Inverse Laplace Transform

Lecture - 8
08/08/17.

• Inverse Laplace transform of integrals.

Let $\bar{f}(s) = \int_s^\infty \bar{g}(u)\, du$. Find $f(t)$

i.e to find $L^{-1}\{\bar{f}(s)\}$.

$$L^{-1}\left[\int_s^\infty \bar{g}(u)\, du\right] = \frac{g(t)}{t}.$$

Ex-1 $\quad L^{-1}\left\{\int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right) du\right\}$.

$$= L^{-1}\left\{\int_s^\infty \bar{g}(u)\, du\right\} = \frac{g(t)}{t}.$$

$$\bar{g}(u) = \frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}.$$

$$g(t) = L^{-1}\{\bar{g}(u)\} = L^{-1}\left\{\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right\}.$$

$$= L^{-1}\left(\frac{u}{u^2+a^2}\right) - L^{-1}\left(\frac{u}{u^2+b^2}\right) = \cos at - \cos bt.$$

$$\therefore L^{-1}\left\{\int_s^\infty \cdots\cdots du\right\} = \frac{g(t)}{t} = \frac{\cos at - \cos bt}{t}.$$

- Multiplication by s.

<u>Thm.</u> If $L\{f(t)\} = \bar{f}(s)$ & $f(0)=0$,

then $L^{-1}\{s\bar{f}(s)\} = f'(t)$

$L\{f'(t)\}$
$= s\bar{f}(s) - f(0)$

<u>Ex-1</u>. Find $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$.

$$= L^{-1}\left\{s \cdot \frac{s}{(s^2+a^2)^2}\right\}$$

$$= L^{-1}\left\{s \cdot \bar{f}(s)\right\}.$$ This will be equal to $= \dfrac{d}{dt}f(t)$, provided $f(0)=0$.

$$\frac{s}{(s^2+a^2)^2}.$$
$$= \left(\frac{d}{ds} \cdot \frac{1}{s^2+a^2}\right) \times -\frac{1}{2}$$
$$= \left(\frac{d}{ds} \cdot \frac{\bar{f}(s)}{a}\right) \times -\frac{1}{2}$$

$$\bar{f}(s) = \frac{s}{(s^2+a^2)^2}.$$

$$= \frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) \times -\frac{1}{2}.$$

$$\underline{\bar{f}(s)} = -\frac{1}{2a}\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \underline{-\frac{1}{2a}\frac{d}{ds}\bar{g}(s)} \longrightarrow (1).$$

$$\bar{g}(s) = \frac{a}{s^2+a^2} \Rightarrow g(t) = \sin at.$$

Taking inverse LT on both sides of (1),

$$L^{-1}\{\bar{f}(s)\} = -\frac{1}{2a}L^{-1}\left\{\frac{d}{ds}\bar{g}(s)\right\}.$$

Remember.

$$L\{t\,g(t)\} = -\frac{d}{ds}\bar{g}(s).$$

so, $$L^{-1}L\{t\,g(t)\} = -L^{-1}\left(\frac{d}{ds}\bar{g}(s)\right\}.$$

$$\therefore L^{-1}\left(\frac{d}{ds}\bar{g}(s)\right) = -t\,g(t) = -t\sin at.$$

From (1) $$\bar{f}(s) = -\frac{1}{2a}\frac{d}{ds}\bar{g}(s).$$

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = -\frac{1}{2a}L^{-1}\left\{\frac{d}{ds}\bar{g}(s)\right\}.$$

$$\Rightarrow f(t) = -\frac{1}{2a}\times -t\sin at = \frac{t\sin at}{2a}.$$

$$\therefore f(0) = 0.$$

So, $$L^{-1}\{s\bar{f}(s)\} = f'(t) = \frac{d}{dt}\left(\frac{t\sin at}{2a}\right)$$

$$= \frac{at\cos at + \sin at}{2a}.$$ //

Alt.

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right\}$$

$= f * g$ convolution of f & g.

Here $g = f$

• Division by powers of s.

Thm. $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\} = \displaystyle\int_0^t f(u)\,du$.

$L^{-1}\left\{\dfrac{\bar{f}(s)}{s^n}\right\} = \underbrace{\displaystyle\int_0^t\int_0^t \cdots \int_0^t}_{n\text{ times}} f(t)\,dt^n$.

Ex. $L^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\}$

Alt: $\displaystyle\int_0^t\int_0^t e^{-t}\,dt^2 = \int dt\left(\int_0^t e^{-t}dt\right)$
$= \displaystyle\int_0^t dt\,(1-e^{-t})\,dt$ — $*$

$= L^{-1}\left\{\dfrac{1}{s^2}\times \bar{f}(s)\right\}$; $\bar{f}(s) = \dfrac{1}{s+1}$.

$* = t+e^{-t}\Big|_0^t = t+e^{-t}-1$. //

$= L^{-1}\left\{\dfrac{1}{s}\cdot\dfrac{1}{s}\bar{f}(s)\right\}$.

$= L^{-1}\left\{\dfrac{1}{s}\cdot\bar{g}(s)\right\}$, say; $\bar{g}(s) = \dfrac{\bar{f}(s)}{s}$.

$= \displaystyle\int_0^t g(u)\,du$. $\longrightarrow$ (1).

$g(t) = \displaystyle\int_0^t f(u)\,du$.

Now, $\bar{f}(s) = \dfrac{1}{s+1}$.

$\therefore f(t) = e^{-t}$

$g(t) = \displaystyle\int_0^t e^{-u}\,du$.
$= \left[e^{-u}\right]_t^0 = 1-e^{-t}$.

Substituting $g(t)$ into (1), get.

$$\cancel{g(t)} = L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

$$= \int_0^t g(u)\,du = \int_0^t \left(1 - e^{-u}\right) du.$$

$$= t + e^{-t} - 1.$$

<u>Ex. 2.</u>

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\cdot\underbrace{\frac{s}{(s^2+a^2)^2}}_{\to\ \bar{f}(s)}\right\}$$

$$= \int_0^t f(u)\,du = \frac{\sin at - at\cos at}{2a^3} \quad //.$$

partial fractions.

(1) $$L^{-1}\left(\frac{3s+7}{s^2-2s-3}\right).$$

$$= L^{-1}\left(\frac{3s+7}{(s-3)(s+1)}\right)$$

$$\frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$$

$$= \frac{(A+B)s + A-3B}{(s-3)(s+1)}.$$

$$3s+7 = (A+B)s + A-3B.$$

Comparing like powers of s,

$$\left.\begin{aligned} A+B &= 3. \\ A-3B &= 7. \end{aligned}\right\} \begin{aligned} B &= -1, \\ A &= 4. \end{aligned}$$

$$L^{-1}\left(\frac{3s+7}{(s-3)(s+1)}\right) = L^{-1}\left(\frac{4}{s-3} - \frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{4}{s-3}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 4e^{3t} - e^{-t}.$$

② Find $L^{-1}\left[\dfrac{s^3+6s^2+14s}{(s+2)^4}\right]$

$$\frac{s^3+6s^2+14s}{(s+2)^4} = \frac{A}{(s+2)^4} + \frac{B}{(s+2)^3} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

$$s^3+6s^2+14s = A + B(s+2) + C(s+2)^2 + D(s+2)^3$$

$s=0, \quad 0 = A + 2B + 4C + 8D.$

$s=-2, \quad (-2)^3+6.4-28 = A = -12$

-8
$-28.$
$+24.$

$s=-1$

$s=1$

$$= \frac{-12}{(s+2)^4} + \frac{2}{(s+2)^3} + \frac{1}{s+2}$$

$L(t^3) = \dfrac{3!}{s^4}$

$$L^{-1}\left(\frac{s^3+6s^2+14s}{(s+2)^4}\right) = -12e^{-2t}L^{-1}\left(\frac{1}{s^4}\right)$$

$$+ 2e^{-2t}L^{-1}\left(\frac{1}{s^3}\right) + e^{-2t}$$

$L(t^2) = \dfrac{2!}{s^3}.$

$$= \left(-\frac{12}{6}.t^3 + \frac{2}{2}.t^2 + 1\right)e^{-2t}$$

$$= (-2t^2 + t^2 + 1)e^{-2t}.$$

(3) $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

Alt.

Note! $2(s^2+2s+2) + (s^2+2s+5)$

$$= 3(s^2+2s)+9 = 3(s^2+2s+3).$$

$$= L^{-1}\left\{\frac{\frac{2}{3}(s^2+2s+2)+\frac{1}{3}(s^2+2s+5)}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= \frac{2}{3}L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}.$$

$$= \frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}.$$

$$= \frac{2}{3}\times\frac{1}{2}e^{-t}L^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

$$= \frac{1}{3}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t.$$

4. Find $L^{-1}\left[\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right]$.

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

5. Find. $L^{-1}\left[\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right]$.

$$\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \frac{As+B}{s^2-2s+2} + \frac{Cs+D}{(s^2-2s+2)^2}.$$

Left as exercises.