-i(y-a)R}\right| = 0$$

Similarly, we get

$$L[e^{-iat}] = \frac{1}{s+ia}.$$

1.5.3 Problem 3

Find the Laplace transform of the unit step function (commonly known as the Heaviside function). This function is given as

$$u(t-a) = \begin{cases} 0 & \textit{if}\ t < a, \\ 1 & \textit{if}\ t \geq a. \end{cases}$$

Solution: Let us find the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^{\infty} = \frac{e^{-as}}{s},$$

where of course $s > 0$ and $a \geq 0$.

1.5.4 Problem 4

Find the Laplace transform of $t^n,\ n = 1, 2, 3, ...$

Solution: Using definition of Laplace transform we get

$$\begin{aligned} L[t^n] = \int_0^\infty e^{-st}t^n\,\mathrm{d}t &= \left[t^n\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}nt^{n-1}\,\mathrm{d}t \\ &= 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,\mathrm{d}t = \frac{n}{s}L[t^{n-1}] \end{aligned}$$

Putting $n = 1$:

$$L[t] = \frac{1}{s}L[1] = \frac{1}{s^2} = \frac{1!}{s^2}$$

Putting $n = 2$:

$$L[t^2] = \frac{2}{s^3} = \frac{2!}{s^3}$$

If we assume $L[t^n] = \frac{n!}{s^{n+1}}$, then

$$L[t^{n+1}] = \frac{n+1}{s} L[t^n] = \frac{(n+1)!}{s^{n+2}} \quad \Rightarrow \quad L[t^n] = \frac{n!}{s^{n+1}}, \ \text{Re}(s) > 0.$$

One can also extend this result for non-integer values of n.

1.5.5 Problem 5

Find $L[t^\gamma]$ for non-integer values of γ.

Solution: Using the definition of Laplace transform we get

$$L[t^\gamma] = \int_0^\infty e^{-st} t^\gamma \, \mathrm{d}t, \ \ (\gamma > -1)$$

Note that the above integral is convergent only for $\gamma > -1$. We substitute $u = st \ \Rightarrow \mathrm{d}u = s\mathrm{d}t$ where $s > 0$. Thus we get

$$L[t^\gamma] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\gamma \frac{1}{s} \, \mathrm{d}u = \frac{1}{s^{\gamma+1}} \int_0^\infty e^{-u} u^\gamma \, \mathrm{d}u$$

We know

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u} \, \mathrm{d}u \ \ (p > 0)$$

Then,

$$L[t^\gamma] = \frac{\Gamma(\gamma+1)}{s^{\gamma+1}}, \ \ \gamma > -1, s > 0$$

Note that for $\gamma = 1, 2, 3, ...$, the above formula reduces to the formula we got in previous example for integer values, i.e., $L[t^\gamma] = \dfrac{\gamma!}{s^{\gamma+1}}$.

1.5.6 Problem 6

Let $f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$. Find $L[f(t)]$.

Solution: Applying the definition of Laplace transform we obtain

$$L[f(t)] = L\left[\sum_{k=0}^{n} a_k t^k\right]$$

Using the linearity of the transform we get

$$L[f(t)] = \sum_{k=0}^{n} L[t^k] = \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}.$$

Remark 2: *For an infinite series $\sum_{n=0}^{\infty} a_n t^n$, it is not possible, in general, to obtain Laplace transform of the series by taking the transform term by term.*