Convolution theorem for Laplace Transform and properties of convolution:

<u>Thm</u>. (Convolution theorem): If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$, then

$$L\{f(t) * g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s)\bar{g}(s) \longrightarrow (1)$$

or, equivalently,

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t) * g(t) \longrightarrow (2)$$

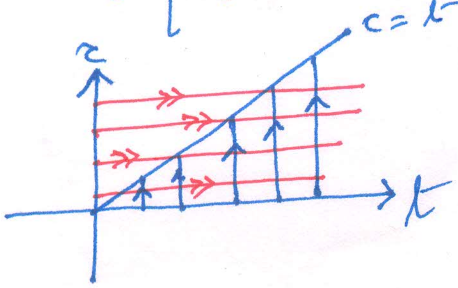
where $f(t) * g(t)$ (or, sometimes written as $(f*g)(t)$ or simply $f*g$) is called the convo-lution of the funct^ns $f(t)$ and $g(t)$ and is defined by the integral (in the context of Laplace transform)

$$f*g = f(t)*g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau. \longrightarrow (3)$$

The integral in (3) is often referred to as the <u>convolution integral</u> (or Faultung).

<u>Proof</u>. We've by definit^n,

$$L\{f(t)*g(t)\} = \int_0^\infty e^{-st}dt\left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right)$$



$$= \int_0^\infty g(\tau)\,d\tau \int_\tau^\infty e^{-st} f(t-\tau)\,dt$$

Put $t-\tau = x$

$$\therefore L\{(f*g)\} = \int_0^\infty g(\tau)\,d\tau \int_0^\infty e^{-s(x+\tau)} f(x)\,dx$$

$$= \underbrace{\int_0^\infty g(\tau) e^{-s\tau} d\tau}_{\bar{g}(s)} \underbrace{\int_0^\infty e^{-sx} f(x)\,dx}_{\bar{f}(s)}.$$

$$= \bar{f}(s)\,\bar{g}(s).$$

<u>Properties of convolution operation for the Laplace Transform:</u>

1. $f(t) * \{g(t) * h(t)\} = \{f(t) * g(t)\} * h(t)$ – (associative)
2. $f(t) * g(t) = g(t) * f(t)$ (commutative)
3. $f(t) * (g(t) + h(t)) = f(t) * g(t) + f(t) * h(t)$ (distributive)
4. $f(t) * \{a\,g(t)\} = \{a\,f(t)\} * g(t) = a(f*g)$; $a \to$ const.
5. $L\{f_1 * f_2 * f_3 * \cdots * f_n\} = \bar{f}_1(s)\,\bar{f}_2(s).\bar{f}_3(s)\cdots\bar{f}_n(s)$
6. $L\{f * f * \cdots * f\} = \bar{f}^n(s).$

First we prove property 2. $\therefore f * g = g * f$.

$$(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau .$$

Put $t - \tau = u$

$$= -\int_t^0 f(u)\, g(t-u)\, du = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t g(t-u)\, f(u)\, du = g * f .$$

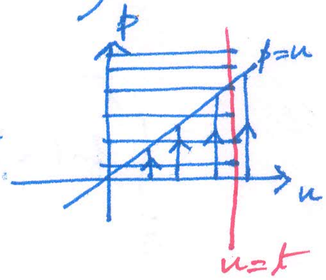
$$\therefore f * g = g * f = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(u)\, g(t-u)\, du .$$

Proof of property 1.

$$(f \star g) \star h = \int_0^t (f \star g)(t-u)\, h(u)\, du .$$

$$= \int_0^t h(t-u)\, (f \star g)(u)\, du .$$

$$= \int_{u=0}^{t} h(t-u)\, du \left(\int_{p=0}^{u} f(p)\, g(u-p)\, dp \right)$$



$$= \int_{p=0}^{t} f(p)\, dp \int_{u=p}^{t} h(t-u)\, \cancel{du}\, g(u-p)\, du$$

Put $u - p = x$

$u = x + p$.

$t - u = t - x - p$

$$= \int_{p=0}^{t} f(p)\, dp \int_{x=0}^{t-p} h(t-x-p)\, g(x)\, dx$$

$$= \int_0^t f(p)\, dp \int_0^{t-p} h(t-p-x)\, g(x)\, dx .$$

$$= \int_0^t f(p)\, (h \star g)(t-p)\, dp .$$

$$= \int_0^t (g \star h)(t-p)\, f(p)\, dp .$$

$$= (g \star h) \star f = f \star (g \star h)$$

$= \text{L.H.S.}$

Examples on convolution theorem

1. Find $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$

<u>Ans.</u> $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \bar{f}(s)\,\bar{g}(s)$, say.

$\bar{f}(s) = \frac{1}{s^2} \Rightarrow f(t) = t$

$\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow g(t) = \sin t.$

$\therefore L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = f * g = \int_0^t f(u)\,g(t-u)\,du$

$= \int_0^t u \sin(t-u)\,du = \int_0^t (\sin u)(t-u)\,du$ $\quad \because f*g = g*f.$

$= (t-u)\cos u \Big|_t^0 - \int_0^t \cos u\,du$

$= t - \left[\sin u\right]_0^t = t - \sin t.$

2. $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

$L^{-1}\left[\frac{1}{s(s+1)}\right] = \int_0^t e^{-u}\,du = \left[e^{-u}\right]_t^0 = (1-e^{-t}).$

$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\bar{f}(s)\cdot\bar{g}(s)\right]$

$\bar{f}(s) = \frac{1}{s(s+1)} \Rightarrow f(t) = 1-e^{-t}$

$\bar{g}(s) = \frac{1}{s+2} \Rightarrow g(t) = e^{-2t}.$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right]$$

$$= (f * g)(t) = \int_0^t (1-e^{-u})\,e^{-2(t-u)}\,du.$$

$$= e^{-2t}\int_0^t (e^{2u} - e^{u})\,du = e^{-2t}\left[\frac{e^{2u}}{2}\Big|_0^t - e^{u}\Big|_0^t\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{2} - \frac{1}{2} - e^{t} + 1\right] = e^{-2t}\left(\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}. \quad //$$

Heaviside's expansion theorem.

If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, then,

$$L^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \sum_{n=1}^{\infty} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \exp(\alpha_k t) \longrightarrow (1)$$

where α_k's are distinct roots of the equation $\bar{q}(s) = 0$.

Proof Without loss of generality let us assume that the leading coefficient of $\bar{q}(s)$ is unity so that

$$\bar{q}(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)\cdots(s-\alpha_n) \longrightarrow (2)$$

Using the rules of partial fraction, we can express $\dfrac{\bar{p}(s)}{\bar{q}(s)}$ as,

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_n}{s-\alpha_n} \longrightarrow (3)$$

$$\therefore \bar{p}(s) = A_1(s-\alpha_2)\cdots(s-\alpha_n) + A_2(s-\alpha_1)(s-\alpha_3)\cdots(s-\alpha_n) + \cdots + A_n(s-\alpha_1)\cdots(s-\alpha_{n-1}) \longrightarrow (4)$$

$$\therefore \bar{p}(\alpha_k) = A_k(\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n) \longrightarrow (5)$$

Diff. (2) w.r.to s we get

$$\bar{q}'(s) = (s-\alpha_2)(s-\alpha_3)\cdots(s-\alpha_k)$$
$$+(s-\alpha_1)(s-\alpha_3)\cdots(s-\alpha_n)+\cdots+(s-\alpha_1)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n)$$
$$\cdots+(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{n-1}).$$

$$\therefore \bar{q}'(\alpha_k) = (\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n) \longrightarrow (6)$$

$\therefore$ From (5),

$$\bar{p}(\alpha_k) = A_k\,\bar{q}'(\alpha_k) \Rightarrow A_k = \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}$$

Substituting this A_k into (3) we get,

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n}\frac{A_k}{s-\alpha_k} = \sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}\cdot\frac{1}{s-\alpha_k}.$$

$$\therefore f(t) = L^{-1}\left(\sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}\cdot\frac{1}{s-\alpha_k}\right)$$
$$= \sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}L^{-1}\left(\frac{1}{s-\alpha_k}\right) = \sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}e^{\alpha_k t}. //$$

Ex. $L^{-1}\left\{\frac{s}{s^2-3s+2}\right\}$.

$s^2-3s+2 = (s-2)(s-1)$.

Here $\bar{p}(s) = s$, $\bar{q}(s) = s^2-3s+2$

$\bar{q}'(s) = 2s-3$.

$$\therefore L^{-1}\left\{\frac{s}{s^2-3s+2}\right\} = \frac{\bar{p}(1)}{\bar{q}'(1)}e^{t} + \frac{\bar{p}(2)}{\bar{q}'(2)}e^{2t}$$
$$= -e^{t} + 2e^{2t} //.$$