① Linear Combination

- $\underline{v}_1, \underline{v}_2, \underline{v}_3 \in V$

  $a, b \in F$

L.C.
$$\begin{cases} \underline{v}_3 = \underline{v}_1 + \underline{v}_2 \\ \underline{v}_2 = a \cdot \underline{v}_1 \\ \underline{v}_3 = a \cdot \underline{v}_1 + b \cdot \underline{v}_2 \end{cases}$$

- In general

$$\underline{v} = \sum_{i=1}^{n} c_i \cdot \underline{v}_i \quad , \quad c_i \in F, \ v_i \in V \ \forall i \in [1, n].$$

$\downarrow$
linear combination of $v_i \in V$.

- e.g.

$$f(x) = x^2 + 4x - 3$$
$$g_1(x) = x^2 - 2x + 5$$
$$g_2(x) = 2x^2 - 3x$$
$$g_3(x) = x + 3.$$

- if $f(x) = a \cdot g_1(x) + b \cdot g_2(x) + c \cdot g_3(x)$,

find a, b, c.

- compare coeffs. of the same powers.

$$\left.\begin{aligned} 1 &= a + 2 \cdot b \\ 4 &= -2a - 3b + c \end{aligned}\right\} \text{solve simult. eqns.}$$

$$-3 = 5a + 3c.$$ (calc.)

$\therefore\ a = -3,\ b = 2,\ c = 4.$

- elegant method - use matrix algebra.

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

$$[\underline{\underline{A}} \mid \underline{b}]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 4 \\ 5 & 0 & 3 & -3 \end{array}\right]$$

- $R_2 \to R_2 + 2\cdot R_1$, $R_3 \to R_3 - 5\cdot R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -10 & 3 & -8 \end{array}\right]$$

- $R_3 \to R_3 + 10 R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 13 & 52 \end{array}\right]$$

$\therefore \quad 13c = 52$

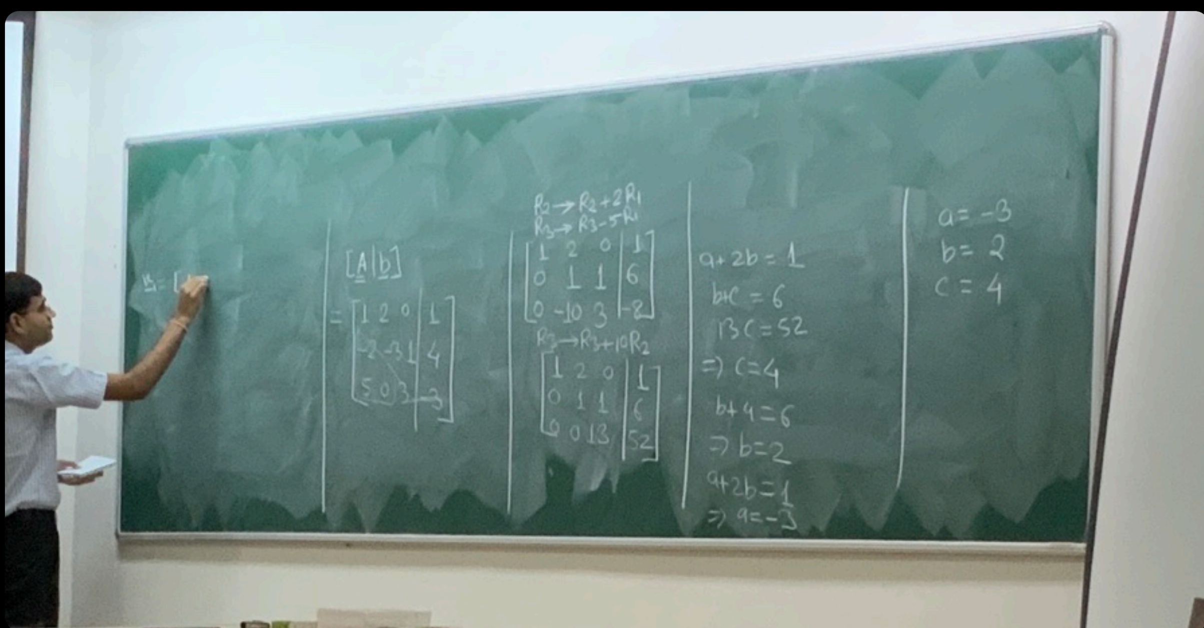
$c = 4$

$\therefore \quad b + c = 6$

$\Rightarrow \quad b = 2$

$\therefore \quad a + 2b = 1.$

$\therefore \quad a = 1 - 4 = -3$



(2) $v_1 = [1 \quad 2 \quad 3]^T$

$v_2 = [2 \quad 3 \quad 1]^T$

$v = [a \quad b \quad c]^T.$

- find the condition on a, b, c s.t.

v can be repr. as an L.C. of v_1 & v_2.

<u>Solution</u>

let $\underline{v} = k_1 \underline{v}_1 + k_2 \underline{v}_2$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_1 + 2k_2 \\ 2k_1 + 3k_2 \\ 3k_1 + k_2 \end{bmatrix}$$

$$\therefore \left[\begin{array}{cc|c} 1 & 2 & a \\ 2 & 3 & b \\ 3 & 1 & c \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & b-2a \\ 0 & -5 & c-3a \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & b-2a \\ 0 & 0 & c-5b+7a \end{array}\right].$$

$$\Rightarrow \boxed{7a - 5b + c = 0.} \rightsquigarrow \text{ans.}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -7k_1 + 5k_2 \end{bmatrix}$$

$$= k_1 \underset{\underline{u}_1}{\begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}} + k_2 \underset{\underline{u}_2}{\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}}$$

$\therefore$ any 3D vector can be repr. as an L.C. of $\underline{u}_1$ & $\underline{u}_2$

$\therefore$ $\underline{u}_1$ & $\underline{u}_2$ span $\mathbb{R}^3$ over $\mathbb{R}$.

(3) Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\underline{v}_1 = [1 \quad 1]^T$, $\underline{v}_2 = [1 \quad 2]^T$

(see photos for proper solution)

if $\underline{v}_1$ & $\underline{v}_2$ span $\mathbb{R}^2$, then

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha \cdot \underline{v}_1 + \beta \cdot \underline{v}_2$$

$$\therefore \quad = \alpha \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

$$\therefore \quad \rightarrow \left[\begin{array}{cc|c} 1 & 1 & \alpha \\ 1 & 2 & \beta \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & \alpha \\ 0 & 1 & \beta-\alpha \end{array}\right]$$

$$\alpha = b, \quad \beta = a-b.$$

since $\alpha \in \mathbb{R}$ & $\beta \in \mathbb{R}$, $b \in \mathbb{R}$

$\Rightarrow a \in \mathbb{R}.$

(4) Show that the polynomials $p_1(x) = 1$, $p_2(x) = 1+x$, $p_3(x) = (1+x)^2$ span the space of polynomials of degree at most 2.

$$p(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2.$$

let $p(x) = \alpha \cdot p_1(x) + \beta \cdot p_2(x) + \gamma \cdot p_3(x)$

$$\therefore \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x^0 & x^1 & x^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta-\alpha \\ 0 & 2 & 1 & \gamma-\alpha \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta - \alpha \\ 0 & 0 & 1 & \alpha - 2\beta + \gamma \end{array}\right]$$

$\therefore \quad \alpha = a_0$

$\beta - \alpha = a_1$

$\Rightarrow \quad \beta = a_0 + a_1$

$\alpha - 2\beta + \gamma = a_2$

$\Rightarrow \quad \gamma = a_2 - a_0 + 2(a_0 + a_1)$

$\gamma = a_0 + 2a_1 + a_2$

(5) verify if the foll. mats $\mathbb{R}^{2\times 2}$ over $\mathbb{R}$ are linearly indep.

$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{\underline{C}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Condition of L. indep. $\Rightarrow$

$\sum_{i=1}^{n} c_i \cdot x_i = 0$ is $c_i = 0 \ \forall i \in [1, n]$.

$$c_1 \cdot \underline{\underline{A}} + c_2 \cdot \underline{\underline{B}} + c_3 \cdot \underline{\underline{C}} = \underline{\underline{0}}$$

$$\therefore \begin{bmatrix} c_1 & c_1 \\ c_1 & c_1 \end{bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} -c_3 & -c_3 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 = -c_3 \quad (1)$$

$$c_1 = -c_3 \quad (2)$$

$$c_1 = 0 \quad (3)$$

$$c_1 + c_2 = 0 \quad (4)$$

$$\Rightarrow \quad c_1 = 0, \quad c_3 = 0, \quad c_2 = 0$$

$\Rightarrow$ lin dep.

- If the three vectors were not lin dep, it would have been possible to write one of them as an l.c. of the other two.
- Only lin dep vectors can span a vector space.

⑥ Determine whether the vectors →

$$f(x) = 2x^3 + x^2 + x + 1$$
$$g(x) = x^3 + 3x^2 + x - 2$$
$$h(x) = x^3 + 2x^2 - x + 3$$

are lindep.

$$c_1 \cdot f + c_2 \cdot g + c_3 \cdot h = 0$$

$$(2c_1 + c_2)x^3 + (c_1 + 3c_2)x^2 + (c_1 + c_2)x$$
$$+ (c_1 - 2 \cdot c_2) = -c_3 x^3 - 2c_3 x^2$$
$$+ c_3 \cdot x - 3c_3.$$

$$\therefore \quad 2c_1 + c_2 = -c_3 \quad (1)$$

$$c_1 + 3c_2 = -2c_3 \quad (2)$$

$$\therefore \quad c_2 - 6c_2 = -c_3 + (+2)(2c_3)$$

$$\therefore \quad -5c_2 = 3c_3$$

$$\therefore \quad c_3 = -\frac{5c_2}{3}. \quad (5)$$

$$c_1 + c_2 = c_3 \quad (3)$$

$$c_1 - 2c_2 = -3c_3 \quad (4)$$

$$3c_2 = 4c_3$$

$$\therefore \quad c_3 = \frac{3}{2}c_2 \quad (6)$$

$\Rightarrow \quad c_3 = c_2 = 0$

$\therefore \quad c_1 = 0.$

7. Check lin dep.

$f(x) = x$

$g(x) = e^{2x}.$

Wronksian Determinant.

If $= 0 \Rightarrow$ not lin dep.

⑧ check whether the foll. vectors form a basis of $\mathbb{R}^4(\mathbb{R})$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\2\end{bmatrix}, \begin{bmatrix}2\\5\\6\\4\end{bmatrix}, \begin{bmatrix}2\\6\\8\\5\end{bmatrix}$$

Basis = set of vectors which span the vector space.

① should be lindep.

② number of vectors = dimension of the vector space.

Midsem syllabus = chapter 1 → 4
&
no repeat in endsem

16th → CT.