11.1 Miscellaneous Example Problems

11.1.1 Problem 1

Using the convolution theorem prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad [m,n>0].$$

Solution: Let $f(t) = t^{m-1},\ g(t) = t^{n-1}$, then

$$(f*g)(t) = \int_0^t \tau^{m-1}(t-\tau)^{n-1}\mathrm{d}\tau,$$

Substituting $\tau = ut$ so that $\mathrm{d}\tau = t\,\mathrm{d}u$ we obtain

$$(f*g)(t) = \int_0^1 t^{m-1}u^{m-1}t^{n-1}(1-u)^{n-1}t\mathrm{d}u$$

We simplify the above expression to get

$$(f*g)(t) = t^{m+n-1}\int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = t^{m+n-1}B(m,n)$$

Taking Laplace transform and using convolution property, we find

$$L[t^{m+n-1}B(m,n)] = L[f(t)]*L[g(t)] = L[t^{m-1}]*L[t^{n-1}] = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Taking inverse Laplace transform,

$$t^{m+n-1}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

Hence, we get the desired result as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

11.1.2 Problem 2

Show that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

Solution: We know

$$L[\sin t] = \frac{1}{s^2+1}$$

Therefore, we get

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \frac{\pi}{2} - \tan^{-1} s.$$

Taking limit as $s \to 0$ (see remarks below for details) we find

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}.$$

Remark 1: *Suppose that f is piecewise continuous on $[0,\infty)$ and $L[f(t)] = F(s)$ exists for all $s > 0$, and $\int_0^\infty f(t)\,\mathrm{d}t$ converges. Then $\lim_{s\to 0+} F(s) = \lim_{s\to 0+} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty f(t)\,\mathrm{d}t$.*

Remark 2: *If f is a piecewise continuous function ans $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s)$ converges uniformly for all $s \in E$, then $F(s)$ is a continuous function on E, that is, for $s \to s_0 \in E$,*

$$\lim_{s\to s_0} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s_0) = \int_0^\infty \lim_{s\to s_0} e^{-st} f(t)\,\mathrm{d}t.$$

Remark 3: *Recall that the integral $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is said to converge uniformly for s in some domain Ω if for any $\epsilon > 0$ there exists some number τ_0 such that if $\tau \geq \tau_0$ then*

$$\left|\int_\tau^\infty e^{-st} f(t)\,\mathrm{d}t\right| < \epsilon$$

for all s in Ω.

11.1.3 Problem 3

Using Laplace transform, evaluate the following integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2+a^2}\,dx$$

Solution: Let

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2 + a^2}\, \mathrm{d}x$$

Taking Laplace transform, we get

$$F(s) = \int_0^\infty \frac{x}{x^2 + a^2} \frac{x}{x^2 + s^2}\, \mathrm{d}x$$

Using the method of partial fractions we obtain

$$F(s) = \int_0^\infty \frac{1}{x^2 + s^2}\, \mathrm{d}x - \frac{a^2}{s^2 - a^2} \int_0^\infty \left(\frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} \right) \mathrm{d}x$$

Evaluating the above integrals we have

$$F(s) = \frac{1}{s} \tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty - \frac{a^2}{s^2 - a^2} \left[\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{s} \tan^{-1}\left(\frac{x}{s}\right) \right]_0^\infty$$

On simplification we obtain

$$F(s) = \frac{1}{2} \frac{\pi}{s + a}$$

Taking inverse Laplace transform we find

$$f(t) = \frac{1}{2} \pi e^{-at}$$

Hence the value of the given integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2 + a^2}\, \mathrm{d}x = 2 \int_0^\infty \frac{x \sin xt}{x^2 + a^2}\, \mathrm{d}x = \pi e^{-at}.$$

11.1.4 Problem 4

Evaluate $\displaystyle\int_0^\infty \frac{\cos tx}{x^2 + 1}\, \mathrm{d}x, \quad t > 0.$

Solution: Let

$$f(t) = \int_0^\infty \frac{\cos tx}{x^2 + 1}\, \mathrm{d}x.$$

Taking Laplace transform on both sides,

$$\begin{aligned} L[f(t)] &= \int_0^\infty \frac{s}{(x^2 + 1)(s^2 + x^2)}\, \mathrm{d}x \\ &= \frac{s}{s^2 + 1} \int_0^\infty \left(\frac{1}{x^2 + 1} - \frac{1}{s^2 + x^2} \right) \mathrm{d}x \\ &= \frac{s}{s^2 - 1} \left[\tan^{-1} x - \frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right) \right]_0^\infty \\ &= \frac{s}{s^2 - 1} \left(\frac{\pi}{2} - \frac{\pi}{2s} \right) = \frac{\pi}{2} \frac{1}{s + 1}. \end{aligned}$$

Taking inverse Laplace transform on both sides,

$$f(t) = \frac{\pi}{2}e^{-t}.$$

11.1.5 Problem 5

Evaluate $\int_0^\infty e^{-x^2}\mathrm{d}x.$

Solution: Let

$$g(t) = \int_0^\infty e^{-tx^2}\,\mathrm{d}x$$

Now taking Laplace on both sides,

$$L[g(t)] = \int_0^\infty \frac{1}{s+x^2}\,\mathrm{d}x = \frac{1}{\sqrt{s}}\arctan\left(\frac{x}{\sqrt{s}}\right)\Big|_0^\infty = \frac{1}{\sqrt{s}}\frac{\pi}{2}$$

Taking inverse Laplace transform we obtain

$$g(t) = \frac{\pi}{2}L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{\pi}{2}\frac{1}{\sqrt{\pi}\sqrt{t}}.$$

Hence for $t = 1$ we get

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$