Solving PDE through integral trans-form.

30/10/17
8 ~ 9:55 a.m
Lecture - 26

$$ar + bs + ct + f(x, y, z, p, q) = 0 \longrightarrow (1)$$

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

$$(a, b, c) \equiv (a, b, c)(x, y).$$

$b^2 - 4ac > 0$ (1) $\rightarrow$ hyperbolic; $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c_0^2}\dfrac{\partial^2 u}{\partial t^2}$. (wave eq.)

$b^2 - 4ac = 0$ (1) $\rightarrow$ parabolic, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$. (heat eq.)

$b^2 - 4ac < 0$ (1) $\rightarrow$ elliptic; $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. (Laplace eq.)

Note: (1) is linear.

(I) Fourier sine transform of $f''(x) = \dfrac{d^2 f}{dx^2}$; $0 < x < \infty$.

$$\mathcal{F}_s[f''(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin \omega x \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[f'(x) \sin \omega x \right]_0^\infty - \omega \int_0^\infty f'(x) \cos \omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(0 - 0) - \omega \int_0^\infty f'(x) \cos \omega x \, dx \right]$$ assuming $f'(x) = 0$ as $x \to \infty$.

$$= -\sqrt{\frac{2}{\pi}}\, \omega \left[\left[f(x) \cos \omega x \right]_0^\infty + \omega \int_0^\infty f(x) \sin \omega x \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}\, \omega \left[\{0 - f(0)\} + \omega \int_0^\infty f(x) \sin \omega x \, dx \right]$$ assuming $f(x) \to 0$ as $x \to \infty$.

So, $\mathcal{F}_s\left[f''(x)\right] = +\sqrt{\frac{2}{\pi}}\,\omega f(0) - \sqrt{\frac{2}{\pi}}\,\omega^2 \int_0^\infty f(x)\sin\omega x\, dx$.

$$\mathcal{F}_s\left[f''(x)\right] = \sqrt{\frac{2}{\pi}}\,\omega f(0) - \omega^2 \mathcal{F}_s\left[f(x)\right].$$

(B) Fourier cosine transform of $f''(x)$; $0 < x < \infty$.

$$\mathcal{F}_c\left[f''(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x)\cos\omega x\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[f'(x)\cos\omega x\right]_0^\infty + \omega\int_0^\infty f'(x)\sin\omega x\, dx\right].$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - f'(0)\right) + \omega\left\{\left[f(x)\sin\omega x\right]_0^\infty - \omega\int_0^\infty f(x)\cos\omega x\, dx\right\}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) + \sqrt{\frac{2}{\pi}}\,\omega\left\{(0-0) - \omega\int_0^\infty f(x)\cos\omega x\, dx\right\}.$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) - \omega^2 F_c(\omega).$$

$$\therefore\ \mathcal{F}_c\left[f''(x)\right] = -\sqrt{\frac{2}{\pi}}\, f'(0) - \omega^2 F_c(\omega)$$

assuming $f(x), f'(x) \to 0$ as $x \to \infty$.

(c) Fourier transform of $f''(x)$, $-\infty < x < \infty$.

$$\mathcal{F}[f''(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(x)\, e^{i\omega x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[f'(x)\, e^{i\omega x} \right]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} f'(x)\, e^{i\omega x}\, dx \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[(0-0) - i\omega \left\{ \left[f(x)\, e^{i\omega x} \right]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \right] \right].$$

$$= -\frac{i\omega}{\sqrt{2\pi}} \left[(0-0) - i\omega \cdot \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \right].$$

$$= (-i\omega)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx.$$

So, $\mathcal{F}[f''(x)] = (-i\omega)^2 F(\omega)$,

assuming $f(x)$, $f'(x) \to 0$ as $|x| \to \infty$.

i.e as $x \to \pm\infty$.

Ex 1. Solve the Boundary Value Problem (BVP)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad -\infty < x < \infty; \; t > 0 \longrightarrow (1)$$

such that

(i) $u(x,0) = f(x)$ for all x . $\rightarrow$ (known function) .

(ii) u and u_x are both zero at $x = \pm\infty$

Solution. Apply Fourier transform w.r.to x on both sides of (1):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t} e^{i\omega x} \, dx .$$

or, $$(-i\omega)^2 U(\omega, t) = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) e^{i\omega x} \, dx .$$

$$= \frac{\partial}{\partial t} U(\omega, t) .$$

where $$U(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\omega x} \, dx .$$

or, $$-\omega^2 U(\omega, t) = \frac{d}{dt} U(\omega, t)$$

$$\frac{d}{dt} U(\omega, t) + \omega^2 U(\omega, t) = 0 .$$

or, $$\frac{dU(\omega, t)}{U(\omega, t)} = -\omega^2 dt .$$

Integrating. $$\ln U(\omega, t) = -\omega^2 t + \ln c .$$

$$\therefore \; U(\omega, t) = c e^{-\omega^2 t} \longrightarrow (*)$$

We have, $u(x,0) = f(x)$.

Take Fourier transform w.r.to x on both sides:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{i\omega x} dx = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx}.$$

or, $U(\omega, 0) \qquad = F(\omega)$.

$\therefore \; U(\omega, 0) = F(\omega)$

In (*) put $t=0$; which gives ~~$U(\omega,t) = c$~~.

$$U(\omega, 0) = c = F(\omega).$$

$$\therefore U(\omega, t) = F(\omega) e^{-\omega^2 t}.$$

So, $u(x,t) =$ inverse F.T. of $U(\omega, t)$,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega, t) e^{-i\omega x} d\omega.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-\omega^2 t} e^{-i\omega x} d\omega.$$

$$\therefore u(x,t) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) e^{i\omega u} du \right) e^{-\omega^2 t} e^{-i\omega x} d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) du \left(\int_{-\infty}^{\infty} e^{-\omega^2 t + i\omega(u-x)} d\omega. \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) du \left[\int_{-\infty}^{\infty} e^{-t\left\{\omega^2 - i\omega\frac{(u-x)}{t}\right\}} d\omega \right].$$

Note. $$\omega^2 - i\omega\frac{(u-x)}{t} = \omega^2 - 2.\omega.\frac{i(u-x)}{2t} - \frac{(u-x)^2}{4t^2} + \frac{(u-x)^2}{4t^2}.$$

$$= \left\{\omega - \frac{i(u-x)}{2t}\right\}^2 + \frac{(u-x)^2}{4t^2}.$$

So, $$e^{-t\left\{\omega^2 - i\omega\frac{(u-x)}{t}\right\}}.$$

$$= e^{-t\left[\left\{\omega - \frac{i(u-x)}{2t}\right\}^2 + \frac{(u-x)^2}{4t^2}\right]}$$

$$= e^{-t\left\{\omega - i\frac{(u-x)}{2t}\right\}^2} e^{-\frac{(u-x)^2}{4t}}.$$

$$\therefore u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du \int_{-\infty}^{\infty} e^{-t\left\{\omega - i\frac{(u-x)}{2t}\right\}^2} e^{-\frac{(u-x)^2}{4t}} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(u-x)^2}{4t}} du \int_{-\infty}^{\infty} e^{-t\left\{\omega - \frac{i(u-x)}{2t}\right\}^2} d\omega.$$

For the inner integral, substitute

$$\sqrt{t}\left\{\omega - \frac{i(u-x)}{2t}\right\} = v.$$

$$\therefore \sqrt{t}\,d\omega = dv. \quad \therefore d\omega = \frac{dv}{\sqrt{t}}.$$

$$\therefore u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(u-x)^2}{4t}} du \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-v^2} dv \to \sqrt{\pi}$$

$$= \frac{1}{2\pi} \times \frac{\sqrt{\pi}}{\sqrt{t}} \int_{-\infty}^{\infty} f(u)\, e^{-\frac{(u-x)^2}{4t}} du.$$

$$\therefore \quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(u)\, e^{-\frac{(u-x)^2}{4t}}\, du .$$

Ex-2. Solve the PDE

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0, \quad k = \text{const}. \qquad (1).$$

such that $u(0,t) = u_0$ (B.C., const.), $t \geq 0$ — This is a B.C. of Dirichlet type.

$u(x,0) = 0 \quad \forall x$. (I.C.)

$$\left[L[f'(t)] = s\bar{f}(s) - f(0) \qquad L\left[\frac{\partial u(x,t)}{\partial t}\right] = s\,\bar{u}(x,s) - u(x,0) \right]$$

Case I Apply Fourier sine transform on both sides of eq. (1).

$$k\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \omega x \, dx$$

$$\text{or, } k\sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \omega x\right]_0^{\infty} - \omega \int_0^{\infty} \frac{\partial u}{\partial x} \cos \omega x \, dx \right] = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial t} \int_0^{\infty} u(x,t) \sin \omega x \, dx.$$

$$\text{or, } k\sqrt{\frac{2}{\pi}} \left[0 - \omega \left\{ \left[u(x,t) \cos \omega x\right]_0^{\infty} + \omega \int_0^{\infty} u(x,t) \sin \omega x \, dx \right\} \right] = \frac{d}{dt} \bar{U}_s(\omega, t)$$

$$\text{or, } -k\sqrt{\frac{2}{\pi}}\, \omega \left[-u(0,t) + \omega \int_0^{\infty} u(x,t) \sin \omega x \, dx \right] = \frac{d}{dt} \bar{U}_s(\omega, t).$$

or, $\sqrt{\frac{2}{\pi}} k\omega u_0 - k\omega^2 U_s(\omega, t)$.

$$= \frac{d}{dt} U_s(\omega, t).$$

where $$U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} u(x, t) \sin \omega x \, dx.$$

or, $$\frac{d}{dt} U_s(\omega, t) + k\omega^2 U_s(\omega, t) = \sqrt{\frac{2}{\pi}} k\omega u_0.$$

$$\equiv \frac{dy}{dx} + P(x) y = Q(x).$$

$$\text{I. F.} = e^{\int P(x)\,dx} = e^{\int k\omega^2 dt} = e^{k\omega^2 t}.$$

$$\frac{d}{dt}\left(U_s(\omega, t) e^{k\omega^2 t}\right) = \sqrt{\frac{2}{\pi}} k\omega u_0 e^{k\omega^2 t}.$$

Integrate, $$U_s(\omega, t) e^{k\omega^2 t} = \sqrt{\frac{2}{\pi}} k\omega u_0 \int e^{k\omega^2 t} dt + C.$$

So, $$U_s(\omega, t) e^{k\omega^2 t} = \sqrt{\frac{2}{\pi}} \cancel{k}\cancel{\omega} u_0 \frac{e^{k\omega^2 t}}{\cancel{k}\omega^{\cancel{2}}} + C.$$

or, $$U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} + C e^{-k\omega^2 t} \longrightarrow (*)$$

Putting $t = 0$, $$U_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} + C.$$

$$\therefore C = U_s(\omega, 0) - \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega}.$$

Now,

$u(x,0)=0,$

$\therefore \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin\omega x\,dx = 0.$

or, $U_s(\omega,0)=0.$

So, $c=-\sqrt{\frac{2}{\pi}}\frac{u_0}{\omega}.$

Subst. c into (*) get

$$U_s(\omega,t)=\sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega}-\sqrt{\frac{2}{\pi}}\cdot\frac{u_0}{\omega}e^{-k\omega^2 t}.$$

$$U_s(\omega,t)=\sqrt{\frac{2}{\pi}}u_0\left[\frac{1-e^{-k\omega^2 t}}{\omega}\right]$$

$$\therefore u(x,t)=\mathcal{F}_s^{-1}\left[U_s(\omega,t)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty U_s(\omega,t)\sin\omega x\,d\omega.$$

$$=\frac{2}{\pi}u_0\int_0^\infty\frac{1-e^{-k\omega^2 t}}{\omega}\sin\omega x\,d\omega.$$

$$=\frac{2}{\pi}u_0\left[\int_0^\infty\frac{\sin\omega x}{\omega}d\omega-\int_0^\infty\frac{e^{-k\omega^2 t}}{\omega}\sin\omega x\,d\omega\right]$$

$$=\frac{2}{\pi}u_0\left[\frac{\pi}{2}-\int_0^\infty\frac{e^{-k\omega^2 t}}{\omega}\sin\omega x\,d\omega\right]$$

$$F\left[e^{-a^2x^2}\right]=F_c\left(e^{-a^2x^2}\right)=\frac{1}{\sqrt{2}a}e^{-\frac{\omega^2}{4a^2}}.$$

$$I(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos \omega x \, dx = \frac{1}{\sqrt{2}\,a} e^{-\frac{\omega^2}{4a^2}}.$$

or, $$I(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2\omega^2} \cos x\omega \, d\omega = \frac{1}{\sqrt{2}\,a} e^{-\frac{x^2}{4a^2}}.$$

Let

$$J(x) = \int_0^\infty \frac{e^{-k\omega^2 t}}{\omega} \sin \omega x \, d\omega.$$

$$\therefore J'(x) = \int_0^\infty e^{-k\omega^2 t} \cos \omega x \, d\omega.$$

$$\int_0^\infty e^{-a^2\omega^2} \cos x\omega \, d\omega = \frac{\sqrt{\pi}}{2a} e^{-\frac{x^2}{4a^2}}.$$

If $a^2 = kt$, then, $$\int_0^\infty e^{-kt\omega^2} \cos x\omega \, d\omega = \frac{\sqrt{\pi}}{2\sqrt{kt}} e^{-\frac{x^2}{4kt}}$$

So, $$J'(x) = \frac{\sqrt{\pi}}{\sqrt{4kt}} e^{-\frac{x^2}{4kt}}.$$

$$\therefore J(x) = \int_0^x \frac{\sqrt{\pi}}{\sqrt{4kt}} e^{-\frac{u^2}{4kt}} \, du.$$

$$= \frac{\sqrt{\pi}}{\sqrt{4kt}} \int_0^x e^{-\frac{u^2}{4kt}} \, du.$$

Remember

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}\, d\alpha .$$

Now,

$$J(x) = \frac{\sqrt{\pi}}{\sqrt{4kt}} \cdot \frac{\sqrt{\pi}}{2} \times \frac{2}{\sqrt{\pi}} \int_0^x e^{-\frac{u^2}{4kt}}\, du .$$

Put, $\dfrac{u}{2\sqrt{kt}} = \alpha$.

$$\therefore\ J(x) = \frac{\pi}{\cancel{2}\sqrt{4kt}} \times \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\alpha^2}\, d\alpha \times \cancel{2}\sqrt{kt} .$$

$$= \frac{\pi}{2} \times \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\alpha^2}\, d\alpha = \frac{\pi}{2}\, \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

$$u(x,t) = \frac{2}{\pi} u_0 \left[\frac{\pi}{2} - \underbrace{\int_0^\infty \frac{e^{-k\omega^2 t}}{\omega} \sin\omega x\, d\omega}_{J(x)}\right].$$

$$= \frac{2}{\pi} u_0 \left[\frac{\pi}{2} - \frac{\pi}{2}\, \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right]$$

$$= u_0 \left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right] = u_0\, \mathrm{erfc}\left(\frac{x}{\sqrt{4kt}}\right).$$

13th Nov → class test.

syllabus →. Fourier series, Fourier transfom.

~~End~~ End - sem → entire syllabus.

<u>Procedure II.</u> Laplace transform.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \longrightarrow (1).$$

Define $\bar{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt.$

Apply Laplace transform on both sides w.r.to. t,

to get $\underline{\int_0^\infty \frac{\partial u}{\partial t}(x,t)e^{-st}dt} = k\int_0^\infty \frac{\partial^2}{\partial x^2}u(x,t)e^{-st}dt.$

$\int_0^\infty f'(t)e^{-st}dt.$
$= s\bar{f}(s) - f(0)$

or,

$$s\bar{u}(x,s) - \underset{\overset{\shortparallel}{0\text{(given)}}}{u(x,0)} = k\frac{d^2}{dx^2}\bar{u}(x,s)$$

or, $\frac{d^2}{dx^2}\bar{u}(x,s) - \frac{s}{k}\bar{u}(x,s) = 0.$

aux. eqn. $m^2 - \frac{s}{k} = 0.$

$m = \pm\sqrt{\frac{s}{k}}.$

$\bar{u}(x,s) = c_1(s)e^{-\sqrt{\frac{s}{k}}x} + c_2(s)e^{\sqrt{\frac{s}{k}}x}.$

As $x \to \infty$, the solution must be bounded.

so, $c_2(s) \equiv 0.$

$\therefore \bar{u}(x,s) = c_1(s)e^{-\sqrt{\frac{s}{k}}x}.$

or, $\bar{u}(0,s) = c_1(s)$

~~as~~. $u(0,t) = u_0$ (given).

$$L[u(0,t)] = L[u_0] = \frac{u_0}{s}.$$

or, $\bar{u}(0,s) = \frac{u_0}{s}$.

$\therefore\ c_1(s) = \frac{u_0}{s}$.

$$\therefore\ \bar{u}(x,s) = \frac{u_0}{s} e^{-\sqrt{\frac{s}{k}}\,x}.$$

$$\therefore\ u(x,t) = u_0\, L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{k}}\,x}}{s}\right] \rightarrow (**).$$

$$L\left[\mathrm{erf}\left(\frac{a}{2\sqrt{t}}\right)\right] = \frac{1-e^{-a\sqrt{s}}}{s}.$$

$$L\left[\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right] = L\left[1 - \mathrm{erf}\left(\frac{a}{2\sqrt{t}}\right)\right].$$

$$= \cancel{\frac{1}{s}} - \cancel{\frac{1}{s}} + \frac{e^{-a\sqrt{s}}}{s} \rightarrow (\#).$$

Compare $(**)$ & $(\#)$.

$$a = \frac{x}{\sqrt{k}}.$$

$$\therefore\ u(x,t) = u_0\, \mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

$$= u_0\, \mathrm{erfc}\left(\frac{x}{\sqrt{4kt}}\right).$$