Lecture 20.

10/10/17

Fourier sine transform (FST) & Fourier cosine transform (FCT)

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$\Big\}$ Fourier transform & inversion formula.

Fourier integral representation:

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \alpha(t-x)\, dt\, d\alpha \; ; \; -\infty < x < \infty.$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \left\{ \cos\alpha t \cos\alpha x + \sin\alpha t \sin\alpha x \right\} dt\, d\alpha.$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos\alpha x \left(\int_{-\infty}^{\infty} f(t) \cos\alpha t\, dt \right) d\alpha + \frac{1}{\pi} \int_{0}^{\infty} \sin\alpha x \left(\int_{-\infty}^{\infty} f(t) \sin\alpha t\, dt \right) d\alpha. \longrightarrow (1)$$

f(x) is even

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos\alpha x\, d\alpha \left(\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos\alpha t\, dt \right) \longrightarrow (2)$$

Define $F_c(\alpha) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_{0}^{\infty} f(t) \cos\alpha t\, dt \longrightarrow (3).$

$\therefore$ (2) is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(\alpha) \cos\alpha x\, dx \longrightarrow (4)$$

$F_c(\alpha) \rightarrow$ Fourier cosine transform of $f(x)$

$f(x)$ given by R.H.S. of (4) is inverse FCT.

f(x) is odd.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin\alpha x\, d\alpha \left(\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin\alpha t\, dt \right) \longrightarrow (5)$$

Define, $F_S(\alpha) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_{0}^{\infty} f(t) \sin\alpha t\, dt. \longrightarrow (6).$

$\therefore$ (5) is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_S(\alpha) \sin\alpha x\, d\alpha \longrightarrow (7).$$

$F_S(\alpha) \rightarrow$ Fourier sine transform of $f(x)$

$f(x)$ given by R.H.S. of (7) is inverse F.S.T.

Suppose given $g(x)$, $\underline{0 < x < \infty}$.

You can find it's FST as well as FCT of $g(x)$.

<u>How?</u>

FST

$$f(x) = \begin{cases} g(x), & 0 < x < \infty. \\ -g(-x), & -\infty < x < 0. \end{cases}$$

Then we can have FST. of $f(x)$, because, $f(x)$ is an odd funct^n.

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin \alpha x \, dx.$$

= F.S.T. of $g(x)$.

FCT

$$f(x) = \begin{cases} g(x), & 0 < x < \infty \\ +g(-x), & -\infty < x < 0. \end{cases}$$

Then $f(x)$ is even & we can have it's FCT.

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \alpha x \, dx.$$

= F.C.T. of $g(x)$.

> Thus any function $f(x)$ defined in $\underline{0 < x < \infty}$ can have F.C.T as well as F.S.T.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

$\underline{-\infty < x < \infty}$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$\underline{0 < x < \infty}$.

Ex. Find Fourier cosine and sine transform of the function $f(x) = x^{p-1}$, $0 < p < 1$.

$$\text{FCT of } f(x) = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \cos \alpha x \, dx$$

$$\text{FST of } f(x) = F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \sin \alpha x \, dx.$$

Consider $I = \int_0^\infty x^{p-1} e^{-\alpha x} \, dx.$

Put $x = iy$, $\quad I = \int_0^\infty (iy)^{p-1} e^{-\alpha i y} \, i \, dy.$

or, $I = i^p \int_0^\infty y^{p-1} e^{-i\alpha y} \, dy.$ $\qquad i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}$

$= e^{\frac{i\pi}{2}}$

or, $\int_0^\infty x^{p-1} e^{-\alpha x} dx = e^{\frac{i\pi p}{2}} \int_0^\infty y^{p-1} e^{-i\alpha y} dy.$ $\qquad (i)^p = e^{\frac{i\pi}{2} \cdot p}.$

$\therefore \int_0^\infty y^{p-1} e^{-i\alpha y} dy = e^{-\frac{i\pi p}{2}} \int_0^\infty x^{p-1} e^{-\alpha x} dx.$ $\qquad \alpha x = v.$

$$= e^{-\frac{i\pi p}{2}} \int_0^\infty \frac{v^{p-1}}{\alpha^{p-1}} e^{-v} \frac{dv}{\alpha}.$$

$$= \frac{e^{-\frac{i\pi p}{2}}}{\alpha^p} \int_0^\infty e^{-v} v^{p-1} \, dv. = \frac{e^{-\frac{i\pi p}{2}}}{\alpha^p} \Gamma(p).$$

or, $\int_0^\infty x^{p-1} (\cos \alpha x - i \sin \alpha x) \, dx = \left(\cos\frac{\pi p}{2} - i \sin\frac{\pi p}{2}\right) \frac{\Gamma(p)}{\alpha^p}$

$$\therefore \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \cos \alpha x \, dx = \alpha^{-p} \Gamma(p) \cos\frac{\pi p}{2} \times \sqrt{\frac{2}{\pi}}$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \sin \alpha x \, dx = \alpha^{-p} \Gamma(p) \sin\frac{\pi p}{2} \times \sqrt{\frac{2}{\pi}}.$$

Q. Find F.C.T. of $f(x) = \frac{1}{1+x^2}$ & hence find F.S.T. of $\frac{x}{1+x^2}$.

Sol. $F_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos \alpha x}{1+x^2} dx$.

[Remember $\int_0^\infty \frac{\cos \alpha x}{x^p} dx$ is absolutely convgt. if $p > 1$.

$\int_0^\infty \frac{\cos \alpha x}{x^2(1+\frac{1}{x^2})}$ $\quad \frac{1}{1+x^2} \sim O\left(\frac{1}{x^2}\right)$

$\therefore \int_0^\infty \frac{\cos \alpha x \, dx}{x^2+1} \sim \int_0^\infty \frac{\cos \alpha x \, dx}{x^2}$].

If $\int_a^\infty |f(x)| dx$ is convgt. then $\int_a^\infty f(x) dx$ is convgt

$|f| = \frac{|\cos \alpha x|}{1+x^2} < \frac{1}{1+x^2} = g$

$\int_0^\infty g \, dx$ is convgt

$\therefore \int |f| dx$ & hence $\int f dx$ are convgt

$I(\alpha) = \int_0^\infty \frac{\cos \alpha x}{1+x^2} dx$.

$\frac{dI}{d\alpha} = -\int_0^\infty \frac{x \sin \alpha x}{1+x^2} dx$.

[$\frac{x}{1+x^2} = \frac{x}{x^2(1+\frac{1}{x^2})} \approx \frac{1}{x}$.

$\int_0^\infty \frac{\sin \alpha x}{x} dx$.

$\frac{d^2 I}{d\alpha^2} = -\int_0^\infty \frac{x^2 \cos \alpha x}{1+x^2} dx$.

$\frac{x^2}{1+x^2} = \frac{x^2}{x^2(1+\frac{1}{x^2})} = 1$

$\approx \int_0^\infty \cos \alpha x \, dx$]

$= -\int_0^\infty \frac{x^2}{x} \frac{\sin \alpha x}{1+x^2} dx$.

$= -\int_0^\infty \frac{(x^2+1-1) \sin \alpha x}{x(1+x^2)} dx$.

$= -\int_0^\infty \frac{\sin \alpha x}{x} dx + \int_0^\infty \frac{\sin \alpha x}{x(1+x^2)} dx$

$\frac{dI}{d\alpha} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin \alpha x}{x(1+x^2)} dx$.

$$\frac{d^2I}{d\alpha^2} = \int_0^\infty \frac{x\cos\alpha x}{x(1+x^2)}\,dx = I(\alpha)$$

$$\therefore \frac{d^2I}{d\alpha^2} - I(\alpha) = 0$$

$$I(\alpha) = c_1 e^{\alpha} + c_2 e^{-\alpha}.$$

$$I(0) = \int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^\infty = \frac{\pi}{2}$$

$I'(0) = 0$; Note, for $\alpha = 0$, $\int_0^\infty \frac{\sin\alpha x}{x}\,dx = 0$.

$$\therefore \quad c_1 + c_2 = \frac{\pi}{2}$$

$$(+) \quad c_1 - c_2 = 0$$

$$2c_1 = \frac{\pi}{2} \Rightarrow c_1 = \frac{\pi}{4} \quad \therefore c_2 = c_1 = \frac{\pi}{4}.$$

$$\therefore I(\alpha) = \frac{\pi}{4}\left(e^{\alpha} + e^{-\alpha}\right)$$

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} \cdot \cosh\alpha = \sqrt{\frac{\pi}{2}}\cosh\alpha.$$

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx = -\sqrt{\frac{2}{\pi}}\frac{dI}{d\alpha}$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{4}\left(e^{\alpha} - e^{-\alpha}\right) = -\sqrt{\frac{\pi}{2}}\sinh\alpha.$$

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