Transform Calculus

(MA-20101)

Assignment-2

1. Let $f(t)$ be defined by

$$f(t) = \begin{cases} \text{t}, & 0 \le t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} \le t \le \pi \\ \pi - \dfrac{t}{2}, & \pi \le t \le 2\pi. \end{cases}$$

   Then sketch the graph of $f(t)$ and determine a Fourier series of $f(t)$ by assuming $f(t) = f(t + 2\pi)$.

2. Determine the Fourier series for $f(x) = H(x)$, the Heaviside Unit step function, in the range $[-\pi, \pi]$, $f(x) = f(x + 2\pi)$. Hence find the value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

3. Find the Fourier Series of the function

$$f(x) = \begin{cases} \sin\left(\dfrac{x}{2}\right), & 0 \le x \le \pi \\ -\sin\left(\dfrac{x}{2}\right), & \pi \le x \le 2\pi \end{cases}$$

   with $f(x) = f(x + 2\pi)$.

4. Determine the Fourier series for the function $f(x) = 1 - x^2$, $f(x) = f(x + 2\pi)$. Suggest the possible value of $f(x)$ at $x = \pi$.

5. Find the Fourier series expansion of the function $f(t)$ where

$$f(t) = \begin{cases} \pi^2, & -\pi < t < 0 \\ (t - \pi)^2, & 0 \le t < \pi \end{cases}$$

   with $f(t) = f(t + 2\pi)$. Hence determine the values of the series

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Determine the two Fourier half-range series for the above function $f(t)$ and sketch the graph of the function in both cases over the range $[-2\pi \leq t \leq 2\pi]$.

6. Given the half-range sine series $f(\pi - t) = \dfrac{8}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^3}$, $0 \leq t \leq \pi$, use Parseval's theorem to deduce the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$. Hence, deduce the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

7. Expand $f(x) = \sin x$, $0 < x < \pi$ in Fourier cosine series.

8. Expand the following functions in Fourier sine series :

   (a) $f(x) = \cos x$, $(0 < x < \pi)$

   (b) $f(x) = x^3$, $(0 \leq x < \pi)$.

9. Graph each of the following functions and find its corresponding Fourier series, using properties of even and odd functions whenever applicable

   (a)
   $$f(x) = \begin{cases} 8, & 0 < x < 2 \\ \text{-}8, & 2 < x < 4, \end{cases}$$
   periodicity of f is 4.

   (b)
   $$f(x) = \begin{cases} 2x, & 0 \leq x \leq 3 \\ 0, & -3 < x < 0, \end{cases}$$
   periodicity of f is 6.

   (c) $f(x) = 4x$, $0 < x < 10$. Periodicity of f is 10 .

10. Find the Fourier Series for e^{-ax}, $-\pi < x < \pi$ and deduce series for $\dfrac{\pi}{\sinh \pi}$.

11. Expand $f(x) = x^2$, $-\pi < x < \pi$ in Fourier series and show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

12. Find the Fourier series for the function f defined by $f(x) = x - x^2$, $-\pi < x < \pi$. Deduce that
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

13. If a is a real number, find Fourier series of the function f defined by $f(x) = e^{ax}$, $-\pi < x < \pi$, $\quad f(x + 2\pi) = f(x)$, $x \in \mathbb{R}$.
Deduce the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2}$.

14. Expand $f(t) = (1 - t^2)$, $-1 \leq t \leq 1$ in Fourier series.

15. Determine Fourier series expansion of the function
$$f(x) = \begin{cases} 2, & 0 < x < \frac{2\pi}{3} \\ 1, & \frac{2\pi}{3} < x < \frac{4\pi}{3} \\ 0, & \frac{4\pi}{3} < x < 2\pi. \end{cases}$$

16. Determine the Fourier series of the square wave function f defined by
$$f(x) = \begin{cases} \text{-k}, & -\pi < x < 0 \\ \text{k}, & 0 < x < \pi, \end{cases}$$
$f(x) = f(x + 2\pi)$. Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.

17. Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series assuming that the function is of period 2π.

18. Find Fourier series for $f(x) = x \sin x$, $0 < x < 2\pi$.

19. Determine half-range sine series for the function f defined by $f(t) = t^2 + t$, $0 \leq t \leq \pi$.

20. Find the Fourier series of the following function :

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ -x^2, & -\pi \leq x \leq 0. \end{cases}$$

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