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Sturm-Liouville Problem

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda w(x)\right]y = 0$$

Homogeneous 2nd order ODE with linear BC

$x = a$, $\alpha_1 y(a) + \beta_1 \frac{dy(a)}{dx} = 0$, $\alpha_1^2 + \beta_1^2 > 0$

$x = b$, $\alpha_2 y(b) + \beta_2 \frac{dy(b)}{dx} = 0$, $\alpha_2^2 + \beta_2^2 > 0$.

Constraint :- $P(x), q(x), w(x)$ are real valued continuous function including $\frac{dP}{dx}$

$P(x), w(x)$ are +ve over $[a, b]$, $q(x) \le 0$

$\lambda \to$ Eigen values (characteristic values)

Corresponding solns called eigen functions (ϕ)

Non trivial soln :- $\lambda_1 \to \phi_1$, $\lambda_2 \to \phi_2$ ... $\lambda_n \to \phi_n$

let λ_n, λ_m be two distinct eigen values $\to$ $\lambda_n \to \phi_n$, $\lambda_m \to \phi_m$.

$$(\lambda_m - \lambda_n)\int_a^b \phi_n(x)\phi_m(x)\, w(x)\, dx = 0 \text{ — derive}$$

$$\Rightarrow \int_a^b \phi_n(x)\,\phi_m(x)\,\underbrace{w(x)}_{\text{weight factor}}\, dx = 0 \to \text{orthogonal.}$$

→ consider a set of functions $\{\phi_n(x),\ n = 0, 1, 2 \dots\}$ orthogonal wrt weight function $w(x)$

$$f(x) = A_0\phi_0(x) + A_1\phi_1(x) + \dots + A_n\phi_n(x) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x)$$

(Arbitrary function)

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$$\int_a^b f(x)\,\phi_n(x)\,w(x)\,dx = \sum_{n=0}^{\infty}\int_a^b A_n\,\phi_n(x)\,\phi_m(x)\,w(x)\,dx.$$

$$A_n = \frac{\int_a^b \phi_n(x)\, f(x)\, w(x)\, dx}{\int_a^b \phi_n^2(x)\, \underset{w(x)}{dx}}$$

$$\int_a^b \phi_n^2(x)\, \underset{w(x)}{dx} = \text{norm of eigen function } \phi_n(x)$$

⟹ Ordinary fourier series : $\frac{d^2y}{dx^2} + \lambda^2 y = 0$ — ① → eigen value problem

$\alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0$, $\alpha_1^2 + \beta_1^2 > 0$

$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0$, $\alpha_2^2 + \beta_2^2 > 0$.

for $p(x) = 1$, $q(x) = 0$, $w(x) = 1$

↳ eqn ① is S-L problem.

complete orthogonal set in $[0, L]$

→ Fourier sine series :- $\frac{d^2y}{dx^2} + \lambda^2 y = 0$

BC :- $\alpha_1 y(0) = 0$

$\alpha_2 y(L) = 0$.

$\phi_n(x) = \sin\lambda_n x \Rightarrow \lambda_n = \frac{n\pi}{L}$, $n = 1, 2, \dots$

Thus, $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$

$$A_n = \frac{\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\underbrace{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx}_{L/2}}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx.$$

$f(x)$ is any function

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→ If $f(x)$ is odd ⇒ $[-L, 0]$ $\xrightarrow{\text{Converge to}}$ $-f(-x)$

for all odd functions, expansion of x will work for $[-L, L]$

Fourier cosine series

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \qquad \phi_n(x) = \cos \lambda_n x$$

$$\lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2 \ldots$$

$$\frac{dy(0)}{dx} = 0, \quad \frac{dy(L)}{dy} = 0$$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x \qquad \text{Period} = 2L$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \qquad \text{even function}$$

Norm for $n = 0$ is L $\qquad N_m = \int_0^L \cos^2 \frac{n\pi}{L} x \, dx$

for $n = 1$ to ∞ → $L/2$

$$A_0 = \frac{1}{L} \int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots$$

for even function → expansion works for $[-L, L]$

Fourier Bessel series → Heat transfer in cylindrical coordinates

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - \nu^2) R = 0 \longrightarrow \text{SL problem (eigen value problem)}$$

$$\alpha_1 R(0) + \beta_1 \frac{dR(0)}{dr} = 0, \quad \alpha_1^2 + \beta_1^2 > 0$$

$$\alpha_2 (R(b)) + \beta_2 \frac{dR(b)}{dr} = 0, \quad \alpha_2^2 + \beta_2^2 > 0$$

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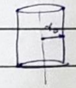
→ $P(r) = r$, $w(r) = r$, $q(r) = -\frac{\nu^2}{r}$ → special case of SL

solid cylinder  domain :- $[0, r_0]$, $R(0)$ = finite

BCL ② $\alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0$

Soln :- $R(r) = C_1 J_\nu(\lambda r) + C_2 Y_\nu(\lambda r)$

$r = 0$, $R(0)$ = finite → $\boxed{R(r) = C_1 J_\nu(\lambda r)}$ $\quad C_2 = 0$

BCL- $\alpha C_1 J_\nu(\lambda r_0) + \beta C_1 \frac{d}{dx} J_\nu(\lambda r_0) = 0$

$$\alpha J_\nu(\lambda r) + \frac{d}{dx} J_\nu(\lambda r_0) = 0$$

eigen function :- $J_\nu(\lambda r)$

eigen value :- λ_n $(n = 1, 2, 3 \ldots)$

⇒ Any arbitrary function $f(r)$ in $[0, r_0]$

$$f(r) = \sum_{n=1}^{\infty} A_n J_\nu(\lambda r)$$

$$A_n = \frac{\int_0^{r_0} f(r) J_\nu(\lambda_n r)\, r\, dr}{N_n} \qquad N_n = \int_0^{r_0} J_\nu^2(\lambda_n r)\, r\, dr$$

Separation of variables :- $\frac{1}{\alpha} \frac{\partial T(r,t)}{\partial t} = \nabla^2 T(r,t)$

Homogeneous, istropic, solid, stationary, opaque body, without heat generation, (const ρ, C_p k)

$$T(r, t=0) = f(r) \text{ in } R$$

$$k \frac{\partial T}{\partial r}(r,t) + hT(r,t) = 0 \text{ on boundary}$$

~~$T(r,t) = \psi(r)\Gamma(t)$~~ $T(r,t) = \psi(r)\,\Gamma(t)$

$$\frac{1}{\alpha} \psi(r) \frac{\partial \Gamma(t)}{\partial t} = \Gamma(t) \nabla^2 \psi(r)$$

$$\frac{1}{\alpha}\frac{1}{\Gamma(t)}\frac{d\Gamma(t)}{dt} = \frac{1}{\psi(r)}\nabla^2\psi(r) = \text{const } (-\lambda^2)$$

$$\Gamma(t) = c_1 e^{-\alpha\lambda^2 t}$$

helmholtz equation
$$\begin{cases} \nabla^2\psi(r) + \lambda^2\psi(r) = 0 \\ k\frac{\partial}{\partial n}\psi(r) + h\psi(r) = 0 \quad \text{(boundary condition)} \end{cases}$$

General soln :- $T(r,t) = \sum C_m e^{-\alpha\lambda_m^2 t}\psi_m(r)$ ①

for $C_m \Rightarrow IC = f(r) = \sum C_m \left. e^{-\alpha\lambda_m^2 t}\right|_{t=0} \psi_m(r)$

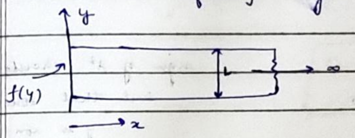
$$C_m = \frac{\int_R f(r)\,\psi_m(r)\,dr}{\int_R \psi_m^2(r)\,dr}$$

→ separation of variable is applicable if
① DE is linear and homogeneous
② BC's are linear, 3 of them are homogeneous
(2-D) (4)

<u>Separation of variables</u>

<u>Examples</u>

① Consider an infinitely long 2D fin of thickness L



Ambient = T_∞

heat transfer coeff = h = very large

Find steady state $T(x,y)$

k = constant $\qquad \dot{q} = 0$.

$$\cancel{-kA_c(x)\left.\frac{dT}{dx}\right|_x + kA_c(x)\left.\frac{dT}{dx}\right|_{x+\Delta x} - kA_y\left.\frac{dT}{dy}\right|_y + kA_y\left.\frac{dT}{dy}\right|_{y+\Delta y} - hA_s(T-T_\infty) = 0}$$

$$\cancel{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

$$\cancel{x=0 \rightarrow T(0,y) = f(y)}$$

$$\cancel{x\to\infty, \quad T = T_\infty \quad \boxed{T(\infty,y) = T_\infty}}$$

$$\cancel{y=0 \quad -k\frac{\partial T}{\partial y}\Big|_{y=0} = h(T-T_\infty)}$$

$$\cancel{y=L \quad -k\frac{\partial T}{\partial y}\Big|_{y=L} = h(T-T_\infty)}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

~~x=0~~ $T(0,y) = f(y)$

$T(\infty,y) = T_\infty$

becoz h is very large
$$\begin{cases} T(x,0) = T_\infty \\ T(x,L) = T_\infty \end{cases}$$

$\theta(x,y) = T(x,y) - T_\infty$

$$\boxed{\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0} \rightarrow \text{homogeneous}$$

separation of variables can be used if atleast 3 BC are homogeneous & DE is linear and homogeneous

$\theta(0,y) = f(y) - T_\infty = F(y)$

$\theta(\infty,y) = 0$

$\theta(x,0) = 0$

$\theta(x,L) = 0$.

$\theta(x,y) = X(x)Y(y)$

$Y\dfrac{d^2X}{dx^2} + X\dfrac{d^2Y}{dy^2} = 0$

$\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \text{const} = \pm\lambda^2$

sign of λ^2 should be such that soln of y is non-trivial (eigen value problem in homogeneous soln)

$y \to$ homogeneous soln (S-L problem)

$\dfrac{d^2Y}{dy^2} + Y\lambda^2 = 0.$

$\boxed{Y = C_1 \sin(\lambda y) + C_2 \cos(\lambda y)}$

$Y(x,0) = 0$

$Y(x,L) = 0$

$\cancel{0 = C_2}$ $\boxed{0 = C_2}$

$0 = C_1 \sin\lambda L$

$\lambda L = n\pi$

$\lambda_n = \dfrac{n\pi}{L}$ $n = 1, 2, \ldots$

$Y = A_n \sin\lambda_n y$

$\lambda_n = \dfrac{n\pi}{L}, \; n = 1, 2, \ldots$

$\rightarrow \dfrac{d^2X}{dx^2} - \lambda^2 X = 0$

$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

$X(0,y) = F(y)$

$X(\infty,y) = 0$

$\boxed{C_1 = 0}$

$\boxed{F(y) = C_2}$

$X = B_n e^{-\lambda_n x}$

$\theta(x,y) = B_n e^{-\lambda_n x} \times A_n \sin\lambda_n y.$

$\theta(x,y) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n x} \sin\lambda_n y$

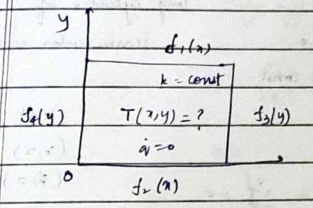
$\theta(0,y) = F(y)$

$F(y) = \sum_{n=0}^{\infty} C_n \sin\lambda_n y.$

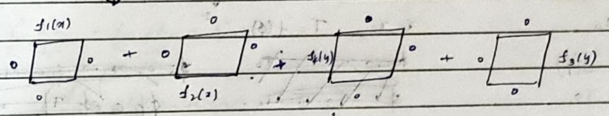
$\int_0^L F(y) \sin\lambda_n y \, dy = C_n \int_0^L \sin^2\lambda_n y \, dy.$

$\boxed{C_n = \dfrac{2}{L} \int_0^L F(y) \sin\lambda_n y \, dy}$

Q2)

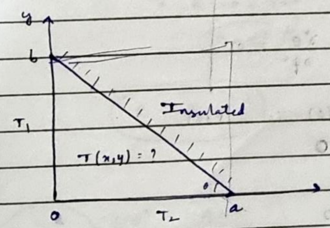


$\rightarrow$ split & take one NH at a time



$\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$

Q)



$\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$

$T(x,0) = T_2$

$T(0,y) = T_1$

k = const

$\dot{q} = 0.$

$\dfrac{x}{a} + \dfrac{y}{b} = 0 \rightarrow \dfrac{\partial T}{\partial x} = 0, \; \dfrac{\partial T}{\partial y} = 0.$

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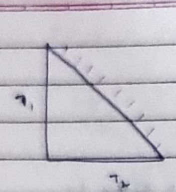
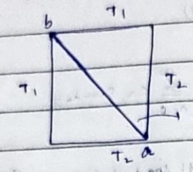


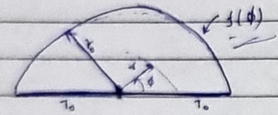
$=$



Thermal symmetry acts as insulated boundary.

thermal symmetry (plane of symmetry)

Q)



$\dot{q} = 0$

$k = \text{const}$

long cylinder of semicircular cross section

(r, ϕ)

(r, z)

(ϕ, z) ✗.

steady state

$T(r, \phi) = ?$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \cancel{\frac{\partial^2 T}{\partial z^2}} = 0$$

$T(0, \phi) = T_0$

$T(r_0, \phi) = f(\phi)$

$T(r, 0) = T_0$

$T(r, \pi) = T_0$

$r = r_0, \quad T = f(\phi)$

$r = 0, \quad T = T_0$

~~periodic BC.~~

$\theta = T - T_0$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

$\theta(r_0, \phi) = f(\phi) - T_0 = F(\phi)$

$\theta(0, \phi) = 0$

$\theta(r, 0) = 0$

$\theta(r, \pi) = 0$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

$\theta(r, \phi) = R(r)\, \psi(\phi)$

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$$\psi \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = \pm\lambda^2$$

$+\lambda^2$

$$-\frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = \lambda^2$$

$$\frac{\partial^2 \psi}{\partial \phi^2} + \lambda^2 \psi = 0$$

$\psi_n = A_n \sin \lambda_n \phi$

$\lambda_n = \frac{n\pi}{\pi}$

$\lambda_n = n$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - \lambda^2 = 0.$$

cauchy euler eqn.

$r^{+\lambda}, r^{-\lambda} \to$ linear combination

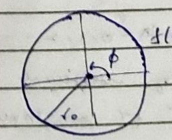
$R = A_1 r^{\lambda} + A_2 r^{-\lambda}$

$R = 0, \; r = 0.$

$A_2 = 0$

$R = A_1 r^{\lambda}$

Q)



$f(\phi)$

$\dot{q} = 0$

$k = \text{const}$

steady state $T(x, y) = ?$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

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- $T(r_0, \phi) = f(\phi)$
  $T(r, 0) = T(r, 2\pi)$
  $\left.\frac{\partial T}{\partial \phi}\right|_{\phi=0} = \left.\frac{\partial T}{\partial \phi}\right|_{\phi=2\pi}$
- $T(0, \phi) = \text{finite}$

$\rightarrow$ $T(r, \phi) = T(r, \phi + 2\pi)$

$\frac{1}{r} \frac{\partial T(r, \phi)}{\partial \phi} = \frac{1}{r} \frac{\partial T(r, \phi + 2\pi)}{\partial \phi}$ } periodic BC

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda^2 R = 0 \qquad \Bigg| \qquad \frac{d^2 \varphi}{d\phi^2} + \lambda^2 \varphi = 0$$

$$R = B_1 r^{\lambda} + B_2 r^{-\lambda}$$

$$\varphi = A_1 \sin\lambda\phi + A_2 \cos\lambda\phi$$

$$A_1 \sin\lambda\phi + A_2 \cos\lambda\phi = A_1 \left(\sin(\lambda(\phi + 2\pi)\right) + A_2 \cos(\lambda(\phi + 2\pi))$$

$$\lambda A_1 \cos\lambda\phi - A_2 \lambda \sin\lambda\phi = \lambda A_1 \cos(\lambda(\phi + 2\pi)) - A_2 \lambda (\sin(\lambda(\phi + 2\pi))$$

$$A_1 \left(\lambda \cos\lambda\phi - \lambda \cos(\lambda(\phi + 2\pi))\right) + A_2 \left(\lambda \sin\lambda(\phi + 2\pi) - \lambda \sin\lambda\phi\right)$$

$$A_1 \left(\sin\lambda\phi - \sin\lambda(\phi + 2\pi)\right) + A_2 \left(\cos\lambda\phi - \cos(\lambda(\phi + 2\pi)\right) = 0.$$

$\lambda = n, \; n = 0, 1, 2 \ldots$

determinante of coefficients should vanish for non-trivial soln.

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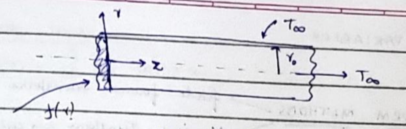
Q)



semiinfinite cylinder of radius r_0. h = large

$\dot{q} = 0$

$k = \text{const}$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)\right) + \frac{\partial^2 T}{\partial z^2} = 0$$

$T(0, z) = T_\infty$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right)\right) + \frac{\partial^2 \theta}{\partial z^2} = 0.$$

$\theta(0, z) = \text{finite}$ — $\theta(0, z) \to \frac{\partial \theta}{\partial r} = 0$ — can be written

$\theta(r_0, z) = 0$

$\theta(r, 0) = F(r)$

$\theta(r, \infty) = 0$.

$$\theta = R(r)\, Z(z)$$

$$\frac{d^2 Z}{dz^2} - \lambda^2 Z = 0.$$

$Z(\infty) = 0$

$Z = B_1 e^{-\lambda z}$.

$\frac{dR(0)}{dr} = 0 \qquad \Big| \qquad R(0) = \text{finite}$

$R(r_0) = 0$

$R = A_1 J_0(\lambda_n r)$

where λ_n as the roots $J_0(\lambda_n r_0) = 0$.

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→ SEPARATION OF VARIABLES

→ INTEGRAL TRANSFORM METHODS → finite fourier transform → rectangular coordinate
→ Hankel transform → cylindrical coordinate
→ laplace transform

(a) laplace transform

(1) 2D PDE $\xrightarrow{FFT}$ ODE $T(x)$ → solve → take inverse → $T(x,y)$

FINITE FOURIER TRANSFORM

Any $f(x)$ which is piecewise differentiable in $[0, L]$ can be expanded in sine series.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx}$$

$$\Rightarrow f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x')\sin\left(\frac{n\pi}{L}x'\right)dx'\right]\sin\left(\frac{n\pi}{L}\right)x$$

rearrange. let $K_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$. → Kernel of transformation → normalized eigen value → $\frac{\text{eigen function}}{\sqrt{N_m}}$ → norm of eigen function

$$\bar{f}_n = \int_0^L f(x)K_n(x)\,dx$$ → finite fourier transform of $f(x)$

~~$K_n(x) = \sum_{n=1}$~~

$$f(x) = \sum_{n=1}^{\infty} K_n(x)\bar{f}_n$$ ⟹ Inversion formula

for $\sin\frac{n\pi}{L}x$ it is $L/2$ Norm

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Cosine series

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$$

$$K_n(x) = \frac{1}{\sqrt{N_m}}\cos\frac{n\pi}{L}x$$

$N_m = L \quad n=0$

$N_m = L/2 \quad n = 1, 2, \ldots$

→ STL

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$\alpha_1 y(0) + \beta_1\frac{dy(0)}{dx} = 0, \quad \alpha_1^2 + \beta_1^2 > 0$$

$$\alpha_2 y(L) + \beta_2\frac{dy(L)}{dx} = 0, \quad \alpha_2^2 + \beta_2^2 > 0$$

$\phi_n(x)$ → eigen function

$$f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x)$$

$$A_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\underbrace{\int_0^L \phi_n^2(x)\,dx}_{N_m}} = \frac{1}{N_m}\int_0^L f(x)\phi_n(x)\,dx$$

$$K_n(x) = \frac{1}{\sqrt{N_m}}\phi_n(x)$$

$$\bar{f}_n = \int_0^L f(x)\,K_n(x)\,dx$$

$$f(x) = \sum_{n=0}^{\infty}\bar{f}_n K_n(x)$$

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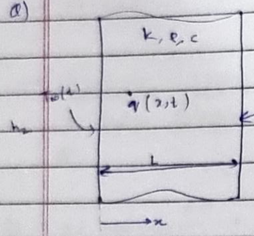

At time $t=0$, $T(x,t) = T_i(x)$

1D unsteady state heat conduction with space, time dependent $\dot{q}$ and environment temperature

[Consider h_1, h_2 high]

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{\dot{q}(x,t)}{k}$$

(Cannot solve using separation of variables)

$T(x,0) = T_i(x)$

$T(0,t) = T_\infty(t)$

$T(L,t) = T_\infty(t)$

Operator is linear $L_1(t)$. you can apply FFT

define transformation of $T(x,t)$

$$\bar{T}_n(t) = \int_0^L T(x,t)\, k_n(x)\, dx$$

Inversion $T(x,t) = \sum_{n=1}^{\infty} \bar{T}_n(t)\, k_n(x)$ Kernel → normalized eigen value function

[how to get kernel]??

convert it into STL — linear homogeneous DE

take homogeneous DE, homogeneous BC

Remove non-homogeneity from PDE & BC

$$\frac{1}{\alpha}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

$\psi(x,0) = T_i(x)$

$\psi(0,t) = 0$

$\psi(L,t) = 0$

get eigen function from this & then get kernel from this

$\psi(x,t) = X(x)\,\Gamma(t)$

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$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$X(0) = 0$

$X(L) = 0$

$X(x) = C_1 \sin\lambda x + C_2 \cos\lambda x$

$C_2 = 0$

$\sin(\lambda L) = 0$

$\lambda L = n\pi$

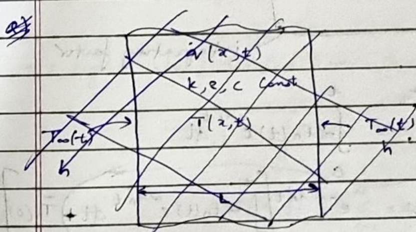
$\lambda = \frac{n\pi}{L}$

~~let, $k_n(x) = \sqrt{\frac{2}{L}} \int_0^L \sin\frac{n\pi x}{L}$~~

let, $k_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$

$k_n'(x) = \sqrt{\frac{2}{L}} \times \frac{n\pi}{L} \cos\frac{n\pi}{L}x$

$k_n''(x) = -\sqrt{\frac{2}{L}} \frac{n^2\pi^2}{L^2} \sin\frac{n\pi x}{L}$



$$\Rightarrow \int_0^L \frac{1}{\alpha} k_n(x) \frac{\partial T(x,t)}{\partial t}\, dx = \int_0^L k_n(x) \frac{\partial^2 T}{\partial x^2}\, dx + \int_0^L k_n(x) \frac{\dot{q}(x,t)}{k}\, dx$$

$\frac{1}{k}\bar{\dot{q}}_n(t)$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^L k_n(x)\, T(x,t)\, dx$$

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt}$$

$$\left[k_n(x)\frac{\partial T}{\partial x}\right]_0^L - \int k_n'(x)\frac{\partial T}{\partial x}\, dx$$

0.

$$\left[k_n'(x)\, T(x,t)\right]_0^L - \int_0^L k_n''(x)\, T(x,t)\, dx$$

$$T_\infty(t)\left[\sqrt{\frac{2}{L}}\frac{n\pi}{L} - \sqrt{\frac{2}{L}}\frac{n\pi}{L}\cos n\pi\right] - \int_0^L k_n''(x)\, T(x,t)\, dx$$

$k_n(0) = 0$

$k_n(L) = 0$

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$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\left[\sqrt{\frac{2}{L}}\underbrace{\frac{n\pi}{L}}_{\lambda_n} - \sqrt{\frac{2}{L}}\frac{n\pi}{L}\cos n\pi\right] - \int_0^L \sqrt{\frac{2}{L}}\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}\,T(x,t)\,dx + \frac{\dot{q}_n(t)}{k}$$

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\left[\sqrt{\frac{2}{L}}\lambda_n - \sqrt{\frac{2}{L}}\lambda_n\cos n\pi\right] - \frac{n^2\pi^2}{L^2}\bar{T}_n(t) + \frac{\bar{\dot{q}}_n(t)}{k}$$

$$\Rightarrow \frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\lambda_n\sqrt{\frac{2}{L}}\left[1-(-1)^n\right] - \lambda_n^2\bar{T}_n(t) + \frac{\bar{\dot{q}}_n(t)}{k}$$

$$\frac{d\bar{T}_n(t)}{dt} + \alpha\lambda_n^2\bar{T}_n(t) = \underbrace{\alpha\left[T_\infty(t)\lambda_n\sqrt{\frac{2}{L}}\left[1-(-1)^n\right] + \frac{\bar{\dot{q}}_n(t)}{k}\right]}_{F_n(t)}$$

$$\boxed{\frac{d\bar{T}_n(t)}{dt} + \alpha\lambda_n^2\bar{T}_n(t) = \alpha F_n(t)}$$ → linear ODE

→ integrating factor

$$\bar{T}_n(t)\,e^{\alpha\lambda_n^2 t} = \int_0^t \alpha F_n(t)\,e^{\alpha\lambda_n^2 t}\,dt$$

$$\bar{T}_n(t) = e^{-\alpha\lambda_n^2 t}\left[\int_0^t \alpha F_n(t)\,e^{\alpha\lambda_n^2 t}\,dt + \bar{T}_n(0)\right]$$

$$\bar{T}_n(0) = \int_0^L T(x,0)\,k_n(x)\,dx = \int_0^L T_i(x)\,k_n(x)\,dx$$

- Invert $\bar{T}_n(t)$ to get $T(x,t)$

$$\boxed{T(x,t) = \sum_{n=1}^{\infty} k_n(x)\,\bar{T}_n(t)}$$

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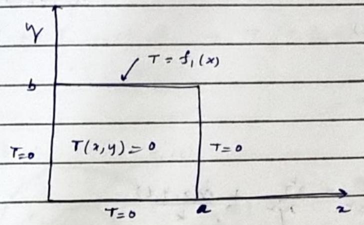
2D steady state



$\dot{q} = 0$

const ρ, c, k

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$x = 0, \quad T(0,y) = 0$

$x = a, \quad T(a,y) = 0$

$y = 0, \quad T(x,0) = 0$

$y = b, \quad T(x,b) = f_1(x)$

→ eigen value problem in x-domain

$$\bar{T}_n(y) = \int_0^a k_n(x)\,T(x,y)\,dx$$

$$T(x,y) = \sum_{n=1}^{\infty} k_n(x)\,\bar{T}_n(y)$$

$\psi = X(x)\,Y(y)$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{X}\frac{d^2X}{dx^2} = \lambda_n^2$$

$\lambda_n = \frac{n\pi}{L}$,

$$\boxed{k_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}}$$

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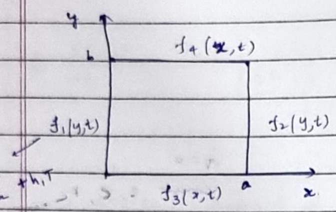
Q)



$f_1, f_2, f_3, f_4 \to$ like convection bc

$$\frac{1}{\alpha}\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}(x,y,t)}{k}$$

FFT-1 $\to$ $T(x,y,t) \to \bar{T}_n(y,t)$ — } $\to$ Invert twice

FFT-2 $\to$ $\bar{T}_n(y,t) \Rightarrow \bar{\bar{T}}_n(t)$

① Rectangular coordinate

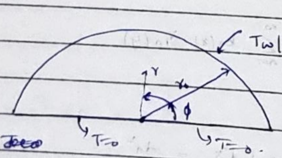
② cylindrical coordinate $\to$ hankel transformation

[kernel is bessel function]

③ ~~1D~~ $\to$ spherical coordinate system $\to$ legendre polynomial

1D $\to T(r,t)$ $\boxed{\theta(r,t) = rT(r,t)}$

Q)



steady state

$0 \le r \le r_0$

$0 \le \phi \le \pi$

$0 \le z < \infty$

$\dot{q}(r,\phi)$

$T(r,\phi,z)$ k, ρ, c const

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}(r,\phi)}{k}$$

$r = 0$, $T(0,\phi,z) = 0$

$r = r_0$, $T(r_0,\phi,z) = T_w(\phi)$

$\phi = 0$, $T(r,0,z) = 0$

$\phi = \pi$, $T(r,\pi,z) = 0$

$T(r,\phi,0) = 0$

$T(r,\phi,\infty) = \infty$

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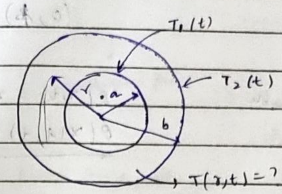
$\to \phi \to [0,\pi]$ $\quad \bar{T}_n(r,z) = \int_0^\pi k_n(\phi)\, T(r,\phi,z)\, d\phi$, $\quad k_n(\phi) \to \sqrt{\frac{2}{\pi}}\sin n\phi$

① FFT wrt ϕ $\Rightarrow$ $T(r,\phi,z) \to \bar{T}_n(r,\phi) \xrightarrow{\text{Hankel}} \bar{\bar{T}}_n(r)$ $\Rightarrow$ solve ODE

$\Downarrow$

Invert $\bar{T}_n(r)$

Q)



At $t = 0$

$T(r,0) = T_i(r)$

1D spherical shell + unsteady state | $k, \rho, c \to$ const, $\dot{q} = 0$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}$$

$T(a,t) = T_1(t)$

$T(b,t) = T_2(t)$

$T(r,0) = T_i(r)$

$\theta(r,t) = rT(r,t)$

$$\frac{\partial \theta}{\partial r} = r\frac{\partial T}{\partial r} + T$$

$$\frac{\partial^2 \theta}{\partial r^2} = r\frac{\partial^2 T}{\partial r^2} + 2\frac{\partial T}{\partial r}$$

$$\frac{\partial \theta}{\partial t} = r\frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{1}{\alpha}\frac{1}{r}\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial^2 \theta}{\partial r^2}$$

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2}$$

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2}$$

$\theta(r,0) = r\,T_i(r)$ ✓

$\theta(a,t) = a\,T_1(t)$

$\theta(b,t) = b\,(T_2(t))$

apply FFT
Convert the domain to $(0, b)$

~~$r = x + a$~~

Solution: change of variable

$r = x + a$.

$\theta(r,t) = r\,T(r,t)$

$\theta(x+a,t) = (x+a)\,T(r,t)$

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$$

$\theta(x,0) = (x+a)\,T_i(x+a)$

~~$\theta(0,t)$~~

~~$\theta(b+a)$~~.

$\theta(0,t) = a\,T_1(t)$

$\theta(b-a,t) = b\,T_2(t)$

$\left(x \to [0, b-a] \right.$
$\left. \hookrightarrow \text{apply FFT} \right)$ ✓

$$\bar{\theta}_n(\lambda) = \int_0^{b-a} k_n(x)\,\theta(x,t)\,dx$$

$$\theta(x,t) = \sum_{n=1}^{\infty} k_n(x)\,\bar{\theta}_n(t)$$

$$\frac{d^2\psi}{dx^2} + \psi\lambda^2 = 0$$

$\psi = c_1 \sin\lambda x + c_2 \cos\lambda x$

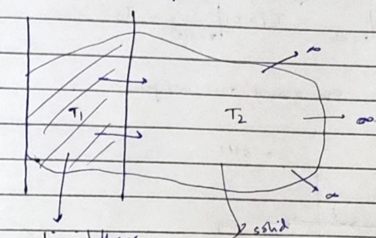
$\sin\lambda(b-a) = 0$

$\lambda_n(b-a) = n\pi$

$$\lambda_n = \frac{n\pi}{b-a}$$

$$k_n(x) = \sqrt{\frac{2}{b-a}}\left[\sin\frac{n\pi}{b-a}x\right]$$

→ solid with semi infinite width



Start heating from one side
↓
solid will melt.

(Boundary is not fixed.)

phase change problem where interface moves with time.

<u>Heat conduction</u> <u>with phase change</u>

location of interface is not known

→ Moving boundary problems.
(interface)

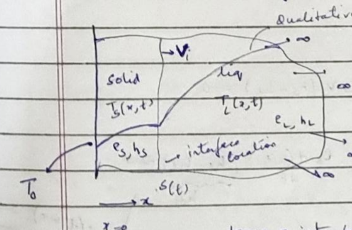
unsteady

<u>Solidification of liquid</u> — (steady state is not possible)

Consider a semi-infinite pool of liquid (1D)

• fusion temp T_f

Initially liquid is at temp $> T_f$



At time $t=0$, bring the surface $x=0$ to $T_0 < T_f$ and maintain it

(liq will start solidifying at $x=0$)

temp of interface $= T_f$

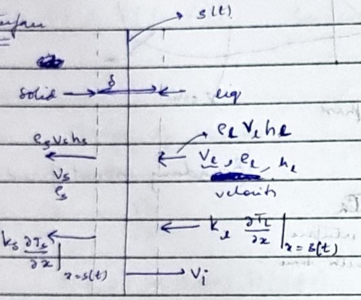
$h \to$ specific enthalpy

Continuity of temp at interface

$$T_S(x,t)\Big|_{x=s(t)} = T_L(x,t)\Big|_{x=s(t)} = T_f$$

→ write energy balance eqn around the interface

location of interface



relative velocity

$$V_i = \frac{ds(t)}{dt} = V_s$$

Mass balance, $\rho_L V_L = \rho_s V_s$ (No accumulation)
$= \rho_s V_i$

$$\rho h_L V_L + k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} = \rho_s V_s h_s + k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)}$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s V_s h_s - \rho_L h_L V_L = 0$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s V_s (h_s - h_L) = 0.$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s \frac{d(s(t))}{dt} \underbrace{(h_s - h_L)}_{h_{SL}} = 0$$

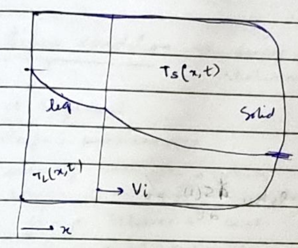
→ Stefan condition
This bc must be satisfied at the interface.

$s(t=0) = 0$.

① if convection dominates at the liq phase

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} \rightarrow h(T_\infty - T_L)$$

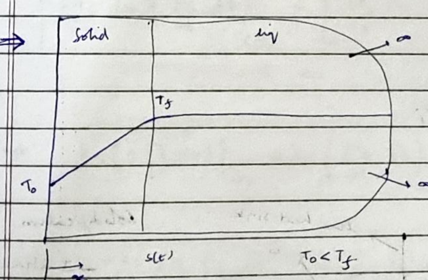
Melting of solid



→ derive & check

find the stefan condition
$\rho_s \rightarrow \rho_L$

⇒



→ Solidification of liq
Initially the liq pool was at fusion temp T_f
→ Single phase problem with phase change

→ Need not solve
$$\frac{1}{\alpha}\frac{\partial T_L}{\partial x} = \frac{\partial^2 T_L}{\partial x^2}$$
→ bc ICs
$T_L(x,t) = T_f$

$$\frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$$

$\cancel{\frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}}$ $T_s(0,t) = T_0$

?

$T_s(x=s(t),t) = T_f$

$$\cancel{k_L \frac{\partial T_L}{\partial x}\Big|_{x=s(t)}} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s h_{SL} \frac{ds(t)}{dt} = 0 \rightarrow s(t=0) = 0$$

0

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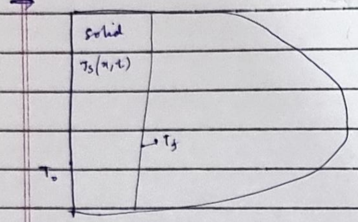


~~Initially the liq~~

$\left.\frac{\partial T}{\partial x}\right|_{s} \to d(s)$

$\rightarrow \frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$

$T_s(0,t) = T_0$

$T_s(x=s(t), t) = T_f$

$k_L \left.\frac{\partial T_L}{\partial x}\right|_{x=s(t)} - k_s \left.\frac{\partial T_s}{\partial x}\right|_{s(t)} + \rho_s h_{sL}\frac{ds(t)}{dt} = 0$

$s(t=0) = 0.$

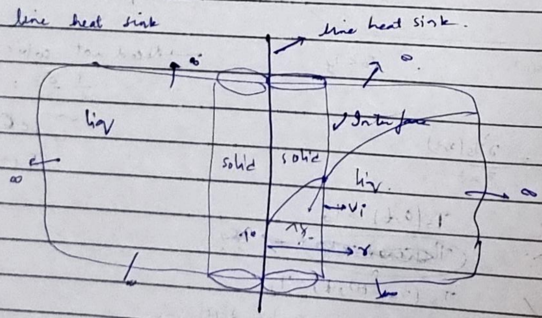
$\rightarrow \frac{1}{\alpha_L}\frac{\partial T_L(x,t)}{\partial t} = \frac{\partial^2 T_L(x,t)}{\partial x^2}$

$T_L(x,0) = T_i$

$T_L(x=s(t), t) = T_f$

$T_L(x=\infty, t) = T_i$

Q) line heat sink



Solidification

→ Initially i have a liquid pool.

$T_0 < T_f$

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$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_L}{\partial r}\right) = \frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}$

$\rightarrow r=0, \; -k_s\frac{\partial T}{\partial r} = q_{line}$

→ stefan condition → linear ? nonlinear? ⇒ It is non linear $\left\{ k_L, k_s, \rho_s, h_{sL}, \rho_L \right.$ ↳ const

h_s, h_L

it says that interface ~~between solid and liq~~ moves with a velocity which is the difference in the jump between the heat flux of solid & liq phase.

$T_s(x,t)\Big|_{s(t)} = T_L(x,t)\Big|_{s(t)} = T_f$

$\left[\frac{\partial T_s}{\partial x}dx + \frac{\partial T_s}{\partial t}dt\right]_{s(t)} = \left[\frac{\partial T_L}{\partial x}dx + \frac{\partial T_L}{\partial t}dt\right]_{s(t)} = 0.$

$\frac{ds}{dt} = \frac{dx}{dt}\Big|_{x=s(t)}$

$\Rightarrow k_L\left(-\frac{\partial T_L}{\partial t}dt\right)\Big|_{s(t)} + k_s\left(\frac{\partial T_s}{\partial t}dt\right) + \rho_s h_{sL}\frac{ds(t)}{dt}dx = 0.$

$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\frac{ds}{dt}\left(\frac{ds(t)}{dt}\right) = 0$

$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\left(\frac{ds}{dt}\right)^2 = 0.$

non-linear

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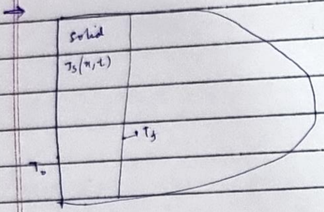


~~Initially the liq~~

$\left.\frac{\partial T}{\partial x}\right|_S \rightarrow d(S)$

$$\frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$$

$T_s(0,t) = T_0$

$T_s(x=s(t), t) = T_f$

$$k_l \left.\frac{\partial T_l}{\partial x}\right|_{x=s(t)} - k_s \left.\frac{\partial T_s}{\partial x}\right|_{s(t)} + \rho_s h_{sl}\frac{ds(t)}{dt} = 0$$

$s(t=0) = 0.$

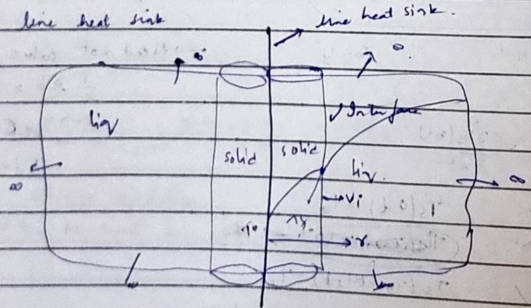
$$\frac{1}{\alpha_l}\frac{\partial T_L(x,t)}{\partial t} = \frac{\partial^2 T_L(x,t)}{\partial x^2}$$

$T_L(x,0) = T_i$

$T_L(x=s(t), t) = T_f$

$T_L(x\to\infty, t) = T_i$

Q) line heat sink



Solidification

→ Initially i have a ~~bio~~ liquid pool.

$T_0 < T_f$

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$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_L}{\partial r}\right) = \frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}$$

→ $r=0$, $-k_s\frac{\partial T}{\partial r} = q_{line}$

⟹ Stefan condition → linear? nonlinear? ⟹ It is non linear

$k_l, k_s, \rho_s, h_{sl}, \rho_l$ → const

h_s, h_l

it says that interface ~~between solid and liq~~ moves with a velocity which is the difference in the jump between the heat flux of solid & liq phase.

$$T_s(x,t)\Big|_{s(t)} = T_L(x,t)\Big|_{s(t)} = T_f$$

$$\left[\frac{\partial T_s}{\partial x}dx + \frac{\partial T_s}{\partial t}dt\right]_{s(t)} = \left[\frac{\partial T_L}{\partial x}dx + \frac{\partial T_L}{\partial t}dt\right]_{s(t)} = 0.$$

$$\frac{ds}{dt} = \left.\frac{dx}{dt}\right|_{x=s(t)}$$

$$\Rightarrow k_L\left(-\frac{\partial T_L}{\partial t}dt\right)\Big|_{s(t)} + k_s\left(\frac{\partial T_s}{\partial t}dt\right) + \rho_s h_{sL}\frac{ds(t)}{dt}dx = 0.$$

$$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\frac{ds}{dt}\left(\frac{ds(t)}{dt}\right) = 0$$

$$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\left(\frac{ds}{dt}\right)^2 = 0.$$

non-linear

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Non-dimensional form

$\theta_s = \dfrac{T_s - T_f}{T_f - T_0}$ $\quad \theta_L = \dfrac{k_L}{k_s}\dfrac{T_L - T_f}{T_f - T_0}$.

& $\xi = \dfrac{x}{L}$, $\tau = \dfrac{Ste\, \alpha_s t}{L^2}$

Ste = stefan number = $\dfrac{c_s (T_f - T_0)}{h_{sL}}$ ← sensible heat; h_{sL} is latent; c_s → specific heat capacity of solid; for solid

if Ste is very small → we can assume quasi steady state approx

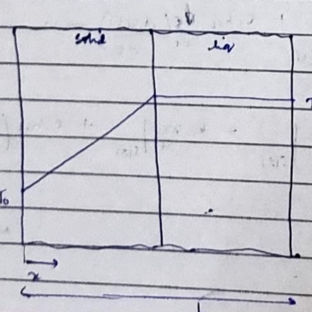
quasi steady state approx
↳ Small stefan number ($Ste \leq 0.1$)

Case 1: c_s is small
thermal diffusivity α_s is very large → means heat propogates with ∞ speed
↳ the moment interface moves SS is achieved

Case 2: latent heat is very high → $\dfrac{ds(t)}{dt} \approx 0$.
↳ Interface is sort of stationary.

∴ $\dfrac{\partial T_s}{\partial t} = 0, \; \dfrac{\partial T_L}{\partial t} = 0$

Q)



1D problem in finite region

quasi SS

$\dfrac{\partial^2 T_s}{\partial x^2} = 0$ $\quad \cancel{\dfrac{\partial^2 T_L}{\partial x^2} = 0}$ $\quad T_L(x,t) = T_f$

$T_L(x,t) = T_f$

$T(x=0,t) = T_0$

$T(x = s(t), t) = T_f$

$-k_s \dfrac{\partial T_s}{\partial x}\Big|_{s(t)} + e_s h_{sL} \dfrac{ds(t)}{dt} = 0$.

$T_L(s(t), t) = T_f$

$T_L(L, t) = T_f$

How much time is needed to solidify the entire liq? — $s(t) = L$ → find time

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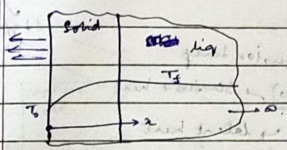
$T_s(x,t) = ax + b$.

→ stefan exact soln
Pennes Bio-heat eqn
Solutions with Laplace transform
Duhamel Principle
Inverse heat transfer.

STEFAN EXACT SOLUTION



solid phase: $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

$T(0,t) = T_0$, $T(x,t)\big|_{x = s(t)} = T_f$

$-k \dfrac{\partial T}{\partial x}\Big|_{x = s(t)} + \rho h_{sL} \dfrac{ds}{dt} = 0$ $\quad s(0) = 0$.

→ This problem accepts a similarity soln in terms of

$\eta = \dfrac{x}{2\sqrt{\alpha t}}$ ← similarity parameter

$\theta(\eta) = T(\eta) - T_0$

$\dfrac{d^2\theta}{d\eta^2} + 2\eta \dfrac{d\theta}{d\eta} = 0$ ⟶ $\theta(\eta) = A + B\,\mathrm{erf}(\eta)$

$\theta(0) = 0$, $\theta(\lambda) = T_f - T_0$, $\lambda = \dfrac{s(t)}{2\sqrt{\alpha t}}$

$A = 0$, $B = \dfrac{T_f - T_0}{\mathrm{erf}(\lambda)}$

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B = const

∴ parameter λ must ~~be~~ also be a const, independent of time.

$$\frac{\theta(\eta)}{T_f - T_0} = \frac{erf(\eta)}{erf(\lambda)}$$

$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{erf\left(\frac{x}{2\sqrt{\alpha t}}\right)}{erf(\lambda)} \qquad \lambda = ??$$

$$-k\frac{\partial T}{\partial x}\Big|_{x=s(t)} + \rho h_{sl}\frac{ds}{dt} = 0 \rightarrow \lambda e^{\lambda^2} erf(\lambda) = \frac{1}{\sqrt{\pi}} \, Ste$$

↳ use stefan BC to find λ.

Ste → stefan number

$$Ste = \frac{c(T_f - T_0)}{h_{sl}}$$

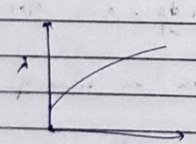
T_f → fusion temp

$c(T_f - T_0)$ → sensible heat

h_{sl} → latent heat

- Ste << 1, solidification process is very rapid
- Ste >> 1 → solidification is slower

$$\lambda e^{\lambda^2} erf(\lambda) = \frac{1}{\sqrt{\pi}} \, Ste$$ → non-linear trancendental algebraic equation and cannot be solved explicitly.



When Ste is small, λ is also small

if λ is known.

$$\lambda = \frac{s(t)}{2\sqrt{\alpha t}} \qquad s(t) = \lambda 2\sqrt{\alpha t}$$

$$q_0'' = k\left(\frac{\partial T}{\partial x}\right)\Big|_{x=0} = \frac{k(T_f - T_0)\, t^{-1/2}}{\sqrt{\pi\alpha}\, erf(\lambda)}$$

decreases with time

(with time solid thickness ↑ ∴ resistance ↑)

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Pennes Bio-heat equation

Heat exchange through the skin

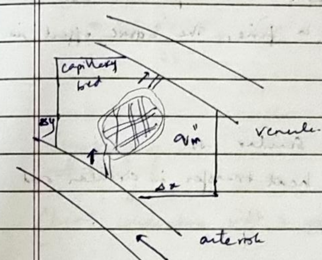
additional source in energy eqn: flow of blood in one heat balance eqn

→ perfusion

→ complex nature of heat transfer in living tissue precludes exact mathematical model

→ Assumptions

1. Equilibration site → heat exchange takes place between blood & tissues only in the capillary
2. Blood perfusion → no effect of direction
3. Vascular architecture ↳
4. Blood temperature → temp in the artery is assumed to be same as body core temp.



Fourier Bioheat eqn + perfusion

$$E_{in} + E_g - E_{out} = \dot{E}$$

Blood perfusion :- the volumetric flow of heat per unit ~~of~~ volume of tissue.

↳ like heat generation

q_b''' = rate of energy added by blood per unit volume of tissue

perfusion

q_m''' = metabolic

$$\dot{E}_g = q''' \, dx\, dy\, dz = (q_b''' + q_m''')\, dx\, dy\, dz$$

$$q_b''' = \rho_b c_b \, \omega_b \, (T_{a0} - T)$$

c_b → specific heat of blood

ρ_b → density of blood

ω_b → blood volumetric flow rate per unit tissue volume

$$q''' = q_m''' + \rho_b c_p \omega_b (T_{a0} - T)$$

$$\nabla \cdot (k \nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$ → Pennes Bio-heat eqn.

$$\nabla \cdot k(\nabla T) + \rho_b c_p \omega_b (T_{a0} - T) + q_m''' \doteq \rho C \frac{\partial T}{\partial t}$$

conduction — Perfusion (convection) — metabolic heat generation — change in Storage.

~~[illegible]~~

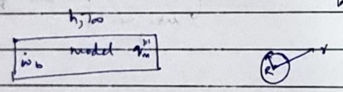
$\rightarrow Q_r(x,t) \rightarrow$ additional heat source term.

$$\rightarrow k\frac{\partial^2 T}{\partial x^2} + \omega_b \rho_b c_b (T_a - T) + Q_m + Q_r(x,t) = \rho c \frac{\partial T}{\partial t}$$

↳ similar to fin

mathematical role of perfusion term in pennes eqn is identical to surface convection in fins. The same effect is seen in ~~[illegible]~~ porous

Temp distribution in the forearm — similar to heat transfer in circular rod



SS

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\rho_b c_b \omega_b}{k}(T_{a0} - T) + \frac{q_m'''}{k} = 0$$

$$\frac{dT(0)}{dr} \rightarrow 0, \quad T(0) = \text{finite}$$

$$-k\frac{dT(R)}{dr} = h(T(R) - T_\infty)$$

$$q = -k\frac{\partial T}{\partial x}$$

$$\rightarrow \rho C \frac{\partial T}{\partial x} = -\frac{\partial q}{\partial x} + (\rho C)_b \omega_b (T_b - T) + Q_m$$

Assumes that

$q(t,x) = -k\nabla T(t,x) \rightarrow$ fourier's law holds

↳ heat flows with ∞ speed.

→ Cattaneo and Vernotte.

$q(t+\tau, x) = -k\frac{\partial T(t,x)}{\partial x} \rightarrow \tau$ is thermal relaxation

fourier law will not work when there is high temperature.

$$q + \tau\frac{\partial q}{\partial t} = -\frac{k\partial T}{\partial x} \rightarrow \text{Non-fourier heat conduction}$$

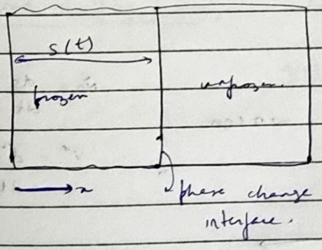
$$\tau\rho C\frac{\partial^2 T}{\partial t^2} + (\rho C + \tau\rho_b c_b \omega_b)\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q_m + \rho_b c_b \omega_b (T_a - T)$$

Cryosurgery
↓
(heat conduction with 2 phase system) ✓

Mushy region → melts at range of temperatures instead of sharp temp change

unfrozen → healthy tissue

frozen → unhealthy (killed cells) (No blood perfusion term) (No blood) (No metabolism)



Frozen:-

$$\rho_f c_f \frac{\partial T_f}{\partial t} = k_f \frac{\partial^2 T_f}{\partial x^2}$$

$$0 \le x \le x_i$$

~~April~~ April

unfrozen

$$\underbrace{\rho_u c_u}_{\text{tissue}} \frac{\partial T_u}{\partial t} = k_u \frac{\partial^2 T_u}{\partial x^2} + q_b + q_m \qquad x_i \le x \le L$$

$$q_b = \underbrace{\rho_b c_b w_b}_{\text{blood}} (T_b - T_u)$$

Conditions at interface

$$T_f(x_i, t) = T_{ph} = T_u(x_i, t)$$

$$k_f \frac{\partial T_f(x_i,t)}{\partial x} - k_u \frac{\partial T_u(x_i,t)}{\partial x} = \rho_u L \frac{\partial x_i}{\partial t}$$

BC: $T_f(0,t) = T_0 = -196^\circ C$ → liq nitrogen frozen region

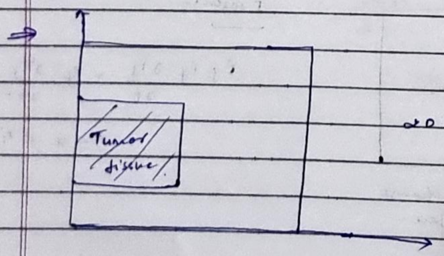
→ Same eqn for laser surgery

Non-fourier eqn

Cryosurgery (hyperbolic)

non-fourier law.

stephen condition will change.



2D eqn.

$$v_n = \frac{ds}{dt}$$

$$v^* = \frac{d^2 s}{dt^2}$$

steady state equation

spherical coordinate system

Laplace transform

$$\bar{f}(p) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$$

Laplace transformation is linear

$$\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}(f_2(t))$$

Existence of laplace transform

It will not exist if

1. $F(t)$ has infinite discontinuities for some values of t,
2. $F(t)$ has singularity at $t \to 0$
3.

First shift property :- $\bar{f}(p) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{e^{at} f(t)\} = \bar{f}(p-a)$$

Second shifting property :- $\bar{f}(p) = \mathcal{L}\{f(t)\}$

$$g(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < 0 \end{cases}$$

$$\mathcal{L}\{g(t)\} = e^{-ap} \bar{f}(p)$$

Change of scale property:

$$\bar{f}(p) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\,\bar{f}(p/a)$$

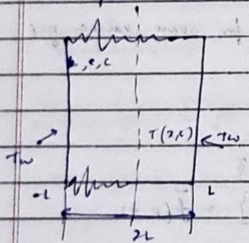
$\Rightarrow$ $\mathcal{L}\left\{\frac{df}{dt}\right\} = p\bar{f}(p) - f(0)$

Convolution theorem

$$\mathcal{L}^{-1}\{\bar{f}(p)\} = \mathcal{L}^{-1}\{\bar{g}(p)\,\bar{h}(p)\} = \int_{t'=0}^{t} g(t-t')\,h(t')\,dt'$$

$$g*h = h*g$$

Q) 1D plane wall



$t=0,\ T=T_i$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \Bigg| \quad \begin{array}{l} x=\pm L,\ T=T_w \\ t=0,\ T=T_i \end{array}$$

take laplace

$$\frac{1}{\alpha}\left(p\bar{T}(x,p) - T_i\right) = \mathcal{L}\left\{\frac{\partial^2 T}{\partial x^2}\right\}$$

$$\frac{\partial^2 \bar{T}}{\partial x^2}$$

~~[illegible]~~

plane of symmetry

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \Bigg| \quad \frac{1}{\alpha}\left(p\bar{T} - T_i\right) = \frac{\partial^2 \bar{T}}{\partial x^2}$$

$$\frac{\partial T(0,t)}{\partial x} = 0, \quad T(L,t) = T_w$$

$$T(x,0) = T_i$$

$$\frac{d^2\bar{T}}{dx^2} - \frac{p\bar{T}}{\alpha} = \frac{-T_i}{\alpha} \qquad \bar{T} = \bar{T}(x,p) = \mathcal{L}(T(x,t)) = \int_0^\infty e^{-pt}\,T(x,t)\,dt$$

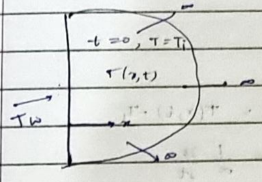
$$\frac{d\bar{T}(0,p)}{dx} = 0, \quad \bar{T}(L,p) = \frac{T_w}{p}$$

$$\bar{T}(x,p) = C_1\cosh mx + C_2\sinh mx + \frac{T_i}{p}$$

$x=0 \to BC \to C_2 = 0$

$x=L \to BC \to C_1 = \dfrac{T_i - T_w}{p\cosh mL}$

$\Rightarrow$ Semi-infinite solid (no phase change)



$$\theta(x,t) = T(x,t) - T_i$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad t=0,\ T(x,0) = T_i$$

$$T(0,t) = T_w$$

$$\boxed{\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}} \qquad \lim_{x\to\infty} T(x,t) = T_i$$

$t=0,\ \theta=0$

$\theta(x,0) = 0$

$\theta(0,t) = T_w - T_i$

$\lim_{x\to\infty}\theta(x,t) = 0$

take laplace transform

$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\bar{\theta}(0,p) = \frac{\theta_w}{p}, \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0$$

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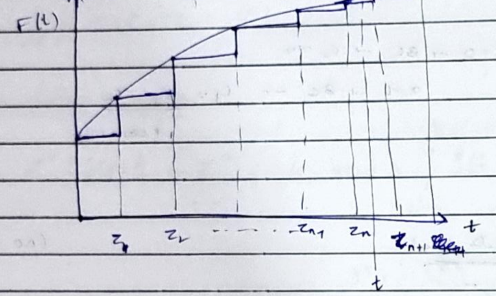
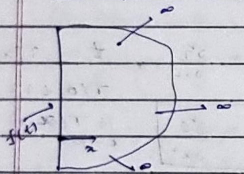
Duhamel method

linear system with time dependent BC or time dependent internal energy generation ...



Semi infinite domain



$$\theta(x,t) = T(x,t) - T_i$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\theta(0,t) = F(t) \text{ and } \lim_{x\to\infty} \theta(x,t) = 0.$$

$$F(t) = f(t) - T_i$$

~~approx~~ approximate $F(t)$ as a piecewise ~~cons~~ function

(till now we have solved direct (or) forward heat conduction)

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Inverse heat conduction

→ Given a solid, I have the temp distribution and want to find BC → This is called inverse heat conduction.

Assume $q(t) = P_0 + P_1 t + P_2 t^2 + \dots + P_n t^n$.