8.1 Example Problems on Convolution

8.1.1 Problem 1

Find the inverse Laplace transform of the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\,\frac{1}{s^2}.$$

Solution: We recognize the two elementary entries

$$L^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \qquad \text{and} \qquad L^{-1}\left[\frac{1}{s^2}\right] = t.$$

Therefore,

$$L^{-1}\left[\frac{1}{s+1}\,\frac{1}{s^2}\right] = \int_0^t \tau e^{-(t-\tau)}\, d\tau$$

On integration by parts we obtain

$$L^{-1}\left[\frac{1}{s+1}\,\frac{1}{s^2}\right] = e^{-t} + t - 1.$$

8.1.2 Problem 2

Use the convolution theorem to evaluate

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right].$$

Solution: Note that

$$L[\sin t] = \frac{1}{s^2+1} \qquad \text{and} \qquad L[\cos t] = \frac{s}{s^2+1}$$

Using convolution theorem,

$$L[\sin t * \cos t] = L[\sin t]L[\cos t] = \frac{s}{(s^2+1)^2}.$$

Therefore, we have

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \int_0^t \sin\tau\cos(t-\tau)\mathrm{d}\tau.$$

Using the trigonometric equality $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ we get

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}\int_0^t [\sin t + \sin(2\tau - t)]\mathrm{d}\tau.$$

On integration we find

$$\begin{aligned} L^{-1}\left[\frac{s}{(s^2+1)^2}\right] &= \frac{1}{2}t\sin t + \frac{1}{2}\left[-\frac{\cos(2\tau-t)}{2}\right]_0^t \\ &= \frac{1}{2}t\sin t + \frac{1}{4}[\cos t - \cos t]. \end{aligned}$$

Finally we have the following result

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}t\sin t.$$

8.1.3 Problem 3

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right]$$

Solution: We know the following elementary transforms

$$L\left[\frac{1}{\sqrt{t}}\right] = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \Rightarrow L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{t\pi}}$$

and

$$L^{-1}\left[\frac{1}{s-1}\right] = e^t.$$

Then by the convolution theorem, we find

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{1}{\sqrt{t\pi}} * e^t = \int_0^t \frac{1}{\sqrt{\tau\pi}}e^{t-\tau}\mathrm{d}\tau.$$

Substitution $u = \sqrt{\tau} \Rightarrow du = \dfrac{1}{2\sqrt{\tau}}d\tau$ gives

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\mathrm{d}\tau = 2\frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\mathrm{d}u.$$

Thus, we have

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = e^t\mathrm{erf}(\sqrt{t}).$$

8.1.4 Problem 4

Use convolution theorem to evaluate

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right].$$

Solution: We know

$$L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2} \text{ and } L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

By the convolution theorem we have

$$\begin{aligned}
L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] &= \frac{1}{2}t^2 * \sin t = \frac{1}{2}\int_0^t \sin\tau\,(t-\tau)^2\,\mathrm{d}\tau \\
&= \frac{1}{2}\left[\left(-\cos\tau(t-\tau)^2\right)\Big|_0^t - 2\int_0^t (t-\tau)\cos\tau\mathrm{d}\tau\right] \\
&= \frac{1}{2}\left[t^2 - 2\left((t-\tau)\sin\tau\right)\Big|_0^t - 2\int_0^t \sin\tau\mathrm{d}\tau\right].
\end{aligned}$$

Finally we get the desired inverse Laplace transform as

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{t^2}{2} + \cos t - 1.$$

We shall continue discussing various properties of inverse Laplace transform. We mainly cover change of scale property, inverse Laplace transform of integrals and derivatives etc.

8.2 Change of Scale Property

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}[F(as)] = \frac{1}{a}f\left(\frac{t}{a}\right)$

8.2.1 Example

If

$$L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t,$$

then find

$$L^{-1}\left[\frac{s}{2s^2-8}\right]$$

Solution: Given that

$$L^{-1}\left[\frac{s}{s^2-16}\right]=\cosh 4t$$

Replacing s by $2s$ and using scaling property we find

$$L^{-1}\left[\frac{2s}{4s^2-16}\right]=\frac{1}{2}\cosh 2t$$

Thus, we obtain

$$L^{-1}\left[\frac{s}{2s^2-8}\right]=\frac{1}{2}\cosh 2t$$

8.3 Inverse Laplace Transform of Derivatives (Derivative Theorem)

If $L^{-1}[F(s)]=f(t)$ *then* $L^{-1}\left[\frac{d^n}{ds^n}F(s)\right]=(-1)^n t^n f(t),\quad n=1,2\ldots$

8.3.1 Example

Find the inverse Laplace transform of

$$(i)\ \frac{2as}{(s^2+a^2)^2}\qquad\qquad (ii)\ \frac{s^2-a^2}{(s^2+a^2)^2}$$

Solution: Note that

$$\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)=\frac{-2as}{(s^2+a^2)^2}\ \text{ and }\ \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=\frac{a^2-s^2}{(s^2+a^2)^2}$$

Direct application of the derivative theorem we obtain

$$(i)\qquad L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right]=(-1)\,t\,L^{-1}\left[-\frac{a}{s^2+a^2}\right]=t\sin at$$

and

$$(ii)\qquad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]=(-1)\,t\,L^{-1}\left[-\frac{s}{s^2+a^2}\right]=t\cos at$$

8.4 Inverse Laplace Transform of Integrals

If $L^{-1}[F(s)] = f(t)$ *then* $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

8.4.1 Example

Find the inverse Laplace transform $f(t)$ *of the function*

$$\int_s^\infty \frac{1}{s(s+1)}\,ds$$

Solution: By the method of partial fraction we obtain

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 1 - e^{-t}.$$

Using the inverse Laplace transform of integrals we get

$$L^{-1}\left[\int_s^\infty \frac{1}{s(s+1)}\right] = \frac{1-e^{-t}}{t}.$$

8.5 Multiplication by Powers of s

If $L^{-1}[F(s)] = f(t)$ *and* $f(0) = 0,$ *then* $L^{-1}[sF(s)] = f'(t)$

8.5.1 Example

Using $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$*, and with the application of above result compute* $L^{-1}\left[\frac{s}{s^2+1}\right]$.

Solution: Direct application of the above result leads to

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}\sin t = \cos t.$$

8.6 Division by Powers of s

If $L^{-1}[F(s)] = f(t)$, *then*

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du.$$

8.7 Example Problems

8.7.1 Problem 1

Compute

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

Solution: we could proceed by applying this integration rule.

$$L^{-1}\left[\frac{1}{s}\,\frac{1}{s^2+1}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+1}\right]\ \mathrm{d}u = \int_0^t \sin\tau\ \mathrm{d}u = 1-\cos t.$$

8.7.2 Problem 2

Find inverse Laplace transform of $\dfrac{1}{(s^2+1)^2}$

Solution: We know that

$$L^{-1}\left[\frac{s}{(1+s^2)^2}\right] = \frac{1}{2}t\sin t.$$

We now apply the above result as

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right] = L^{-1}\left[\frac{1}{s}\,\frac{s}{(1+s^2)^2}\right] = \frac{1}{2}\int_0^t t\sin t\,\mathrm{d}t.$$

Evaluating the above integral we get

$$L^{-1}\left[\frac{1}{(1+s^2)^2}\right] = \frac{1}{2}(-t\cos t + \sin t).$$

8.7.3 Problem 3

Find inverse Laplace transform of $\dfrac{s-1}{s^2(s^2+1)}$.

Solution: It is easy to compute

$$L^{-1}\left[\frac{s-1}{(s^2+1)}\right] = L^{-1}\left[\frac{s}{(s^2+1)}\right] - L^{-1}\left[\frac{1}{(s^2+1)}\right] = \cos t - \sin t.$$

Now repeated application of the above result we get

$$L^{-1}\left[\frac{s-1}{s(s^2+1)}\right] = \int_0^t (\cos t - \sin t)\,\mathrm{d}t = \sin t + \cos t - 1.$$

Finally, we obtain the desired transform as

$$L^{-1}\left[\frac{s-1}{s^2(s^2+1)}\right] = \int_0^t (\sin t + \cos t - 1)\,\mathrm{d}t = 1 - t + \sin t - \cos t.$$