① 2D Parabolic well posed problem.

(continued from last lecture, see those notes)

② Example 1.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

$$\begin{cases} u(t=0) = u_0. \\ u(x=0) = 0. \\ \left.\dfrac{\partial u}{\partial x} + \beta \cdot u \right|_{x=1} = 0 \end{cases}$$

let $u(x,t) = X(x) \cdot T(t)$.

$$\therefore \quad \frac{1}{T} \cdot \frac{dT}{dt} = \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = -\alpha^2$$

$$\therefore \quad \frac{d^2 X}{dx^2} + \alpha^2 \cdot X$$

$$X(x=0) = 0$$

$$\left.\frac{dX}{dx} + \beta \cdot X \right|_{x=1} = 0$$

$\therefore \quad X_n = c_1 \cdot \sin(\alpha_n \cdot x)$

$\alpha_n + \beta \cdot \tan(\alpha_n) = 0$

the eigen values are the roots of this equation.

$|||ly \quad T_n = c_2 \cdot e^{-\alpha_n^2 \cdot t}$

$\therefore \quad u_n = c_n \cdot \sin(\alpha_n \cdot x) \cdot e^{-\alpha_n^2 \cdot t}$.

$\therefore \quad u = \sum_{n=1}^{\infty} c_n \cdot \sin(\alpha_n \cdot x) \cdot e^{-\alpha_n^2 \cdot t}$

$u_0 = \sum_{n=1}^{\infty} c_n \cdot \sin(\alpha_n \cdot x)$.

$\int_0^1 u_0 \cdot \sin(\alpha_n \cdot x)\, dx = c_n \int_0^1 \sin^2(\alpha_n \cdot x)\, dx$

$\therefore \quad u_0 \cdot \left(\frac{1 - \cos(\alpha_n)}{\alpha_n} \right) = \frac{c_n}{2} \left[1 - \frac{\sin(2\alpha_n)}{2\alpha_n} \right]$

by using

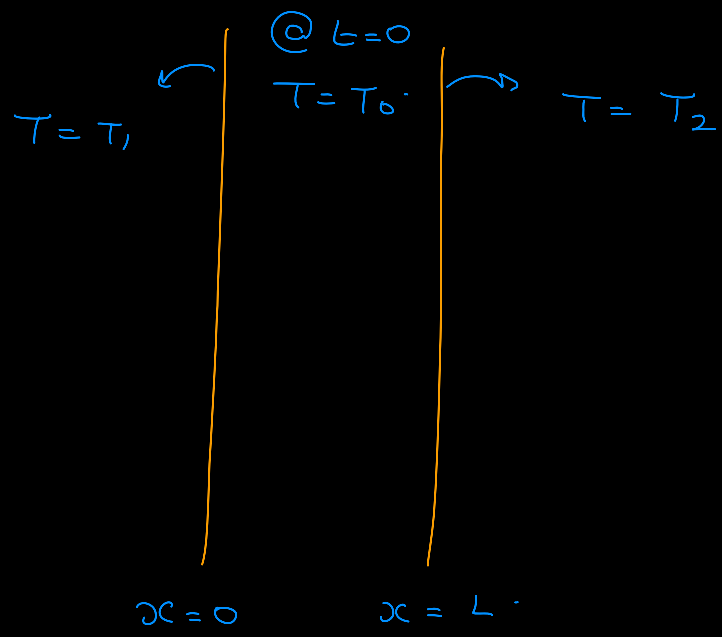
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$\therefore \quad u_0 \cdot \left(\frac{1 - \cos(\alpha_n)}{\alpha_n} \right) = \frac{c_n}{2} \left[1 - \frac{1}{2\alpha_n} \cdot \frac{2 \tan \alpha_n}{1 + \tan^2 \alpha_n}\right.$

use $\tan(\alpha_n) = \dfrac{-\alpha_n}{\beta}$.

$$\therefore \quad C_n = 2 \cdot u_0 \cdot \left(\frac{1 - \cos(\alpha_n)}{\alpha_n}\right) \cdot \left(\frac{\beta^2 + \alpha_n^2}{\beta^2 + \beta + \alpha_n^2}\right)$$

(3) 2D heat conduction problem.



$$\rho \cdot c_p \cdot \frac{\partial T}{\partial t} = K \cdot \frac{\partial^2 T}{\partial x^2}.$$

heat accumulation.

from the heat diffusion equation.

$$T(x, 0) = T_0$$

$$T(0, t) = T_1$$

$$T(L, t) = T_2.$$

- we need to transform its eqn & B.C.s to look like the standard case

$\therefore$ let $\theta = \dfrac{T - T_1}{T_2 - T_1}$, $x^* = \dfrac{x}{L}$.

$$\rho \cdot c_p \cdot \frac{\partial \theta}{\partial t} = \frac{k}{L^2} \cdot \frac{\partial^2 \theta}{\partial x^{*2}}.$$

$$\therefore \rho \cdot c_p \cdot \frac{L^2}{k} \cdot \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^{*2}}$$

$$\therefore \frac{L^2}{\alpha} \cdot \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^{*2}}.$$

- take $\rightarrow$ $\tau = \dfrac{t \cdot \alpha}{L^2}$ $\rightsquigarrow$ time scale

$$\therefore \frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{\partial \theta}{\partial \tau} \cdot \frac{\alpha}{L^2}.$$

$$\therefore \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^{*2}}.$$

$$
\begin{cases}
\vartheta(\tau=0) = \dfrac{T_0 - T_1}{T_1} = \vartheta_0 . \\
x^* = 0 , \quad \vartheta = 0 . \\
x^* = 1 , \quad \vartheta = \dfrac{T_2 - T_1}{T_1} = \vartheta_1
\end{cases}
$$

but this is an ill-posed problem, so we convert into a well-posed one.

split ϑ into a well posed & an ill posed part. split the ill posed part into a steady state & a transient problem.

(see photos).

- write $\vartheta = \vartheta_1 + \vartheta_2$.

→ Class test tomorrow.

→ surprise next time.