① Generalized 2nd Order Operator.

- $L(u) = a_0 \frac{d^2u}{dx^2} + a_1 \cdot \frac{du}{dx} + a_0 \cdot u$

$\downarrow$

"Sturm Louville operator".

- we have demonstrated that $L = L^*$.
- Now we'll test if $B = B^*$.

$$\left.\begin{aligned} Lu &= -\lambda \cdot r \cdot u \\ \left(\alpha_1 \frac{du}{dx} + \alpha_2 \cdot u\right)_{x=0} &= 0 \\ \left(\beta_1 \frac{du}{dx} + \beta_2 \cdot u\right)_{x=1} &= 0 \end{aligned}\right\} B.$$

- $J(u, v) = \left[v \cdot a_0 \cdot u' - v' \cdot a_0 \cdot u - v \cdot \cancel{a_0'} \cdot u + \cancel{a_1} v \cdot u \right]_a^b$

$\because a_0' = a_1$

where
$a_0 = P$
$a_1 = P'$
$a_2 = q$
$a_3 = r$

$$= \left[v \cdot a_0 \cdot u' - v' \cdot a_0 \cdot u \right]_a^b$$

$$= v(a) \cdot a_0(a) \cdot u'(a) - v(b) \cdot a_0(b) \cdot u'(b)$$
$$- v'(a) \cdot a_0(a) \cdot u(a) + v'(b) \cdot a_0(b) \cdot u(b)$$

$$= a_0(b) \cdot (-u(b)) \left[\frac{\beta_2}{\beta_1} \cdot v(b) + v'(b) \right]$$

$$+ \ a_0(a) \cdot u(a) \cdot \left[\frac{\alpha_2}{\alpha_1} \cdot v(a) + v'(a) \right]$$

$$\left.\begin{aligned} \therefore \ \left(\alpha_1 \cdot \frac{dv}{dx} + \alpha_2 \cdot v \right)_{x=a} &= 0 \\ \left(\beta_1 \cdot \frac{dv}{dx} + \beta_2 \cdot v \right)_{x=b} &= 0 \end{aligned}\right] B^*$$

$$\therefore \quad B = B^*$$

$\therefore$ $L \equiv$ self adjoint.

② Theorem 1 : For S-L op., there are infinite eigenvalues for each set of B.C.

but why? Investigate.

Q what are eigenvalues, actually? I understand the math def., but what is the intuition in this context?

③ orthogonal functions

$\varphi_m(x)$ & $\varphi_n(x)$ are orthogonal w.r.t. weight $r(x)$ iff $\langle \varphi_m , \varphi_n \rangle = 0$

$$\Rightarrow \int_x \varphi_m(x) \cdot \varphi_n(x) \cdot dx = 0.$$

④ Theorem 2

if λ_m & λ_n are distinct eigenvalues
& y_m & y_n are the corresp. eigen-
-functions, then y_m & y_n are orthogonal
w.r.t $r(x)$.

$$Ly = -\lambda \cdot r \cdot y$$

③ = take inner product w.r.t y_m ← $L y_n = -\lambda_n \cdot r \cdot y_n$ ①

④ = take inner product w.r.t y_n ← $L y_m = -\lambda_m \cdot r \cdot y_m$ ②

③ − ④

$$\Rightarrow \int y_m \cdot (L y_n)\, dx - \int y_n \cdot (L y_m)\, dx = (\lambda_m - \lambda_n) \cdot \int y_m \cdot y_n \cdot r \cdot dx$$

$$\Rightarrow \int y_n \cdot \cancel{L^*} y_m\, dx + J(u, v) - \int y_n \cdot \cancel{(L y_m)}\, dx = \text{''}$$

$L = L^*$ & $J(u, v) = 0$,

$$\Rightarrow \underbrace{(\lambda_m - \lambda_n)}_{\neq 0} \cdot \int y_m \cdot y_n \cdot r \cdot dx = 0$$

$\neq 0$ (∵ distinct by def.)

$$\therefore \int y_m \cdot y_n \cdot r \cdot dx = 0$$

∴ y_m & y_n are orthogonal

(5) Summary

1. Characteristics of PDEs
2. Various B.Cs
3. Definition of well & ill posed problems
   - conversion of ill posed problem to well posed problem.
4. Special ODEs
   - corresp. eigenvalue problems.
5. Adjoint operator & its B.Cs
6. S-L operator
7. Assorted theorems.

Cartesian Coordinates / Parabolic PDEs.

(1) Well posed problem.

(i) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t=0) = u_0$$

$$u(0, t) = 0$$

$$u(1, t) = 0$$

$\Rightarrow$ well posed.

② $\rho \cdot c_p \cdot \frac{\partial T}{\partial x} = k \cdot \frac{\partial^2 T}{\partial x^2}$

$T(x, 0) = T_0$ ⇒ ill posed.

$T(a, t) = T_1$ ⇒ can be converted to a well posed problem by change of vars.

$T(b, t) = T_2$

② Separation of variables

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(x, t=0) = u_0$

$u(0, t) = 0$

$u(1, t) = 0$

Assume : $u = X(x) \cdot T(t)$

$\therefore X \cdot \frac{dT}{dt} = T \cdot \frac{d^2 X}{dx^2}$

$$\therefore \underbrace{\frac{1}{T} \frac{dT}{dt}}_{f(t)} = \underbrace{\frac{1}{X} \cdot \frac{d^2 X}{dx^2}}_{f(x)} = -\lambda$$

constant

$\therefore$ LHS = RHS = const.

Case 1 : let $\lambda = \alpha^2$ (i.e. positive)

$$\therefore \frac{d^2 X}{dx^2} + \alpha^2 \cdot X = 0 .$$

$$X(x=0) = 0$$
$$X(x=1) = 0$$

$$\therefore X_n = c_n \cdot \sin(n\pi x),$$

where $\alpha_n = n\pi$, $n = 1, 2, \ldots$

$$\therefore X = \sum_{n=1}^{\infty} X_n \quad \text{(superposition)}$$

$$= \sum_{n=1}^{\infty} c_n \cdot \sin(n\pi x) .$$

Similarly for $T \rightarrow$

$$\frac{1}{T_n} \cdot \frac{dT_n}{dt} = -\alpha^2 = -n^2 \cdot \pi^2$$

$$\therefore T_n = c_1 \cdot \exp(-n^2 \cdot \pi^2 \cdot t^2)$$

$$\therefore \quad u_n(x,t) = X_n \cdot T_n$$

$$= c_n' \sin(n\pi x) \cdot \exp(-n^2 \pi^2 \cdot t^2)$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} u_n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n' \cdot \sin(n\pi x) \cdot \exp(-n^2 \pi^2 \cdot t^2)}$$

Applying the I.C. $\rightarrow$

$$u(x,0) = u_0 .$$

$$\therefore \quad u_0 = \sum c_n' \cdot \sin(n\pi x)$$

- multiply by $\sin(m\pi x)$

$$\therefore \quad u_0 \cdot \int_0^1 \sin(m\pi x)\,dx = \sum_{n=1}^{\infty} c_n \cdot \int \sin(m\pi x) \cdot \sin(n\pi x)\,dx$$

$$=$$

case 2 : $\lambda = -\alpha_n^2$

$$\therefore \quad \frac{d^2 X}{dx^2} + \alpha_n^2 \cdot X = 0$$

$$\left.\frac{dX}{dx}\right|_{x=0} = 0$$

$$X(x=0) = 1$$

$$\Rightarrow \quad \alpha = (2n-1)\cdot\frac{\pi}{2}.$$

$$\frac{1}{T_n}\cdot\frac{dT}{dt} = -\alpha_n^2$$

$$\therefore \quad T_n = c_2 \cdot e^{-\alpha_n^2 \cdot t}$$

$$\therefore \quad U_n = C_n \cdot \cos(\alpha_n \cdot x) \cdot e^{-\alpha_n^2 \cdot t}$$

$$\therefore \quad U = \sum_{n=1}^{\infty} C_n \cdot \cos(\alpha_n \cdot x) \cdot e^{-\alpha_n^2 \cdot t}.$$

- apply the I.C. $U = u_0$ @ $t = 0$

$$\therefore \quad u_0 = \sum_{n=1}^{\infty} C_n \cdot \cos(\alpha_n \cdot x) \cdot 1$$

- multiply by $\cos(\alpha_m \cdot x)$ & integrate

$$\therefore \int_0^1 u_0 \cdot \cos(\alpha_m \cdot x)\,dx = \int_0^1 C_n \cdot \cos(\alpha_n \cdot x) \cdot \cos(\alpha_m \cdot x)\,dx$$

- put $m \rightarrow n$

$$\therefore \int_0^1 u_0 \cdot \cos(\alpha_n \cdot x)\, dx = c_n \underbrace{\int_0^1 \cos^2(\alpha_n x)\, dx}_{= 1/2}$$

(evaluate)

$$\therefore \int_0^1 u_0 \cdot \cos(\alpha_n \cdot x)\, dx = \frac{c_n}{2}.$$

$$\therefore \quad c_n = \frac{2 \cdot u_0 \cdot \sin(\alpha_n \cdot x)}{\alpha_n}.$$

$$\therefore \boxed{u(x,t) = 2 \cdot u_0 \cdot \sum_{n=1}^{\infty} \frac{\sin(\alpha_n \cdot x)}{\alpha_n} \cdot \cos(\alpha_n \cdot x) \cdot e^{-\alpha_n^2 \cdot t}}$$