Applications to PDEs

Ex: Heat equation: $K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$; $-\infty < x < \infty, t > 0$

BCs: $u(x,t)$ and $u_x(x,t)$ both $\to 0$ as $|x| \to \infty$

IC: $u(x,0) = f(x)$ $-\infty < x < \infty$

Sol: Taking Fourier transform w.r.t x

$$-K\alpha^2 \hat{u}(\alpha,t) = \frac{d\hat{u}}{dt} \Rightarrow \frac{d\hat{u}}{dt} + K\alpha^2 \hat{u}(\alpha,t) = 0$$

Not that B.Cs are already used

The solution of the ODE:

$$\hat{u}(\alpha,t) = c\, e^{-K\alpha^2 t} \quad \text{--- (1)}$$

The Fourier transform of the initial condition gives:

$$\hat{u}(\alpha,0) = \hat{f}(\alpha)$$

We use this condition in (1) to get c as

$$\hat{f}(\alpha) = c$$

$$\Rightarrow \hat{u}(\alpha,t) = \hat{f}(\alpha)\, e^{-K\alpha^2 t}$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-K\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

Not easy to get explicit form.

We would like to have $f(x)$ in the solution but not $\hat{f}(\alpha)$..

Recall: $F\{f*g\} = \sqrt{2\pi}\, \hat{f}(\alpha)\, \hat{g}(\alpha)$

Product form $\hat{f}(\alpha)\, e^{-k\alpha^2 t}$ suggest that we can use convolution theorem.

Let $e^{-k\alpha^2 t}$ be the Fourier transform of $g(x)$; then

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k\alpha^2 t}\, e^{-i\alpha x}\, d\alpha \qquad \text{— (2)}$$

Consider the integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2 - 2bx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}\,x + \frac{b}{\sqrt{a}}\right)^2 + \frac{b^2}{a}}\, dx$$

$$= e^{\left(\frac{b^2}{a}\right)} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}\,x + \frac{b}{\sqrt{a}}\right)^2}\, dx$$

Subst. $\sqrt{a}\,x + \frac{b}{\sqrt{a}} = t \Rightarrow dx = \frac{dt}{\sqrt{a}}$

$$\Rightarrow \quad I = e^{b^2/a} \int_{-\infty}^{\infty} e^{-t^2}\, \frac{dt}{\sqrt{a}} = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{b^2/a}$$

Let $a = kt$ & $b = ix/2$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-kt\alpha^2 - ix\alpha}\, d\alpha = \frac{\sqrt{\pi}}{\sqrt{kt}}\, e^{-\frac{x^2}{4kt}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{x^2}{4kt}} = \frac{1}{\sqrt{2kt}} e^{-\frac{x^2}{4kt}}$$

Now using convolution theorem we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\left[f(x) * g(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\beta)\, g(x-\beta)\, d\beta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\beta) \frac{1}{\sqrt{2kt}} e^{-\left(\frac{(x-\beta)^2}{4kt}\right)} d\beta$$

Subst. $z = -\frac{(x-\beta)}{\sqrt{4kt}} \Rightarrow dz = \frac{d\beta}{\sqrt{4kt}}$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + \sqrt{4kt}\, z)\, e^{-z^2}\, dz$$

ANS.

Ex: Solve the following heat equation

$$K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 < x < \infty \qquad t > 0$$

BC. $u(0,t) = u_0 \qquad t \geq 0$

IC. $u(x,0) = 0 \qquad 0 < x < \infty$

u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$.

Sol: Since u is specified at $x = 0$ and $0 < x < \infty$, the Fourier sine transform is applicable to this problem.

Taking Fourier sine transform.

$$\sqrt{\frac{2}{\pi}}\, K \cdot \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx = \frac{d}{dt} \hat{u}_s(\alpha, t)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\, K \cdot \left\{ \frac{\partial u}{\partial x} \cdot \sin \alpha x \Big|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x} \cdot \cos \alpha x \cdot \alpha \, dx \right. = \frac{d \hat{u}_s}{dt}$$

$$\Rightarrow K \cdot \sqrt{\frac{2}{\pi}} \left[-\alpha \cdot \left\{ u \cos \alpha x \Big|_0^\infty + \int_0^\infty u \cdot \sin \alpha x \, (\alpha)\, dx \right\} \right. = \frac{d \hat{u}_s}{dt}$$

$$\Rightarrow K \cdot \sqrt{\frac{2}{\pi}} \left[\alpha u(0) - \alpha^2 \int_0^\infty u \sin \alpha x \, dx \right] = \frac{d \hat{u}_s}{dt}$$

$$\Rightarrow K\alpha \sqrt{\frac{2}{\pi}}\, u_0 - K\alpha^2\, \hat{u}_s(\alpha, t) = \frac{d \hat{u}_s}{dt}$$

$$\Rightarrow \frac{d \hat{u}_s}{dt} + K\alpha^2\, \hat{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}}\, K\alpha\, u_0 .$$

$$\text{I.F.} = e^{K\alpha^2 t}$$

$$\hat{u}_s \cdot e^{K\alpha^2 t} = \int \sqrt{\frac{2}{\pi}} \cdot K\alpha\, u_0\, e^{+K\alpha^2 t}\, dt + C$$

$$\Rightarrow \quad \hat{u}_s = \left(\sqrt{\frac{2}{\pi}}\ \frac{1}{\alpha} u_0 \int k\alpha^2 e^{+k\alpha^2 t}\, dt\right) e^{-k\alpha^2 t} + c\, e^{-k\alpha^2 t}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha} \cdot u_0 \cdot e^{k\alpha^2 t} \cdot e^{-k\alpha^2 t} + c\, e^{-k\alpha^2 t}$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}}\ \frac{u_0}{\alpha} + c\, e^{-k\alpha^2 t}$$

IC: $\quad u(x,0) = 0 \Rightarrow \hat{u}_s(\alpha, 0) = 0$

$$\Rightarrow \quad \hat{u}_s(\alpha, 0) = 0 = \sqrt{\frac{2}{\pi}}\ \frac{u_0}{\alpha} + c$$

$$\Rightarrow \quad c = -\sqrt{\frac{2}{\pi}}\ \frac{u_0}{\alpha}$$

$$\Rightarrow \quad \hat{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}}\ \frac{u_0}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)$$

Inverting it:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_s(\alpha, t) \sin \alpha x \, d\alpha$$

$$= \frac{2}{\pi} u_0 \int_0^\infty \frac{\sin \alpha x}{\alpha} \left(1 - e^{-k\alpha^2 t}\right) d\alpha.$$

$\square$

Solve. $\dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}$ subject to the conditions

$$u(x,0) = 0, \quad x \geq 0$$

$$u_x(0,t) = -\mu \text{ (constant)}, \quad t > 0$$

u & $\dfrac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$

Sol: Since u_x is specified at $x=0$, the Fourier cosine transform is applicable to this problem

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = K F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} \Rightarrow$$

$$\frac{d}{dt}\hat{u}_c = K\left\{-\sqrt{\frac{2}{\pi}}\, u_x(0,t) - \alpha^2 F_c\{u\}\right\}$$

$$\Rightarrow \frac{d\hat{u}_c}{dt} + K\alpha^2 \hat{u}_c = K\mu\sqrt{\frac{2}{\pi}}$$

$$\text{I.F.} = e^{K\alpha^2 t}$$

So. $\hat{u}_c \cdot e^{K\alpha^2 t} = \int \sqrt{\frac{2}{\pi}}\, K\mu\, e^{K\alpha^2 t}\, dt + c$

$$\Rightarrow \hat{u}_c\, e^{K\alpha^2 t} = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{\alpha^2}\, e^{K\alpha^2 t} + c$$

$$\boxed{u(x,0) = 0 \Rightarrow \hat{u}_c(\alpha,0) = 0 \Rightarrow 0 = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{\alpha^2} + c}$$

$$\Rightarrow \hat{u}_c = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{\alpha^2} + c\, e^{-K\alpha^2 t} = \sqrt{\frac{2}{\pi}}\,\frac{\mu}{\alpha^2}\left(1 - e^{-K\alpha^2 t}\right).$$

inverting it: $u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_c(\alpha,t)\cos\alpha x\, d\alpha$

$$= \frac{2\mu}{\pi}\int_0^\infty \frac{\cos\alpha x}{\alpha^2}\left(1 - e^{-K\alpha^2 t}\right) d\alpha.$$

$\square$

Solution of Wave equation:

Ex: Solve the wave equation described by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty$$

ICs:

$$u(x,0) = f(x) \qquad -\infty < x < \infty$$

$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

BCs: u & $\frac{\partial u}{\partial x}$ both tends to zero at $|x| \to \infty$.

Sol: Taking F.T. of PDE, we have:

$$\frac{d^2 \hat{u}(\alpha,t)}{dt^2} = c^2\left(-\alpha^2 \hat{u}(\alpha,t)\right)$$

$$\Rightarrow \frac{d^2\hat{u}}{dt^2} + c^2\alpha^2 \hat{u}(\alpha,t) = 0$$

Its general solution $\hat{u}(\alpha,t) = C_1 \cos(c\alpha t) + C_2 \sin(c\alpha t)$

F.T. of initial condition $u(x,0) = f(x) \Rightarrow \hat{u}(\alpha,0) = \hat{f}(\alpha)$

$$\& \; u_t(x,0) = 0 \Rightarrow \frac{d\hat{u}}{dt}(\alpha,0) = 0$$

$$\Rightarrow C_1 = \hat{f}(\alpha) \; \& \; \frac{d\hat{u}}{dt} = -C_1 \sin(c\alpha t)(c\alpha) + C_2 \cos(c\alpha t)(c\alpha)$$

$$\Rightarrow 0 = C_2 c\alpha. \Rightarrow C_2 = 0.$$

$$\Rightarrow \hat{u}(\alpha,t) = \hat{f}(\alpha)\cos(c\alpha t)$$

Taking inverse Fourier transform.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \cos(c\alpha t)\, e^{-i\alpha x}\, d\alpha$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \left\{ \frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2} \right\} e^{-i\alpha x} \, d\alpha$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha(x-ct)} \, d\alpha + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha(x+ct)} \, d\alpha \right]$$

$$= \frac{1}{2} \left[f(x-ct) + f(x+ct) \right]$$

This is known as D' Alembert's solution of the wave equation.

Solve $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \; ; \quad 0 < x < \infty \quad t > 0$

ICs: $\quad u(x,0) = f(x)$

$\quad\quad u_t(x,0) = g(x)$

BCs: $u(0,t) = 0$; both u & $\dfrac{\partial u}{\partial x}$ tends to 0 as $x \to \infty$

Sol: Taking Fourier sine transform of PDE, we have

$$\frac{d^2}{dt^2} \hat{u}_s(\alpha,t) = c^2 \left\{ \sqrt{\frac{2}{\pi}} \, \alpha u(0,t) - \alpha^2 \hat{u}_s(\alpha,t) \right\}$$

$$\Rightarrow \frac{d^2}{dt^2} \hat{u}_s(\alpha,t) + \alpha^2 c^2 \hat{u}_s(\alpha,t) = 0$$

Its general solution:

$$\hat{u}_s(\alpha,t) = C_1 \cos(c\alpha t) + C_2 \sin(c\alpha t) .$$

At $t=0$: $\hat{u}_s(\alpha,0) = \hat{f}_s(\alpha)$ & $\frac{d}{dt}\hat{u}_s(\alpha,0) = \hat{g}_s(\alpha)$

$$\Rightarrow \quad C_1 = \hat{f}_s(\alpha) \quad \& \quad \frac{d\hat{u}_s}{dt} = -C_1 \sin(c\alpha t)(c\alpha) + C_2 \cos(c\alpha t)(c\alpha)$$

$$\Rightarrow \quad \hat{g}_s(\alpha) = C_2(c\alpha)$$

$$\Rightarrow \quad \hat{u}_s(\alpha,t) = \hat{f}_s(\alpha)\cos(c\alpha t) + \frac{\hat{g}_s(\alpha)}{c\alpha}\sin(c\alpha t)$$

Taking inverse:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_s(\alpha,t) \sin\alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\hat{f}_s(\alpha)\cos(c\alpha t)\sin\alpha x + \frac{\hat{g}_s(\alpha)}{c\alpha}\sin(c\alpha t)\sin\alpha x\right] d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\hat{f}_s(\alpha)}{2}\left\{\sin(x+ct)\alpha + \sin(x-ct)\alpha\right\} d\alpha$$
$$+ \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\hat{g}_s(\alpha)}{2c\alpha}\left\{\cos(x-ct)\alpha - \cos(x+ct)\alpha\right\} d\alpha \quad \text{—(2)}$$

Since $g(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}_s(\alpha) \sin\alpha u \, d\alpha$

$$\Rightarrow \int_{x-ct}^{x+ct} g(u)\, du = \sqrt{\frac{2}{\pi}} \int_{x-ct}^{x+ct} \int_0^\infty \hat{g}_s(\alpha) \sin\alpha u \, d\alpha \, du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}_s(\alpha) \int_{x-ct}^{x+ct} \sin\alpha u \, du \, d\alpha$$

$$\int_{x-ct}^{x+ct} g(u)\,du = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}_s(\alpha) \left\{ -\frac{\cos \alpha u}{\alpha} \right\}_{x-ct}^{x+ct} d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\hat{g}_s(\alpha)}{\alpha} \left\{ \cos(x-ct)\alpha - \cos(x+ct)\alpha \right\} d\alpha$$

This the solution is given as

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du. \qquad \square$$

Solution of Laplace equation :

Ex: Solve the following BVP:

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty \;,\; y > 0$$

BCs: $u(x,0) = f(x) \qquad -\infty < x < \infty$

u is bounded as $y \to \infty$; u & $\frac{\partial u}{\partial x}$ both vanish as $|x| \to \infty$

Sol: Taking Fourier transform w.r.t. x:

$$-\alpha^2 \hat{u}(\alpha,y) + \frac{d^2}{dy^2}\hat{u}(\alpha,y) = 0$$

its solution $\; -\alpha^2 \hat{u}(\alpha,y) + \frac{d^2}{dy^2}\hat{u}(\alpha,y) = 0$

$$\hat{u}(\alpha,y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$$

Since u is bounded as $y \to \infty \Rightarrow \hat{u}(\alpha,y)$ must be bounded as $y \to \infty$

$\Rightarrow c_1 = 0$ for $\alpha > 0$ & $c_2 = 0$ if $\alpha < 0$.

This for any α

$$\hat{u}(\alpha, y) = c.\, e^{-|\alpha| y}$$

Using B.C. : $\hat{u}(\alpha, 0) = \hat{f}(\alpha)$

$$\Rightarrow c = \hat{f}(\alpha)$$

$$\Rightarrow \hat{u}(\alpha, y) = \hat{f}(\alpha)\, e^{-|\alpha| y}$$

hence: $u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-|\alpha| y} e^{-i\alpha x}\, d\alpha.$

It does not look good to have sol. interms of $\hat{f}(\alpha)$.

Let $g(x) = F^{-1}\{e^{-|\alpha| y}\}$

Then by convol. theorem: $F\{f*g\} = \sqrt{2\pi}\, \hat{f}(\alpha) \cdot \hat{g}(\alpha)$

$$\Rightarrow F^{-1}\{\hat{f}(\alpha) \cdot \hat{g}(\alpha)\} = \frac{1}{\sqrt{2\pi}} (f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\beta)\, g(x-\beta)\, d\beta$$

So: $g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-|\alpha| y}}_{\text{even.}} \; \underbrace{e^{-i\alpha x}}_{= (\cos\alpha x - i \underbrace{\sin\alpha x}_{\text{odd}})} d\alpha$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot \int_0^{\infty} e^{-\alpha y} \cos\alpha x \, d\alpha$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-\alpha y} \cos \alpha x \, d\alpha}_{I}$$

$$I = \int_0^\infty e^{-\alpha y} \cos \alpha x \, d\alpha = \frac{e^{-\alpha y}}{-y} \cos \alpha x \Big|_0^\infty$$

$$- \int_0^\infty \frac{e^{-\alpha y}}{-y} \cdot (-\sin \alpha x)(x) \, d\alpha$$

$$= \frac{1}{y} - \frac{x}{y} \int_0^\infty e^{-\alpha y} \sin \alpha x \, d\alpha \qquad y > 0$$

$$= \frac{1}{y} - \frac{x}{y} \left[\frac{e^{-\alpha y}}{-y} \cdot \sin \alpha x \Big|_0^\infty - \int_0^\infty \frac{e^{-\alpha y}}{-y} \cdot \cos \alpha x \, (x) \, d\alpha \right]$$

$$= \frac{1}{y} - \frac{x}{y} \cdot \frac{x}{y} I$$

$$\Rightarrow \quad I = \frac{1}{y} \cdot \frac{y^2}{x^2+y^2} = \frac{y}{x^2+y^2}$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \left(\frac{y}{x^2+y^2} \right)$$

$$\Rightarrow \quad u(x,y) = F^{-1}\{\hat{f}(\alpha) \cdot \underbrace{e^{-|\alpha| y}}_{\hat{g}(\alpha)}\} = \frac{1}{\sqrt{2\pi}} \cdot f * g$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty f(\beta) \cdot \frac{y}{(x-\beta)^2+y^2} \cdot d\beta$$

$$\boxed{u(x,y) = \frac{1}{\pi} \int_{-\infty}^\infty f(\beta) \frac{y}{(x-\beta)^2+y^2} \, d\beta}$$

→ This solution is a well-known Poisson integral formula

Ex: Solve two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad -\infty < x < \infty \ ; \ 0 < y < \infty$$

subject to the conditions

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial y} = 0 \text{ at } \underline{y=0}$$

u & u_x both vanish as $|x| \to \infty$.

Sol: Taking Fourier transform:

$$\frac{d^2}{dy^2} \hat{u}(\alpha, y) - \alpha^2 \hat{u}(\alpha, y) = 0$$

Its solution: $\hat{u}(\alpha, y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$

Now: $u(x,0) = f(x) \Rightarrow \hat{u}(\alpha, 0) = \hat{f}(\alpha)$

$$\Rightarrow \hat{f}(\alpha) = c_1 + c_2 \quad (1)$$

$u_y = 0 \Rightarrow \frac{d}{dy} \hat{u}(\alpha, 0) = 0 = \left\{\alpha c_1 e^{\alpha y} - c_2 \alpha e^{-\alpha y}\right\}_{y=0}$

$$\Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

$$\Rightarrow c_1 = c_2 = \frac{\hat{f}(\alpha)}{2}$$

$\Rightarrow$ Solution: (transformed) $\hat{u}(\alpha, y) = \frac{\hat{f}(\alpha)}{2}\left[e^{\alpha y} + e^{-\alpha y}\right]$

taking inverse Fourier transform

$$\varphi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\alpha)}{2} \left(e^{\alpha y} + e^{-\alpha y}\right) e^{-i\alpha x} \, d\alpha$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \left[e^{-i\alpha(x-iy)} + e^{-i\alpha(x+iy)}\right]$$

$$= \frac{1}{2}\left[f(x-iy) + f(x+iy)\right]$$