References: 1. Bacquette
2. Oggunaike
3. Stephanopoulos

PAD

08.01.21 Process Dynamics (and Control)

Process Modeling
Math. eqs. and assumptions

→ Process Dynamics
- How fast do the changes occur?
- Where am I going?

→ Process Control
- Process demand
- Process safety

Bosch Haber process: Synthesis of NH_3 from N_2 and H_2
~500 bar, 650°C + catalyst → process demand

The importance of process dynamics is to identify the extremities of the system.
For example, in the above case, the two extremes are -
1. T and P fall - T = 27°C, P = 1 atm → against process demand
2. Reactor blows up → against process safety

We won't focus on modeling. Model eqs. will be provided.

Process Dynamics

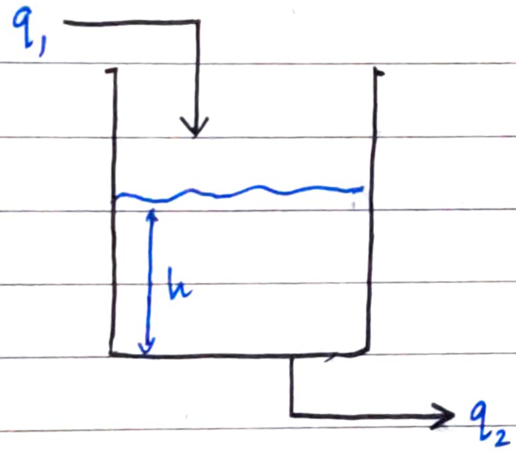
→ State-Space domain
Tells everything that can happen in the system - all extremeties.

→ Transform domain
B/w two states - how to reach a state from another.

classmate

We always start from the state-space domain to understand the possibilities and then move to transform domain to reach goals.

Dynamical System – Has atleast one variable which varies in time.



Liquid-level problem.

Consider a tank

$q_1, q_2 \rightarrow$ volumetric flow rates

$h \rightarrow$ level of liq.

$A \rightarrow$ const. cross-section area.

Mass Conservation –

Input – Output = Accumulation + time rate of change (+/–) generation.

$$q_1 - q_2 = A\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{q_1}{A} - \frac{q_2}{A} \rightarrow \text{model eq.}$$

Here, the dynamical variable is h.

State - space representation of a dynamical system

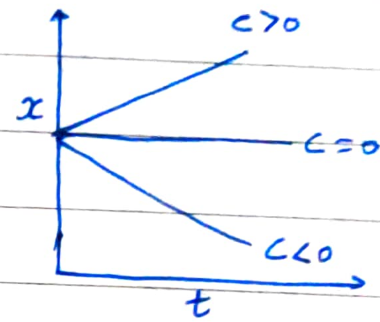
Case 1: $q_1 = \text{constant} = c_1$

$q_2 = \text{constant} = c_2$

$$\therefore \frac{dx}{dt} = \frac{c_1 - c_2}{A} = \text{constant} = c. \quad , \text{ where } x = h(t)$$

Consider 3 cases.

$c > 0; \; c = 0, \; c < 0$



What happens as $t \to \infty$? – Depends on c.

Case 2 : $q_2 = f(h)$ $\rightarrow$ gravity driven flow

$q_1 = \text{constant} = c$

$$\frac{dx}{dt} = \frac{c - f(h)}{A}$$

$$\Rightarrow \frac{dx}{dt} + \frac{1}{A} f(h) = \frac{c}{A} \quad \rightarrow \text{can be solved using I.F.}$$

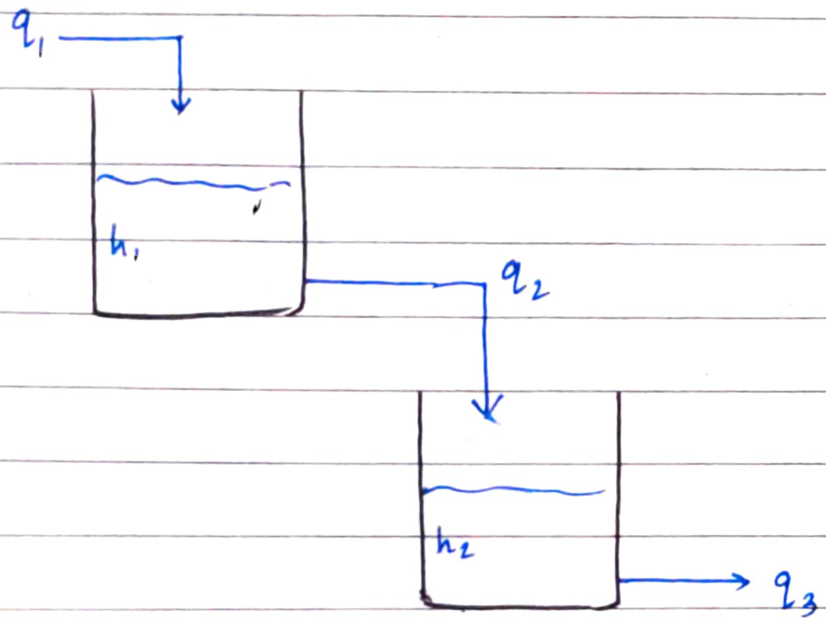
Case 3 : $q_2 = g(h)$ $\rightarrow$ q_2 altered by placing a valve

$q_1 = f(t)$ $\rightarrow$ q_1 altered as function of time by placing a pump.

$$\frac{dx}{dt} = \frac{-g(x) + f(t)}{A}$$

$$\frac{dx}{dt} + \frac{g(x)}{A} = \frac{f(t)}{A}$$

Now, let's complicate it more.



Here, there are 2 dynamical variables h_1 and h_2

The state of the system is described by a "vector" : state vector $[h_1 \; h_2]^T$

hence, state space domain

$$A_1 \frac{dh_1}{dt} = q_1 - q_2 \quad ; \quad A_2 \frac{dh_2}{dt} = q_2 - q_3$$

Cases:

1. $q_1, q_2, q_3 \rightarrow$ const.
2. $q_1 = f(t)$, $q_2 = f(h_1)$, $q_3 = f(h_2)$
3. $q_1 = f(h_1)$, $q_2 = f(h_1, h_2)$, $q_3 = f(h_2)$

What is the "order" of the two-tank liquid level system?

$$\frac{dh_1}{dt} = \frac{q_1 - q_2}{A} \quad ; \quad \frac{dh_2}{dt} = \frac{q_2 - q_3}{A}$$

Order of a system is equal to the number of 1st order ODEs used to model the system.

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b \rightarrow \text{II order.}$$

Let $\frac{dx}{dt} = y \Rightarrow a_2 \frac{dy}{dt} + a_1 y + a_0 x = b$

Hence,

$$\left\{ \begin{array}{ll} \frac{dy}{dt} = \frac{b}{a_2} - \frac{a_0}{a_2}x - \frac{a_1}{a_2}y & \text{and} \rightarrow \text{I order} \\ \frac{dx}{dt} = y & \rightarrow \text{I order.} \end{array} \right.$$

Together, a II order system.

So, a general n^{th} order system is represented as -

$$\left. \begin{array}{l} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ \frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N \end{array} \right\} N \text{ 1st order systems.}$$

classmate

This is equivalent to,

$$\frac{d}{dx}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ a_{N1} & \cdots & \cdots & a_{NN} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Eigen value problem

State vector whose variation with time is of interest.

Example :

Consider a fluid reservoir at $T = T_\infty$, in which an object of temperature T_0 is immersed at $t = 0$. The time rate of change of temp. of object is governed by Newton's law of cooling, according to which rate of change of temp of a body is proportional to the diff. in temp of the body and the surroundings. Hence,

$$\frac{dT}{dt} = -h(T - T_\infty)$$

Determine the equilibrium temperature(s) of the system.

classmate

Given, $\frac{dT}{dt} = -h(T - T_\infty)$

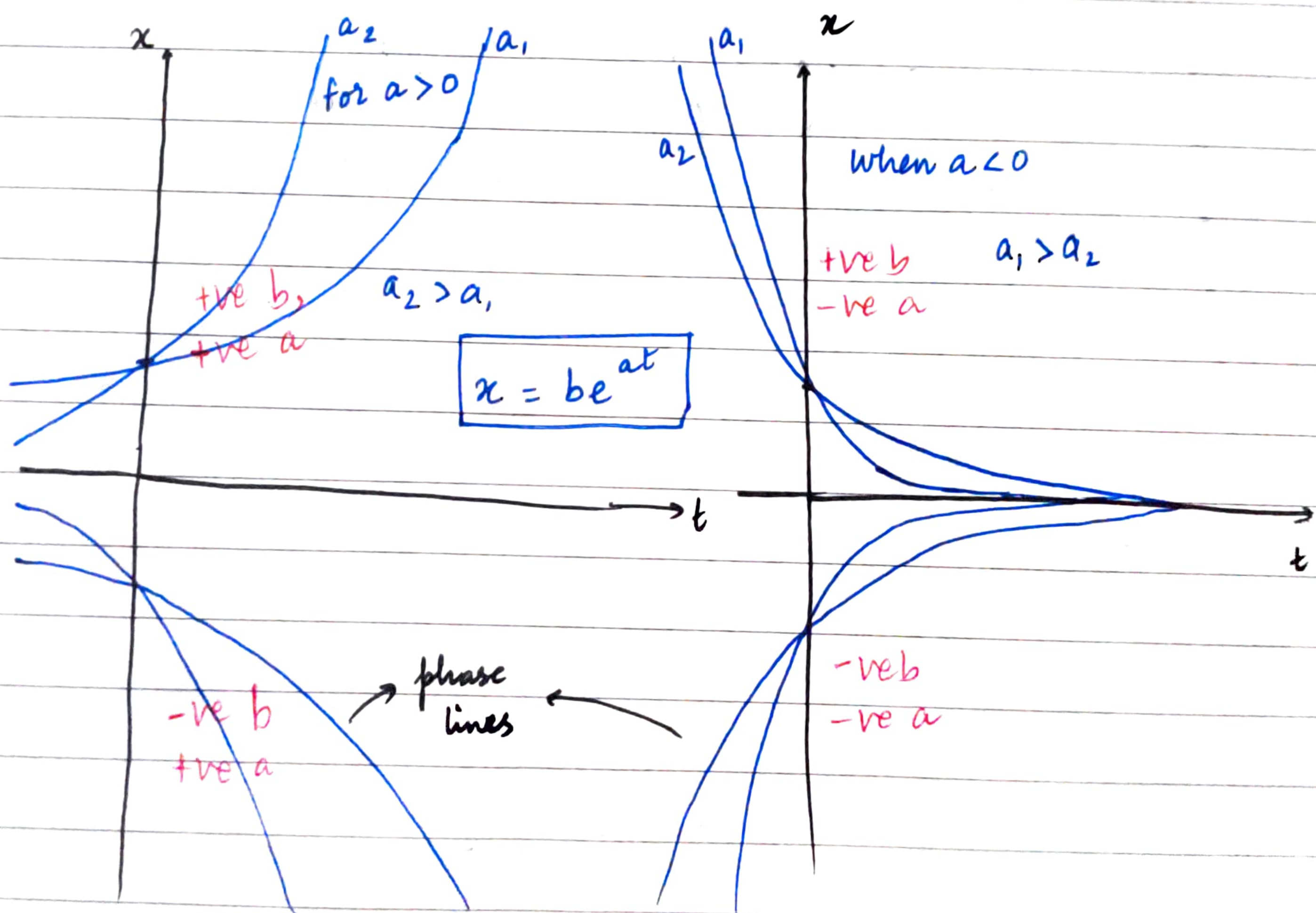
Let us consider the simplest 1^{st} order system modelled as -

$$\frac{dx}{dt} = ax$$

$\Rightarrow x(t) = x(0) \cdot e^{at}$.

↳ Initial condition.

Let $x(0) = 1$; then $x(t) = e^{at}$



Phase portrait. – Collection of solutions of system that represents all possibilities.

Q: What is the effect of 'a' on the system?

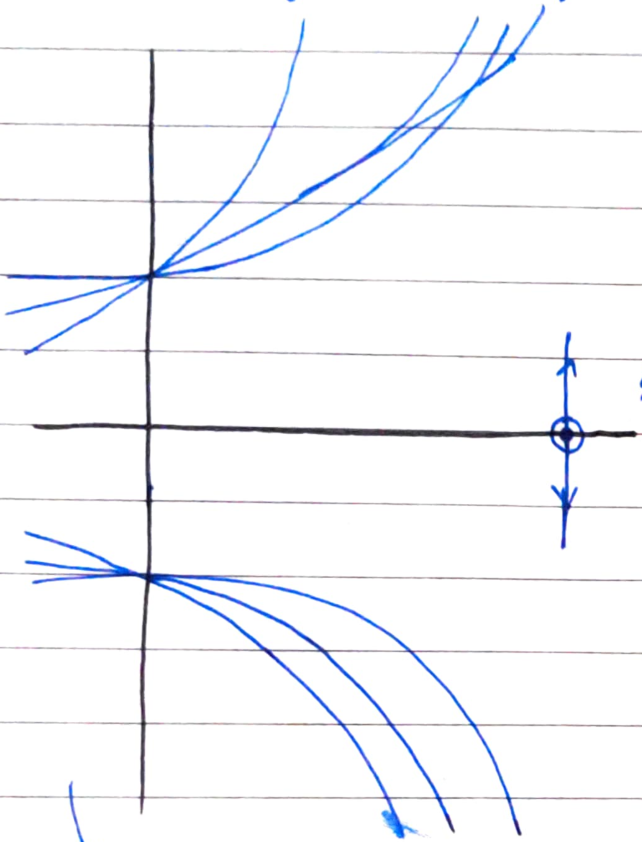
→ Large 'a' means quicker dynamics.

Q: What is the fate of the system? (or) What happens to the system as $t \to \infty$?

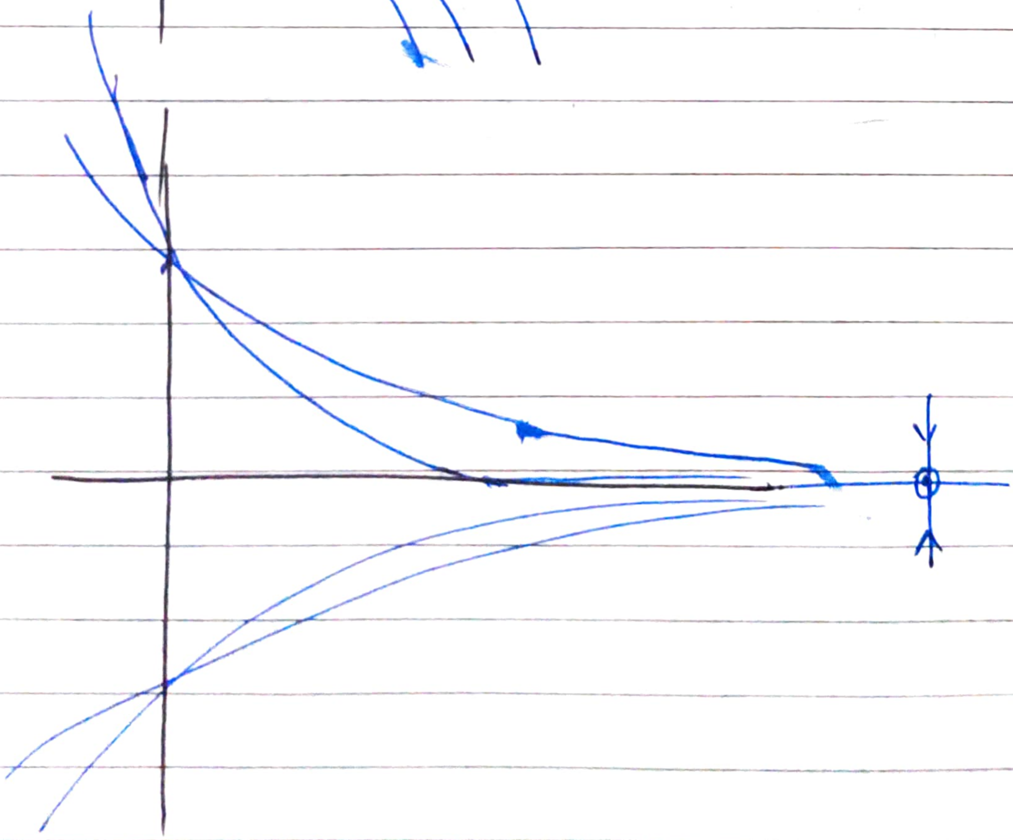
→ If $a > 0$, $x \to \infty$; If $a < 0$, $x \to 0$ irrespective of magnitude of a.

→ splits into 2.

And, the system has bifurcation at $a = 0$.



solutions diverging from this point. } called $\boxed{\text{source solution}}$



solutions converging called $\boxed{\text{sink solution}}$.

$$\frac{dT}{dt} = -h(T - T_\infty)$$

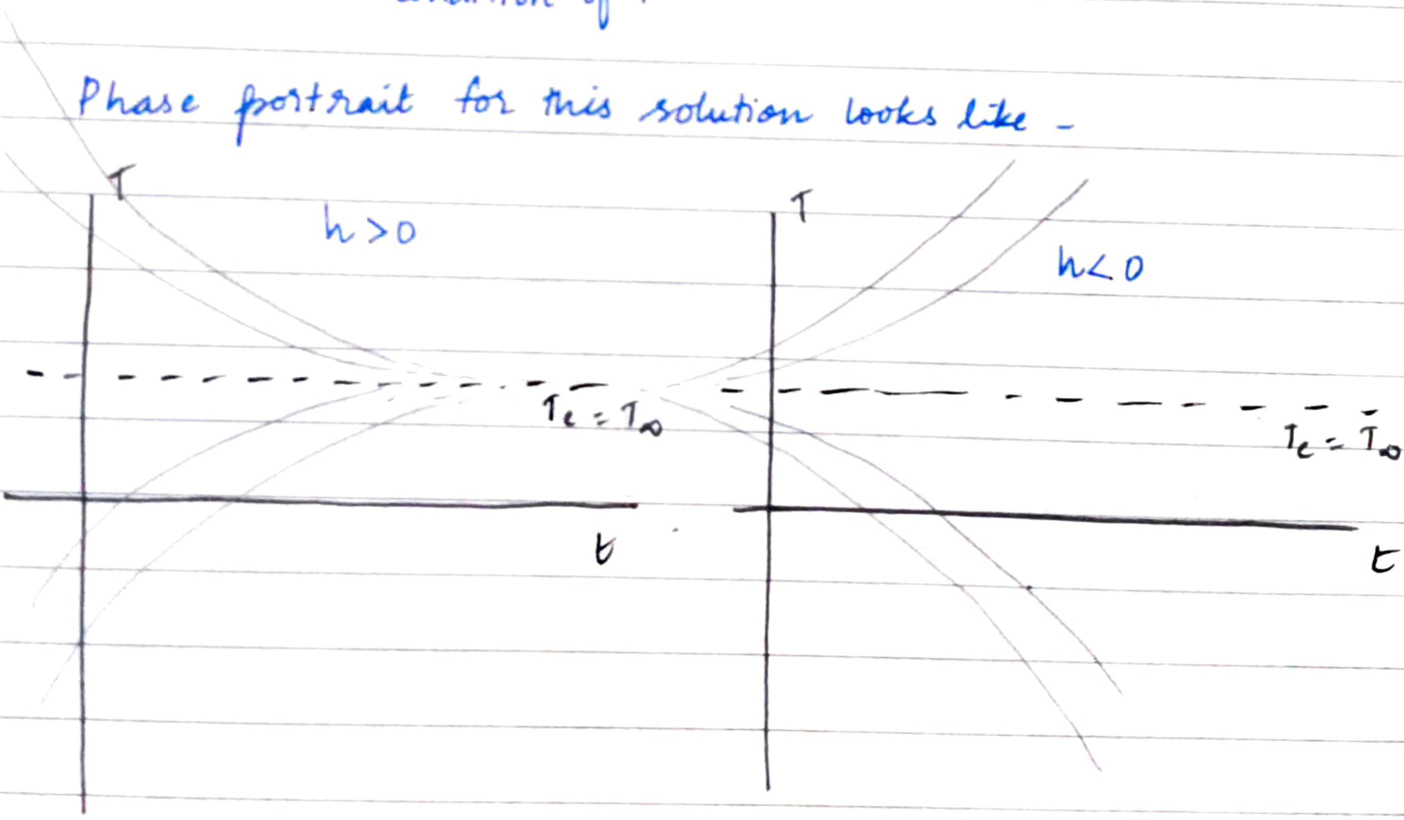
Let $T - T_\infty = T^*$

$$\frac{dT^*}{dt} = -hT^*$$

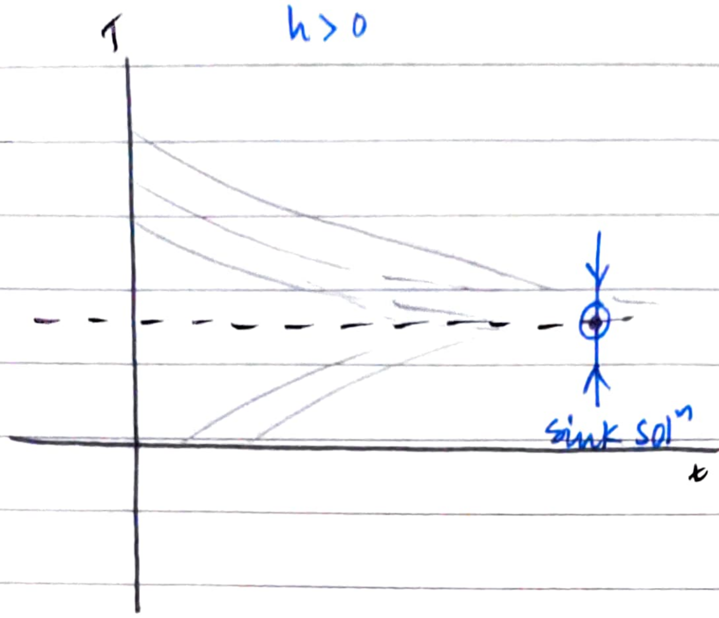
$$T^* = T^*(0) \cdot e^{-ht}$$

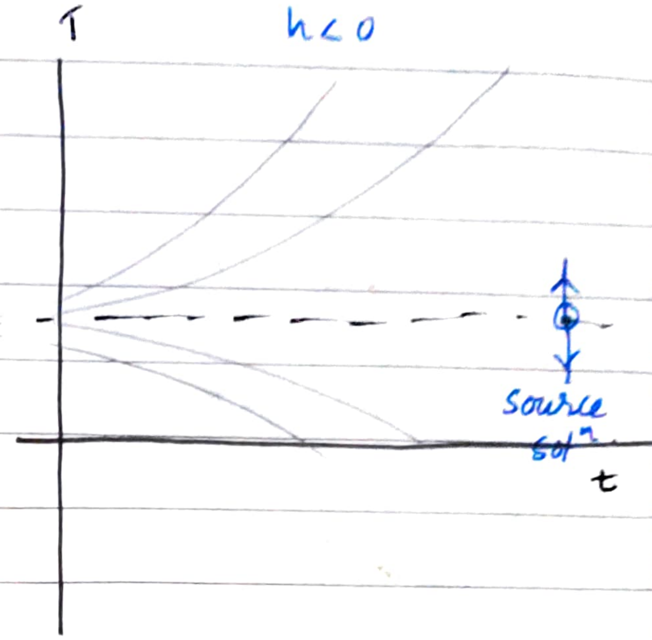
$$\boxed{T = T_\infty + \underbrace{(T_0 - T_\infty)}_{\text{initial condition of } T^*} e^{-ht}} \rightarrow \text{At } t \to \infty,\ T = T_\infty \rightarrow \text{eq. condition}$$

Phase portrait for this solution looks like -



How to reach phase portrait without the analytical solution? - PTO





Start from eq. solution.

$$\frac{dT}{dt} = -h(T - T_\infty) = 0 \Rightarrow T = T_\infty \rightarrow \text{equilibrium sol.}$$

For $h > 0$

When $T > T_\infty$, $\frac{dT}{dt} < 0$ and when $T < T_\infty$, $\frac{dT}{dt} > 0$

For $h < 0$,

When $T < T_\infty$, $\frac{dT}{dt} < 0$ and when $T > T_\infty$, $\frac{dT}{dt} > 0$.

Consider a case when reservoir temp. is a function of time ($T_\infty = f(t)$). Analyse dynamics of the system for -

1) Linear decay ; $T_\infty = a - bt$
2) Linear rise ; $T_\infty = a + bt$
3) Exponential decay ; $T_\infty = a e^{-bt}$
4) Exponential rise ; $T_\infty = a e^{bt}$

15.01.21

Dynamics of higher order systems

A governing eq. which is an ODE of n^{th} order can be converted to a system of 'n' 1^{st} order ODEs.

First order: $\frac{dx}{dt} = ax$ where, $x \rightarrow$ dynamical variable
$a \rightarrow$ parameter.

N^{th} order:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N$$

$$\vdots$$

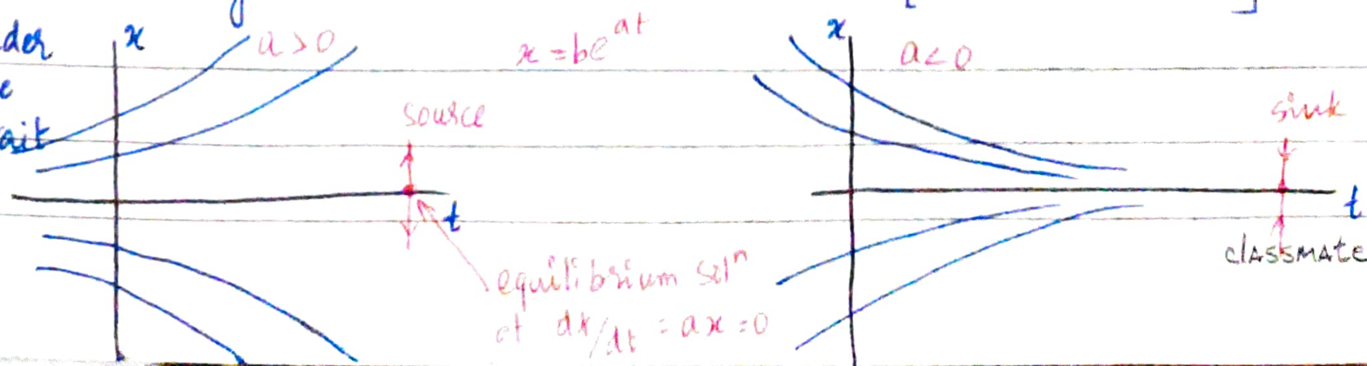
$$\frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \dots + a_{NN}x_N$$

This dynamical eq. can be represented as a matrix eq as -

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & \dots & \dots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{N1} & \dots & \dots & a_{NN} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Here, dynamical variable is a vector $\rightarrow [x_1\ x_2 \dots x_N]^T$

1^{st} order phase portrait



In 1^{st} order systems as shown in the above phase diagram, the fate of the system and its dynamics depended on the value of 'a'.

For an ~~N^{th}~~ N^{th} order system, the dynamics of the system depend on the $N \times N$ coefficient matrix, particularly on the eigen values of this matrix.

In $\frac{dx}{dt} = ax$; a is the eigen value of the differential operator d/dx, with x as the eigen function.

What is the solution to the system of equations $\frac{d}{dt}\underline{x} = \underline{\underline{A}}\,\underline{x}$?
(Eigen value problem)

vector matrix

If λ_i s are the eigen values of $\underline{\underline{A}}$ with corresponding eigen vectors as $\underline{V_i}$ s, then the solution is given as

$$\boxed{\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{V_i}} \longrightarrow (\star)$$

Consider a second order system given by –

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \quad ; \quad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2$$

$\therefore$ Dynamical variable : $[x_1 \ \ x_2]^T$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

classmate

So, it is req. to analyse the eigen values of $\underline{\underline{A}}$

$$\boxed{\underline{\text{Case I}}: \quad \underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}$$

Then $\lambda_1 = a; \ \lambda_2 = b$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

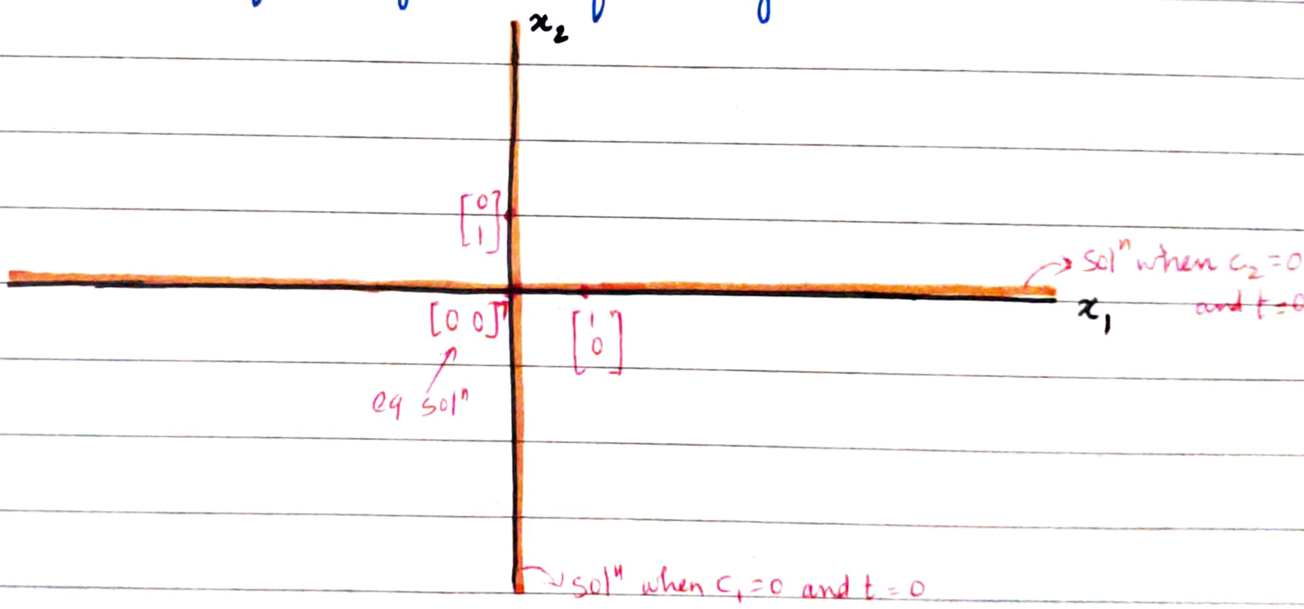
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

from eq (*)

In this solution, if i) $c_1 = 1; c_2 = 0$ and $t = 0$; $[x_1 \ x_2]^T = [1 \ 0]^T$

ii) $c_1 = 0; c_2 = 1$ and $t = 0$; $[x_1 \ x_2]^T = [0 \ 1]^T$

Now, the aim is to be able to draw the phase portait, i.e., determine the fate / dynamics of the system.



The equilibrium solutions are obtained by setting

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

** That is, equilibrium solutions will lie in the "null space" of A or

$$\underline{\underline{A}} \ \underline{x} = \underline{0} \ ; \text{ie.}; \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}} \rightarrow \text{eq. sol}^n$$

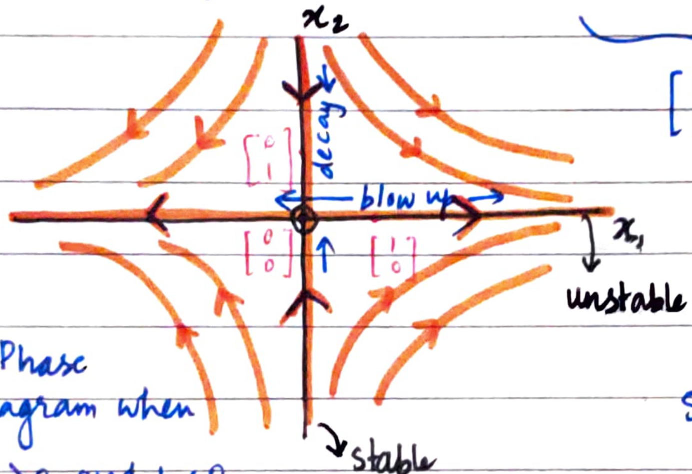
$\therefore$ $[0 \ 0]^T$ is the eq. solution.

We saw that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the solution with equilibrium at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

case i) $a > 0,\ b < 0$

Then $c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will blow up and $\underbrace{c_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ will decay.}}$

SADDLE SOLUTIONS



Phase diagram when $a > 0$ and $b < 0$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to x_2 axis and since it decays, it's the stable axis.

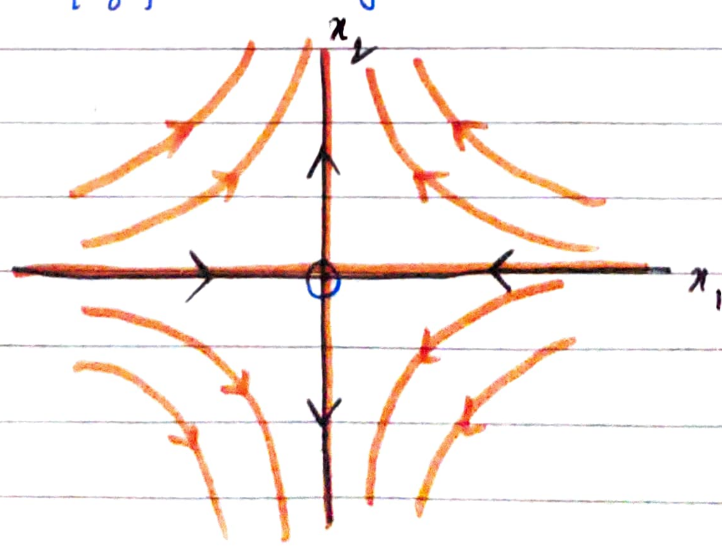
Similarly, x_1 is the unstable axis.

Solutions diverge from x_1 and converge from x_2 onto equil. sol?

This solution is called a '*saddle solution*'.

case ii) $a < 0,\ b > 0$

Then $c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will decay ($x_1 \to$ stable) and $c_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will blow up ($x_2 \to$ uns-table)



This is also a saddle solutions.

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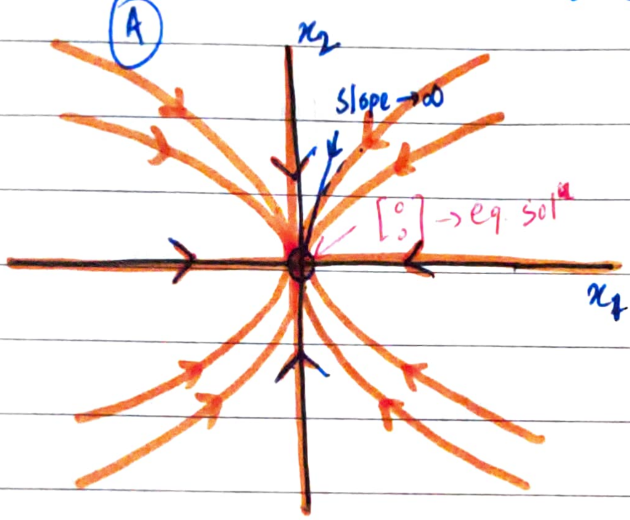
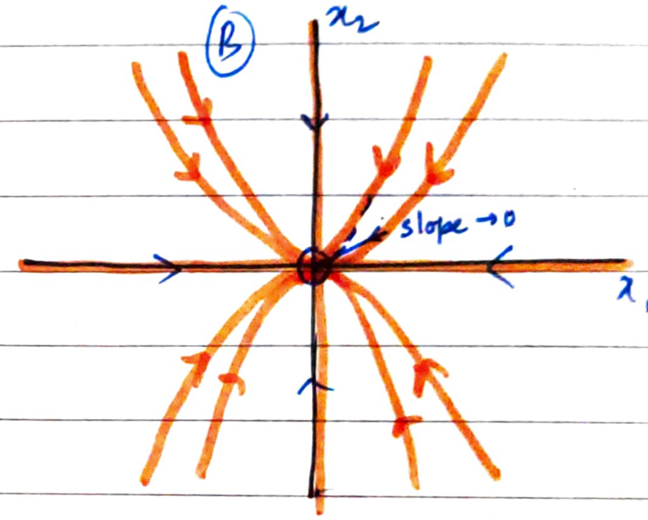
Both case (i) and (ii) lead to unstable solutions as atleast one of (a, b) is '>0' and the solution eventually blows up as $t \to \infty$.

Case (iii): $a < 0$, $b < 0$

Both $c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will decay. So both axes are stable. and both axes are converging. In this case, there are 2 possibilities.





Every solution converges to equilibrium solution, that is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unlike in the previous cases. But, how do we determine which is the correct phase portrait, A or B? For that, we need to find -

$$x_1 = x_{10} e^{at} \quad \text{and} \quad x_2 = x_{20} e^{bt}$$

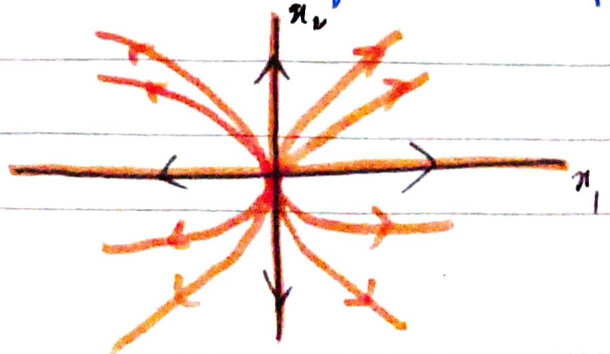
$$\therefore \frac{dx_2}{dx_1} = \frac{x_{20} b}{x_{10} a} \cdot e^{(b-a)t}$$

when $b > a$; $\frac{dx_2}{dx_1} \to \infty$ as $t \to \infty$ → (A)

when $b < a$; $\frac{dx_2}{dx_1} \to 0$ as $t \to \infty$ → (B)

Case (iv) : $a > 0$, $b > 0$

Both axes are unstable. System blows up as $t \to \infty$.



classmate

$$\boxed{\underline{\underline{\text{Case II}}}: \quad \underline{\underline{A}} = \begin{bmatrix} 0 & +a \\ -a & 0 \end{bmatrix}}$$

Then $\lambda_1 = ia$ and $v_1 = [1 \quad i]^T$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{iat}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{Put } e^{iat} = \cos at + i \sin at.$$

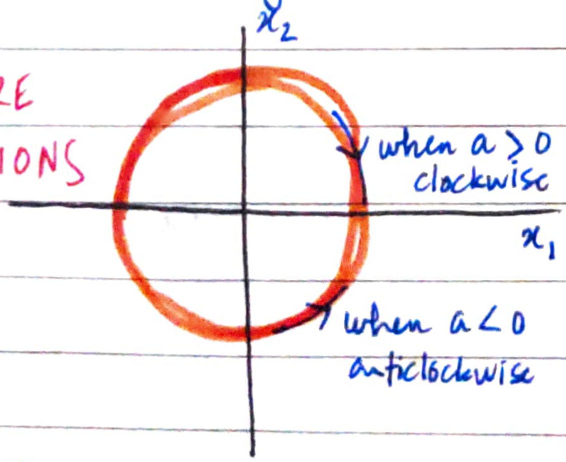
$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos at + i \sin at \\ -\sin at + i \cos at \end{bmatrix} = \underbrace{\begin{bmatrix} \cos at \\ -\sin at \end{bmatrix}}_{\text{Real part}} + i \underbrace{\begin{bmatrix} \sin at \\ \cos at \end{bmatrix}}_{\text{Imaginary part}}$$

Therefore, Re and Im are individually the solutions.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} \cos at \\ -\sin at \end{bmatrix} + c_2\begin{bmatrix} \sin at \\ \cos at \end{bmatrix} = \begin{bmatrix} c_1 \cos at + c_2 \sin at \\ -c_1 \sin at + c_2 \cos at \end{bmatrix}$$

upon plotting $(c_1 \cos at + c_2 \sin at, \; -c_1 \sin at + c_2 \cos at)$ on <u>www.desmos.com</u>, you'll see that it is a circle, whose radius changes with change in c_1 or c_2.

CENTRE
SOLUTIONS



The moment you have imaginary eigen values (can happen even for real), your system shows oscillations that are periodic in nature. The combination of x_1 and x_2 will keep on repeating.

The solutions won't decay or blow up. But they'll keep oscillating about the same value. Such solutions are called "centre solutions"

classmate

$$\boxed{\text{Case III}: \quad \underline{\underline{A}} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}}$$

$\lambda_1 = (a+ib)$ and $v_1 = [1 \ \ i]^T$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{(a+ib)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{at} \cdot e^{ibt}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{at}\left[\underbrace{\begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix}}_{\text{Real}} + i\underbrace{\begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}}_{\text{Imaginary}}\right]$$

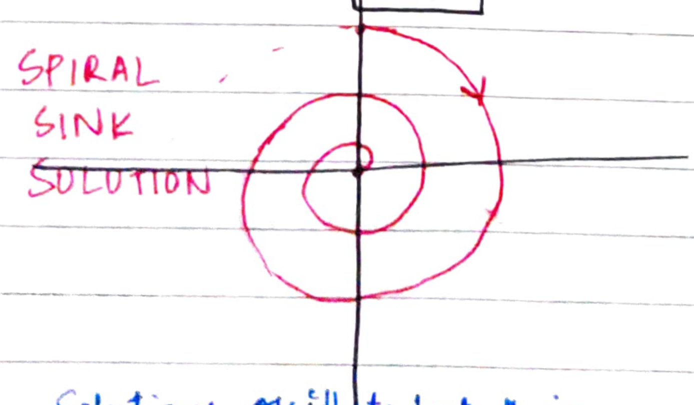
Therefore, Re and Im parts are individually the sol^{n}s.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} + c_2 e^{at}\begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}$$

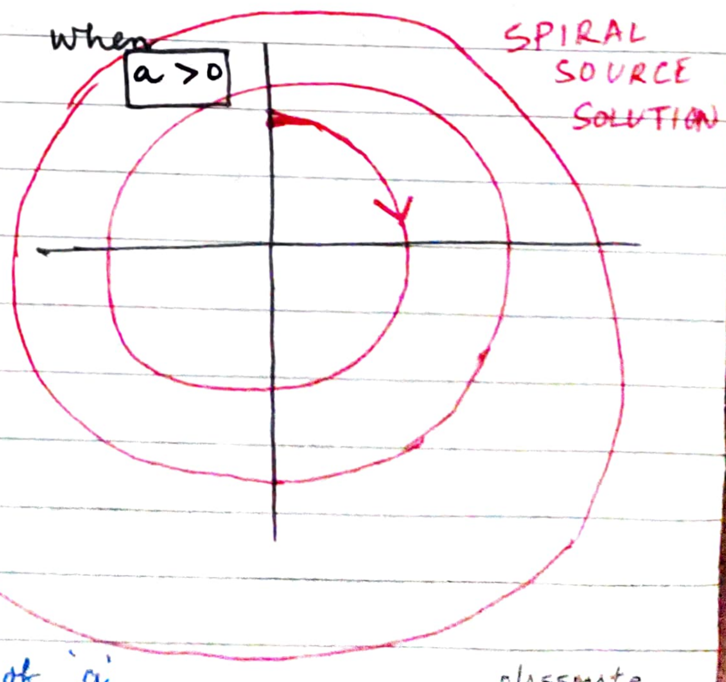
$$= \begin{bmatrix} e^{at}(c_1 \cos bt + c_2 \sin bt) \\ e^{at}(-c_1 \sin bt + c_2 \cos bt) \end{bmatrix}$$

→ How does this phase diagram look?

When $\boxed{a < 0}$



When $\boxed{a > 0}$



Solutions oscillate but their values could be decaying or blowing up with every oscillation based on the value (sign) of 'a'.

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So, conclusions:

- First order systems are monotonous. They either decay ~~[illegible]~~ or blow up monotonously.
- Higher order systems introduce complexity. HO systems with complexity (imaginary terms) oscillate. Whether the oscillations die out, sustain or blow up depends on the eigen value. If eigen value is purely imaginary, then oscillations sustain without change in magnitude. If eigen value is complex with real part '>0', then oscillations blow up, If real part '<0' then oscillations decay.

Example: Consider system of elementary reactions in series - $A \rightarrow B \rightarrow C$. The kinetics are given by -

III order system

$$\begin{cases} \dfrac{dC_A}{dt} = -k_1 C_A \\ \dfrac{dC_B}{dt} = k_1 C_A - k_2 C_B \\ \dfrac{dC_C}{dt} = k_2 C_B \end{cases}$$

The reactions are carried out in batch reactor with initial concs as C_{A0}, C_{B0}, C_{C0}.

a) By sequentially solving the three eqs, determine expressions for the time evolutions of C_A, C_B, C_C and their concentrations as $t \rightarrow \infty$

$$\frac{dC_A}{dt} = -k_1 C_A \Rightarrow C_A = C_{A0} e^{-k_1 t}$$

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$$\frac{dC_B}{dt} + k_2 C_B = k_1 C_{A0} e^{-k_1 t}$$ → solved using I.F. method.

Similarly, C_C can be found by substituting C_B into the ODE.

b) Convert the above system of eqs to a matrix equation and solve for the concentrations. Check if the solutions match the ones obtained from (a).

$$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v_i}$$ → We need to find eigen values & vectors

Using Wolfram alpha; we get;

$\lambda_1 = 0$; $V_1 = [0 \quad 0 \quad 1]^T$

$\lambda_2 = -k_1$; $V_2 = \left[+\left(\frac{k_1 - k_2}{k_2}\right) \quad -\frac{k_1}{k_2} \quad 1\right]^T$

$\lambda_3 = -k_2$; $V_3 = [0 \quad -1 \quad 1]^T$

$$\therefore \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-k_1 t} \begin{bmatrix} -\left(\frac{-k_1 + k_2}{k_2}\right) \\ -k_1/k_2 \\ 1 \end{bmatrix} + c_3 e^{-k_2 t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore C_A = \underbrace{c_2 \left(\frac{-k_2 + k_1}{k_2}\right)}_{\approx C_{A0}} e^{-k_1 t}$$ → same result → decay (monotonous)

$$C_B = c_2 \left(\frac{\cancel{k_2} - k_1}{k_2}\right) e^{-k_1 t} - c_3 e^{-k_2 t}$$

$$C_C = c_1 + c_2 e^{-k_1 t} + c_3 e^{-k_2 t}$$

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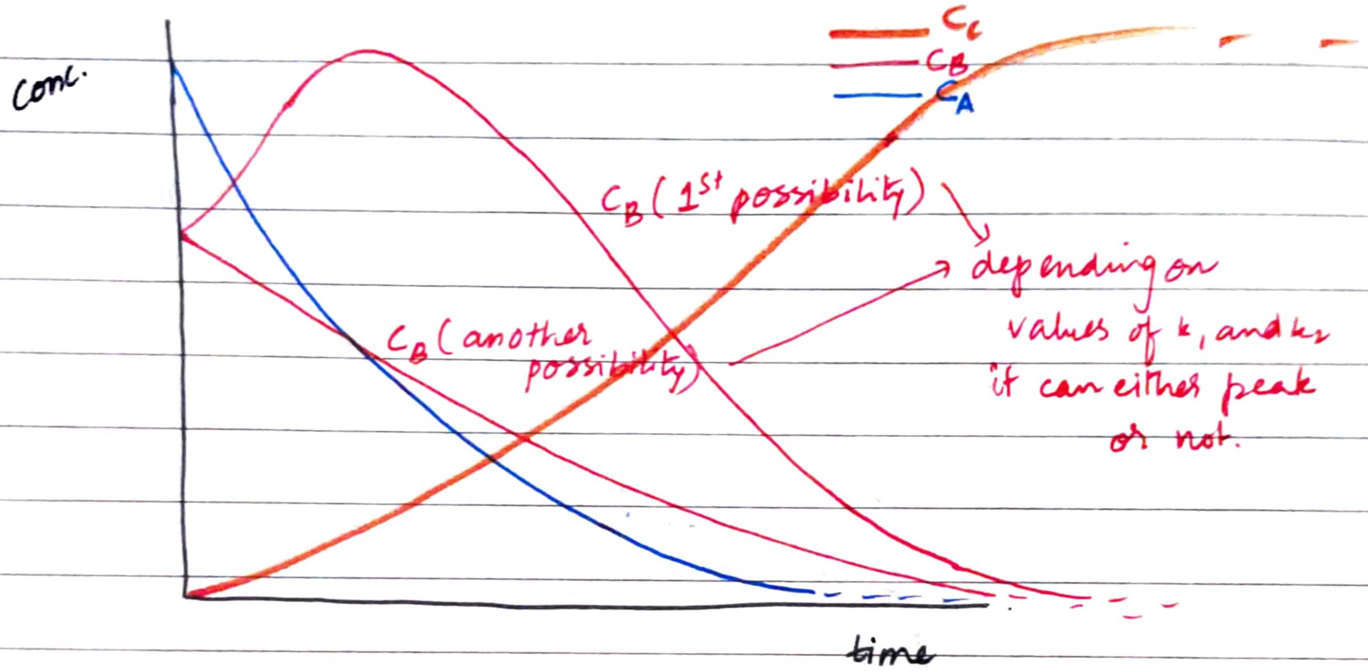
desmos.com/calculator

We got $C_B = C_2\left(\frac{-k_1}{k_2}\right)e^{-k_1 t} - C_3 e^{k_2 t}$

$C_B = A e^{-k_1 t} + B e^{k_2 t}$ → what are its dynamics?



c) Determine the equilibrium solutions.

Equilibrium sol^{n}s lie in the null space of A ⇒ $\underline{\underline{A}}\,\underline{x} = 0$

$$\underline{\underline{A}} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-reduced form (Step 1)

Step 2: Convert matrix to equations.

$$\left.\begin{aligned} -k_1 C_A &= 0 \\ -k_2 C_B &= 0 \end{aligned}\right\} \text{from } \underline{\underline{A}}\,\underline{x} = 0$$

$$\boxed{\therefore C_{Ae} = C_{Be} = 0}$$

The is no constraint for C_c → It means that, whatever amount of A and B were in the system determines

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d) Comment on the stability of the solutions. Analyse your comment in reference to the mag. and sings of k_i's.

Stability is governed by eigen values. They were 0, $-k_1$ and $-k_2$.

When eigen values are +ve, unstable and vice versa.

$\lambda_1 = 0$, $\lambda_2 < 0$, $\lambda_3 < 0$ ⟶ System is stable as none of the eigen values are '>0'.

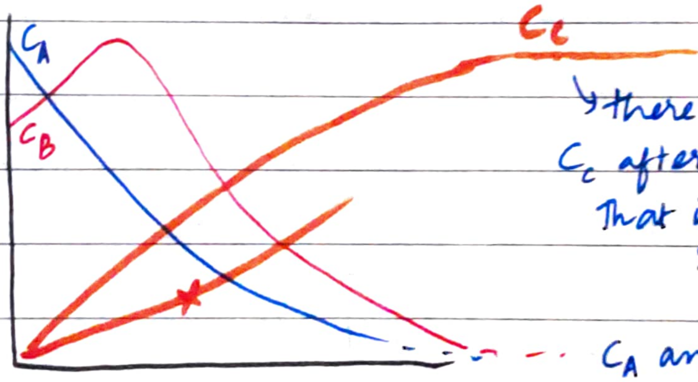
what does this mean?

Physically, this means that the system won't blow up; i.e., concentrations won't blow up to ∞.

-ve eigen value means system squeezes or decays.

+ve eigen value means system expands or blows up.

'0' eigen value means system is rigid.

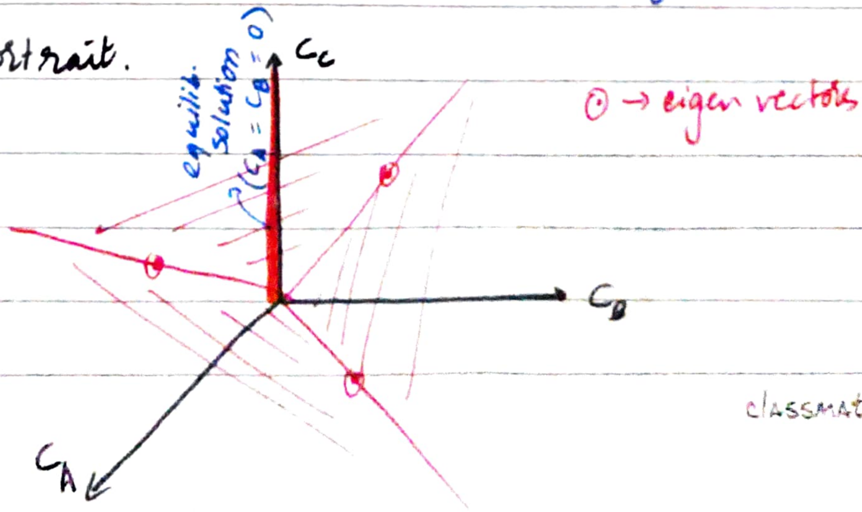


there is no dynamics of C_c after it reaches this phase. That is the significance of $\lambda_1 = 0$ here.

C_A and $C_B \to 0$ as $t \to \infty$

Similarly, C_c becomes equal to C_{final} and it remains there. Of course there are minor changes due to small changes in A and B; but there is no dynamic in C.

e) Draw the phase portrait.



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22.01.21

Consider the case of a single linear spring of spring constant k with mass m attached to it such that the motion is confined along spring axis. The following questions govern the dynamics of the system.

once disturbed, you leave the spring alone. No persistant external force.

Free undamped system :

(A) : $$m \frac{d^2x}{dt^2} + kx = 0$$

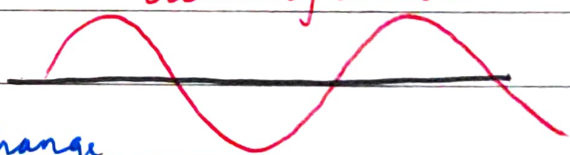
$$\Rightarrow \frac{d^2x}{dt^2} + a^2 x = 0 \quad ; \quad a^2 = \frac{k}{m} \rightarrow \text{always positive}$$

→ (1)

Solution 1 $\Rightarrow x(t) = c_1 \sin at + c_2 \cos at$ – sinusoidal undamped

As a increases, the time period of oscillations changes but amplitude is the same.

Oscillatory solution



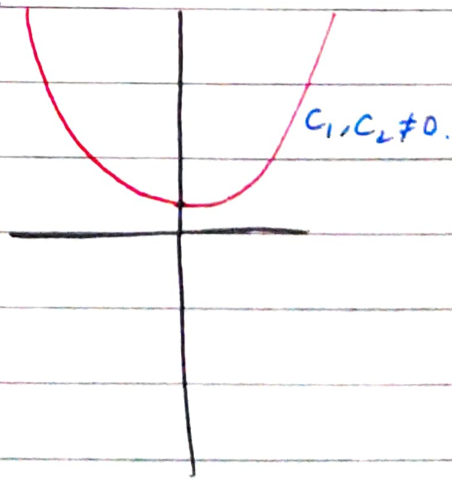
The system has bifurcation about 'a'. (Check recording). As c_1/c_2 change, the initial values and amplitude change.

Solution 2 $\Rightarrow x(t) = c_1 e^{at} + c_2 e^{-at}$ → (2)

If $c_1 = 0$; decay behaviour

$c_2 = 0$; blows up monotonously

$c_1, c_2 \neq 0$; blows up eventually



Free vibration with damping:

(B): $$m \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0 \quad ; \quad a = \frac{c}{m}, \; b = \frac{k}{m}$$

$$D^2 + aD + b = 0$$

$$\Rightarrow D = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$x(t) = c_1 \exp\left[\left(\frac{-a + \sqrt{a^2-4b}}{2}\right)t\right] + c_2 \exp\left[\left(\frac{-a - \sqrt{a^2-4b}}{2}\right)t\right]$$

- When $\sqrt{a^2 - 4b}$ is real and is equal to c;

$$x(t) = c_1 \exp\left(\frac{-a+c}{2}t\right) + c_2 \exp\left(\frac{-a-c}{2}t\right)$$

oscillatory solution

Same as the solution of free undamped system, solution 2. The decay/rise will happen like in the previous case. Qualitatively same as eq ①.

- When $\sqrt{a^2-4b}$ is imaginary and is equal to ic;

$$x(t) = c_1 \exp\left(\frac{-a+ic}{2}t\right) + c_2 \exp\left(\frac{-a-ic}{2}t\right)$$

$$= e^{-\frac{at}{2}}\left[c_1 \exp^{ict/2} + c_2 \exp^{-ict/2}\right]$$

$$= e^{-at/2}\left[c_1 \cos\left(\frac{ct}{2}\right) + c_2 \cos\left(\frac{ct}{2}\right) + i\left[c_1 \sin\left(\frac{ct}{2}\right) - c_2 \sin\left(\frac{ct}{2}\right)\right]\right]$$

$$= e^{-at/2}\left[\cos\left(\frac{ct}{2}\right).\underbrace{(c_1 + c_2)}_{\alpha} + i \sin\left(\frac{ct}{2}\right).\underbrace{(c_1 - c_2)}_{\beta}\right]$$

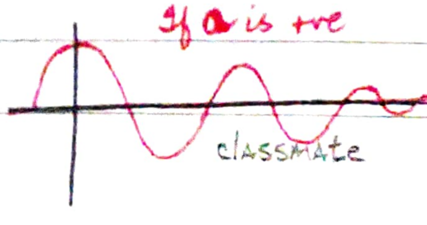
$$x(t) = e^{-at/2}\left[\alpha \cos\frac{ct}{2} + i\beta \sin\frac{ct}{2}\right] \rightarrow Re + i\,Im$$

Re and Im are itself sol's individually.

$$\therefore x(t) = c_1 e^{-at} \cos\left(\frac{ct}{2}\right) + c_2 e^{-at} \sin\left(\frac{ct}{2}\right)$$

$$x(t) = e^{-at}\left[c_1 \cos\left(\frac{ct}{2}\right) + c_2 \sin\left(\frac{ct}{2}\right)\right]$$

If a is +ve



So, in a general 2^{nd} order system; three things can happen -

- exponential decay
- exponential growth
- oscillatory
  - → constant amplitude
  - → decaying amplitude
  - → growing amplitude

Forced vibration without damping:

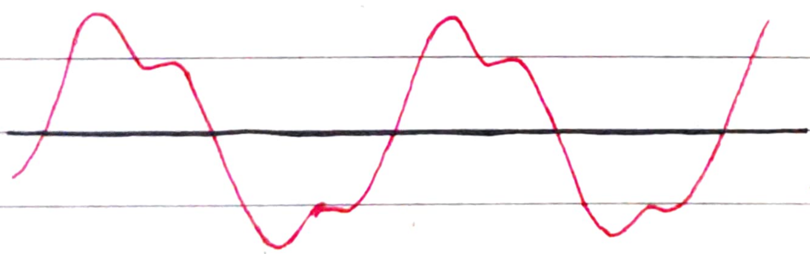
C : $m\dfrac{d^2x}{dt^2} + kx = F_0 \sin \omega t$ → external force

Solve using wolfram alpha

$\Rightarrow \dfrac{d^2x}{dt^2} + a^2 x = b \sin \omega t$; $a^2 = k/m$ & $b = F_0/m$

We get $x(t) = c_1 \cos(at) + c_2 \sin(at) + \dfrac{b \sin(\omega t)}{a^2 - \omega^2}$

Plot on desmos.



Inherent dynamics ($c_1 \cos at + c_2 \sin at$) are being superposed by forcing function ($b \sin \omega t$)

If we analyse forced vibration with damping, we would get

$$x(t) = e^{-ct}\left(c_1 \cos at + c_2 \sin at + \frac{b \sin(\omega t)}{a^2 - \omega^2}\right)$$

e^{-ct} → will cause damping.

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a) Solve (A) using the matrix method, phase diagram. Find equilibrium solutions.

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + ax = 0 \quad ; \quad a = \frac{k}{m}$$

Let $\frac{dx}{dt} = y \Rightarrow \frac{dy}{dt} = -ax.$

$$\therefore \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

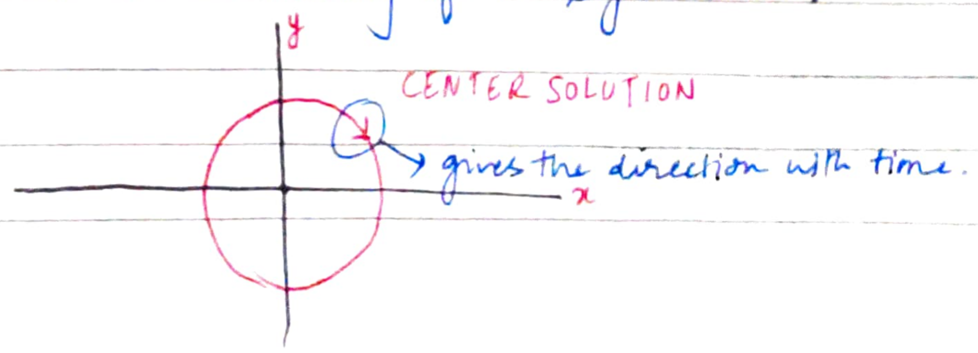
We get, $\lambda_1 = -i\sqrt{a}$; $v_1 = \begin{bmatrix} i/\sqrt{a} & 1 \end{bmatrix}^T$

$\lambda_2 = +i\sqrt{a}$; $v_2 = \begin{bmatrix} -i/\sqrt{a} & 1 \end{bmatrix}^T$

$$\therefore \underline{x} = c_1 e^{-i\sqrt{a}t}\begin{bmatrix} i/\sqrt{a} \\ 1 \end{bmatrix} + c_2 e^{i\sqrt{a}t}\begin{bmatrix} -i/\sqrt{a} \\ 1 \end{bmatrix}$$

$$\underline{x} = c_1\begin{bmatrix} \frac{i}{\sqrt{a}}\cos(\sqrt{a}t) + \frac{1}{\sqrt{a}}\sin(\sqrt{a}t) \\ \cos\sqrt{a}t - i\sin\sqrt{a}t \end{bmatrix} + c_2\begin{bmatrix} \frac{-i}{\sqrt{a}}\cos(\sqrt{a}t) + \frac{1}{\sqrt{a}}\sin(\sqrt{a}t) \\ \cos\sqrt{a}t + i\sin\sqrt{a}t \end{bmatrix}$$

Observations: Eigen values are purely imaginary. Ossillations with center solution are sustained. Eigen vectors only give magnitude of solutions. Everything we need to know qualitatively about the solutions and phase portrait can be obtained from just an understanding of the eigen values.



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Equilibrium solutions: $\underline{\underline{A}}\,\underline{x} = 0$

$$\underline{\underline{A}}\underline{x} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = 0$ and $-ax = 0$

$\therefore$ $x = 0$ and $y = 0$ are equilibrium solutions.

The above solution holds for $k > 0$, $m > 0$; i.e.; $a > 0$

Consider for any hypothetical scenario, $a < 0$. Then,

$\left.\begin{aligned} \cancel{\lambda_1 = -\sqrt{a}} \\ \cancel{\lambda_2 = \sqrt{a}} \end{aligned}\right\}$ ~~we get saddle solutions.~~ ??

b) Solve (B) using matrix

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

$$\frac{dx}{dt} = y \quad ; \quad \frac{dy}{dt} + ay + bx = 0$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{aligned} \lambda_1 &= \tfrac{1}{2}\left(-\sqrt{a^2-4b} - a\right) \\ \lambda_2 &= \tfrac{1}{2}\left(\sqrt{a^2-4b} - a\right) \end{aligned}$$

~~Ans~~: If $\sqrt{a^2-4b}$ is real, λ_1 is real & λ_2 is real $\rightarrow$ Saddle solutions.

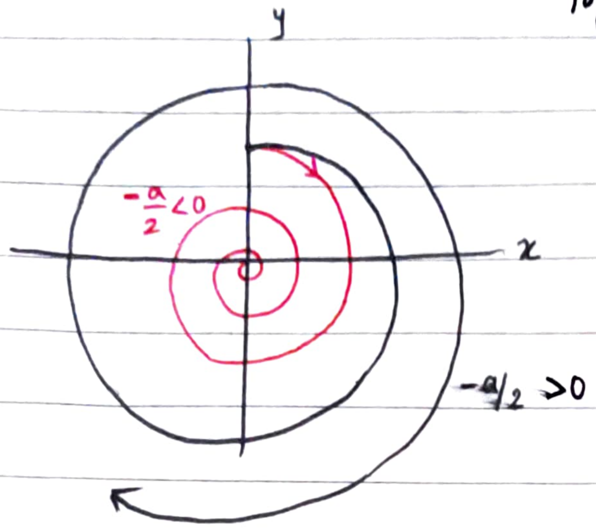
If $\sqrt{a^2-4b}$ is imaginary, λ_1 and λ_2 would be complex. Solution would be oscillatory. It might be centre or spiral based upon the value of $-a/2$; the real part. If $a=0$, then centre solutions. If $-a/2 < 0$, then spiral sink.

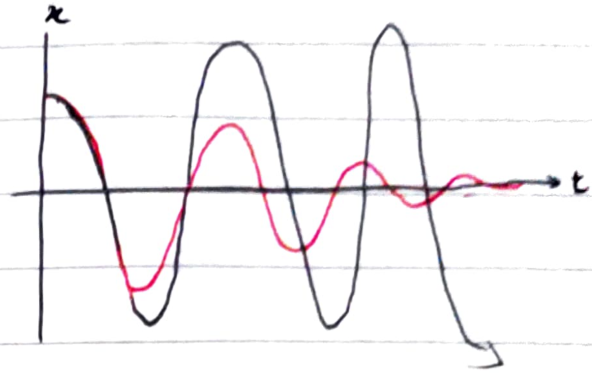
If $-a/2 > 0$, then spiral source.

Quiz 1: Jan 31, Sunday 9:00 - 10:00 pm

Pen & paper test → scan to PDF → email.

Topic: Linear dynamics — Topic 1, 2, 3





c) Solve forced damping.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = c\sin\omega t$$

$$\Rightarrow \frac{dx}{dt} = y \; ; \quad \frac{dy}{dt} + ay + bx = c\sin\omega t.$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ c\sin\omega t \end{bmatrix}$$

$$\Rightarrow \frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{b}$$ → How to solve this?

↓

Refer to class notes. Last 10 mins of the class.

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