In this lesson we discuss differentiation and integration of the Fourier series of a function. We can get some idea of the complexity of the new series if looking at the terms of the series. In the case of differentiation we get terms like $n\sin(nx)$ and $n\cos(nx)$, where presence of n as product makes the magnitude of the terms larger then the original and therefore convergence of the new series becomes more difficult. This is exactly other way round in the case of integration where n appears in division and new terms become smaller in magnitude and thus we expect better convergence in this case. We shall deal these two case separately in next sections.

9.1 Differentiation

We first discuss term by term differentiation of the Fourier series. Let f be a piecewise continuous with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{9.1}$$

Can we differentiate term by term the Fourier series of a function f in order to obtain the Fourier series of f'? In other words, is it true that

$$f'(x) \sim \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)]? \tag{9.2}$$

In general the answer to this question is no.

Let us consider the Fourier series of $f(x) = x$ in $[-\pi, \pi]$. This is an odd function and therefore Fourier series will be $x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. If we differentiate the series term by term we get $\sum_{n=1}^{\infty} 2(-1)^{n+1} \cos(nx)$. Note that this is not the Fourier series of $f'(x) = 1$ since the Fourier series of $f(x) = 1$ is simply 1.

We consider one more simple example to illustrate this fact. Consider the half range sine series for $\cos x$ in $(0, \pi)$

$$\cos x \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2nx)}{(4n^2-1)}$$

If we differentiate this series term by term then we obtain the series

$$\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n^2 \cos(2nx)}{(4n^2-1)}$$

This series can not be the Fourier series of $-\sin x$ because it diverges as

$$\lim_{n\to\infty} \frac{16}{\pi} \frac{n^2 \cos(2nx)}{(4n^2-1)} \neq 0$$

For the term by term differentiation we have the following result

9.1.1 Theorem

If f is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, f' is piecewise continuous on $[-\pi, \pi]$, and if

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

(in fact in this case we can replace $\sim$ by $=$) is the Fourier series of f, then the Fourier series of f' is given by

$$f'(x) \sim \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] \, .$$

Moreover, at a point x, we have

$$\frac{f'(x+) + f'(x-)}{2} = \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] \, .$$

If f' is continuous at a point x then

$$f'(x) = \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] \, .$$

Proof: Since f' is piecewise continuous and this is sufficient condition for the existence of Fourier series of f'. So we can write Fourier series of as

$$f'(x) \sim \frac{\bar{a}_0}{2} + \sum_{n=1}^{\infty} \left[\bar{a}_n \cos(nx) + \bar{b}_n \sin(nx)\right] \tag{9.3}$$

where

$$\bar{a}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos(nx) \, \mathrm{d}x, \quad \bar{b}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) \, \mathrm{d}x$$

Now we simplify coefficients $\bar{a}_n$ and $\bar{b}_n$ and write them in terms of a_n and b_n. Using the condition $f(-\pi) = f(\pi)$, we can easily show that

$$\bar{a}_0 = 0, \quad \bar{a}_n = nb_n, \quad \bar{b}_n = -na_n$$

Now the Fourier series of f' (9.3) reduces to

$$f'(x) \sim \sum_{n=1}^{\infty} [nb_n \cos(nx) - na_n \sin(nx)]$$

Convergence of this series to $\dfrac{f'(x+) + f'(x-)}{2}$ or $f'(x)$ is a direct consequence of convergence theorem of Fourier series. ■

9.2 Integration

In general, for an infinite series uniform convergence is required to integrate the series term by term. In the case of Fourier series we do not even have to assume the convergence of the Fourier series to be integrated. However, integration term by term of a Fourier series does not, in general, lead to a Fourier series. The main results can be summarize as:

9.2.1 Theorem

Let f be piecewise continuous function and have the following Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{9.4}$$

Then no matter whether this series converges or not we have for each $x \in [-\pi, \pi]$,

$$\int_{-\pi}^{x} f(t)dt = \frac{a_0(x+\pi)}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \Big(\cos(nx) - \cos n\pi \Big) \right] \tag{9.5}$$

and the series on the right hand side converges uniformly to the function on the left.

Proof: We define

$$g(x) = \int_{-\pi}^{x} f(t)dt - \frac{a_0}{2}x$$

Since f is piecewise continuous function, it is easy to prove that g is continuous. Also

$$g'(x) = f(x) - \frac{a_0}{2} \tag{9.6}$$

at each point of continuity of f. This implies that g' is piecewise continuous and further we see that

$$g(-\pi) = \frac{a_0 \pi}{2}$$

and

$$g(\pi) = \int_{-\pi}^{\pi} f(t)dt - \frac{a_0}{2}\pi = \pi a_0 - \frac{a_0}{2}\pi = \frac{a_0\pi}{2}$$

Hence, the Fourier series of the function g converges uniformly to g on $[-\pi, \pi]$. Thus we have

$$g(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} [\alpha_n \cos(nx) + \beta_n \sin(nx)]$$

Using Theorem 9.1.1 we have the following result for the Fourier series of g' as

$$g'(x) \sim \sum_{n=1}^{\infty} [-n\alpha_n \sin(nx) + n\beta_n \cos(nx)]$$

Fourier series of f and the relation (9.6) gives

$$g'(x) = f(x) - \frac{a_0}{2} \sim \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Now comparing the last two equations we get

$$n\beta_n = a_n \qquad -n\alpha_n = b_n \quad n = 1, 2, \ldots$$

Substituting these values in the Fourier series of g we obtain

$$g(x) = \int_{-\pi}^{x} f(t)dt - \frac{a_0}{2}x = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right]$$

We can rewrite this to get

$$\int_{-\pi}^{x} f(t)dt = \frac{a_0}{2}x + \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right] \tag{9.7}$$

To obtain α_0 we set $x = \pi$ in the above equation

$$\alpha_0 = a_0\pi + \sum_{n=1}^{\infty} \frac{2b_n}{n}\cos(n\pi)$$

Substituting α_0 in the equation (9.7) we obtain the required result (9.5). ■

Remark 1: *Note that the series on the right hand side of (9.5) is not a Fourier series due to presence of x.*

Remark 2: *The above Theorem on integration can be established in a more general sense as:*
If f be piecewise continuous function in $-\pi \leq x \leq \pi$ and if

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

is its Fourier series then no matter whether this series converges or not, it is true that

$$\int_a^x f(t)dt = \frac{a_0}{2} \int_a^x a_0 \, \mathrm{d}x + \sum_{n=1}^{\infty} \int_a^x [a_n \cos(nx) + b_n \sin(nx)] \, \mathrm{d}x$$

where $-\pi \leq a \leq x \leq \pi$ and the series on the right hand side of converges uniformly in x to the function on the left for any fixed value of a.