Pennes Bio-Heat Equation

The complex nature of heat transfer in living tissue precludes exact mathematical modelling. Following FOUR simplifying assumptions are made by Pennes.

Assumptions:

Equilibration Site: The principal heat exchange between blood and tissue takes place in the capillary beds, the arterioles supplying blood to the capillaries and the venules draining it. Thus all pre-arteriole and post-venule heat transfer between blood and tissue is neglected.

Blood Perfusion: The flow of blood in the small capillaries is assumed to be isotropic. This neglects the effect of blood flow directionality.

Vascular Architecture: Larger blood vessels in the vicinity of capillary beds play no role in the energy exchange between tissue and capillary blood. Thus the Pennes model does not consider the local vascular geometry.

Blood Temperature: Blood is assumed to reach the arterioles supplying the capillary beds at the body core temperature T_{a0}. It instantaneously exchanges energy and equilibrates with the local tissue temperature T.

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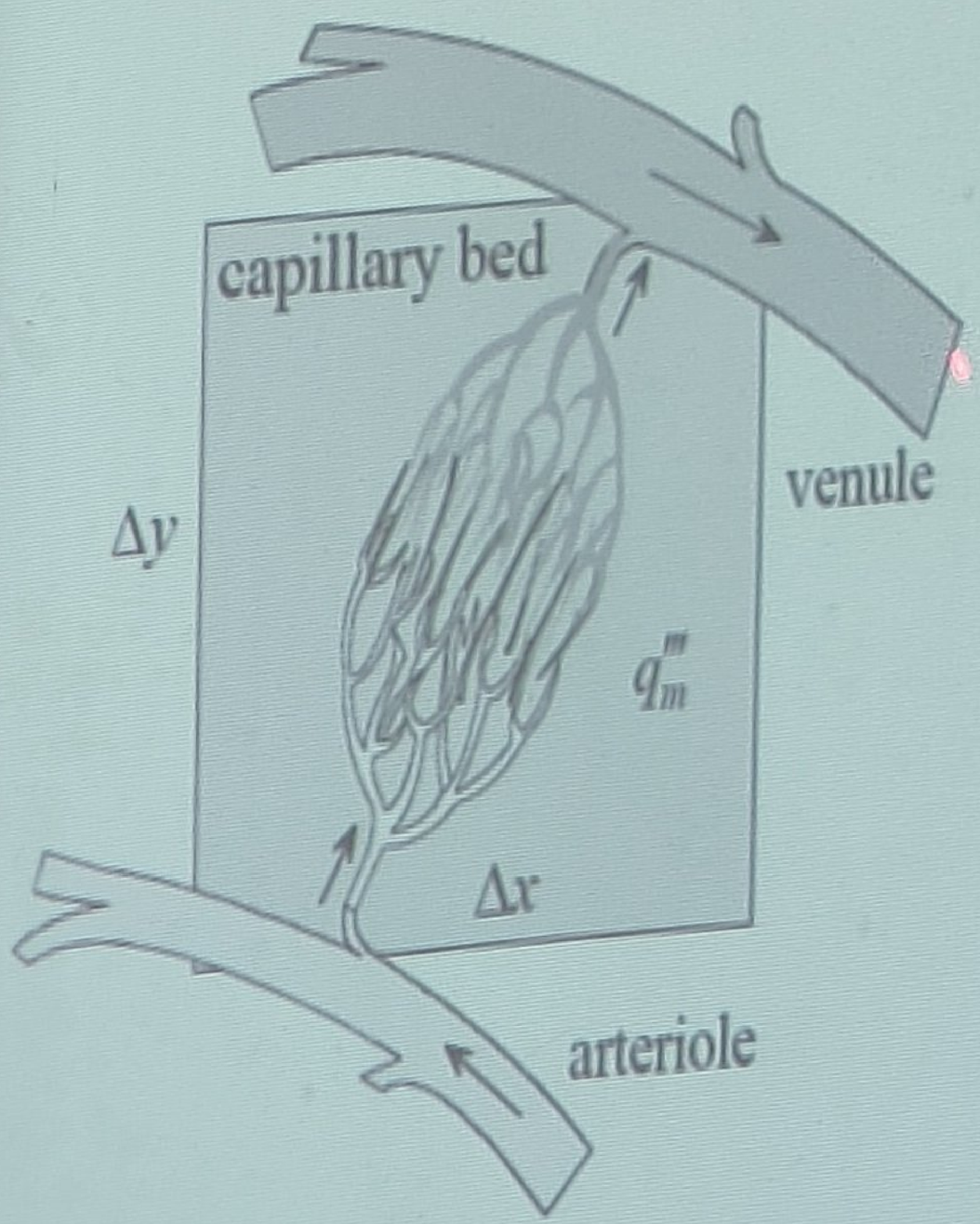
Pennes Bio-Heat Equation

Based on the FOUR assumptions, Pennes modeled blood effect as an isotropic heat source or sink which is proportional to blood flow rate and the difference between body core temperature T_{a0} and local tissue temperature T.

In this model, blood originating at temperature T_{a0} does not experience energy loss or gain as it flows through long branching arteries leading to the arterioles and capillaries.



Blood perfused tissue element. This tissue-vessels matrix is treated as a continuum whose collective temperature is T.

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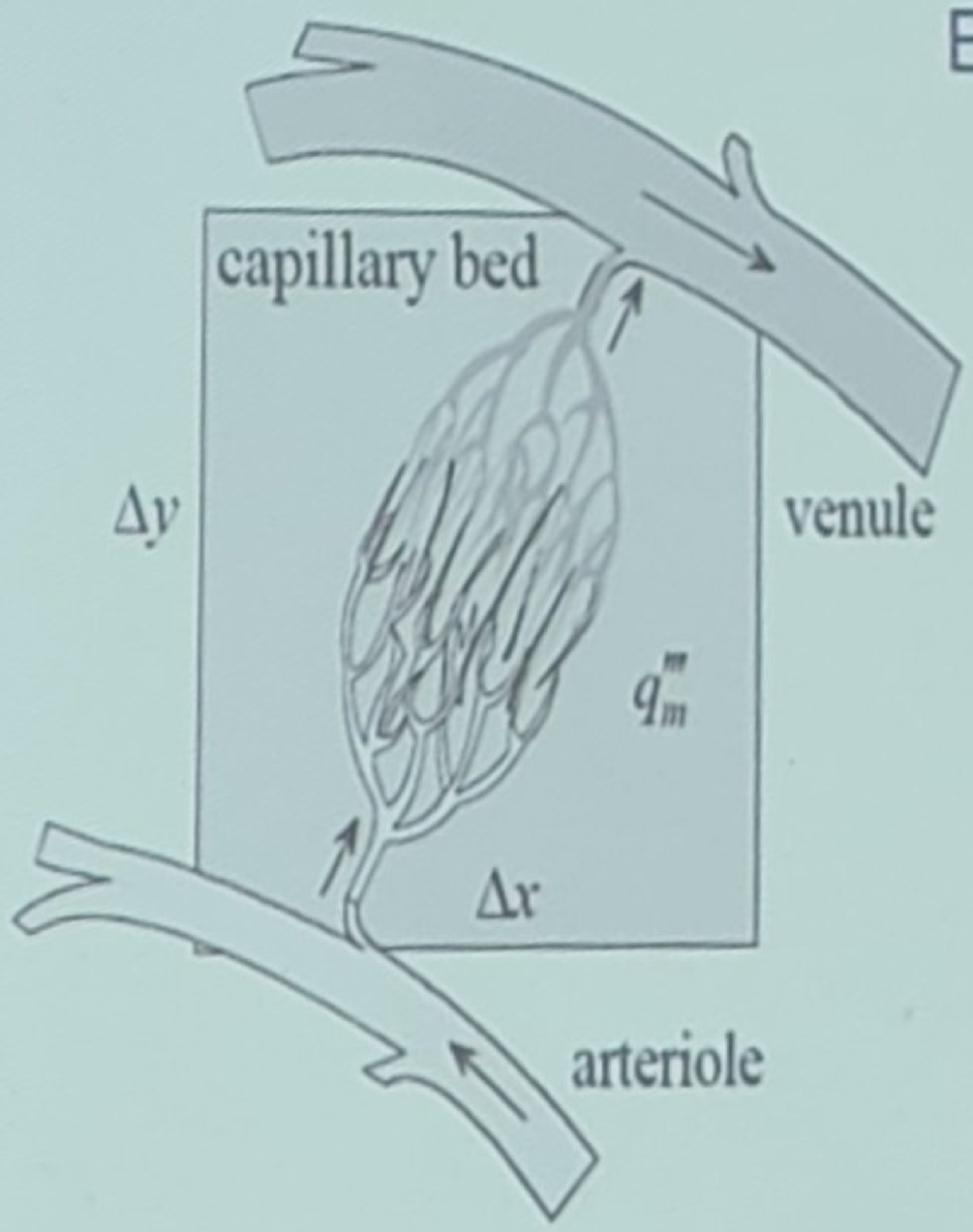
Pennes Bio-Heat Equation



Energy conservation for the element is:

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}$$

The rate of energy added to the element is by conduction and convection (mass motion). Here the convection component is eliminated and replaced by energy added due to blood perfusion. The simplest way to account for this effect is to treat it as energy generation.

q_b''' = net rate of energy added by the blood per unit volume of tissue

q_m''' = rate of metabolic energy production per unit volume of tissue

$$\dot{E}_g = q'''dxdydz = (q_b''' + q_m''')dxdydz\ .$$

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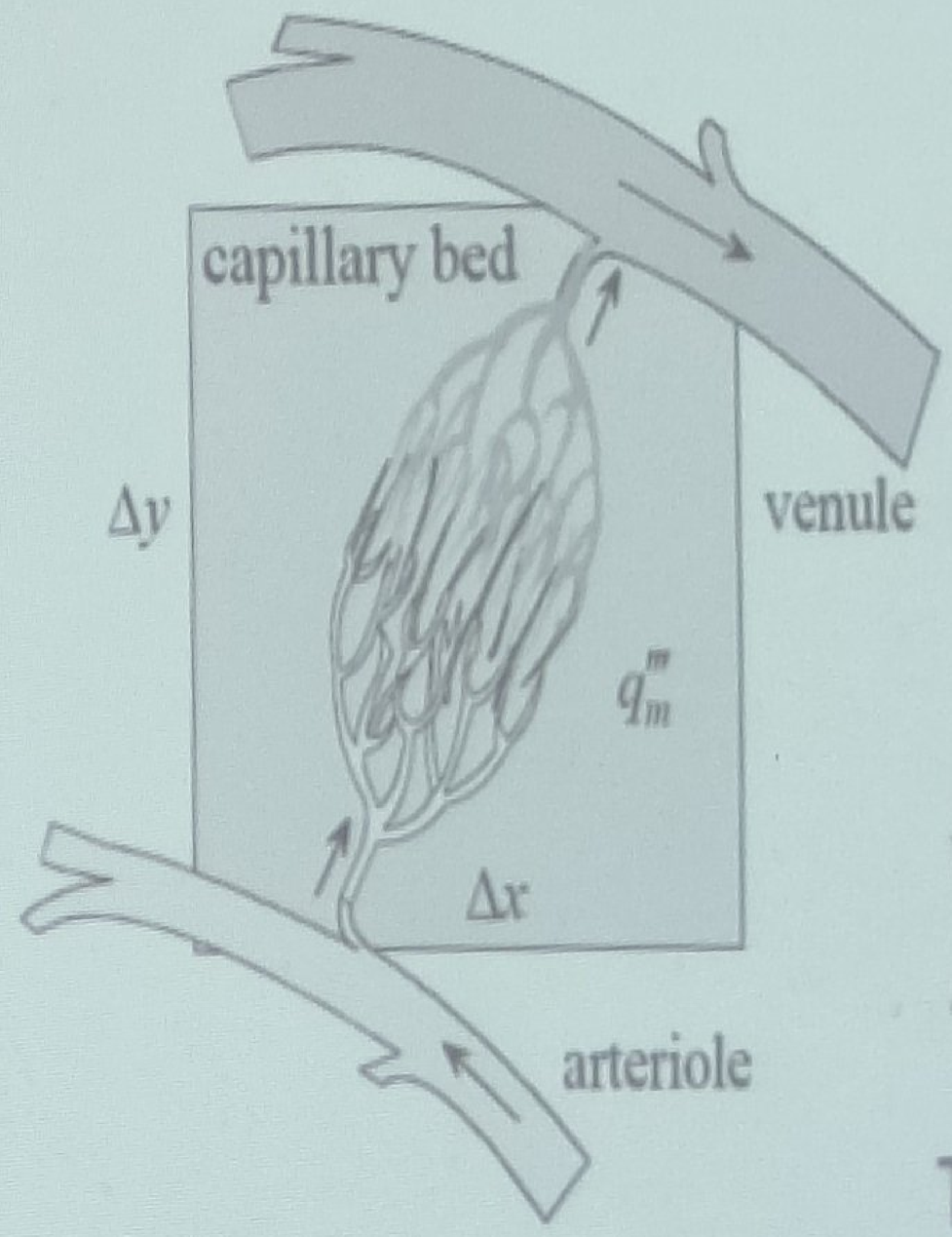
Pennes Bio-Heat Equation



According to Pennes, blood enters the element at the body core temperature T_{a0}, it instantaneously equilibrates and exists at the temperature of the element T. Thus,

$$q_b''' = \rho_b c_b \dot{w}_b (T_{a0} - T)$$

c_b = specific heat of blood

$\dot{w}_b$ = blood volumetric flow rate per unit tissue volume

ρ_b = density of blood

Thus, $q''' = q_m''' + \rho_b c_b \dot{w}_b (T_{a0} - T)$

Recall:

$$\nabla \cdot (k \nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t} \Rightarrow \nabla \cdot k \nabla T + \rho_b c_b \dot{w}_b (T_{a0} - T) + q_m''' = \rho c \frac{\partial T}{\partial t}$$

c = specific heat of tissue

k = thermal conductivity of tissue

ρ = density of tissue

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Pennes Bio-Heat Equation

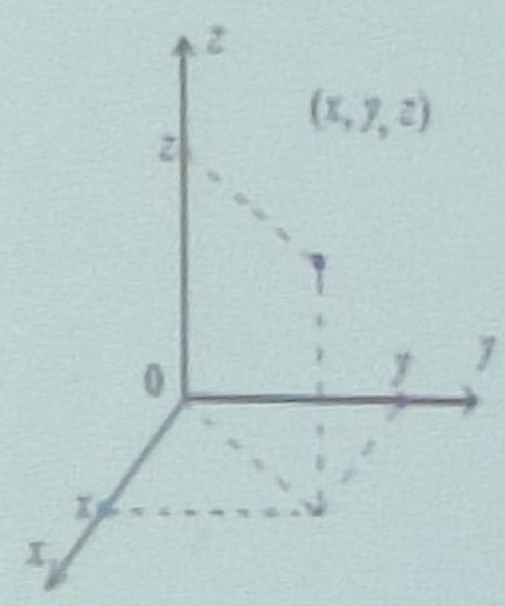
$$\nabla \cdot k\nabla T + \rho_b c_b \dot{w}_b (T_{a0} - T) + q_m''' = \rho c \frac{\partial T}{\partial t}$$

Conduction | Perfusion (convection) | Metabolic heat generation | Change in storage

$$\nabla \cdot k\nabla T =$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

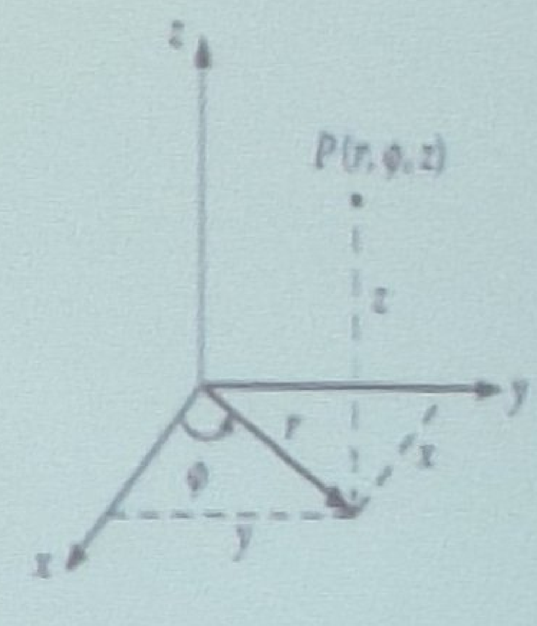
Spherical coordinates



$$\nabla \cdot k\nabla T =$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

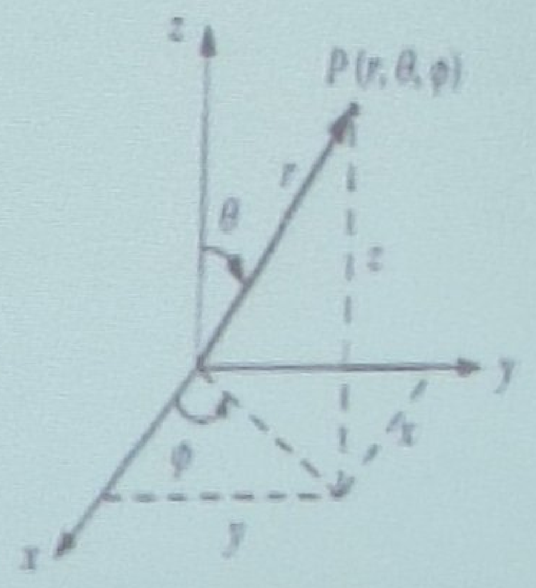
Cylindrical coordinates



$$\nabla \cdot k\nabla T =$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)$$

Spherical coordinates



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Pennes Bio-Heat Equation

The generalized Pennes equation (1D):

$$K\frac{\partial^2 T}{\partial x^2} + \omega_b \rho_b c_b (T_a - T) + Q_m + Q_r(x,t) = \rho c \frac{\partial T}{\partial t}$$

- ρ, c, K are respectively the density, the specific heat, and the thermal conductivity of the tissue
- ρ_b, c_b denote density and specific heat of blood
- ω_b the blood perfusion [(vol/time)/vol] (volumetric flow of blood perfusion per unit tissue volume)
- T_a the arterial temperature which is treated as a constant
- T the tissue temperature
- Q_m is the metabolic heat generation, and $Q_r(x,t)$ the heat source due to spatial heating (external heat source)

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Pennes Bio-Heat Equation: Criticism

Pennes Bio-heat equation is the most widely applied model for temperature distribution in the living biological tissues. However, the following criticism about the FOUR assumptions of Pennes's model have been made.

Equilibration Site:

Thermal equilibration does not occur in the capillaries, as Pennes assumed. Instead it takes place in pre-arteriole and post-venule vessels having diameters ranging from 70 – 500 μm.

This conclusion is based on theoretical determination of the *thermal equilibration length*, L_e, which is the distance blood travels along a vessel for its temperature to equilibrate with the local tissue temperature. For arterioles and capillaries this distance is much shorter than their length. Vessels for which $L_e/L > 1$ are commonly referred to as *thermally significant.*

Blood Perfusion:

Directionality of blood perfusion is an important factor in the interchange of energy between vessels and tissue. The Pennes equation does not account for this effect.

Capillary blood perfusion is not isotropic but proceeds from arterioles to venules.

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Pennes Bio-Heat Equation: Criticism

Vascular Architecture:

Pennes equation does not consider the local vascular geometry. Thus significant features of the circulatory system are not accounted for. This includes energy exchange with large vessels, countercurrent heat transfer between artery-vein pairs and vessel branching and diminution.

Blood Temperature:

The arterial temperature varies continuously from the deep body temperature of the aorta to the secondary arteries supplying the arterioles, and similarly for the venous return. Thus, contrary to Pennes' assumption, pre-arteriole blood temperature is not equal to body core temperature Tao and vein return temperature is not equal to the local tissue temperature T. Both approximations overestimate the effect of blood perfusion on local tissue temperature.

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Applications:

Despite these shortcomings of the Pennes equation, it has been successfully applied to many practical situations such as hyperthermia therapy, blood perfusion measurements, cryosurgery and thermal simulation of whole body.

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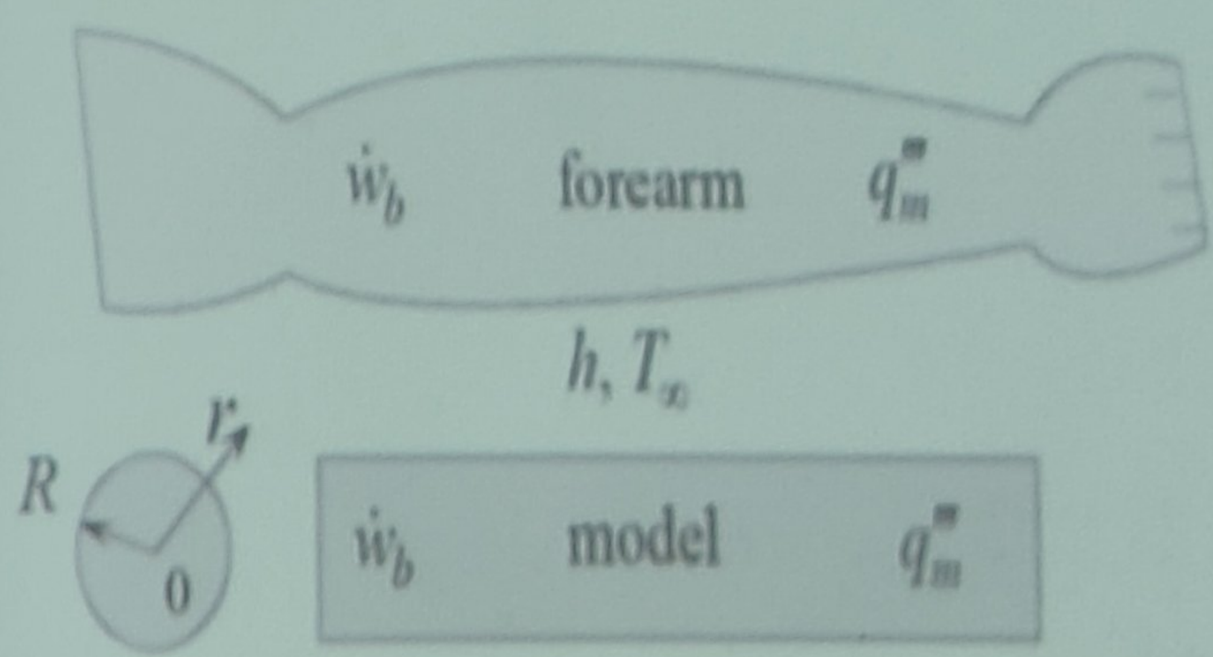
Pennes Bio-Heat Equation

Temperature Distribution in the Forearm



Model the forearm as a cylinder of radius R with volumetric blood perfusion rate per unit tissue volume $\dot{w}_b$ and metabolic heat production q_m'''.

Use Pennes Bio-heat equation to determine the steady state one-dimensional temperature distribution in the arm.

Governing Equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\rho_b c_b \dot{w}_b}{k}(T_{a0}-T)+\frac{q_m'''}{k}=0$$

Boundary Conditions:

$$\frac{dT(0)}{dr}=0, \text{ or } T(0)=\text{finite},$$

$$-k\frac{dT(R)}{dr}=h\left[T(R)-T_\infty\right].$$

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Pennes Bio-Heat Equation

$$\nabla \cdot k\nabla T + \rho_b c_b \dot{w}_b (T_{a0} - T) + q_m''' = \rho c \frac{\partial T}{\partial t}$$

Conduction | Perfusion (convection) | Metabolic heat generation | Change in storage

$$\left.\begin{aligned} \rho c \frac{\partial T}{\partial t} &= -\frac{\partial q}{\partial x} + (\rho c)_b \dot{w}_b (T_b - T) + Q_m, \\ q(t,x) &= -k\nabla T(t,x) \end{aligned}\right]$$

Fourier's Law

$$\rho C_p \frac{\partial T}{\partial t} = k\nabla^2 T + \omega_b \rho_b c_b (T_b - T) + Q_m$$

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Pennes Bio-Heat Equation

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} + (\rho c)_b w_b (T_b - T) + Q_m, \qquad q(t,x) = -k\nabla T(t,x)$$

Fourier's law assumes that heat flux q and temperature change ∇T at any point x appear at the same time instant t.

This implies that thermal signals propagate with an infinite speed. In fact, heat is always found to propagate at finite speed.

On the other hand, biological tissue are inhomogeneous where heat flux responds to temperature gradient via relaxation behavior.

This has led to work of searching for new constitutive models. Cattaneo and Vernotte simultaneously suggested a modified heat flux model as

$$q(t+\tau,x) = -k\frac{\partial T(t,x)}{\partial x}$$ where τ is thermal relaxation time for heat flux.

The first order Taylor expansion of q in the above equation gives:

$$q + \tau\frac{\partial q}{\partial t} = -k\frac{\partial T}{\partial x}$$ This is Cattaneo-Vernotte's constitutive relation for Non-Fourier Heat Conduction.

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Pennes Bio-Heat Equation

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} + (\rho c)_b w_b (T_b - T) + Q_m,$$

$$q(t,x) = -k \nabla T(t,x)$$

Instead of using Fourier's equation,

$$q + \tau \frac{\partial q}{\partial t} = -k \frac{\partial T}{\partial x}$$

Use Cattaneo-Vernotte's constitutive relation

Then we get,

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} + (\rho c + \tau \rho_b c_b w_b) \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q_m + \rho_b c_b w_b (T_b - T).$$

Hyperbolic bio-heat equation

For 2D:

$$\tau_q \rho c \frac{\partial^2 T}{\partial t^2} + (\rho c + \tau_q \rho_b c_b w_b) \frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \rho_b c_b w_b (T_b - T) + Q_m.$$

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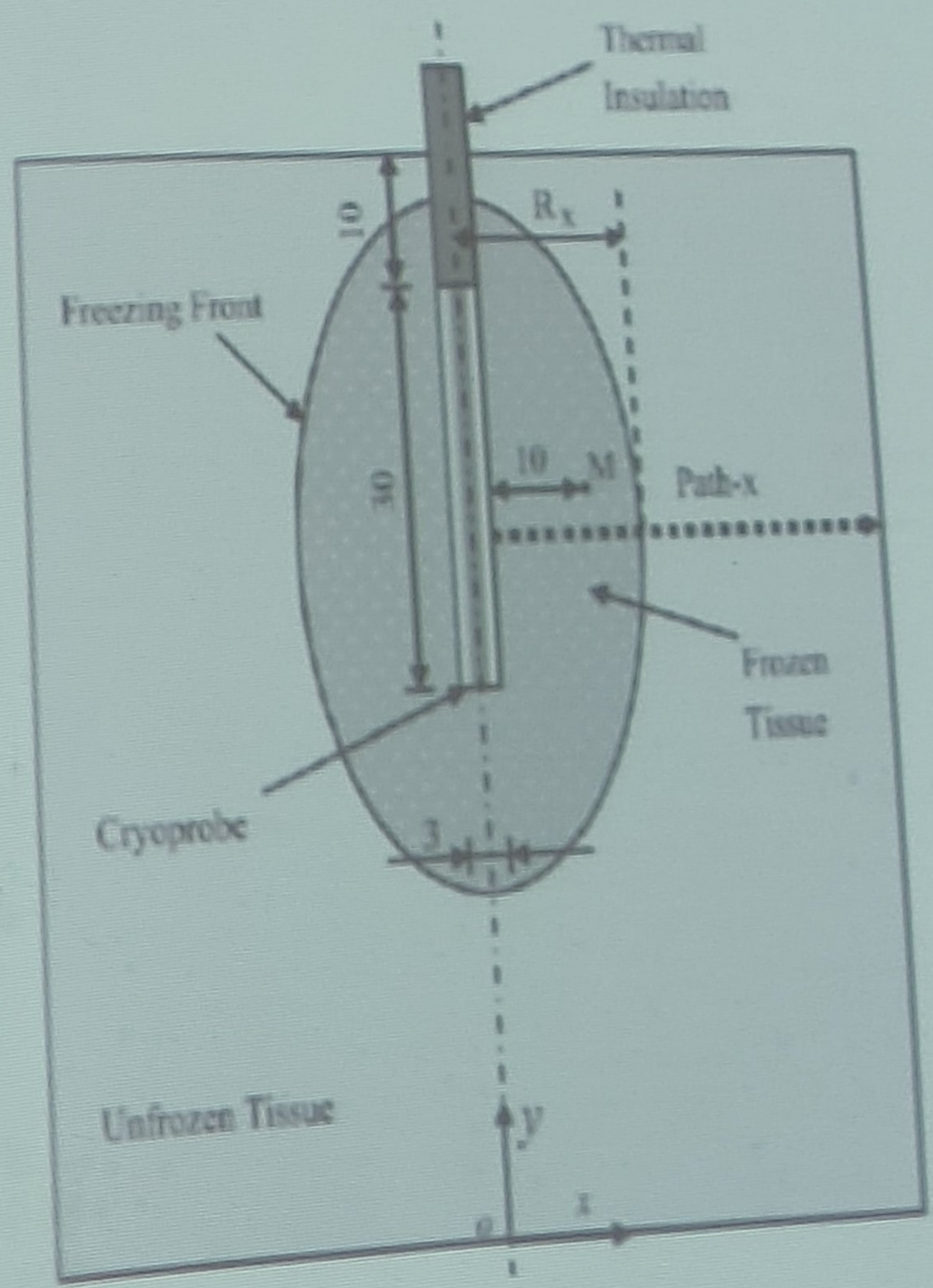
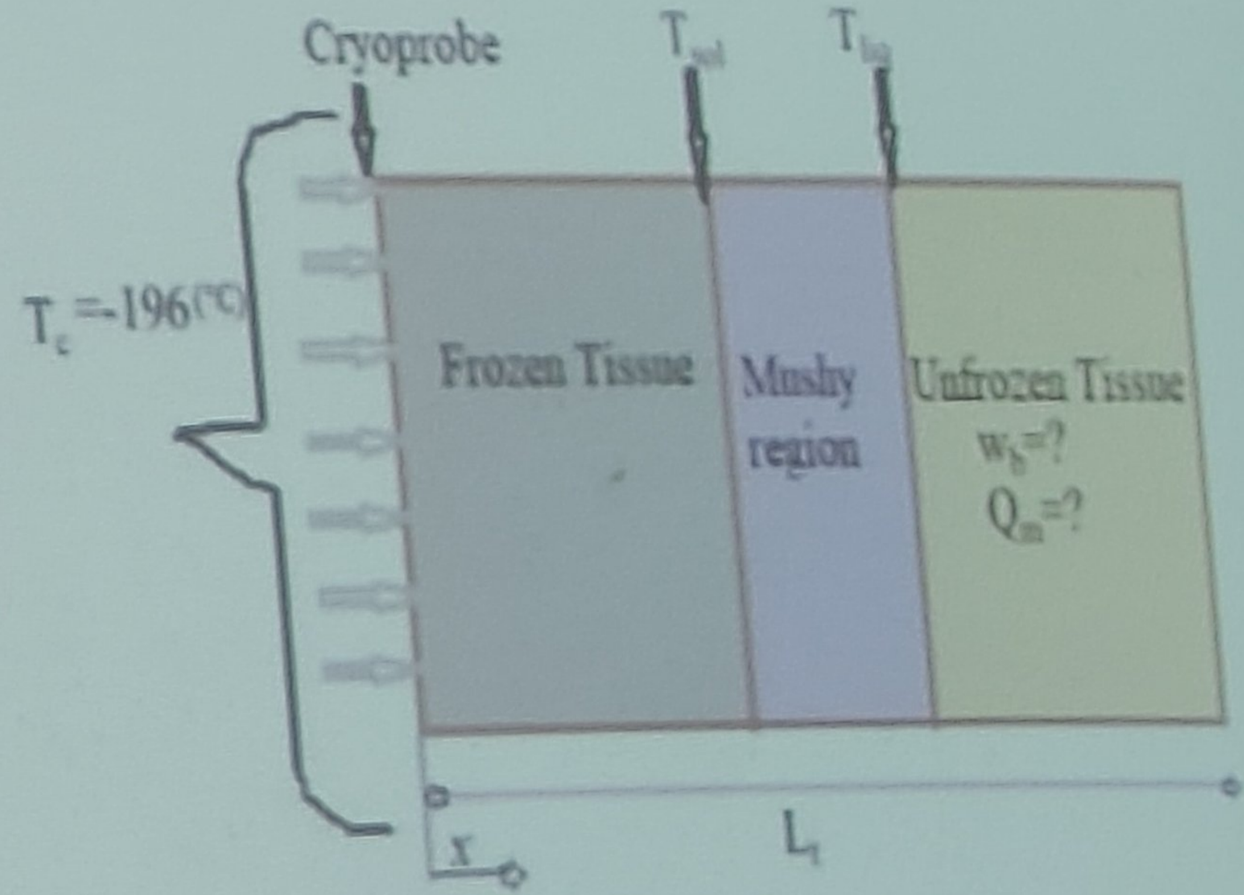
Cryosurgery





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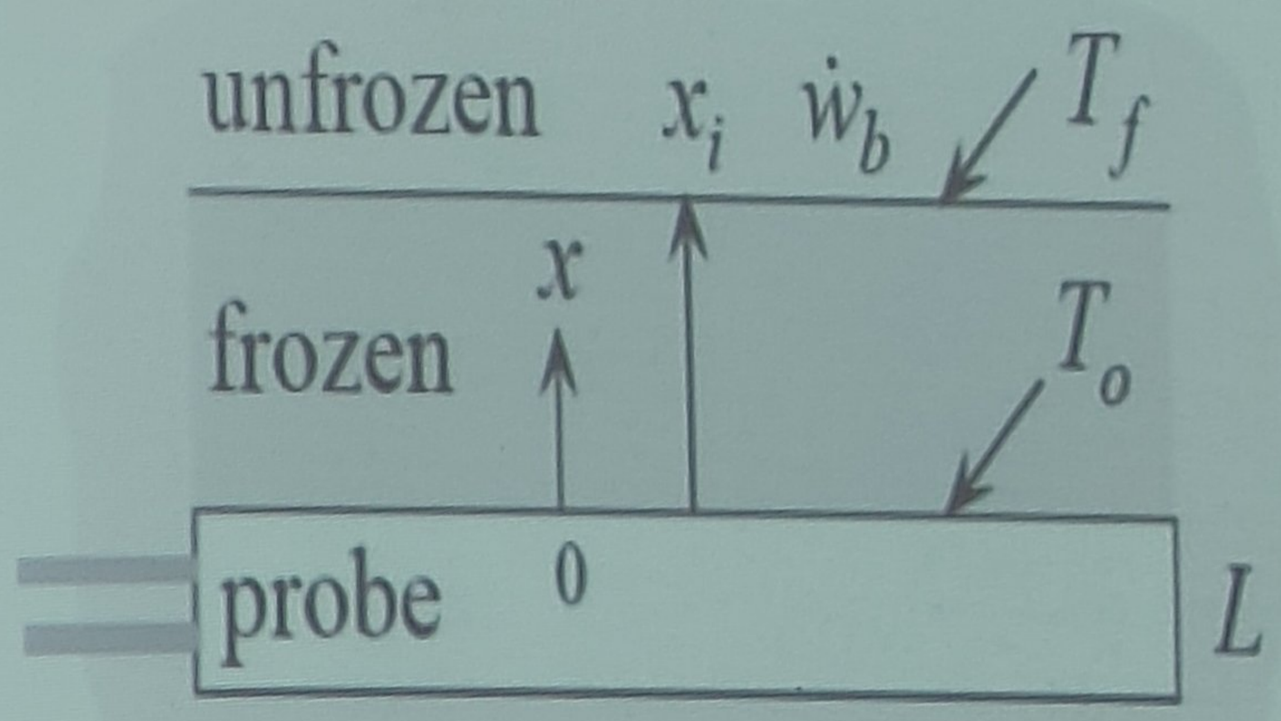
Cryosurgery



Cryosurgical probes are used in medical procedures to selectively freeze and destroy diseased tissue.

The cryoprobe surface is maintained at a temperature below tissue freezing temperature causing a frozen front to form at the surface and propagate outward.

Knowledge of the maximum frozen layer or lesion size is helpful to the surgeon in selecting the proper settings for the cryoproble.

Maximum lesion size corresponds to the steady state temperature distribution.

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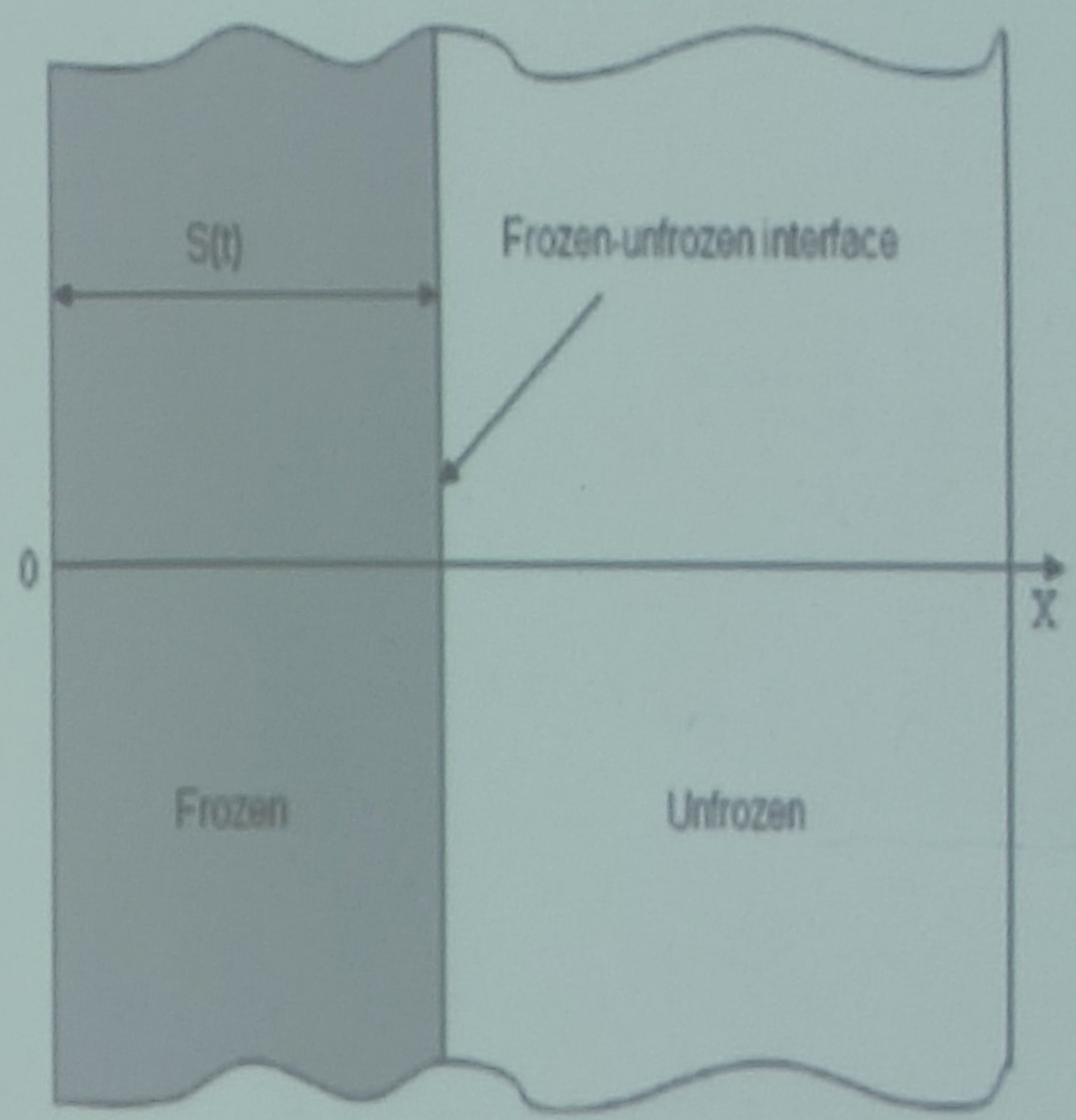
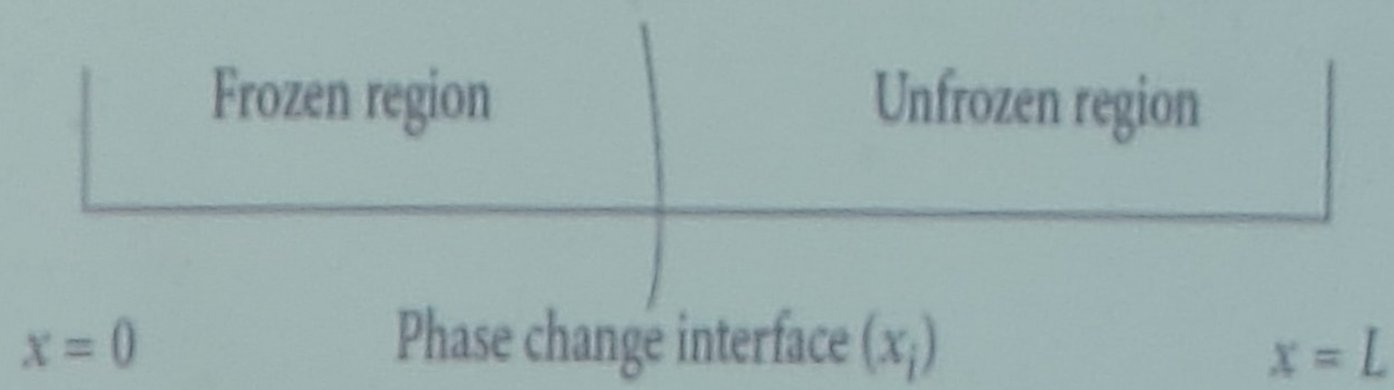
Cryosurgery: 1D Model, Sharp Boundary

Parabolic Heat Conduction
(Fourier Law holds)





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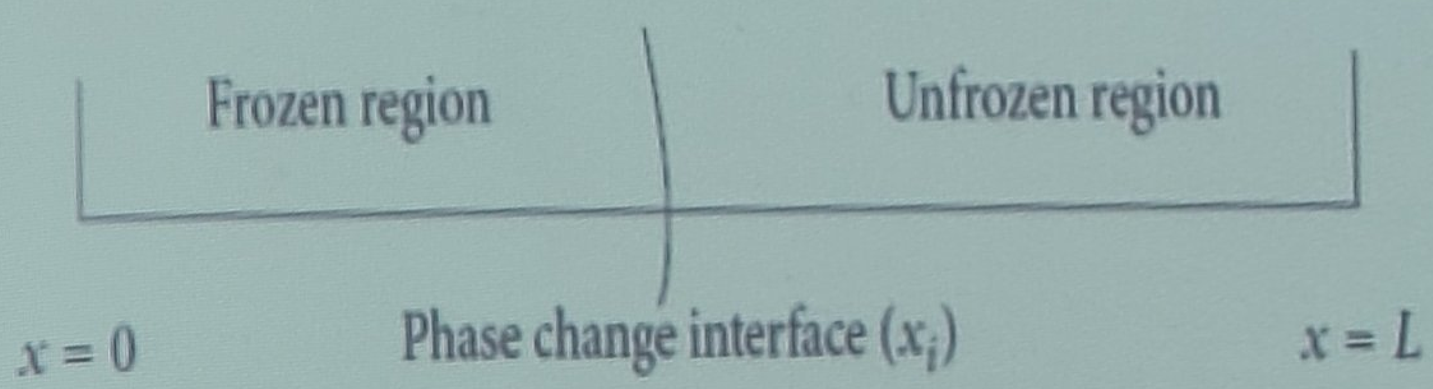
Cryosurgery: 1D Model, Sharp Boundary

Parabolic Heat Conduction
(Fourier Law holds)



In frozen region:

$$\rho_f c_f \frac{\partial T_f}{\partial t} = k_f \frac{\partial^2 T_f}{\partial x^2} \quad \text{for } 0 \le x \le x_i.$$

In unfrozen region:

$$\rho_u c_u \frac{\partial T_u}{\partial t} = k_u \frac{\partial^2 T_u}{\partial x^2} + q_b + q_m \quad \text{for } x_i \le x \le L.$$

$$q_b = \rho_b c_b \omega_b (T_b - T_u)$$

(Blood perfusion)

Conditions at phase change interface:

$$T_f(x_i, t) = T_{ph} = T_u(x_i, t),$$

$$k_f \frac{\partial T_f(x_i, t)}{\partial x} - k_u \frac{\partial T_u(x_i, t)}{\partial x} = \rho_u l \frac{dx_i}{dt},$$

Initial conditions:

$$T_u(x, 0) = T_I = 37^\circ\text{C},$$

$$T_f(x, 0) = T_I = 37^\circ\text{C}.$$

Subscripts u, f and ph denote frozen, unfrozen and phase change, respectively.

Boundary conditions:

$$T_f(0, t) = T_0 = -196^\circ\text{C},$$

$$\frac{\partial T_u(L, t)}{\partial x} = 0.$$

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Cryosurgery: 1D Model, Sharp Boundary

Hyperbolic Heat Conduction (Non-Fourier)

In frozen region:

$$\tau \rho_f c_f \frac{\partial^2 T_f}{\partial t^2} + \rho_f c_f \frac{\partial T_f}{\partial t} = k_f \frac{\partial^2 T_f}{\partial x^2} \quad \text{for } 0 \le x < s(t).$$

In unfrozen region:

$$\tau \rho_u c_u \frac{\partial^2 T_u}{\partial t^2} + (\rho_u c_u + \tau \rho_b c_b \omega_b) \frac{\partial T_u}{\partial t} = k_u \frac{\partial^2 T_u}{\partial x^2} + Q_m + \rho_b c_b \omega_b (T_b - T_u)$$

$$\text{for } s(t) < x \le l$$

Conditions at phase change interface $x = s(t)$ are:

$$k_f \frac{\partial T_f(s,t)}{\partial x} - k_u \frac{\partial T_u(s,t)}{\partial x} = \rho L \frac{ds}{dt} + \tau \rho L \frac{d^2 s}{dt^2}$$

$$T_f(s(t),t) = T_u(s(t),t) = T_{ph},$$

where subscripts u, f and ph denote frozen, unfrozen and phase change respectively.

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Cryosurgery: 2D Model, Sharp Boundary

Hyperbolic Heat Conduction (Non-Fourier)

In frozen region:

$$\tau_q \rho_f c_f \frac{\partial^2 T_f}{\partial t^2} + \rho_f c_f \frac{\partial T_f}{\partial t} = k_f \left(\frac{\partial^2 T_f}{\partial x^2} + \frac{\partial^2 T_f}{\partial y^2} \right),$$

Subscripts f, u, b denote frozen, unfrozen and blood of tissue, respectively.

In unfrozen region:

$$\tau_q \rho_u c_u \frac{\partial^2 T_u}{\partial t^2} + (\rho_u c_u + \tau_q \rho_b c_b w_b) \frac{\partial T_u}{\partial t} = k_u \left(\frac{\partial^2 T_u}{\partial x^2} + \frac{\partial^2 T_u}{\partial y^2} \right)$$

$$+ \rho_b c_b w_b (T_b - T_u) + Q_m,$$

Conditions at phase change interface $x = s(t)$ are:

$$k_f \frac{\partial T_f(S, t)}{\partial n} - k_u \frac{\partial T_u(S, t)}{\partial n} = \rho L v_n + \tau_q \rho L \dot{v}_n,$$

$$T_u(S, t) = T_f(S, t) = T_{ph}. \qquad \text{where } v_n = \frac{\partial S}{\partial t}, \ \dot{v}_n = \frac{\partial^2 S}{\partial t^2}$$

Here subscript ph denote phase change

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Bio-Heat: Pennes Equation

Thermal damage during retinal laser irradiation

- Non-Fourier heat conduction is laser irradiation of the retina
- Here a high heat flux is typically applied for a short period of time
- Bio-Pennes dual phase lag model

The dual phase lag constitutive relation: $$\mathbf{q} + \tau_q \frac{\partial \mathbf{q}}{\partial t} = -k\nabla T - k\tau_T \frac{\partial \nabla T}{\partial t} \quad (1)$$

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + Q_m + Q_l + (w\rho c)_b (T_b - T) \quad (2)$$

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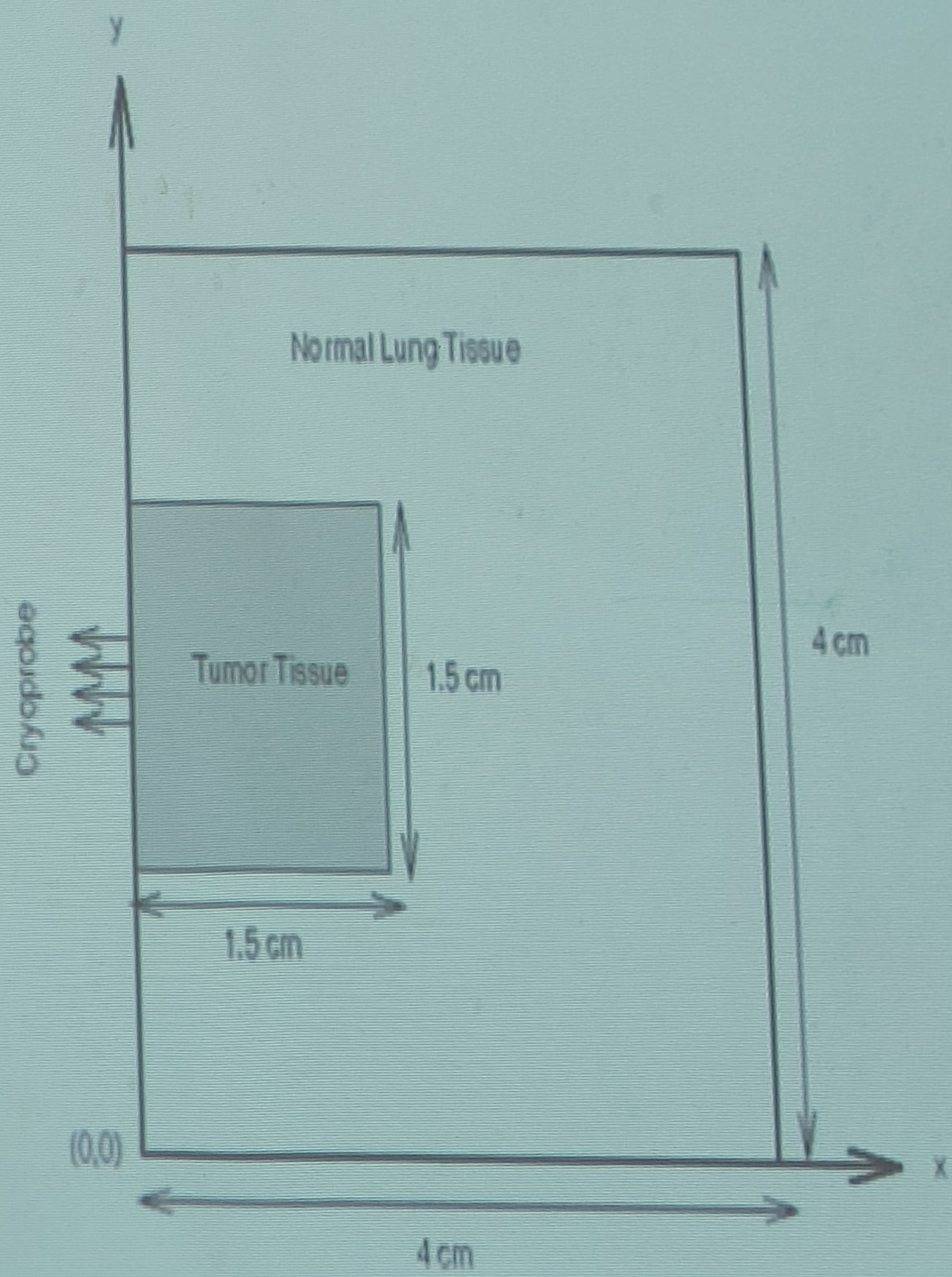
Cryosurgery: 2D Model, Sharp Boundary

Hyperbolic Heat Conduction
(Non-Fourier)



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Cryosurgery: 2D Model, Sharp Boundary

Hyperbolic Heat Conduction (Non-Fourier)

In frozen region:

$$\tau_q \rho_f c_f \frac{\partial^2 T_f}{\partial t^2} + \rho_f c_f \frac{\partial T_f}{\partial t} = k_f \left(\frac{\partial^2 T_f}{\partial x^2} + \frac{\partial^2 T_f}{\partial y^2} \right).$$

Subscripts f, u, b denote frozen, unfrozen and blood of tissue, respectively.

In unfrozen region:

$$\tau_q \rho_u c_u \frac{\partial^2 T_u}{\partial t^2} + (\rho_u c_u + \tau_q \rho_b c_b w_b) \frac{\partial T_u}{\partial t} = k_u \left(\frac{\partial^2 T_u}{\partial x^2} + \frac{\partial^2 T_u}{\partial y^2} \right)$$

$$+ \rho_b c_b w_b (T_b - T_u) + Q_m.$$

Conditions at phase change interface $x = s(t)$ are:

$$k_f \frac{\partial T_f(S, t)}{\partial n} - k_u \frac{\partial T_u(S, t)}{\partial n} = \rho L v_n + \tau_q \rho L \dot{v}_n.$$

$$T_u(S, t) = T_f(S, t) = T_{ph}. \qquad \text{where } v_n = \frac{dS}{dt}, \ \dot{v}_n = \frac{d^2 S}{dt^2}$$

Here subscript ph denote phase change

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Pennes Bio-Heat Equation

Steady-State Analysis: Find temperature distribution before heating/cooling

The initial temperature field of the biological bodies can be obtained by solving steady-state version subjected to boundary conditions:

$$K\frac{d^2T_0(x)}{dx^2} + \omega_b\rho_b c_b[T_a - T_0(x)] + Q_m = 0$$

$$T_0(x) = T_c,\ x = L \qquad\qquad -K\frac{dT_0(x)}{dx} = h_0[T_f - T_0(x)],\ \ x = 0$$

- Here, the skin surface is defined at $x = 0$, while the body core at $x = L$
- $T(x, 0) = T_0(x)$ is steady-state temperature fields before heating
- T_c is the body core temperature
- h_0 is the apparent heat convection coefficient between the skin surface and the surrounding air under physiologically basal state and is an overall contribution from natural convection and radiation
- T_f is the surrounding air temperature.

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Bio-Heat: Pennes Equation

For 1-D spherical coordinate system (constant thermo-physical properties):

$$\rho c \frac{\partial T}{\partial t} = k \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r} \right) + \omega_b \rho_b c_b (T_a - T) + q_m + Q(r,t),$$

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=0} = 0,$$

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R_1} = h_f \left[T - f(t) \right] \Big|_{r=R_1},$$

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Laplace Transforms

Laplace transformation is linear.

$$\pounds\{C_1 f_1(t) + C_2 f_2(t)\} = C_1 \pounds\{f_1(t) + C_2 \pounds(f_2(t)\}$$
$$= C_1 \bar{f}_1(p) + C_2 \bar{f}_2(p)$$

Existence of Laplace Transform:

The Laplace transform of a function F(t) may not always exist. This is because the integral may not exist if

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Laplace Transforms

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Existence of Laplace Transform:

The Laplace transform of a function F(t) may not always exist. This is because the integral may not exist if

1. $F(t)$ has infinite discontinuities for some values of t;
2. $F(t)$ has a singularity as $t \to 0$
3. $F(t)$ diverges exponentially for large t

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Laplace Transforms: Important Properties

First shifting property. If $\bar{f}(p) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\left\{e^{at} f(t)\right\} = \bar{f}(p-a)$$

Second shifting property, if $\bar{f}(p) = \mathcal{L}\{f(t)\}$ and

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

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Laplace Transforms: Important Properties

First shifting property. If $\bar{f}(p) = \pounds\{f(t)\}$, then

$$\pounds\left\{e^{at} f(t)\right\} = \bar{f}(p-a)$$

Second shifting property, if $\bar{f}(p) = \pounds\{f(t)\}$ and

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

then

$$\pounds\left\{g(t)\right\} = e^{-ap}\bar{f}(p)$$

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Laplace Transforms: Important Properties

Change of scale property. If $\bar{f}(p) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$$

Laplace transform of derivatives. Let $f(t)$ be continuous with a piecewise continuous derivative df/dt over every interval $0 \le t \le t_1$. Also, let $f(t)$ be of exponential order for $t > t_1$. Then

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = p\,\bar{f}(p) - f(0)$$

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Laplace Transforms: Convolution Theorem

Let,

$$\bar{f}(p) = \bar{g}(p)\bar{h}(p)$$

where $\bar{g}(p)$ and $\bar{h}(p)$ are the transforms of the known functions $g(t)$ and $h(t)$, respectively. In this case

$$\pounds^{-1}\left\{\bar{f}(p)\right\} = \pounds^{-1}\{\bar{g}(p)\bar{h}(p)\} = \int_{t'=0}^{t} g(t-t')h(t')dt'$$

This integral, sometimes denoted by g * h, is called the convolution of g(t) and h(t).

It has the following property

$$g * h = h * g$$

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Laplace Transforms: Heat Conduction Problems

Formulation of the problem for $T(x, t)$
- Differential equation
- Initial and boundary conditions

↓

Laplace transform of the problem with respect to t

↓

Ordinary differential equation and boundary conditions for $\bar{T}(x, p)$

↓

Ordinary differential equation theory

↓

Solution for $\bar{T}(x, p)$

↓

Inversion of $\bar{T}(x, p)$

↓

Solution for $T(x, t)$

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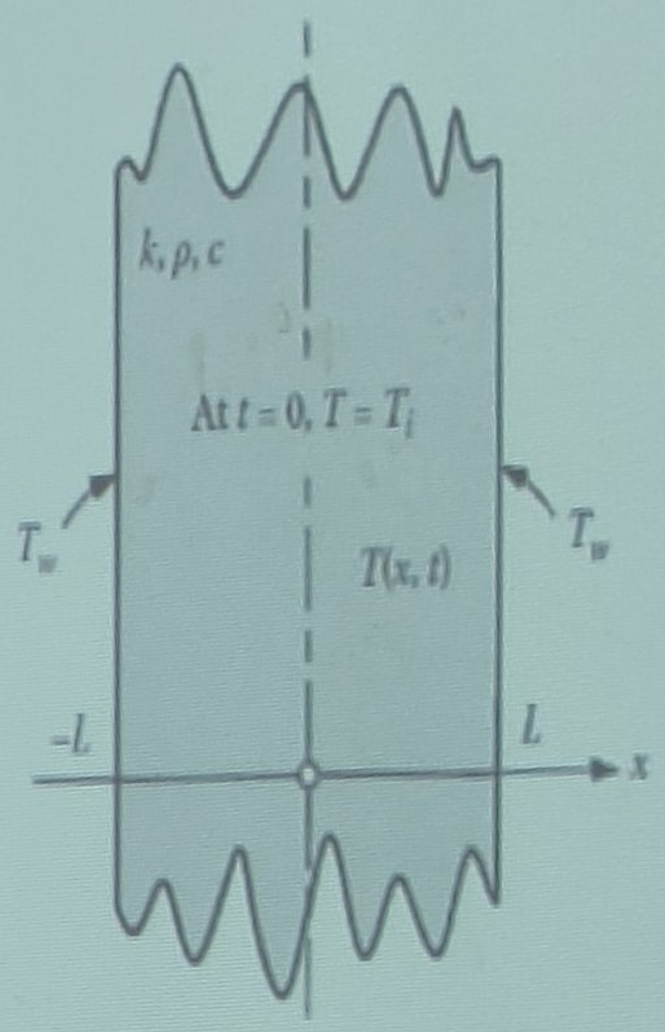
Laplace Transforms: Heat Conduction Problems

1D Plane Wall:



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{and} \quad T(L,t) = T_w$$

Laplace transform of PDE:

$$\pounds\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\alpha}\pounds\left\{\frac{\partial T}{\partial t}\right\}$$

$$\pounds\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{\partial^2 \bar{T}}{\partial x^2}$$

$$\pounds\left\{\frac{\partial T}{\partial t}\right\} = p\bar{T}(x,p) - T_i$$

$$\bar{T} = \bar{T}(x,p) = \pounds\{T(x,t)\} = \int_0^\infty e^{-pt}T(x,t)dt$$

Laplace transformed PDE:

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T} = -\frac{T_i}{\alpha}$$

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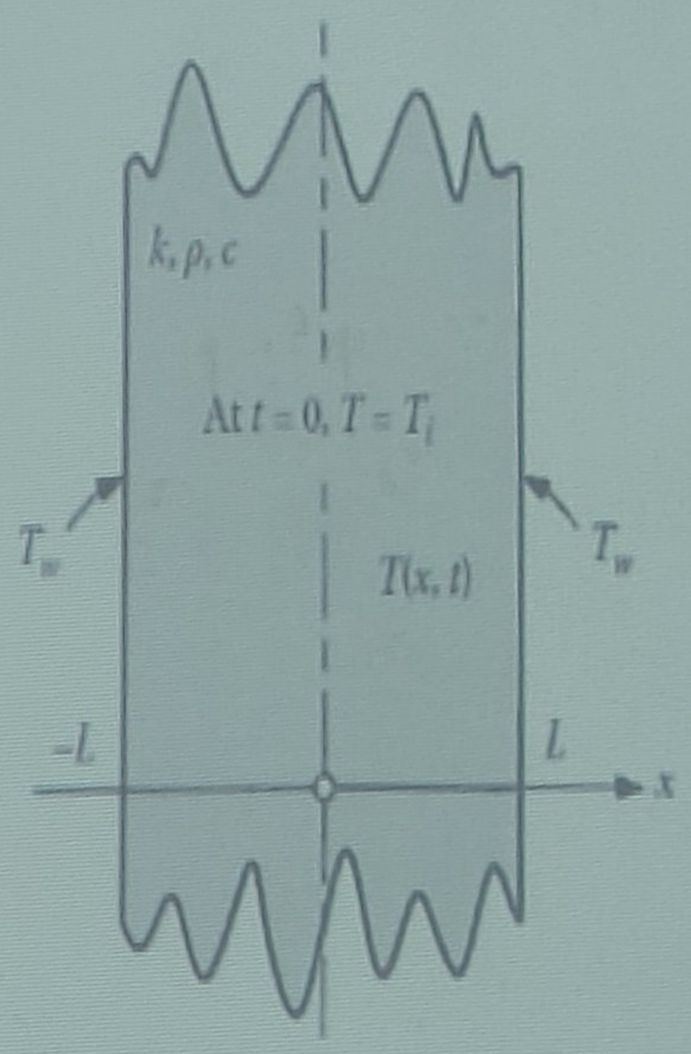
Laplace Transforms: Heat Conduction Problems

1D Plane Wall:



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{and} \quad T(L,t) = T_w$$

Laplace transform of PDE:

$$\mathcal{L}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\alpha}\mathcal{L}\left\{\frac{\partial T}{\partial t}\right\}$$

$$\mathcal{L}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{\partial^2 \bar{T}}{\partial x^2}$$

$$\mathcal{L}\left\{\frac{\partial T}{\partial t}\right\} = p\bar{T}(x,p) - T_i$$

$$\bar{T} = \bar{T}(x,p) = \mathcal{L}\{T(x,t)\} = \int_0^\infty e^{-pt}T(x,t)\,dt$$

Laplace transformed PDE:

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T} = -\frac{T_i}{\alpha}$$

Laplace transform of BC:

$$\frac{\bar{T}(0,p)}{dx} = 0 \quad \text{and} \quad \bar{T}(L,p) = \frac{T_w}{p}$$

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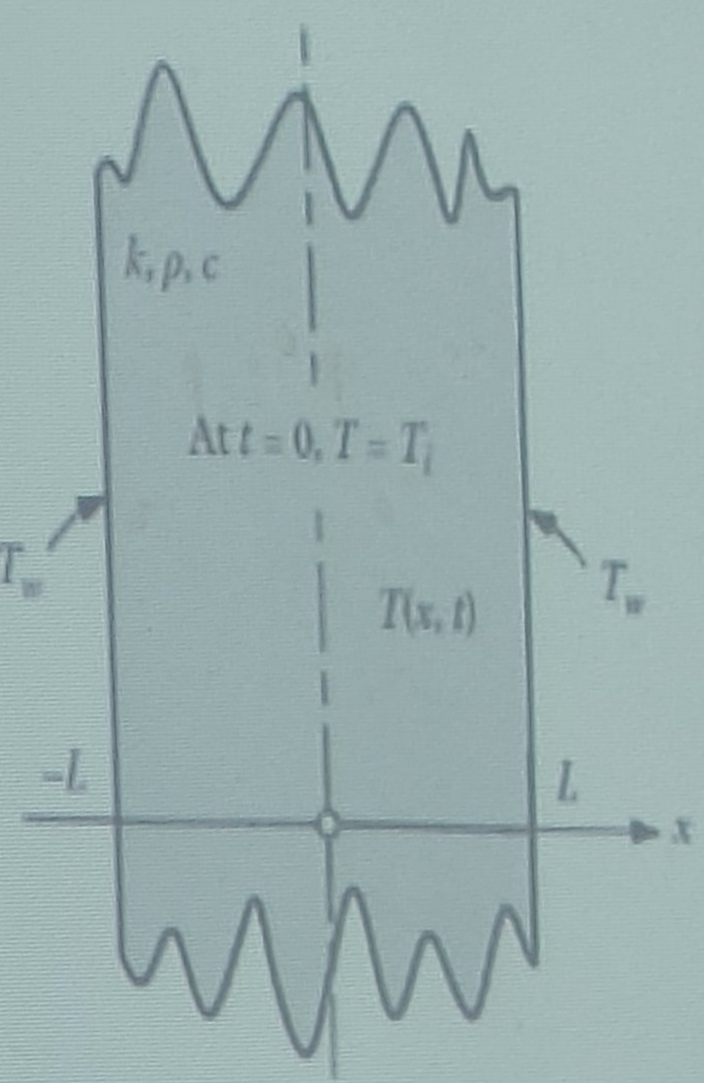
Laplace Transforms: Heat Conduction Problems

1D Plane Wall:



Laplace transformed PDE: $$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T} = -\frac{T_i}{\alpha}$$

Laplace transform of BC: $$\frac{\bar{T}(0,p)}{dx} = 0 \quad \text{and} \quad \bar{T}(L,p) = \frac{T_w}{p}$$

The solution of the ODE:

$$\bar{T}(x,p) = C_1 \cosh mx + C_2 \sinh mx + \frac{T_i}{p}$$

where $m^2 = p/\alpha$, and C_1 and C_2 are the constants of integration to be determined from the BC.

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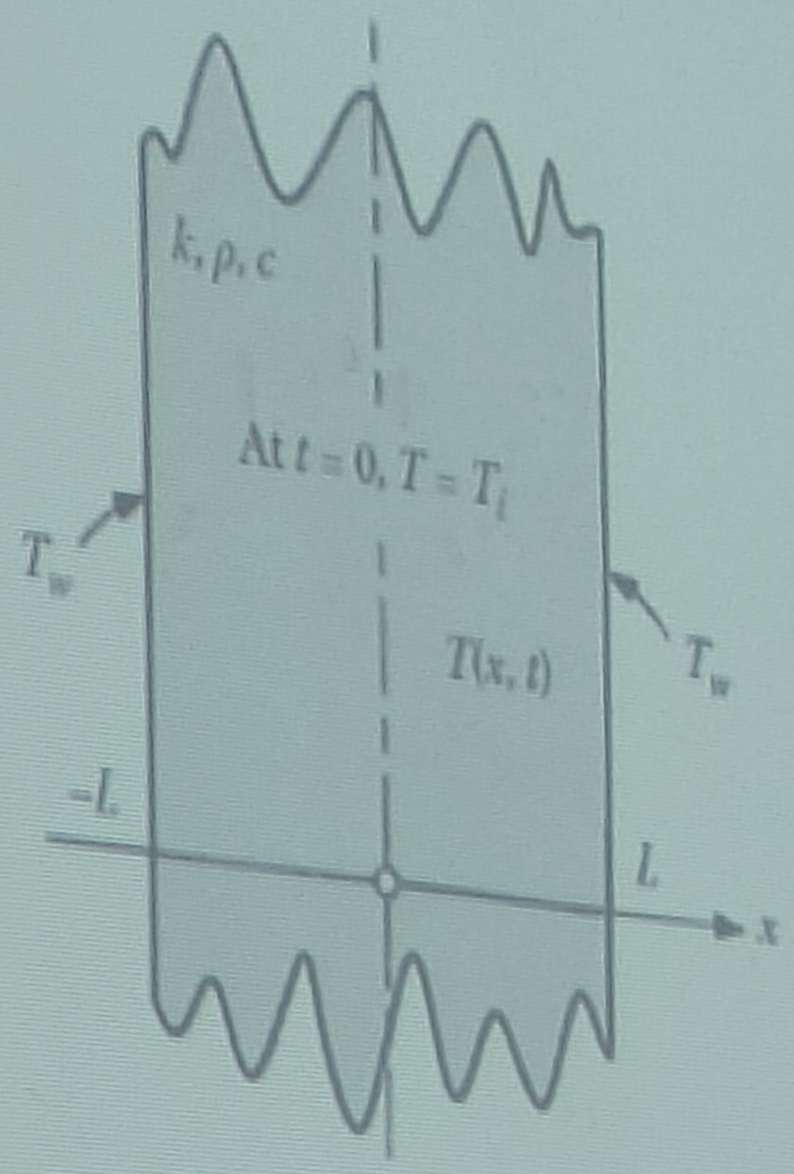
1D Plane Wall:

Heat Conduction Problems



Thus, the solution for $\bar{T}(x,p)$ may now be written as

$$\bar{T}(x,p)=\frac{T_i}{p}-(T_i-T_w)\frac{\cosh mx}{p\cosh mL}$$

Inversion:

$$T(x,p)=T_i\mathcal{L}^{-1}\left\{\frac{1}{p}\right\}-(T_i-T_w)\mathcal{L}^{-1}\left\{\frac{\cos mx}{p\cosh mL}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\right\}=1$$

$$\mathcal{L}^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\}=1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2\alpha t}{4L^2}\right]\cos\frac{(2n-1)\pi x}{2L}$$

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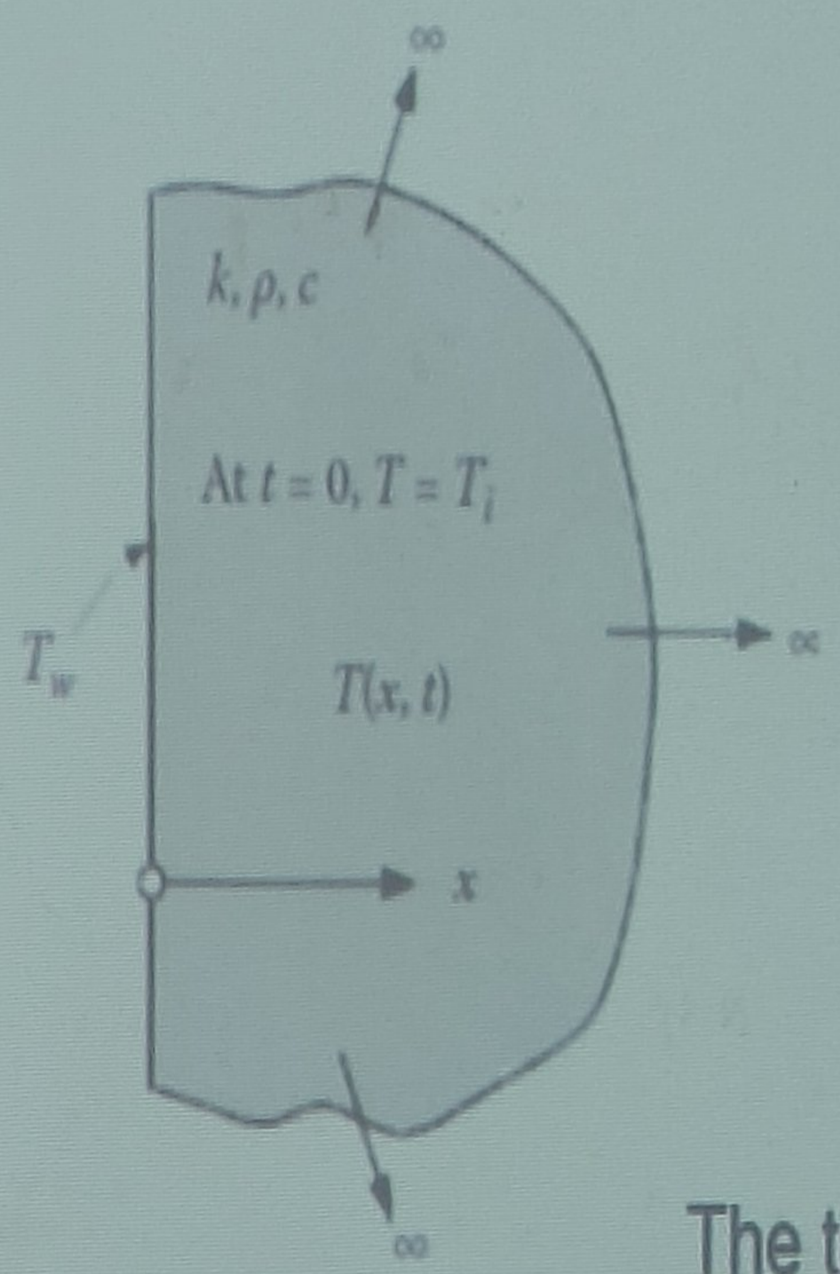
Laplace Transforms: Heat Conduction Problems

Semi-infinite Solid:

Define: $\theta(x,t) = T(x,t) - T_i$



$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\theta(0,t) = T_w - T_i = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = 0$$

The transformed differential equation and BC are:

$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\bar{\theta}(0,p) = \frac{\theta_w}{p} \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0$$

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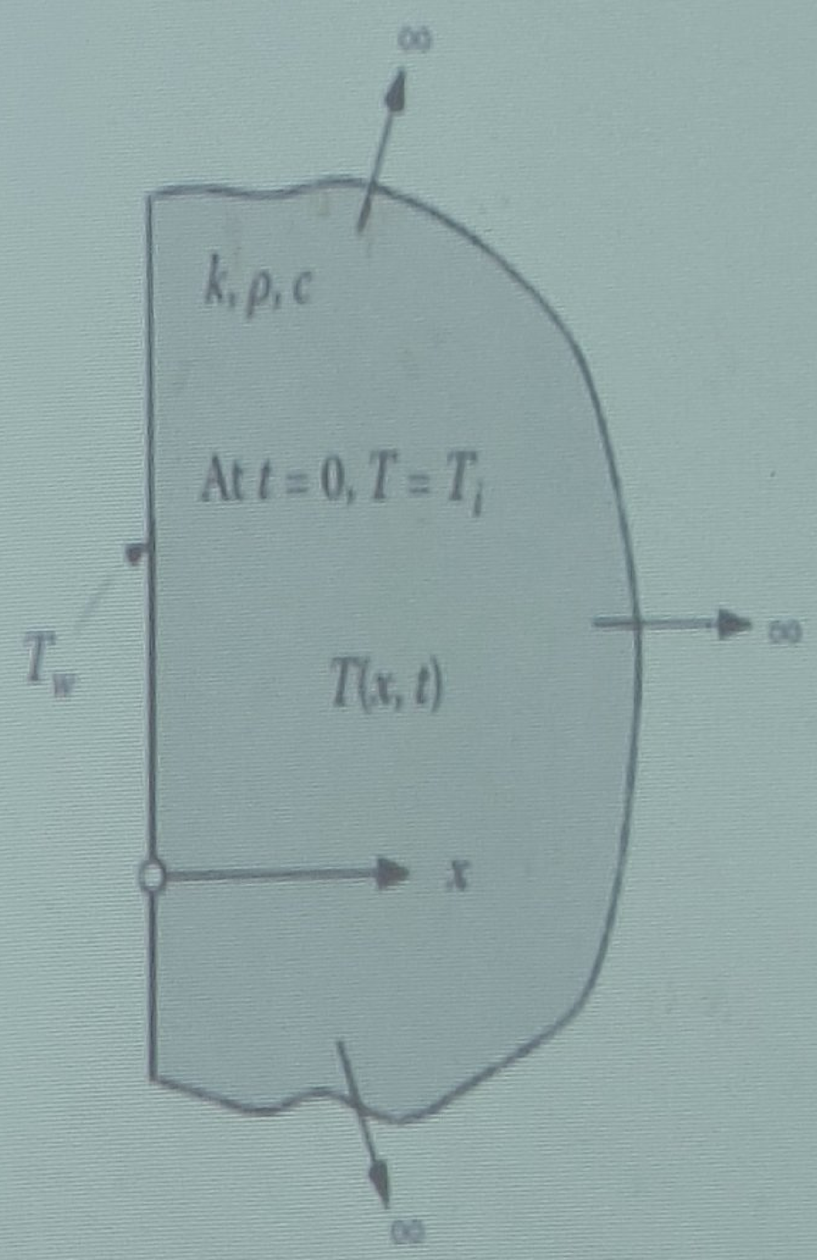
Laplace Transforms: Heat Conduction Problems

Semi-infinite Solid:

$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0$$



$$\bar{\theta}(0,p) = \frac{\theta_w}{p} \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0$$

The general solution: $\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx}$

where C_1 and C_2 are the constants of integration, and $m^2 = p/\alpha$.

$\lim_{x\to\infty}\bar{\theta}(x,p) = 0$ gives $C_2 = 0$

$$\frac{\bar{\theta}(x,p)}{\theta_w} = \frac{e^{-mx}}{p} = \frac{-e^{x\sqrt{p/\alpha}}}{p}$$

$\bar{\theta}(0,p) = \dfrac{\theta_w}{p}$ gives $C_1 = \theta_w/p$.

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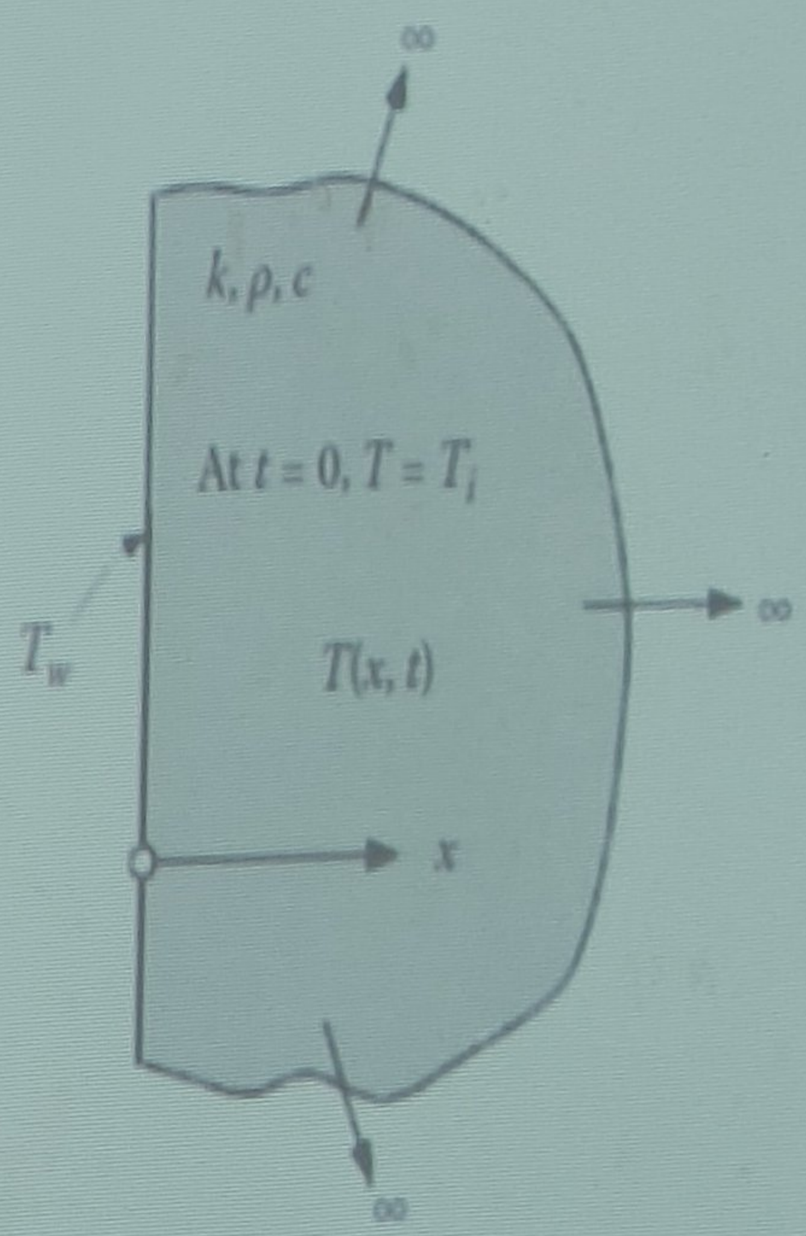
Laplace Transforms: Heat Conduction Problems

Semi-infinite Solid:

$$\frac{\bar{\theta}(x,p)}{\theta_w} = \frac{e^{-mx}}{p} = \frac{-e^{x\sqrt{p/\alpha}}}{p}$$



Inversion:

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t)-T_i}{T_w-T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Equivalently

$$\frac{T(x,t)-T_w}{T_i-T_w} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

because erfc(x) = 1 – erf(x).

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Duhamel's Principle

There are many engineering problems such as heat transfer in internal combustion engine walls and space reentry problems in which the boundary condition functions are *time dependent*.

In nuclear reactor fuel elements during power transients, the energy generation rate in the fuel elements varies with time.

Duhamel's method is a superposition technique that can be used to obtain the solution of linear heat conduction problems with time-dependent boundary conditions and/or time dependent internal energy generation from the solution of the same problem with time independent boundary conditions and/or time-independent internal energy generation.

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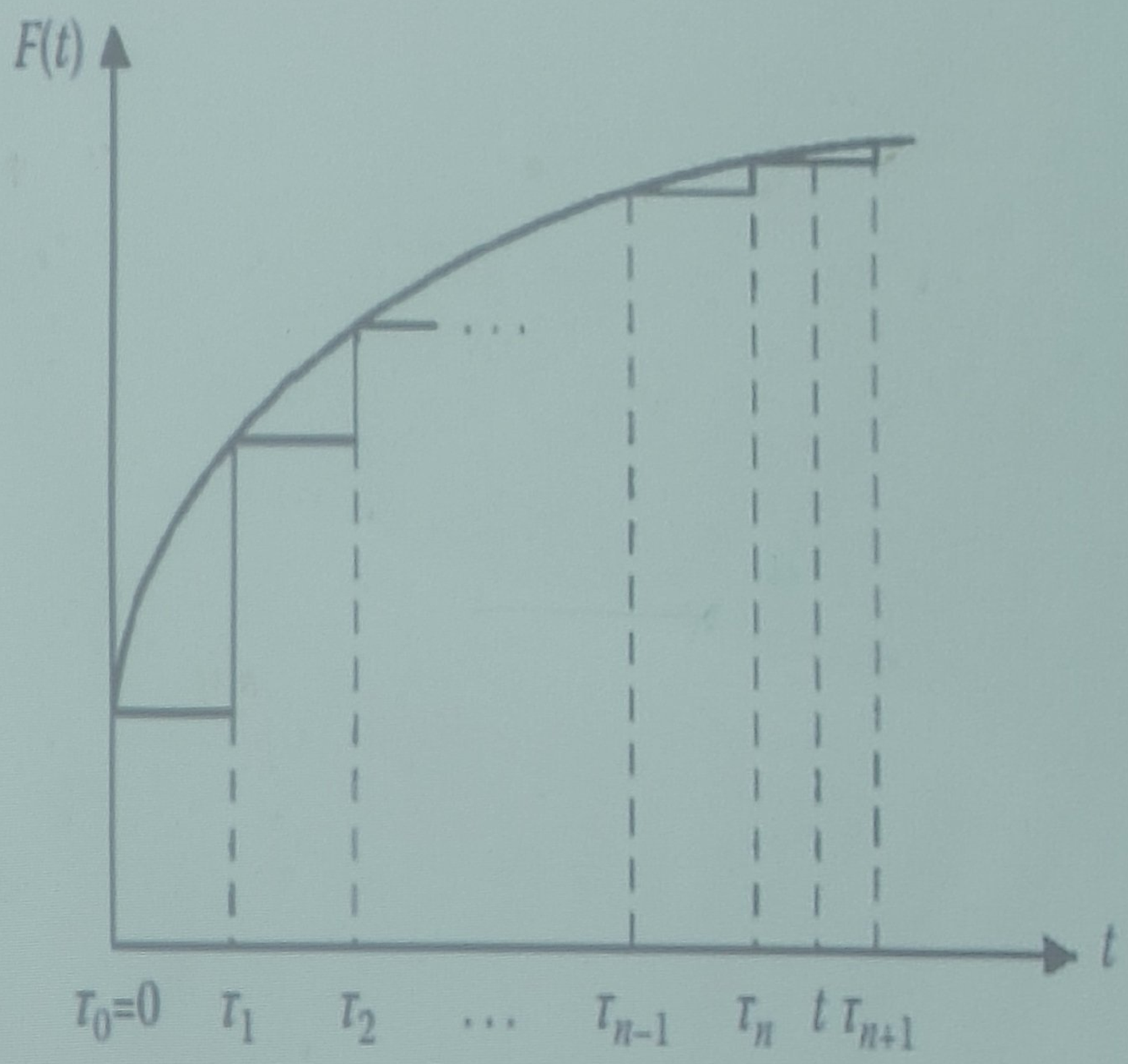
Duhamel's Principle



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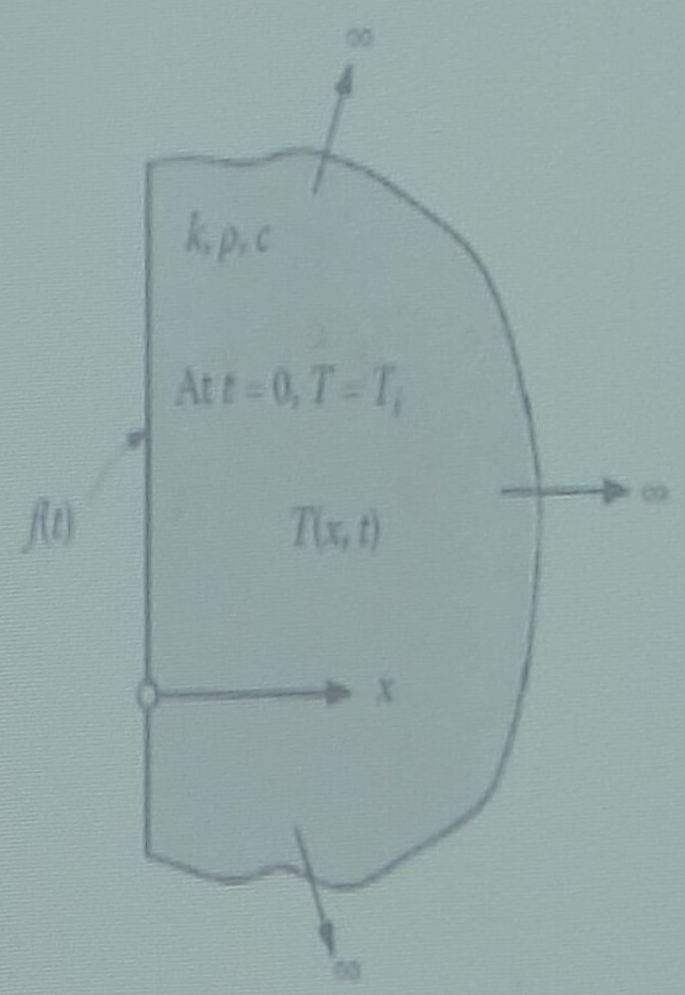
Duhamel's Principle

Consider a semi-infinite solid, $0 \le x < \infty$, initially at a uniform temperature Ti.



$$\theta(x,t) = T(x,t) - T_i$$

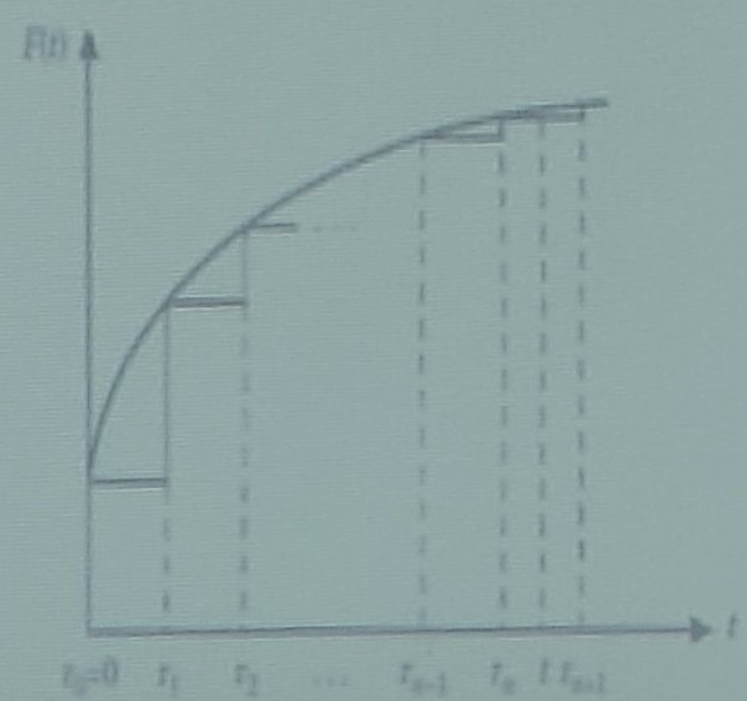
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\theta(0,t) = F(t) \quad \text{and} \quad \lim_{x \to \infty} \theta(x,t) = 0$$

$$\theta(x,0) = 0$$

$$F(t) = f(t) - T_i.$$

Since the problem is linear, it is possible to construct an approximate solution for $\theta(x, t)$ by first merely breaking the function $F(t)$ up into a number of constant temperature steps as illustrated in Figure and then superposing the solutions over each step.



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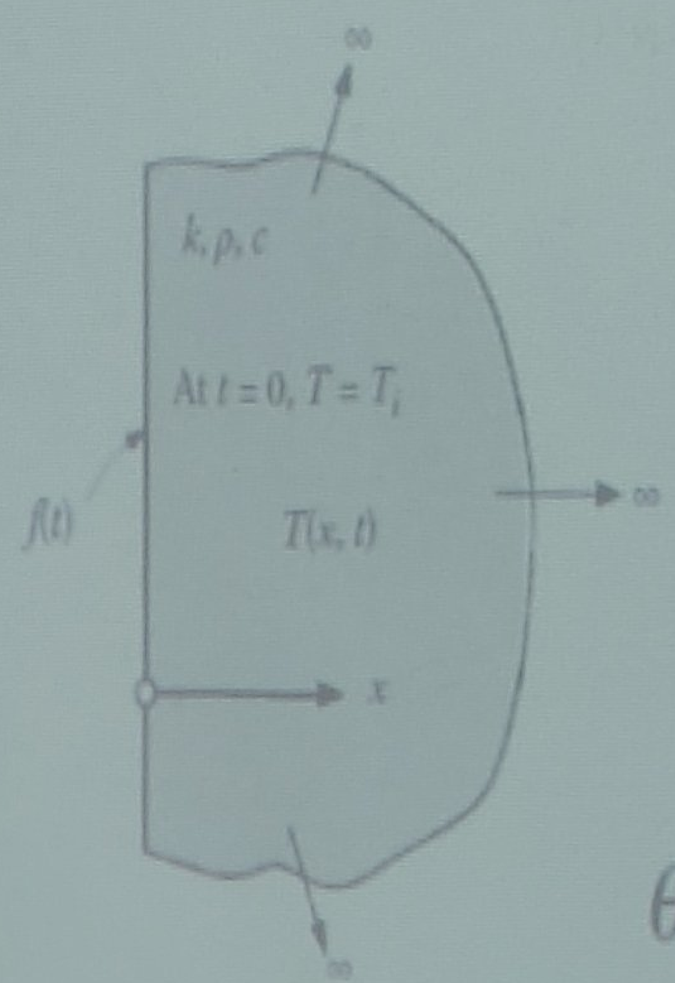
Duhamel's Principle

Accordingly, we now approximate $F(t)$ over a time interval $(0, t)$ a

$$F(t) \cong F(0)H(t) + \sum_{i=1}^{n} \underbrace{[F(\tau_i) - F(\tau_i - 1)]}_{f(\tau_i) - f(\tau_i - 1)} H(t - \tau_i)$$

where $\tau_0 = 0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n < t < \tau_{n+1}$, and $H(t)$ is the so-called *Heaviside step function*, defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



With the approximated BC at x = 0, the solution to the problem at any instant t can now be written in the form:

$$\theta(x,t) \cong F(0)\phi(x,t) + \sum_{i=1}^{n} [f(\tau_i) - f(\tau_i - 1)]\phi(x, t - \tau_i)$$

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Direct Heat Conduction Problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=\rho c_p\frac{\partial T}{\partial t} \text{ in } 0<x<L, \text{ for } t>0$$

$$-k\frac{\partial T}{\partial x}=q(t) \text{ at } x=0, \text{ for } t>0$$

$$T=T_L \text{ at } x=L, \text{ for } t>0$$

$$T=F(x) \text{ for } t=0, \text{ in } 0<x<L$$

Find T(x,t)

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Inverse Heat Conduction Problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=\rho c_p\frac{\partial T}{\partial t} \text{ in } 0<x<L, \text{ for } 0<t\le t_f$$

$$-k\frac{\partial T}{\partial x}=q(t)=?\,(\text{unknown}) \text{ at } x=0, \text{ for } 0<t\le t_f$$

$$T=T_L \text{ at } x=L, \text{ for } 0<t\le t_f$$

$$T=F(x) \text{ for } t=0 \text{ in } 0<x<L$$

with temperature measurements taken at an interior location x_{meas}, at different times t_i, given by:

$$Y\left(x_{meas},t_i\right)\equiv Y_i \text{ at } x=x_{meas}, \text{ for } t=t_i\left(i=1,2,\ldots,I\right)$$

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Inverse Heat Conduction Problem

Another example of parameter estimation is the recovering of unknown constant thermo-physical properties, such as the thermal conductivity k or the volumetric heat capacity ρc_p, appearing in the PDE.

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