Spreading of a spot in a resting fluid

Spot of tracer $\Rightarrow$ $\quad C_1 = C_0\,\delta(z) = \dfrac{M}{A}\,\delta(z)$ .... Initial condition

M = total amount of solute in the system

A = cross-sectional area over which diffusion occurs

$\delta(z)$ = Dirac function

Governing Eqn. $\quad \dfrac{\partial c_1}{\partial t} = D\,\dfrac{\partial^2 c_1}{\partial z^2}$

Boundary Conditions: Far from the pulse, the solute conc^n is zero

$$\Rightarrow \quad t > 0, \quad z = \infty, \quad c_1 = 0$$

Also $\displaystyle\int_{-\infty}^{\infty} c_1 A\,dz = \int_{-\infty}^{\infty} \frac{M}{A}\,\delta(z) A\,dz = M$ (Property of Dirac Function)

Symmetry in conc^n profile across $z = 0$.

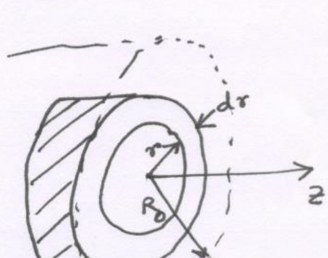
<u>Solution</u>

$$c_1 = \frac{M/A}{\sqrt{4\pi D t}}\, e^{-\frac{z^2}{4Dt}}$$

when the pulse moves through a capillary at uniform velocity $\bar{u}$,

$$c_1 = \frac{M/A}{\sqrt{4\pi D t}}\, e^{-\frac{(z-\bar{u}t)^2}{4Dt}}$$

When the velocity profile becomes non-uniform, the governing equation changes, along with BCs



$$\frac{\partial c_1}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_1}{\partial r}\right) - 2v_0\left[1-\left(\frac{r}{R_0}\right)^2\right]\frac{\partial c_1}{\partial z}$$

$t = 0$, all z, $\quad c_1 = \left(\dfrac{M}{\pi R_0^2}\right)\delta(z)$

$t > 0$, $r = R_0$, $\quad \dfrac{\partial c_1}{\partial r} = 0$ (no flow)

$t > 0$, $r = 0$, $\quad \dfrac{\partial c_1}{\partial r} = 0$ (symmetry)

$$\bar{c}_1(z) = \frac{1}{\pi R_0^2}\int_0^{R_0} 2\pi r\, c_1(r,z)\,dr$$

Solution

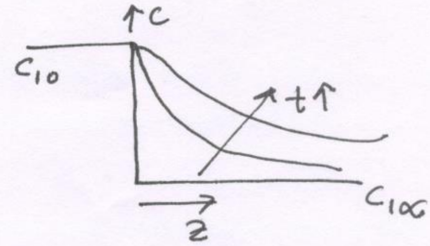
$$\bar{c}_1 = \frac{M/(\pi R_0^2)}{\sqrt{4\pi E_z t}}\, e^{-\frac{(z-v_0 t)^2}{4E_z t}}$$

(Taylor Dispersion)

Here, E_z = Dispersion coefficient $= \dfrac{(R_0 v_0)^2}{48D}$

Diffusion of a concentration front

Spreading of a front, separating two regions: One occupied by a tracer with diffusion coefficient D initially. Other region is with no tracer initially



Governing Equation remains same as before

$$\frac{\partial c_1}{\partial t} = D \frac{\partial^2 c_1}{\partial z^2}$$

$t = 0$, all z, $c_1 = c_{1\infty}$

$t > 0$, $z = 0$, $c_1 = c_{10}$

$t > 0$, $z = \infty$, $c_1 = c_{1\infty}$

Solution

$$\frac{c_1 - c_{10}}{c_{1\infty} - c_{10}} = \operatorname{erf}\left(\frac{z}{\sqrt{4Dt}}\right)$$

when $c_{1\infty} = 0$, $c_1(z,t) = c_0\left[1 - \operatorname{erf}\left(\frac{z}{\sqrt{4Dt}}\right)\right]$

the diffusion in negative z will be mirror image, which will reduce $c_1(z,t)$ by half.

Instead of static system, when the front moves at average velocity $\bar{u}$, z will be replaced by $(z - \bar{u}t)$.

Further, when the velocity profile is laminar (parabolic), D is to be replaced by $E_z \equiv D_{ct} = \frac{R^2 \bar{u}^2}{48D}$

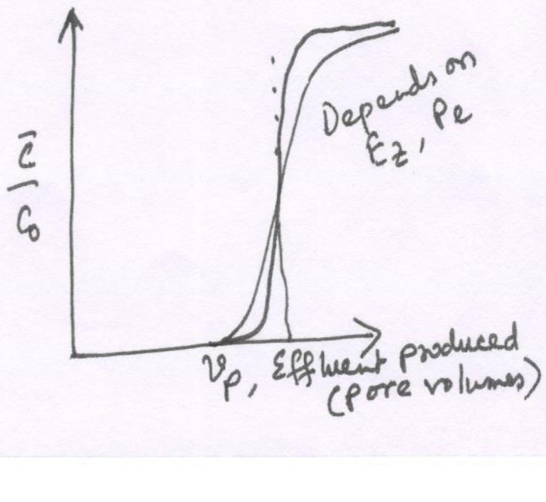
where R is the radius of the capillary.

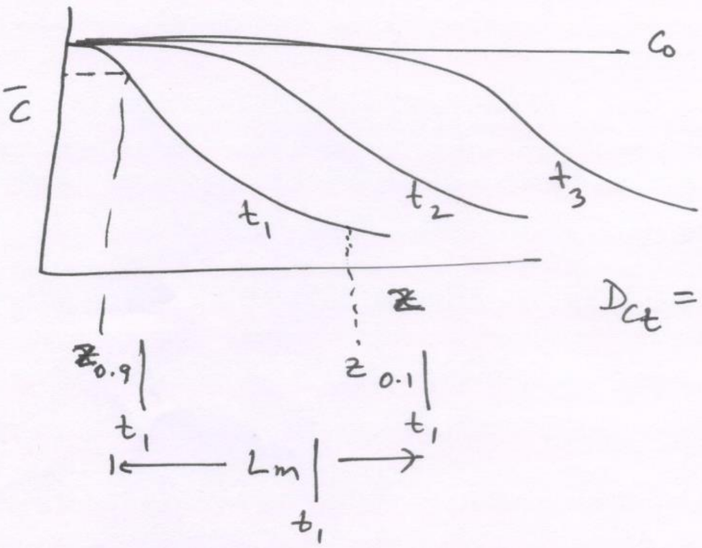
Aris has shown $D_{ct} = D + \frac{R^2 \bar{u}^2}{48D}$, when axial diffusion over and above radial diffusion is considered.

Finally $$\frac{\bar{c}}{c_0} = \frac{1}{2}\left[1 - \operatorname{erf} \frac{z - \bar{u}t}{\sqrt{4 D_{ct} t}}\right]$$

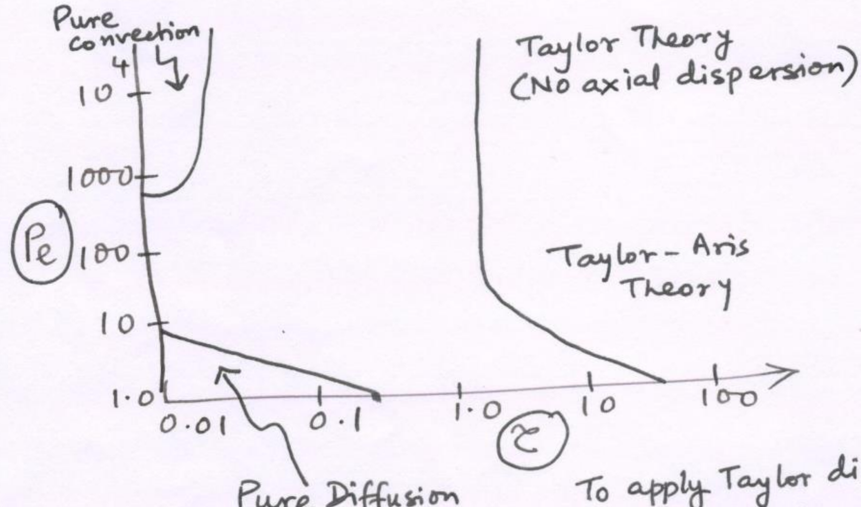
$$\text{Peclet No.} = \frac{\text{Convective transport}}{\text{diffusive transport}} = \frac{2R\bar{u}}{D}$$

Dimensionless time $\tau = \frac{tD}{R^2}$





$$D_{ct} = \frac{1}{t}\left[\frac{L_m}{3.625}\right]^2$$



To apply Taylor dispersion, velocity profile has to reach the walls (fully developed flow). This is possible only when $\tau > 0.8$

Dispersion coefficient for channels with non-circular cross-section

→ Flow between two parallel plates $D_{ct} = D + \frac{8}{945}\frac{u_{max}^2 h^2}{D}$

where h is half-width of the channel

→ Flow in concentric annulus $D_{ct} = D + \frac{8}{945}\left[\frac{R_0^2 u_{max}^2 (1-k^2)}{4D}\right]$

Where $\frac{R_0}{R_i} = \frac{1}{k} \leq 1.5$ (narrow gap)

so that the equation for parallel plate can be extended

When $\frac{1}{k} > 100$, R_i is negligible, compared to R_0

$\Rightarrow D_{ct}$ approaches value of a tube.

→ Equation for stagnant region in capillary wall