3.1 Existence of Laplace Transform

In this lesson we shall discuss existence theorem on Laplace transform. Since every Laplace integral is not convergent, it is very important to know for which functions Laplace transform exists.

Consider the function $f(t) = e^{t^2}$ and try to evaluate its Laplace integral. In this case we realize that

$$\lim_{R\to\infty} \int_0^R e^{t^2-st}\,\mathrm{d}t = \infty, \text{ for any choice of } s$$

Naturally question arises in mind that for which class of functions, the Laplace integral converges? So before answering this question we go through some definition.

3.2 Piecewise Continuity

A function f is called piecewise continuous on $[a,b]$ if there are finite number of points $a < t_1 < t_2 < \ldots < t_n < b$ such that f is continuous on each open subinterval $(a,t_1), (t_1,t_2), \ldots, (t_n,b)$ and all the following limits exists

$$\lim_{t\to a+} f(t),\ \lim_{t\to b-} f(t),\ \lim_{t\to t_j+} f(t), \text{ and } \lim_{t\to t_j-} f(t),\ \forall j.$$

Note: A function f is said to be piecewise continuous on $[0,\infty)$ if it is piecewise continuous on every finite interval $[0,b],\ b \in R_+$.

3.2.1 Example 1

The function defined by

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1; \\ 3-t, & 1 < t \le 2; \\ t+1, & 2 < t \le 3; \end{cases}$$

is piecewise continuous on $[0,3]$.

3.2.2 Example 2

The function defined by

$$f(t) = \begin{cases} \frac{1}{2-t}, & 0 \leq t < 2; \\ t+1, & 2 \leq t \leq 3; \end{cases}$$

is not piecewise continuous on $[0, 3]$.

3.3 Example Problems

3.3.1 Problem 1

Discuss the piecewise continuity of

$$f(t) = \frac{1}{t-1}$$

.

Solution: $f(t)$ is not piecewise continuous in any interval containing 1 since

$$\lim_{t \to 1\pm} f(t)$$

do not exists.

3.3.2 Problem 2

Check whether the function

$$f(t) = \begin{cases} \frac{1-e^{-t}}{t}, & t \neq 0; \\ 0, & \textit{otherwise} \end{cases}$$

is piecewise continuous or not.

Solution: The given function is continuous everywhere other than at 0. So we need to check limits at this point. Since both the left and right limits

$$\lim_{t \to 0-} f(t) = 1 \quad \text{and} \quad \lim_{t \to 0+} f(t) = 1$$

exists, the given function is piecewise continuous.

3.4 Functions of Exponential Orders

A function f is said to be of exponential order α if there exist constant M and α such that for some $t_0 \geq 0$

$$|f(t)| \leq Me^{\alpha t} \text{ for all } t \geq t_0$$

Equivalently, a function $f(t)$ is said to be of exponential order α if

$$\lim_{t \to \infty} e^{-\alpha t} |f(t)| = \text{ a finite quantity}$$

Geometrically, it means that the graph of the function f on the interval (t_0, ∞) does not grow faster than the graph of exponential function $Me^{\alpha t}$

3.5 Example Problems

3.5.1 Problem 1

Show that the function $f(t) = t^n$ has exponential order α for any value of $\alpha > 0$ and any natural number n.

Solution: We check the limit

$$\lim_{t \to \infty} e^{-\alpha t} t^n$$

Repeated application of L'hospital rule gives

$$\lim_{t \to \infty} e^{-\alpha t} t^n = \lim_{t \to \infty} \frac{n!}{\alpha^n e^{\alpha t}} = 0$$

Hence the function is of exponential order.

3.5.2 Problem 2

Show that the function $f(t) = e^{t^2}$ is not of exponential order.

Solution: For given function we have

$$\lim_{t \to \infty} e^{-\alpha t} e^{t^2} = \lim_{t \to \infty} e^{t(t-\alpha)} = \infty$$

for all values of α. Hence the given function is not of exponential order.

3.5.3 Theorem (Sufficient Conditions for Laplace Transform)

If f is piecewise continuous on $[0, \infty)$ and of exponential order α then the Laplace transform exists for $Re(s) > \alpha$. Moreover, under these conditions Laplace integral converges absolutely.

Proof: Since f is of exponential order α, then

$$|f(t)| \leq M_1 e^{\alpha t}, \;\; t \geq t_0 \tag{3.1}$$

Also, f is piecewise continuous on $[0, \infty)$ then

$$|f(t)| \leq M_2, \;\; 0 \leq t \leq t_0 \tag{3.2}$$

From equation (3.1) and (3.2) we have

$$|f(t)| \leq M e^{\alpha t}, \;\; t \geq 0$$

Then

$$\int_0^R |e^{-st} f(t)| dt \leq \int_0^R |e^{-(x+iy)t} M e^{\alpha t}| dt$$

Here we have assumed s to be a complex number so that $s = x + iy$. Noting that $|e^{-iy}| = 1$ we find

$$\int_0^R |e^{-st} f(t)| dt \leq M \int_0^R e^{-(x-\alpha)t} dt$$

On integration we obtain

$$\int_0^R |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha} - \frac{M}{x-\alpha} e^{-(x-\alpha)R}$$

Letting $R \to \infty$ and noting $Re(s) = x > \alpha$, we get

$$\int_0^\infty |e^{-st} f(t)| dt \leq \frac{M}{x-\alpha}$$

Hence the Laplace integral converges absolutely and thus converges. This implies the existence of Laplace transform. For piecewise continuous functions of exponential order, the Laplace transform always exists. Note that it is a sufficient condition, that means if a function is not of exponential order or piecewise continuous then the Laplace transform may or may not exist. ■

Remark 1: *We have observed in the proof of existence theorem that*

$$\left|\int_0^\infty e^{-st} f(t)dt\right| \leq \int_0^\infty |e^{-st} f(t)|dt \leq \frac{M}{Re(s)-\alpha} \quad \textit{for } Re(s) > \alpha$$

We now deduce two important conclusions with this observation:

- $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \to 0$ *as* $Re(s) \to \infty$

- *if* $L[f(t)] \not\to 0$ *as* $s \to \infty$ *(or* $Re(s) \to \infty$*) then* $f(t)$ *cannot be piecewise continuous function of exponential order. For example functions such as* $F_1(s) = 1$ *and* $F_2(s) = s/(s+1)$ *are not Laplace transforms of piecewise continuous functions of exponential order, since* $F_1(s) \not\to 0$ *and* $F_2(s) \not\to 0$ *as* $s \to \infty$.

Remark 2: *It should be noted that the conditions stated in existence theorem are sufficient rather than necessary conditions. If these conditions are satisfied then the Laplace transform must exist. If these conditions are not satisfied then Laplace transform may or may not exist. We can observe this fact in the following examples:*

- *Consider, for example,*

  $$f(t) = 2te^{t^2}\cos(e^{t^2})$$

  Note that $f(t)$ *is continuous on* $[0,\infty)$ *but not of exponential order, however the Laplace transform of* $f(t)$ *exists, since*

  $$L[f(t)] = \int_0^\infty e^{-st} 2te^{t^2}\cos(e^{t^2})dt$$

  Integration by parts leads to

  $$L[f(t)] = e^{-st}\sin(e^{t^2})\Big|_0^\infty + s\int_0^\infty e^{-st}\sin(e^{t^2})dt$$

  Using the definition of Laplace transform we obtain

  $$L[f(t)] = -\sin(1) + sL[\sin(e^{t^2})]$$

  Note that $L[\sin(e^{t^2})]$ *exists because the function* $\sin(e^{t^2})$ *satisfies both the conditions of existence theorem. This example shows that Laplace transform of a function which is not of exponential order exists.*

- *Consider another example of the function*

$$f(t) = \frac{1}{\sqrt{t}},$$

which is not piecewise continuous since $f(t) \to \infty$ *as* $t \to 0$*. But we know that*

$$L[f(t)] = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

This example shows that Laplace transform of a function which is not piecewise continuous exists. These two examples clearly shows that the conditions given in existence theorem are sufficient but not necessary.