2.1 Basic Properties (Linearity)

Now we compute the Laplace transform of some elementary functions, before discussing the restriction that have to be imposed on $f(t)$ so that it has a Laplace transform. With the help of Laplace transform of elementary function we can get Laplace transform of complicated function using properties of the transform that will be discussed later. Another important aspect of the finding Laplace transform of elementary function relies on using them for getting inverse Laplace transform.

2.2 Example Problems

2.2.1 Problem 1

Find Laplace transform of (i) $\cosh \omega t$*, (ii)* $\cos \omega t$*, (iii)* $\sinh \omega t$ *(iv)* $\sin \omega t$.

Solution: (i) Using the definition of Laplace transform we get

$$L[\cosh \omega t] = L\left[\frac{e^{\omega t} + e^{-\omega t}}{2}\right]$$

Using linearity of the transform we obtain

$$L[\cosh \omega t] = \frac{1}{2}\Big(L\left[e^{\omega t}\right] + L\left[e^{-\omega t}\right]\Big)$$

Applying the Laplace transform of exponential function we obtain

$$L[\cosh \omega t] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s+\omega}\right] = \frac{s}{s^2-\omega^2}$$

(ii) Following similar steps we obtain

$$L[\cos \omega t] = L\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

Using linearity, we obtain

$$L[\cos \omega t] = \frac{1}{2}L\left[e^{i\omega t}\right] + \frac{1}{2}L\left[e^{-i\omega t}\right]$$

We know the Laplace transform of exponential functions which can be used now to get

$$L[\cos \omega t] = \frac{1}{2}\left\{\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right\} = \frac{1}{2}\frac{2s}{s^2+\omega^2}$$

Thus we have

$$L[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

Similarly we get the last two cases (iii) and (iv) as

$$L[\sinh \omega t] = \frac{\omega}{s^2 - \omega^2} \quad \text{and} \quad L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

2.2.2 Problem 2

Find the Laplace transform of $(3 + e^{6t})^2$.

Solution: We determine the Laplace transform as follows

$$L(3 + e^{6t})^2 = L(3 + e^{6t})(3 + e^{6t}) = L(9 + 6e^{6t} + e^{12t})$$

Using linearity we get

$$\begin{aligned} L(3 + e^{6t})^2 &= L(9) + L(6e^{6t}) + L(e^{12t}) \\ &= 9L(1) + 6L(e^{6t}) + L(e^{12t}) \end{aligned}$$

Using the Laplace transform of elementary functions appearing above we obtain

$$L(3 + e^{6t})^2 = \frac{9}{s} + \frac{6}{s-6} + \frac{1}{s-12}$$

2.2.3 Problem 3

Find the Laplace transform of $\sin^3 2t$.

Solution: We know that

$$\sin 3t = 3 \sin t - 4 \sin^3 t$$

This implies that we can write

$$\sin^3 2t = \frac{1}{4}\left(3 \sin 2t - \sin 6t\right)$$

Applying Laplace transform and using its linearity property we get

$$L[\sin^3 2t] = \frac{1}{4}\left(3L[\sin 2t] - L[\sin 6t]\right)$$

Using the Laplace transforms of $\sin at$ we obtain

$$L[\sin^3 2t] = \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\frac{6}{s^2+36}$$

Thus we get

$$L[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$$

2.2.4 Problem 4

Find Laplace transform of the function $f(t) = 2^t$.

Solution: First we rewrite the given function as

$$f(t) = 2^t = e^{\ln 2^t} = e^{t\ln 2}$$

Now $f(t)$ is function of the form e^{at} and therefore

$$L[f(t)] = \frac{1}{s-\ln 2}, \quad \text{for } s > \ln 2$$

2.2.5 Problem 5

Find (a) $L[t^3 - 4t + 5 + 3\sin 2t]$ *and (b)* $L[H(t-a) - H(t-b)]$.

Solution: (a) Using linearity of the transform we get

$$L[t^3 - 4t + 5 + 3\sin 2t] = L[t^3] - 4L[t] + L[5] + 3L[\sin 2t]$$

Using Laplace transform evaluated in previous previous examples, we have

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{6}{s^4} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{(s^2+4)}$$

On simplification we find

$$L[t^3 - 4t + 5 + 3\sin 2t] = \frac{(5s^5 + 2s^4 + 20s^3 10s^2 + 24)}{[s^4(s^2+4)]}$$

(b) Using Linearity property we get

$$L[H(t-a) - H(t-b)] = L[H(t-a)] - L[H(t-b)]$$

Applying the definition of Laplace transform we obtain

$$\begin{aligned}L[H(t-a)-H(t-b)] &= \int_0^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_0^\infty H(t-b)e^{-st}\,\mathrm{d}t \\ &= \int_a^\infty H(t-a)e^{-st}\,\mathrm{d}t - \int_b^\infty H(t-b)e^{-st}\,\mathrm{d}t\end{aligned}$$

Integration gives

$$L[H(t-a)-H(t-b)] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

This implies

$$L[H(t-a)-H(t-b)] = \frac{e^{-as}-e^{-bs}}{s}$$

2.2.6 Problem 6

Find Laplace transform of the following function

$$f(t) = \begin{cases} t/c, & \text{if } 0 < t < c\,; \\ 1, & \text{if } t > c. \end{cases}$$

Here c is some constant.

Solution: Using the definition of Laplace transform we have

$$L[f(t)] = \int_0^c e^{-st}\left(\frac{t}{c}\right)\,\mathrm{d}t + \int_c^\infty e^{-st}\,\mathrm{d}t$$

Integrating by parts we find

$$L[f(t)] = \left[\frac{t}{c}\left(-\frac{e^{-st}}{s}\right) - \frac{1}{c}\left(-\frac{e^{-st}}{s^2}\right)\right]_0^c + \left[-\frac{e^{-st}}{s}\right]_c^\infty$$

On simplifications we obtain

$$L[f(t)] = \frac{1-e^{sc}}{cs^2}$$

2.2.7 Problem 7

Find Laplace transform of the function $f(t)$ given by

$$f(t) = \begin{cases} 0, & \text{if } 0 < t < 1\,; \\ t, & \text{if } 1 < t < 2; \\ 0, & \text{if } t > 2. \end{cases}$$

Solution: By the definition of Laplace transform we have

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_1^2 e^{-st} t\,\mathrm{d}t$$

Integrating by parts we obtain

$$\begin{aligned} L[f(t)] &= \left[t\left(-\frac{e^{-st}}{s}\right)\right]_1^2 + \int_1^2 \frac{e^{-st}}{s}\,\mathrm{d}t \\ &= -\frac{2e^{-2s} - e^{-s}}{s} - \frac{e^{-2s} - e^{-s}}{s^2} \end{aligned}$$

2.2.8 Problem 8

Find Laplace transform of $\sin\sqrt{t}$.

Solution: We have

$$\sin\sqrt{t} = t^{1/2} - \frac{1}{3!}t^{3/2} + \frac{1}{5!}t^{5/2} - \frac{1}{7!}t^{7/2} + ...$$

Then, taking the Laplace transform of each term in the series we get

$$\begin{aligned} L[\sin\sqrt{t}] &= L[t^{1/2}] - \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] + \dots \\ &= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{s^{9/2}} + \dots \end{aligned}$$

Further simplifications leads to

$$\begin{aligned} L[\sin\sqrt{t}] &= \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{3!}\frac{3}{2}\frac{1}{s} + \frac{1}{5!}\frac{5}{2}\frac{3}{2}\frac{1}{s^2} - \frac{1}{7!}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{s^3} + \dots\right] \\ &= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{2^2 s} + \frac{1}{2!}\frac{1}{(2^2 s)^2} - \frac{1}{3!}\frac{1}{(2^2 s)^3} + \dots\right] \end{aligned}$$

Thus, we have

$$L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}.$$