In previous lessons we have evaluated Laplace transforms and inverse Laplace transform of various functions that will be used in this and following lessons to solve ordinary differential equations. In this lesson we mainly solve initial value problems.

12.1 Solving Differential/Integral Equations

We perform the following steps to obtain the solution of a differential equation.

(i) Take the Laplace transform on both sides of the given differential/integral equations.

(ii) Obtain the equation $L[y] = F(s)$ from the transformed equation.

(iii) Apply the inverse transform to get the solution as $y = L^{-1}[F(s)]$.

In the process we assume that the solution is continuous and is of exponential order so that Laplace transform exists. For linear differential equations with constant coefficients one can easily prove that under certain assumption that the solution is continuous and is of exponential order. But for the ordinary differential equations with variable coefficients we should be more careful. The whole procedure of solving differential equations will be clear with the following examples.

12.2 Example Problems

12.2.1 Problem 1

Solve the following initial value problem

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y^{'}(0) = 0.$$

Solution: Take the Laplace transform on both sides, we get

$$L[y^{''}] + L[y] = L[1]$$

Using derivative theorems we find

$$s^2 L[y] - sy(0) - y^{'}(0) + L[y] = L[1]$$

We plug in the initial conditions now to obtain

$$L[y]\left(1 + s^2\right) = \frac{1}{s} \Rightarrow L[y] = \frac{1}{s\left(1 + s^2\right)}$$

Using partial fractions we obtain

$$L[y] = \frac{1}{s} - \frac{s}{1+s^2}$$

Taking inverse Laplace transform we get

$$y(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s}{1+s^2}\right] = 1 - \cos t$$

12.2.2 Problem 2

Solve the initial value problem

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

Solution: We will take the Laplace transform on both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} L[x''(t) + x(t)] &= L[\cos(2t)], \\ s^2X(s) - sx(0) - x'(0) + X(s) &= \frac{s}{s^2+4}. \end{aligned}$$

Plugging the initial conditions, we obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

We now solve for $X(s)$ as

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}$$

We use partial fractions to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

12.2.3 Problem 3

Solve the following initial value problem

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}, \quad y(0) = 2,\, y'(0) = 6.$$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y'(0) - 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{2}{(s-3)^3}$$

Using initial values we obtain

$$s^2Y(s) - 2s - 6 - 6\left[sY(s) - 2\right] + 9Y(s) = \frac{2}{(s-3)^3}$$

We solve for $Y(s)$ to get

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2(s-3)}{(s-3)^2}$$

Taking inverse Laplace transform, we find

$$y(t) = \frac{2}{4!}t^4e^{3t} + 2e^{3t} = \frac{1}{12}t^4e^{3t} + 2e^{3t}.$$

12.2.4 Problem 4

Solve

$$y'' + y = CH(t-a), \quad y(0) = 0,\, y'(0) = 1.$$

Solution: Taking Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = C\int_a^\infty e^{-st}\,\mathrm{d}t$$

We substitute the given initial values to obtain

$$(s^2+1)Y(s) = 1 + C\frac{e^{-as}}{s}$$

Solve for $Y(s)$ as

$$Y(s) = \frac{1}{s^2+1} + C\frac{e^{-as}}{s(s^2+1)}$$

Method of partial fractions leads to

$$y(t) = \sin t + CL^{-1}\left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-as}\right]$$

By inverse Laplace transform we obtain

$$y(t) = \sin t + CH(t-a)[1-\cos(t-a)].$$

12.2.5 Problem 5

Solve the following initial value problem

$$x''(t) + x(t) = H(t-1) - H(t-5), \quad x(0) = 0, \quad x'(0) = 0,$$

Solution: We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We can easily show that

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t.$$

In other words $L[1-\cos t] = \dfrac{1}{s(s^2+1)}$. So using the shifting theorem we find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-s}L[1-\cos t]\right] = \left[1-\cos(t-1)\right]H(t-1).$$

Similarly, we have

$$L^{-1}\left[\frac{e^{-5s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-5s}L[1-\cos t]\right] = \left[1-\cos(t-5)\right]H(t-5).$$

Hence, the solution is

$$x(t) = \left[1-\cos(t-1)\right]H(t-1) - \left[1-\cos(t-5)\right]H(t-5).$$