Analysis of Vibrations

Consider the general equation of motion of a single linear spring of spring constant k with mass m attached to it such that the motion of the mass is confined only along the direction of the spring axis.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 u(t) \tag{1}$$

where x is the unstretched length of the spring and c is called the damping coefficient. $F_0 u(t)$ is the external force acting on the system. We have seen previously that the system is called free for $F_0 = 0$ while it is called undamped for $c = 0$.

(a) Analyze the system in transform domain and determine the natural period of oscillation, damping coefficient and the steady state gain.

(b) Write the expression for the transfer function of the system.

(c) Derive the condition(s) for the system to exhibit underdamped, critically damped and overdamped response.

(d) Analyze the response of the system to a step input of magnitude A in case of underdamped, critically damped and overdamped response.

(e) For the underdamped response, determine the expressions for overshoot, decay ratio, frequency of oscillations and time period of oscillations.

(f) For the underdamped response, examine the effect of the parameters c, k and m on the rise time and response time.

(g) Repeat parts (d)-(f) for pulse, impulse, ramp and sinusoidal inputs.

Response of a body towards changes in reservoir temperature

We previously considered the cooling of a body immersed in a fluid reservoir of temperature T_∞ when the temperature of the body was $T_\circ$ at time $t = 0$. The time rate of change of temperature of the object was governed by Newton's law of cooling.

$$\frac{dT}{dt} = -h(T - T_\infty)$$

We established the steady state temperature of the system to be T_∞. Let us now consider a situation in which the reservoir temperature is not constant. We wish to know the response of the system with changes in reservoir temperature as the reservoir temperature changes following ideal forcing functions.

(a) Establish the response of the system when subjected to an ideal step function of magnitude A. Identify the time constant and static gain of the system.

(b) Repeat part (a) with an ideal pulse function, impulse function, ramp function and a sinusoidal function.

(c) Repeat parts (a) and (b) if the system has a very small time constant.

(d) Repeat parts (a) and (b) if the system has an extremely large time constant.

Competitive Population Growth with Multiple Species

A bioreactor uses two types of bacteria for a process such that the growth of bacteria involves possible consumption of one species by the other. The system is modelled by the following set of coupled nonlinear ordinary differential equations.

$$\frac{dx}{dt} = \alpha x - \gamma xy$$
$$\frac{dy}{dt} = \epsilon\gamma xy - \beta y$$

where x and y are the populations of the first and second type of bacteria, respectively. All the constants are non-zero positive constants.

(a) Show that the non-dimensional population equations, scaled with respect to the non-trivial equilibrium populations, are given by the following equations.

$$\frac{dx^*}{dt} = \alpha(1 - y^*)x^*$$
$$\frac{dy^*}{dt} = -\beta(1 - x^*)y^*$$

(b) For $\alpha = \beta = 1$, develop the linearized phase portrait for the non-dimensional model.

(c) Determine the condition for bifurcation in the system.