Parseval's relation

Lecture-25
24/10/17 (Tue)

1) $$\int_{-\infty}^{\infty} F(\omega)\,\overline{G(\omega)}\,d\omega = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx.$$

$F(\omega) \to$ complex fourier transform of $f(x)$.

$G(\omega) \to$ " " " " $g(x)$.

2) $$\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |f(x)|^2\,dx.$$

3). $$\int_{0}^{\infty} F_c(\omega)\,G_c(\omega)\,d\omega = \int_{0}^{\infty} F_s(\omega)\,G_s(\omega)\,d\omega = \int_{0}^{\infty} f(x)\,g(x)\,dx.$$

Ex 1. Find the inverse fourier transform of $e^{-a|\omega|}$ $(a>0)$. Hence evaluate $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^4}$

$\dfrac{x}{x^2+a^2} \cdot \dfrac{x}{x^2+a^2}$

$$\mathcal{F}^{-1}\left(e^{-a|\omega|}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|\omega|}\, e^{-i\omega x}\,d\omega.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{a\omega - i\omega x}\,d\omega + \int_{0}^{\infty} e^{-a\omega - i\omega x}\,d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a-ix)\omega}\,d\omega + \int_{0}^{\infty} e^{-(a+ix)\omega}\,d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+x^2}.$$

$\mathcal{F}^{-1}\left(e^{-a|\omega|}\right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+x^2}$.

If $f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+x^2}$, $F(\omega) = e^{-a|\omega|}$.

$f'(x) = \frac{-1}{\sqrt{2\pi}} \cdot \frac{4ax}{(x^2+a^2)^2}$

$\mathcal{F}[f'(x)] = (-i\omega) F(\omega)$. provided $f(x) \to 0$ as $|x| \to \infty$.

or. Now, $f(x) = \frac{2a}{\sqrt{2\pi}} \cdot \frac{1}{a^2+x^2} \to 0$ as $x \to \pm\infty$. or, $|x| \to \infty$.

$\therefore \mathcal{F}\left[\cancel{+}\frac{1}{\sqrt{2\pi}} \cdot \frac{4ax}{(x^2+a^2)^2}\right] = (\cancel{+} i\omega)\, e^{-a|\omega|}$

So, $\mathcal{F}\left[\frac{x}{(x^2+a^2)^2}\right] = \frac{\sqrt{2\pi}\, i\omega}{4a} e^{-a|\omega|}$.

If $g(x) = \frac{x}{(x^2+a^2)^2}$, then $G(\omega) = \frac{\sqrt{2\pi}\, i\omega}{4a} \cdot e^{-a|\omega|}$.

Now apply Parseval's relation on $g(x)$ & $G(\omega)$.

i.e $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$.

or, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^4} dx = \int_{-\infty}^{\infty} \frac{2\pi\, \omega^2}{16a^2} e^{-2a|\omega|} d\omega$.

$= \frac{4\pi}{16a^2} \int_{0}^{\infty} \omega^2 e^{-2a\omega} d\omega$.

L.T. of ω^2 w.r.t $s = 2a$.

$\therefore 2\int_0^{\infty} \frac{x^2\, dx}{(x^2+a^2)^4} = \frac{\pi}{4a^2}\left[\frac{2!}{(2a)^3}\right]$.

So, $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^4} = \frac{\pi}{32a^5}$ //

Q. Find F.C. transform of x^{p-1} $(0<p<1)$ & e^{-ax}

Hence find the value of the integral $\displaystyle\int_0^\infty \frac{x^{-p}\,dx}{x^2+a^2}$

Sol. F.C.T. of $x^{p-1} = f(x)$ $= \sqrt{\frac{2}{\pi}}\,\omega^{-p}\,\Gamma(p)\cos\frac{\pi p}{2} = F_c(\omega)$

F.C.T. of $e^{-ax} = g(x)$ $= \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+\omega^2} = G_c(\omega)$

Now, $\displaystyle\int_0^\infty F_c(\omega)\,G_c(\omega)\,d\omega = \int_0^\infty f(x)\,g(x)\,dx$

or, $\displaystyle\frac{2}{\pi}\int_0^\infty \omega^{-p}\,\Gamma(p)\cos\frac{\pi p}{2}\cdot\frac{a}{a^2+\omega^2}\,d\omega = \int_0^\infty x^{p-1}e^{-ax}\,dx$

or, $\displaystyle\frac{2}{\pi}\cdot\Gamma(p)\cos\frac{\pi p}{2}\cdot a\int_0^\infty \frac{\omega^{-p}}{a^2+\omega^2}\,d\omega = \int_0^\infty x^{p-1}e^{-ax}\,dx$

$\therefore \displaystyle\int_0^\infty \frac{\omega^{-p}}{a^2+\omega^2}\,d\omega = \frac{\pi}{2}\cdot\frac{[\Gamma(p)]^{-1}\sec\frac{\pi p}{2}}{a}\cdot\int_0^\infty x^{p-1}e^{-ax}\,dx$

$\displaystyle = \frac{\pi}{2}\cdot\frac{\sec\frac{\pi p}{2}}{a^{p+1}}$ //

~~Prob.~~ Modulation property of Fourier Transform.

Let $\mathcal{F}[f(x)] = F(\omega)$.

Then

~~find~~ $\mathcal{F}[f(x)\cos ax]$ ~~in terms of F~~ $= \frac{1}{2}\{F(\omega+a) + F(\omega-a)\}$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{i\omega x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{iax} + e^{-iax}}{2} e^{i\omega x} dx.$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\omega+a)x} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\omega-a)x} dx\right]$$

$$= \frac{1}{2}\left[F(\omega+a) + F(\omega-a)\right].$$

Ex- Find F.T. of e^{-4x^2}. Hence find F.T. of $e^{-4x^2}\cos 4x$.

F.T. of $e^{-a^2x^2} = \frac{1}{a\sqrt{2}} \cdot e^{-\frac{\omega^2}{4a^2}}$.

$\therefore$ F.T. of $e^{-4x^2} = \frac{1}{2\sqrt{2}} e^{-\frac{\omega^2}{16}}$.

$\therefore$ by modulation property $F(e^{-4x^2}\cos 4x)$

$$= \frac{F(\omega+4) + F(\omega-4)}{2}.$$

$$= \frac{1}{2}\left[\frac{1}{2\sqrt{2}} e^{-\frac{(\omega+4)^2}{16}} + \frac{1}{2\sqrt{2}} e^{-\frac{(\omega-4)^2}{16}}\right].$$

Partial Differential Equations

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 3, \quad y \equiv y(x)$$

→ ordinary differential equation.

$z(x, y)$ → function of two variables.

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^3 z}{\partial x^3}, -$$

x, y → independent variables.

z → dependent variable.

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{z(x+h, y) - z(x, y)}{h}.$$

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{z(x, y+k) - z(x, y)}{k}.$$

2nd order P.D.E's

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2} = 2z + \frac{\partial z}{\partial x}.$$ an example of PDE.

$$\left(\frac{\partial^2 z}{\partial x^2}\right)^2 - \frac{\partial z}{\partial x} \cdot x = z^2 + y^2$$ an example of PDE.

$$z_x + z_y = 4x + y + z$$

↳ 1st order PDE

A general linear 2nd order PDE is of the form

$$ar + bs + ct + dp + eq + f = 0.$$

$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, $p = \frac{\partial z}{\partial x}$,

$q = \frac{\partial z}{\partial y}$ &

a, b, c, d, e, f functions of x & y.

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = F(x, y, z, z_x, z_y) \longrightarrow (1).$$

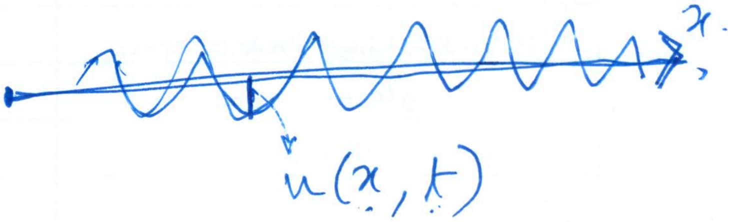
Classification of 2nd order linear PDE.

(1) is hyperbolic, if $b^2 - 4ac > 0$.

(1) " parabolic if $b^2 - 4ac = 0$.

(1) " elliptic if $b^2 - 4ac < 0$.

A. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2}$. $\rightarrow$ wave equation.



$c_0^2 = \frac{T}{\rho}$ $T \rightarrow$ tension. $\rho \rightarrow$ density.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0 \longrightarrow (A).$$

Compare (1) & (A) : $a = 1$, $b = 0$, $c = -\frac{1}{c_0^2}$.

$$b^2 - 4ac = 0 - 4 \cdot 1 \times -\frac{1}{c_0^2} = \frac{4}{c_0^2} > 0.$$

$\therefore$ (A) is of hyperbolic type.

B. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K} \cdot \frac{\partial u}{\partial t}$. $\longrightarrow$ (B).

$\rightarrow$ 1 dimensional heat conduction equation.

$u(x,t) \rightarrow$ temperature at any pt. x & at any time t

Compare (B) with (1),

$a=1, \quad b=0=c. \quad \therefore b^2-4ac=0.$

So, (B) is of parabolic type.

C. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad \phi = \phi(x,y).$ $\rightarrow$ (C).

$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = F$

Compare (C) with (1).

$a=1, \quad b=0, \quad c=1.$

$\therefore b^2-4ac = 0-4.1.1. = -4 < 0.$

So, (C) is of elliptic type.