Lectures - 29 & 30.
6/11/17 &
7/11/17.

Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Compare with

$$a u_{xx} + b u_{xy} + c u_{yy} + f(x, y, u, p, q) = 0.$$

$$a = 1, \ b = 0, \ c = 1. \quad p = u_x, \ q = u_y.$$

$$b^2 - 4ac = -4 < 0.$$

$\therefore$ Lap. eqn. is of elliptic type.

Ex. Solve $u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \ y > 0. \rightarrow (1)$

subject to, $u(x,0) = f(x) \ \forall \ x \in (-\infty, \infty)$.

u is bdd. as $y \to \infty$, $u, u_x \to 0$ as $|x| \to \infty \rightarrow (*)$

Sol. Apply F.T. w.r.to x on both sides of (1):

$$\mathcal{F}[u_{xx}] + \mathcal{F}[u_{yy}] = 0.$$

or $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx} e^{i\omega x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{yy} e^{i\omega x} dx = 0.$$

or, $$(-i\omega)^2 U(\omega, y) + \frac{d^2}{dy^2} U(\omega, y) = 0$$

where $$U(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{i\omega x} dx.$$

$$\therefore \frac{d^2 U}{dy^2}(\omega, y) - \omega^2 U(\omega, y) = 0.$$

$$U(\omega, y) = A_1 e^{-\omega y} + A_2 e^{\omega y}$$

Taking condition (*) into consideration express $U(\omega, y)$ as

$$U(\omega, y) = A e^{-|\omega| y}.$$

$$U(\omega, y) = A(\omega) e^{-|\omega| y} \longrightarrow (**)$$

$e^{-|\omega| y} = e^{-\omega y}, \ \omega > 0.$

$e^{-|\omega| y} = e^{\omega y}, \ \omega < 0.$

$$u(x, 0) = f(x).$$

ake F.T. on both sides of $f(x)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx.$$

or,

$$U(\omega, 0) = F(\omega), \text{ say}.$$

From $(**)$, $U(\omega, 0) = A(\omega) = F(\omega)$.

$$\therefore U(\omega, y) = F(\omega) e^{-|\omega| y}.$$

$$\therefore u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega, y) e^{-i\omega x} d\omega.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-|\omega| y} e^{-i\omega x} d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) e^{i\omega u} du \right) e^{-|\omega| y} e^{-i\omega x} d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) du \left(\int_{-\infty}^{\infty} e^{i\omega u - |\omega| y - i\omega x} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} e^{-\{|\omega| y + i(x-u)\omega\}} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \left[\int_{-\infty}^{0} e^{-(-\omega y + i(x-u)\omega)}\,d\omega + \int_{0}^{\infty} e^{-(\omega y + i(x-u)\omega)}\,d\omega \right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \left\{ \int_{-\infty}^{0} e^{\omega\{y - i(x-u)\}}\,d\omega + \int_{0}^{\infty} e^{-\omega\{y+i(x-u)\}}\,d\omega \right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \left[\left.\frac{e^{\omega\{y-i(x-u)\}}}{y-i(x-u)}\right|_{-\infty}^{0} + \left.\frac{e^{-\omega\{y+i(x-u)\}}}{y+i(x-u)}\right|_{0}^{\infty} \right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \left(\frac{1}{y-i(x-u)} + \frac{1}{y+i(x-u)} \right).$$

$$= \frac{2y}{2\pi}\int_{-\infty}^{\infty} \frac{f(u)\,du}{y^2 + (x-u)^2}.$$

$$\therefore u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)\,du}{y^2+(x-u)^2}$$

$\rightarrow$ Poisson's integral for the Dirichlet-BVP

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0.$$

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

Ex 2. Solve.

$$u_{xx} + u_{yy} = 0 \quad \to (1) \quad -\infty < x < \infty,\ y > 0.$$

$$u_y(x, 0) = f(x) \quad \to (2), \quad -\infty < x < \infty.$$

$u(x,y)$ is bounded as $y \to \infty$, $u, u_x \to 0$ as $|x| \to \infty$

Solution. Let us define.

$$\phi(x,y) = \frac{\partial u}{\partial y}(x,y) \text{ ; where } u(x,y) \text{ satisfies the eqn (1) } ~~Dirichlet problem~~$$

Then

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}\right).$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right).$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial}{\partial y}(0) = 0.$$

Also, $\phi(x, 0) = \frac{\partial}{\partial y} u(x, 0) = f(x)$. from (2).

$\therefore$ $\phi(x, y)$ satisfies $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$, $-\infty < x < \infty$, $y > 0$

& $\phi(x, 0) = f(x)$, $-\infty < x < \infty$).

$\therefore$ $\phi(x, y)$ satisfies a Dirichlet problem in $-\infty < x < \infty$.

Hence its solution is given by,

$$\phi(x,y) = \frac{\partial u}{\partial y}(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)\,du}{y^2 + (x-u)^2}$$

$$u(x,y) = \int \frac{\partial u}{\partial y}(x,y)\, dy + C.$$

$$= \int \frac{y}{\pi}\left(\int_{-\infty}^{\infty} \frac{f(u)\, du}{y^2 + (x-u)^2} \right) dy + C$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u)\, du \int \frac{y\, dy}{y^2 + (x-u)^2} + C.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \ \ln \left| y^2 + (x-u)^2 \right| + C.$$

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \left| y^2 + (x-u)^2 \right| f(u)\, du + C.$$

$u(x,y)$ will exist if $\int_{-\infty}^{\infty} f(x,y)\, dx = 0.$

(1) $\rightarrow$ Nec. condition for existence of sol. of Newmann problem.

(2) $\rightarrow$ Solution to Newmann problem is not unique.

Ex. Solve, using F.T. w.r.to x

$$u_{tt} + u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t > 0 \quad \to (1)$$

$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad -\infty < x < \infty.$

Apply F.T. w.r.to x on (1),

$$(-i\omega)^4 \cancel{U}(\omega,t) + \frac{d^2}{dt^2} U(\omega,t) = 0.$$

$\mathcal{F}[f^{(n)}(x)] = (-i\omega)^n F(\omega)$

$$\therefore \frac{d^2 U(\omega,t)}{dt^2} + \omega^4 U(\omega,t) = 0.$$

a.e: $m^2 + \omega^4 = 0$

$m = \pm i\omega^2$

$$U(\omega,t) = A_1 \cos(\omega^2 t) + A_2 \sin(\omega^2 t).$$

$$A_1 = F(\omega) \qquad A_2 = 0.$$

$$U(\omega,t) = F(\omega) \cos \omega^2 t.$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \cos \omega^2 t \, e^{-i\omega x} \, d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) e^{i\omega u} \, du \right) \cos \omega^2 t \, e^{-i\omega x} \, d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \left(\int_{-\infty}^{\infty} e^{i\omega u - i\omega x} \cos \omega^2 t \, d\omega. \right)$$

$$U(\omega, t) = F(\omega) \cos \omega^2 t$$

$$u(x,t) = (f * g)(x,t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u, t)\, du.$$

where $g(x,t) = F^{-1}(\cos \omega^2 t)$.

<u>Q.</u> Find $F^{-1}(\cos \omega^2 t)$; given $\mathcal{F}\left[\cos \frac{\omega^2}{2}; x\right] = \cos\left(\frac{\pi}{4} - \frac{x^2}{2}\right)$

$$F^{-1}(\cos \omega^2 t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos \omega^2 t\, e^{-i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos \omega^2 t\, e^{i\omega x} d\omega \equiv F\left\{\cos \omega^2 t; x\right\}.$$

Now, $\mathcal{F}\left[\cos \frac{\omega^2}{2}; x\right] = \cos\left(\frac{\pi}{4} - \frac{x^2}{2}\right) = F(x)$, say.

If $f(\omega) = \cos \frac{\omega^2}{2}$

then $\cos \omega^2 t = \cos\left(\frac{\omega^2}{2} \cdot 2t\right) = f(\sqrt{2t}\, \omega)$. $\mathcal{F}[f(a\omega)] = \frac{1}{|a|} F\left(\frac{x}{a}\right)$

$$\therefore \mathcal{F}[\cos \omega^2 t] = \mathcal{F}[f(\sqrt{2t}\, \omega)] = \frac{1}{\sqrt{2t}} F\left(\frac{x}{a}\right)$$

$$= \frac{1}{\sqrt{2t}} \cos\left(\frac{\pi}{4} - \frac{x^2}{2a^2}\right)$$

$$= \frac{1}{\sqrt{2t}} \cos\left(\frac{\pi}{4} - \frac{x^2}{4t}\right)$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(u) \cos\left\{\frac{\pi}{4} - \frac{(x-u)^2}{4t}\right\} du$$

Solve applying Laplace Transform w. r. to t.

Ex 1. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$; $x>0,\ t>0 \longrightarrow (1)$.

$u(x,0) = x$, $u(0,t) = t$.
$\longrightarrow (2)$ $\longrightarrow (3)$

Let $\bar{u}(x,s) = \int_0^\infty u(x,t)\, e^{-st}\, dt$.

Apply LT w. r. to t on both sides of (1).

$$L\left[\frac{\partial u}{\partial x}\right] = L\left[\frac{\partial u}{\partial t}\right].$$

or, $\int_0^\infty \frac{\partial u}{\partial x} e^{-st} dt = \int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt$.

or, $\frac{d}{dx}\bar{u}(x,s) = s\bar{u}(x,s) - u(x,0)$.

$L[f'(t)] = s\bar{f}(s) - f(0)$.

$\therefore \frac{d}{dx}\bar{u}(x,s) - s\bar{u}(x,s) = -x$.

I.F: $e^{-\int s\,dx} = e^{-sx}$.

$\frac{dy}{dx} + P(x)y = Q(x)$

I.f $= e^{\int P(x)\,dx}$.

$\therefore \frac{d}{dx}\bar{u}(x,s)\, e^{-sx} = -x e^{-sx}$.

$\therefore \bar{u}(x,s)\, e^{-sx} = -\int x\, e^{-sx}\, dx + C(s)$

$$= \frac{e^{-sx}}{s}\cdot x - \int \frac{e^{-sx}}{s}\, dx + C(s)$$

$\bar{u}(x,s) = e^{sx}\left[\frac{x}{s} e^{-sx} + \frac{e^{-sx}}{s^2}\right] + C(s) e^{sx}$.
$\longrightarrow (4)$.

$u(0,t) = t$.

$L[u(0,t)] = \frac{1}{s^2} \Rightarrow \bar{u}(0,s) = \frac{1}{s^2}$.

In (4) subst; $x = 0$.

$\frac{1}{s^2} = \bar{u}(0,s) = \frac{1}{s^2} + c(s)$.

$c(s) = 0$

$\bar{u}(x,s) = \frac{x}{s} + \frac{1}{s^2}$.

$\therefore u(x,t) = L^{-1}\left(\frac{x}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right)$.

$= x\,L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) = x + t$.

2. Solve. $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x$; $x > 0,\ t > 0$. $\longrightarrow (1)$

$u(x,0) = 0,\ x > 0$ & $u(0,t) = 0$; $t > 0$.

Apply LT w.r.to t on both sides of (1):

$\frac{d}{dx}\bar{u}(x,s) + \frac{s}{x}\bar{u}(x,s) = \frac{1}{s}$.

I.F. $= x^s$

$\frac{d}{dx}\left(\bar{u}(x,s)\,x^s\right) = \frac{x^s}{s}$.

$\therefore \bar{u}(x,s)\,x^s = \frac{x^{s+1}}{s(s+1)} + c$.

Put $x=0$ $0 \cdot 0 = 0 + c \Rightarrow c = 0$.

$u(0,t) = 0$.

$\therefore \bar{u}(0,s) = 0$.

$$\bar{u}(x,s) = \frac{x}{s(s+1)} = \left(\frac{1}{s} - \frac{1}{s+1}\right) x ,$$

$$u(x,t) = x\left(1 - e^{-t}\right).$$

Ex. Solve

$$u_t + u_x = 0 , \quad x > 0, \ t > 0 .$$

$$u(x,0) = \sin x , \quad u(0,t) = 0$$

by applying LT w.r.to t.

$$\frac{d}{dx} \bar{u}(x,s) + s\bar{u}(x,s) = \sin x .$$

$$\frac{d}{dx} \bar{u}(x,s) e^{sx} = \sin x \, e^{sx}$$

$$\bar{u}(x,s) = \frac{s \sin x - \cos x}{1+s^2} + c e^{-xs}$$

$$c = \frac{1}{1+s^2} .$$

$$\bar{u}(x,s) = \frac{s \sin x - \cos x}{1+s^2} + \frac{e^{-xs}}{1+s^2} .$$

$$u(x,t) = \sin x \cos t - \cos x \sin t + \sin(t-x) H(t-x)$$

$$= \sin(x-t) - \sin(x-t) H(t-x) .$$

$$(x,t) = \ldots, x < \ldots \quad = \begin{cases} \sin(x-t) - \sin(x-t) , & t - x > 0 \Rightarrow x < t \\ \sin(x-t) - 0 , & t - x < 0 , \ x > t . \end{cases}$$

$$= \ldots, x > \ldots$$