In this lesson we continue the application of Laplace transform for solving initial and boundary value problems. In this lesson we will also look for differential equations with variable coefficients and some boundary value problems.

13.0.1 Problem 1

Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t-3)H(t-3), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution: Recall the second shifting theorem

$$L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

We now apply the Laplace transform to the differential equation to get

$$s^2Y(s) - sy(0) - y^{'}(0) + 2\left[sY(s) - y(0)\right] + 2Y(s) = e^{-3s}$$

Plugging the initial values we find

$$\left[s^2 + 2s + 2\right]Y(s) = e^{-3s}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{1}{[(s+1)^2 + 1]}e^{-3s}$$

Taking inverse Laplace transform with the use of first and second shifting properties we obtain

$$y(t) = L^{-1}\left[\frac{1}{[(s+1)^2+1]}e^{-3s}\right] = H(t-3)e^{-(t-3)}\sin(t-3).$$

13.0.2 Problem 2

Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

Solution: We first apply the Laplace transform to the equation. Denoting the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual, we have

$$s^2X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2+\omega_0^2}.$$

We know

$$L^{-1}\left[\frac{1}{s^2+\omega_0^2}\right] = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore, using the convolution theorem, we find

$$x(t) = \int_0^t f(\tau)\frac{\sin\big(\omega_0(t-\tau)\big)}{\omega_0}\,d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0 t)}{\omega_0} f(t-\tau)\,d\tau.$$

13.0.3 Problem 3

Find the general solution of

$$y'' + y = e^{-t}.$$

Solution: Taking Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s+1}$$

Denoting $y(0)$ by y_0 and $y'(0)$ by y_1 we find

$$(s^2+1)Y(s) - sy_0 - y_1 = \frac{1}{s+1}$$

Now we solve for $Y(s)$ to obtain

$$Y(s) = \frac{1}{(s+1)(s^2+1)} + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Method of partial fractions leads to

$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s-1}{s^2+1}\right] + \frac{sy_0}{s^2+1} + \frac{y_1}{s^2+1}$$

Taking the inverse transform we get

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t + y_0\cos t + y_1\sin t$$

This can be rewritten as

$$y(t) = \frac{1}{2}e^{-t} + \left(y_0 - \frac{1}{2}\right)\cos t + \left(y_1 + \frac{1}{2}\right)\sin t$$

Note that y_0 and y_1 are arbitrary, so the general solution is given by

$$y(t) = \frac{1}{2}e^{-t} + C_0\cos t + C_1\sin t.$$

13.0.4 Problem 4

Solve the following boundary value problem

$$y'' + y = \cos t, \quad y(0) = 1, y\left(\frac{\pi}{2}\right) = 1.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2Y(s) - sy(0) - y^{'}(0) + Y(s) = \frac{s}{s^2+1}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{s}{(s^2+1)^2} + \frac{s}{s^2+1} + \frac{y^{'}(0)}{s^2+1}$$

Taking inverse Laplace transform on both sides we get,

$$y(t) = \frac{1}{2}t\sin t + \cos t + y^{'}(0)\sin t.$$

Given $y(\frac{\pi}{2}) = 1$, therefore

$$1 = \frac{1}{2}\frac{\pi}{2} + 0 + y^{'}(0) \Rightarrow y^{'}(0) = \left(1 - \frac{\pi}{4}\right).$$

Hence, we obtain the solution as

$$y(t) = \frac{1}{2}t\sin t + \cos t + \left(1 - \frac{\pi}{4}\right)\sin t.$$

13.0.5 Problem 5

Solve the following fourth order initial value problem

$$\frac{d^4y}{dx^4} = -\delta(x-1),$$

with the initial conditions

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

Solution: We apply the transform and get

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) + C_1 x + \frac{C_2}{6} x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6} 2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3} C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6} 3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

We now demonstrate the potential of Laplace transform for solving ordinary differential equations with variable coefficients.

13.0.6 Problem 6

Solve the initial value problem

$$y^{''} + ty^{'} - 2y = 4; \qquad y(0) = -1,\ y^{'}(0) = 0.$$

Solution: Taking Laplace transform on both sides we get,

$$s^2Y(s) - sy(0) - y^{'}(0) + \left(-\frac{d}{ds}L[y^{'}]\right) - 2Y(s) = 4L[1]$$

Using the given initial values and applying derivative theorem once again, we get

$$s^2Y(s) + s - \frac{d}{ds}\left(sY(s) - y(0)\right) - 2Y(s) = \frac{4}{s}$$

On simplification we find the following differential equation

$$\frac{dY}{ds} + \left(\frac{3}{s} - s\right)Y(s) = -\frac{4}{s^2} + 1$$

Integrating factor of the above differential equation is given as

$$e^{\int\left(\frac{3}{s}-s\right)\mathrm{d}s} = s^3e^{-\frac{s^2}{2}}$$

Hence, the solution of the differential equation can be written as

$$Y(s)s^3e^{-\frac{s^2}{2}} = \int\left(-\frac{4}{s^2} + 1\right)s^3e^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

On integration we find

$$Y(s)s^3e^{-\frac{s^2}{2}} = 4e^{-\frac{s^2}{2}} - \left(s^2e^{-\frac{s^2}{2}}\right) + \int 2se^{-\frac{s^2}{2}}\,\mathrm{d}s + c$$

We can simplify the above expression to get

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \left(\frac{c}{s^3}\right)e^{\frac{s^2}{2}}$$

Since, $Y(s) \to 0$ as $s \to \infty, c$ must be zero. Putting $c = 0$ and taking inverse Laplace transform we get the desired solution as

$$y(t) = t^2 - 1$$

13.0.7 Problem 7

Solve the initial value problem

$$ty^{''} + y^{'} + ty = 0; \;\; y(0) = 1, \; y^{'}(0) = 0$$

Solution: Taking Laplace transform on both sides we get,

$$-\frac{d}{ds}L[y^{''}] + L[y^{'}] + \left(-\frac{d}{ds}L[y]\right) = 0$$

Application of derivative theorem leads to

$$-\frac{d}{ds}\left\{s^2 Y(s) - sy(0) - y^{'}(0)\right\} + \{sY(s) - y(0)\} - \frac{d}{ds}Y(s) = 0$$

Plugging initial values, we find

$$\left(s^2 + 1\right) Y^{'}(s) + sY(s) = 0$$

On integration we get

$$Y(s) = \frac{c}{\sqrt{1 + s^2}}$$

Taking inverse Laplace transform we find

$$y(t) = cJ_0(t)$$

Noting $y(0) = 1, \; J_0(0) = 1$, we find $c = 1$. Thus, the required solution is

$$y(t) = J_0(t).$$