① Recap

- For op. $L = a_0 \cdot \frac{d^2}{dx^2} + a_1 \cdot \frac{d}{dx} + a_2$

$$\rightarrow L^* = a_0 \cdot \frac{d^2}{dx^2} + (2a_0' - a_1)\frac{d}{dx}$$
$$+ (a_0'' - a_1' + a_2)$$

- $\langle v, Lu \rangle = J(u, v) + \langle u, L^* v \rangle$

- if $L = L^*$ & $B = B^*$ ← B.C. of adj.

$\Rightarrow L =$ self adjoint op.

② Example 1

$$L = \frac{d}{dx^2}$$

$$\left.\begin{array}{l} u(x=0) = 0 \\ u(x=1) = 0 \end{array}\right\} \text{B.C.}$$

$$\langle v, Lu \rangle = \int_0^1 v \cdot \frac{d^2u}{dx^2}\, dx$$

$$= v \cdot \frac{du}{dx}\Big|_0^1 - \int_0^1 \frac{dv}{dx} \cdot \frac{du}{dx} \cdot dx$$

$$= v \cdot \frac{du}{dx}\Big|_0^1 - \frac{dv}{dx} \cdot u\Big|_0^1$$

$$+\int_0^1 u \cdot \frac{d^2 v}{dx^2}\, dx \Rightarrow L^* = \frac{d^2}{dx^2}.$$

$$= \langle u , L^* v \rangle$$

$$= \overbrace{\left[v \frac{du}{dx} - u \frac{dv}{dx} \right]_0^1}^{J(u,v)}$$

$$+ \langle u , L^* v \rangle$$

$$\therefore J(u,v) = v(1) \cdot \frac{du}{dx}\bigg|_{x=1} - v(0) \cdot \frac{du}{dx}\bigg|_{x=0}$$

$$- \cancel{u(1)} \frac{dv}{dx}\bigg|_{x=1} + \cancel{u(0)} \cdot \frac{dv}{dx}\bigg|_{x=0}$$

$\because$ given B.C.

$$\therefore J(u,v) = v(1) \cdot \frac{du}{dx}\bigg|_{x=1} - v(0) \cdot \frac{du}{dx}\bigg|_{x=0}$$

We want $J(u,v) = 0$

$$\therefore \left.\begin{aligned} v(x=1) &= 0 \\ v(x=0) &= 0 \end{aligned}\right\} B^*$$

$$\therefore B = B^*$$

$$L = L^* = \frac{d^2}{dx^2}.$$

③ Example 2

④ Example 3

$$L = \frac{d^2}{dx^2}$$

$$\left.\begin{array}{l} u(x=0) = 0 \\ \left.\dfrac{du}{dx} + \beta u\right|_{x=1} = 0 \end{array}\right\} B.C.$$

find L^*, B^*

$$J(u,v) = v(1)\cdot\left.\frac{du}{dx}\right|_{x=1} - v(0)\cdot\left.\frac{du}{dx}\right|_{x=0}$$

same as before to get this

$\because$ given B.C.

$$- u(1)\left.\frac{dv}{dx}\right|_{x=1} + \cancel{u}(0)\cdot\left.\frac{dv}{dx}\right|_{x=0}$$

$$= v(1)\cdot(-\beta\cdot u(1)) - v(0)\cdot\left.\frac{du}{dx}\right|_{x=0}$$

$$- u(1)\cdot\left.\frac{dv}{dx}\right|_{x=1}$$

$$= - u(1)\cdot\left.\left(\frac{dv}{dx} + \beta\cdot v\right)\right|_{x=1} - v(0)\cdot\left.\frac{du}{dx}\right|_{x=0}$$

Set $J = 0$ → why do we do this?

$$\Rightarrow \left.\beta\cdot v + \frac{dv}{dx}\right|_{x=1} = 0 \quad \Big\} \; B^*$$

$\&\quad u(0) = 0$

$$L = L^* = \frac{d^2}{dx^2}$$

$\Rightarrow$ Self adjoint.

(5) Generalized 2nd Order Operator.

- $L(u) = a_0 \frac{d^2 u}{dx^2} + a_1 \cdot \frac{du}{dx} + a_0 \cdot u$

$\downarrow$

"Sturm Louville operator".

- During separation of variables

$$Lu = -\lambda \cdot a_3 \cdot u$$

$$\therefore \quad a_0 \cdot u'' + a_1 \cdot u' + a_2 \cdot u + \lambda \cdot a_3 \cdot u = 0 \quad (1)$$

$$\frac{d}{dx}\left(p(x) \cdot \frac{du}{dx}\right) + q(x) \cdot u + \lambda \cdot r(x) \cdot u = 0 \quad (2)$$

this can be rewritten as

where $p(x) = e^{\int \frac{a_1(x)}{a_0(x)} dx}$

$$q(x) = \frac{a_2(x)}{a_0(x)} \cdot p$$

$$r(x) = \frac{a_3(x)}{a_0(x)} \cdot p$$

But how? $\rightarrow$

$$(2) \rightarrow \quad p' \cdot u' + p \cdot u'' + \frac{a_2}{a_0} \cdot p \cdot u$$

$$+ \lambda \cdot \frac{a_3}{a_0} \cdot p \cdot u = 0$$

$$\text{but} \rightarrow p' = \frac{a_1}{a_0} \cdot p.$$

$$\therefore \quad \frac{a_1}{a_0} \cdot u' + u'' + \frac{a_2}{a_0} \cdot u + \lambda \cdot \frac{a_3}{a_0} \cdot u = 0$$

$$\therefore \quad a_1 \cdot u' + u'' + a_2 \cdot u + \lambda \cdot a_3 \cdot u = 0$$

$$= (1).$$

Q.E.D.

- $$p(x) = e^{\int \frac{a_1}{a_0} \cdot dx}.$$

$$\therefore \quad \ln(p) = \int \frac{a_1}{a_0} \cdot dx$$

$$\therefore \quad \frac{1}{p} \cdot \frac{dp}{dx} = \frac{a_1}{a_0}$$

$$\therefore \quad \frac{dp}{dx} = \frac{a_1}{a_0} \cdot p.$$

- adjoint

$$\frac{d}{dx}\left(p(x)\cdot\frac{du}{dx}\right) + q(x)\cdot u + \lambda\cdot r(x)\cdot u = 0$$

take $L = \frac{d}{dx}\left(p\cdot\frac{d}{dx}\right) + q$

$$= p\cdot\frac{d^2}{dx^2} + p'\cdot\frac{d}{dx} + q$$

we know that -

if $L = a_0\cdot\frac{d^2}{dx^2} + a_1\cdot\frac{d}{dx} + a_2$

$$\Rightarrow L^* = a_0\cdot\frac{d^2}{dx^2} + (2a_0' - a_1)\frac{d}{dx}$$

$$+ (a_0'' - a_1' + a_2)$$

$\therefore$ $L =$