Solving ~~...~~. integral equation using Fourier Transform.

Lecture - 24
23/10/17 (5-6 V4)

1. Solve $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$.

or, $\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \underline{\lambda} x \, dx = \sqrt{\frac{2}{\pi}} e^{-\lambda}$.

or, $F_c(f(x); \lambda) = \sqrt{\frac{2}{\pi}} e^{-\lambda} = F_c(\lambda)$.

$\therefore$ The inverse Fourier cosine transform is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-\lambda} \cos \lambda x \, d\lambda .$$

$$= \frac{2}{\pi} \underbrace{\int_0^\infty e^{-\lambda} \cos \lambda x \, d\lambda}_{}.$$

= L.T. of $\cos \lambda x$ w.r.to the transform variable $s = 1$.

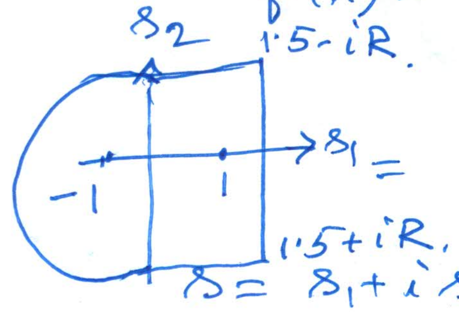
$$f(x) = \frac{2}{\pi} \frac{1}{x^2+1}.$$

$\operatorname{Re} s > \underline{a}$.

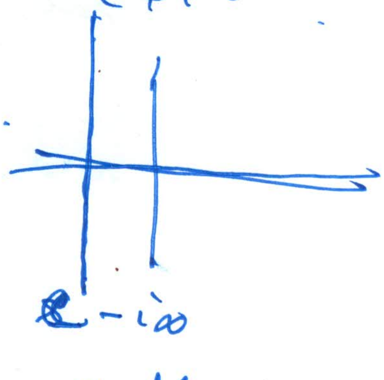
$\int_0^\infty e^{-s\lambda} \cos \lambda p \, d\lambda = \frac{s}{s^2+p^2}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt = \bar{f}(s)$$

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s) e^{st} \, ds .$$



$= \int \frac{1}{s^2-1} e^{st} \, ds .$



s-plane.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx . \quad \omega \text{ here is real. } -\infty < \underline{\omega} < \infty .$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$F_{\substack{c\\s}}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \begin{matrix}\cos\omega x\\ \sin\omega x\end{matrix} dx ; \quad 0 < \underline{\omega} < \infty .$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_{\substack{c\\s}}(\omega) \begin{matrix}\cos\omega x\\ \sin\omega x\end{matrix} d\omega$$

$$\mathcal{F}\left[\frac{1}{1+x^2}\right] = \mathcal{F}_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{\pi}{2}} e^{-\omega}$$

$$\mathcal{F}_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{\pi}{2}} e^{-\omega} = \sqrt{\frac{\pi}{2}} e^{-|\omega|} , \quad \underline{0 < \omega < \infty} .$$

$$\mathcal{F}\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{\pi}{2}} e^{-|\omega|} , \quad -\infty < \omega < \infty .$$

$$\mathcal{F}^{-1}\left[\sqrt{\frac{\pi}{2}} e^{-|\omega|}\right] = \frac{1}{1+x^2}$$

$$L.H.S = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} e^{-|\omega|} e^{-i\omega x} d\omega .$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-(1+ix)|\omega|} d\omega .$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{+(1+ix)\omega} d\omega + \int_0^{\infty} e^{-(1+ix)\omega} d\omega \right]$$

Ex 1. Solve the I E.

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda & \text{for } 0\le\lambda\le 1. \\ 0 & \text{for } \lambda>1. \end{cases}$$

Sol. $$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx = F_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), & 0\le\lambda\le 1 \\ 0, & \lambda>1. \end{cases}$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-\lambda)\cos\lambda x\,d\lambda.$$

$$= \frac{2}{\pi}\left[(1-\lambda)\frac{\sin\lambda x}{x}\Big|_0^1 + \int_0^1 \frac{\sin\lambda x}{x}\,d\lambda\right].$$

$$= \frac{2}{\pi x}\int_0^1 \sin\lambda x\,d\lambda = \frac{2}{\pi x}\left[\frac{\cos\lambda x}{x}\right]_1^0$$

$$= \frac{2}{\pi}\cdot\frac{1-\cos x}{x^2}. //$$

Parseval's relation

Thm 1. If $\mathcal{F}\{f(x)\} = F(\omega)$ & $\mathcal{F}\{g(x)\} = G(\omega)$,

then $\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(\omega)\,\overline{G(\omega)}\,d\omega$.

Pf. $\int_{-\infty}^{\infty} F(\omega)\,\overline{G(\omega)}\,d\omega = \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx\right)\left(\frac{1}{\sqrt{2\pi}}\overline{\int_{-\infty}^{\infty} g(u)e^{i\omega u}du}\right)d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx\right)\left(\int_{-\infty}^{\infty}\overline{g(u)}\,e^{-i\omega u}du\right)d\omega$.

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,dx\int_{-\infty}^{\infty}\overline{g}(u)\,du\left(\underline{\int_{-\infty}^{\infty} e^{-i\omega(u-x)}d\omega}\right) \rightarrow (1)$

Now, $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{i\omega x}dx = \frac{1}{\sqrt{2\pi}}\times e^{i\omega\cdot 0}$

$= \frac{1}{\sqrt{2\pi}}$.

So,

$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\right] = \delta(x)$.

$\therefore \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-i\omega x}\,d\omega = \delta(x)$.

or, $\underline{\int_{-\infty}^{\infty} e^{-i\omega y}\,d\omega} = 2\pi\,\delta(y) \longrightarrow (2)$.

By virtue of (2), (1) becomes,

$\int_{-\infty}^{\infty} F(\omega)\,\overline{G(\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,dx\int_{-\infty}^{\infty}\overline{g(u)}\,du \times \delta(u-x) \times 2\pi$

$$\therefore \int_{-\infty}^{\infty} F(\omega)\,\overline{G(\omega)}\,d\omega = \int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} \overline{g(u)}\,\delta(u-x)\,du.$$

$$= \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx \rightarrow (3).$$

Thm 2. Put $g(x) = f(x)$ in (3). Then,

$$\int_{-\infty}^{\infty} F(\omega)\,\overline{F(\omega)}\,d\omega = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx.$$

Or, $$\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |f(x)|^2\,dx,$$

Thm 3. $$\int_0^{\infty} f(x)\,g(x)\,dx = \int_0^{\infty} F_c(\omega)\,G_c(\omega)\,d\omega.$$

$$= \int_0^{\infty} F_s(\omega)\,G_s(\omega)\,d\omega.$$

$$\int_0^{\infty} f(x)\,g(x)\,dx = \int_0^{\infty} f(x)\left(\sqrt{\frac{2}{\pi}}\int_0^{\infty} G_c(\omega)\cos\omega x\,d\omega\right)dx$$

$$\cancel{= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\,dx}.$$

$$= \int_0^{\infty} G_c(\omega)\,d\omega\left(\int_0^{\infty}\sqrt{\frac{2}{\pi}}\,f(x)\cos\omega x\,dx\right).$$

$$= \int_0^{\infty} G_c(\omega)\,F_c(\omega)\,d\omega.$$

Q. Find Fourier FCT of e^{-ax}.

Hence evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$.

$$\mathcal{F}_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\omega x\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+\omega^2}.$$

$$\mathcal{F}_c\left[e^{-ax};\omega\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+\omega^2}.$$

or, $\mathcal{F}_c\left[e^{-at};x\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2} = F_c(x).$

$$\mathcal{F}_c\left[e^{-bt};x\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{b}{b^2+x^2} = G_c(x), \text{ say}.$$

$$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}.$$

$$= \frac{\pi}{2}\int_0^\infty \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{b}{b^2+x^2}\,dx.$$

$$= \frac{\pi}{2ab}\int_0^\infty F_c(x)\,G_c(x)\,dx.$$

$$= \frac{\pi}{2ab}\int_0^\infty f(t)\,g(t)\,dt = \frac{\pi}{2ab}\int_0^\infty e^{-at}e^{-bt}\,dt$$

$$= \frac{\pi}{2ab(a+b)}.$$

2. Find

(i) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ (ii) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$.

$$F_s\left[e^{-at}\,;\,x\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{x}{x^2+a^2} = F_s(x).$$

$$\frac{\pi}{2}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{x}{x^2+a^2}\sqrt{\frac{2}{\pi}}\frac{x}{x^2+a^2}\,dx.$$

$$= \frac{\pi}{2}\int_0^\infty f(t)\cdot f(t)\,dt = \frac{\pi}{4a}.$$

(ii) $\frac{\pi}{2.a.a(a+a)} = \frac{\pi}{4a^3}$.