We have seen that piecewise continuity of a function is sufficient for the existence of the Fourier series. We have not yet discussed the convergence of the Fourier series. Convergence of the Fourier series is a very important topic to be explored in this lesson.

In order to motivate the discussion on convergence, let us construct the Fourier series of the function

$$f(x) = \begin{cases} -\cos x, & -\pi/2 \le x < 0; \\ \cos x, & 0 \le x \le \pi/2. \end{cases} \qquad f(x+\pi) = f(x).$$

In this case the function is an odd function and therefore $a_n = 0$, $n = 0, 1, 2, \ldots$. We compute the Fourier coefficient b_n by

$$b_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin(2nx)\, \mathrm{d}x = \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin(2nx)\, \mathrm{d}x = \frac{8}{\pi} \frac{n}{(4n^2-1)}$$

The Fourier series is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(2nx) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2nx)}{4n^2-1}.$$

Note that the Fourier series at $x = 0$ converges to 0. So the Fourier series of f does not converge to the value of the function at $x = 0$.

With this example we pose the following questions in connection to the convergence of the Fourier series

1. Does the Fourier series of a function $f(x)$ converges at a point $x \in [-L, L]$.
2. If the series converges at a point x, is the sum of the series equal to $f(x)$.

The answers of these questions are in the negative because

1. There are Lebesgue integrable functions on $[-L, L]$ whose Fourier series diverge everywhere on $[-L, L]$.
2. There are continuous functions whose Fourier series diverge at a countable number of points.
3. We have already seen in the above examples that the Fourier series converges at a point but the sum is not equal to the the value of the function at that point.

We need some additional conditions to ensure that the Fourier series of a function $f(x)$ converges and it converges to the function $f(x)$. Though, we have several notions of convergence like pointwise, uniform, mean square, etc. we first stick to the most common notion of convergence, that is, pointwise convergence. Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a, b]$ if for each $x \in [a, b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. A more formal definition of pointwise convergence will be given later.

3.1 Convergence Theorem (Dirichlet's Theorem, Sufficient Conditions)

Theorem Statement: *Let f be a **piecewise continuous function** on $[-L, L]$ and the **one sided derivatives** of f, that is,*

$$\lim_{h\to 0+} \frac{f(x+h) - f(x+)}{h} \text{ in } x \in [-L, L) \quad \& \quad \lim_{h\to 0+} \frac{f(x-) - f(x-h)}{h} \text{ in } x \in (-L, L] \tag{3.1}$$

exist (and are finite), then for each $x \in (-L, L)$ the Fourier series converges and we have

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{k\pi x}{L} + b_n \sin\frac{k\pi x}{L} \right]$$

At both endpoints $x = \pm L$ the series converges to $[f(L-) + f((-L)+)]/2$, thus we have

$$\frac{f(L-) + f((-L)+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n a_n$$

Remark 1: *If the function is continuous at a point x, that is, $f(x+) = f(x-)$ then we have*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_n \cos\frac{k\pi x}{L} + b_n \sin\frac{k\pi x}{L} \right] \tag{3.2}$$

In other words, if f is continuous with $f(-L) = f(L)$ and one sided derivatives (3.1) exist then equality (3.2) holds for all x.

Remark 2: *In the above theorem condition on f are sufficient conditions. One may replace these conditions (piecewise continuity and one sided derivatives) by slightly more restrictive conditions of piecewise smoothness. A function is said to be **piecewise smooth** on $[-L, L]$ if it is piecewise continuous and has a piecewise continuous derivative. The* *difference between the two similar restrictions on f will be clear from the example of the function*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

It can easily easily be shown that derivative of the function exist everywhere and thus the function has one sided derivatives and satisfy the conditions of the convergence Theorem (3.1). However the function is not piecewise smooth because the $\lim_{x\to 0} f'(x)$ does not exist as

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

If a function is piecewise smooth then it can easily be shown that left and right derivatives exist. Let f be a piecewise smooth function on $[-L, L]$ then $\lim_{x\to a\pm} f'(x)$ exists for all $a \in [-L, L]$. This implies

$$\lim_{x\to a+} f'(x) = \lim_{x\to a+} \left(\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h} \right)$$

Interchanging the two limits on the right hand side we obtain

$$\lim_{x\to a+} f'(x) = \lim_{h\to 0+} \left(\lim_{x\to a+} \frac{f(x+h) - f(x)}{h} \right) = \lim_{h\to 0+} \frac{f(a+h) - f(a+)}{h}$$

Similarly one can shown the existence of left derivative. This example confirms that piecewise smoothness is stronger condition than piecewise continuity with existence of one sided derivatives.

3.2 Different Notions of Convergence

3.2.1 Mean Square Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. Let f be defined on $[a, b]$. We say that the sequence $\{f_m\}_{m=1}^{\infty}$ converges in the mean square sense to f on $[a, b]$ if

$$\lim_{m\to\infty} \int_a^b |f(x) - f_m(x)|^2 \, \mathrm{d}x = 0$$

3.2.2 Pointwise Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a, b]$ if for each $x \in [a, b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. That is, for each $x \in [a, b]$ and $\varepsilon > 0$ there is a natural number $N(\varepsilon, x)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon, x)$$

3.2.3 Uniform Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on $[a, b]$ if for each $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon), \text{ and for all } x \in [a, b]$$

There is one more interesting fact about the uniform convergence. If $\{f_m\}_{m=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function to f on $[a, b]$, then f is continuous.

3.2.4 Example 1

Let $u_n = x^n$ on $[0, 1)$. Clearly, the sequence $\{u_n\}_{n=1}^{\infty}$ converges pointwise to 0, that is, for fixed $x \in [0, 1)$ we have $\lim\limits_{n\to\infty} u_n = 0$. But it does not converge uniformly to 0 as we shall show that for given ε there does not exist a natural number N independent of x such that $|u_n - 0| < \varepsilon$. Suppose that the series converges uniformly, then for a given ε with

$$|u_n - 0| < \varepsilon, \tag{3.3}$$

we seek for a natural number $N(\varepsilon)$ such that relation (3.3) holds for $n > N$. Note that relation (3.3) holds true if

$$x^n < \varepsilon \Longleftrightarrow n > \frac{\ln \varepsilon}{\ln x}$$

It should be evident now that for given x and ε one can define

$$N := \left[\frac{\ln \varepsilon}{\ln x}\right], \qquad \textit{where [ ] gives integer rounded towards infinity}$$

It once again confirms pointwise convergence. However if x is not fixed then $\ln \varepsilon / \ln x$ grows without bounds for $x \in [0, 1)$. Hence it is not possible to find N which depends only on ε and therefore the sequence u_n does not converge uniformly to 0.

3.2.5 Example 2

Let $u_n = \dfrac{x^n}{n}$ on $[0, 1)$. This sequence converges uniformly and of course pointwise to 0. For given $\varepsilon > 0$ take $n > N := \left[\frac{1}{\varepsilon}\right]$ then noting $\left[\frac{1}{\varepsilon}\right] > \frac{1}{\varepsilon}$ we have $|u_n - 0| < x^n/n < 1/n < \varepsilon$ for all $n > N$ Hence the sequence u_n converges uniformly.