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Sturm - Liouville Problem

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda w(x)\right]y = 0$$

Homogeneous 2nd order ODE with linear BC

$x = a$, $\alpha_1 y(a) + \beta_1 \frac{dy(a)}{dx} = 0$, $\alpha_1^2 + \beta_1^2 > 0$

$x = b$, $\alpha_2 y(b) + \beta_2 \frac{dy(b)}{dx} = 0$, $\alpha_2^2 + \beta_2^2 > 0$.

Constraint :- $P(x), q(x), w(x)$ are real valued continuous function including $\frac{dP}{dx}$

$P(x)$, $w(x)$ are +ve over $[a, b]$, $q(x) \le 0$

$\lambda \rightarrow$ Eigen values (characteristic values)

Corresponding soln called eigen functions (ϕ)

Non trivial soln :- $\lambda_1 \rightarrow \phi_1$, $\lambda_2 \rightarrow \phi_2$ ... $\lambda_n \rightarrow \phi_n$

let λ_n, λ_m be two distinct eigen values $\rightarrow$ $\lambda_n \rightarrow \phi_n$

$\lambda_m \rightarrow \phi_m$.

$$(\lambda_m - \lambda_n)\int_a^b \phi_n(x)\phi_m(x)\, w(x)\, dx = 0 \text{ — derive}$$

$$\Rightarrow \int_a^b \phi_n(x)\,\phi_m(x)\,\underbrace{w(x)}_{\text{weight factor}}\, dx = 0 \rightarrow \text{orthogonal}.$$

$\rightarrow$ consider a set of functions $\{\phi_n(x); n = 0, 1, 2 \dots\}$ orthogonal wrt weight function $w(x)$

$$f(x) = A_0\phi_0(x) + A_1\phi_1(x) + \dots + A_n\phi_n(x) + \dots$$

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

(Arbitrary function)

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$$\int_a^b f(x)\,\phi_n(x)\,w(x)\,dx = \sum_{n=0}^{\infty}\int_a^b A_n\,\phi_n(x)\,\phi_m(x)\,w(x)\,dx.$$

$$A_n = \frac{\int_a^b \phi_n(x)\, f(x)\, w(x)\, dx}{\int_a^b \phi_n^2(x)\, \underset{w(x)}{dx}}$$

$\int_a^b \phi_n^2(x) \underset{w(x)}{dx}$ = norm of eigen function $\phi_n(x)$

$\Rightarrow$ Ordinary fourier series : $\frac{d^2y}{dx^2} + \lambda^2 y = 0$ — ① $\rightarrow$ eigen value problem

$\alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0$, $\alpha_1^2 + \beta_1^2 > 0$

$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0$, $\alpha_2^2 + \beta_2^2 > 0$.

for $p(x) = 1$, $q(x) = 0$, $w(x) = 1$

$\hookrightarrow$ eqn ① is S-L problem.

complete orthogonal set in $[0, L]$

$\rightarrow$ Fourier sine series :- $\frac{d^2y}{dx^2} + \lambda^2 y = 0$

BC :- $\alpha_1 y(0) = 0$

$\alpha_2 y(L) = 0$.

$\phi_n(x) = \sin\lambda_n x \Rightarrow \lambda_n = \frac{n\pi}{L}$, $n = 1, 2, \dots$

Thus, $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$

$$A_n = \frac{\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\underbrace{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx}_{L/2}}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx.$$ $f(x)$ is any function

→ If f(x) is odd ⇒ [−L, 0] $\xrightarrow{\text{converge to}}$ $-f(-x)$

for all odd functions, expansion of x will work for [−L, L]

Fourier cosine series

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$\phi_n(x) = \cos\lambda_n x$

$\lambda_n = \frac{n\pi}{L}$, $n = 0, 1, 2 \ldots$

$\frac{dy(0)}{dx} = 0$, $\frac{dy(L)}{dy} = 0$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$$

Period = 2L

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

even function

Norm for n = 0 is L
for n = 1 to ∞ → L/2

$$N_m = \int_0^L \cos^2\frac{n\pi}{L}x\,dx$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1, 2, \ldots$$

for even function → expansion works for [−L, L]

Fourier Bessel series → Heat transfer in cylindrical coordinates

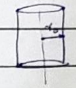
$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - \nu^2)R = 0$$ → SL problem (eigen value problem)

$\alpha_1 R(0) + \beta_1 \frac{dR(0)}{dr} = 0$, $\alpha_1^2 + \beta_1^2 > 0$

$\alpha_2 (R(b)) + \beta_2 \frac{dR(b)}{dr} = 0$, $\alpha_2^2 + \beta_2^2 > 0$

→ $P(r) = r$, $w(r) = r$, $q(r) = -\frac{\nu^2}{r}$ → special case of SL

solid cylinder



domain :- $[0, r_0]$, $R(0)$ = finite

BC ② $\alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0$

Soln :- $R(r) = C_1 J_\nu(\lambda r) + C_2 Y_\nu(\lambda r)$

$r = 0$, $R(0)$ = finite → $\boxed{R(r) = C_1 J_\nu(\lambda r)}$ $C_2 = 0$

BC :- $\alpha C_1 J_\nu(\lambda r_0) + \beta C_1 \frac{d}{dx} J_\nu(\lambda r_0) = 0$

$\alpha J_\nu(\lambda r) + \frac{d}{dx} J_\nu(\lambda r_0) = 0$

eigen function :- $J_\nu(\lambda r)$

eigen value :- λ_n $(n = 1, 2, 3 \ldots)$

⇒ Any arbitrary function $f(r)$ in $[0, r_0]$

$$f(r) = \sum_{n=1}^{\infty} A_n J_\nu(\lambda r)$$

$$A_n = \frac{\int_0^{r_0} f(r) J_\nu(\lambda_n r)\, r\,dr}{N_n}$$

$$N_n = \int_0^{r_0} J_\nu^2(\lambda_n r)\, r\,dr$$

Separation of variables :- $\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \nabla^2 T(r,t)$

Homogeneous, isotropic, solid, stationary, opaque body, without heat generation, (const ρ, Cp k)

$T(r, t=0) = f(r)$ in R

$k\frac{\partial T}{\partial n}(r,t) + hT(r,t) = 0$ on boundary

~~$T(r,t) = \psi(r)\Gamma(t)$~~ $T(r,t) = \psi(r)\,\Gamma(t)$

$$\frac{1}{\alpha}\psi(r)\frac{\partial \Gamma(t)}{\partial t} = \Gamma(t)\nabla^2\psi(r)$$

$$\frac{1}{\alpha}\frac{1}{\Gamma(t)}\frac{d\Gamma(t)}{dt} = \frac{1}{\psi(r)}\nabla^2\psi(r) = \text{const } (-\lambda^2)$$

$$\Gamma(t) = c_1 e^{-\alpha\lambda^2 t}$$

helmholtz equation
$$\begin{cases} \nabla^2\psi(r) + \lambda^2\psi(r) = 0 \\ k\frac{\partial}{\partial n}\psi(r) + h\psi(r) = 0 \quad \text{(boundary condition)} \end{cases}$$

General soln : $T(r,t) = \sum C_m e^{-\alpha\lambda_m^2 t}\,\psi_m(r)$

for $C_m \Rightarrow IC = f(r) = \sum C_m \cancel{e^{-\alpha\lambda_m^2 t}}\,\psi_m(r)$ 1 $t=0$

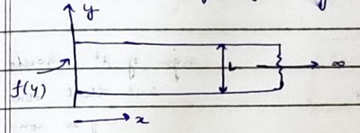
$$C_m = \frac{\int_R f(r)\,\psi_m(r)\,dr}{\int_R \psi_m^2(r)\,dr}$$

→ separation of variable is applicable to if
1. DE is linear and homogeneous
2. BC's are linear, 3 of them are homogeneous
(2-D) (4)

Separation of variables

Examples

1. Consider an infinitely long 2D fin of thickness L



Ambient = T_∞

heat transfer coeff = h = very large

Find steady state $T(x,y)$

k = constant $\dot{q} = 0$.

~~$-kA_c(x)\frac{dT}{dx}\Big|_x + kA_c(x)\frac{dT}{dx}\Big|_{x+\Delta x} - kA_y\frac{dT}{dy}\Big|_y + kA_y\frac{dT}{dy}\Big|_{y+\Delta y}$~~

~~$-hA_s(T-T_\infty) = 0$~~

~~$kA_c\frac{d^2T}{dx^2} + kA_y\frac{d^2T}{dy^2} - hA_s$~~

~~$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$~~

~~$x=0 \rightarrow T(0,y) = f(y)$~~

~~$x\to\infty,\ T = T_\infty \quad \boxed{T(\infty,y) = T_\infty}$~~

~~$y=0 \quad -k\frac{\partial T}{\partial y}\Big|_{y=0} = h(T-T_\infty)$~~

~~$y=L \quad -k\frac{\partial T}{\partial y}\Big|_{y=L} = h(T-T_\infty)$~~

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$x=0$ $T(0,y) = f(y)$

$T(\infty,y) = T_\infty$

because h is very large
$$\begin{cases} T(x,0) = T_\infty \\ T(x,L) = T_\infty \end{cases}$$

$\theta(x,y) = T(x,y) - T_\infty$

$$\boxed{\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0} \longrightarrow \text{homogeneous}$$

separation of variables can be used if atleast 3 BC are homogeneous & DE is linear and homogeneous

$\theta(0,y) = f(y) - T_\infty = F(y)$

$\theta(\infty,y) = 0$

$\theta(x,0) = 0$

$\theta(x,L) = 0$.

$$\theta(x,y) = X(x)\,Y(y)$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \text{const} = \pm\lambda^2$$

sign of λ^2 should be such that soln of Y is non-trivial (eigen value problem in homogeneous solns) $Y \to$ homogeneous soln (S-L problems)

$$\frac{d^2Y}{dy^2} + Y\lambda^2 = 0$$

$$\boxed{Y = c_1 \sin(\lambda y) + c_2 \cos(\lambda y)}$$

$Y(x,0) = 0$

$Y(x,L) = 0$

$\boxed{0 = c_2}$

$0 = c_1 \sin\lambda L$

$\lambda L = n\pi$

$\lambda_n = \frac{n\pi}{L}$ $\quad n = 1, 2, \ldots$

$\boxed{Y = A_n \sin\lambda_n y}$

$\lambda_n = \frac{n\pi}{L},\ n = 1, 2 \ldots$

$$\rightarrow \frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$X(0,y) = F(y)$

$X(\infty,y) = 0$

$\boxed{c_1 = 0}$

$\boxed{F(y) = c_2}$

$X = B_n e^{-\lambda_n x}$

$\theta(x,y) = B_n e^{-\lambda_n x} \times A_n \sin\lambda_n y$

$$\theta(x,y) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n x} \sin\lambda_n y$$

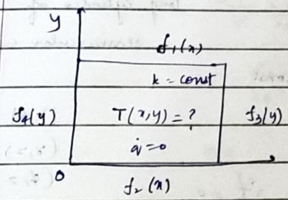
$\theta(0,y) = F(y)$

$$F(y) = \sum_{n=0}^{\infty} C_n \sin\lambda_n y$$

$$\int_0^L F(y)\sin\lambda_n y\, dy = C_n \int_0^L \sin^2\lambda_n y\, dy$$

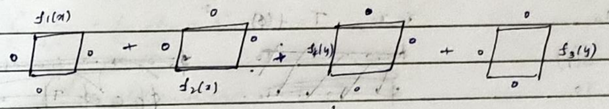
$$\boxed{C_n = \frac{2}{L}\int_0^L F(y)\sin\lambda_n y\, dy}$$

d2)



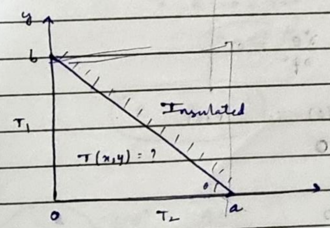
$\rightarrow$ split & take one NH at a time

$\downarrow$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

e)



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$T(x,0) = T_2$

$T(0,y) = T_1$

$k = \text{const}$

$\dot{q} = 0$

$\frac{x}{a} + \frac{y}{b} = 0 \rightarrow \frac{\partial T}{\partial x} = 0,\ \frac{\partial T}{\partial y} = 0.$

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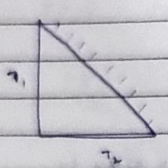

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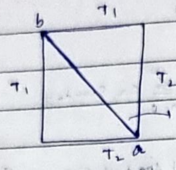
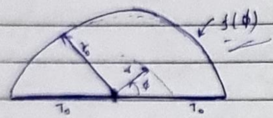


Thermal symmetry acts as insulated boundary.

→ thermal symmetry (plane of symmetry)

Q)



$\dot{q} = 0$

$k = \text{const}$

long cylinder of semicircular cross section

steady state

$T(r, \phi) = ?$

(r, ϕ)

(r, z)

(ϕ, z) ⊗.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \cancel{\frac{\partial^2 T}{\partial z^2}} = 0$$

$T(0, \phi) = T_0$

$T(r_0, \phi) = f(\phi)$

$T(r, 0) = T_0$

$T(r, \pi) = T_0$

$r = r_0, \quad T = f(\phi)$

$r = 0, \quad T = T_0$

~~periodic BC~~

$\theta = T - T_0$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

$\theta(r_0, \phi) = f(\phi) - T_0 = F(\phi)$

$\theta(0, \phi) = 0$

$\theta(r, 0) = 0$

$\theta(r, \pi) = 0$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

$\theta(r, \phi) = R(r)\,\psi(\phi)$

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$$\psi\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = \pm\lambda^2$$

$+\lambda^2$

$$-\frac{1}{\psi}\frac{\partial^2 \psi}{\partial \phi^2} = \lambda^2$$

$$\frac{\partial^2 \psi}{\partial \phi^2} + \lambda^2 \psi = 0$$

$\psi_n = A_n \sin \lambda_n \phi$

$\lambda_n = \frac{n\pi}{\pi}$

$\lambda_n = n$

$$\frac{r^2}{R}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - \lambda^2 = 0$$

cauchy euler eqn.

$r^{+\lambda}, r^{-\lambda}$ → linear combination

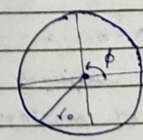
$R = A_1 r^{\lambda} + A_2 r^{-\lambda}$

$R \neq 0, \; r \to 0$

$A_2 = 0$

$R = A_1 r^{\lambda}$

Q)



$f(\phi)$

$\dot{q} = 0$

$k = \text{const}$

steady state $T(x, y) = ?$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2} = 0$$

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- $T(r_0, \phi) = f(\phi)$
- $T(r, 0) = T(r, 2\pi)$
- $\cancel{\left.\frac{\partial T}{\partial \phi}\right|_{\phi=0} = \left.\frac{\partial T}{\partial \phi}\right|_{\phi=2\pi}}$ $\quad \left.\frac{\partial T}{\partial \phi}\right|_{\phi=0} = \left.\frac{\partial T}{\partial \phi}\right|_{\phi=2\pi}$
- $T(0, \phi) = \text{finite}$

$\Rightarrow$ $T(r,\phi) = T(r, \phi + 2\pi)$

$\left.\begin{array}{l} \dfrac{1}{r}\dfrac{\partial T(r,\phi)}{\partial \phi} = \dfrac{1}{r}\dfrac{\partial T(r, \phi+2\pi)}{\partial \phi} \end{array}\right\}$ periodic BC

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0 \qquad \Bigg| \qquad \frac{d^2\varphi}{d\phi^2} + \lambda^2 \varphi = 0$$

$$R = B_1 r^{\lambda} + B_2 r^{-\lambda}$$

$$\varphi = A_1 \sin\lambda\phi + A_2 \cos\lambda\phi$$

$$A_1 \sin\lambda\phi + A_2 \cos\lambda\phi = A_1 \sin(\lambda(\phi+2\pi)) + A_2 \cos(\lambda(\phi+2\pi))$$

$$\lambda A_1 \cos\lambda\phi - A_2 \lambda \sin\lambda\phi = \lambda A_1 \cos(\lambda(\phi+2\pi)) - A_2 \lambda \sin(\lambda(\phi+2\pi))$$

$$A_1(\lambda\cos\lambda\phi - \lambda\cos(\lambda(\phi+2\pi))) + A_2(\lambda\sin\lambda(\phi+2\pi) - \lambda\sin\lambda\phi)$$

$$A_1(\sin\lambda\phi - \sin\lambda(\phi+2\pi)) + A_2(\cos\lambda\phi - \cos(\lambda\phi+2\pi)) = 0.$$

$\lambda = n, \quad n = 0, 1, 2 \ldots$

determinants of coefficients should vanish for non-trivial soln.

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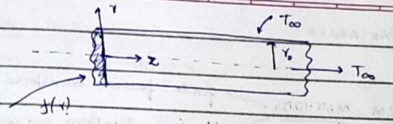

$h = \text{large}$

semi-infinite cylinder of radius r_0.

$\dot{q} = 0$

$k = \text{const}$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) + \frac{\partial^2 T}{\partial z^2} = 0$$

$\cancel{T(0, z) = T_\infty}$

$\cancel{T(r_0, z) = T_\infty}$

$\cancel{T(r, 0) = f(r)}$

$\cancel{T(r, \infty) = T_\infty}$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)\right) + \frac{\partial^2 \theta}{\partial z^2} = 0.$$

$\theta(0, z) = \text{finite}$ $\qquad$ $\theta(0,z) \rightarrow \frac{\partial \theta}{\partial r} = 0 \rightarrow$ can be written

$\theta(r_0, z) = 0$

$\theta(r, 0) = F(r)$

$\theta(r, \infty) = 0.$

$\theta = R(r)\, Z(z)$ $\qquad$ $\frac{d^2 Z}{dz^2} - \lambda^2 Z = 0.$

$\frac{dR(0)}{dr} = 0$ $\quad\Big|\quad$ $R(0) = \text{finite}$ $\qquad$ $Z(\infty) = 0$

$R(r_0) = 0$ $\qquad$ $Z = B_1 e^{-\lambda z}.$

$R = A_1 J_0(\lambda_n r)$

where λ_n is the roots

$J_0(\lambda_n r_0) = 0.$

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→ SEPARATION OF VARIABLES

→ INTEGRAL TRANSFORM METHODS → finite fourier transform → rectangular coordinate
→ Hankel transform → cylindrical coordinate
→ laplace transform

(a) laplace transform

① 2D PDE $\xrightarrow{FFT}$ ODE $T(x)$
$T(x,y)$ ← take inverse ← solve

FINITE FOURIER TRANSFORM

Any $f(x)$ which is piecewise differentiable in $[0, L]$ can be expanded in sine series.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right)x \, dx}$$

$$\Rightarrow f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x') \sin\left(\frac{n\pi}{L}x'\right)dx'\right]\sin\left(\frac{n\pi}{L}\right)x$$

rearrange. let $K_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$ → Kernel of transformation
↳ normalized eigen value ↳ $\frac{\text{eigen function}}{\sqrt{N_m}}$ → norm of eigen function

for $\sin\frac{n\pi}{2}x$ it is $\frac{L}{2}$ ↳ Norm

$\bar{f}_n = \int_0^L f(x) K_n(x)\, dx$ → finite fourier transform of $f(x)$

$f(x) = \sum_{n=1}^{\infty} K_n(x)\, \bar{f}_n$ → Inversion formula

Cosine series

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$$

$$K_n(x) = \frac{1}{\sqrt{N_m}}\cos\frac{n\pi}{L}x$$

$N_m = L \quad n=0$

$N_m = L/2 \quad n = 1, 2, \ldots$

→ STL

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$\alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0, \quad \alpha_1^2 + \beta_1^2 > 0$$

$$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0, \quad \alpha_2^2 + \beta_2^2 > 0$$

$\phi_n(x)$ → eigen function

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

$$A_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\underbrace{\int_0^L \phi_n^2(x)\,dx}_{N_m}} = \frac{1}{N_m}\int_0^L f(x)\phi_n(x)\,dx$$

$$K_n(x) = \frac{1}{\sqrt{N_m}}\phi_n(x)$$

$$\bar{f}_n = \int_0^L f(x)\, K_n(x)\, dx$$

$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n\, K_n(x)$$

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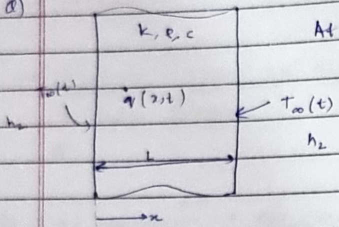


At time $t=0$, $T(x,t) = T_i(x)$

1D unsteady state heat conduction with space, time dependent $\dot q$ and environment temperature

[Consider h_1, h_2 high]

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{\dot q(x,t)}{k}$$

(Cannot solve using separation of variables)

$T(x,0) = T_i(x)$

$T(0,t) = T_\infty(t)$

$T(L,t) = T_\infty(t)$

Operator is linear

L, t. you can apply FFT

define transformation of $T(x,t)$

$$\bar T_n(t) = \int_0^L T(x,t)\, K_n(x)\, dx$$

Inversion $\quad T(x,t) = \sum_{n=1}^{\infty} \bar T_n(t)\, K_n(x) \quad$ Kernel → normalized eigen value function

[how to get kernel]??

convert it into STL — linear homogeneous DE

take homogeneous DE, homogeneous BC

Remove non-homogeneity from PDE & BC

$$\frac{1}{\alpha}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

$\psi(x,0) = T_i(x)$

$\psi(0,t) = 0$

$\psi(L,t) = 0$

$\psi(x,t) = X(x)\,\Gamma(t)$

get eigen function from this & then get kernel from this

Page No. Date.

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$X(0) = 0$

$X(L) = 0$

$X(x) = C_1 \sin\lambda x + C_2 \cos\lambda x$

$C_2 = 0$

$\sin(\lambda L) = 0$

$\lambda L = n\pi$

$\lambda = \frac{n\pi}{L}$

Let, ~~$K_n(x) = \sqrt{\frac{2}{L}} \int_0^L \sin\frac{n\pi x}{L}$~~

Let, $K_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$

$K_n'(x) = \sqrt{\frac{2}{L}} \times \frac{n\pi}{L}\cos\frac{n\pi}{L}x$

$K_n''(x) = -\sqrt{\frac{2}{L}}\,\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}$

Q)

$$\Longrightarrow \int_0^L \frac{1}{\alpha} K_n(x)\frac{\partial T(x,t)}{\partial t}\,dx = \int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}\,dx + \int_0^L K_n(x)\frac{\dot q(x,t)}{k}\,dx$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^L K_n(x)\,T(x,t)\,dx = \frac{1}{\alpha}\frac{d\bar T_n(t)}{dt}$$

$$\left[K_n(x)\frac{\partial T}{\partial x}\right]_0^L - \int K_n'(x)\frac{\partial T}{\partial x}\,dx$$

0.

$$\left[K_n'(x)\,T(x,t)\right]_0^L - \int_0^L K_n''(x)\,T(x,t)\,dx$$

$$T_\infty(t)\left[\sqrt{\frac{2}{L}}\frac{n\pi}{L} - \sqrt{\frac{2}{L}}\frac{n\pi}{L}\cos n\pi\right] - \int_0^L K_n''(x)\,T(x,t)\,dx$$

$$\frac{1}{k}\bar{\dot q}_n(t)$$

$K_n(0) = 0$

$K_n(L) = 0$

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\left[\sqrt{\frac{2}{L}}\underbrace{\frac{n\pi}{L}}_{\lambda_n} - \sqrt{\frac{2}{L}}\frac{n\pi}{L}\cos n\pi\right] - \int_0^L \sqrt{\frac{2}{L}}\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}\,T(x,t)\,dx + \frac{\dot{\bar{q}}_n(t)}{k}$$

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\left[\sqrt{\frac{2}{L}}\lambda_n - \sqrt{\frac{2}{L}}\lambda_n\cos n\pi\right] - \cancel{\sqrt{\frac{2}{L}}\lambda_n^2} - \frac{n^2\pi^2}{L^2}\bar{T}_n(t) + \frac{\dot{\bar{q}}_n(t)}{k}$$

$$\Rightarrow \frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = T_\infty(t)\lambda_n\sqrt{\frac{2}{L}}\left[1-(-1)^n\right] - \lambda_n^2\bar{T}_n(t) + \frac{\dot{\bar{q}}_n(t)}{k}$$

$$\frac{d\bar{T}_n(t)}{dt} + \alpha\lambda_n^2\bar{T}_n(t) = \underbrace{\alpha\left[T_\infty(t)\lambda_n\sqrt{\frac{2}{L}}\left[1-(-1)^n\right] + \frac{\dot{\bar{q}}_n(t)}{k}\right]}_{F_n(t)}$$

$$\boxed{\frac{d\bar{T}_n(t)}{dt} + \alpha\lambda_n^2\bar{T}_n(t) = \alpha F_n(t)}$$

→ linear ODE

→ integrating factor

$$\bar{T}_n(t)\,e^{\alpha\lambda_n^2 t} = \int_0^t \alpha F_n(t)\,e^{\alpha\lambda_n^2 t}\,dt$$

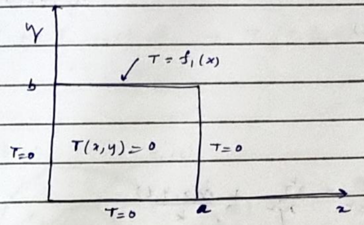
$$\bar{T}_n(t) = e^{-\alpha\lambda_n^2 t}\left[\int_0^t \alpha F_n(t)\,e^{\alpha\lambda_n^2 t}\,dt + \bar{T}_n(0)\right]$$

$$\bar{T}_n(0) = \int_0^L T(x,0)\,k_n(x)\,dx = \int_0^L T_i(x)\,k_n(x)\,dx$$

- Invert $\bar{T}_n(t)$ to get $T(x,t)$

$$\boxed{T(x,t) = \sum_{n=1}^{\infty} k_n(x)\,\bar{T}_n(t)}$$

2D steady state



$\dot{q} = 0$

const ρ, c, k

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$x = 0$, $T(0,y) = 0$

$x = a$, $T(a,y) = 0$

$y = 0$, $T(x,0) = 0$

$y = b$, $T(x,b) = f_1(x)$

→ eigen value problem in x-domain

$$\bar{T}_n(y) = \int_0^a k_n(x)\,T(x,y)\,dx$$

$$T(x,y) = \sum_{n=1}^{\infty} k_n(x)\,\bar{T}_n(y)$$

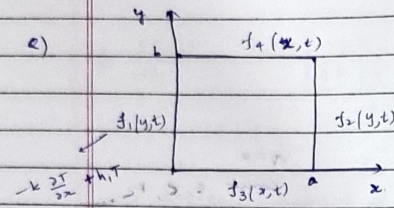
$$\psi = X(x)\,Y(y)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{X}\frac{d^2X}{dx^2} = \lambda_n^2$$

$$\lambda_n = \frac{n\pi}{L},$$

$$\boxed{k_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}}$$

Q)



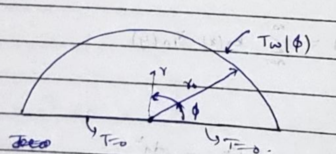
$f_1, f_2, f_3, f_4 \to$ like convection bc

$$\frac{1}{\alpha}\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}(x,y,t)}{k}$$

FFT-1 $\to$ $T(x,y,t) \to \bar{T}_n(y,t)$
FFT-2 $\to$ $\bar{T}_n(y,t) \to \bar{T}_n(t)$ } $\to$ Invert twice

① Rectangular coordinate
② cylindrical coordinate → hankel transformation
[kernel is bessel function]
③ ~~1D~~ → spherical coordinate system → legendre polynomial
1D → $T(r,t)$ $\boxed{\theta(r,t) = rT(r,t)}$

Q)



steady state
$0 \le r \le r_0$
$0 \le \phi \le \pi$
$0 \le z < \infty$

$\dot{q}(r,\phi)$
$T(r,\phi,z)$ k, ρ, c const

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}(r,\phi)}{k}$$

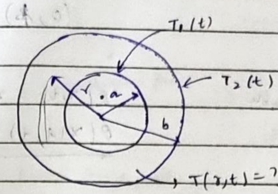
$r = 0$, $T(0,\phi,z) = 0$
$r = r_0$, $T(r_0,\phi,z) = T_w(\phi)$
$\phi = 0$, $T(r,0,z) = 0$
$\phi = \pi$, $T(r,\pi,z) = 0$

$T(r,\phi,0) = 0$
$T(r,\phi,\infty) = \infty$

→ $\phi \to [0,\pi]$ $\quad \bar{T}_n(r,z) = \int_0^\pi K_n(\phi)\, T(r,\phi,z)\, d\phi$; $K_n \to \sqrt{\frac{2}{\pi}}\sin n\phi$

① FFT wrt ϕ $\Rightarrow$ $T(r,\phi,z) \to \bar{T}_n(r,\phi) \xrightarrow{\text{Hankel}} \bar{\bar{T}}_n(?)$ ODE $\Rightarrow$ solve $\Downarrow$ Invert $\bar{T}_n(?)$

Q)



At $t = 0$
$T(r,0) = T_i(r)$

1D spherical shell + unsteady state | $k, \rho, c \to$ const, $\dot{q} = 0$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}$$

$T(a,t) = T_1(t)$
$T(b,t) = T_2(t)$
$T(r,0) = T_i(r)$

$\theta(r,t) = rT(r,t)$

$$\frac{\partial \theta}{\partial r} = r\frac{\partial T}{\partial r} + T$$

$$\frac{\partial^2 \theta}{\partial r^2} = r\frac{\partial^2 T}{\partial r^2} + 2\frac{\partial T}{\partial r}$$

$$\frac{\partial \theta}{\partial t} = r\frac{\partial T}{\partial t}$$

$$\to \frac{1}{\alpha}\frac{1}{r}\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial^2 \theta}{\partial r^2}$$

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2}$$

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2}$$

$$\theta(r,0) = r\,T_i(r)$$

$$\theta(a,t) = a\,T_1(t)$$

$$\theta(b,t) = b\,(T_2(t))$$

apply FFT

Convert the domain to $(0, b)$

~~$r-a$~~

Change of variable

$r = x + a$

$$\theta(r,t) = r\,T(r,t)$$

$$\theta(x+a,t) = (x+a)\,T(r,t)$$

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$$

$$\theta(x,0) = (x+a)\,T_i(x+a)$$

~~$\theta(0,t)$~~

~~$\theta(b+a)$~~

$$\theta(0,t) = a\,T_1(t)$$

$$\theta(b-a,t) = b\,T_2(t)$$

($x \to [0, b-a]$

↳ apply FFT)

$$\bar{\theta}_n(t) = \int_0^{b-a} K_n(x)\,\theta(x,t)\,dx$$

$$\theta(x,t) = \sum_{n=1}^{\infty} K_n(x)\,\bar{\theta}_n(t)$$

$$\frac{d^2\psi}{dx^2} + \psi\lambda^2 = 0$$

$$\psi = C_1\sin\lambda x + C_2\cos\lambda x$$

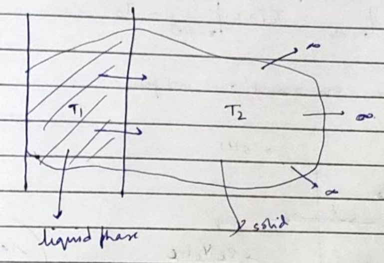
$$\sin\lambda(b-a) = 0$$

$$\lambda_n(b-a) = n\pi$$

$$\lambda_n = \frac{n\pi}{b-a}$$

$$K_n(x) = \sqrt{\frac{2}{b-a}}\left[\sin\frac{n\pi}{b-a}x\right]$$

→ solid with semi infinite width



Start heating from one side

↓

solid will melt.

(Boundary is not fixed.)

phase change problem where interface moves with time.

Heat conduction with phase change → location of interface is not known

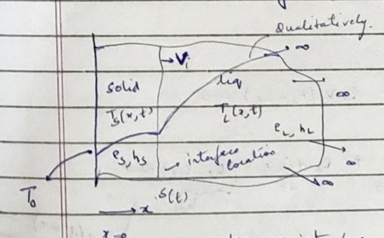
→ Moving boundary problems (interface) — unsteady

Solidification of liquid → (steady state is not possible)

Consider a semi infinite pool of liquid (1D)

- fusion temp T_f

Initially liquid is at temp $> T_f$



At time $t=0$, bring the surface $x=0$ to $T_0 < T_f$ and maintain it

(liq will start solidifying at $x=0$)

temp of interface $= T_f$

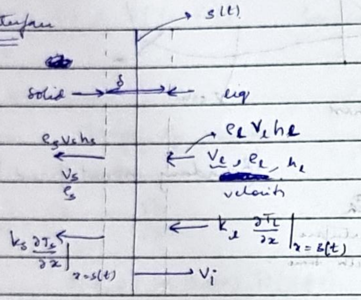
$h \to$ specific enthalpy

Continuity of temp at interface

$$T_s(x,t)\Big|_{x=s(t)} = T_L(x,t)\Big|_{x=s(t)} = T_f$$

→ write energy balance eqn around the interface

location of interface



relative velocity

$$V_i = \frac{ds(t)}{dt} = V_s$$

Mass balance, $\rho_L V_L = \rho_s V_s$ (No accumulation)

$$= \rho_s V_i$$

$$\rho h_L V_L + k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} = \rho_s V_s h_s + k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)}$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s V_s h_s - \rho_L h_L V_L = 0$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s V_s (h_s - h_L) = 0.$$

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s \frac{d(s(t))}{dt} \underbrace{(h_s - h_L)}_{h_{sL}} = 0$$

stefan condition

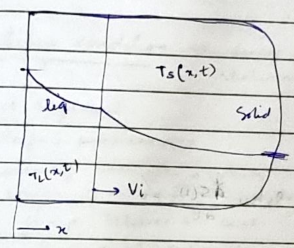
This bc must be satisfied at the interface.

$s(t=0) = 0$

① if convection dominates at the liq phase

$$k_L \frac{\partial T_L}{\partial x}\Big|_{s(t)} \rightarrow h(T_\infty - T_L)$$

Melting of solid

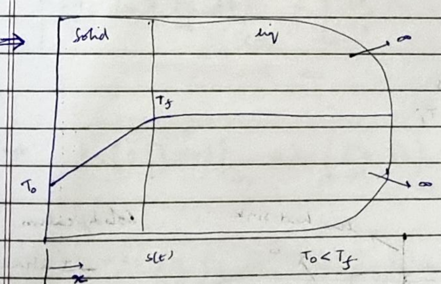


→ derive & check

find the stefan condition

$\rho_s \rightarrow \rho_L$

⇒



→ Solidification of liq

Initially the liq pool was at fusion temp T_f

→ Single phase problem with phase change

→ Need not solve

$$\frac{1}{\alpha}\frac{\partial T_L}{\partial x} = \frac{\partial^2 T_L}{\partial x^2}$$

↳ bc & ICs

$T_L(x,t) = T_f$

$$\frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$$

$\cancel{\frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}}$

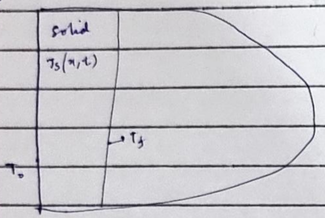
$T_s(0,t) = T_0$

~~$T_s(x,0)$~~ → ??

$T_s(x = s(t), t) = T_f$

$$\cancel{k_L \frac{\partial T_L}{\partial x}}\Big|_{x=s(t)} - k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s h_{sL} \frac{ds(t)}{dt} = 0 \longrightarrow s(t=0) = 0$$

0



~~Initially the liq~~

$$\left.\frac{\partial T}{\partial x}\right|_s \rightarrow d(s)$$

$$\frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$$

$$T_s(0,t) = T_o$$

$$T_s(x=s(t), t) = T_f$$

$$k_L \left.\frac{\partial T_L}{\partial x}\right|_{x=s(t)} - k_s \left.\frac{\partial T_s}{\partial x}\right|_{s(t)} + \rho_s h_{sL}\frac{ds(t)}{dt} = 0$$

$$s(t=0) = 0.$$

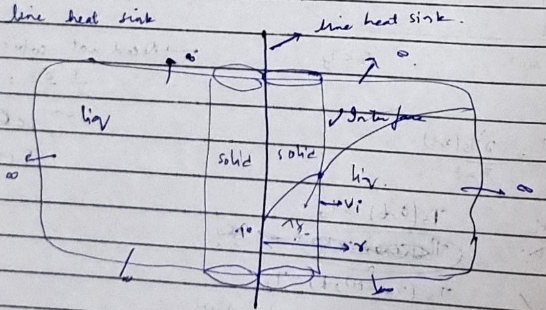
$$\frac{1}{\alpha_L}\frac{\partial T_L(x,t)}{\partial t} = \frac{\partial^2 T_L(x,t)}{\partial x^2}$$

$$T_L(x,0) = T_i$$

$$T_L(x=s(t), t) = T_f$$

$$T_L(x=\infty, t) = T_i$$

Q) line heat sink



Solidification

→ Initially i have a liquid pool.

$T_o < T_f$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_L}{\partial r}\right) = \frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}$$

$$r=0, \quad -\frac{k_s \partial T}{\partial r} = q_{line}$$

→ stefan condition → linear ? Nonlinear ? ⟹ It is non linear

h_s, h_L

$k_L, k_s, \rho_s, h_{sL}, \rho_L$ → const

it says that interface ~~between solid and liq~~ move with a velocity which is the difference in the jump between the heat flux of solid & liq phase.

$$T_s(x,t)\big|_{s(t)} = T_L(x,t)\big|_{s(t)} = T_f$$

$$\left[\frac{\partial T_s}{\partial x}dx + \frac{\partial T_s}{\partial t}dt\right]_{s(t)} - \left[\frac{\partial T_L}{\partial x}dx + \frac{\partial T_L}{\partial t}dt\right]_{s(t)} = 0.$$

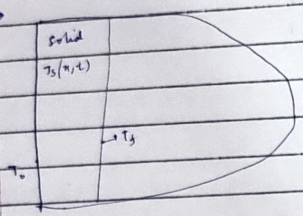
$$\frac{ds}{dt} = \left.\frac{dx}{dt}\right|_{x=s(t)}$$

$$\Rightarrow k_L\left(-\frac{\partial T_L}{\partial t}dt\right)\bigg|_{s(t)} + k_s\left(\frac{\partial T_s}{\partial t}dt\right) + \rho_s h_{sL}\frac{ds(t)}{dt}dx = 0.$$

$$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\frac{ds}{dt}\left(\frac{ds(t)}{dt}\right) = 0$$

$$-k_L\left.\frac{\partial T_L}{\partial t}\right|_{s(t)} + k_s\left.\frac{\partial T_s}{\partial t}\right|_{s(t)} + \rho_s h_{sL}\left(\frac{ds}{dt}\right)^2 = 0$$

non-linear



~~Initially the liq~~

$$\frac{\partial T}{\partial x}\Big|_s \to d(s)$$

$$\frac{1}{\alpha_s}\frac{\partial T_s(x,t)}{\partial t} = \frac{\partial^2 T_s(x,t)}{\partial x^2}$$

$T_s(0,t) = T_0$

$T_s(x=s(t),t) = T_f$

$$k_l\frac{\partial T_l}{\partial x}\Big|_{x=s(t)} - k_s\frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s h_{sl}\frac{ds(t)}{dt} = 0$$

$s(t=0) = 0.$

$$\frac{1}{\alpha_l}\frac{\partial T_l(x,t)}{\partial t} = \frac{\partial^2 T_l(x,t)}{\partial x^2}$$

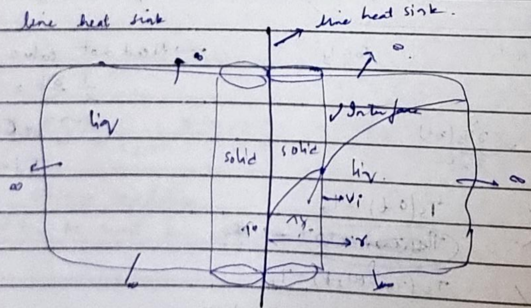
$T_l(x,0) = T_i$

$T_l(x=s(t),t) = T_f$

$T_l(x\to\infty,t) = T_i$

Q) line heat sink

Solidification



→ Initially i have a ~~bio~~ liquid pool.

$T_0 < T_f$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_L}{\partial r}\right) = \frac{1}{\alpha_L}\frac{\partial T_L}{\partial t}$$

→ $r=0$, $-k_s\frac{\partial T}{\partial r} = q_{line}$

⟹ stefan condition → linear? nonlinear? ⟹ It is non linear

$k_l, k_s, \rho_s, h_{sl}, \rho_l$ ↳ const

h_s, h_l

it says that interface ~~between solid and liq~~ moves with a velocity which is the difference in the jump between the heat fluxes of solid & liq phases.

$$T_s(x,t)\Big|_{s(t)} = T_L(x,t)\Big|_{s(t)} = T_f$$

$$\left[\frac{\partial T_s}{\partial x}dx + \frac{\partial T_s}{\partial t}dt\right]_{s(t)} = \left[\frac{\partial T_L}{\partial x}dx + \frac{\partial T_L}{\partial t}dt\right]_{s(t)} = 0.$$

$\frac{ds}{dt} = \frac{dx}{dt}$ at $x = s(t)$

$$\Rightarrow k_L\left(-\frac{\partial T_L}{\partial t}dt\right)\Big|_{s(t)} + k_s\left(\frac{\partial T_s}{\partial t}dt\right) + \rho_s h_{sL}\frac{ds(t)}{dt}dx = 0.$$

$$-k_L\frac{\partial T_L}{\partial t}\Big|_{s(t)} + k_s\frac{\partial T_s}{\partial t}\Big|_{s(t)} + \rho_s h_{sL}\frac{ds}{dt}\left(\frac{ds(t)}{dt}\right) = 0$$

$$-k_L\frac{\partial T_L}{\partial t}\Big|_{s(t)} + k_s\frac{\partial T_s}{\partial t}\Big|_{s(t)} + \rho_s h_{sL}\left(\frac{ds}{dt}\right)^2 = 0$$

non-linear

Non-dimensional form

$\theta_s = \frac{T_s - T_f}{T_f - T_0}$ $\qquad \theta_L = \frac{k_L}{k_s}\frac{T_L - T_f}{T_f - T_0}$.

& $\eta = \frac{x}{L}$, $\tau = Ste \frac{\alpha_s t}{L^2}$

Ste = stefan number = $\frac{\overbrace{c_s (T_f - T_0)}^{\text{sensible heat}}}{h_{sL}}$

for solid

specific heat capacity of solid

h_{sL} is latent

if Ste is very small → we can assume quasi steady state approx

quasi steady state approx

↳ Small stefan number ($Ste \leq 0.1$)

Case 1: Ste is small

thermal diffusivity α_s is very large → means heat propogates with ∞ speed

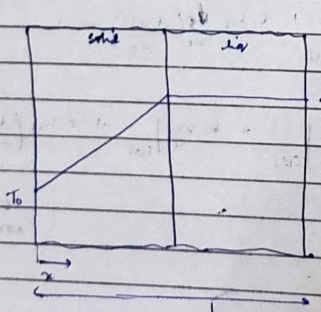
↳ the moment interface moves SS is achieved

Case 2: latent heat is very high → $\frac{ds(t)}{dt} \approx 0$.

↳ Interface is sort of stationary.

$\frac{\partial T_s}{\partial t} \to 0$, $\frac{\partial T_L}{\partial t} \to 0$

Q)



1D problem in finite region

quasi SS

$\frac{\partial^2 T_s}{\partial x^2} = 0$ $\qquad \cancel{\frac{\partial^2 T_L}{\partial x^2} = 0}$ $\qquad T_L(x,t) = T_f$

$T_L(x,t) = T_f$

$T(x=0,t) = T_0$

$T(x=s(t),t) = T_f$

$-k_s \frac{\partial T_s}{\partial x}\Big|_{s(t)} + \rho_s h_{sL} \frac{ds(t)}{dt} = 0$.

$T_L(s(t),t) = T_f$

$T_L(L,t) = T_f$

How much time is needed to solidify the entire liq? — $s(t) = L$ → find t

$T_s(x,t) = ax + b$.

→ stefan exact soln

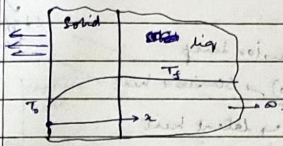
Pennes Bio-heat eqn

Solutions with laplace transform

Duhamel Principle

Inverse heat transfer.

STEFAN EXACT SOLUTION



solid phase: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

$T(0,t) = T_0$, $T(x,t)\big|_{x=s(t)} = T_f$

$-k \frac{\partial T}{\partial x}\Big|_{x=s(t)} + \rho h_{sL} \frac{ds}{dt} = 0$ $\qquad s(0) = 0$.

→ This problem accepts a similarity soln in terms of

$\eta = \frac{x}{2\sqrt{\alpha t}}$

similarity parameter

$\theta(\eta) = T(\eta) - T_0$

$\frac{d^2\theta}{d\eta^2} + 2\eta \frac{d\theta}{d\eta} = 0$ ⟶ $\theta(\eta) = A + B\,\mathrm{erf}(\eta)$

$A = 0$, $B = \frac{T_f - T_0}{\mathrm{erf}(\lambda)}$

$\theta(0) = 0$, $\theta(\lambda) = T_f - T_0$, $\lambda = \frac{s(t)}{2\sqrt{\alpha t}}$

B = const

∴ parameter λ must also be a const, independent of time.

$$\frac{\theta(\eta)}{T_f - T_0} = \frac{erf(\eta)}{erf(\lambda)}$$

$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{erf\left(\frac{x}{2\sqrt{\alpha t}}\right)}{erf(\lambda)} \qquad \lambda = ??$$

$$-k\frac{\partial T}{\partial x}\Big|_{x=s(t)} + \rho h_{sl}\frac{ds}{dt} = 0 \rightarrow \lambda e^{\lambda^2} erf(\lambda) = \frac{1}{\sqrt{\pi}} Ste$$

↳ use stefan BC to find λ.

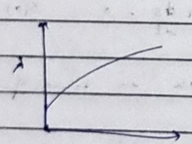
Ste → stefan number

$$Ste = \frac{c(\overbrace{T_f}^{\text{fusion temp}} - T_0) \rightarrow \text{sensible heat}}{h_{sl} \rightarrow \text{latent heat}}$$

→ Ste << 1, solidification process is very rapid

→ Ste >> 1 → solidification is slower

$$\lambda e^{\lambda^2} erf(\lambda) = \frac{1}{\sqrt{\pi}} Ste$$ → non-linear transcendental algebraic equation and cannot be solved explicitly.



When Ste is small, λ is also small

if λ is known.

$$\lambda = \frac{s(t)}{2\sqrt{\alpha t}} \qquad s(t) = \lambda 2\sqrt{\alpha t}$$

$$q_0'' = k\left(\frac{\partial T}{\partial x}\right)\Big|_{x=0} = \frac{k(T_f - T_0)\, t^{-1/2}}{\sqrt{\pi\alpha}\, erf(\lambda)}$$

↳ decreases with time

(with time solid thickness ↑ ∴ resistance ↑)

Pennes Bio-heat equation

Heat exchange through the skin

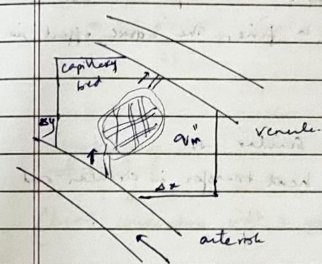
additional term in energy eqn (flow of blood) in the heat balance eqn

→ perfusion

→ complex nature of heat transfer in living tissue precludes exact mathematical model

→ Assumptions

1. Equilibration site → heat exchange takes place between blood & tissues only in the capillary
2. Blood perfusion → no effect of direction
3. Vascular architecture →
4. Blood temperature → temp in the artery is assumed to be same as body core temp



Fourier Bio eqn + perfusion

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}$$

Blood perfusion :- the volumetric flow of heat per unit volume of tissue.

↳ like heat generation

→ q_b''' = rate of energy added by blood per unit volume of tissue

perfusion q_m''' = metabolic

$$\dot{E}_g = q''' \, dx\,dy\,dz = (q_b''' + q_m''')\, dx\,dy\,dz$$

$$q_b''' = \rho_b c_b \omega_b (T_{a0} - T)$$

c_b → specific heat of blood

ρ_b → density of blood

ω_b → blood volumetric flow rate per unit tissue volume

$$q''' = q_m''' + \rho_b c_p \omega_b (T_{a0} - T)$$

$$\nabla\cdot(k\nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$ → Pennes Bio-heat eqn.

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$$\underbrace{\nabla\cdot k(\nabla T)}_{\text{conduction}} + \underbrace{\rho_b c_p \omega_b (T_{a0} - T)}_{\text{Perfusion (convection)}} + \underbrace{q_m'''}_{\text{metabolic heat generation}} \doteq \underbrace{\rho C \frac{\partial T}{\partial t}}_{\text{change in Storage.}}$$

~~[illegible]~~

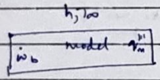
$\rightarrow Q_r(x,t) \rightarrow$ additional heat source term.

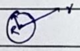
$$\rightarrow k\frac{\partial^2 T}{\partial x^2} + \omega_b \rho_b c_b (T_a - T) + Q_m + Q_r(x,t) = \rho c \frac{\partial T}{\partial t}$$

↳ similar to fin

mathematical role of perfusion term in pennes's eqn is identical to surface convection in fins. The same effect is seen in ~~[illegible]~~ porous

Temp distribution in the forearm — similar to heat transfer in circular rod





SS

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\rho_b c_p \omega_b}{k}(T_{a0} - T) + \frac{q_m'''}{k} = 0$$

$$\frac{dT(0)}{dr} = 0, \quad T(0) = \text{finite}$$

$$-k\frac{dT(R)}{dr} = h(T(R) - T_\infty)$$

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$$q = -k\frac{\partial T}{\partial x}$$

$$\Rightarrow \rho C\frac{\partial T}{\partial x} = -\frac{\partial q}{\partial x} + (\rho C)_b \omega_b (T_b - T) + Q_m$$

Assumes that

$q(t,x) = -k\nabla T(t,x) \rightarrow$ fourier's law holds

$\rightarrow$ heat flows with ∞ speed.

$\rightarrow$ Cattaneo and Vernotte.

$$q(t+\tau, x) = -k\frac{\partial T(t,x)}{\partial x} \rightarrow \tau \text{ is thermal relaxation}$$

fourier law will not work when there is high temperatures.

$$q + \tau\frac{\partial q}{\partial t} = -k\frac{\partial T}{\partial x} \rightarrow \text{Non-fourier heat conduction}$$

$$\tau \rho C\frac{\partial^2 T}{\partial t^2} + (\rho C + \tau \rho_b C_b \omega_b)\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q_m + \rho_b C_b \omega_b (T_a - T)$$

Cryosurgery

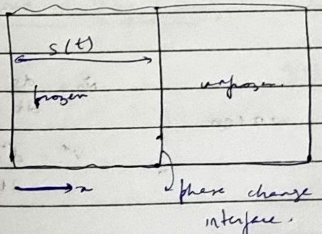
↓

(heat conduction with 2 phase system) ✓

Mushy region $\rightarrow$ melts at range of temperatures instead of sharp temp change

unfrozen $\rightarrow$ healthy tissue

frozen $\rightarrow$ unhealthy (killed cells) (No blood perfusion term) (No blood) (No metabolism)



Frozen:-

$$\rho_f C_f \frac{\partial T_f}{\partial t} = k_f \frac{\partial^2 T_f}{\partial x^2}$$

$$0 \le x \le x_i$$

~~April's~~ April 5

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unfrozen

$$\underset{\text{tissue}}{\rho_u c_u} \frac{\partial T_u}{\partial t} = k_u \frac{\partial^2 T_u}{\partial x^2} + q_b + q_m \qquad x_i \le x \le L$$

$$q_b = \underset{\text{blood}}{\rho_b c_b \omega_b}(T_b - T_u)$$

Conditions at Interface

$$T_f(x_i, t) < T_{ph} = T_u(x_i, t)$$

$$k_f \frac{\partial T_f(x_i,t)}{\partial x} - k_u \frac{\partial T_u(x_i,t)}{\partial x} = \rho_u L \frac{\partial x_i}{\partial t}$$

BC $\quad T_f(0,t) = T_0 = -196°c \rightarrow$ Liq nitrogen (frozen region)

∂T_u

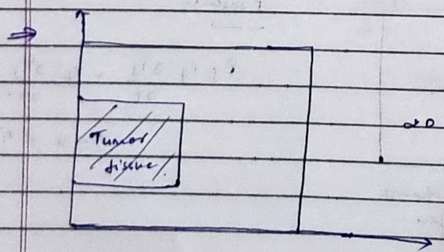
→ Same eqn for laser surgery

Non-fourier eqn

Cryosurgery (hyperbolic)

non-fourier law.

stephan condition will change.



2D eqn.

$$v_n = \frac{ds}{dt}$$

$$v^* = \frac{d^2 s}{dt^2}$$

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steady state equation

spherical coordinate system

Laplace transform

$$\bar{f}(p) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$$

Laplace transformation is linear

$$\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}(f_2(t))$$

Existence of laplace transfor

It will not exist if

① $F(t)$ has infinite discontinuities for some values of t,
② $F(t)$ has singularity at $t \to 0$
③

First shift property :- $\bar{f}(p) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{e^{at} f(t)\} = \bar{f}(p-a)$$

Second shifting property :- $\bar{f}(p) = \mathcal{L}\{f(t)\}$

$$g(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < 0 \end{cases}$$

$$\mathcal{L}\{g(t)\} = e^{-ap} \bar{f}(p)$$

change of scale property:

$$\bar f(p) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\bar f(p/a)$$

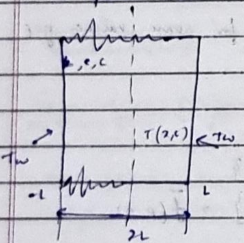
$$\Rightarrow \mathcal{L}\left\{\frac{df}{dt}\right\} = p\bar f(p) - f(0)$$

Convolution theorem

$$\mathcal{L}^{-1}\{\bar f(p)\} = \mathcal{L}^{-1}\{\bar g(p)\bar h(p)\} = \int_{t'=0}^{t} g(t-t')\, h(t')\, dt'$$

$$g*h = h*g$$

Q) 1D plane wall



$t=0,\ T=T_i$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \Big|\quad x=\pm L,\ T=T_w;\quad t=0,\ T=T_i$$

take laplace

$$\frac{1}{\alpha}\left(p\bar T(x,p) - T_i\right) = \mathcal{L}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{\partial^2 \bar T}{\partial x^2}$$

~~[illegible]~~

plane of symmetry

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T(0,t)}{\partial x} = 0,\quad T(L,t) = T_w$$

$$T(x,0) = T_i$$

$$\frac{1}{\alpha}\left(p\bar T - T_i\right) = \frac{\partial^2 \bar T}{\partial x^2}$$

$$\frac{d^2\bar T}{dx^2} - \frac{p}{\alpha}\bar T = \frac{-T_i}{\alpha}$$

$$\bar T = \bar T(x,p) = \mathcal{L}(T(x,t)) = \int_0^\infty e^{-pt}\, T(x,t)\, dt$$

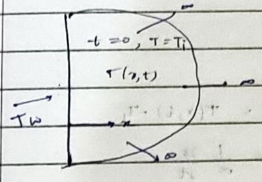
$$\frac{d\bar T(0,p)}{dx} = 0,\quad \bar T(L,p) = \frac{T_w}{p}$$

$$\bar T(x,p) = C_1\cosh mx + C_2\sinh mx + \frac{T_i}{p}$$

$x=0 \to BC \to C_2 = 0$

$x=L \to BC \to C_1 = \dfrac{T_i - T_w}{p\cosh mL}$

→ Semi-infinite solid (no phase change)



$$\theta(x,t) = T(x,t) - T_i$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2},\quad t=0,\ T(x,0) = T_i$$

$$T(0,t) = T_w$$

$$\lim_{x\to\infty} T(x,t) = T_i$$

$$\boxed{\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}}$$

$$t=0,\ \theta=0$$

$$\theta(x,0) = 0$$

$$\theta(0,t) = T_w - T_i$$

$$\lim_{x\to\infty}\theta(x,t) = 0$$

take laplace transform

$$\frac{d^2\bar\theta}{dx^2} - \frac{p}{\alpha}\bar\theta = 0$$

$$\bar\theta(0,p) = \frac{\theta_w}{p},\quad \lim_{x\to\infty}\bar\theta(x,p) = 0$$

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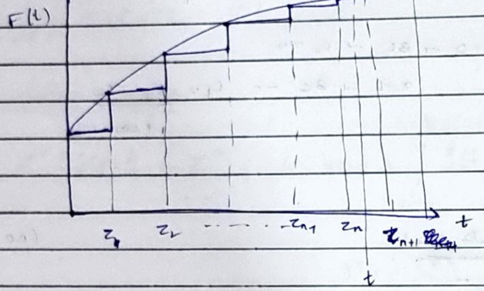
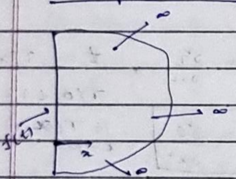
Duhamel method

linear system with time dependent BC or time dependent internal energy generation ...



Semi infinite domain



$\theta(x,t) = T(x,t) - T_i$

$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$

$\theta(x,0) = 0$

$\theta(0,t) = F(t)$ and $\lim_{x\to\infty}\theta(x,t) = 0.$

$F(t) = f(t) - T_i$

~~approximate~~ approximate F(t) as a piecewise ~~const~~ function

(till now we have solved direct (or) forward heat conduction) ✓

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Inverse heat conduction

→ Given a solid, I have the temp distribution and want to find BC → This is called inverse heat conduction.

Assume $q(t) = P_0 + P_1 t + P_2 t^2 + \dots + P_n t^n$.