Lecture 6

01/08/17

Tuesday

1. L.T. of derivatives.

$$L[f'(t)] = s\bar{f}(s) - f(0) \quad ; \quad \bar{f}(s) = L[f(t)]$$

2. L.T. of integral.

$$L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}.$$

3. Derivative of Laplace Transform.

$$-\frac{d}{ds}\bar{f}(s) = L[t\,f(t)]$$

$$(-1)^n \frac{d^n}{ds^n}\bar{f}(s) = L[t^n f(t)].$$

4. Integral of LT.

$$\int_s^\infty \bar{f}(s)\,ds = L\left[\frac{f(t)}{t}\right]$$

Ex-1. Find $L\{t\cos at\}$.

$$L\{\cos at\} = \frac{s}{s^2+a^2} = \bar{f}(s)$$

$$L\{t\cos at\} = -\frac{d}{ds}\bar{f}(s) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Ex-2. Find $L\left\{\frac{\sin at}{t}\right\}$

Here $f(t) = \sin at$. To find $L\left\{\frac{f(t)}{t}\right\}$

$$\bar{f}(s) = L[f(t)] = \frac{a}{s^2+a^2} \quad \therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds.$$

$$\int_s^\infty \frac{a}{s^2+a^2}\,ds = \cancel{a}\left[\frac{1}{\cancel{a}}\tan^{-1}\frac{s}{a}\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a}.$$

$$= \cot^{-1}\frac{s}{a} = \tan^{-1}\frac{a}{s}.$$

3. Find $L\{t^n e^{-at}\}$. Hence find $L\left\{\int_0^t \tau^n e^{-a\tau}\,d\tau\right\}$

n is a +ve integer.

step-1

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

step-2

$$L\{\underbrace{t^n e^{-at}}_{f(t)}\} = \frac{n!}{(s+a)^{n+1}} = \bar{f}(s)$$

1st shifting th.

If $L[f(t)] = \bar{f}(s)$

$L[f(t)\,e^{at}] = \bar{f}(s-a)$

step 3

To find $L\left[\int_0^t f(\tau)\,d\tau\right] = \frac{\bar{f}(s)}{s}.$

$$\therefore L\left[\int_0^t \tau^n e^{-a\tau}\,d\tau\right] = \frac{1}{s}\cdot\frac{n!}{(s+a)^{n+1}}.$$

4. Evaluate $L\left\{2\sqrt{\frac{t}{\pi}}\right\}$. Hence find $L\left\{\frac{1}{\sqrt{\pi t}}\right\}$.

$$L\left\{\frac{2}{\sqrt{\pi}}\sqrt{t}\right\} = \frac{2}{\sqrt{\pi}}L(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\cdot\frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}}.$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}}}\cdot\frac{\frac{1}{\cancel{2}}\cdot\cancel{\sqrt{\pi}}}{s^{3/2}} = \frac{1}{s^{3/2}}.$$

$L(t^a)$

$= \frac{\Gamma(a+1)}{s^{a+1}}.$

Way 1. $\frac{d}{dt}\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$.

If $f(t) = 2\sqrt{\frac{t}{\pi}}$, $f'(t) = \frac{1}{\sqrt{\pi t}}$.
$f(0) = 0$.

$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = L[f'(t)] = s\bar{f}(s) - f(0)$.

$= s \cdot \times \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$.

Way 2. Check. $f(t) = 2\sqrt{\frac{t}{\pi}}$.

$\frac{f(t)}{t} = \frac{2}{\sqrt{\pi t}}$. Make sure, you're getting same answer.

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$.

5. Find $L\left\{\int_0^t\int_0^t\int_0^t \cos au\,du\,du\,du\right\}$.

$f(t) = \cos at$. $L\{f(t)\} = \frac{s}{s^2+a^2}$.

$L\left[\underbrace{\int_0^t \cos au\,du}_{f_1(t)}\right] = \frac{\bar{f}(s)}{s} = \frac{1}{s^2+a^2}$.

$f_1(t) = \int_0^t \cos au\,du$ $L[f_1(t)] = \bar{f}_1(s)$

Then,

$$L\left[\int_0^t \underbrace{\int_0^t \cos au\, du}\, du\right]$$

$$= L\left[\int_0^t f_1(t)\, du\right] = \frac{1}{s}\cdot\frac{1}{s^2+a^2}.$$

$$L\left[\int_0^t\int_0^t\int_0^t \cos au\, du\, du\, du\right] = \frac{1}{s^2}\cdot\frac{1}{s^2+a^2}.$$

6. $L\left\{\int_0^t \frac{\sin u}{u}\, du\right\}$

Step 1 Find. $L\{f(t)\} = \frac{1}{s^2+1}$

Step 2. Find. $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1}\frac{1}{s}.$

Step 3 Find. $L\left[\int_0^t \frac{f(t)}{t}\, dt\right] = \frac{1}{s}\tan^{-1}\frac{1}{s}.$

Q. Evaluate. $\int_{t=0}^{\infty}\int_{u=0}^{t} \frac{e^{-t}\sin u}{u}\, du\, dt$ using Laplace transform.

$$= \int_{t=0}^{\infty} e^{-t}\left(\int_0^t \frac{\sin u}{u}\, du\right) dt = L\left[\int_0^t \frac{\sin u}{u}\, du, 1\right]$$

$$= \frac{1}{1}\tan^{-1} 1 = \frac{\pi}{4}.$$

7. Evaluate $\int_0^{\infty} t^2 e^{-4t} \sin 2t \, dt$ using Laplace Transform.

$$= \int_0^{\infty} \underbrace{(t^2 \sin 2t)}_{f(t)} e^{-4t} dt. \qquad = \frac{11}{500} \text{ Ans.}$$

$$= L\{f(t); 4\}$$

$$L\{f(t)\} = L\{t^2 \sin 2t\}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s).$$

$$\bar{f}(s) = L\{\sin 2t\} = \frac{2}{s^2+4}.$$

$$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2} \cdot \frac{2}{s^2+4}$$

Lokenath Debnath - Integral Transform.

Exercises

1. Find $L\left\{\int_0^t u e^{-u} \sin 4u \, du\right\}$

2. Find $L\left\{\sinh ct \int_0^t e^{au} \sinh bu \, du\right\}$

3. Find $L\left\{t e^{-2t} \sin t\right\}$

4. Find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

5. Evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt$. (using L.T.)

It may or may not be mentioned.