Lecture - 5
31/08/17
Monday.

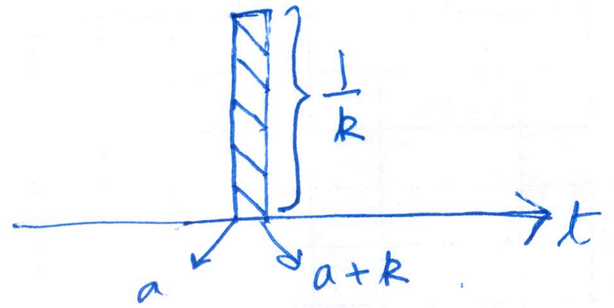
LT. of Dirac Deltā Function.

An impulsive force can be expressed as.

$$f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k. \\ 0, & \text{otherwise}. \end{cases} \longrightarrow (1)$$

$$\lim_{k\to 0} f_k(t-a) = \cancel{\delta(t-a)}\ \delta(t-a).$$

↳ Dirac Deltā funct^n



$$\int_a^{a+k} \frac{1}{k}\, dt = 1.$$

↳ Impulse of the force $f_k(t-a)$

or, $$\int_a^{a+k} f_k(t-a)\, dk = 1. \longrightarrow (2)$$

Dirac deltā funct^n:

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

From (1), $$\delta(t-a) = \begin{cases} \infty, & t = a. \\ 0, & \text{otherwise}. \end{cases}$$

Also, the definition can be taken as

$$\int_0^\infty \delta(t-a)\, dt = 1$$

This occurs by virtue of (2).

Dirac deltā funct^n is an example of generalized funct^n.

§ If $f(t)$ be any <u>continuous</u> funct^n then

$$\int_0^\infty f(t)\,\delta(t-a)\, dt = f(a) \longrightarrow (3)$$

L.T. of $\delta(t-a)$.

$$= \int_0^\infty e^{-st}\delta(t-a)\,dt.$$

Comparing with (3), $f(t) = e^{-st}$

$$\therefore \int_0^\infty e^{-st}\delta(t-a)\,dt = f(a) = e^{-sa}. //.$$

$|f(t)| < Me^{at}$

$f(t)$ is piece-wise continuous.

① L.T. of derivatives.

Thm. If $L\{f(t)\} = \bar{f}(s)$, then,

$$L\{f'(t)\} = s\bar{f}(s) - f(0).$$

$$L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0).$$

$$\vdots$$

$$L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Pf. $L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt.$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}f(t)\,dt.$$

$$= \lim_{t\to\infty} e^{-st}f(t) - \lim_{t\to 0} e^{-st}f(t) + s\int_0^\infty e^{-st}f(t)\,dt.$$

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$\cancel{\lim_{t\to\infty} e^{-2t}e^{2t}\cos t \neq 0}$.

$= 0 - f(0) + s\bar{f}(s).$

Note: 1st term is zero because $f(t)$ is of exponential order 'a', say and then in order the LT of $f(t)$ exists, it is assumed $\mathrm{Re}\, s > a$.

$\therefore L\{f'(t)\} = s\bar{f}(s) - f(0).$

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② LT of integrals.

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{\bar{f}(s)}{s} \quad \text{where } \bar{f}(s) = L\{f(t)\}$$

Leibnitz rule

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(\alpha,t)\,dt = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(\alpha,t)\,dt + \frac{db}{d\alpha} f(\alpha, b(\alpha)) - \frac{da}{d\alpha} f(\alpha, a(\alpha))$$

Pf. Let, $g(t) = \int_0^t f(u)\,du.$

$g'(t) = f(t)$ & $g(0) = 0.$

Let $f(t)$ is of exponential order a as $t \to \infty$. i.e. $\exists\ M > 0$ & a large no. t_0 : $|f(t)| < Me^{at}$ whenever $t > t_0$.

$$|g(t)| = \left|\int_0^t f(u)\,du\right| \le \int_0^t |f(u)|\,du < \int_0^t Me^{au}\,du.$$

Now, $\int_0^t Me^{au}\,du = \frac{M}{a}\left[e^{au}\right]_0^t = \frac{M}{a}\left(e^{at}-1\right) = M_0\left(e^{at}-1\right)$, $M_0 = \frac{M}{a}$.

$\therefore |g(t)| < M_0(e^{at}-1) < M_0 e^{at}$

$\therefore g(t)$ is of exp. order 'a'. Thus, we can have LT. of $g(t)$ & $g'(t)$.

$\therefore\ L[g'(t)] = s\bar{g}(s) - g(0)$,

where $\bar{g}(s) = L\{g(t)\}$.

Also, $g(0) = 0$.

$\therefore\ L[g'(t)] = s\bar{g}(s) = s\,L[g(t)]$

or, $L[f(t)] = s\,L\left[\int_0^t f(u)\,du\right]$

$$\therefore\ L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}.$$

Derivative of Laplace Transform.

(3) $-\dfrac{d}{ds}\bar{f}(s) = L[t\,f(t)]$

$(-1)^2\dfrac{d^2}{ds^2}\bar{f}(s) = L[t^2 f(t)]$.

$\vdots$

$(-1)^n\dfrac{d^n}{ds^n}\bar{f}(s) = L[t^n f(t)]$.

Pf. $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$.

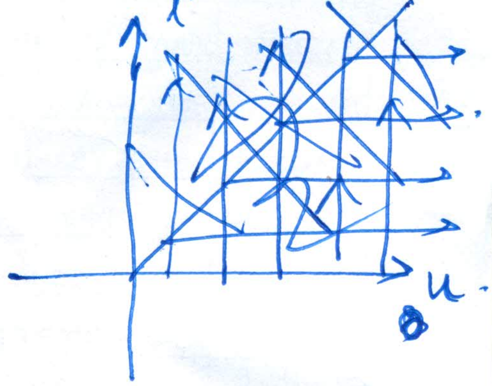
$$\bar{f}'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right)dt.$$

$$= \int_0^\infty -t\,e^{-st} f(t)\,dt = -\int_0^\infty \left(t\,f(t)\right)e^{-st}\,dt$$

or, $-\dfrac{d}{ds}\bar{f}(s) = L(t\,f(t))$

(4) Integral of Laplace transform.

$$L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{F}(s)}{s}.$$



$$\int_s^\infty \bar{F}(u)\,du = L\left[\frac{f(t)}{t}\right].$$

$$\int_s^\infty \bar{F}(u)\,du = \int_s^\infty du \left(\int_0^\infty e^{-ut} f(t)\,dt\right).$$

$$= \int_0^\infty f(t)\,dt \left(\int_s^\infty e^{-ut}\,du\right)$$

$$= \int_0^\infty f(t)\,dt \left(\frac{e^{-ut}}{t}\Bigg|_\infty^s\right) = \int_0^\infty f(t)\,dt \cdot \frac{e^{-st}}{t}$$

$$= \int_0^\infty \left(\frac{f(t)}{t}\right) e^{-st}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

Ex1. Find $L\left\{\sin\sqrt{t}\right\}$. Hence find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

$$L\left\{\sin\sqrt{t}\right\} = \int_0^\infty \sin\sqrt{t}\, e^{-st}\, dt$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \int_0^\infty \left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right] e^{-st}\, dt$$

$$= \int_0^\infty t^{1/2} e^{-st}\, dt - \frac{1}{3!}\int_0^\infty t^{3/2} e^{-st}\, dt + \frac{1}{5!}\int_0^\infty t^{5/2} e^{-st}\, dt - \cdots$$

Use, $L\left[t^a\right] = \frac{\Gamma(a+1)}{s^{a+1}}$.

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx$$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} + \frac{1}{5!}\frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{5}{2}+1}} - \cdots$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}{s^{3/2}\cdot s} + \frac{1}{5!}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}{s^{3/2}\cdot s^2}$$

$$- \frac{1}{7!}\cdot\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}{s^{3/2}\cdot s^3} + \cdots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}}\left[1 - \frac{1}{3!}\cdot\frac{\frac{3}{2}}{s} + \frac{1}{5!}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}}{s^2} - \frac{1}{7!}\cdot\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}{s^3} + \cdots\right].$$

$$= \frac{\Gamma(\frac{3}{2})}{s^{3/2}}\left[1 - \frac{1}{\cancel{3}\cdot 2}\cdot\frac{\cancel{3}}{2\cdot s} + \frac{1}{\cancel{5}\cdot\underbrace{4\cdot\cancel{3}\cdot 2}}\cdot\frac{\cancel{5}\cdot\cancel{3}}{\underbrace{2\cdot 2\cdot s^2}_{2\cdot 16 s^2}} - \frac{1}{\underbrace{\cancel{7}\cdot 6\cdot\cancel{5}\cdot 4\cdot\cancel{3}\cdot 2}_{3!\ 2^3}}\cdot\frac{\cancel{7}\cdot\cancel{5}\cdot\cancel{3}}{\underbrace{2^3 s^3}_{2^3 s^3}}\right].$$

$$= \frac{\Gamma(\frac{3}{2})}{s^{3/2}}\left[1 - \frac{1}{4s} + \underbrace{\frac{1}{2}}_{=2!}\cdot\frac{1}{(4s)^2} - \underbrace{\frac{1}{6}}_{=3!}\cdot\frac{1}{(4s)^3} + \underbrace{\frac{1}{24}}_{=4!}\cdot\frac{1}{(4s)^4} - \cdots\right]$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}}e^{-\frac{1}{4s}} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}} = L(\sin\sqrt{t}).$$

$$f(t) = \sin\sqrt{t}, \quad f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}; \quad f(0) = 0.$$

$$L[f'(t)] = s\bar{f}(s) - f(0) = s \times L[\sin\sqrt{t}]$$

$$= \frac{\sqrt{\pi}}{2\sqrt{s}}e^{-\frac{1}{4s}}$$

$$\Rightarrow L\left[f'(t)\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}.$$

$$\text{or, } L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}.$$

$$\text{or, } \frac{1}{2} L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

$$\Rightarrow L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}.$$

Ex-2. Find $L\left\{\int_0^t \underbrace{\frac{1-e^{-u}}{u}}_{f(u)} du\right\}$.

step 1. Find $L\{f(u)\}\ \left(= \bar{f}(s)\right)$

step 2. Find $L\left\{\int_0^t f(u)\, du\right\}\ \left(= \frac{\bar{f}(s)}{s}\right)$

$$L\{f(u)\} = \int_0^\infty e^{-su} f(u)\, du.$$

$$= \int_0^\infty e^{-su} \cdot \frac{1-e^{-u}}{u}\, du.$$

$$= \int_0^\infty \frac{e^{-su} - e^{-(s+1)u}}{u}\, du.$$

$$I(s) = \int_0^\infty \frac{e^{-su} - e^{-(s+1)u}}{u} du.$$

$$\frac{dI(s)}{ds} = \int_0^\infty \frac{-ue^{-su} + ue^{-(s+1)u}}{u} du$$

$$= \int_0^\infty \left(e^{-(s+1)u} - e^{-su}\right) du.$$

$$= \frac{1}{s+1} - \frac{1}{s}.$$

$$I(s) = \int \frac{1}{s+1} ds - \int \frac{1}{s} ds + C.$$

$$= \ln \frac{s+1}{s} + C.$$

$$= \ln\left(1+\frac{1}{s}\right).$$

$$I(\infty) = 0. \Rightarrow 0 = \ln 1 + C. \Rightarrow C = 0.$$

$$\therefore I(s) = \ln \frac{s+1}{s}.$$

$$\therefore L\left\{f(u)\right\} = \ln \frac{s+1}{s}.$$

$$\therefore L\left\{\int_0^t f(u) du\right\} = \frac{\bar{f}(s)}{s} = \frac{1}{s} \ln\left(1+\frac{1}{s}\right).$$