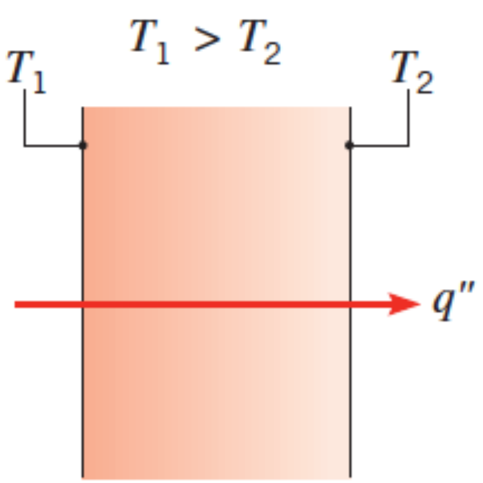
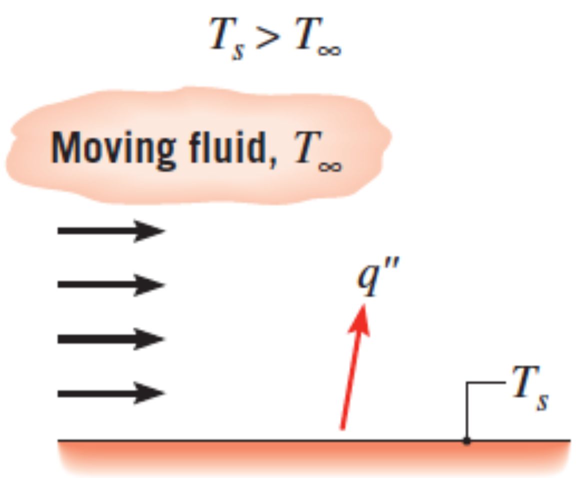
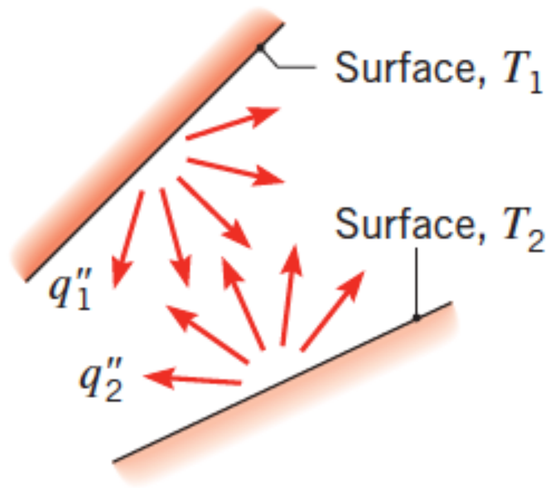
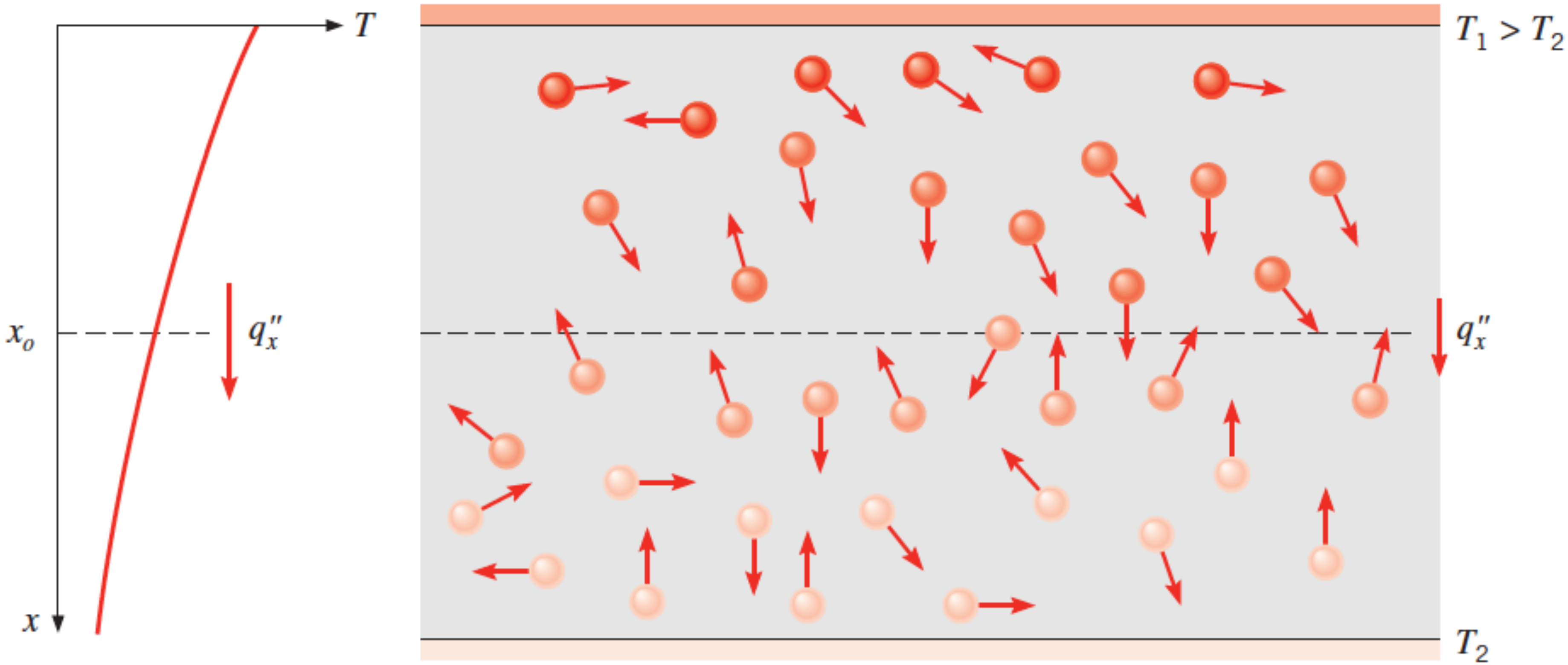
Introduction

- Thermodynamics deals with the end states of the process during which an interaction occurs.
- Provides no information concerning the nature of the interaction or the time rate at which it occurs.
- modes of heat transfer
- heat transfer rates
- **Heat transfer (or heat) is thermal energy in transit due to a spatial temperature difference.**

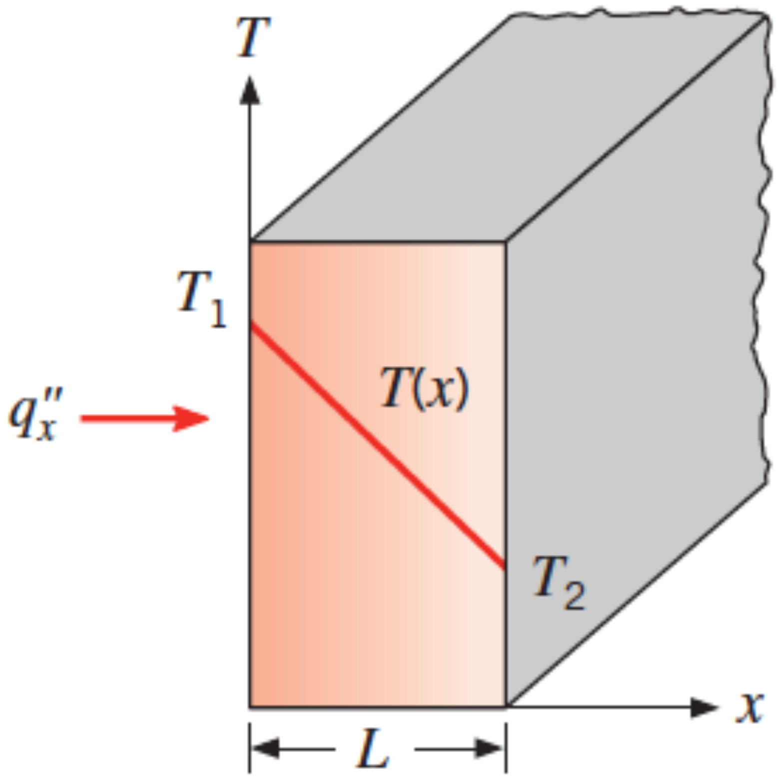
Conduction through a solid or a stationary fluid	Convection from a surface to a moving fluid	Net radiation heat exchange between two surfaces
		

Conduction

- Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.



Rate equation

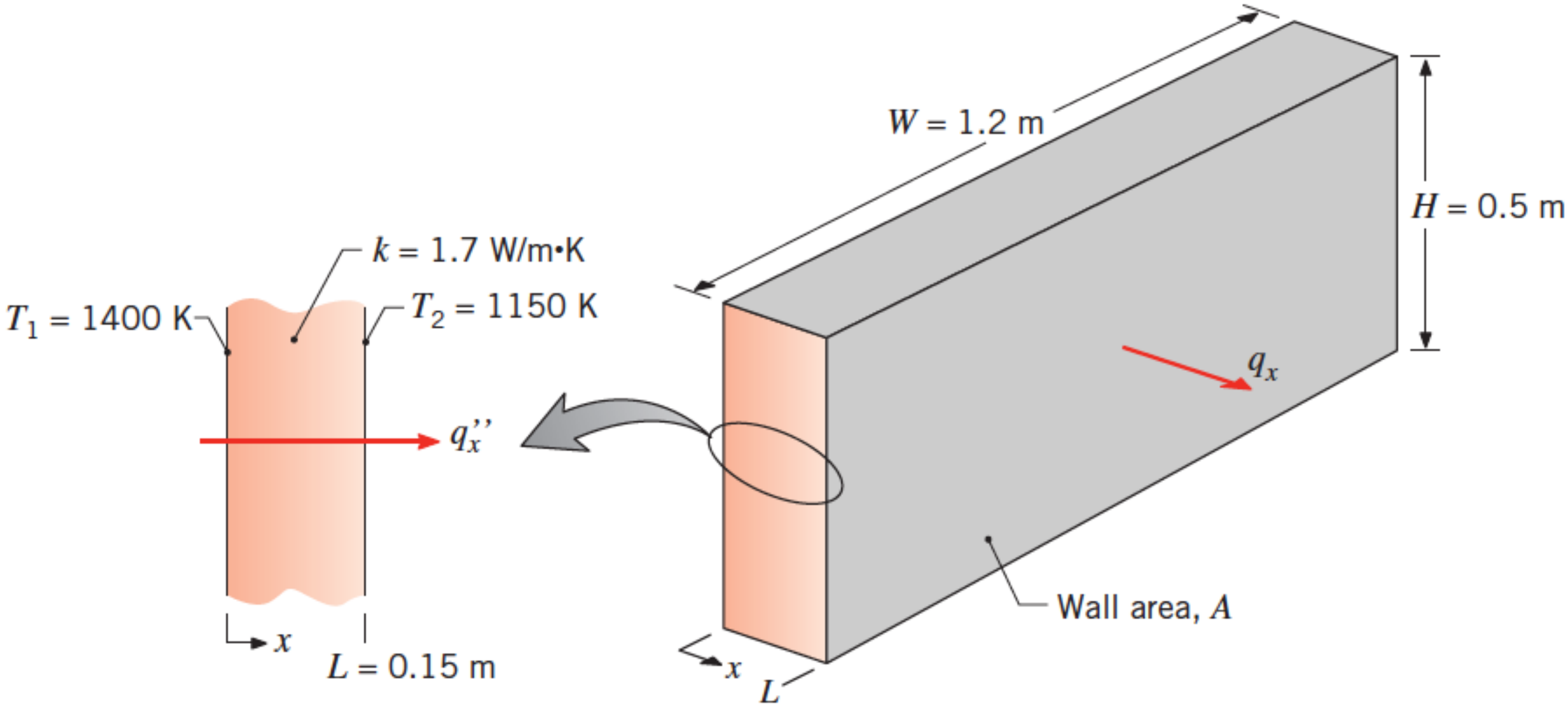


$$q''_x = -k\frac{dT}{dx}$$

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

$$q''_x = -k\frac{T_2 - T_1}{L}$$

$$\boxed{q''_x = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L}}$$



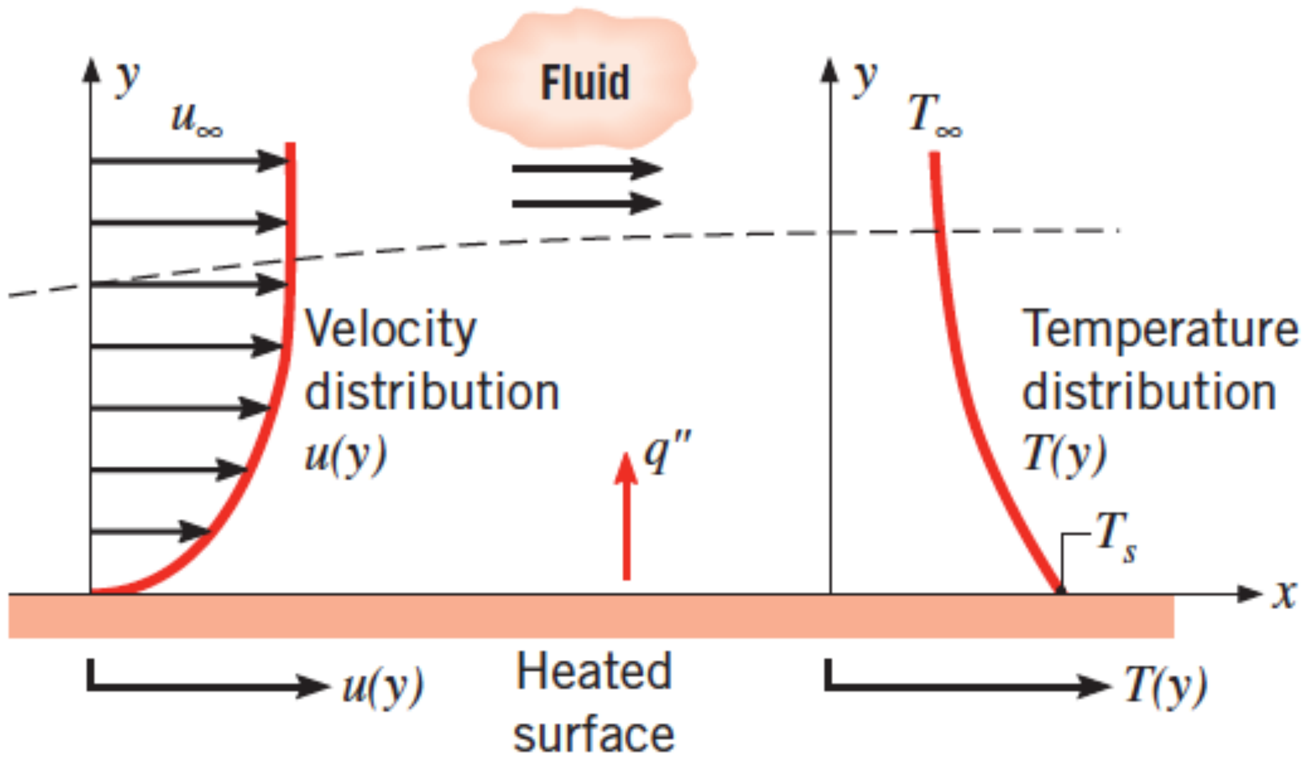
$k = 1.7$ W/m•K
$T_1 = 1400$ K
$T_2 = 1150$ K
q_x''
x
$L = 0.15$ m
$W = 1.2$ m
$H = 0.5$ m
q_x
Wall area, A
x
L

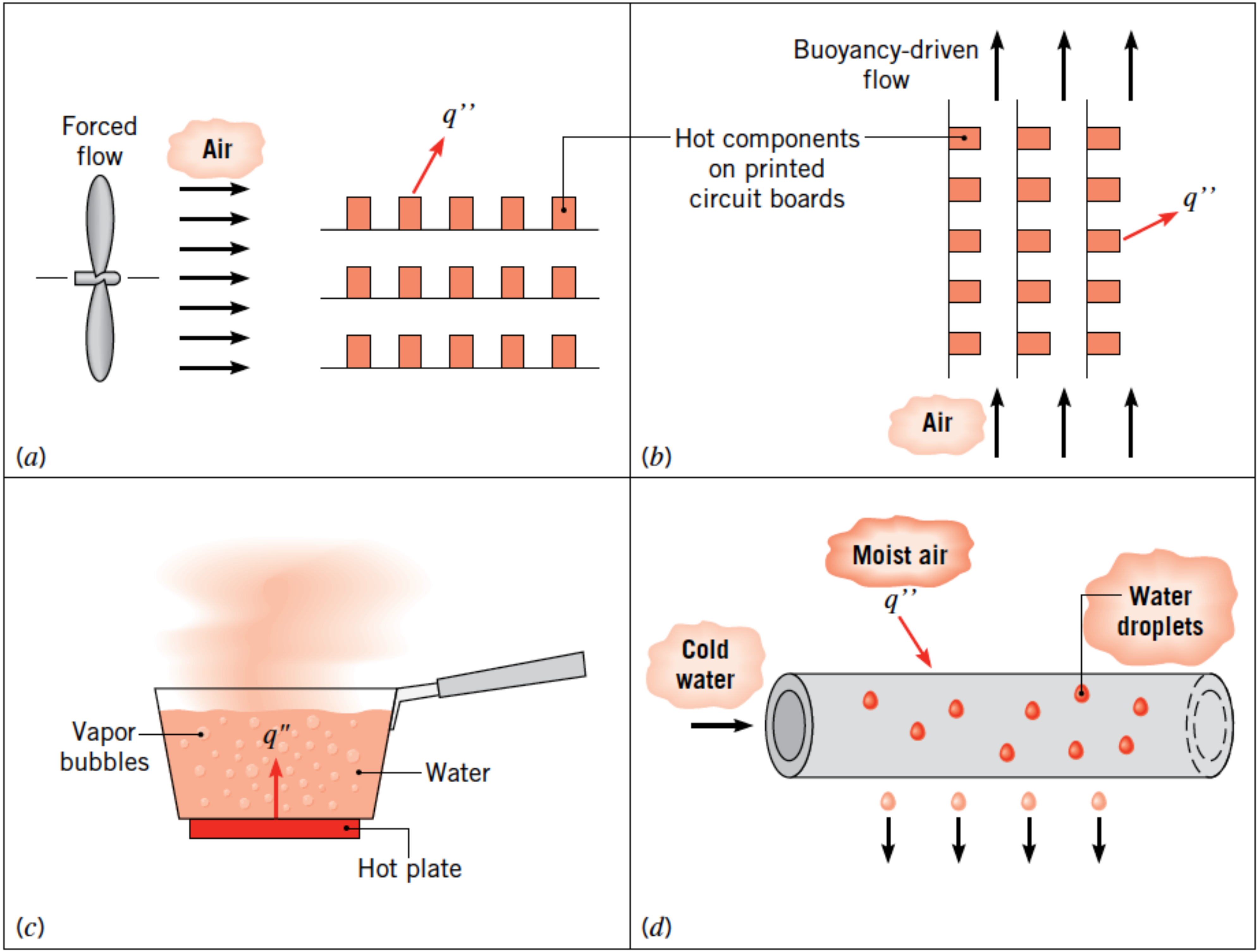
$$q''_x = k\,\frac{\Delta T}{L} = 1.7\ \mathrm{W/m\cdot K} \times \frac{250\ \mathrm{K}}{0.15\ \mathrm{m}} = 2833\ \mathrm{W/m^2}$$

$$q_x = (HW)\,q''_x = (0.5\ \mathrm{m} \times 1.2\ \mathrm{m})\ 2833\ \mathrm{W/m^2} = 1700\ \mathrm{W}$$

Covection

- random molecular motion (diffusion)
- bulk or macroscopic motion of the fluid
- **Convection & Advection**





Forced flow
Air
q''
Hot components on printed circuit boards
(a)
Buoyancy-driven flow
q''
Air
(b)
Vapor bubbles
q''
Water
Hot plate
(c)
Moist air
q''
Water droplets
Cold water
(d)

Rate equation

$$q'' = h(T_s - T_\infty)$$

Radiation

- Thermal radiation is energy emitted by matter that is at a nonzero temperature.
- The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons).
- While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum.

Heat Transfer

Lecture - II

Conduction

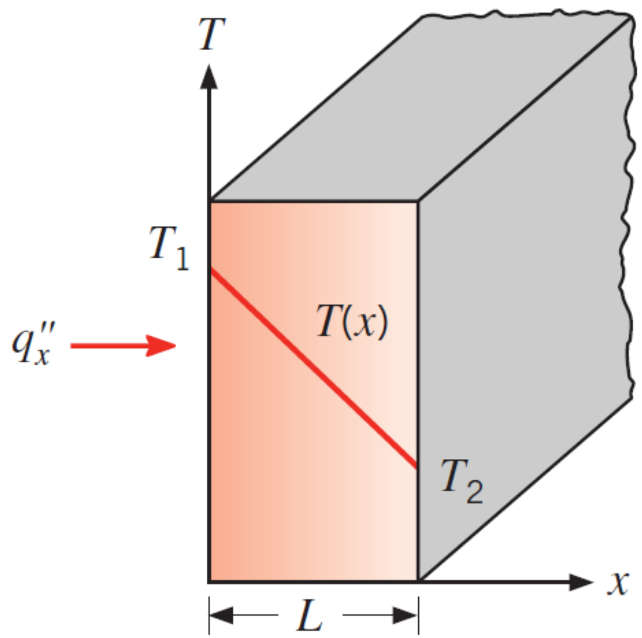
- Transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles

- Equations used to compute the amount of energy being transferred per unit time

- Fourier's Law

$$q''_x = -k\frac{dT}{dx}$$

$$\boxed{q''_x = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L}}$$



The wall of an industrial furnace is constructed from 0.15-m-thick fireclay brick having a thermal conductivity of 1.7 W/m·K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall that is 0.5 m × 1.2 m on a side?

Assumptions:

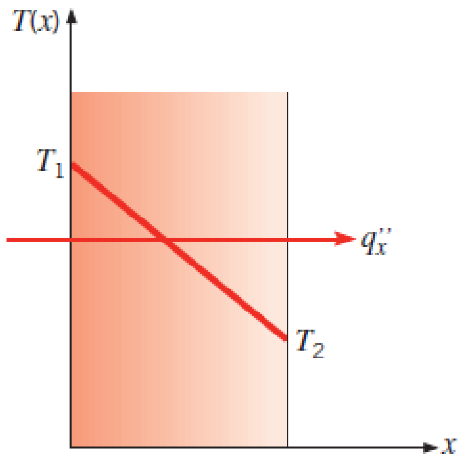
1. Steady-state conditions.
2. One-dimensional conduction through the wall.
3. Constant thermal conductivity.

$$q''_x = k\frac{\Delta T}{L} = 1.7\ \mathrm{W/m \cdot K} \times \frac{250\ \mathrm{K}}{0.15\ \mathrm{m}} = 2833\ \mathrm{W/m^2}$$

$$q_x = (HW)\, q''_x = (0.5\ \mathrm{m} \times 1.2\ \mathrm{m})\, 2833\ \mathrm{W/m^2} = 1700\ \mathrm{W}$$

The Conduction Rate Equation

- Fourier's law is phenomenological

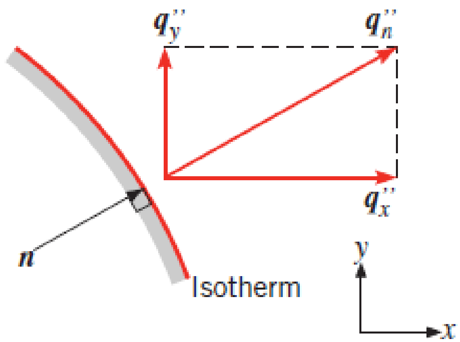


$$q_x \propto A \frac{\Delta T}{\Delta x}$$

$$q_x = kA \frac{\Delta T}{\Delta x}$$

$$q_x = -kA \frac{dT}{dx}$$

$$q''_x = \frac{q_x}{A} = -k \frac{dT}{dx}$$



$$\boldsymbol{q}'' = \boldsymbol{i}q''_x + \boldsymbol{j}q''_y + \boldsymbol{k}q''_z$$

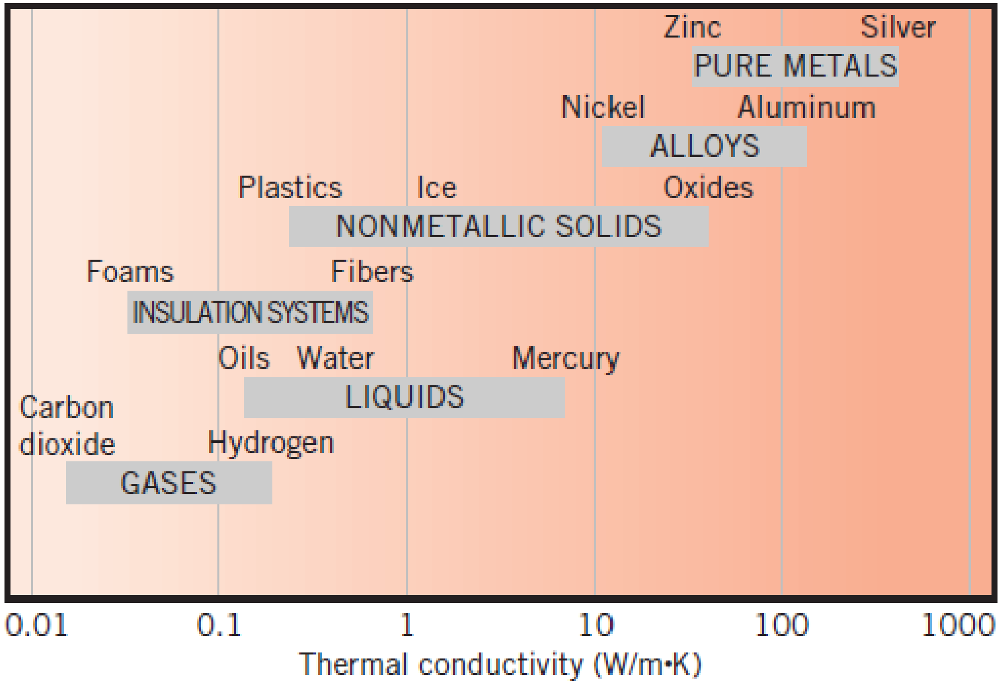
$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial x} + \boldsymbol{j}\frac{\partial T}{\partial y} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

$$q''_x = -k\frac{\partial T}{\partial x} \qquad q''_y = -k\frac{\partial T}{\partial y} \qquad q''_z = -k\frac{\partial T}{\partial z}$$

Thermal Conductivity

- In general, the thermal conductivity of a solid is larger than that of a liquid, which is larger than that of a gas

$$k_x \equiv -\frac{q''_x}{(\partial T/\partial x)}$$



Other Properties

- Transport
  - thermal conductivity
  - kinematic viscosity
- Thermodynamic
  - Density
  - specific heat
- Volumetric heat capacity
  - ability of a material to store thermal energy

Thermal diffusivity

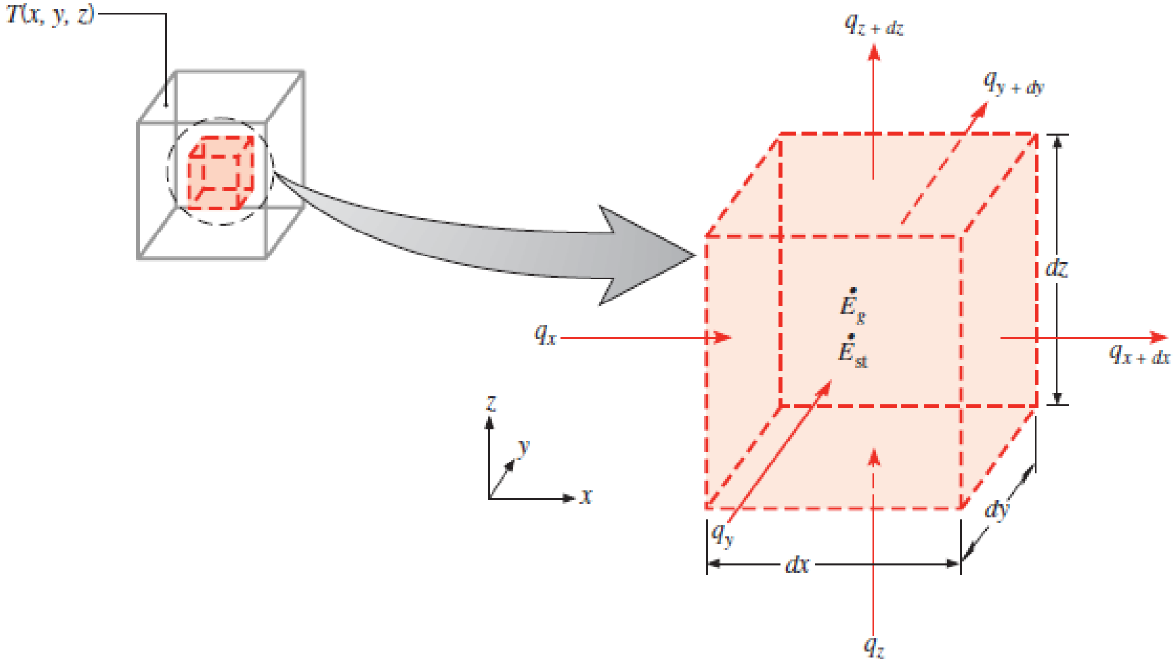
- Ratio of the thermal conductivity to the heat capacity

$$\alpha = \frac{k}{\rho c_p}$$

- Measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy
- Materials of large α will respond quickly to changes in their thermal environment, whereas materials of small α will respond more sluggishly

- Major objective in a conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries

- Consider a homogeneous medium within which there is no bulk motion



$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

$$E_g = q\, dx\, dy\, dz$$

q is the rate at which energy is generated per unit volume of the medium

$$E_{\mathrm{st}} = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$E_{\mathrm{in}} + E_g - E_{\mathrm{out}} = E_{\mathrm{st}}$$

$$q_x + q_y + q_z + q\, dx\, dy\, dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + q\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$q_x = -k\,dy\,dz\,\frac{\partial T}{\partial x}$$

$$q_y = -k\,dx\,dz\,\frac{\partial T}{\partial y}$$

$$q_z = -k\,dx\,dy\,\frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q = \rho c_p \frac{\partial T}{\partial t}$$

Heat diffusion equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q = 0$$

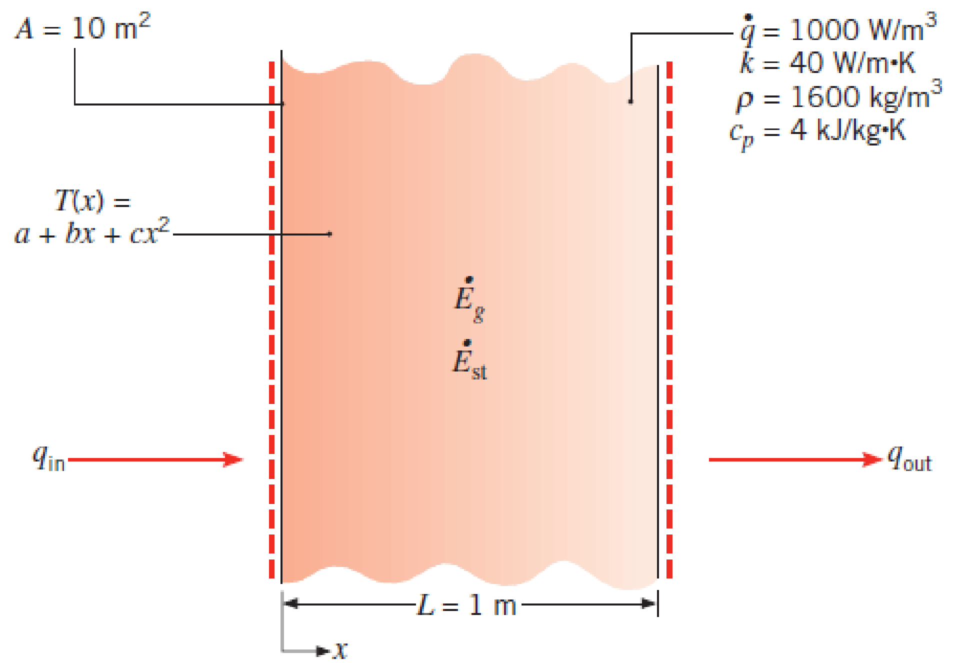
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in degrees Celsius and x is in meters, while $a = 900°\text{C}$, $b = -300°\text{C/m}$, and $c = -50°\text{C/m}^2$. A uniform heat generation, $\dot{q} = 1000\ \text{W/m}^3$, is present in the wall of area $10\ \text{m}^2$ having the properties $\rho = 1600\ \text{kg/m}^3$, $k = 40\ \text{W/m·K}$, and $c_p = 4\ \text{kJ/kg·K}$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1$ m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0$, 0.25, and 0.5 m.



Assumptions:

1. One-dimensional conduction in the x-direction.
2. Isotropic medium with constant properties.
3. Uniform internal heat generation, $\dot{q}$ (W/m^3).

$$\boxed{q_x = -kA\frac{dT}{dx}}$$

$$q_{\text{in}} = q_x(0) = -kA\left.\frac{\partial T}{\partial x}\right|_{x=0} = -kA(b + 2cx)_{x=0}$$

$$q_{\text{in}} = -bkA = 300°\text{C/m} \times 40\ \text{W/m}\cdot\text{K} \times 10\ \text{m}^2 = 120\ \text{kW}$$

$$q_{\text{out}} = q_x(L) = -kA\left.\frac{\partial T}{\partial x}\right|_{x=L} = -kA(b + 2cx)_{x=L}$$

$$q_{\text{out}} = -(b + 2cL)kA = -[-300°\text{C/m}$$

$$+ 2(-50°\text{C/m}^2) \times 1\ \text{m}] \times 40\ \text{W/m}\cdot\text{K} \times 10\ \text{m}^2 = 160\ \text{kW}$$

$$E_{\text{in}} + E_g - E_{\text{out}} = E_{\text{st}}$$

where $E_g = qAL$, it follows that

$$E_{\text{st}} = E_{\text{in}} + E_g - E_{\text{out}} = q_{\text{in}} + qAL - q_{\text{out}}$$

$$E_{\text{st}} = 120\ \text{kW} + 1000\ \text{W/m}^3 \times 10\ \text{m}^2 \times 1\ \text{m} - 160\ \text{kW}$$

$$E_{\text{st}} = -30\ \text{kW}$$

$$\boxed{\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{q}{\rho c_p}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(b + 2cx) = 2c = 2(-50^\circ\text{C/m}^2) = -100^\circ\text{C/m}^2$$

$$\frac{\partial T}{\partial t} = \frac{40\ \mathrm{W/m \cdot K}}{1600\ \mathrm{kg/m^3} \times 4\ \mathrm{kJ/kg \cdot K}} \times (-100^{\circ}\mathrm{C/m^2})$$

$$+ \frac{1000\ \mathrm{W/m^3}}{1600\ \mathrm{kg/m^3} \times 4\ \mathrm{kJ/kg \cdot K}}$$

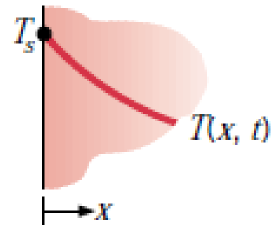
$$\frac{\partial T}{\partial t} = -6.25 \times 10^{-4\circ}\mathrm{C/s} + 1.56 \times 10^{-4\circ}\mathrm{C/s}$$

$$= -4.69 \times 10^{-4\circ}\mathrm{C/s}$$

Boundary and Initial Conditions

1. Constant surface temperature

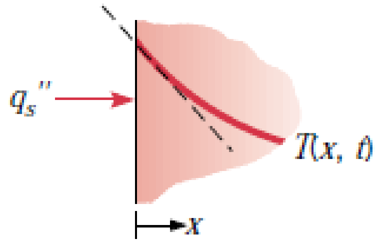
$$T(0, t) = T_s$$



2. Constant surface heat flux

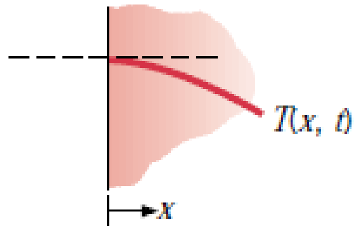
(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$



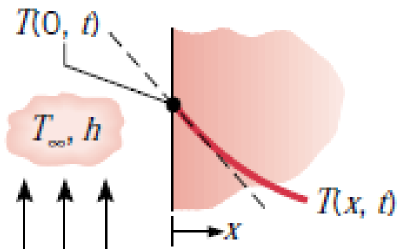
(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

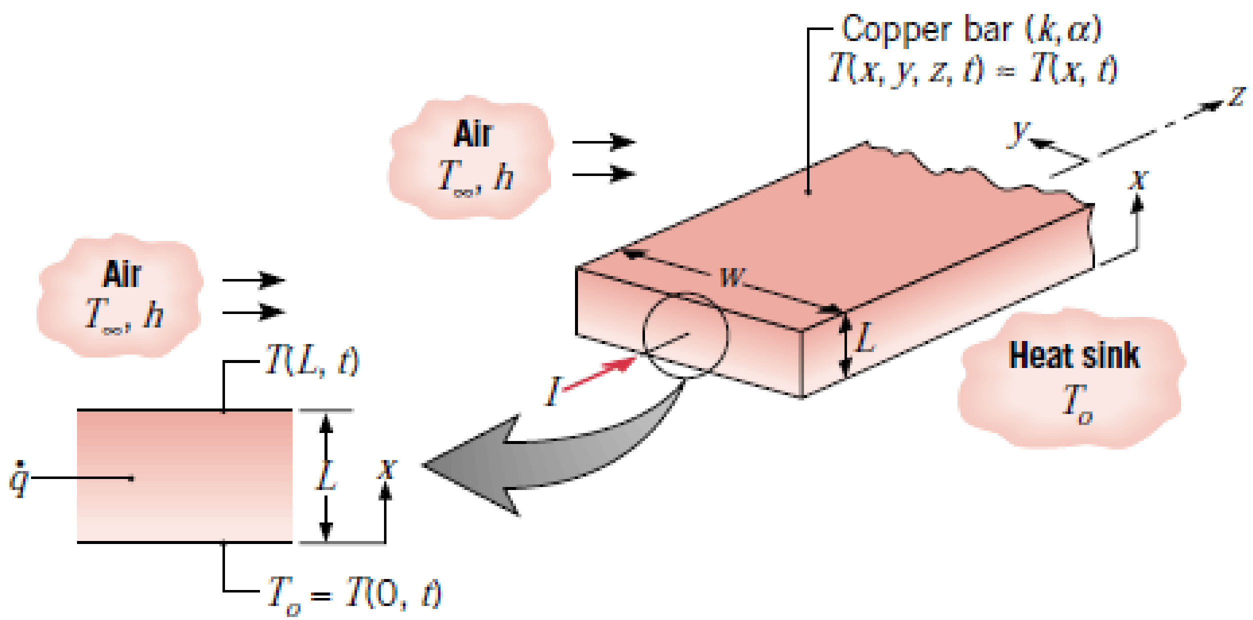


3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)]$$



A long copper bar of rectangular cross section, whose width w is much greater than its thickness L, is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o. Suddenly, an electric current is passed through the bar and an airstream of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o. Obtain the differential equation and the boundary and initial conditions that could be solved to determine the temperature as a function of position and time in the bar.



Assumptions:

1. Since the bar is long and $w \gg L$, end and side effects are negligible and heat transfer within the bar is primarily one dimensional in the x direction.
2. Uniform volumetric heat generation, $\dot{q}$.
3. Constant properties.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

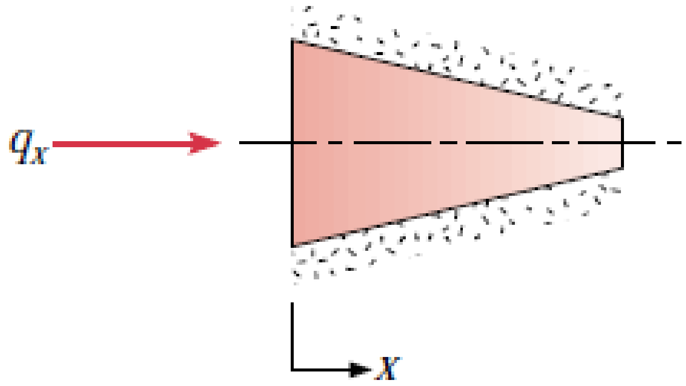
$$T(0, t) = T_o$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h[T(L, t) - T_\infty]$$

$$T(x, 0) = T_o$$

Lecture - III

Assume steady-state, one-dimensional heat conduction through the symmetric shape shown.



Assuming that there is no internal heat generation, derive an expression for the thermal conductivity $k(x)$ for these conditions: $A(x) = (1 - x)$, $T(x) = 300(1 - 2x - x^3)$, and $q = 6000$ W, where A is in square meters, T in kelvins, and x in meters.

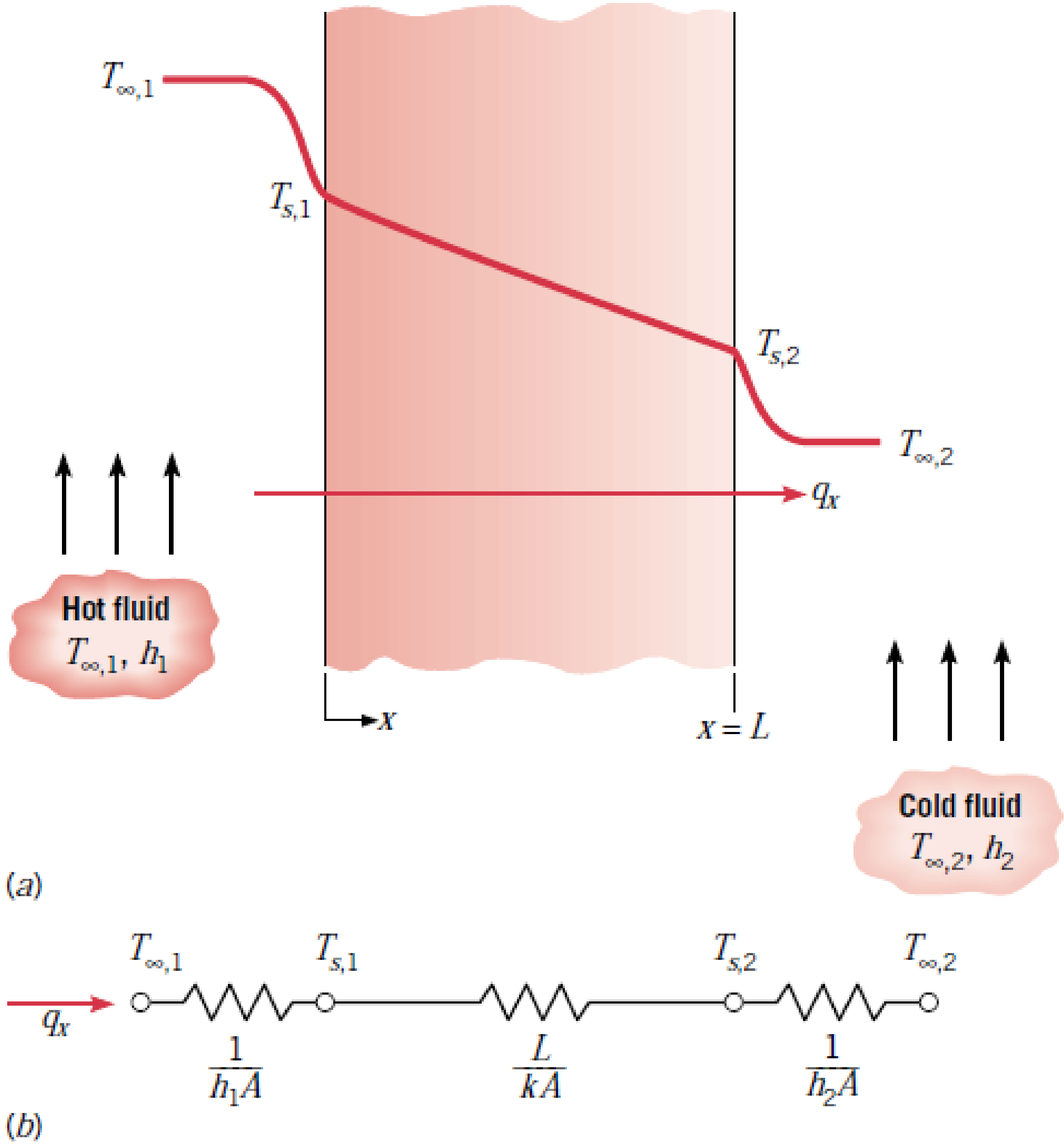
$$q_x = \text{Constant} \neq f(x)$$

$$q_x = -k\, A_x \frac{dT}{dx}$$

$$6000\text{W} = -k \cdot (1\text{-}x)\,\text{m}^2 \cdot \frac{d}{dx}\left[300\left(1 - 2x\text{-}x^3\right)\right]\frac{\text{K}}{\text{m}}$$

$$k = \frac{-6000}{(1\text{-}x)\left[300\left(-2 - 3x^2\right)\right]} = \frac{20}{(1-x)\left(2+3x^2\right)}$$

One-Dimensional, Steady-State Conduction



$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

$$T(x) = C_1 x + C_2$$

$$T(0) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

$$T_{s,1} = C_2$$

$$T_{s,2} = C_1 L + C_2 = C_1 L + T_{s,1}$$

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

$$q''_x = \frac{q_x}{A} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right)$$

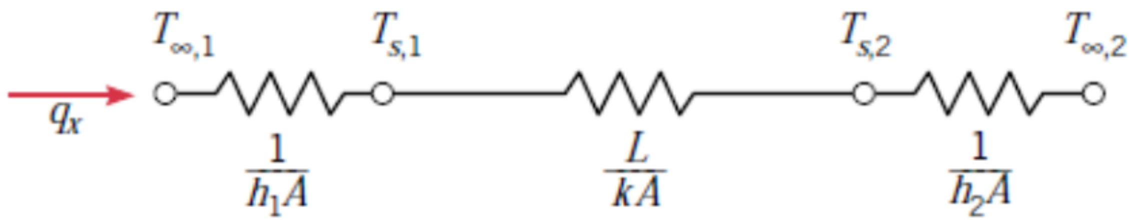
Thermal Resistance

- resistance as the ratio of a driving potential to the corresponding transfer rate

$$R_{t,\,\mathrm{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$

$$q = hA(T_s - T_\infty)$$

$$R_{t,\,\mathrm{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

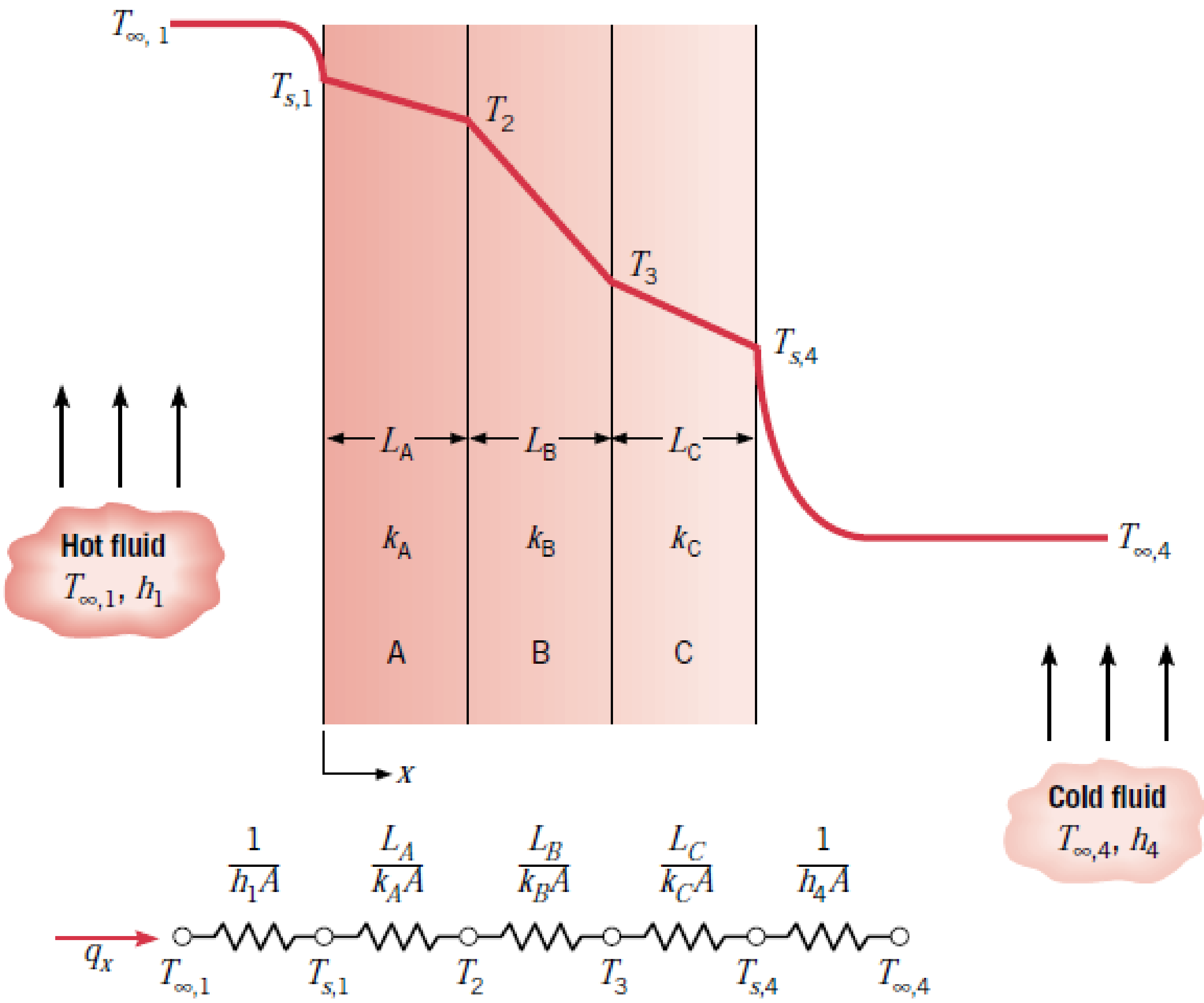


$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\mathrm{tot}}}$$

$$R_{\mathrm{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

The Composite Wall



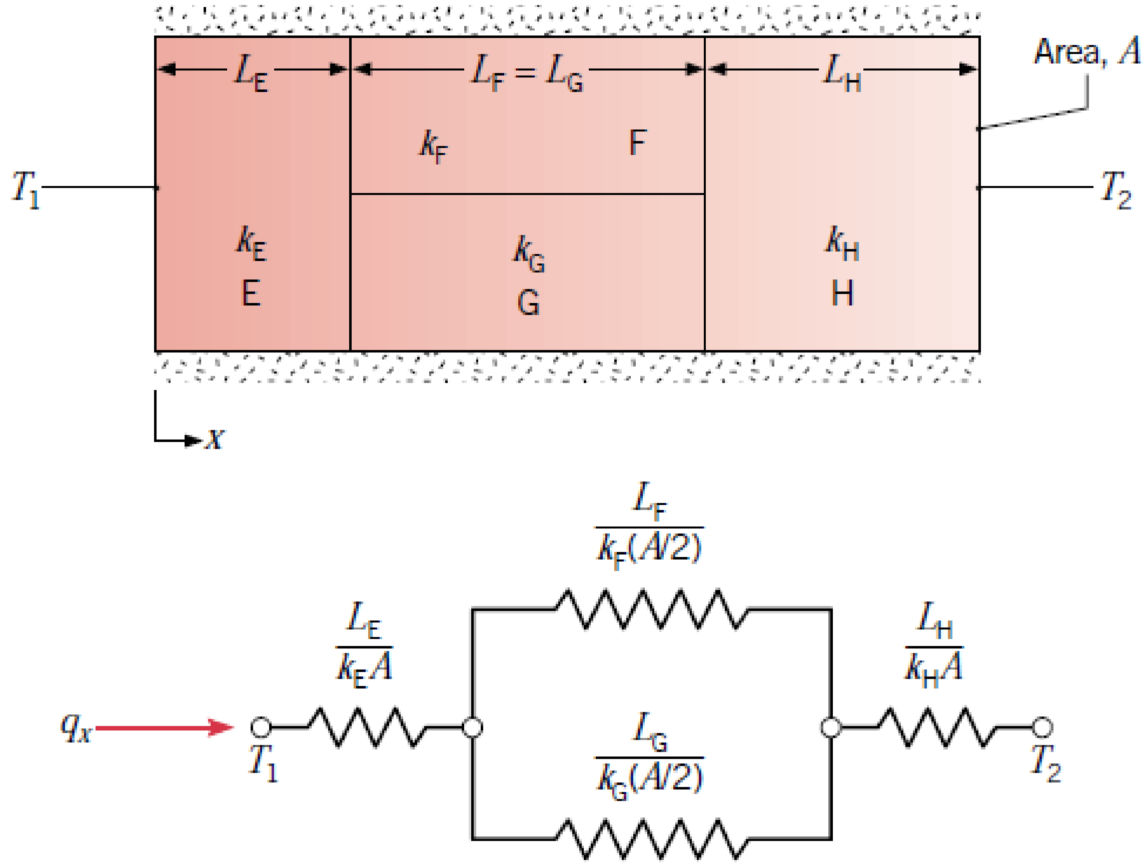
$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_3}{(L_B/k_B A)} = \cdots$$

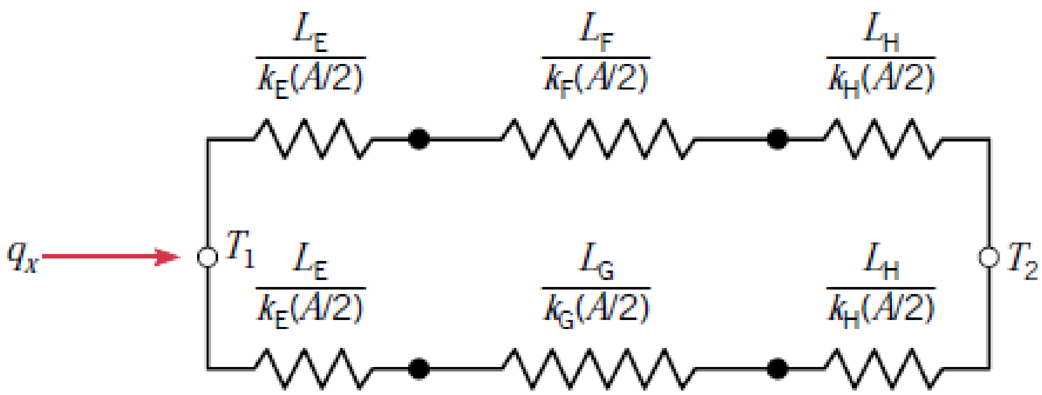
$$\boxed{q_x \equiv UA\,\Delta T}$$

$$U = \frac{1}{R_{tot} A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$



Area, A
L_E
$L_F = L_G$
L_H
k_F
F
T_1
T_2
k_E
E
k_G
G
k_H
H
x
$\frac{L_F}{k_F(A/2)}$
$\frac{L_E}{k_E A}$
$\frac{L_H}{k_H A}$
q_x
T_1
$\frac{L_G}{k_G(A/2)}$
T_2

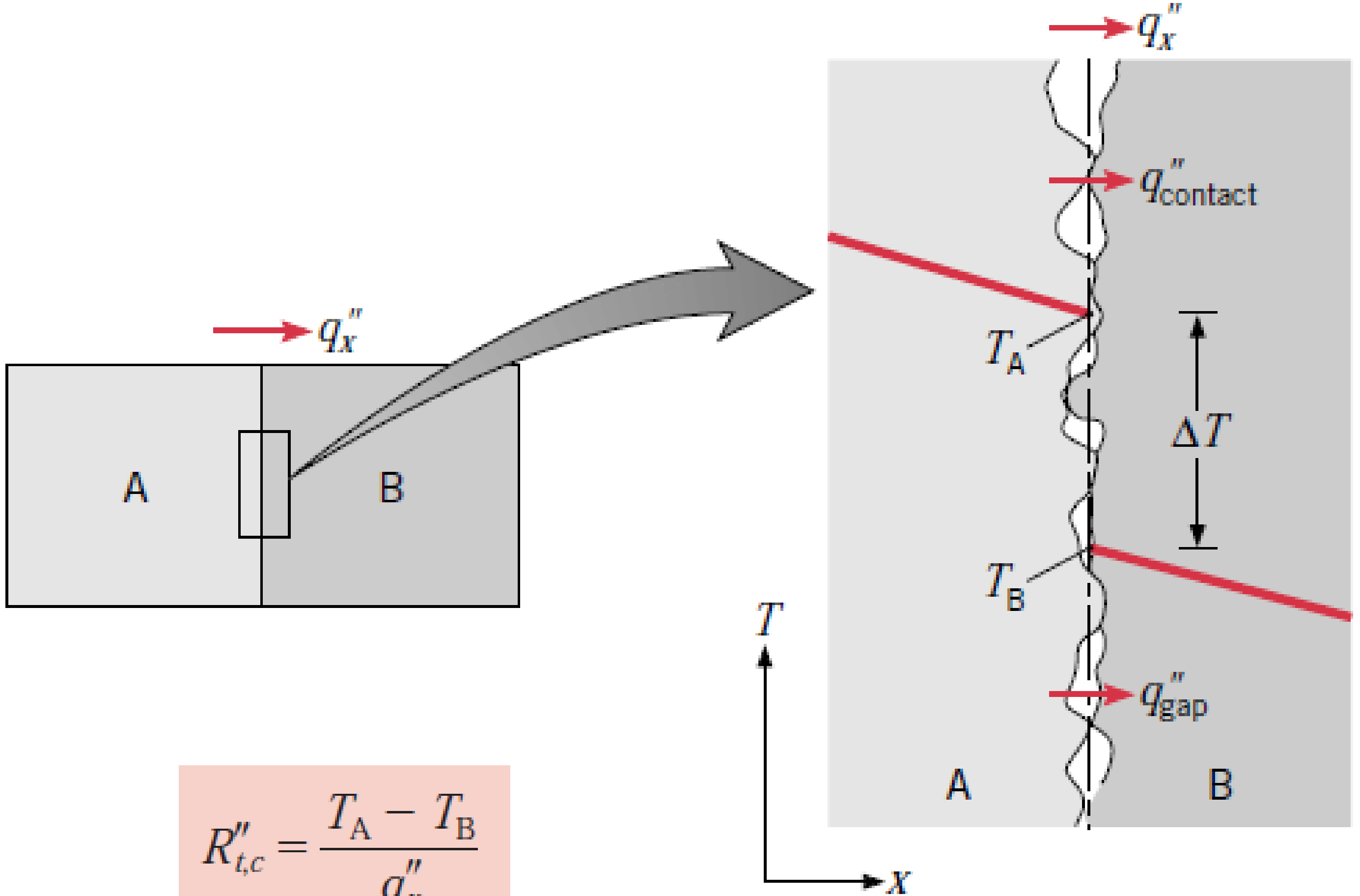
(a)



$\frac{L_E}{k_E(A/2)}$
$\frac{L_F}{k_F(A/2)}$
$\frac{L_H}{k_H(A/2)}$
q_x
T_1
$\frac{L_E}{k_E(A/2)}$
$\frac{L_G}{k_G(A/2)}$
$\frac{L_H}{k_H(A/2)}$
T_2

(b)

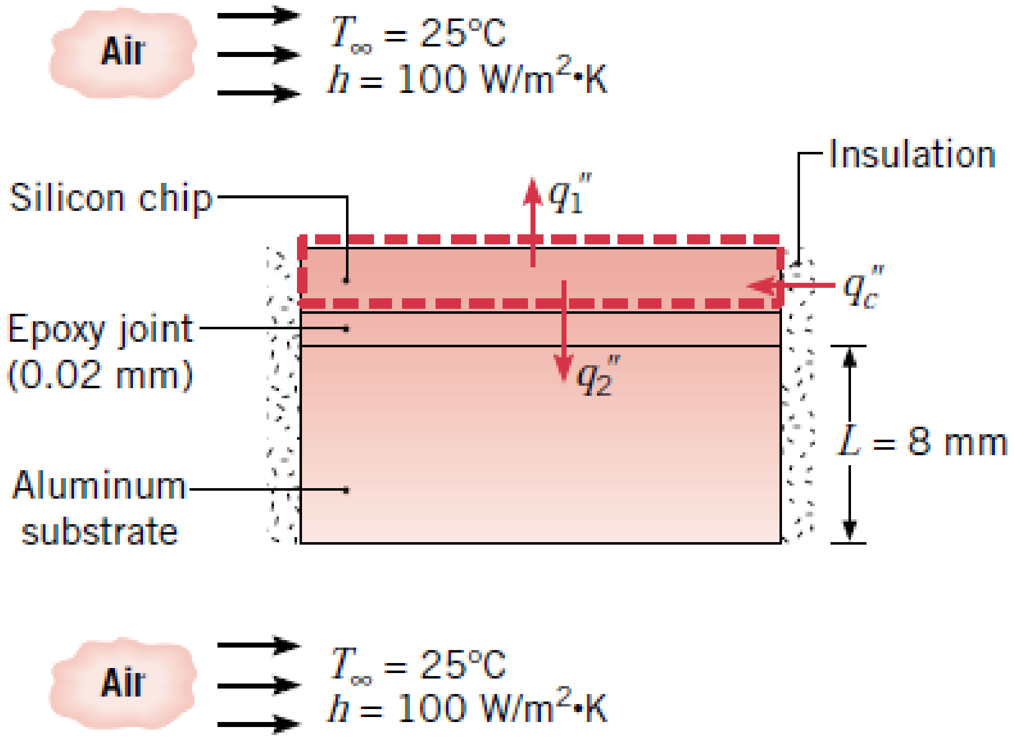
Contact Resistance



$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

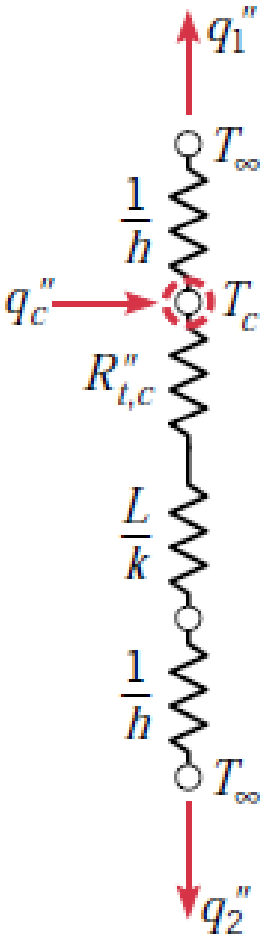
Lecture - IV

A thin silicon chip and an 8-mm-thick aluminum substrate are separated by a 0.02-mm-thick epoxy joint. The chip and substrate are each 10 mm on a side, and their exposed surfaces are cooled by air, which is at a temperature of 25°C and provides a convection coefficient of 100 $W/m^2 \cdot K$. If the chip dissipates 10^4 W/m^2 under normal conditions, will it operate below a maximum allowable temperature of 85°C?



$$k = 239 \text{ W/m} \cdot \text{K}$$

$$R''_{t,c} = 0.9 \times 10^{-4} \text{ m}^2 \cdot \text{K/W}$$



$$R_{t,c}'' = 0.9 \times 10^{-4}$$

Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction (negligible heat transfer from sides of composite).
3. Negligible chip thermal resistance (an isothermal chip).
4. Constant properties.
5. Negligible radiation exchange with surroundings.

$$q_c'' = q_1'' + q_2''$$

$$q_c'' = \frac{T_c - T_\infty}{(1/h)} + \frac{T_c - T_\infty}{R_{t,c}'' + (L/k) + (1/h)}$$

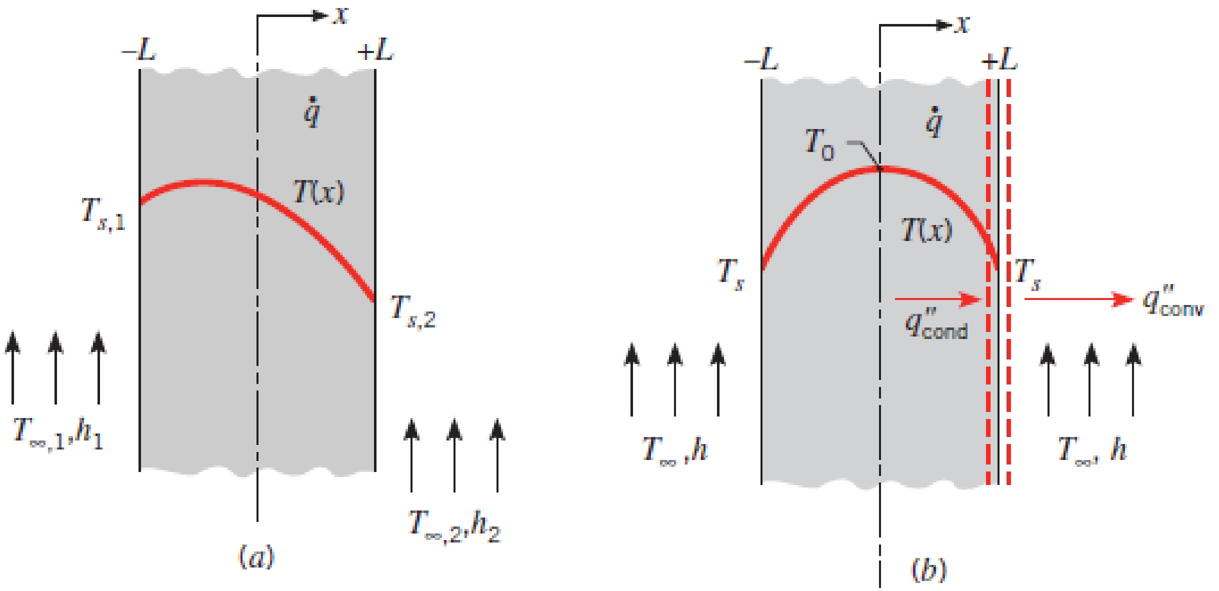
$$T_c = T_\infty + q''_c\left[h + \frac{1}{R''_{t,c} + (L/k) + (1/h)}\right]^{-1}$$

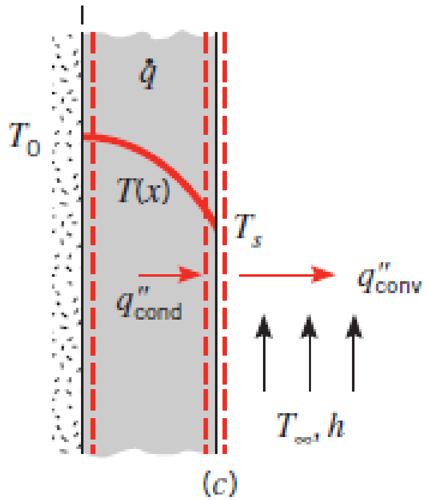
$$T_c = 25°\text{C} + 10^4\ \text{W/m}^2$$

$$\times\left[100 + \frac{1}{(0.9 + 0.33 + 100) \times 10^{-4}}\right]^{-1} \text{m}^2 \cdot \text{K/W}$$

$$T_c = 25°\text{C} + 50.3°\text{C} = 75.3°\text{C}$$

Conduction with Thermal Energy Generation





$$\frac{d^2T}{dx^2} + \frac{q}{k} = 0$$

$$T = -\frac{q}{2k}x^2 + C_1x + C_2$$

$$T(-L) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \qquad \text{and} \qquad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,1} + T_{s,2}}{2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

when both surfaces are maintained at a common temperature

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s$$

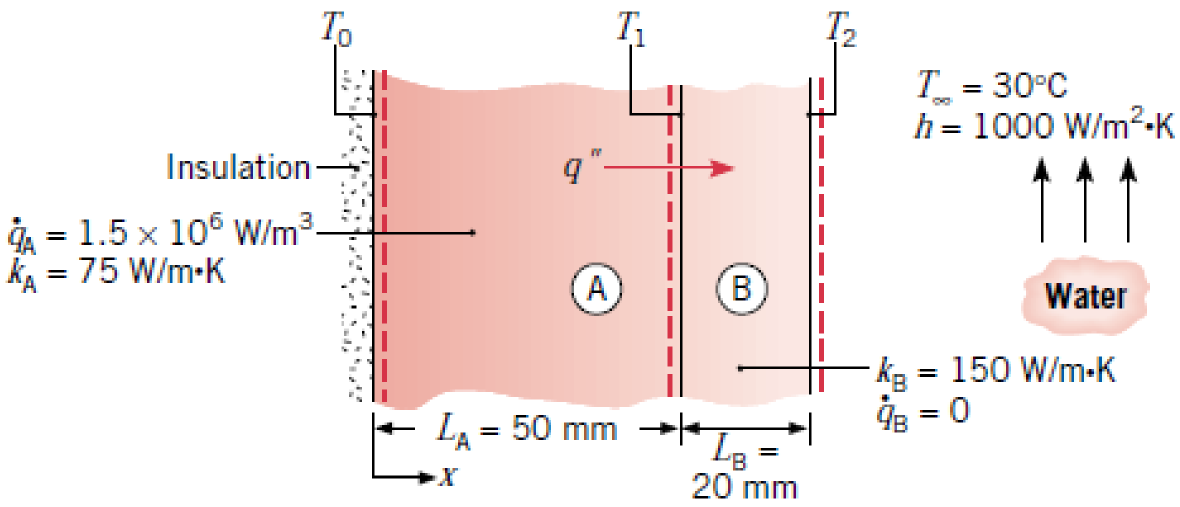
$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

$$-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}L}{h}$$

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m · K, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°$C and $h = 1000$ W/m^2 · K.

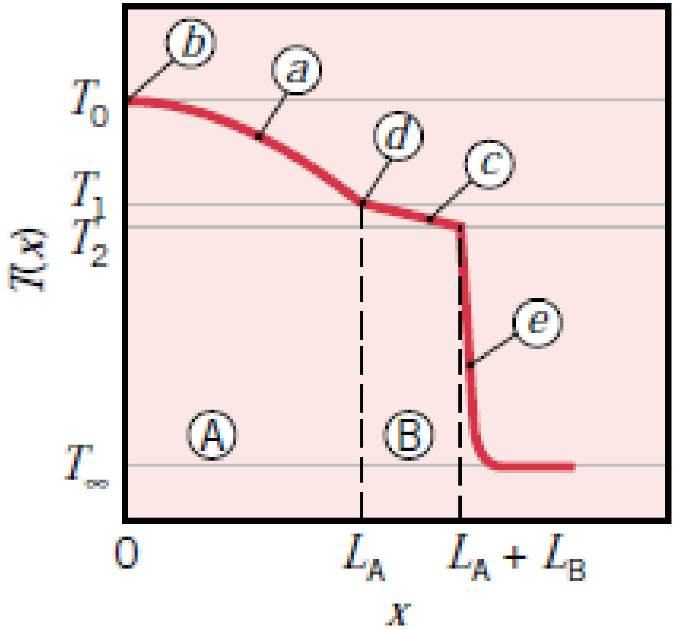
1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

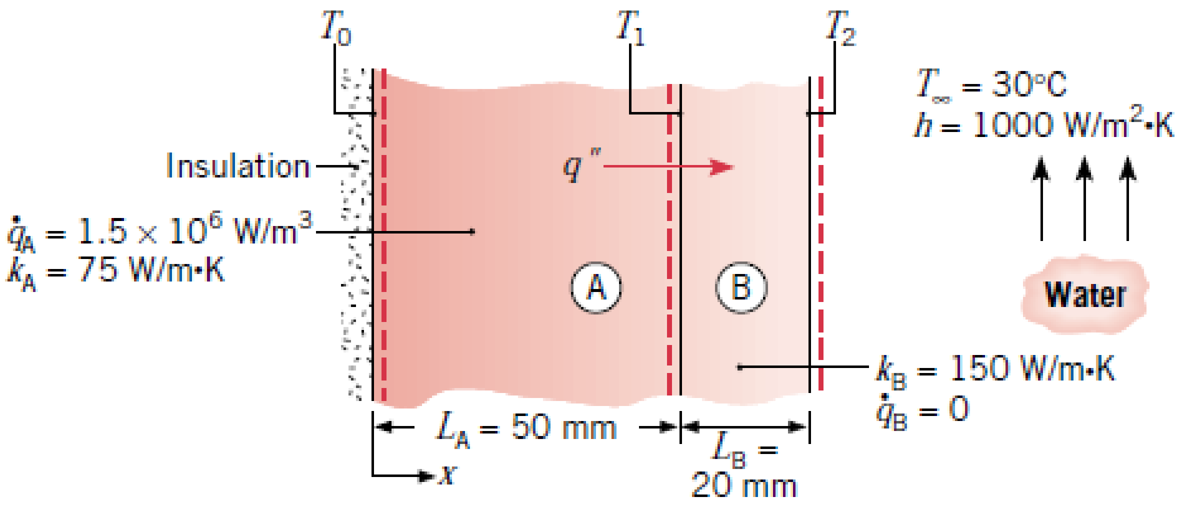


Assumptions

- Steady-state conditions
- One-dimensional conduction in x-direction
- Negligible contact resistance between walls
- Inner surface of A adiabatic
- Constant properties for materials A and B

(a) Parabolic in material A.
(b) Zero slope at insulated boundary.
(c) Linear in material B.
(d) Slope change $= k_B/k_A = 2$ at interface.
The temperature distribution in the water is characterized by
(e) Large gradients near the surface.





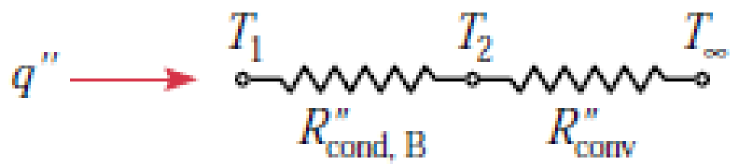
$$q'' = h(T_2 - T_\infty)$$

$$\dot{q}L_A = q''$$

$$T_2 = T_\infty + \frac{\dot{q}L_A}{h}$$

$$T_2 = 30°C + \frac{1.5 \times 10^6 \text{ W/m}^3 \times 0.05 \text{ m}}{1000 \text{ W/m}^2 \cdot \text{K}} = 105°C$$

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s$$



$$T_1 = T_\infty + (R''_{\text{cond,B}} + R''_{\text{conv}})\, q''$$

$$R''_{\text{cond, B}} = \frac{L_{\text{B}}}{k_{\text{B}}} \qquad R''_{\text{conv}} = \frac{1}{h}$$

$$T_1 = 30^\circ\text{C} + \left(\frac{0.02\ \text{m}}{150\ \text{W/m}\cdot\text{K}} + \frac{1}{1000\ \text{W/m}^2\cdot\text{K}}\right)$$
$$\times 1.5 \times 10^6\ \text{W/m}^3 \times 0.05\ \text{m}$$
$$T_1 = 30^\circ\text{C} + 85^\circ\text{C} = 115^\circ\text{C}$$

$$T_0 = \frac{1.5 \times 10^6 \text{ W/m}^3 (0.05 \text{ m})^2}{2 \times 75 \text{ W/m} \cdot \text{K}} + 115°\text{C}$$

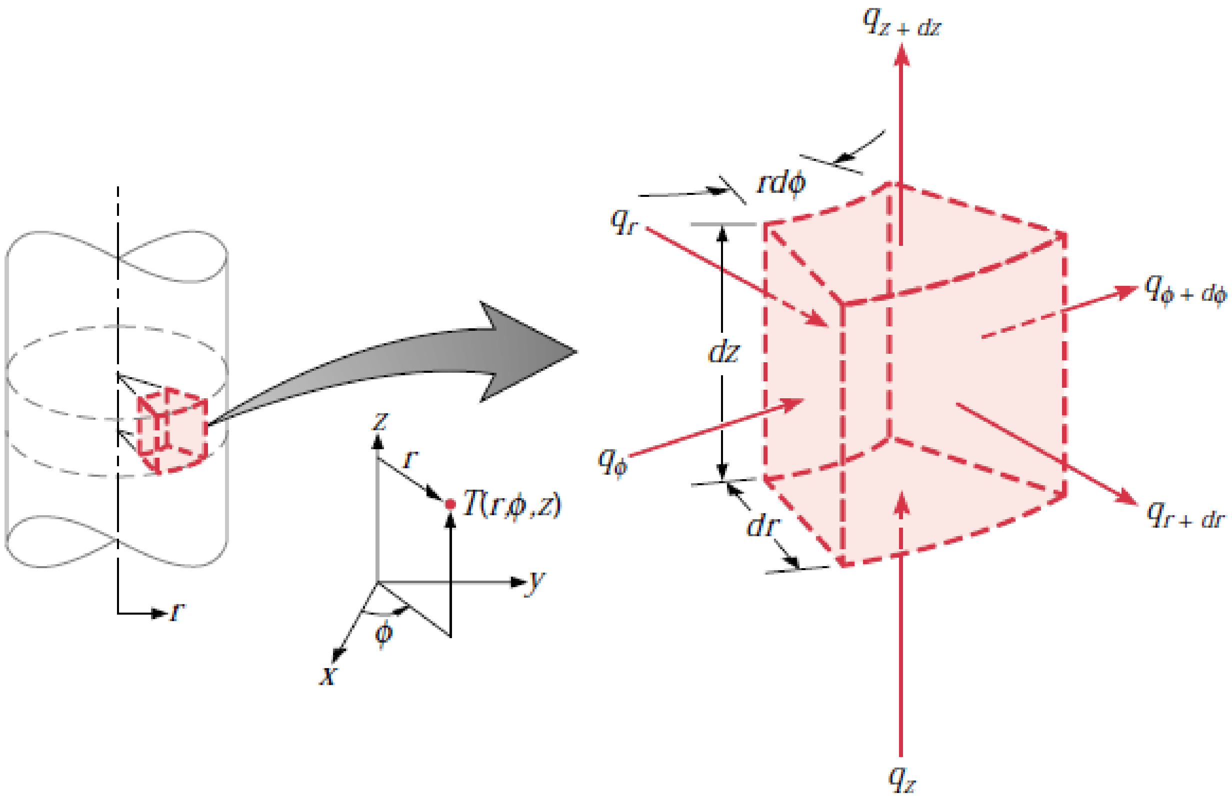
$$T_0 = 25°\text{C} + 115°\text{C} = 140°\text{C}$$

Material A, having heat generation, cannot be represented by a thermal circuit element.

Cylindrical & Spherical Coordinates

$$q'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \phi} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

$$q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\phi = -\frac{k}{r}\frac{\partial T}{\partial \phi} \qquad q''_z = -k\frac{\partial T}{\partial z}$$



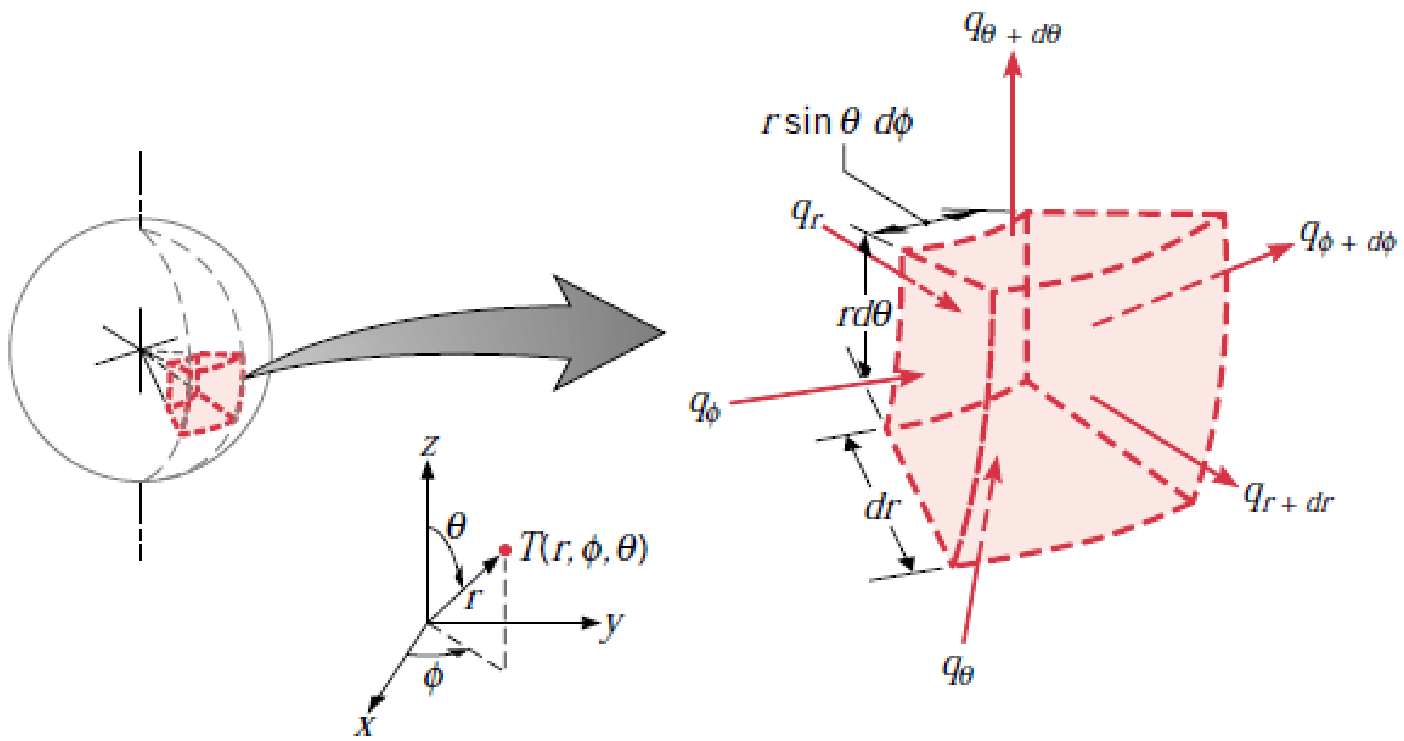
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)$$

$$+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\boldsymbol{q}''=-k\nabla T=-k\left(\boldsymbol{i}\frac{\partial T}{\partial r}+\boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \theta}+\boldsymbol{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$

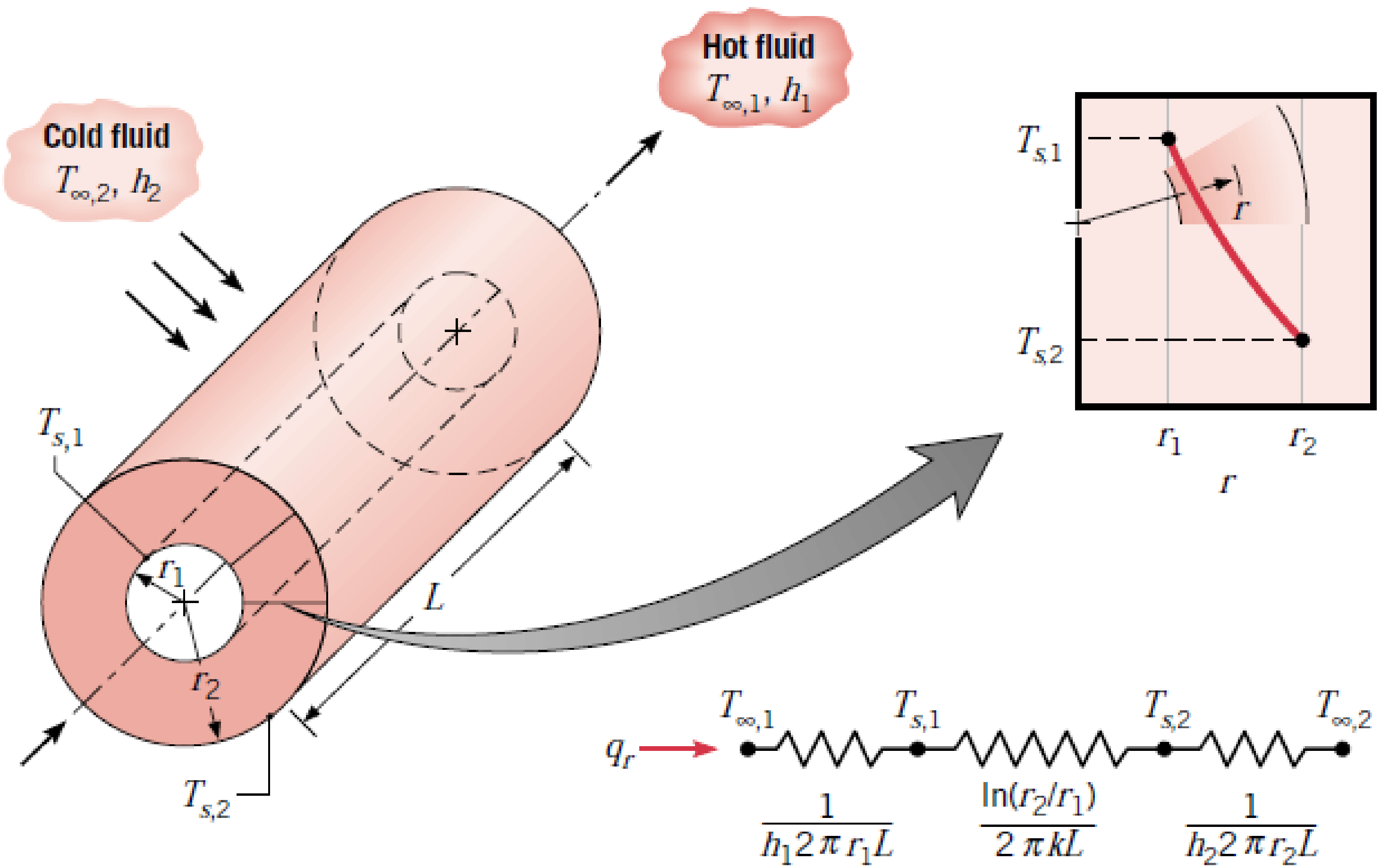
$$q''_r=-k\frac{\partial T}{\partial r} \qquad q''_\theta=-\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad q''_\phi=-\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$



Cylindrical surface

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$



$$\boxed{q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}}$$

$$T(r) = C_1 \ln r + C_2$$

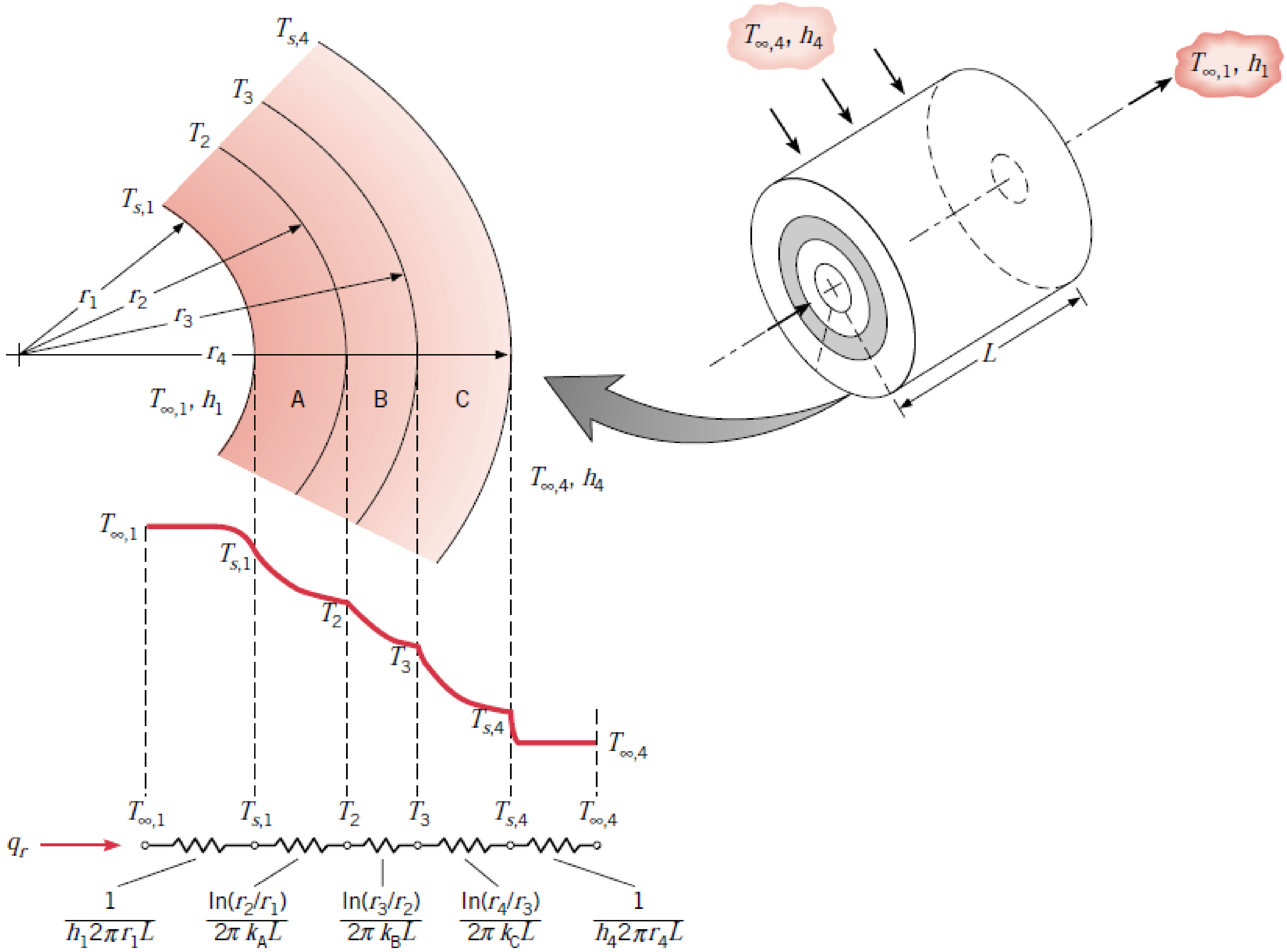
$$T(r_1) = T_{s,1} \qquad \text{and} \qquad T(r_2) = T_{s,2}$$

$$T_{s,1} = C_1 \ln r_1 + C_2 \qquad \text{and} \qquad T_{s,2} = C_1 \ln r_2 + C_2$$

$$\boxed{T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}}$$

$$\boxed{q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}}$$

$$\boxed{R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk}}$$

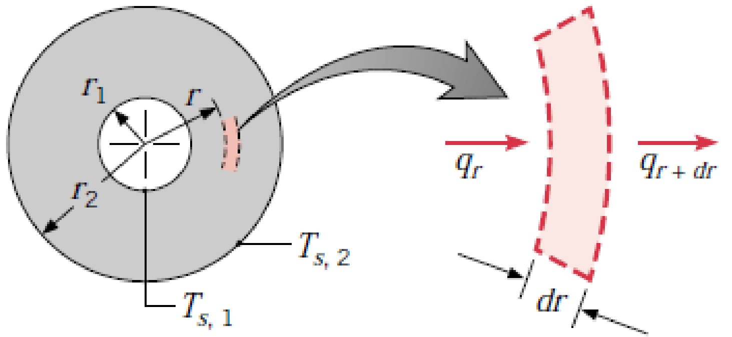


$T_{\infty,4}$, h_4
$T_{\infty,1}$, h_1
L
$T_{s,4}$
T_3
T_2
$T_{s,1}$
r_1
r_2
r_3
r_4
$T_{\infty,1}$, h_1
A
B
C
$T_{\infty,4}$, h_4
$T_{\infty,1}$
$T_{s,1}$
T_2
T_3
$T_{s,4}$
$T_{\infty,4}$
$T_{\infty,1}$
$T_{s,1}$
T_2
T_3
$T_{s,4}$
$T_{\infty,4}$
q_r
$\frac{1}{h_1 2\pi r_1 L}$
$\frac{\ln(r_2/r_1)}{2\pi k_A L}$
$\frac{\ln(r_3/r_2)}{2\pi k_B L}$
$\frac{\ln(r_4/r_3)}{2\pi k_C L}$
$\frac{1}{h_4 2\pi r_4 L}$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln (r_2/r_1)}{2\pi k_A L} + \dfrac{\ln (r_3/r_2)}{2\pi k_B L} + \dfrac{\ln (r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = UA(T_{\infty,1} - T_{\infty,4})$$

$$U_1 = \frac{1}{\dfrac{1}{h_1} + \dfrac{r_1}{k_A}\ln\dfrac{r_2}{r_1} + \dfrac{r_1}{k_B}\ln\dfrac{r_3}{r_2} + \dfrac{r_1}{k_C}\ln\dfrac{r_4}{r_3} + \dfrac{r_1}{r_4}\dfrac{1}{h_4}}$$



$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{T_0}^{T} k(T)\, dT$$

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\, dT$$

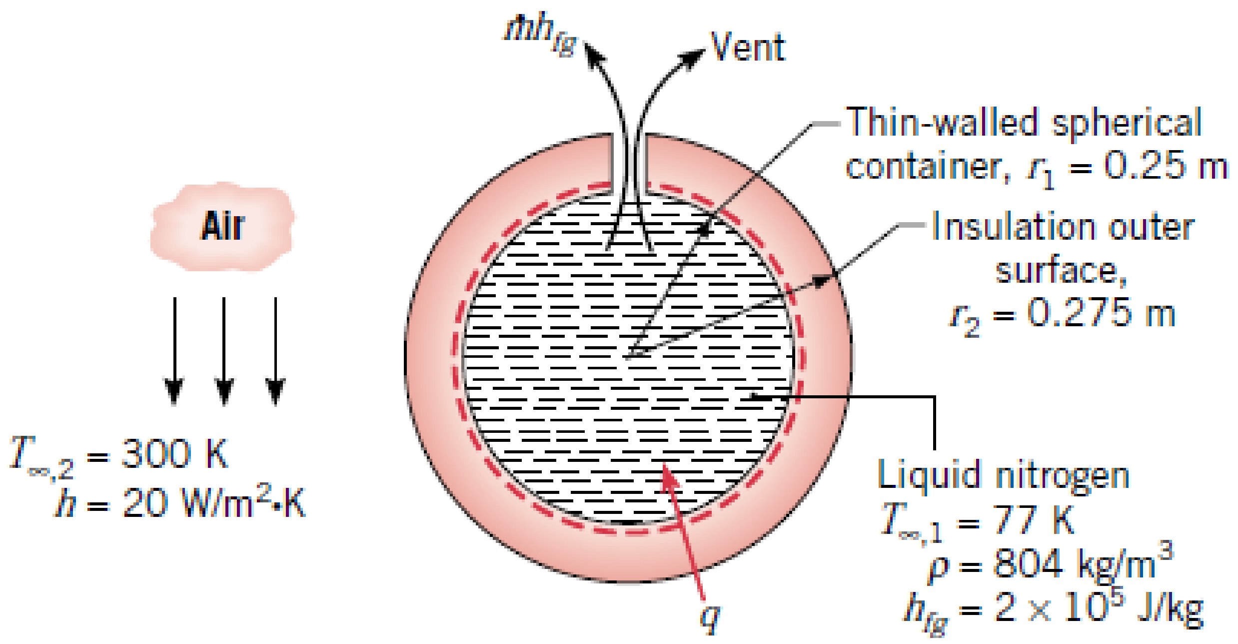
$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

$$R_{t,\,\mathrm{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

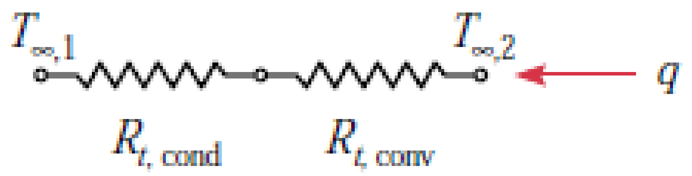
A spherical, thin-walled metallic container is used to store liquid nitrogen at 77 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 300 K. The convection coefficient is known to be 20 $W/m^2 \cdot K$. The latent heat of vaporization and the density of liquid nitrogen are 2×10^5 J/kg and 804 kg/m^3, respectively.

1. What is the rate of heat transfer to the liquid nitrogen?
2. What is the rate of liquid boil-off?

evacuated silica powder (300 K): $k = 0.0017$ $W/m \cdot K$.



$\dot{m}h_{fg}$
Vent
Thin-walled spherical container, r_1 = 0.25 m
Insulation outer surface, r_2 = 0.275 m
Air
$T_{\infty,2}$ = 300 K
h = 20 W/m^2·K
Liquid nitrogen
$T_{\infty,1}$ = 77 K
ρ = 804 kg/m^3
h_{fg} = 2 × 10^5 J/kg
q



$T_{\infty,1}$
$T_{\infty,2}$
q
$R_{t,\text{ cond}}$
$R_{t,\text{ conv}}$

$$R_{t,\,\mathrm{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad R_{t,\mathrm{conv}} = \frac{1}{h4\pi r_2^2}$$

$$q = \frac{T_{\infty,2} - T_{\infty,1}}{(1/4\pi k)[(1/r_1) - (1/r_2)] + (1/h4\pi r_2^2)}$$

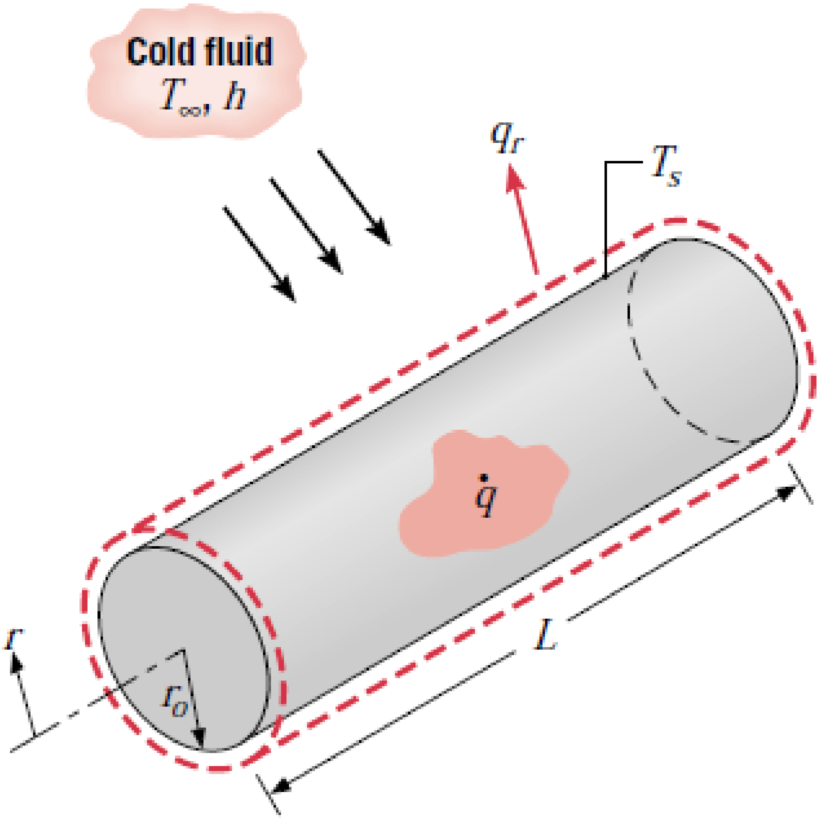
$$q = \frac{223}{17.02 + 0.05}\,\mathrm{W} = 13.06\ \mathrm{W}$$

$$\dot{E}_{\mathrm{in}} - \dot{E}_{\mathrm{out}} = 0$$

$$q - \dot{m}h_{fg} = 0$$

$$\dot{m} = \frac{13.06\ \mathrm{J/s}}{2 \times 10^5\ \mathrm{J/kg}} = 6.53 \times 10^{-5}\ \mathrm{kg/s}$$

Radial Systems with heat generation



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$$

$$T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 \ln r + C_2$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

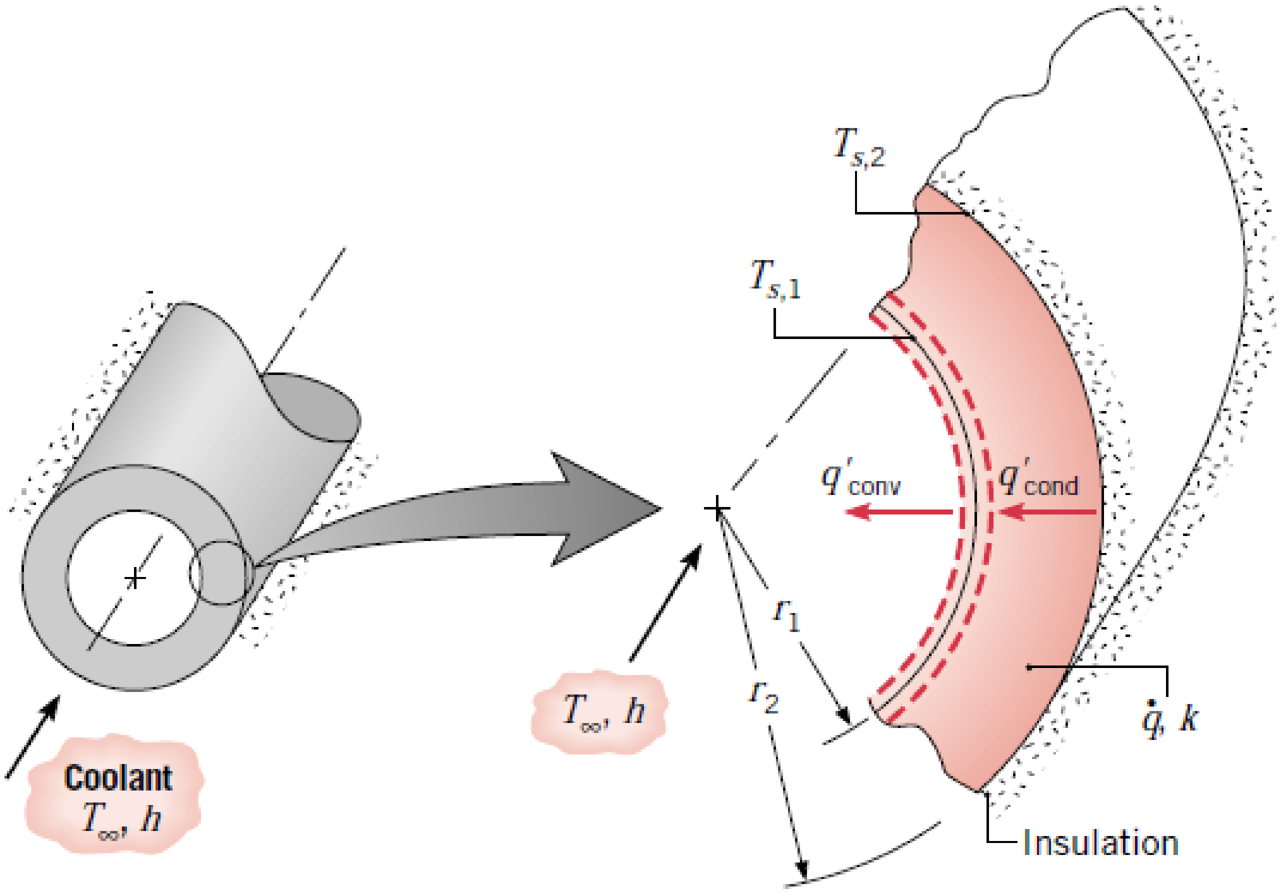
$$T(r_o) = T_s$$

$$T(r) = \frac{\dot{q}r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

A thin electrical heater is wrapped around the outer surface of a long cylindrical tube whose inner surface is maintained at a temperature of 5°C. The tube wall has inner and outer radii of 25 and 75 mm, respectively, and a thermal conductivity of 10 W/m · K. The thermal contact resistance between the heater and the outer surface of the tube (per unit length of the tube) is $R'_{t,c} = 0.01\ \mathrm{m \cdot K/W}$. The outer surface of the heater is exposed to a fluid with $T_\infty = -10°\mathrm{C}$ and a convection coefficient of $h = 100\ \mathrm{W/m^2 \cdot K}$. Determine the heater power per unit length of tube required to maintain the heater at $T_o = 25°\mathrm{C}$.

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.

1. Obtain the general solution for the temperature distribution in the tube.
2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_2$). Specifying this limit as $T_{s,2}$, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.
3. Determine the heat removal rate per unit length of tube.
4. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of $T_{s,2}$ and $\dot{q}$.



Assumptions:

1. Steady-state conditions.
2. One-dimensional radial conduction.
3. Constant properties.
4. Uniform volumetric heat generation.

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln r + C_2$$

$$T(r_2) = T_{s,2}$$

$$\left.\frac{dT}{dr}\right|_{r_2} = 0$$

$$T_{s,2} = -\frac{\dot{q}}{4k} r_2^2 + C_1 \ln r_2 + C_2$$

$$r\frac{dT}{dr} = -\frac{\dot{q}}{2k} r^2 + C_1$$

$$C_1 = \frac{\dot{q}}{2k} r_2^2$$

$$C_2 = T_{s,2} + \frac{\dot{q}}{4k} r_2^2 - \frac{\dot{q}}{2k} r_2^2 \ln r_2$$

$$T(r) = T_{s,2} + \frac{\dot{q}}{4k}(r_2^2 - r^2) - \frac{\dot{q}}{2k} r_2^2 \ln \frac{r_2}{r}$$

$$q'_r = -k2\pi r\frac{dT}{dr}$$

$$q'_r(r_1) = -k2\pi r_1\left(-\frac{\dot{q}}{2k}r_1 + \frac{\dot{q}}{2k}\frac{r_2^2}{r_1}\right) = -\pi\dot{q}(r_2^2 - r_1^2)$$

$$\dot{E}_g - \dot{E}_{\text{out}} = 0$$

$$\dot{E}_g = \dot{q}\pi(r_2^2 - r_1^2)L$$

$$E_{\text{out}} = q'_{\text{cond}}L = -q'_r(r_1)L$$

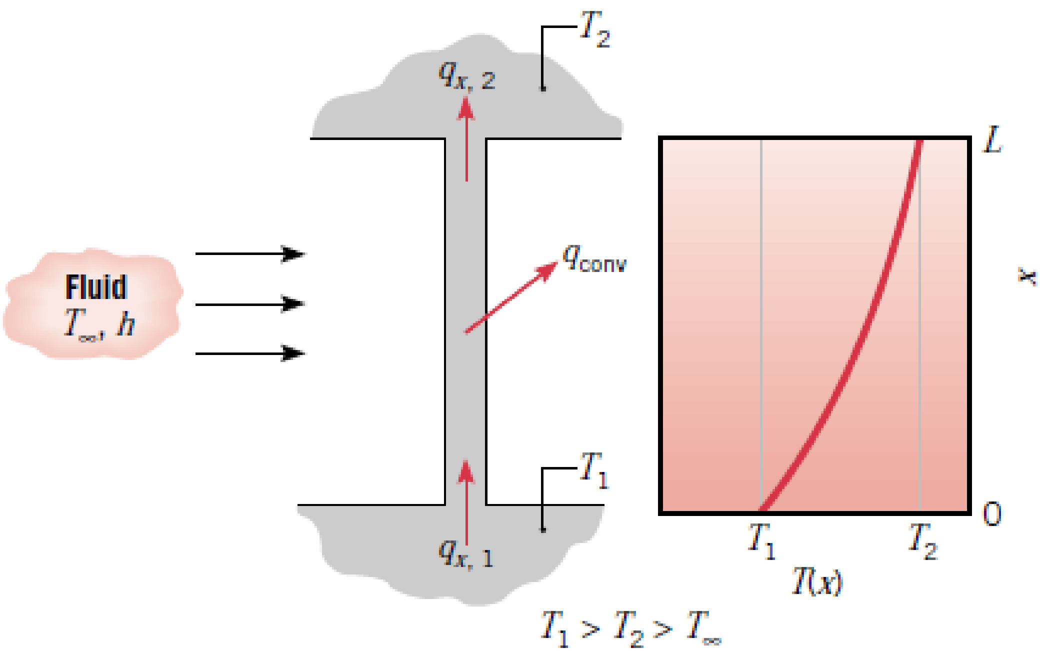
$$q'_{\mathrm{cond}} = q'_{\mathrm{conv}}$$

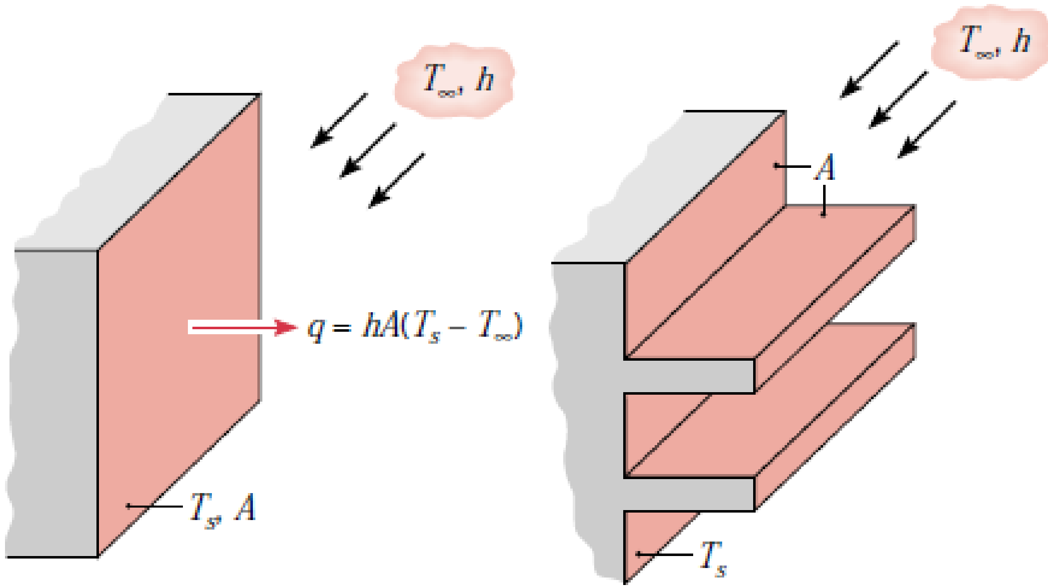
$$\pi\dot{q}(r_2^2 - r_1^2) = h2\pi r_1(T_{s,1} - T_\infty)$$

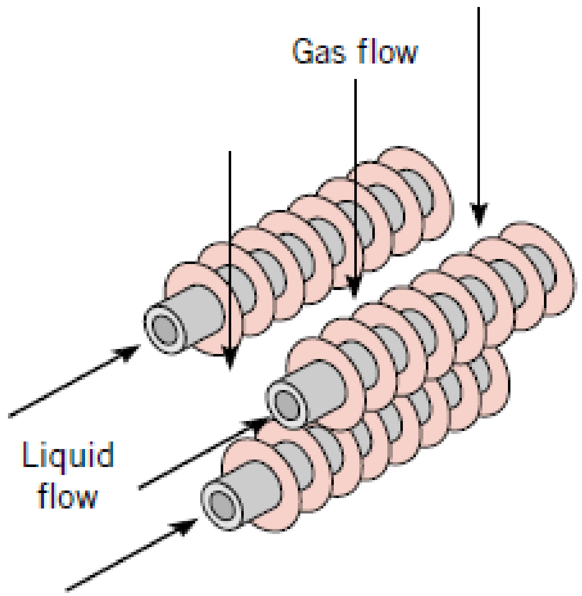
$$\boxed{h = \frac{\dot{q}(r_2^2 - r_1^2)}{2r_1(T_{s,1} - T_\infty)}}$$

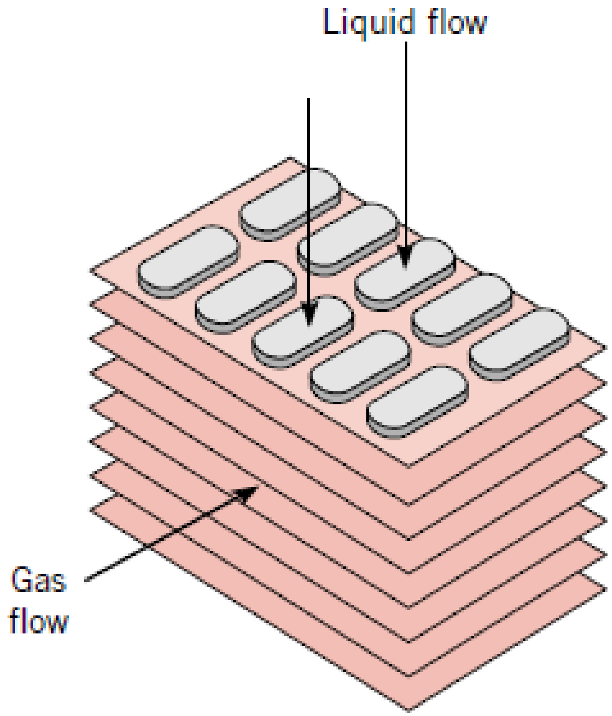
A thin electrical heater is wrapped around the outer surface of a long cylindrical tube whose inner surface is maintained at a temperature of 5°C. The tube wall has inner and outer radii of 25 and 75 mm, respectively, and a thermal conductivity of 10 W/m · K. The thermal contact resistance between the heater and the outer surface of the tube (per unit length of the tube) is $R'_{t,c} = 0.01$ m · K/W. The outer surface of the heater is exposed to a fluid with $T_\infty = -10$°C and a convection coefficient of $h = 100$ W/m^2 · K. Determine the heater power per unit length of tube required to maintain the heater at $T_o = 25$°C.

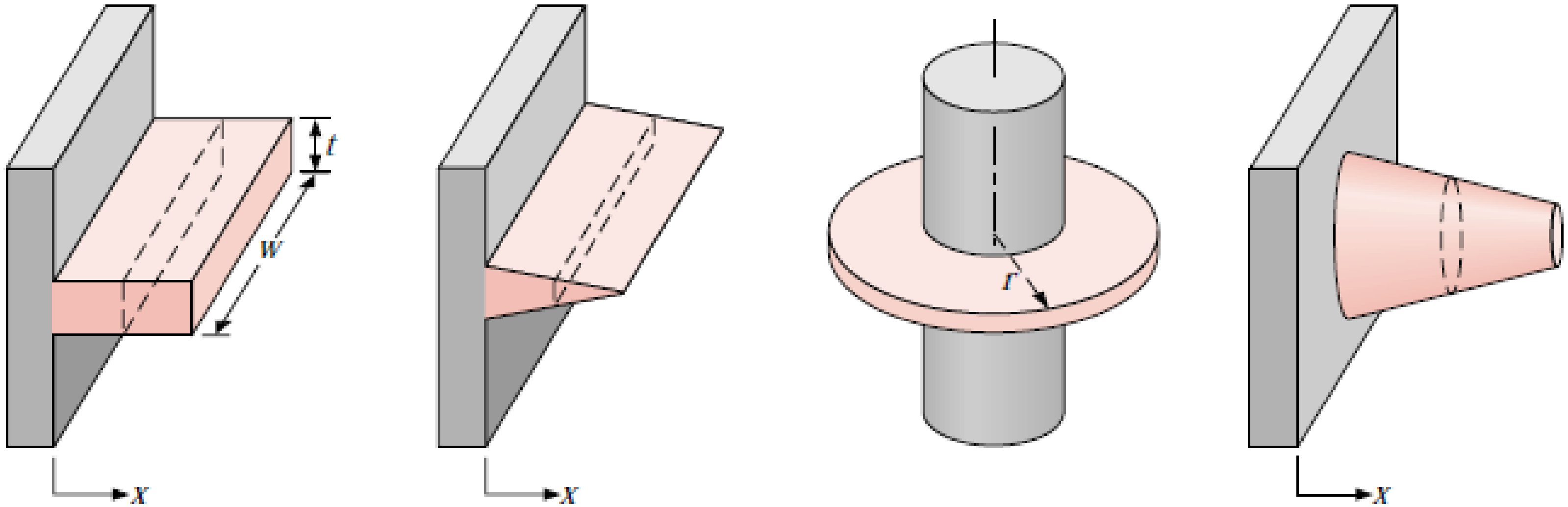
Heat Transfer from Extended Surfaces






Gas flow
Liquid
flow


Liquid flow
Gas
flow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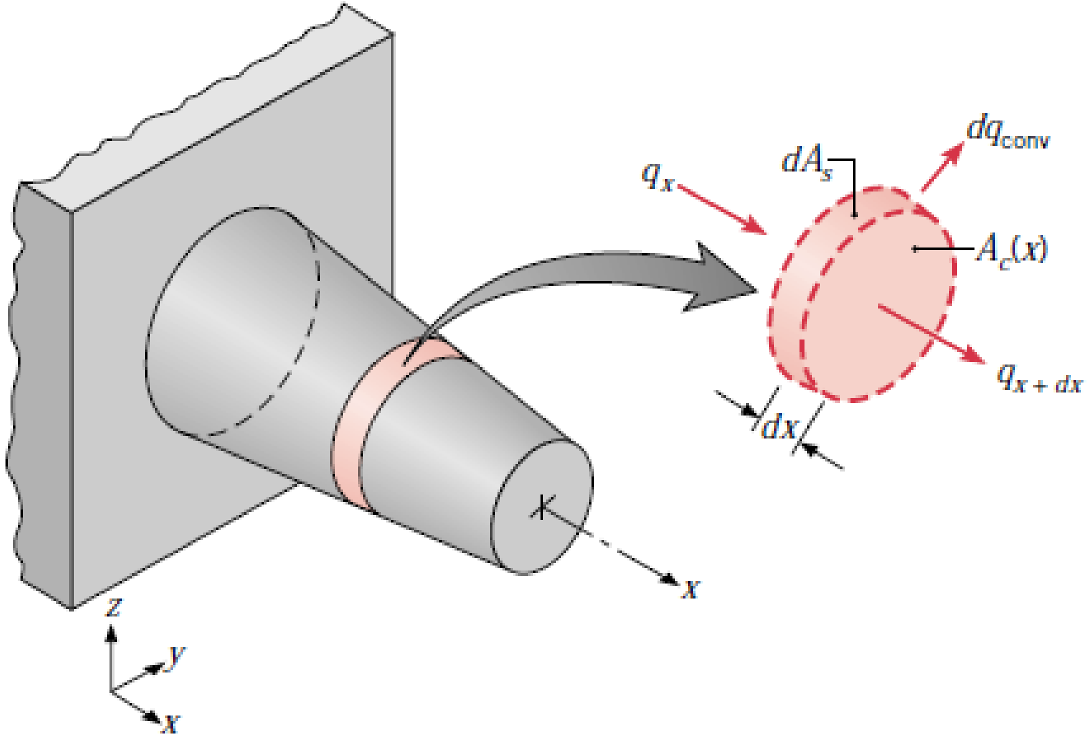


one-dimensional conditions

temperature is uniform across the fin thickness

thermal conductivity is constant

radiation from the surface is negligible

heat generation effects are absent

convection heat transfer coefficient is uniform over the surface

$$q_x = q_{x+dx} + dq_{\text{conv}}$$

$$q_x = -kA_c \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx}\, dx$$

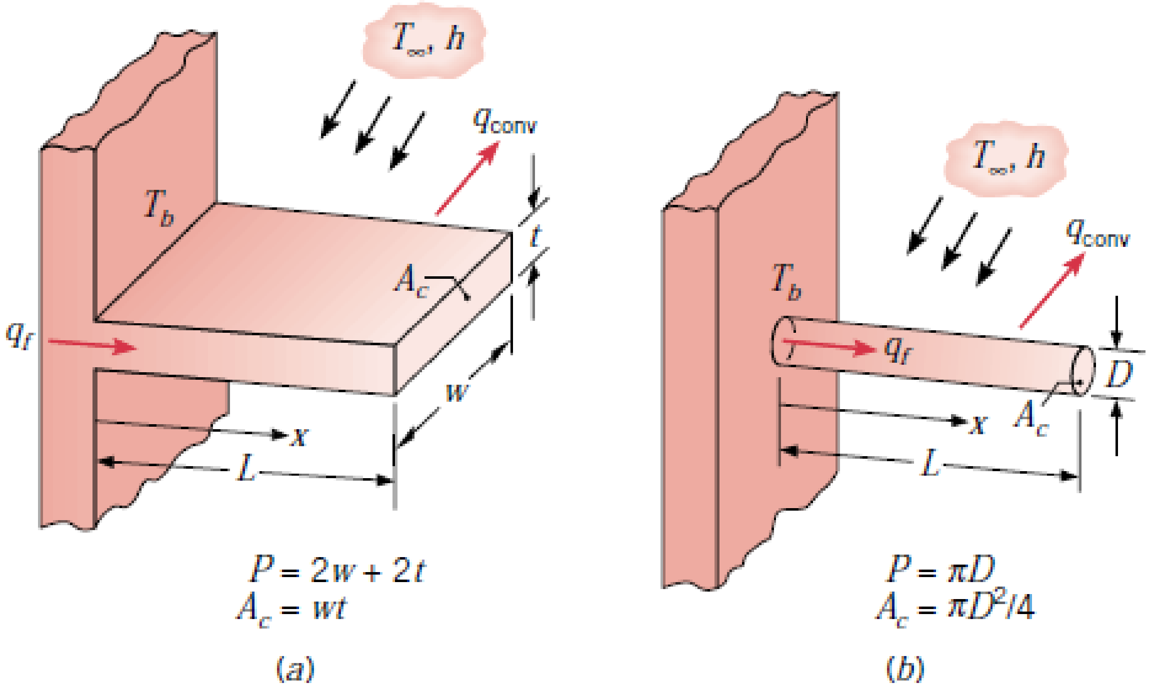
$$q_{x+dx} = -kA_c \frac{dT}{dx} - k\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx$$

$$dq_{\text{conv}} = h\, dA_s (T - T_\infty)$$

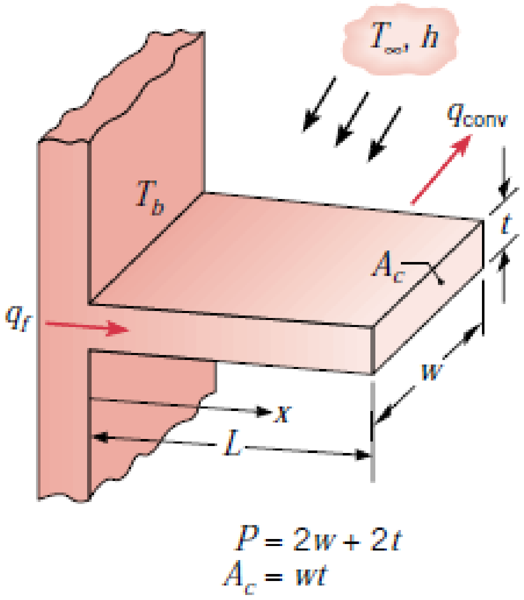
$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

general form of the energy equation for an extended surface



$A_s = Px$ $\qquad$ $dA_s/dx = P$,



(a)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$$\theta(x) \equiv T(x) - T_\infty$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$m^2 \equiv \frac{hP}{kA_c}$$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

Case	Tip Condition ($x = L$)
A	Convection heat transfer: $h\theta(L) = -k\, d\theta/dx\vert_{x=L}$
B	Adiabatic $d\theta/dx\vert_{x=L} = 0$
C	Prescribed temperature: $\theta(L) = \theta_L$
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$

$$hA_c[T(L) - T_\infty] = -kA_c \left.\frac{dT}{dx}\right|_{x=L}$$

$$h\theta(L) = -k \left.\frac{d\theta}{dx}\right|_{x=L}$$

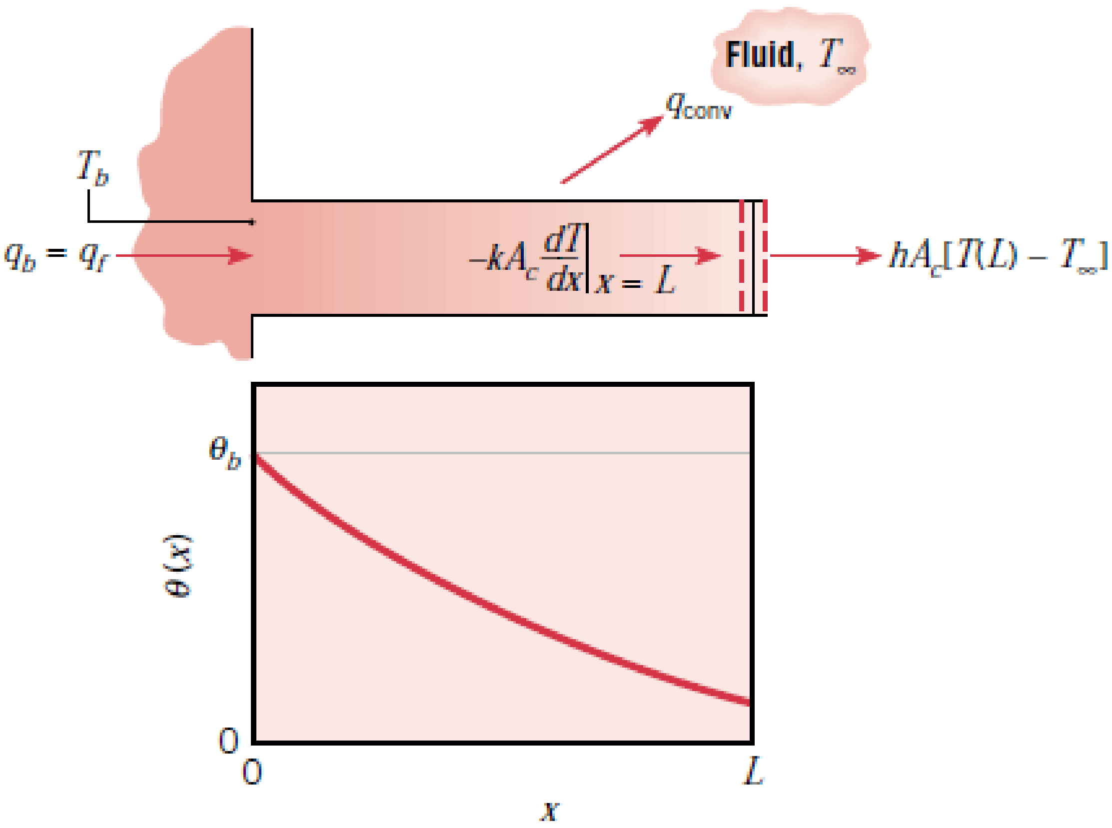
$$\theta_b = C_1 + C_2$$

$$h(C_1 e^{mL} + C_2 e^{-mL}) = km(C_2 e^{-mL} - C_1 e^{mL})$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{du}{dx} \qquad \frac{d}{dx}(\cosh u) = (\sinh u)\frac{du}{dx} \qquad \frac{d}{dx}(\tanh u) = \left(\frac{1}{\cosh^2 u}\right)\frac{du}{dx}$$



$$q_f = q_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

$$q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s$$

$$q_f = \int_{A_f} h\theta(x)\, dA_s$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL}$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$C_1 e^{mL} - C_2 e^{-mL} = 0$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x)}{\cosh mL}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL$$

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$
B	Adiabatic $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M \tanh mL$
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$
D	Infinite fin ($L \to \infty$): $\theta(L) = 0$	e^{-mx}	M

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

Fin Performance

- ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

$$R_{t,f} = \frac{\theta_b}{q_f}$$

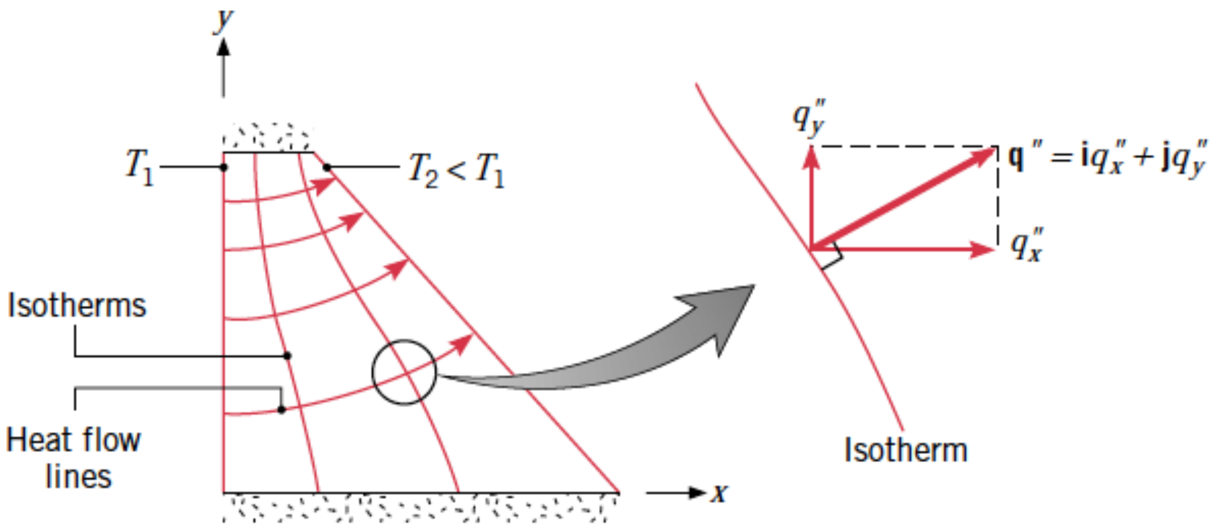
$$R_{t,b} = \frac{1}{hA_{c,b}}$$

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}}$$

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$

Two-Dimensional, Steady-State Conduction

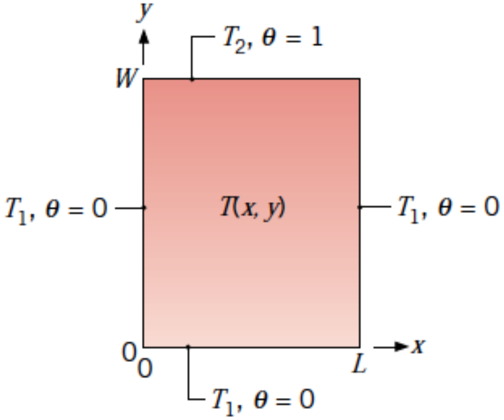
$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial x} + \boldsymbol{j}\frac{\partial T}{\partial y} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$



$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$q_x'' = -k\frac{\partial T}{\partial x} \qquad q_y'' = -k\frac{\partial T}{\partial y} \qquad q_z'' = -k\frac{\partial T}{\partial z}$$



$$\theta \equiv \frac{T - T_1}{T_2 - T_1}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$\theta(0, y) = 0 \quad \text{and} \quad \theta(x, 0) = 0$$
$$\theta(L, y) = 0 \quad \text{and} \quad \theta(x, W) = 1$$

$$\theta(x, y) = X(x) \cdot Y(y)$$

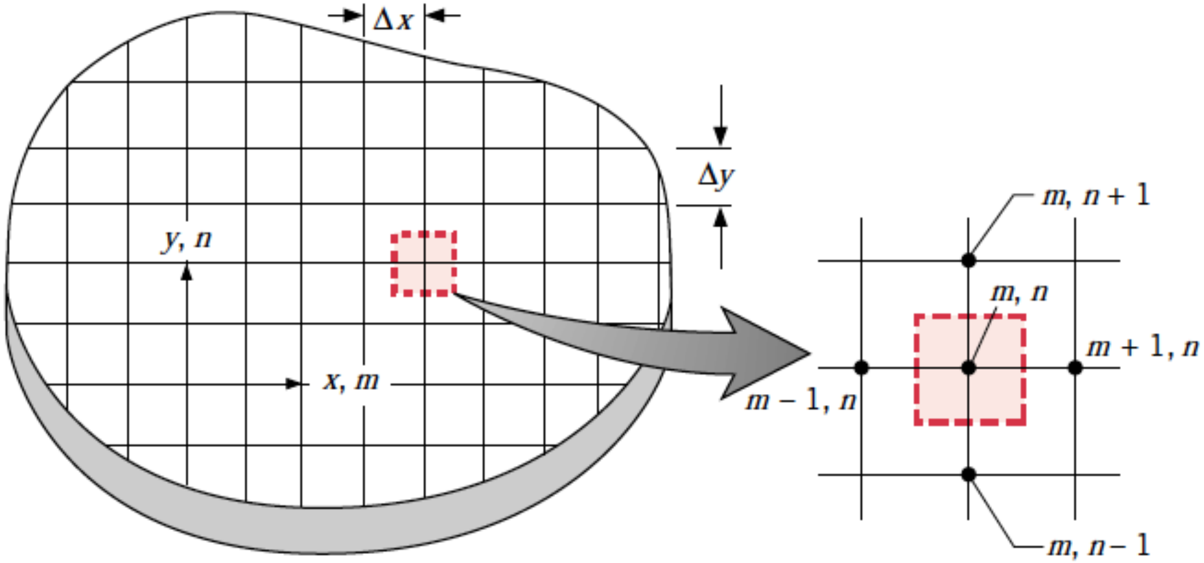
$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$
$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$
$$Y = C_3 e^{-\lambda y} + C_4 e^{+\lambda y}$$

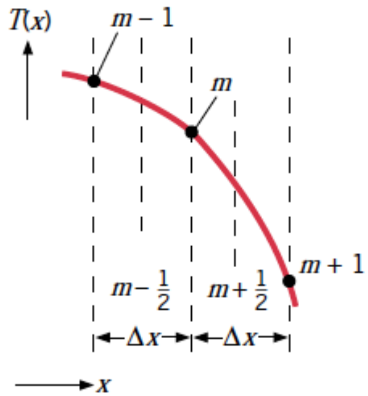
$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

Finite-Difference Equations



$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n} - T_{m-1,\,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\partial T/\partial x|_{m+1/2,n} - \partial T/\partial x|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

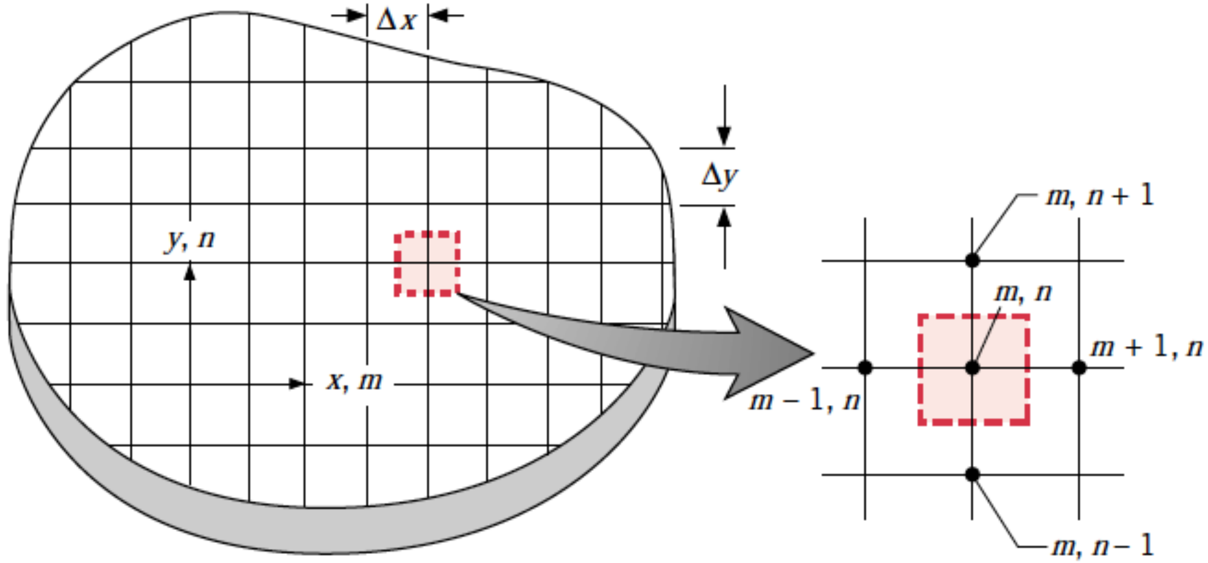
$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

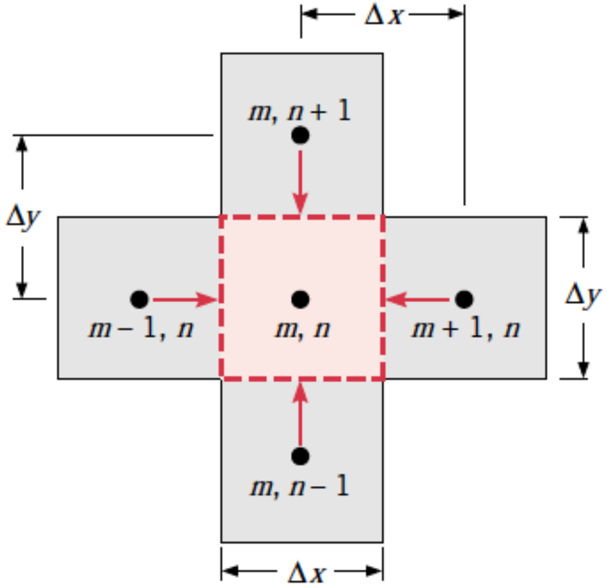
$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{\partial T/\partial y|_{m,n+1/2} - \partial T/\partial y|_{m,n-1/2}}{\Delta y}$$

$$\approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$



$$\dot{E}_{\text{in}} + \dot{E}_g = 0$$



$$\sum_{i=1}^{4} q_{(i) \to (m,n)} + \dot{q}(\Delta x \cdot \Delta y \cdot 1) = 0$$

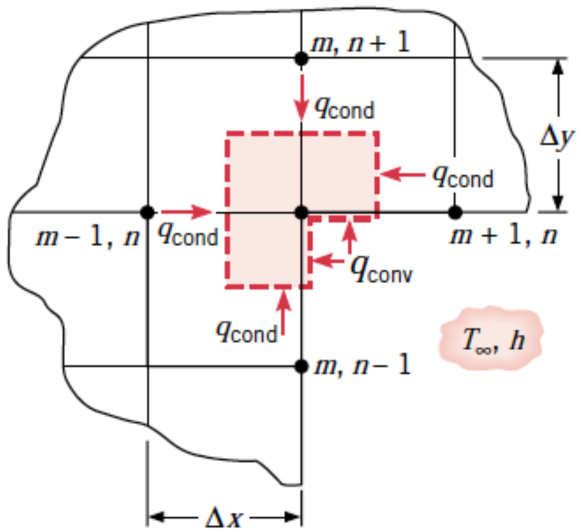
$$q_{(m-1,n) \to (m,n)} = k(\Delta y \cdot 1)\, \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m+1,n) \to (m,n)} = k(\Delta y \cdot 1)\, \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1) \to (m,n)} = k(\Delta x \cdot 1)\, \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m,n-1) \to (m,n)} = k(\Delta x \cdot 1)\, \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

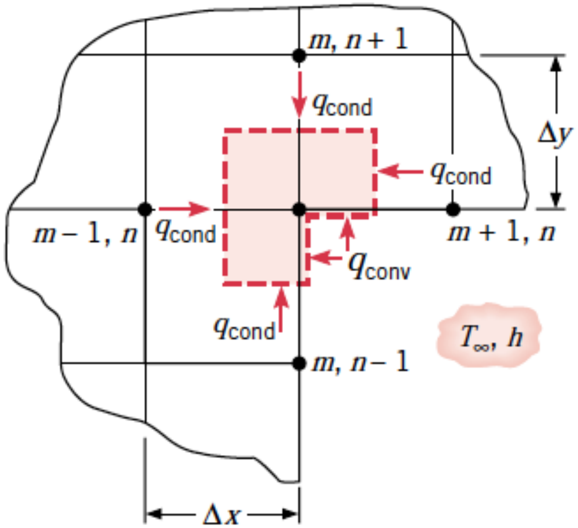


$$q_{(m-1,n)\rightarrow(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1)\rightarrow(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m+1,n)\rightarrow(m,n)} = k\left(\frac{\Delta y}{2} \cdot 1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n-1)\rightarrow(m,n)} = k\left(\frac{\Delta x}{2} \cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$



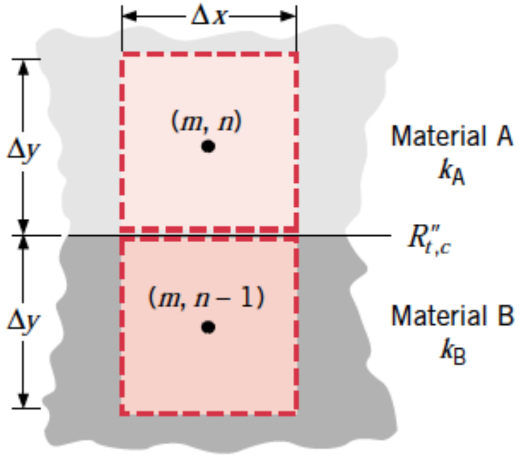
$$q_{(\infty)\to(m,n)} = h\left(\frac{\Delta x}{2}\cdot 1\right)(T_\infty - T_{m,n}) + h\left(\frac{\Delta y}{2}\cdot 1\right)(T_\infty - T_{m,n})$$

$$T_{m-1,n} + T_{m,n+1} + \frac{1}{2}(T_{m+1,n} + T_{m,n-1}) + \frac{h\Delta x}{k}T_\infty - \left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

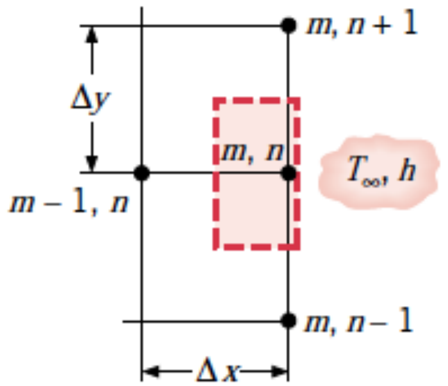
$$q_{(m-1,n)\to(m,n)} = \frac{T_{m-1,n} - T_{m,n}}{R_{t,cond}} = \frac{T_{m-1,n} - T_{m,n}}{\Delta x/k\,(\Delta y \cdot 1)}$$

$$q_{(m,n)\to(m,n-1)} = \frac{T_{m,n} - T_{m,n-1}}{R_{tot}}$$

$$q_{(\infty)\to(m,n)} = \frac{T_\infty - T_{m,n}}{R_{t,conv}} = \frac{T_\infty - T_{m,n}}{\{h[(\Delta x/2)\cdot 1 + (\Delta y/2)\cdot 1]\}^{-1}}$$

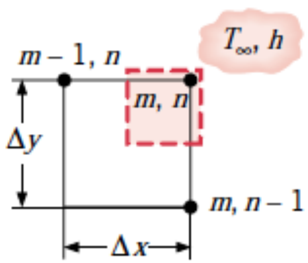


$$R_{tot} = \frac{\Delta y/2}{k_A(\Delta x \cdot 1)} + \frac{R''_{t,c}}{\Delta x \cdot 1} + \frac{\Delta y/2}{k_B(\Delta x \cdot 1)}$$



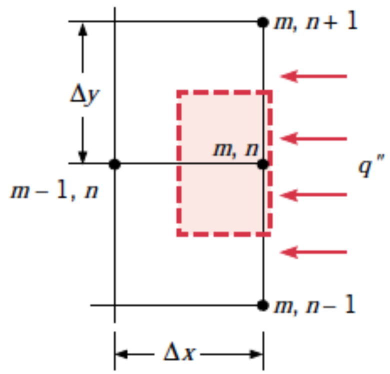
Node at a plane surface with convection

$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right)T_{m,n} = 0$$



Node at an external corner with convection

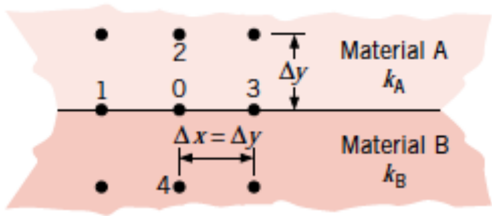
$$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$



Node at a plane surface with uniform heat flux

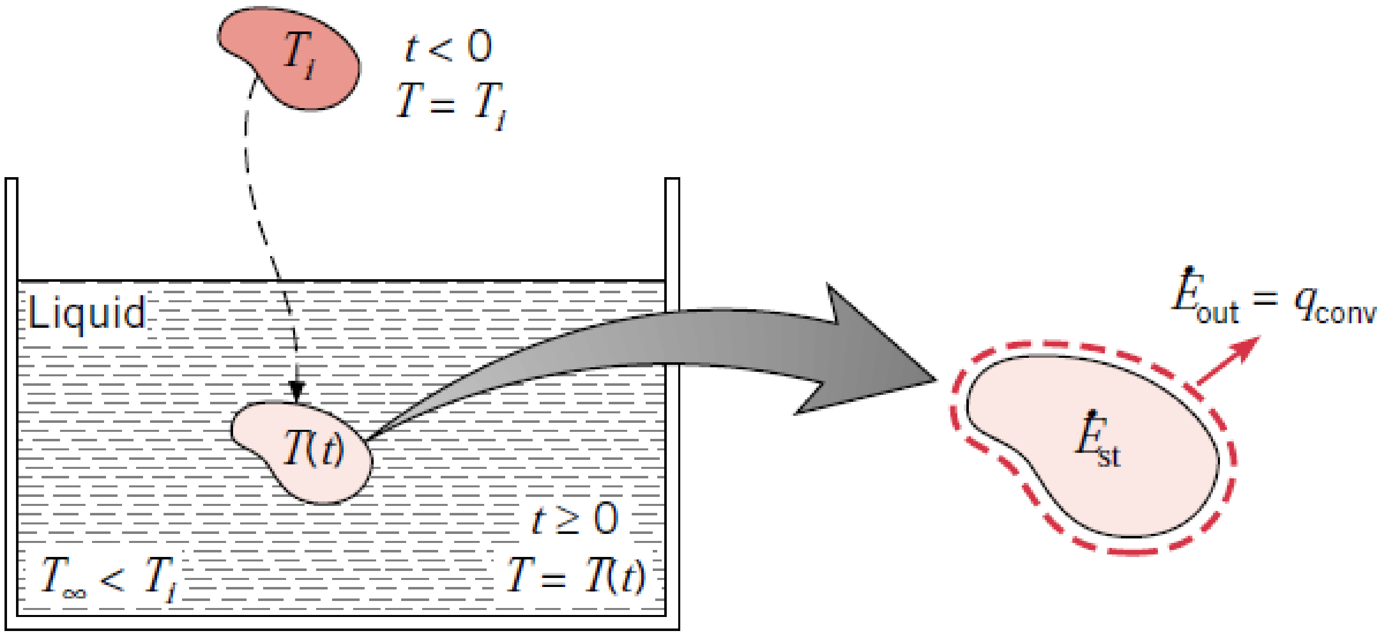
$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\Delta x}{k} - 4T_{m,n} = 0$$

Consider the nodal point 0 located on the boundary between materials of thermal conductivity k_A and k_B.



Derive the finite-difference equation, assuming no internal generation.

Transient Conduction



$$-\dot{E}_{out} = \dot{E}_{st}$$

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$

$$\theta \equiv T - T_\infty \qquad \frac{\rho V c}{hA_s}\frac{d\theta}{dt} = -\theta$$

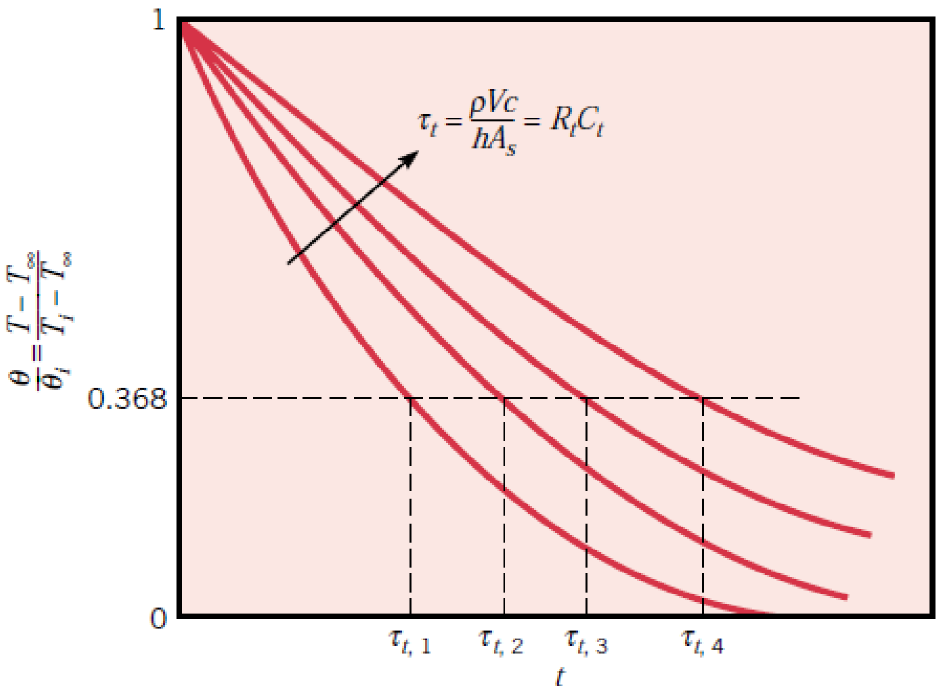
$$\frac{\rho V c}{h A_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

$$\theta_i \equiv T_i - T_\infty$$

$$\frac{\rho V c}{h A_s}\ln\frac{\theta_i}{\theta} = t$$

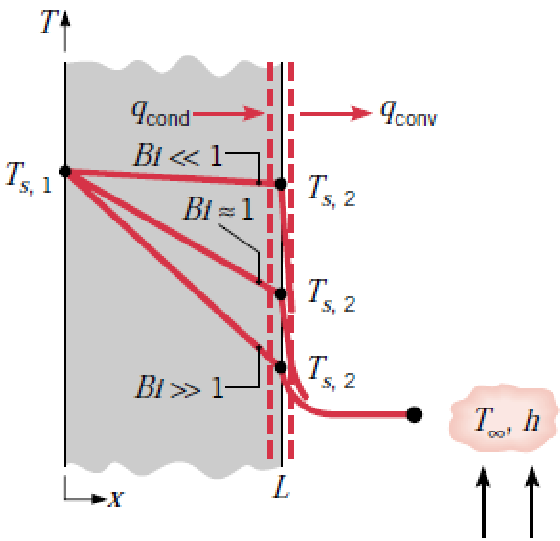
$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right)t\right]$$

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c) = R_t C_t$$



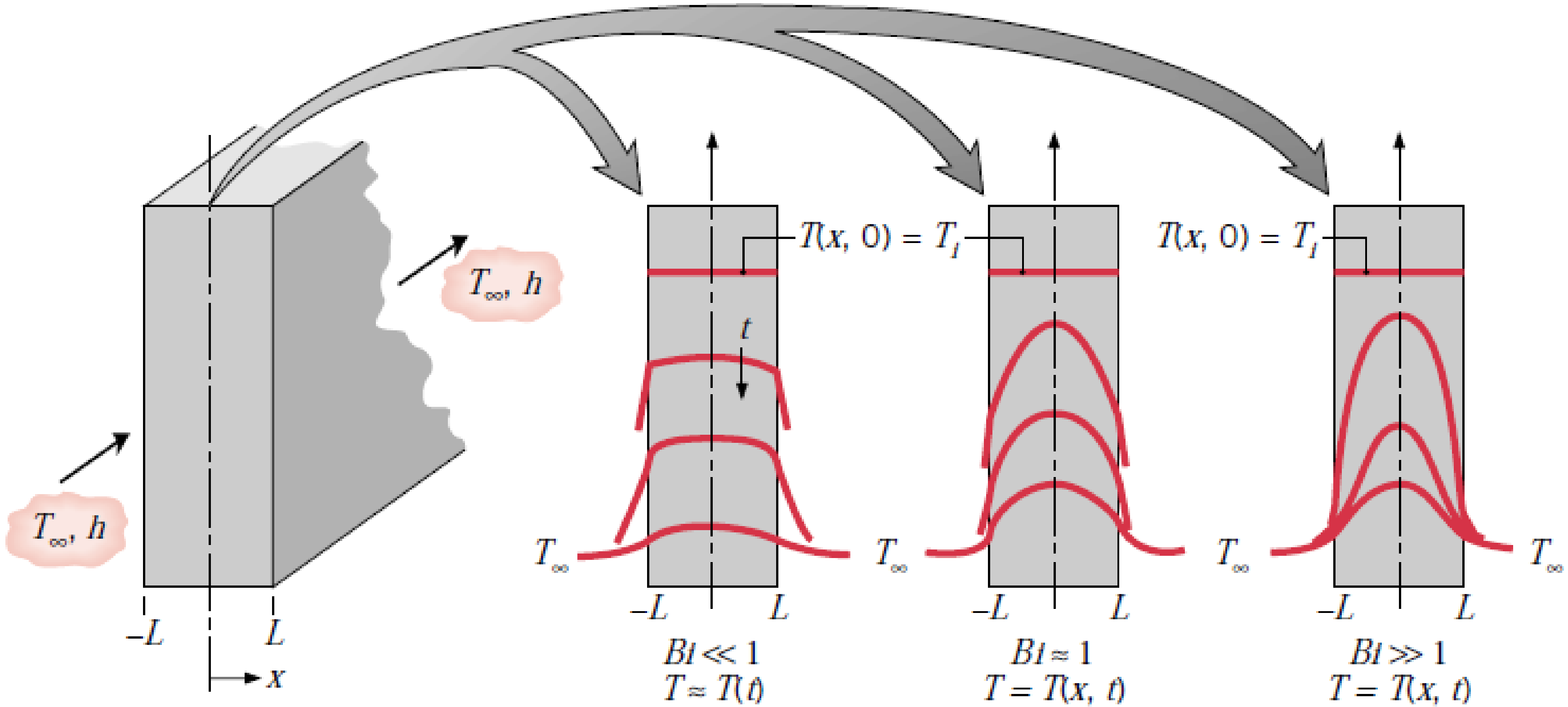
$$Q = \int_0^t q\,dt = hA_s \int_0^t \theta\,dt$$

$$\boxed{Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right]}$$



$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} \equiv Bi$$



$$\boxed{Bi = \frac{hL_c}{k} < 0.1}$$

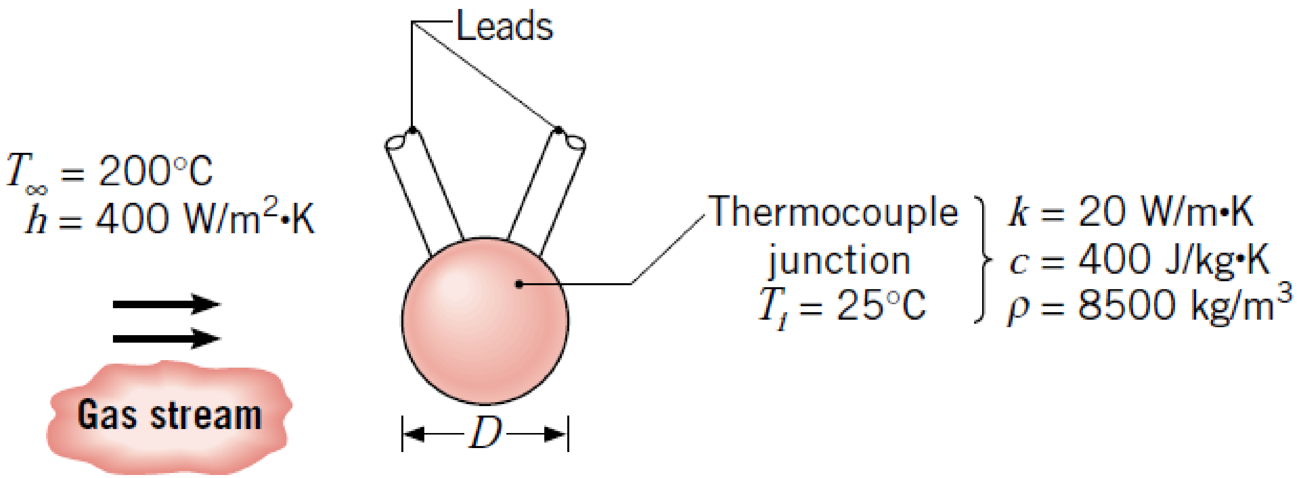
$$L_c \equiv V/A_s$$

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo \qquad \boxed{Fo \equiv \frac{\alpha t}{L_c^2}}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \mathrm{W/m^2 \cdot K}$, and the junction thermophysical properties are $k = 20\ \mathrm{W/m \cdot K}$, $c = 400\ \mathrm{J/kg \cdot K}$, and $\rho = 8500\ \mathrm{kg/m^3}$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?



$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400\ \mathrm{W/m^2 \cdot K} \times 1\ \mathrm{s}}{8500\ \mathrm{kg/m^3} \times 400\ \mathrm{J/kg \cdot K}} = 7.06 \times 10^{-4}\ \mathrm{m}$$

$$Bi = \frac{h(r_o/3)}{k} = \frac{400\ \mathrm{W/m^2 \cdot K} \times 3.53 \times 10^{-4}\ \mathrm{m}}{3 \times 20\ \mathrm{W/m \cdot K}} = 2.35 \times 10^{-3}$$

$$t = \frac{\rho(\pi D^3/6)\,c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6h} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{8500\ \mathrm{kg/m^3} \times 7.06 \times 10^{-4}\ \mathrm{m} \times 400\ \mathrm{J/kg \cdot K}}{6 \times 400\ \mathrm{W/m^2 \cdot K}} \ln \frac{25 - 200}{199 - 200}$$

$$t = 5.2\ \mathrm{s} \approx 5\tau_t$$

Spatial Effects

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x, 0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L, t) - T_\infty]$$

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$$

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

$$x^* \equiv \frac{x}{L}$$

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

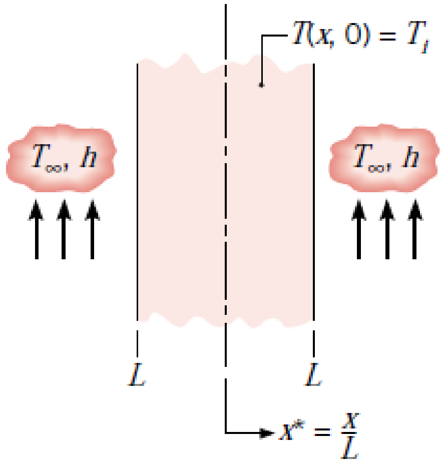
$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo}$$

$$\theta^*(x^*, 0) = 1$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*)$$

$$\theta^* = f(x^*, Fo, Bi)$$



$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)}$$

$$\zeta_n \tan \zeta_n = Bi$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$t = p\,\Delta t$$

$$\left.\frac{\partial T}{\partial t}\right|_{m,\,n} \approx \frac{T_{m,\,n}^{p+1} - T_{m,\,n}^{p}}{\Delta t}$$

$$\frac{1}{\alpha}\frac{T_{m,\,n}^{p+1} - T_{m,\,n}^{p}}{\Delta t} = \frac{T_{m+1,\,n}^{p} + T_{m-1,\,n}^{p} - 2T_{m,\,n}^{p}}{(\Delta x)^2} + \frac{T_{m,\,n+1}^{p} + T_{m,\,n-1}^{p} - 2T_{m,\,n}^{p}}{(\Delta y)^2}$$

$$T_{m,\,n}^{p+1} = Fo\left(T_{m+1,\,n}^{p} + T_{m-1,\,n}^{p} + T_{m,\,n+1}^{p} + T_{m,\,n-1}^{p}\right) + (1 - 4Fo)\,T_{m,\,n}^{p}$$

$$Fo = \frac{\alpha\,\Delta t}{(\Delta x)^2}$$

$$T_m^{p+1} = Fo(T_{m+1}^p + T_{m-1}^p) + (1 - 2Fo)\,T_m^p$$

$$(1 - 2Fo) \geq 0$$

$$Fo \leq \frac{1}{2}$$

$$(1 - 4Fo) \geq 0$$

$$Fo \leq \frac{1}{4}$$

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2}$$

$$(1 + 4Fo)\,T_{m,n}^{p+1} - Fo\left(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}\right) = T_{m,n}^{p}$$