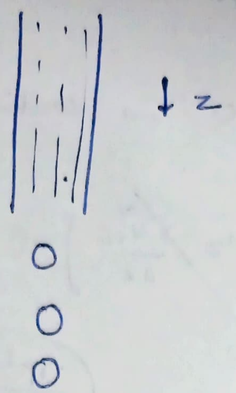
Capillary jet instability



~~fluid is flowing through the~~

jet of liquid is falling vertically down. Eventually it will pinch of and form drops.

→ Because of the surface tension effect, the breaking of droplets is taking place

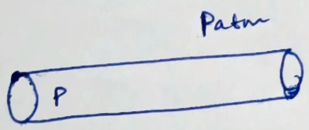
→ Surface tension is important and has to be included.

→ Viscosity is not causing the break up. (it will only change the growth rate) (time of break up)

Eg: honey will take a long time to break

→ Assume → ① Inviscid

→ ② we will consider the jet to be stationary or a horizontal circular thread!



Will the thread break-up and disintegrate into droplets?

What will be the size of droplets?

The base state is $\vec{u}^{ss} = 0$.

$$\boxed{P - P_{atm} = \frac{\gamma}{R}} \neq \gamma(\nabla \cdot n)$$

- Axisymmetry → θ symmetric
→ Infinite extent in z-direction

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} \doteq 0$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial z} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial r} + \underset{\text{viscous}}{0} + \underset{\text{gravity}}{0}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$

Assume $u_z = u_z^{bs} + \epsilon \tilde{u}_z$

$u_z^{bs} = 0$ → perturbation

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}_r) + \frac{\partial}{\partial z}(\tilde{u}_z) = 0 \rightarrow \text{continuity}, \rightarrow O(\epsilon)$$

$$\rho\left(\epsilon \frac{\partial \tilde{u}_r}{\partial t} + \cancel{\epsilon \tilde{u}_r \frac{\partial \epsilon \tilde{u}_r}{\partial r}} + \cancel{\epsilon \tilde{u}_z \frac{\partial \epsilon \tilde{u}_r}{\partial z}}\right) = -\frac{\partial (p^{bs} + \epsilon \tilde{p})}{\partial r}$$

order of ϵ

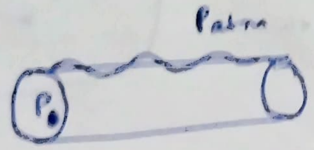
$\frac{\partial p^{bs}}{\partial r} = 0$ → uniform

→ base velocity = 0

$$\rho \frac{\partial \tilde{u}_r}{\partial t} = -\frac{\partial \tilde{p}}{\partial r} \rightarrow O(\epsilon)$$

$$\rho \frac{\partial \tilde{u}_z}{\partial t} = -\frac{\partial \tilde{p}}{\partial z} \rightarrow \text{BC need not worry}$$

because it is ∞ long.



As a result of disturbance, the surface deforms.

$r = a$ → base state

perturbed state → $r = a(1 + \epsilon f(z))$ → deflection of interface

$$\boxed{p - P_{atm} = \gamma \nabla \cdot n}$$ → normal stress BC.

→ curvature

normal stress balance

$\vec{n} = \frac{\nabla F}{|\nabla F|}$ ← absolute value

$F = r - a(1 + \epsilon f(z))$

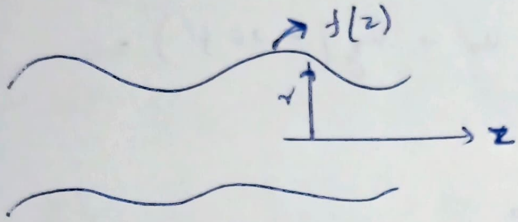
$$\nabla f = \begin{array}{c} \\ \begin{bmatrix} 1 \\ -a\epsilon f'(z) \end{bmatrix} \end{array} \begin{array}{l} \rightarrow r \\ \rightarrow z \end{array}$$

$$\vec{n} = \frac{e_r - a\epsilon f' e_z}{\sqrt{1 + a^2 \epsilon^2 f'^2}}$$

$$\to \nabla\cdot n = \left(\frac{e_r}{r}\frac{\partial (r}{\partial r} + e_z\frac{\partial}{\partial z}\right)\cdot\left(\frac{e_r - a\epsilon f' e_z}{\sqrt{1+a^2\epsilon^2 f'^2}}\right)$$

$$\nabla\cdot n = \frac{1}{r\sqrt{1+a^2\epsilon^2 f'^2}} + \frac{\partial}{\partial z}\left(\frac{-a\epsilon f'}{\sqrt{1+a^2\epsilon^2 f'^2}}\right)$$

$$\nabla\cdot n = \underbrace{\frac{1}{r\sqrt{1+a^2\epsilon^2 f'^2}}}_{\text{radial curvature}} - \underbrace{\frac{a\epsilon f''}{(1+a^2\epsilon^2 f'^2)^{3/2}}}_{\text{axial curvature}}$$



the curvature along the plane of the book (r-z plane) is axial curvature

In r-θ plane  → this is radial curvature.

$$(P_1^{ss} + \epsilon\tilde{P}_1 - P_{atm}) = \gamma\left(\frac{1}{r\sqrt{1+a^2\epsilon^2 f'^2}}\right) - \frac{a\epsilon f''\times\gamma}{(1+a^2\epsilon^2 f'^2)^{3/2}}$$

use binomial expansion.

$$= \frac{\gamma}{\underset{a+a\epsilon f}{r}}\left(1 - \frac{1}{2}a^2\epsilon^2 f'^2 - -\right) - \gamma a\epsilon f''\left[1 - \frac{3}{2}a^2\epsilon^2 f'^2 - -\right]$$

$$= \frac{\gamma}{a}(1 - \epsilon f)\left(1 - \frac{1}{2}a^2\epsilon^2 f'^2\right) - \gamma a\epsilon f''\times\gamma.$$

$$= \frac{\gamma}{a} - \frac{\gamma}{a} f\epsilon - a\epsilon f''\times\gamma$$

$$\boxed{P_1^{ss} - P_{atm} = \frac{\gamma}{a}}$$

$\cancel{\tilde{P} = \frac{\gamma}{a} f}$

P_{atm}

$$\boxed{\tilde{P}_1 = -\frac{\gamma}{a} f - a\gamma f''}$$

γ → surface tension

$F = r - a(1 + \epsilon f(z,t)) = 0$

interface

$\frac{DF}{Dt} = 0 \Rightarrow \frac{\partial F}{\partial t} + v \cdot \nabla F = 0$

kinematic BC.

??

$-a\epsilon \frac{\partial f}{\partial t} + u_r \frac{\partial F}{\partial r} + u_z \frac{\partial F}{\partial z} = 0$

??

$-a\epsilon \frac{\partial f}{\partial t} + u_r + u_z(-a\epsilon f') = 0$

$u_r = u_r^{ss} + \tilde{u}_r$

$-a\epsilon \frac{\partial f}{\partial t} + \epsilon \tilde{u}_r - a \cancel{\epsilon^2} \tilde{u}_z \frac{\partial f}{\partial z} = 0$

$O(\epsilon)$. $\boxed{\tilde{u}_r = \frac{\partial f}{\partial t} a}$

→ Make dimensionless

$l_c = a$ (radius of unperturbed jet)

$P_c = \frac{\gamma}{a}$

$u_c = \sqrt{\frac{P_c}{\rho}} = \sqrt{\frac{\gamma}{a\rho}}$

$t_c = \frac{l_c}{u_c} = a^{3/2}\sqrt{\frac{\rho}{\gamma}}$

no velocity is characterizing the flow - No characteristic velocity (whatever velocity is there is due to γ)

→ $\rho \frac{\partial \tilde{u}_r}{\partial t} = -\frac{\partial \tilde{p}}{\partial r}$

$u_r^* = \frac{u_r}{u_{r,c}}$

$\frac{\partial \hat{u}_r}{\partial t^*} = -\frac{\partial p^*}{\partial r^*}$

BC ✓ :

$$-\frac{\partial u_r^*}{\partial t} = \frac{-\partial p^*}{\partial r^*}$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{-\partial p^*}{\partial z^*}$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{\partial}{\partial z^*} u_z^* = 0.$$

$$u_r^* = -\frac{\partial f}{\partial t^*}$$

$f \rightarrow$ dimensionless

highest growth rate means no breakup or linear regime

$$p_1^* = -f - f''$$

→ let $f(z,t) = c\, e^{\sigma t} \sin kz$

$$p_1^* = -c\, e^{\sigma t} \sin kz + k^2 c\, e^{\sigma t} \sin kz$$

$$= c\, e^{\sigma t} (\sin kz)(k^2 - 1)$$

↳ at $\underline{\underline{r^* = 1}}$ since it is from normal stress BC

$$\boxed{\frac{\partial \bar{u}}{\partial t} = -\nabla p}$$ → take $\underline{\underline{\nabla \cdot}}$

$$\boxed{\nabla^2 p = 0}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial^2 p}{\partial z^2} = 0$$

$$p = (A(t)\cos kz + B(t)\sin kz)\, I_0(kr)$$ ✓

from bc at $r = 1$

$$\boxed{p = B(t) \sin kz\, I_0(kr)}$$

at $r = 1$

$$c\, e^{\sigma t} \sin (k^2 - 1) = B(t) \sin k_z\, I_0 (k)$$

$$B(t) = \frac{c\, e^{\sigma t} (k^2 - 1)}{I_0 k}$$

$$p = \frac{c\, e^{\sigma t} (k^2 - 1) \sin k_z\, I_0 (kr)}{I_0 (k)}$$

find $\frac{\partial p}{\partial r}$

$$\frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r}$$

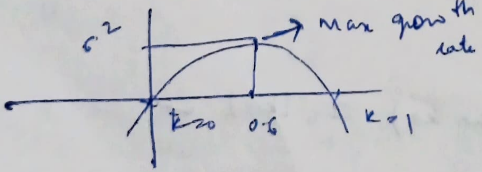
$$u_r = \frac{-c\, e^{\sigma t}\, k (k^2 - 1) \sin k_z\, I_1 (kr)}{\sigma\, I_0 (k)}$$

k = wave number

at $r = 1$, $u_r = \frac{\partial f}{\partial t}$

$u_r = c\, \sigma\, e^{\sigma t} \sin k_z$ → at $r = 1$

$$\sigma^2 = \frac{(1 - k^2)\, k\, I_1 (k)}{I_0 (k)}$$ → for non-zero c



$0 < k < 1$, $\sigma^2 < 0$

$$\lambda = \frac{2\pi}{k}$$

actual $\lambda = \frac{2\pi a}{0.7}$

jet is unstable when λ are greater than $2\pi a$

$k > 1$, $\sigma^2 < 0$, neutrally stable.

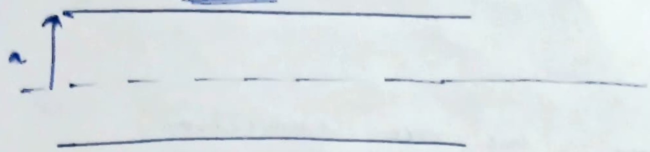
Rayleigh's

state $\xrightarrow{\text{perturbed}}$ Nearby state.

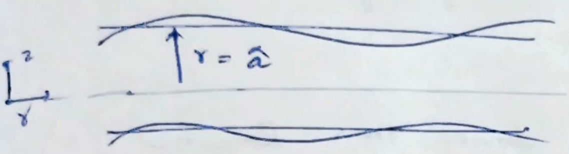
Work is done ~~to~~ by the system to move from state 1 to state 2

→ State 1 is unstable

state 1



state - 2



$r = \hat{a} + \epsilon \cos kz.$

$k_c \sim 1/a$

$k > k_c$

$\underbrace{\Delta A}_{\text{change in area}} < 0$

more area of jet (jet is stable)

↳ upon perturbation jet has gone to higher area.

upon perturbation (unstable)

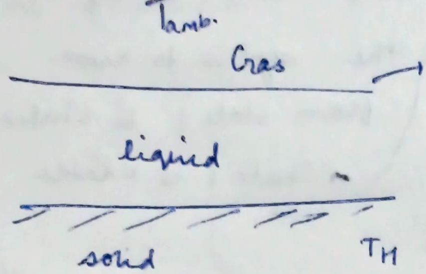
$k < k_c$

perturbed jet has lower area

↳ jet is unstable

Rayleigh ??

Marangoni instability



as the temperature increases, we see convection

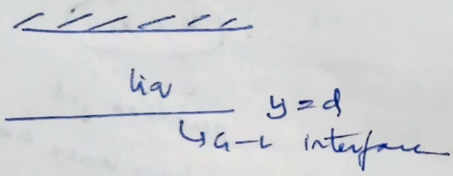
$Ra > Ra_{critical}$. This is when buoyancy is driving force

$Ra \propto d^3 \rightarrow$ film thickness

even when Ra is $O(1)$, much lower than $Ra_{critical}$, convection has been observed.

→ Convection occurs in the absence of gravity. Even when solid wall is on top & hot, interface is below, we see convection.

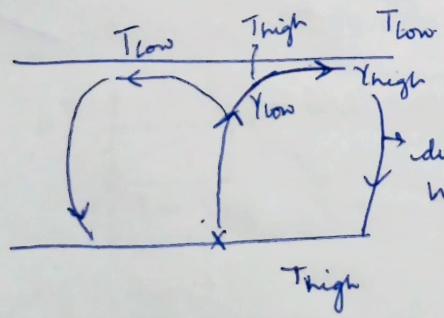
Add surfactant
↓
surface tension ↓



What is causing it? surface tension gradient

Direction of Marangoni flow
↓
low to high surface tension

→



due to conservation of mass, it has to flow down.

Surface tension decreases with an increase in temperature. If we have a hot packet of fluid rising due to disturbance, this is dragged on both sides of cold fluid, which has high surface tension. By continuity, cold fluid comes down & we see convection.

if motion is induced by surface tension gradient → marangoni convection

→ BC at L-L interface

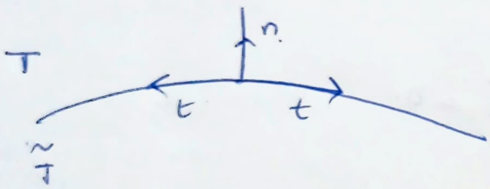
Net force at the ~~inf~~ interface = 0.
↳ being ~~...~~ mass is negligible
(of an interface)

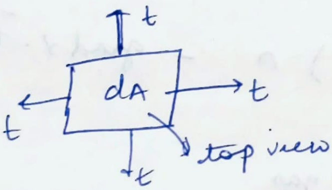
liq 2

liq 2

Thermodynamic perspective is energy per unit area

→ Mechanical perspective
↳ force / Length





T → total stress tensor in upper liq

$\tilde{T}$ " " " in lower liq

— force exerted by upper liq = $T \cdot n$ → along dA

$\tilde{n} = -n$

" " " lower liq = $\tilde{T} \cdot \tilde{n}$

$= -\tilde{T} \cdot \hat{n}$ → along dA

surface tension force = $\gamma \underset{t}{} \, dl$ → along the perimeter

~~Later~~

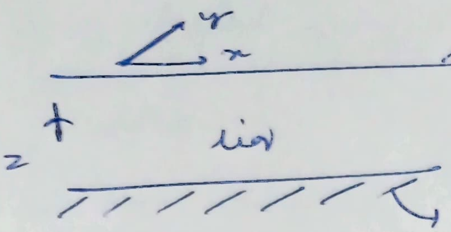
$$\iint_A (T - \tilde{T}) \cdot n \, dA + \int_C \gamma_t \, dL = 0$$

$$(T - \tilde{T}) \cdot n + \nabla \gamma - \gamma n \nabla \cdot n = 0 \checkmark$$

$$(\tilde{P} - P) + \underbrace{n \cdot (\tau - \tilde{\tau}) \cdot n}_{= 0 \text{ if no flow}} - \gamma \nabla \cdot n = 0$$ → normal stress BC.

tangential BC

↳ grad γ is induced by gradients of c, T



→ G–L interface, ↳ stress bc is modified

↳ solid wall ($T = T_0$, no slip, impermeable wall)

vertical component of vel = 0

$-k \frac{\partial T}{\partial z} = h(\underset{\sim}{\Delta T})$

$T - T_{amb}$

$$\gamma = \gamma_0 (1 - \gamma_T (T - T_0))$$

$$\underline{\underline{\gamma_T > 0}}$$

→ $t \cdot (\tau - \tilde{\tau}) \cdot n$ − grad $\gamma \cdot \vec{t} = 0$.

$\tau = 0$ → gas

$\tilde{\tau} < \begin{matrix} \tau_{zx} \\ \tau_{zy} \end{matrix}$

$$\left\{ \begin{aligned} \tau_{zx} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\partial \gamma}{\partial x} \\ \tau_{zy} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{\partial \gamma}{\partial y} \end{aligned} \right\}$$

$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} = 0$

Solution procedure for onset of convection

① Continuity, NS, Energy balance.

② BC → $-k\frac{dT}{dz} = h(T - T_{amb})$ → at the interface.

velocity eqn do not depend on temp.

$\tau_{zx} = -\frac{\partial \sigma}{\partial x}$ $\tau_{zy} = -\frac{\partial \sigma}{\partial y}$.

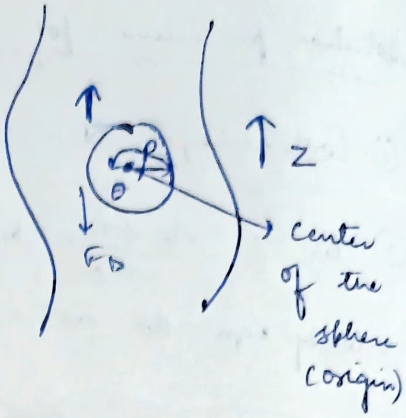
Marangoni → BSL

→ check P ~~BC~~

BC

→ Pg 106 → BSL

Rising Bubble



→ Assumptions

① steady state

② Bubble is spherical

③ No change in diameter

④ Creeping flow.

⑤ No circulation inside the bubble.

→ for describing flow

$$\mu \nabla^2 u = \nabla (P + \rho g z)$$

$$\text{div } u = 0 .$$

→ for temperature

$$(u \cdot \text{grad}) T = \alpha \nabla^2 T$$

$$\nabla^2 T = 0 \rightarrow \text{creeping flow.}$$

→ force balance on the bubble

$$m \frac{dv}{dt} = F_B - F_g - F_D .$$