Fourier Transform.

Lecture-18
9/10/17

$$\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dx = \bar{f}(s) \rightarrow \text{Laplace transform}$$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = F(\omega) \rightarrow \text{Fourier transform}$$

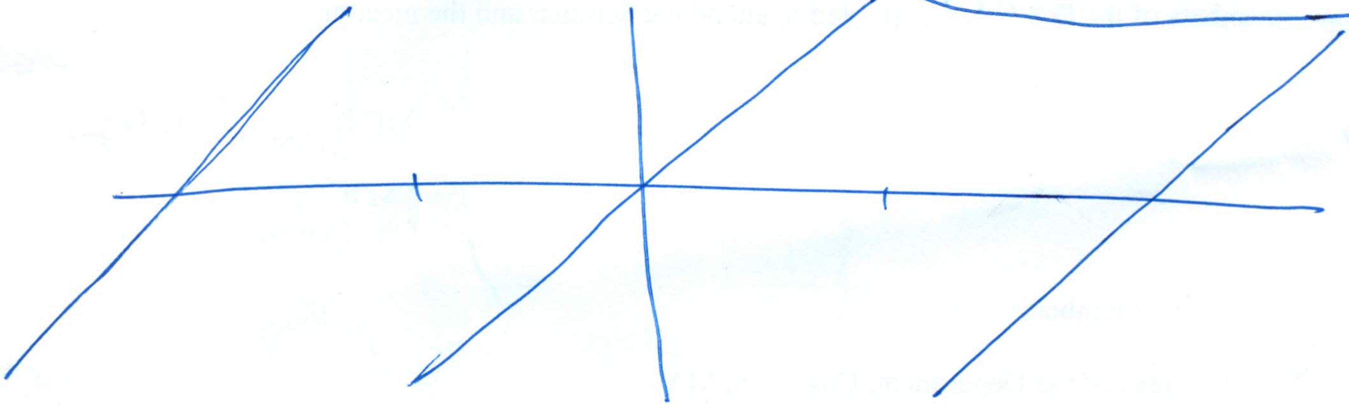
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad -l<x<l.$$

(Fourier series of $f(x)$ in $-l<x<l$)

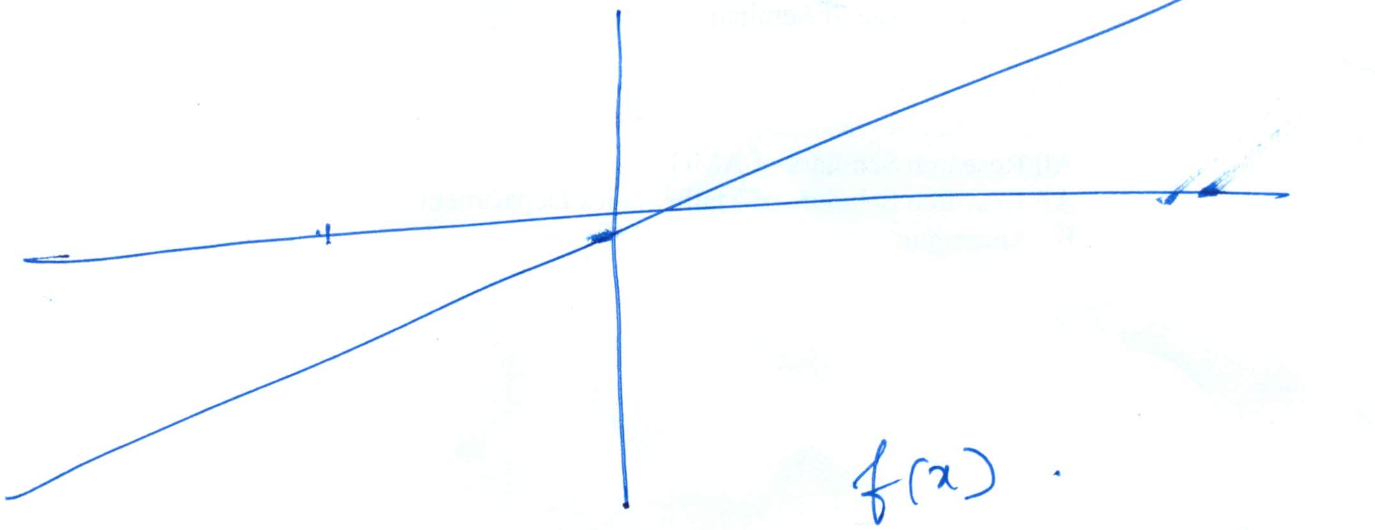
$$\frac{n\pi}{l} = \omega.$$

$f(x) = x, \quad -l<x<l.$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega x + \sum_{n=1}^{\infty} b_n \sin \omega x$$



graph of $\tilde{f}(x)$ ($=$ ~~P~~ Periodic extension of $f(x)$).



$f(x)$.

Fourier integral theorem.

Statement: A function $f(x)$ which is piecewise continuous in every finite interval and is absolutely integrable on the x-axis, can be represented by a Fourier integral

$$f(x) = \int_0^\infty \left[A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \right] d\alpha$$

which is valid at all points of continuity.

The functions $A(\alpha)$ & $B(\alpha)$ are given by

$$A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \alpha t \, dt$$

$$B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \alpha t \, dt .$$

Note: If $x = x_0$ is a point of discontinuity of $f(x)$, then.

$$\int_0^\infty \left[A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \right] d\alpha = \frac{1}{2} \left\{ f(x_0+0) + f(x_0-0) \right\}$$

Absolute convergence of $\int_a^b f(x)\,dx$ / absolute integrability of $f(x)$.

If $\int_a^b |f(x)|\,dx < \infty$ then $\int_a^b f(x)\,dx$ is absolutely ~~con~~ convergent. or $f(x)$ is absolutely integrable.

Proof - Consider the Fourier series expansion of $f(x)$ in $[-l, l]$ i.e

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow (1).$$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\, dt.$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\, dt, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\, dt.$$

Substituting a_0, a_n, b_n into (1) get

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\, dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}\, dt.$$

$$+ \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\, dt.$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\, dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}(t-x)\, dt.$$

Let $\alpha_n = \frac{n\pi}{l}$, $\quad \Delta\alpha_n = \alpha_{n+1} - \alpha_n = (n+1)\frac{\pi}{l} - \frac{n\pi}{l}$

or, $\boxed{\Delta\alpha_n = \frac{\pi}{l}} \Rightarrow \frac{1}{l} = \frac{1}{\pi}\Delta\alpha_n.$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\, dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\alpha_n\int_{-l}^{l} f(t)\cos\alpha_n(t-x)\, dt. \longrightarrow (2)$$

Define

$$F_l(u, x) = \int_{-l}^{l} f(t) \cos u(t-x)\, dt$$

Then (2), becomes,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\, dt + \frac{1}{\pi}\sum_{n=1}^{\infty} \Delta\alpha_n F_l(\alpha_n, x) \longrightarrow (2A)$$

What happens when $l \to \infty$?

$$\left| \frac{1}{2l}\int_{-l}^{l} f(t)\, dt \right| \leq \frac{1}{2l}\int_{-l}^{l} |f(t)|\, dt \longrightarrow (3).$$

Now, $\int_{-l}^{l} |f(t)|\, dt < \int_{-\infty}^{\infty} |f(t)|\, dt < M$, where M is a +ve const. (since $f(t)$ is absolutely integrable).

∴ (3) becomes,

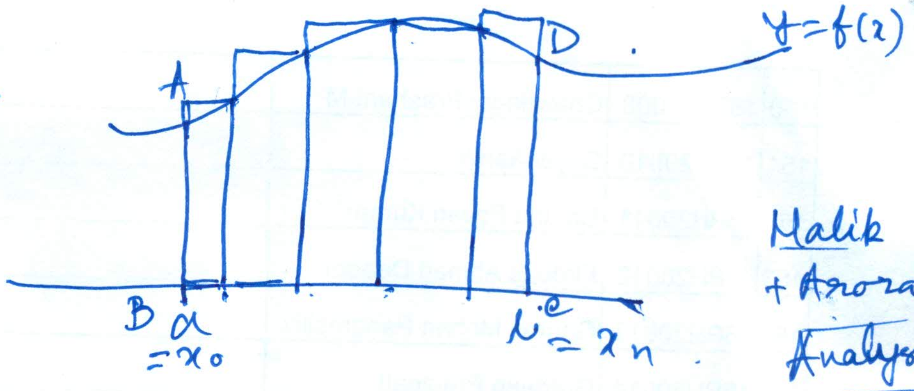
$$\left| \frac{1}{2l}\int_{-l}^{l} f(t)\, dt \right| < \frac{M}{2l} \to 0 \text{ as } l \to \infty.$$

∴ As $l \to \infty$, 1st term in the r.h.s of (2A) vanishes.

Computation of definite integral.

$$\int_a^b f(x)\,dx$$

= area bounded by the lines AB, BC, CD & the curved line DA.



Malik + Arora Analysis.

$$\int_a^b f(x)\,dx = \lim_{\delta \to 0} \sum_{n=1}^{N} f(\xi_n)\,\delta_n$$

$[a, b]$ is partitioned into sub intervals $[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{N-1}, x_N]$ of lengths δ_n.

$\delta \to$ greatest of $\|\delta_n\|$ (length of δ_n).

$$= \lim_{\Delta x \to 0} \Delta x \sum_{n=1}^{N} f(a + n\Delta x) \qquad x_{n-1} \leq \xi_n < x_n$$

Take equal subintervals each of length Δx.

i.e $\Delta x = \delta_n = \delta$

$$\boxed{\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \Delta x \sum_{n=1}^{N} f(a + n\Delta x)}$$

Second term of (2A) is

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \Delta\alpha_n\, F_l(\alpha_n, x).$$

At $l \to \infty$, $\Delta\alpha_n = \frac{\pi}{l} \to 0$.

$$\therefore \frac{1}{\pi} \lim_{l \to \infty} \sum_{n=1}^{\infty} F_l(n\Delta\alpha_n, x)\,\Delta\alpha_n$$

$$= \frac{1}{\pi} \lim_{\Delta\alpha_n \to 0} \sum_{n=1}^{\infty} F_\infty(n\Delta\alpha_n, x)\,\Delta\alpha_n.$$

$$= \frac{1}{\pi} \lim_{N \to \infty} \underbrace{\lim_{\Delta\alpha_n \to 0} \sum_{n=1}^{N} F_\infty(n\Delta\alpha_n, x)\,\Delta\alpha_n}$$

$$= \frac{1}{\pi} \lim_{N \to \infty} \int_0^{l_N} F_\infty(t, x)\, dt. \,,$$ where l_N is some large quantity which $\to \infty$ as $N \to \infty$.

$$= \frac{1}{\pi} \int_0^{\infty} F_\infty(t, x)\, dt.$$

$\therefore$ As $l \to \infty$, (from (2)).

$$f(x) = \cancel{\frac{1}{\pi} \sum_{n=1}^{\infty}} \frac{1}{\pi} \int_0^{\infty} F_\infty(t, x)\, dt.$$

$$= \frac{1}{\pi} \int_0^{\infty} F_\infty(u, x)\, du.$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(t - x)\, dt\, du.$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \{\cos ut \cos ux + \sin ut \sin ux\}\, dt\, du.$$

$$= \int_0^{\infty} \left[\left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ut\, dt \right) \cos ux + \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin ut\, dt \right) \sin ux \right] du.$$

or,

$$f(x) = \int_0^{\infty} \{A(u) \cos ux + B(u) \sin ux\}\, du.$$

where $$A(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ut\, dt.$$

$$B(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin ut\, dt \quad //.$$

Fourier cosine integral of $f(x)$.

Suppose $f(x)$ defined in $[-\infty, \infty]$ is an even funct.

Then, $B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin u\alpha \, du = 0$.

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos u\alpha \, du = \frac{2}{\pi}\int_{0}^{\infty} f(u)\cos u\alpha \, du \longrightarrow (1).$$

$$\therefore \quad f(x) = \int_{0}^{\infty} A(\alpha)\cos\alpha x \, d\alpha \longrightarrow (2).$$

where $A(\alpha)$ is given in (1)

(2) is known as fourier cosine integral.

Fourier sine integral of $f(x)$.

Suppose $f(x)$ defined in $[-\infty, \infty]$ is an odd funct.

Then $A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos u\alpha \, du = 0$.

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin u\alpha \, du = \frac{2}{\pi}\int_{0}^{\infty} f(u)\sin u\alpha \, du \longrightarrow (3).$$

$$\therefore \quad f(x) = \int_{0}^{\infty} B(\alpha)\sin\alpha x \, d\alpha \rightarrow (4)$$

where $B(\alpha)$ is given in (3).

(4) is known as fourier sine integral.

Fourier integral in complex form.

$$f(x) = \int_0^{\infty} \left(A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \right) d\alpha \longrightarrow (1).$$

$$\left(A(\alpha), B(\alpha) \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left(\cos u\alpha , \sin u\alpha \right) du.$$

Subst. in (1),

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ \left(\int_{-\infty}^{\infty} f(u) \cos u\alpha \cos \alpha x \, du \right) + \left(\int_{-\infty}^{\infty} f(u) \sin u\alpha \sin \alpha x \, du \right) \right\} d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \cos \alpha (u - x) \, du \, d\alpha . \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \left\{ e^{i\alpha(u-x)} + e^{-i\alpha(u-x)} \right\} du \, d\alpha .$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \int_0^{\infty} e^{i\alpha(u-x)} d\alpha + \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \int_0^{\infty} e^{-i\alpha(u-x)} d\alpha. \rightarrow (2)$$

Let I be the 2nd term of (2).

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \int_0^{\infty} e^{-i\alpha(u-x)} d\alpha$$

$$\alpha = -\alpha'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \int_0^{-\infty} e^{+i\alpha'(u-x)} \times -d\alpha' = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \, du \int_{-\infty}^{0} e^{i\alpha(u-x)} d\alpha.$$

Subst. I into (2),

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \int_{0}^{\infty} e^{i\alpha(u-x)}\,d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \int_{-\infty}^{0} e^{i\alpha(u-x)}\,d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \int_{-\infty}^{\infty} e^{i\alpha(u-x)}\,d\alpha.$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(u-x)}\,du\,d\alpha.$$

Complex form of Fourier integral.

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,du\,d\alpha.$$

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1. Find the Fourier integral representation of $f(x)$ where $f(x) = \begin{cases} x, & |x|<1 \\ 0, & |x|>1. \end{cases}$

$$f(x) = \int_{0}^{\infty}\left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right)dx.$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos u\alpha\,du = \frac{1}{\pi}\int_{-1}^{1} u\cos u\alpha\,du = 0$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin u\alpha\,du = \frac{1}{\pi}\int_{-1}^{1} u\sin u\alpha\,du$$

$$= \frac{2}{\pi}\int_0^1 u \sin u\alpha \, du .$$

$$B(\alpha) = \frac{2}{\pi}\left[-u\frac{\cos u\alpha}{\alpha}\Big|_1^0 + \frac{1}{\alpha}\int_0^1 \cos u\alpha \, du \right].$$

$$= \frac{2}{\pi}\left[-\frac{\cos\alpha}{\alpha} + \frac{1}{\alpha^2}\left[\sin u\alpha\right]_0^1 \right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\sin\alpha x \, d\alpha .$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\sin\alpha x \, d\alpha = \begin{cases} x, & |x|<1. \\ 0, & |x|>1. \end{cases}$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\sin\alpha x \, d\alpha = \frac{1+0}{2}, \; x=1.$$

$$= \frac{-1+0}{2}, \; x=-1.$$

Use Fourier Int. thm to show that.

$$\int_0^\infty \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\sin\alpha x \, d\alpha = \begin{cases} \frac{\pi}{2}x, & |x|<1. \\ \frac{\pi}{\cancel{4}}, & \cancel{|x|=1}\; x=1 \\ -\frac{\pi}{\cancel{4}}, & x=-1. \\ 0, & |x|>1 \end{cases}$$

Ex. Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\alpha x + \alpha \sin\alpha x}{1+\alpha^2}\, d\alpha = \begin{cases} 0, & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0. \\ \pi e^{-x} & \text{if } x>0. \end{cases}$$

Sol. $f(x) = \begin{cases} e^{-x}, & x>0. \\ 0, & x<0 \end{cases}$.

$$f(x) = \int_0^\infty \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right) d\alpha.$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos u\alpha\, du = \frac{1}{\pi}\int_0^\infty e^{-u}\cos u\alpha\, du.$$

$$= \frac{1}{\pi} L\left[\cos u\alpha ; s=1\right] = \frac{1}{\pi}\cdot\frac{1}{\alpha^2+1}.$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin u\alpha\, du = \frac{1}{\pi}\int_0^\infty e^{-u}\sin u\alpha\, du.$$

$$= \frac{1}{\pi} L\left[\sin u\alpha ; s=1\right] = \frac{1}{\pi}\cdot\frac{\alpha}{\alpha^2+1}.$$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{\alpha^2+1}\, d\alpha.$$

$$\therefore \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{\alpha^2+1}\,d\alpha = f(x), \quad x \neq 0 \quad \text{~~at a pt. of continuity~~}$$

$$= \frac{f(0-0)+f(0+0)}{2} \quad \text{at } x=0.$$

$$= \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \\ \frac{1}{2}, & x=0 \end{cases}$$

$$\therefore \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{\alpha^2+1}\,d\alpha = \begin{cases} 0, & \underline{x<0} \\ \frac{\pi}{2}, & x=0 \\ \pi e^{-x}, & \underline{x>0} \end{cases}$$

<u>Ex.</u> Using Fourier integral representation show that

$$\int_0^\infty \frac{\sin s \cos xs}{s}\,ds = \begin{cases} \frac{\pi}{2}, & \text{if } \cancel{0 \le x < 1}\ |x|<1 \\ \frac{\pi}{4}, & \text{if } |x|=1 \\ 0, & \text{if } |x|>1 \end{cases}$$

<u>Soln.</u> Take

$$f(x) = \begin{cases} 1, & \text{if } \cancel{0 \le}\ |x|<1 \\ 0, & \text{if } x>1. \end{cases}$$