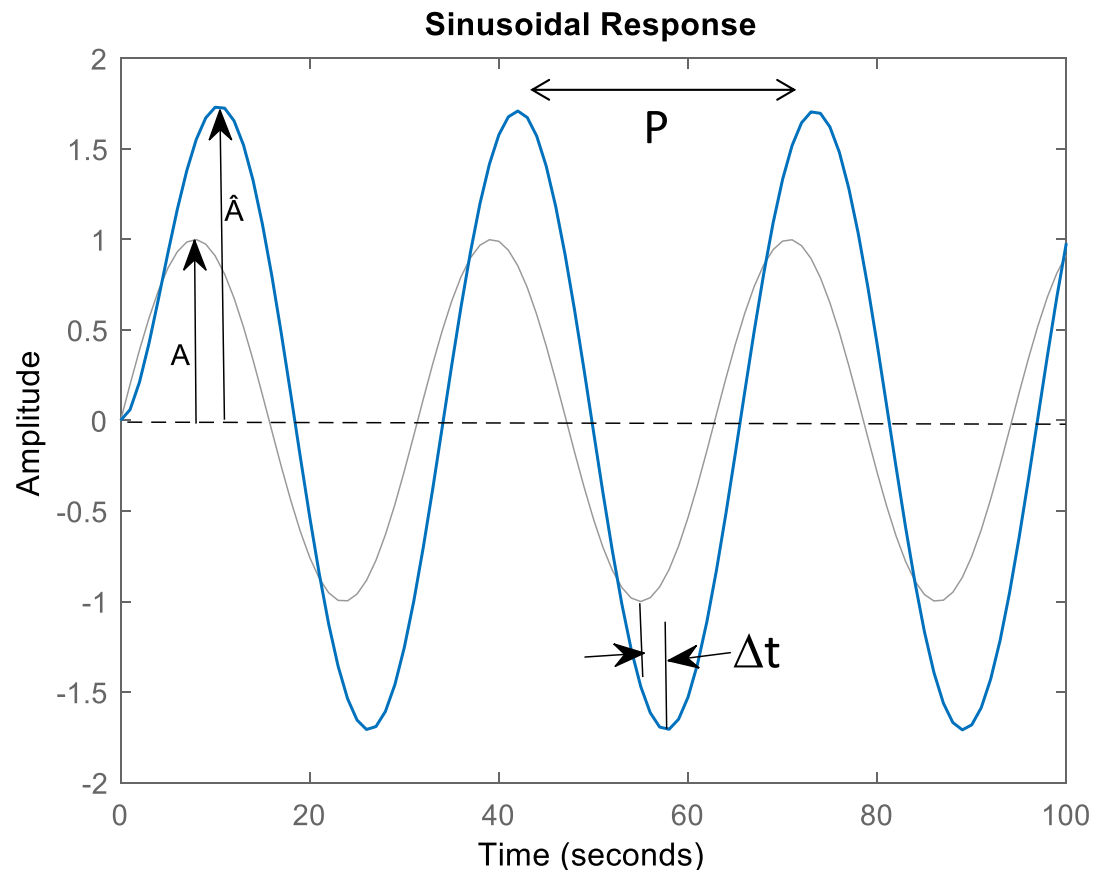
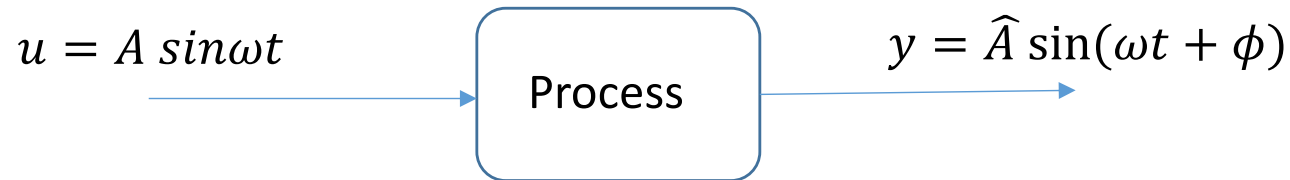


Review of Feedback Control System

Frequency Response



Amplitude Ratio (AR) = \hat{A}/A
Phase lag (ϕ) = $-\Delta t/P \cdot 360$

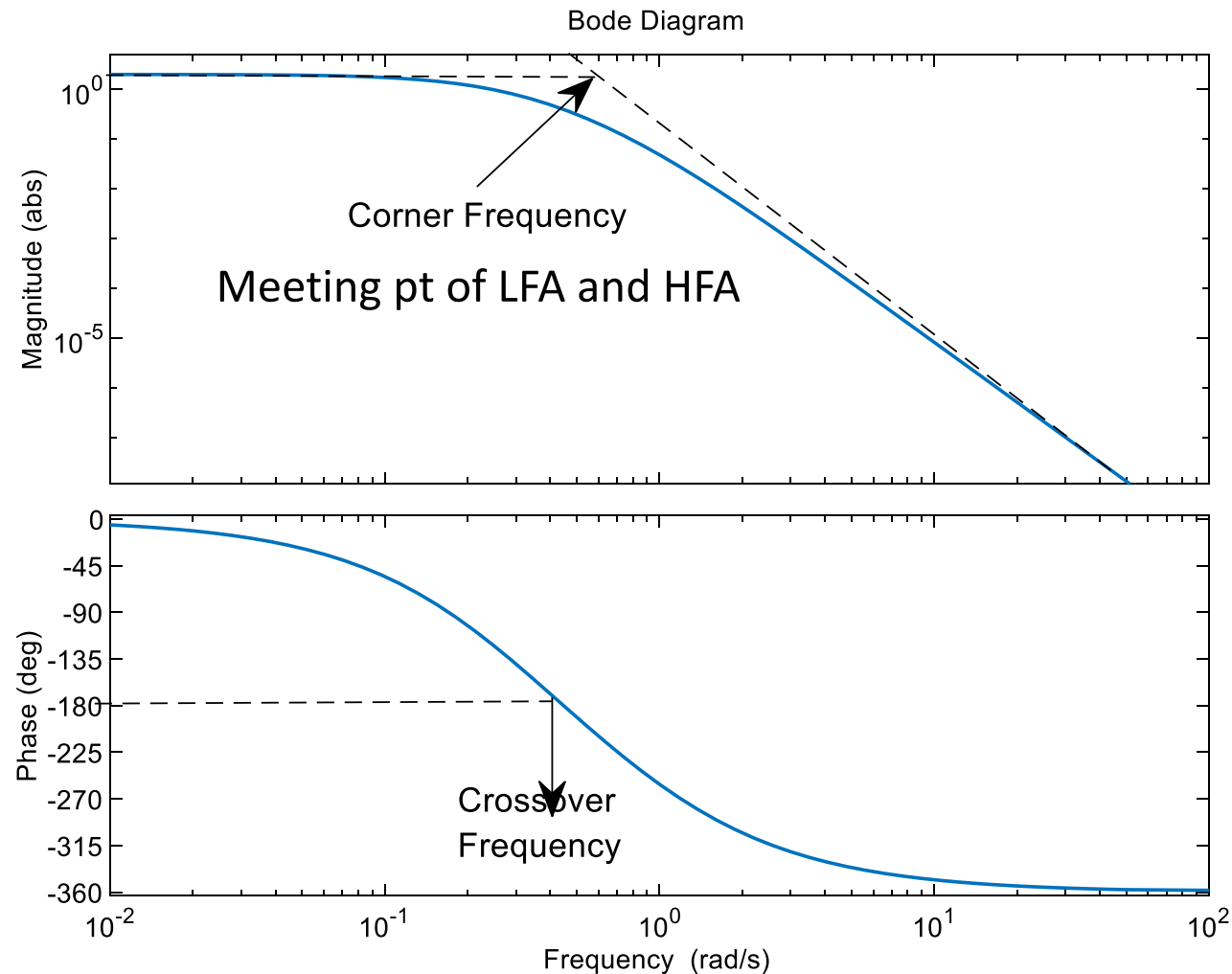
The process acts as a “filter” or “amplifier” that attenuates or amplifies the input at frequency ω .

The frequency response of a process $G(s)$ at a specific frequency ω is a complex number and can be obtained by substituting $s = j\omega$

Frequency Response

- $G(j\omega) = Re(\omega) + j Im(\omega) = \rho(\omega)e^{j\phi(\omega)}$
- $\rho(\omega) = AR = \sqrt{Re^2(\omega) + Im^2(\omega)} = |G(j\omega)|$
- $\phi = \text{phase difference} = \arctan\left(\frac{Im(\omega)}{Re(\omega)}\right) = \angle G(j\omega)$
- Two ways of representing the frequency response
 - Bode Plot
 - Log-Log plot of AR vs ω and semi-log plot ϕ vs ω
 - Nyquist Plot
 - Plot of $Re(\omega)$ vs $Im(\omega)$ for various values of ω .
 - Frequency does not appear explicitly in this plot.

Bode Plot

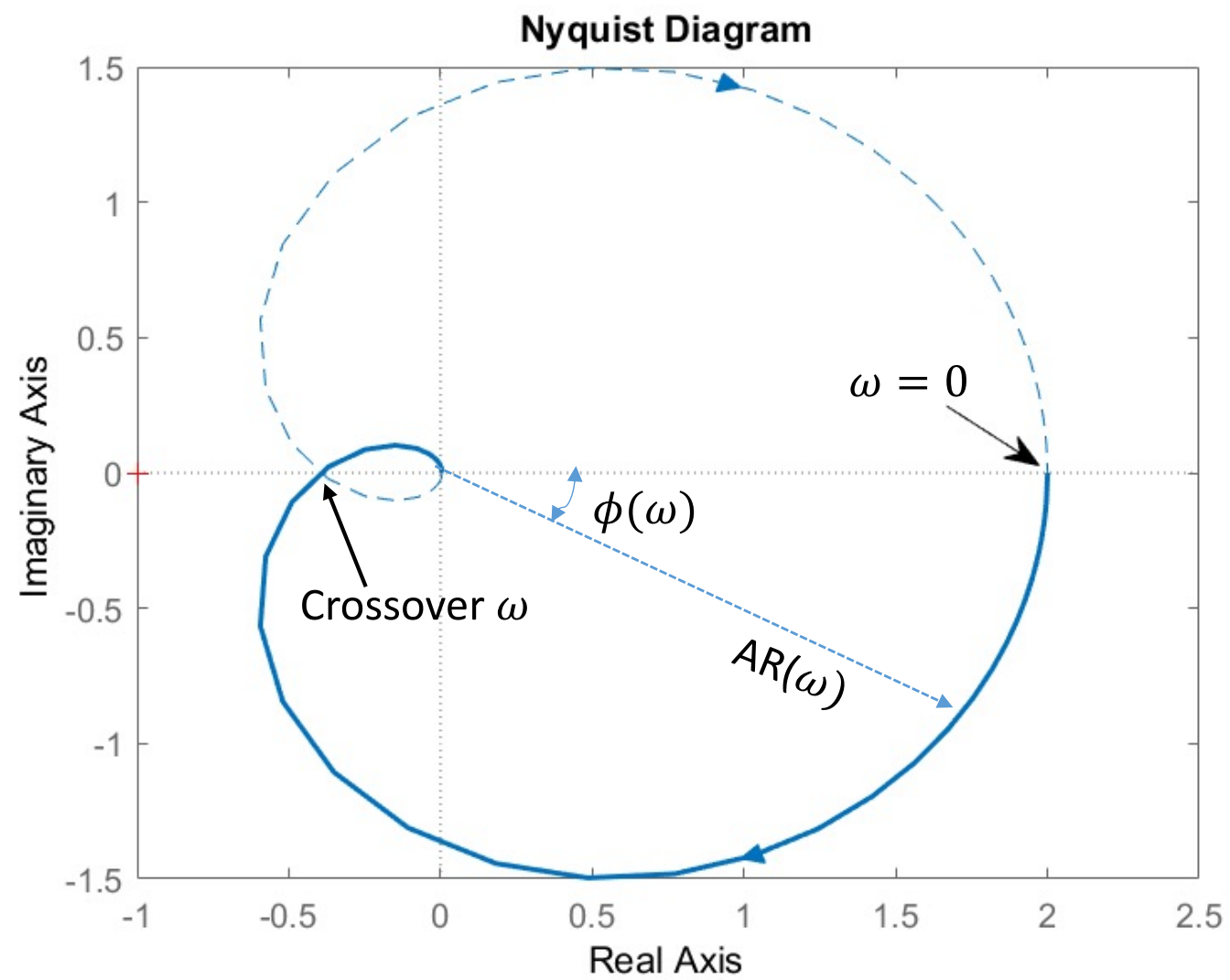


Steady state gain:

$$K = \rho(0)e^{j\phi(0)}$$

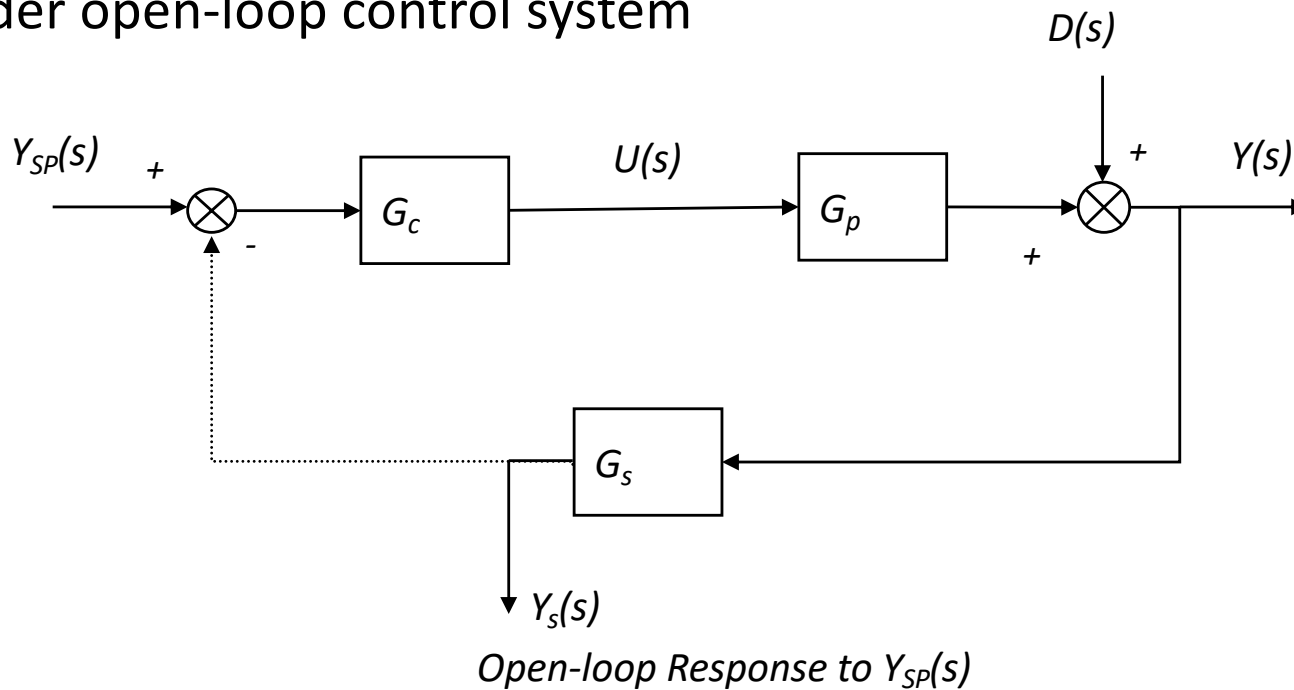
Bandwidth, ω_b , defines the highest frequency at which the process displays no attenuation. It is calculated based on the frequency when amplitude reaches 0.707 K.

Roll-off is slope of decaying AR at high frequencies.



Bode Stability Criterion

Consider open-loop control system



1. Introduce sinusoidal input in setpoint ($D(s)=0$) and observe sinusoidal output
2. Fix gain such $AR=1$ and input frequency such that $\phi=-180$
3. At same time, connect close the loop and set $Y_{SP}(s)=0$

Q: What happens if $AR < 1$, 1 and > 1 ?

Bode Stability Criterion

A closed-loop system is unstable if the frequency of the response of the open-loop G_{OL} has an amplitude ratio greater than one at the critical frequency. Otherwise it is stable.

Strategy:

1. Solve for ω in $\arg(G_{OL}(j\omega)) = -\pi$

2. Calculate AR $AR = G_{OL}(j\omega)$

Bode Stability Criterion

To check for stability:

1. Compute open-loop transfer function
2. Solve for ω in $\phi = -\pi$
3. Evaluate AR at ω
4. If $AR > 1$ then process is unstable

Find ultimate gain:

1. Compute open-loop transfer function without controller gain
2. Solve for ω in $\phi = -\pi$
3. Evaluate AR at ω
4. Let $K_{cu} = \frac{1}{AR}$

Bode Criterion

Consider the transfer function and controller

$$G(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)} \quad G_c(s) = 0.4 \left(1 + \frac{1}{0.1s} \right)$$

- Open-loop transfer function

$$G_{OL}(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)} 0.4 \left(1 + \frac{1}{0.1s} \right)$$

- Amplitude ratio and phase shift

$$AR = \frac{5}{\sqrt{1+\omega^2}} \frac{1}{\sqrt{1+0.25\omega^2}} 0.4 \sqrt{1 + \frac{1}{0.01\omega^2}}$$

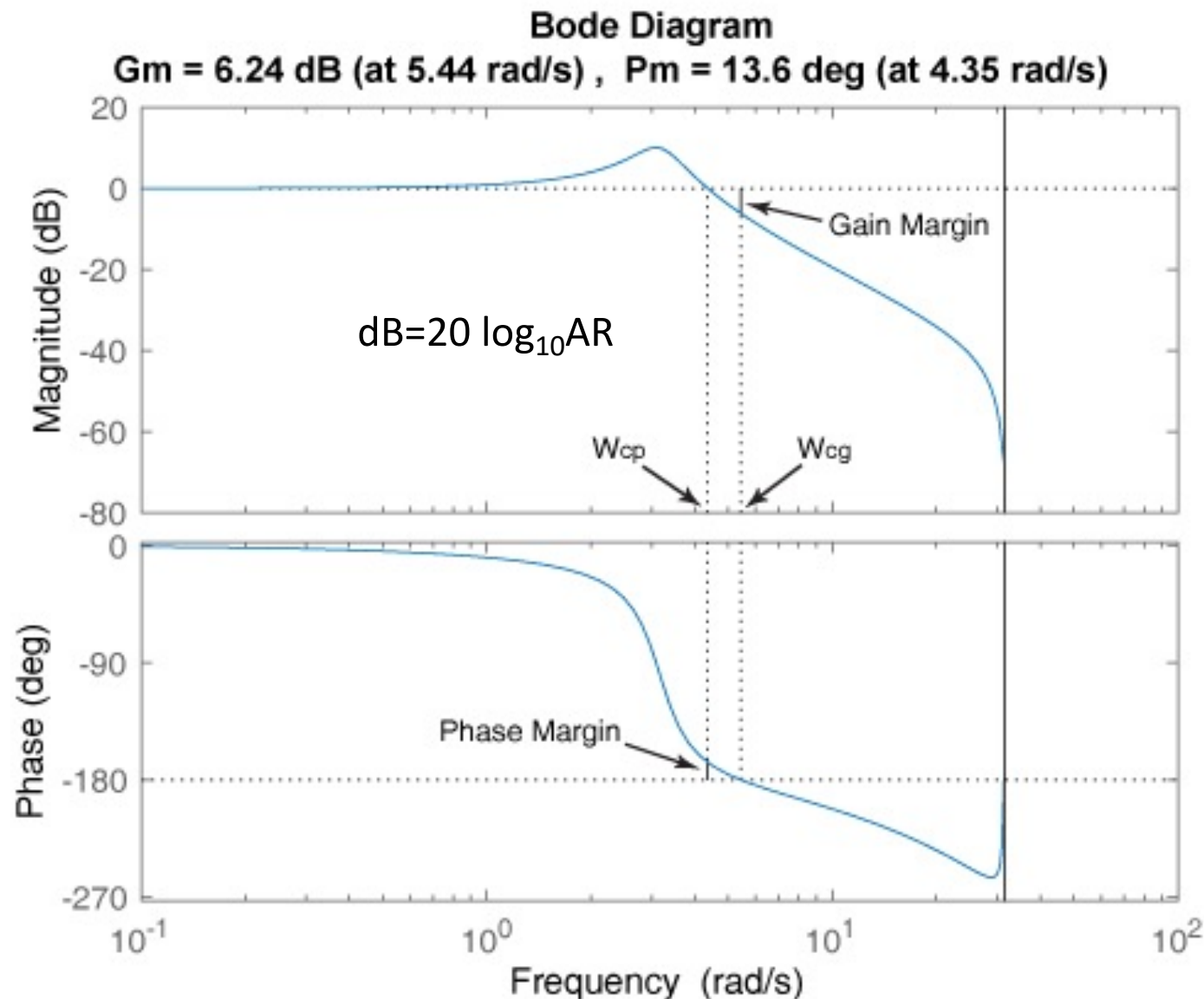
$$\phi = -0.1\omega - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\frac{1}{0.1\omega}\right)$$

- At $\phi = -\pi$, $\omega = 1.4128$, $AR = 6.746$

Bode Stability

- Phase Crossover frequency (ω_{gc}) is the frequency at which phase lag is 180^0 .
- Gain Crossover frequency (ω_{pc}) is the frequency at which $AR = 1$
- Gain Margin (GM) is the amount of gain required for the AR to reach 1 at phase crossover frequency.
 $GM = 1/AR_c$, where AR_c is the Amplitude ratio at gain crossover frequency.
- Phase Margin (PM) is the amount of phase difference required for phase lag to reach 180^0 at gain crossover frequency. $PM = 180 + \phi_{pc}$ where ϕ_{pc} is phase lag at phase crossover frequency.

Gain Margin and Phase Margin



W_{cg} is the frequency where the gain margin is measured, which is a -180° phase crossing frequency.

W_{cp} is the frequency where the phase margin is measured, which is a 0-dB gain crossing frequency.

Nyquist Stability Criterion

If N is the number of times that the Nyquist plot encircles the point $(-1,0)$ in the complex plane in the clockwise direction, and P is the number of open-loop poles of G_{OL} that lie in the right-half plane, then $Z=N+P$ is the number of unstable roots of the closed-loop characteristic equation.

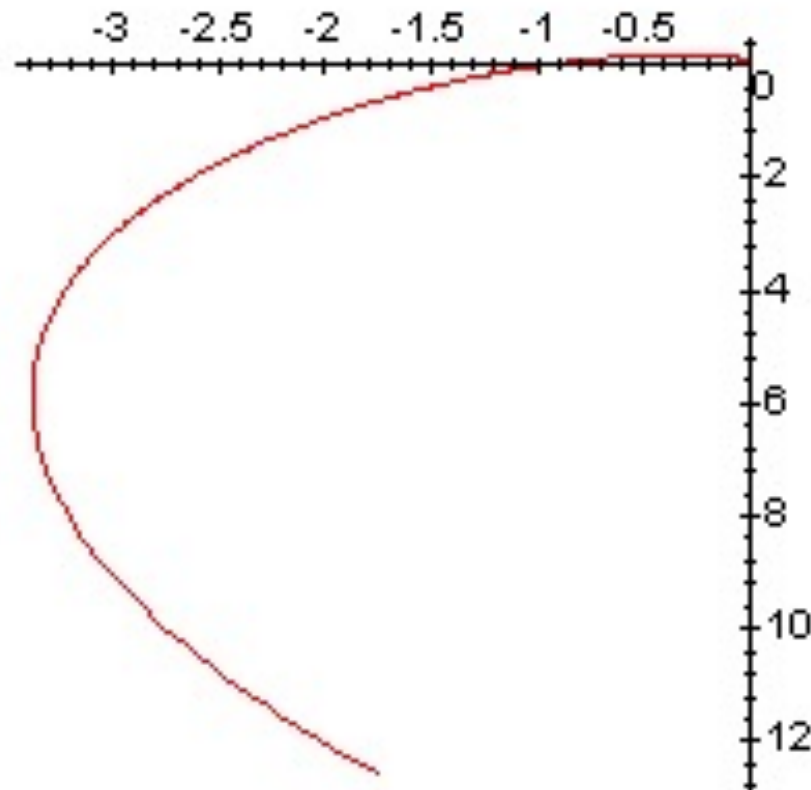
Strategy

1. Substitute $s=j\omega$ in $G_{OL}(s)$
2. Plot $G_{OL}(j\omega)$ in the complex plane
3. Count encirclements of $(-1,0)$ in the clockwise direction

Nyquist Criterion

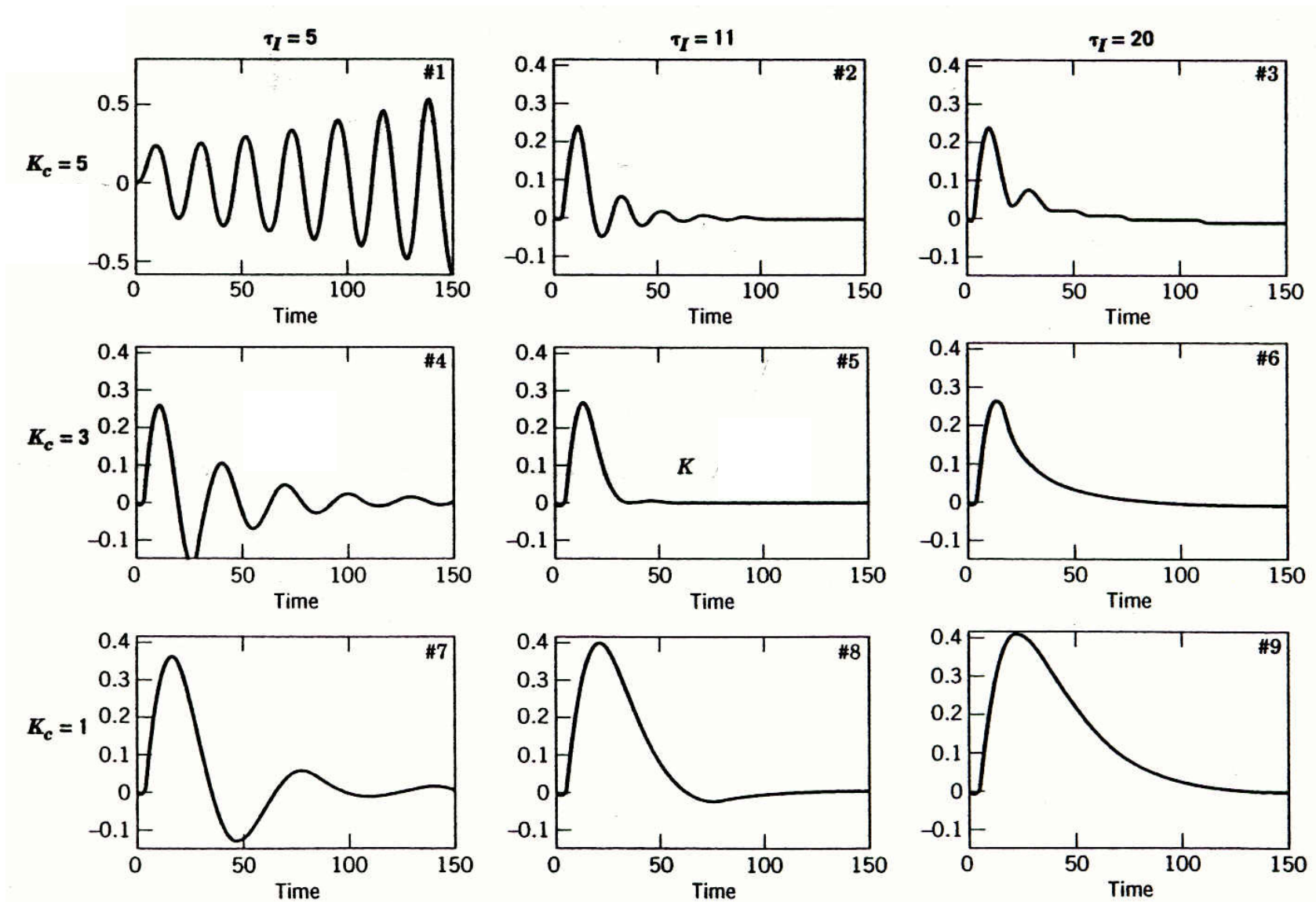
Consider the transfer function $G(s) = \frac{5e^{-0.1s}}{(s+1)(0.5s+1)}$

and the P controller $G_c(s) = 3.2$



PID Controller Tuning

Controller Tuning: A Motivational Example



Unit-step disturbance responses for the candidate controllers (FOTD Model: $K = 1, \tau = 20, \theta = 4$)

Performance Criteria For Closed-Loop Systems

1. The closed-loop system must be stable.
2. The effects of disturbances are minimized, providing good *disturbance rejection*.
3. Rapid, smooth responses to set-point changes are obtained, that is, good *set-point tracking*.
4. Steady-state error (offset) is eliminated.
5. Excessive control action is avoided.
6. The control system is robust, that is, insensitive to changes in process conditions and to inaccuracies in the process model.

Measure of Control Performance

- Overshoot.
- Decay ratio.
- Rising time or Response time.
- Stabilization time or Settling time.
- Integral Error
 - Integral Absolute Error (IAE) $\int_0^{\infty} |e(t)| dt$
 - Integral Square Error (ISE) $\int_0^{\infty} e^2(t) dt$
 - Integral time absolute error (ITAE) $\int_0^{\infty} t |e(t)| dt$
- Phase Margin
- Gain Margin

PID controller settings can be determined by a number of alternative techniques:

1. Empirical Tuning correlations using FODT model
 1. Step response FODT model
 2. Continuous Cycling
2. Direct Synthesis (DS) method
3. Internal Model Control (IMC) method
4. Optimization method
5. Frequency response techniques
6. On-line tuning after the control system is installed.


Empirical Tuning Correlations:

Controller : $G(c) = K_c \left(1 + \frac{1}{\tau_s} + \tau_D s \right)$

- For PID with derivative on output:

- $K'_c = \frac{2\tau_I K_c}{\tau_I + \sqrt{\tau_I(\tau_I - 4\tau_D)}} ; \tau'_I = \frac{\tau_I + \sqrt{\tau_I(\tau_I - 4\tau_D)}}{2} ; \tau'_D = \frac{\tau_I - \sqrt{\tau_I(\tau_I - 4\tau_D)}}{2}$

Step Response Model : $G(s) = \frac{K}{\tau s + 1} e^{-\theta s}$

- Ziegler-Nichols relations 
- Cohen-Coon
- Chien-Hrones-Reswick (CHR)

Controller Type	From Step response		
	K_c	τ_I	τ_D
P	$\frac{\tau}{\theta K}$		
PI	$0.9 \frac{\tau}{\theta K}$	3.3θ	
PID	$1.2 \frac{\tau}{\theta K}$	2θ	0.5θ

Cohen-Coon formula

- Z-N type empirical tuning formula using process model $G(s) = \frac{K}{\tau s + 1} e^{-\theta s}$
- Designed to achieve ¼ decay ratio
- Define $a = \frac{\tau}{\theta K}$

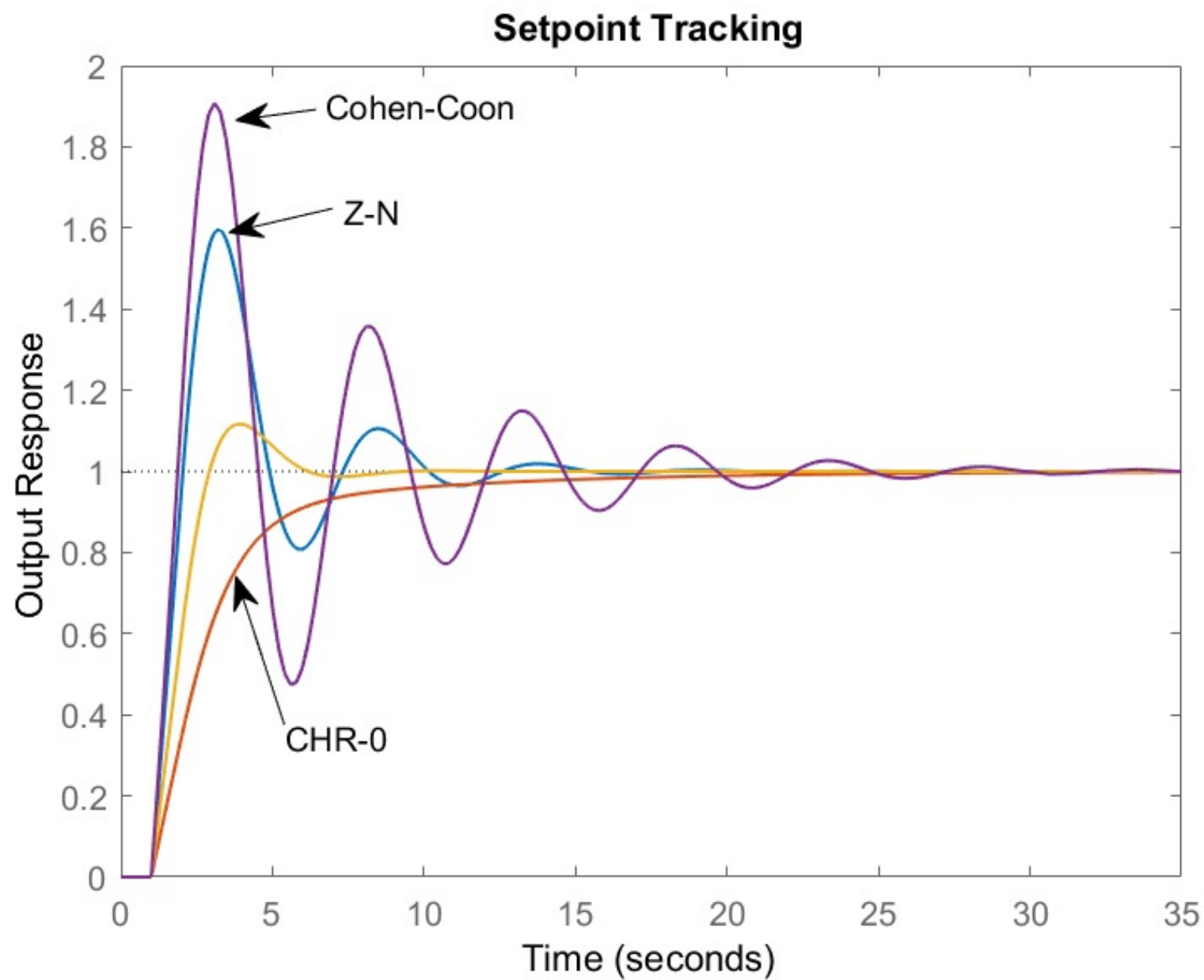
Controller Type	Cohen-Coon		
	K_C	τ_I	τ_D
P	$a \left(1 + \frac{\theta}{3\tau} \right)$		
PI	$a \left(0.9 + \frac{\theta}{12\tau} \right)$	$\frac{\theta [30 + 3(\theta/\tau)]}{9 + 20(\theta/\tau)}$	
PID	$a \left(\frac{3\theta + 16\tau}{12\tau} \right)$	$\frac{\theta [32 + 6(\theta/\tau)]}{13 + 8(\theta/\tau)}$	$\frac{4\theta}{11 + 2(\theta/\tau)}$

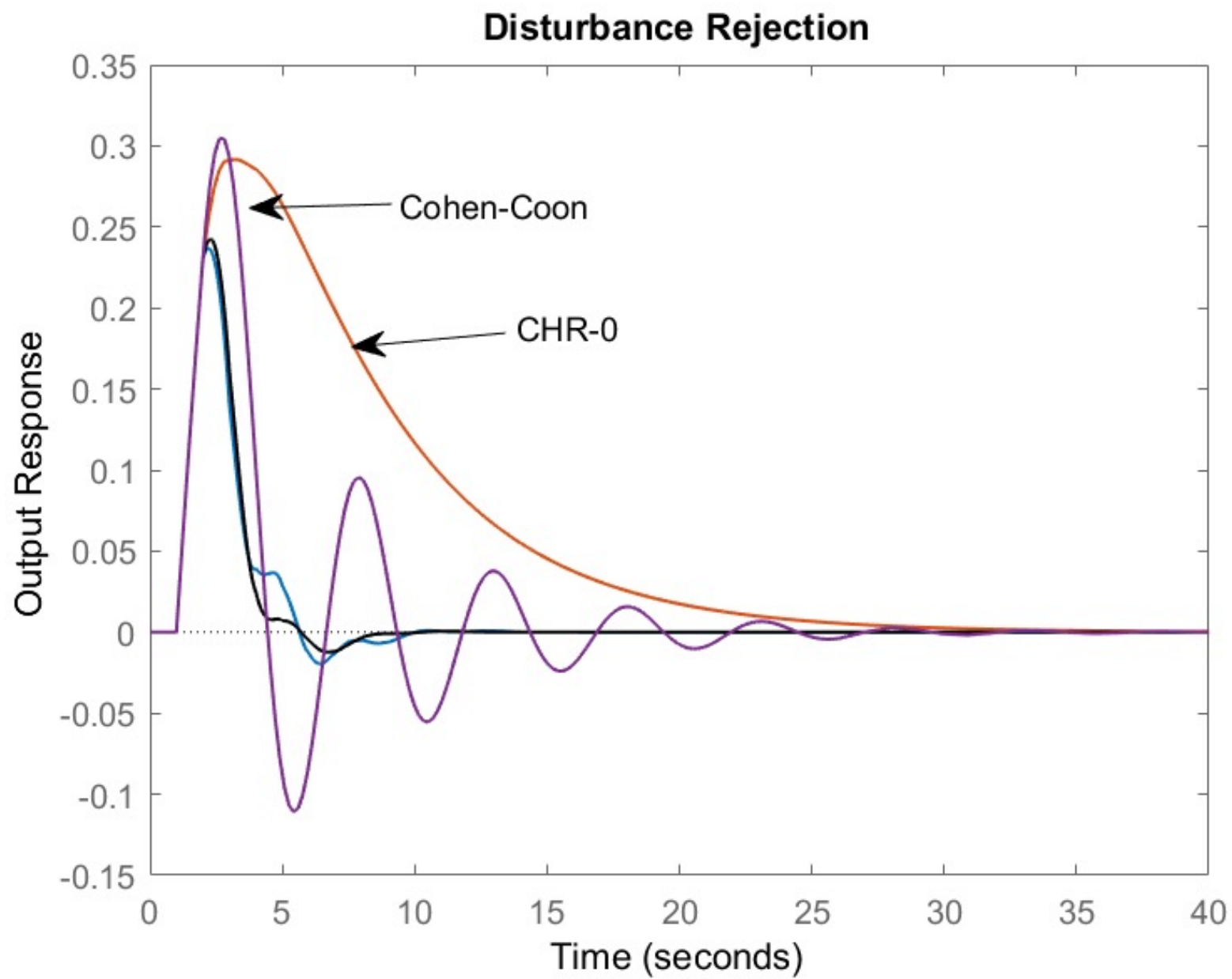
Chien–Hrones–Reswick (CHR) tuning formula

- Z-N type tuning formula

1. for known parameters FODT process $G(s) = \frac{K}{\tau s + 1} e^{-\theta s}$
2. Emphasizes setpoint tracking or disturbance rejection
3. Accommodates speed of response and overshoot qualitatively
4. Define $a = \frac{\tau}{\theta K}$

Controller Type	0% Overshoot Setpoint (Disturbance)			20% Overshoot Setpoint (Disturbance)		
	K_C	τ_I	τ_D	K_C	τ_I	τ_D
P	0.3 a (0.3 a)			0.7 a (0.7 a)		
PI	0.35 a (0.6 a)	1.2 τ (4 θ)		0.6 a (0.7 a)	τ (2.3 θ)	
PID	0.6 a (0.95 a)	τ (2.4 θ)	0.5 θ (0.42 θ)	0.95 a (1.2 a)	1.4 τ (2 θ)	0.47 θ (0.42 θ)





Continuous Cycling Method

Ziegler and Nichols (1942) introduced the *continuous cycling method* for controller tuning. It is based on the following trial-and-error procedure:

Step 1. After the process has reached steady state (at least approximately), eliminate the integral and derivative control action.

Step 2. Set K_c equal to a small value and place the controller in the automatic mode.

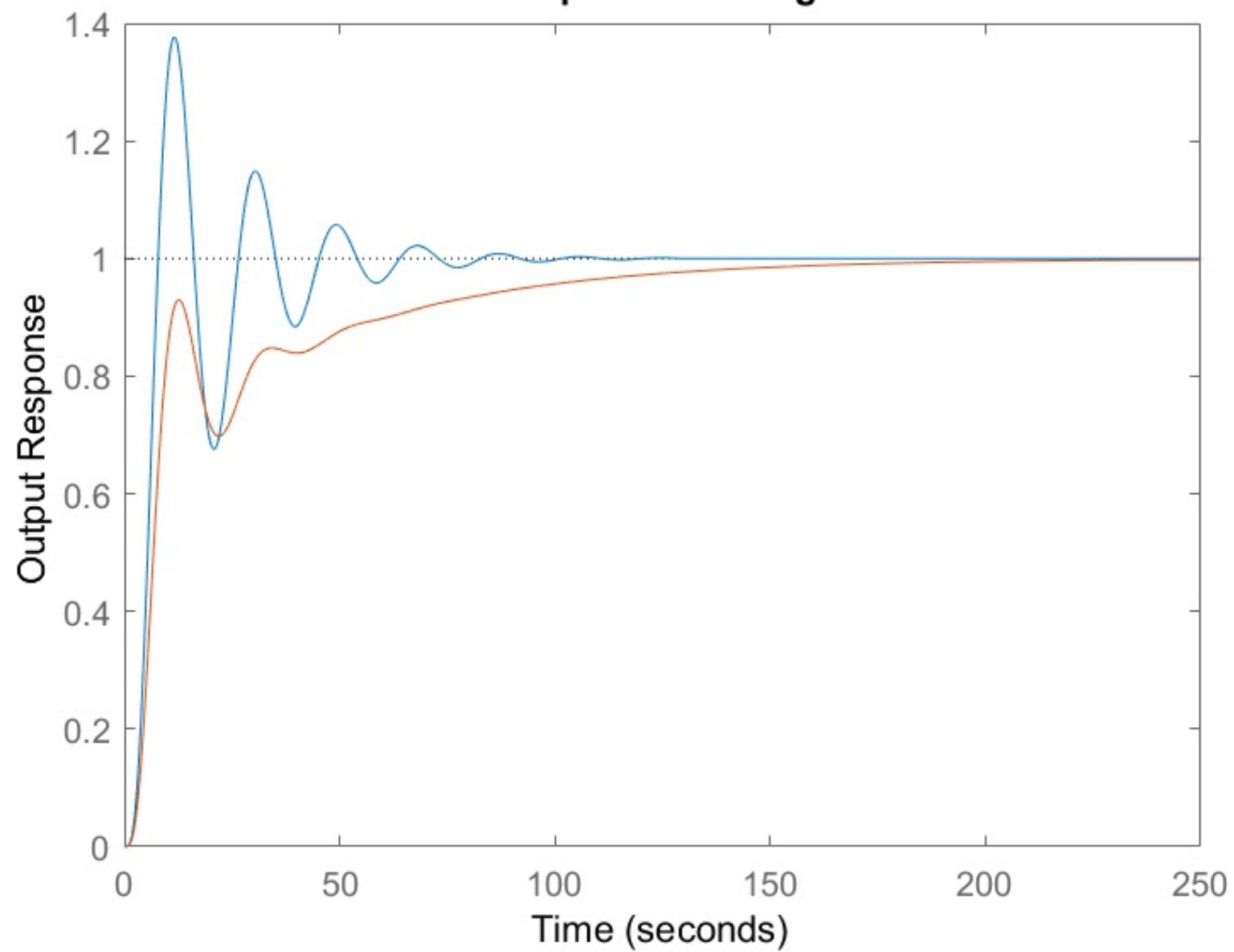
Step 3. Introduce a small set-point step change so that the controlled variable moves away from the set point. Gradually increase K_c in small increments until continuous cycling occurs.

The term continuous cycling refers to a sustained oscillation with a constant amplitude. The numerical value of K_c that produces continuous cycling (for proportional-only control) is called the *ultimate gain*, K_{cu} . The period of the corresponding sustained oscillation is referred to as the *ultimate period*, P_u .

Step 4. Calculate the PID controller settings using the tuning relations in Table.

Controller Type	Tyreus-Luyben			Ziegler-Nichols		
	K_C	τ_I	τ_D	K_C	τ_I	τ_D
P	-			$K_{cu} / 2$		
PI	$K_{cu} / 3.2$	$2.2 P_u$		$K_{cu} / 2.2$	$P_u / 1.2$	
PID	$K_{cu} / 2.2$	$2.2 P_u$	$P_u / 6.3$	$K_{cu} / 1.7$	$P_u / 2$	$P_u / 8$

Setpoint Tracking



Example

- Process Transfer function

$$G(s) = \frac{5}{(4s + 1)(3s + 1)(2s + 1)(s + 1)}$$

Characteristic Eqn. $1 + G_{OL} = 1 + K_c G = 0$

$$\text{So, } 24s^4 + 50s^3 + 35s^2 + 10s + 1 + 5K_c = 0$$

Routh Array

$$\begin{array}{ccc} 24 & 35 & 1+5K_c \end{array}$$

$$\begin{array}{cc} 50 & 10 \end{array}$$

$$\begin{array}{cc} 30.2 & 1+5K_c \end{array}$$

$$\begin{array}{ccc} 10 & -50(1+5K_c)/30.2 & 0 \end{array}$$

$$\begin{array}{c} 1+5K_c \end{array}$$



$$K_{cu} = -0.2 \text{ or } 1.008$$

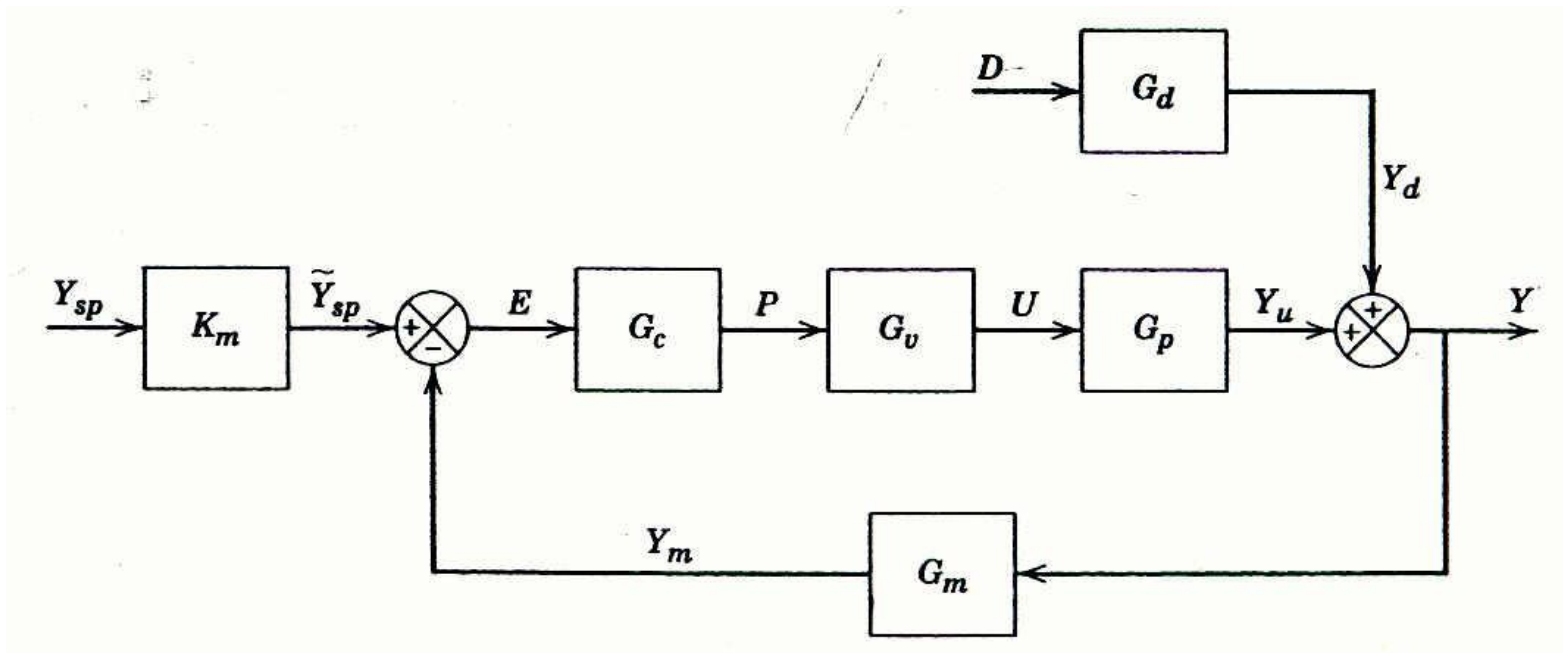
$$\omega_c = 0.446$$

$$P_u = 14.0879$$

Direct Synthesis (DS) method

In the Direct Synthesis (DS) method, the controller design is based on a **process model** and a **desired closed-loop transfer function**.

Consider the block diagram of a feedback control system in Figure



The closed-loop transfer function for set-point changes is:

$$\frac{Y}{Y_{sp}} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} \quad (1)$$

Direct Synthesis (DS) method

For simplicity, let $G \cong G_v G_p G_m$ and assume that $G_m = K_m$.

Then Eq. 1 reduces to:
$$\frac{Y}{Y_{sp}} = \frac{G_c G}{1 + G_c G}$$

Rearranging and solving for G_c gives an expression for the feedback controller:

$$G_c = \frac{Y/Y_{sp}}{G(1 - Y/Y_{sp})}$$

Above equation cannot be used for controller design because the closed-loop transfer function Y/Y_{sp} is *not known*.

Direct Synthesis (DS) method [cont.]

Also, it is useful to distinguish between the actual process G and the model, that provides an approximation of the process behavior.

A practical design equation can be derived by replacing the unknown G by \tilde{G} , and Y/Y_{sp} by a desired closed-loop transfer function, $(Y/Y_{sp})_d$:

$$G_c = \frac{(Y/Y_{sp})_d}{\tilde{G}(1 - Y/Y_{sp})_d}$$

Note that the controller transfer function contains the **inverse of the process model**.

Direct Synthesis (DS) method [cont.]

For processes without time delays, the following first-order model is a reasonable choice,

$$\left(\frac{Y}{Y_{sp}} \right)_d = \frac{1}{\tau_c s + 1}$$

Where τ_c is the desired closed loop time constant.

Because the steady-state gain is one, **no offset** occurs for set-point changes.

Direct Synthesis (DS) method [cont.]

By substituting

$$\left(\frac{Y}{Y_{sp}}\right)_d = \frac{1}{\tau_c s + 1} \xrightarrow{\text{into}} G_c = \frac{\left(Y/Y_{sp}\right)_d}{\tilde{G}(1 - Y/Y_{sp})_d}$$





Solving for G_c , the controller design equation becomes:

$$G_c = \frac{1}{\tilde{G}} \frac{1}{\tau_c s}$$

The term $\frac{1}{\tau_c s}$ provides integral control action and thus eliminates offset.

Design parameter τ_c provides a convenient controller tuning parameter that can be used to make the controller more aggressive (small τ_c) or less aggressive (large τ_c).

Direct Synthesis (DS) method [cont.]

- | Process Model, $\tilde{G}(s)$ | | Controller, $G_c(s)$ |
|--|--|--|
| • K |  | $\frac{1}{K\tau_c s}$ I-Controller |
| • $\frac{K}{s}$ |  | $\frac{1}{K\tau_c}$ P-Controller |
| • $\frac{K}{\tau s + 1}$ |  | $\frac{\tau}{K\tau_c} \left(1 + \frac{1}{\tau s}\right)$ PI |
| • $\frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$ |  | $\frac{\tau_1 + \tau_2}{K\tau_c} \left(\frac{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1}{(\tau_1 + \tau_2)s} \right)$ |

Direct Synthesis (DS) method [cont.]

If the process transfer function contains a known time delay θ , a reasonable choice for the desired closed-loop transfer function is:

$$\left(\frac{Y}{Y_{sp}} \right)_d = \frac{e^{-\theta s}}{\tau_c s + 1}$$

The time-delay term in above is essential because it is physically impossible for the controlled variable to respond to a set-point change at $t = 0$.

Combining reference trajectory with controller equation :

$$\left(\frac{Y}{Y_{sp}} \right)_d = \frac{e^{-\theta s}}{\tau_c s + 1} \quad G_c = \frac{\left(\frac{Y}{Y_{sp}} \right)_d}{\tilde{G} \left(1 - \left(\frac{Y}{Y_{sp}} \right)_d \right)}$$

Direct Synthesis (DS) method [cont.]

Combining reference trajectory with controller equation :

$$\left(\frac{Y}{Y_{sp}} \right)_d = \frac{e^{-\theta s}}{\tau_c s + 1} \quad G_c = \frac{\left(Y/Y_{sp} \right)_d}{\tilde{G} \left(1 - Y/Y_{sp} \right)_d}$$

Gives:

$$G_c = \frac{1}{\tilde{G}} \frac{e^{-\theta s}}{\tau_c s + 1 - e^{-\theta s}}$$

Approximating the time-delay term with a truncated Taylor series expansion: $e^{-\theta s} = 1 - \theta s$ we get

$$G_c = \frac{1}{\tilde{G}} \frac{e^{-\theta s}}{(\tau_c + \theta)s}$$

Note that this controller also contains integral control action.

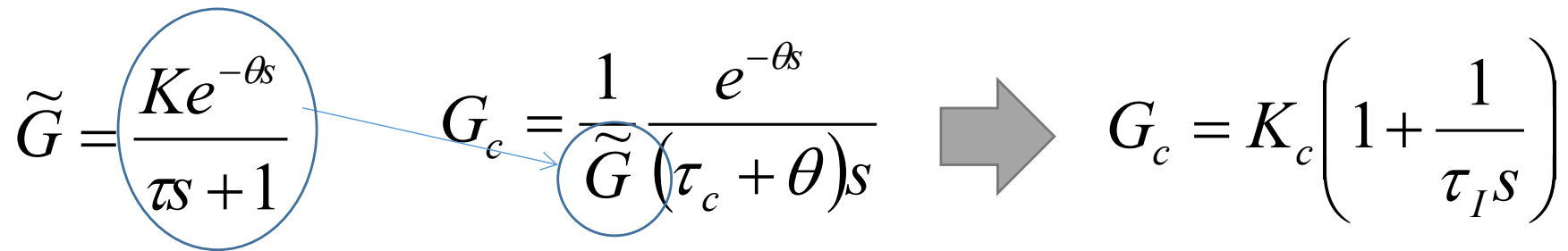
Direct Synthesis (DS) method [cont.]

I. First Order with Dead Time (FOTD) Process

Consider the standard FOTD model,

$$\tilde{G} = \frac{Ke^{-\theta s}}{\tau s + 1}$$

Substituting FOTD model into Controller equation and rearranging gives a PI controller;


$$\tilde{G} = \frac{Ke^{-\theta s}}{\tau s + 1} \quad G_c = \frac{1}{\tilde{G} (\tau_c + \theta)s} e^{-\theta s} \quad \Rightarrow \quad G_c = K_c \left(1 + \frac{1}{\tau_I s} \right)$$

with the following controller settings:

$$K_c = \frac{1}{K} \frac{\tau}{\theta + \tau_c} \quad \tau_I = \tau$$

Example 12.1

Use the DS design method to calculate PID controller settings for the process:

$$G = \frac{2e^{-s}}{(10s + 1)(5s + 1)}$$

Consider three values of the desired closed-loop time constant: .

$$\tau_c = 1, 3, 10$$

Evaluate the controllers for unit step changes in both the set point and the disturbance, assuming that $G_d = G$.

Repeat the evaluation for two cases:

- The process model is perfect ($\tilde{G} = G$).
- The model gain is incorrect, $\tilde{K} = 0.9$, instead of the actual value, $K = 2$.

Direct Synthesis (DS) method [cont.]

2. Second-Order-plus-Time-Delay (SOTD) Model

Consider a second-order-plus-time-delay model,

$$\tilde{G} = \frac{Ke^{-\theta s}}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

Substitution into G_c eqn. and rearranging gives a PID controller;

$$\tilde{G} = \frac{Ke^{-\theta s}}{(\tau_1 s + 1)(\tau_2 s + 1)} \rightarrow G_c = \frac{1}{\tilde{G}} \frac{e^{-\theta s}}{(\tau_c + \theta)s} \rightarrow G_c = K_c \left(1 + \frac{1}{\tau_I s} + \tau_D s \right)$$

with the following controller settings:

$$K_c = \frac{1}{K} \frac{\tau_1 + \tau_2}{\tau_c + \theta} \quad \tau_I = \tau_1 + \tau_2 \quad \tau_D = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}$$

Example 12.1- Solution

Use the DS design method to calculate PID controller settings for the for two cases:

$$\begin{aligned}\tilde{G} &= \frac{2e^{-\theta s}}{(10s+1)(5s+1)} \\ G_c &= \frac{1}{\tilde{G}} \frac{e^{-\theta s}}{(\tau_c + \theta)s}\end{aligned} \quad \Rightarrow \quad G_c = \frac{(10s+1)(5s+1)}{2e^{-\theta s}} \frac{e^{-\theta s}}{(\tau_c + \theta)s}$$

Comparing with standard PID controller;

$$\begin{aligned}G_c &= \frac{(10s+1)(5s+1)}{2(\tau_c + \theta)s} \\ &= \frac{50s^2 + 15s + 1}{2(\tau_c + \theta)s}\end{aligned} \quad \xleftrightarrow{\text{comparing}} \quad \begin{aligned}G_c &= K_c \left(1 + \frac{1}{\tau_I s} + \tau_D s \right) \\ &= K_c \left(\frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s} \right)\end{aligned}$$

Thus;

$$\tau_I = 15 \quad \tau_D = 50/15 = 3.33 \quad K_c = 15/[2(\tau_c + 1)]$$

Example 12.1- Solution

The controller settings are as follows:

$$\tau_I = 15 \quad \tau_D = 50/15 = 3.33 \quad K_c = 15/[2(\tau_c + 1)] \quad \text{(a) For } K = 2$$

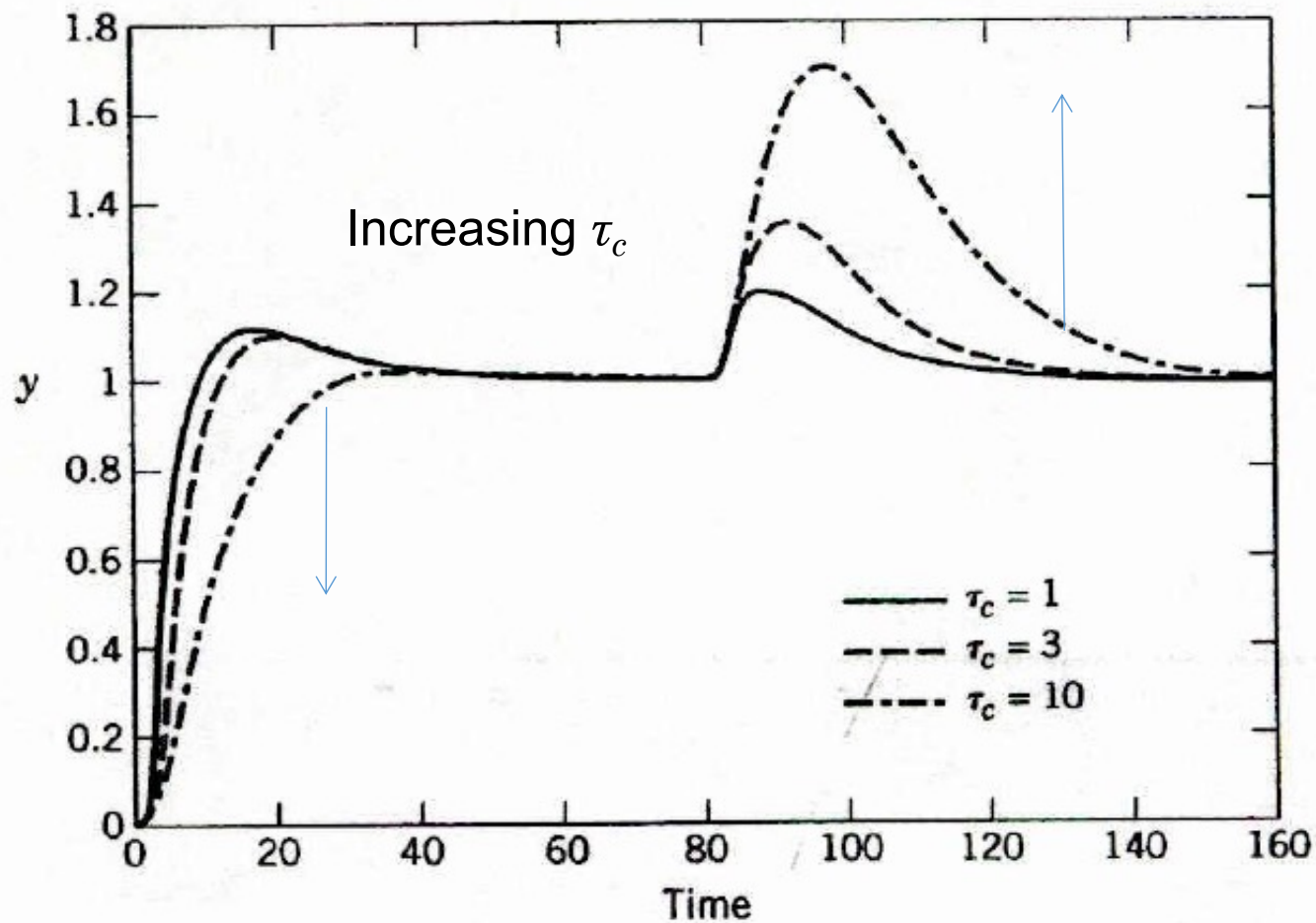
$$K_c = 15/[0.9(\tau_c + 1)] \quad \text{(b) For } K = 0.9$$

	$\tau_c = 1$	$\tau_c = 3$	$\tau_c = 10$
$K_c (\tilde{K} = 2)$	3.75	1.88	0.682
$K_c (\tilde{K} = 0.9)$	8.33	4.17	1.51
τ_I	15	15	15
τ_D	3.33	3.33	3.33

The values of K_c decrease as τ_c increases, but the values of τ_I and τ_D do not change

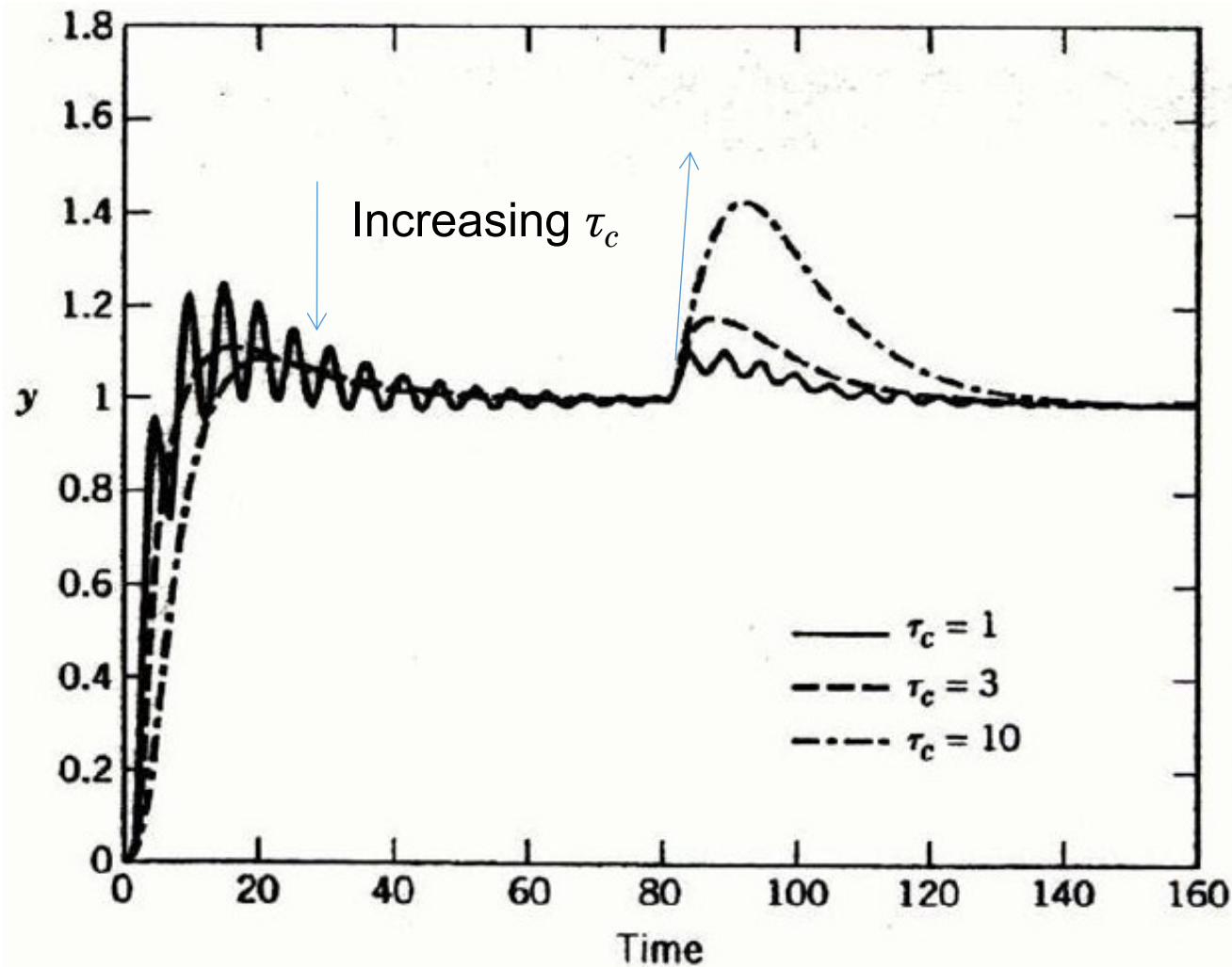
Example 12.1- Solution

Simulation results for (a) ($\tilde{G} = G$), $\tilde{K} = 2$



Example 12.1- Solution

Simulation results for (b) ($\tilde{K} = 0.9$).

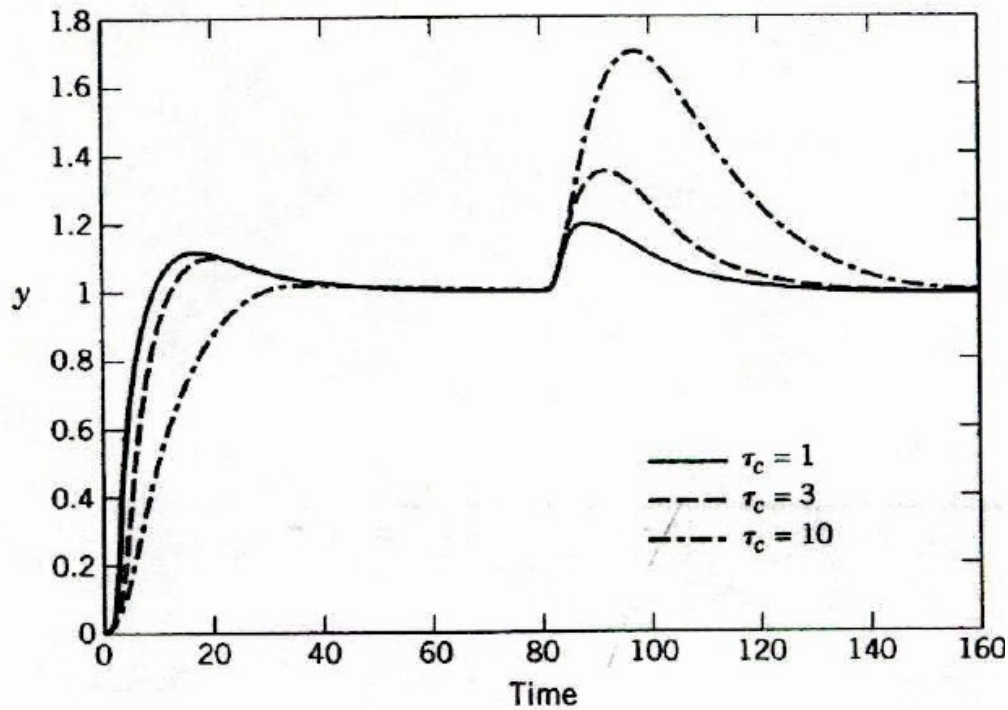


As τ_c increases, the responses become more sluggish

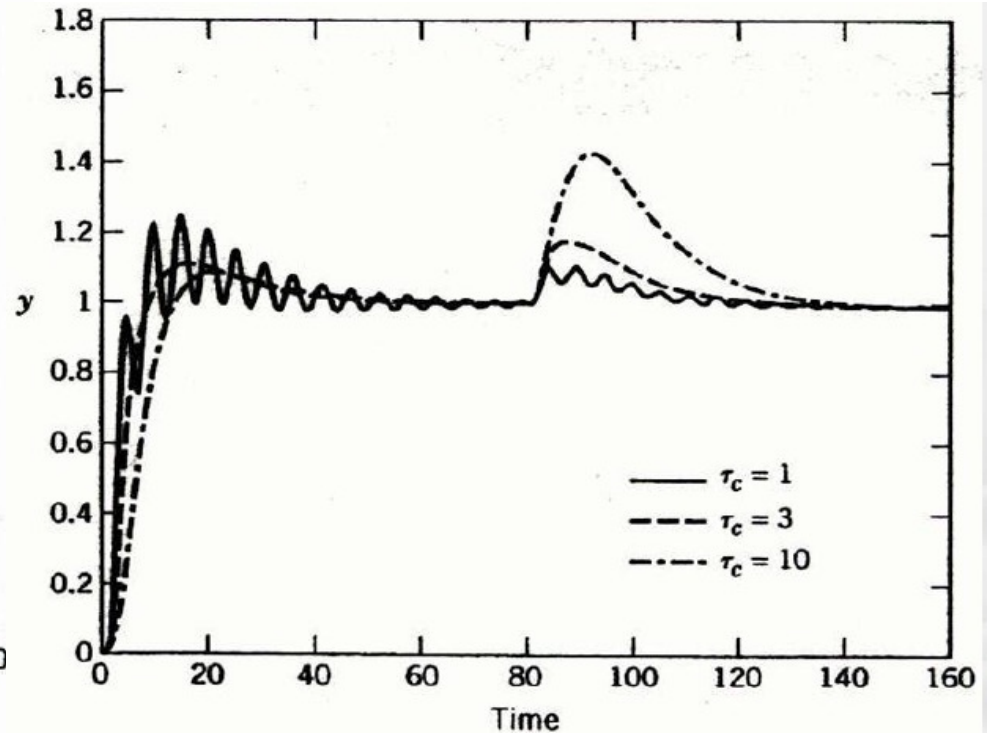
Example 12.1- Solution

Which one is better???WHY?

$$\tilde{K} = 2$$



$$\tilde{K} = 0.9$$



$$K_c = \frac{1}{K} \frac{\tau_1 + \tau_2}{\tau_c + \theta}$$

Thank you

Direct Synthesis Method

- In the Direct Synthesis (DS) method, the controller design is based on a process model and a desired closed-loop transfer function.
- The latter is usually specified for set-point changes, but responses to disturbances can also be utilized (Chen and Seborg, 2002).
- Although these feedback controllers do not always have a PID structure, the DS method does produce PI or PID controllers for common process models.
- As a starting point for the analysis, consider the block diagram of a feedback control system in Figure 12.2. The closed-loop transfer function for set-point changes was derived in Section 11.2:

$$\frac{Y}{Y_{sp}} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} \quad (12-1)$$

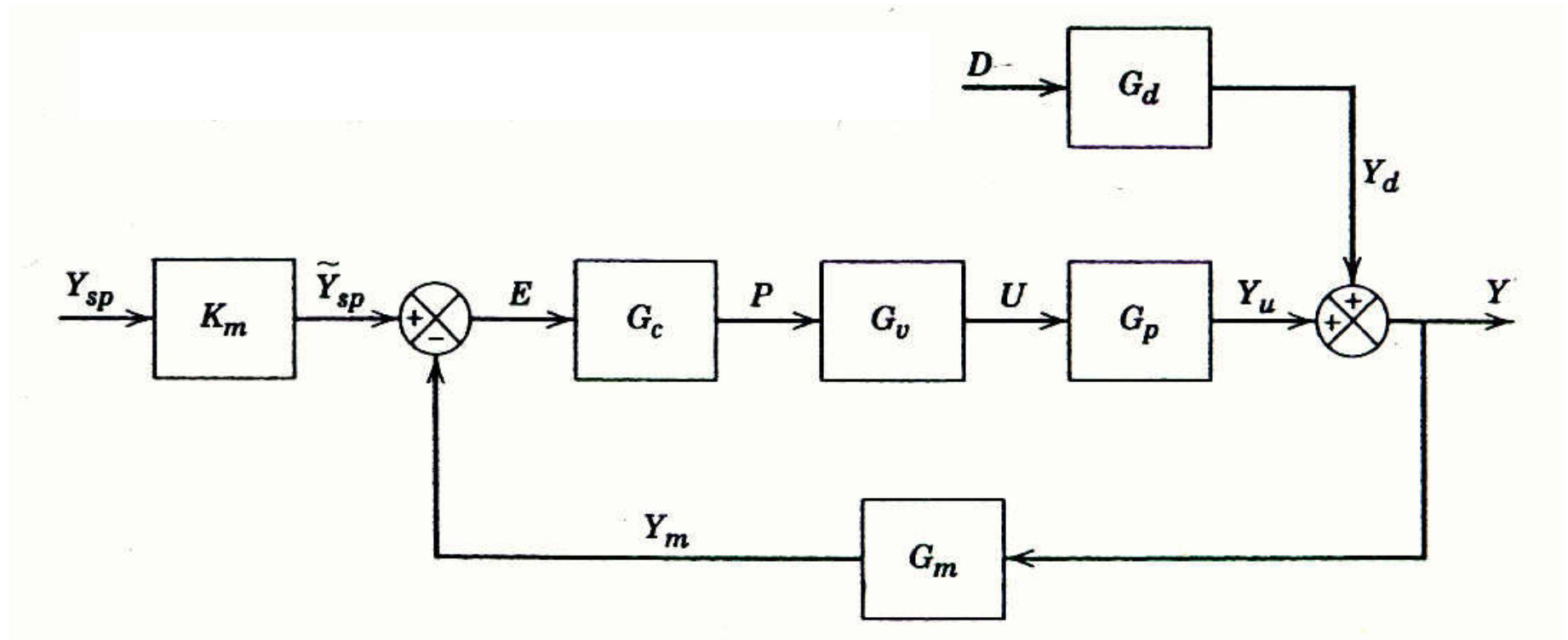


Fig. 12.2. Block diagram for a standard feedback control system.

For simplicity, let $G \triangleq G_v G_p G_m$ and assume that $G_m = K_m$. Then Eq. 12-1 reduces to

$$\frac{Y}{Y_{sp}} = \frac{G_c G}{1 + G_c G} \quad (12-2)$$

Rearranging and solving for G_c gives an expression for the feedback controller:

$$G_c = \frac{1}{G} \left(\frac{Y / Y_{sp}}{1 - Y / Y_{sp}} \right) \quad (12-3a)$$

- Equation 12-3a cannot be used for controller design because the closed-loop transfer function Y/Y_{sp} is not known *a priori*.
- Also, it is useful to distinguish between the actual process G and the model, \tilde{G} , that provides an approximation of the process behavior.
- A practical design equation can be derived by replacing the unknown G by \tilde{G} , and Y/Y_{sp} by a *desired closed-loop transfer function*, $(Y/Y_{sp})_d$:

$$G_c = \frac{1}{\tilde{G}} \left[\frac{(Y/Y_{sp})_d}{1 - (Y/Y_{sp})_d} \right] \quad (12-3b)$$

- The specification of $(Y/Y_{sp})_d$ is the key design decision and will be considered later in this section.
- Note that the controller transfer function in (12-3b) contains the inverse of the process model owing to the $1/\tilde{G}$ term.
- This feature is a distinguishing characteristic of model-based control.

Desired Closed-Loop Transfer Function

For processes without time delays, the first-order model in Eq. 12-4 is a reasonable choice,

$$\left(\frac{Y}{Y_{sp}} \right)_d = \frac{1}{\tau_c s + 1} \quad (12-4)$$

- The model has a settling time of $\sim 4\tau_c$, as shown in Section 5. 2.
- Because the steady-state gain is one, no offset occurs for set-point changes.
- By substituting (12-4) into (12-3b) and solving for G_c , the controller design equation becomes:

$$G_c = \frac{1}{\tilde{G}} \frac{1}{\tau_c s} \quad (12-5)$$

- The $1/\tau_c s$ term provides integral control action and thus eliminates offset.
- Design parameter τ_c provides a convenient controller tuning parameter that can be used to make the controller more aggressive (small τ_c) or less aggressive (large τ_c).