13.1 Example Problems

13.1.1 Problem 1

Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & \textit{when } |x| < a \\ 0, & \textit{when } |x| > a \end{cases}$$

and hence evaluate

$$(i) \int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \, \mathrm{d}\alpha, \quad (ii) \int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} \, \mathrm{d}\alpha \;\; \textit{and} \;\; (iii) \int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, \mathrm{d}x.$$

Solution: (i) Let $\hat{f}(\alpha)$ be the Fourier transform of $f(x)$. Then, by the definition of Fourier transform

$$\begin{aligned} \hat{f}(\alpha) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx \\ =& \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} \left(e^{i\alpha a} - e^{-i\alpha a} \right) dx \end{aligned}$$

This gives

$$\hat{f}(\alpha) = \frac{2}{\sqrt{2\pi}} \frac{\sin a\alpha}{\alpha}$$

From the definition of inverse Fourier transform we also know that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} \, \mathrm{d}\alpha$$

This implies that

$$\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} \, \mathrm{d}\alpha = \sqrt{2\pi} f(x) = \begin{cases} \sqrt{2\pi}, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

Substituting $\hat{f}(\alpha)$ in the above equation we get

$$\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin a\alpha}{\alpha} (\cos \alpha x - i \sin \alpha x) \, \mathrm{d}\alpha = \begin{cases} \sqrt{2\pi}, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

We now split the left hand side into real and imaginary parts to get

$$\int_{-\infty}^{\infty} \frac{\sin a\alpha \cos x\alpha}{\alpha} \, \mathrm{d}\alpha - i \int_{-\infty}^{\infty} \frac{\sin \alpha a \sin \alpha x}{\alpha} \, \mathrm{d}\alpha = \begin{cases} \pi, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

Equating real part on both sides we get the desired result as

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \, \mathrm{d}\alpha = \begin{cases} \pi, \text{ when } |x| < a \\ 0, \text{ when } |x| > a \end{cases}$$

(ii) If we set $x = 0$ and $a = 1$ in the above results, we get

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} \, \mathrm{d}\alpha = \pi, \quad \text{Since } |x| < a$$

Since the integrand is an even function, we get the the desired results

$$\int_{0}^{\infty} \frac{\sin \alpha}{\alpha} = \frac{\pi}{2}$$

(ii) We now apply Parseval's identity for Fourier transform

$$\int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2 \, \mathrm{d}\alpha = \int_{-\infty}^{\infty} |f(\alpha)|^2 \, \mathrm{d}\alpha$$

Substituting the function $f(x)$ and its Fourier transform we get

$$\int_{-\infty}^{\infty} \frac{4}{2\pi} \frac{\sin^2 a\alpha}{\alpha^2} \, \mathrm{d}\alpha = \int_{-a}^{a} \mathrm{d}\alpha = 2a$$

This implies

$$\int_{-\infty}^{\infty} \frac{\sin^2 a\alpha}{\alpha^2} \, \mathrm{d}\alpha = \pi a$$

Since the integrand is an even function we have the desired result as

$$\int_{0}^{\infty} \frac{\sin^2 a\alpha}{\alpha^2} \, \mathrm{d}\alpha = \frac{\pi}{2} a$$

13.1.2 Problem 2

Evaluate the Fourier transform of the rectangular pulse function

$$\Pi(t) = \begin{cases} 1, & \textit{if } |t| < 1/2; \\ 0, & \textit{otherwise.} \end{cases}$$

Apply the convolution theorem to evaluate the Fourier transform of the triangular pulse function

$$\Lambda(t) = \begin{cases} 1 - |t|, & \textit{if } |t| < 1; \\ 0, & \textit{otherwise.} \end{cases}$$

Solution: It is well known result that $\Lambda = \Pi * \Pi$. It can easily be sheen by observing

$$(\Pi * \Pi)(t) = \int_{-\infty}^{\infty} \Pi(y)\Pi(t-y)dy = \begin{cases} \int_{-1/2}^{t+1/2} 1 \cdot 1 dy, & \text{if } -1 < t < 0; \\ \int_{t-1/2}^{1/2} 1 \cdot 1 dy, & \text{if } 0 < t < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have

$$(\Pi * \Pi)(t) = \int_{-\infty}^{\infty} \Pi(y)\Pi(t-y)dy = \begin{cases} 1+t, & \text{if } -1 < t < 0; \\ 1-t, & \text{if } 0 < t < 1; \\ 0 & \text{otherwise.} \end{cases} = \Lambda(t)$$

Using $a = 1/2$ in the previous example we have

$$F(\Pi) = \frac{2}{\sqrt{2\pi}} \frac{\sin(\alpha/2)}{\alpha}$$

Now using convolution result we get

$$F[\Lambda(t)] = F[(\Pi * \Pi)(t)] = \sqrt{2\pi} F(\Pi) F(\Pi) = \frac{4}{\sqrt{2\pi}} \frac{\sin^2(\alpha/2)}{\alpha^2}.$$

In this lesson we provide some miscellaneous examples of Fourier transforms. One of the major applications of Fourier transforms for solving partial differential equations will not be discussed in this module. However, we shall highlights some other applications like evaluating special integrals and the idea of solving ordinary differential equations.

13.2 Example Problems

13.2.1 Problem 1

Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2, & \textit{when } |x| < 1 \\ 0, & \textit{when } |x| > 1 \end{cases}$$

and hence evaluate

$$\int_0^\infty \frac{-x\cos x + \sin x}{x^3} \cos\frac{x}{2}\, \mathrm{d}x.$$

Solution: Using the definition of Fourier transform we get

$$\begin{aligned}\hat{f}(\alpha) = F\left[f(x)\right] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, \mathrm{d}x \\ =& \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\alpha x} \left(1 - x^2\right)\, \mathrm{d}x\end{aligned}$$

Integrating by parts we obtain

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha x}}{i\alpha}(1 - x^2)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{i\alpha x}}{i\alpha}(-2x)\, \mathrm{d}x$$

Again, the application of integration by parts gives

$$\hat{f}(\alpha) = \frac{2}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{(i\alpha)^2} x\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{i\alpha x}}{(i\alpha)^2}\, \mathrm{d}x\right]$$

Further simplifications leads to

$$\begin{aligned}\hat{f}(\alpha) =& \frac{2}{\sqrt{2\pi}} \left[-\frac{1}{\alpha^2}\left(e^{i\alpha} + e^{-i\alpha} - \frac{e^{i\alpha x}}{i\alpha}\Big|_{-1}^{1}\right)\right] \\ =& -\frac{1}{\sqrt{2\pi}} \frac{2}{\alpha^2} \left[e^{i\alpha} + e^{-i\alpha} - \frac{e^{i\alpha}}{i\alpha} + \frac{e^{-i\alpha}}{i\alpha}\right]\end{aligned}$$

Using Euler's equality we obtain

$$\begin{aligned}\hat{f}(\alpha) &= -\frac{1}{\sqrt{2\pi}}\frac{4}{\alpha^2}\left[\cos\alpha - \frac{\sin\alpha}{\alpha}\right]\\ &= \frac{1}{\sqrt{2\pi}}\frac{4}{\alpha^3}\left[-\alpha\cos\alpha + \sin\alpha\right]\end{aligned}$$

We know from the Fourier inversion formula that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

This implies

$$f(x) = \frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}e^{-i\alpha x}\,\mathrm{d}\alpha$$

Equating real parts, on both sides we get

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}\cos\alpha x\,\mathrm{d}\alpha = \frac{\pi}{2}f(x)$$

Substituting the value of the function we obtain

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}\cos\alpha x\,\mathrm{d}\alpha = \begin{cases}\frac{\pi}{2}(1-x^2), & \text{when } |x|<1\\ 0, & \text{when } |x|>1\end{cases}$$

Substitution $x = 1/2$ gives

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{\pi}{2}\left(1-\frac{1}{4}\right),$$

This implies

$$2\int_{0}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{3\pi}{8}$$

Hence we get the desired result as

$$\int_{0}^{\infty}\frac{-\alpha\cos\alpha+\sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{3\pi}{16}$$

13.2.2 Problem 2

Find the Fourier transformation of the function $f(t) = e^{-at}H(t), a > 0$ *where*

$$H(t) = \begin{cases} 0, & \textit{when } t < 0 \\ 1, & \textit{when } t \geq 0 \end{cases}$$

Solution: Using the definition of Fourier transform

$$\begin{aligned} F[f(t)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t}\,\mathrm{d}t \\ =& \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at} e^{i\alpha t}\,\mathrm{d}t \end{aligned}$$

Solving integral leads to

$$F[f(t)] = \frac{1}{\sqrt{2\pi}} \frac{e^{(-a+i\alpha)t}}{(-a+i\alpha)}\Big|_0^\infty$$

Since we know that

$$\lim_{t\to\infty} e^{-at} e^{i\alpha t} = \lim_{t\to\infty} e^{-at}(\cos \alpha t + i \sin \alpha t) = 0$$

We get the required transform as

$$F[f(t)] = -\frac{1}{\sqrt{2\pi}} \frac{1}{(-a+i\alpha)} = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a - i\alpha}\right).$$

13.2.3 Problem 3

Find the Fourier transform of Dirac-Delta function $\delta(t-a)$.

Solution: Recall that the Dirac-Delta function can be thought as

$$\delta(t-a) = \lim_{\epsilon\to 0} \delta_\epsilon(t-a) = \begin{cases} 0, & \text{when } t < a, \quad a > 0 \\ \frac{1}{\epsilon}, & \text{when } a \leq t \leq a+\epsilon \\ 0, & \text{when } t > a+\epsilon \end{cases}$$

Applying the definition of Fourier transform we get

$$\begin{aligned} F[\delta(t-a)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t-a) e^{i\alpha t}\,\mathrm{d}t \\ =& \frac{1}{\sqrt{2\pi}} \int_{a}^{a+\epsilon} \lim_{\epsilon\to 0} \frac{1}{\epsilon} e^{i\alpha t}\,\mathrm{d}t \end{aligned}$$

On integrating we obtain

$$\begin{aligned}
F\left[\delta(t-a)\right] &= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{\epsilon}\frac{e^{i\alpha t}}{i\alpha}\Big|_{a}^{a+\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{\epsilon}\frac{1}{i\alpha}\left(e^{i\alpha(a+\epsilon)}-e^{i\alpha a}\right)\\
&=\frac{1}{\sqrt{2\pi}}e^{i\alpha a}\lim_{\epsilon\to 0}\frac{e^{i\alpha\epsilon}-1}{i\alpha\epsilon}=\frac{1}{\sqrt{2\pi}}e^{i\alpha a}
\end{aligned}$$

With this results we deduce that $F^{-1}(1)=\sqrt{2\pi}\delta(t)$.

13.2.4 Problem 4

Find the Fourier transform of

$$f(t)=e^{-a|t|},\quad -\infty<t<\infty,\ a>0.$$

Solution: Using the definition of Fourier transform we have

$$\begin{aligned}
F\left[e^{-a|t|}\right] &=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{at}e^{i\alpha t}\,\mathrm{d}t+\int_{0}^{\infty}e^{-at}e^{i\alpha t}\,\mathrm{d}t\right]\\
&=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a+i\alpha)t}}{a+i\alpha}\Big|_{-\infty}^{0}+\frac{e^{(-a+i\alpha)t}}{-a+i\alpha}\Big|_{0}^{\infty}\right]\\
&=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\alpha}+(-1)\frac{1}{-a+i\alpha}\right]\\
&=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\alpha}+\frac{1}{a-i\alpha}\right]=\frac{1}{\sqrt{2\pi}}\frac{2a}{a^2+\alpha^2}.
\end{aligned}$$

13.2.5 Problem 5

Find the inverse Fourier transform of $\hat{f}(\alpha)=\dfrac{1}{2\pi(a-i\alpha)^2}$.

Solution: Writing the given function as a product of two functions as

$$F^{-1}\left[\hat{f}(\alpha)\right]=F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\,\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]$$

Application of convolution theorem gives

$$f(t)=\frac{1}{\sqrt{2\pi}}F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]*F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]=\frac{1}{\sqrt{2\pi}}\left[e^{-at}H(t)*e^{-at}H(t)\right]$$

Evaluating the convolution

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax} H(x) e^{-a(t-x)} H(t-x) dx = \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) H(t-x) dx$$

Note that $H(x)H(t-x) = 0$ when $x < 0$ or when $t - x < 0$, i.e.,

$$H(x)H(t-x) = \begin{cases} 1, & \text{if } 0 < x < t; \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$f(t) = \frac{e^{at}}{\sqrt{2\pi}} \int_{0}^{t} dx = \begin{cases} \frac{te^{-at}}{\sqrt{2\pi}}, & \text{if } t > 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Thus we get

$$f(t) = \frac{te^{-at}}{\sqrt{2\pi}} H(t).$$

13.2.6 Problem 6

Using Fourier transform, find the solution of the differential equation

$$y^{'} - 2y = H(t)e^{-2t}, \quad -\infty < t < \infty, \quad y \to 0 \text{ as } |t| \to \infty$$

Solution: Taking Fourier transform on both sides we get

$$F\left[y^{'}\right] - 2F[y] = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{-2 + i\alpha}\right)$$

Aplying the property of Fourier transform of derivatives we get

$$-i\alpha\hat{y} - 2\hat{y} = -\frac{1}{\sqrt{2\pi}} \left(\frac{1}{-2 + i\alpha}\right)$$

Simple algebraic calculation gives the value of transformed variable as

$$\hat{y} = -\frac{1}{\sqrt{2\pi}} \frac{1}{4 + \alpha^2}$$

Taking inverse Fourier transform we get the desired solution as $y = -\frac{1}{4} e^{-2|t|}$.