04/08/2022

Ch E. (PDE) Parabolic = ~~Transient~~ Unsteady state.
(PDE) Elliptical = Steady state.
(PDE) Hyperbolic = very rare for Ch Engg
relevant for civil engineers.

Ex ① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ ($\nabla^2 u = 0$
($\nabla^2 \rightarrow$ Laplacian operator

$x = x_1,\ y = x_2,\ z = x_3$

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{13}\frac{\partial^2 u}{\partial x_1 \partial x_3}$$
$$+ a_{21}\frac{\partial^2 u}{\partial x_2 \partial x_1} + a_{12}\frac{\partial^2 u}{\partial x_2^2} + a_{13}\frac{\partial^2 u}{\partial x_2 \partial x_3}$$
$$+ a_{31}\frac{\partial^2 u}{\partial x_3 \partial x_1} + a_{32}\frac{\partial^2 u}{\partial x_3 \partial x_2} + a_{33}\frac{\partial^2 u}{\partial x_3^2}$$

$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\lambda = 1, 1, 1$

Elliptical PDE

② $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$x = x_1,\ y = x_2,$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda = 1, 1, 0$

parabolic PDE

③ $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$x = x_1,\ y = x_2,\ t = x_3$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

Hyperbolic PDE. $\rightarrow$ Mixed sign.

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Linear operater

If $L(\alpha u+\beta v)$

$= \alpha L(u) + \beta L(v)$

α, β are scalars

u, v are certain functi.

then, L is linear operater

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$\hookrightarrow L$

$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$L = \frac{\partial}{\partial t} - \nabla^2$

$L = \frac{\partial^2}{\partial t^2} - \nabla^2$

Linear operater.

if operater is Linear → Advantageous, we can use principle of linear superposition → Helps in solution of PDES.

ODE: $\frac{d^2u}{dx^2} = \underline{x}$ (if $u=0$ then Homogen)

at, $x=0$, $u=\underline{1}$

$x=1$, $u=\underline{2}$ (if $u=1$ then homogen)

→ 3 sources of Non-homogeneity.

$u = u_1 + u_2 + u_3$

1 NonH at a time Forcing others to be zero.

⇒ u_1 : $\frac{d^2u_1}{dx^2} = \cancel{x}$

at $x=0$ } $u_1=0$
$=1$ }

u_2 : $\frac{d^2u_2}{dx^2} = 0$

at $x=0$, $u_2=1$

$x=1$, $u_2=0$

u_3 : $\frac{d^2u_3}{dx^2} = 0$

at $x=0$, $u_3=0$

$x=1$, $u_3=2$

$\frac{d^2}{dx^2}(u_1+u_2+u_3) = x$: $\frac{d^2u_1}{dx^2} + \frac{d^2u_2}{dx^2} + \frac{d^2u_3}{dx^2} = x$.

at $x=0$, $u_1+u_2+u_3 = 1$

at $x=1$, $u_1+u_2+u_3 = 2$.

u_1 : $\frac{du_1}{dx} = \frac{x^2}{2} + c_1$

$u_1 = \frac{x^3}{6} + c_1x + c_2$

$c_2 = 0$

$u_1 = \frac{x^3}{6} + c_1 x$

$0 = \frac{1}{6} + c_1 \Rightarrow \boxed{c_1 = -\frac{1}{6}}$

$u_1 = \frac{x^3}{6} - \frac{x}{6}$

u_2 : $\frac{du_2}{dx} = c_1$

$u_2 = c_1 x + c_2$.

~~$u_2 = 0$~~

$c_2 = 1$

~~$u_2 = c_1 x$~~

$0 = c_1 + 1$

$c_1 = -1$

$\boxed{u_2 = -x + 1}$

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u_3: $\frac{du_3}{dx} = c_1$

$u_3 = c_1 x + c_2$

$c_2 = 0$

$2 = c_1 + 0$ $\boxed{c_1 = 2}$

$\boxed{u_3 = 2x}$

$u = u_1 + u_2 + u_3$

$= \frac{x^3}{6} - \frac{x}{6} - x + 1 + 2x$

$= \frac{x^3}{6} - \frac{x}{6} + x + 1$

$\boxed{u = \frac{x^3}{6} - \frac{5x}{6} + 1}$

If we directly solving the problem.

$\Rightarrow$ $\frac{du}{dx} = \frac{x^2 c_1}{2}$

$u = \frac{x^3 c_1}{6} + x c_2$

$\boxed{1 = c_2}$

$\Rightarrow$ $u = \frac{x^3 c_1}{6} + 1$

$2 = c_1 + 1$

$\boxed{c_1 = 1}$

$\Rightarrow$ $u = \frac{x^3}{6} + 1$

$\frac{du}{dx} = \frac{x^2}{2} + c_1$

$u = \frac{x^3}{6} + c_1 x + c_2$ $\boxed{c_2 = 1}$ $u = \frac{x^3}{6} + c_1 x + 1$

$2 = \frac{1}{6} + c_1 + 1$

$2 - \frac{7}{6}$

$\boxed{c_1 = \frac{5}{6}}$ $\boxed{u = \frac{x^3}{6} + \frac{5x}{6} + 1}$

Q// Parabolic PDE (Principle of lin superposition)

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

at $t = 0$, $u = u_0$

at $x = 0$, $u = 0$

at $x = 1$, $u = 1$

2 sources of nh

$\therefore$ $u = u_1 + u_2$

u_1: $\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}$

at $t = 0$, $u_1 = u_0$

at $x = 0$, $= 1$ $\}$ $u_1 = 0$

↳ well behaved/ well posed problem.

solvable directly

u_2: $\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}$

at $t = 0$, $u_2 = 0$

at $x = 0$, $u_2 = 0$

at $x = 1$, $u_2 = 1$

↳ Ill posed problem.

initial cond is 0

~~and one BC's~~

~~are Non homo~~

All BC's are

~~Non~~ homogen.

$\Rightarrow$ ILL POSED PROB

to be splited

into (1) time dependent

two subprob (2) Time independent

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$(x,t) = u_2^s(x) + u_2^t(x,t)$

Steady state prob

$\frac{\partial}{\partial t}(u_2^s + u_2^t) = \frac{\partial^2}{\partial x^2}(u_2^s + u_2^t)$

$\frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^s}{\partial x^2} + \frac{\partial^2 u_2^t}{\partial x^2}$

at $x=1$, $u_2=1$

$u_2^t + u_2^s = 1$

u_2^s:

$\frac{d^2 u_2^s}{dx^2} = 0$

at $x=0$, $u_2^s = 0$.

at $x=1$, $u_2^s = 1$

$\frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^t}{\partial x^2}$

at $x=0$, $u_2^t = 0$

at $x=1$, $u_2^t = 0$

at $t=0$, $u_2 = 0$

$u_2^s + u_2^t = 0$

$\boxed{u_2^t = -u_2^s(x)}$

$= -x$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$: at $t=0$, $u = u_{01}$

$x=0$, $\frac{\partial u}{\partial x} + \alpha u = u_{02}$

$x=1$, $\frac{\partial u}{\partial x} = u_{03}$.

$u = u_1 + u_2 + u_3$ (Formulate problems s.t. there exist one NH at a time).

u_1: $\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}$ at $t=0$, $u_1 = u_{01}$

$x=0$, $\frac{\partial u_1}{\partial x} + \alpha u_1 = 0$

$x=1$, $\frac{\partial u_1}{\partial x} = 0$

NH IC, H BCs — well posed prob

HEL

u_3: $\frac{\partial u_3}{\partial t} = \frac{\partial^2 u_3}{\partial x^2}$: at $x=0$, $\frac{\partial u_3}{\partial x} + \alpha u_3 = 0$

at $x=1$, $\frac{\partial u_3}{\partial x} = u_{03}$.

at $t=0$, $u_3 = 0$

Ill posed

u_2: $\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}$

at $t=0$, $u_2 = 0$

at $t=0$, $\frac{\partial u_2}{\partial x} + \alpha_2 u_2 = \ldots$

at $x=1$, $\frac{\partial u_2}{\partial x} = 0$

Ill posed

$u_2 = u_2^s(x) + u_2^t(x,t)$

u_2^s:

$\boxed{\frac{d^2 u_2^s}{dx^2} = 0}$

u_2^t: $\frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^t}{\partial x^2}$

at $x=0$, $\frac{\partial u_2^t}{\partial x} + \alpha u_2^t \ldots$

at $x=1$, $\frac{\partial u_2^t}{\partial x} = 0$

at $x=1$, $\frac{\partial}{\partial x}(u_2^s + u_2^t) = \ldots$

at $x=0$, $\boxed{\frac{du_2^s}{dx} + \frac{\partial u_2^t}{\partial x} + \alpha_1 u_2^s + \alpha u_2^t = u_{02}}$

$\frac{du_2^s}{dx} + \alpha u_2^s = u_{02}$ @ $x=0$

$\frac{du_2^s}{dx} + \frac{\partial u_2^t}{\partial x} = 0$.

11/08/

of the form

Defines Standard value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad ; \quad \lambda \text{ is a scalar.}$$

Subject to at $\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\} y=0$

$$L = \frac{d^2}{dx^2}$$

$\Rightarrow Ly + \lambda y = 0$

$$\boxed{Ly = -\lambda y}$$

Equivalent to Discrete domain $\boxed{AX = \lambda X}$ ← Eigen value problem in cont. domain.

A: Sq. matrix; X: Vector; λ: Eigen value.

Case 1: $\frac{d^2y}{dx^2} + \lambda y = 0$

Sub to $y=0$ at $\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\}$

$\Rightarrow$ Standard value problem.

Case a: $y=0$ is always a trivial sol^n but we aren't looking for a trivial solution

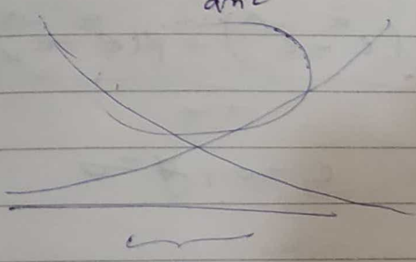
$\boxed{\lambda = 0}$

$$\frac{d^2y}{dx^2} = 0$$

$\Rightarrow \boxed{y=0}$ Trivial solution

Case b: $\lambda = -ve = -\alpha^2$

$$\frac{d^2y}{dx^2} - \alpha^2 y = 0 \quad ; \quad y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$



$0 = c_1 + c_2$ — (1)

$0 = c_1 e^{\alpha} + c_2 e^{-\alpha}$

$y = c_1 (\underbrace{e^{\alpha x} - e^{-\alpha x}}_{+ve})$

Case c: $\lambda = +ve = \alpha^2$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0 \Rightarrow y = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$$

$0 = c_2$

$y = c_1 \sin(\alpha x)$

$0 = c_1 \sin(\alpha)$, if $c_1 = 0$ Trivial sol^n ($y=0$) we aren't looking for that.

For $c_1 \neq 0 \Rightarrow \alpha = 0$

$\boxed{\alpha_n = n\pi}$, $n = 1, 2, \ldots \infty - \{0\}$

$\boxed{y_n = c_1 \sin(n\pi x)}$ ← Eigen function

Case 2: $\frac{d^2y}{dx^2} + \lambda y = 0$ sub to at $x=0$ $\frac{dy}{dx} = 0$

at $x = 1$, $y = 0$

Case a: $\lambda = 0$

$y = c_1 x + c_2$

$y = 0$ Trivial sol^n

$\boxed{\lambda \neq 0}$

Case b: $\lambda = -ve = -\alpha^2$

$$\frac{d^2y}{dx^2} - \alpha^2 y = 0$$

at $x = 0$, $y' = 0$

$$\frac{dy}{dx} = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$$

$0 = c_1\alpha - c_2\alpha$
$\Rightarrow c_1 = c_2$

$y = c_1(e^{\alpha x} + e^{-\alpha x})$

$0 = c_1 \times (e^{\alpha} + e^{-\alpha}) \Rightarrow c_1 = c_2 = 0$

Case C: $\lambda = +ve = \alpha^2$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

$y = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

$\frac{dy}{dx} = c_1\alpha\cos(\alpha x) - c_2\alpha\sin(\alpha x)$

$0 = c_1\alpha\cos(\alpha)$, $c_1 = 0, \alpha \neq 0$

$y = c_2\cos(\alpha x)$

$0 = c_2 \cos\alpha$

$\cos\alpha = 0 \Rightarrow \alpha_n = (2n-1)\pi/2$

$n = 1, 2, \ldots \infty$

Eigen funct

$y_n = C_n\cos\left[(2n-1)\frac{\pi}{2}x\right]$

Case B:

$\frac{d^2y}{dx^2} + \lambda y = 0$; sub to at $x = 0, y = 0$

at $x = 1$, $\boxed{\frac{dy}{dx} + \beta y = 0}$

$\hookrightarrow$ scalar

Case a: $\lambda = 0$

$y = 0$ is soln

Case b: $\lambda = -ve = -\alpha^2$

$y(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$y(x) = c_1(e^{\alpha x} - e^{-\alpha x})$

$\cancel{Ke^{-\beta}} = c_1(e^{\alpha} - e^{-\alpha})$

$\boxed{\cancel{c_1 = \frac{Ke^{-\beta}}{e^{\alpha} - e^{-\alpha}}}}$

Case C: $\lambda = +ve = \alpha^2$

$\frac{dy}{y} = -\beta dx$

$\ln y = -\beta x + c$

$\boxed{y = Ke^{-\beta x}}$

$c_1(\alpha e^{\alpha x} + \alpha e^{-\alpha x}) + \beta c_1(e^{\alpha x} - e^{-\alpha x})$

$c_1[\underbrace{\alpha(e^{\alpha} + e^{-\alpha}) + \beta(e^{\alpha} - e^{-\alpha})}_{+ve}] = 0$

$c_1 = 0, y = 0$

Case C: $\lambda = +ve = \alpha^2$

$y = c_1\sin(\alpha x) + c_2\cos(\alpha x)$

$0 = c_2$

$y = c_1\sin(\alpha x)$

$c_1\alpha\cos\alpha x + \beta c_1\sin(\alpha x)\Big]_{x=1} = 0$

$c_1(\alpha\cos\alpha + \beta\sin\alpha) = 0$

$c_1(\alpha + \beta\tan\alpha) = 0$

For $c_1 \neq 0 \Rightarrow \boxed{\alpha_n + \beta\tan\alpha_n = 0}$ Transcendental equation in α.
$\rightarrow$ NR method

$\boxed{y_n = c_1\sin(\alpha_n x)} \rightarrow$ Eigen functn.

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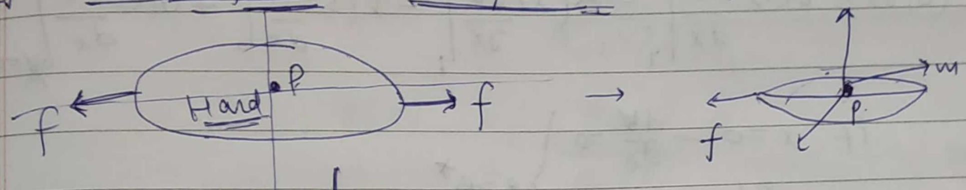
$\# \ \frac{d^2y}{dx^2} + \lambda y = 0$ } Std. eigenvalue Prob → common in solution of PDE's in Cartesian/Rectangular coord.

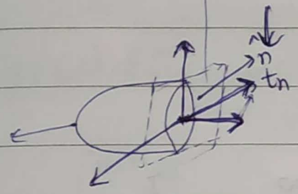
Sub to Hom BC's

BC	E value	E funct
Dirichlet	$n\pi$	$\sin(n\pi x)$
Neumann + Dirichlet	$(2n-1)\frac{\pi}{2}$	$\cos\left[(2n-1)\frac{\pi}{2}x\right]$
Dirichlet + Robin mix	$\alpha_n + \beta \tan\alpha_n = 0$ ↑ Roots	$\sin(\alpha_n x)$

$\boxed{11/08/22}$: AFD (Prof. HK) Stress/strain (Tensor)



$\hat{t}_m =$



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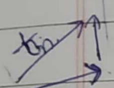
$\hat{t}_n = (\ ) + (\ ) + (\ ).$

$T_{n-} \quad T_{n-} \quad T_{n-}$

$\boxed{T_{y\ell} = \gamma - m\gamma}$



\# $\boxed{24/08/22}$ AMT

For operator $L = a_0 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$

$$L^* = a_0\frac{d^2}{dx^2} + (2a_0' - a_1)\frac{d}{dx} + (a_0'' - a_1' + a_2)$$

$$\langle v, Lu\rangle = J(u,v) + \langle u, L^* v\rangle$$

if $L = L^*$
& $B = B^*$
$L \to$ self adjoint operator.

Ex: 1 $L = \frac{d^2}{dx^2}$, BC's on $u \Rightarrow$ at $x=0$, $x=1$ } $u=0$] B.

$L^* = ?$ $B^* = ?$

$$\langle v, Lu\rangle = \int_0^1 v\frac{d^2u}{dx^2}\,dx = v\frac{du}{dx}\Big|_0^1 - \int_0^1 \frac{dv}{dx}\frac{du}{dx}\,dx$$

$$= v\frac{du}{dx}\Big|_0^1 - \frac{dv}{dx}u\Big|_0^1 + \int_0^1 u\frac{d^2v}{dx^2}\,dx$$

$$= \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + \langle u, L^* v\rangle \qquad L^* = L = \frac{d^2}{dx^2}$$

$$J(u,v) = v(1)\frac{du}{dx}\Big|_1 - \cancel{u(1)}\frac{dv}{dx}\Big|_1 - v(0)\frac{du}{dx}\Big|_0 + \cancel{u(0)}\frac{dv}{dx}\Big|_{x=0}$$

if $\left.\begin{matrix} x=0 \\ =1 \end{matrix}\right\} v=0 \Big\} B^*$

In this particular problem.
$$\left[\begin{matrix} B = B^* \\ L = L^* \end{matrix}\right]$$

Ex: 2 $\quad L = \frac{d^2}{dx^2}$; BC's on $u \Rightarrow$ at $x=0$, $du/dx = 0$; $=1$, $u=0$ $\Big] B$

—— Same as previous ——

$$\therefore J(u,v) = v(1)\frac{du}{dx}\Big|_1 - u(1)\frac{\partial v}{\partial x}\Big|_1 - v(0)\frac{du}{dx}\Big|_0 + u(0)\frac{dv}{dx}\Big|_{x=0}$$

if $x=0 - \frac{dv}{dx} = 0$; $=1$ $\quad v=0$ $\Big\} B = B^*$

Ex: 3 $\quad L = \frac{d^2}{dx^2}$, BC's on $u \Rightarrow$ at $x=0$, $u=0$; $=1$, $\frac{du}{dx} + \beta u = 0$ $\Big] B$

$$\therefore \underbrace{\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1}_{J(u,v)} + \langle u, L^* v\rangle$$

$L^* = L = \frac{d^2}{dx^2}$

$B = B^*$

$$J(u,v) = v(1)\frac{du}{dx}\Big| - u(1)\frac{dv}{dx} - v(0)\frac{du}{dx}\Big|_0 + \cancel{u(0)\frac{du}{dx}}\Big|_{x=0}$$

$$J(u,v) = v(1) * (-\beta)u(x=1) - u(1)\frac{dv}{dx}\Big|_{x=1} - v(0)\frac{du}{dx}\Big|_{x=0}$$

$$= u(1)\left[\frac{dv}{dx}\Big|_{x=1} + \beta v\Big|_{x=1}\right] - v(x=0)\frac{du}{dx}\Big|_{x=0}$$

$\left\{\begin{matrix} L = L^* \\ B = B^* \end{matrix}\right.$

$L \rightarrow$ self adjoint operator.

at $x=0$, $v=0$
at $x=1$, $\frac{dv}{dx} + \beta v = 0$ $\Big\} B^*$

$\rightarrow$ Generalised second order operator.

$$L(u) = a_0\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_2(x)u$$

During separation of variables:

$$Lu = -\lambda a_3 u \quad \text{[Generalized Eigenvalue Problem]}$$

$$\boxed{a_0 u'' + a_1 u' + a_2 u + \lambda a_3 u = 0} \quad \text{----(1)}$$

Equation (1) can be re-written

$$\frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + q(x)u + \lambda r(x)u = 0 \quad \text{---(2)}$$

where, $P(x) = e^{\int \frac{a_1(x)}{a_0(x)} dx}$

$$q(x) = \frac{a_2(x)}{a_0(x)} P$$

$$r(x) = \frac{a_3(x)}{a_0(x)} P$$

Eqs (1) & (2) are identical. [P.T]

$$\frac{d}{dx}\left[e^{\int \frac{a_1(x)}{a_0(x)} dx} \frac{du}{dx}\right] + \frac{a_2(x)Pu}{a_0(x)} + \lambda \frac{a_3(x)Pu}{a_0(x)} = 0$$

$$+ \frac{a_2(x)}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx} u + \frac{\lambda a_3(x)}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx} u = 0$$

$$\frac{a_1(x)}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx} \frac{du}{dx} + e^{\int \frac{a_1(x)}{a_0(x)} dx} \frac{d^2u}{dx^2} + \frac{a_2(x)}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx} u + \quad = 0$$

$$a_0(x) u'' + a_1 u' + a_2 u + \lambda a_3 u = 0 \rightarrow \boxed{a_0 u'' + a_1 u' + a_2 u + \lambda a_3 u = 0}$$

M-II

$$\ln P = \int \frac{a_1}{a_0} dx \Rightarrow \frac{1}{P}\frac{dP}{dx} = \frac{a_1}{a_0} \Rightarrow \frac{dP}{dx} = \frac{a_1}{a_0} P$$

$$\frac{d}{dx}\left[P\frac{du}{dx}\right] + qu + \lambda r u = 0$$

$$L = \frac{d}{dx}\left[P\frac{d}{dx}\right] + q.$$

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$$a_0 u'' + a_1 u' + a_2 u.$$

$$(a_0'' - a_1' + a_2)v.$$

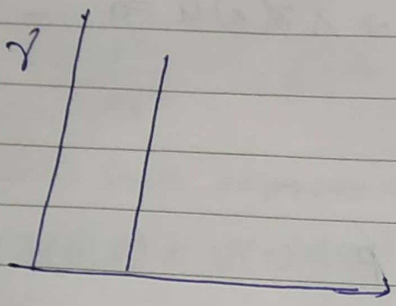
$a_0 = p_j \quad a_1 = p_j' \quad a_2 = q$

$$L^* = p\frac{d^2}{dx^2} + (2p' - p')\frac{d}{dx} + (p'' - p'' + q)$$

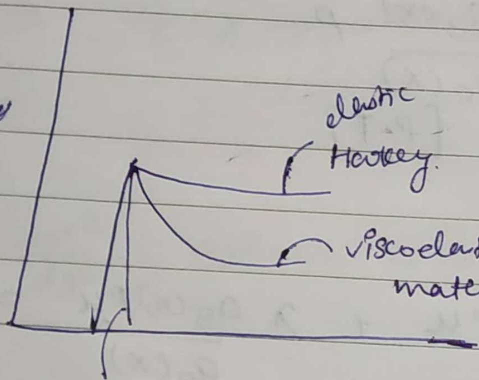
$$= p\frac{d^2}{dx^2} + p'\frac{d}{dx} + q.$$

#24/08/22 AFD : Prof. HK.

Time (strain) dependence



elastic solid | simple liquid | Complex fluid.



→ Draw these on cartesian coordinate and study their behaviour. Do the study on graphs.

→ $G(t) = \frac{\tau(t)}{\gamma}$

$M(t) = -\frac{dG}{dt}$

memory function.

$$\int_0^{\tau} \tau(t) = \int_{-\infty}^{t} M(t)\,\gamma\,dt$$

$(t - t') = s$

Viscous liquid. —viscoelastic method→ elastic solid.

Measure the viscoelastic

(1) stress Relaxation.

(2) Creep

(3) ~~stead~~ sinusoidal oscillation.

25/08/24 Advanced mathematical techniques

Generalized Sturm-Louiville operater;

$$L = a_{0(x)}\frac{d^2}{dx^2} + a_{1(x)}\frac{d}{dx} + a_{2(x)}$$

$\boxed{L = L^*}$ $Lu = -\lambda r u$

$$\left.\begin{aligned} &\text{at } x=a, \ \alpha_1 \frac{du}{dx} + \alpha_2 u = 0 \\ &x = b, \ \beta_1 \frac{du}{dx} + \beta_2 u = 0 \end{aligned}\right\} B$$

$$J(u,v) = \left[v a_0 u' - v' a_0 u - v a_0' u + a_1 v u \right]_a^b$$

$a_0 = p$ $\Rightarrow -[a_0 v u' - a_0 v' u]_a^b$

$a_1 = p'$

$a_2 = q$ $\Rightarrow a_0(b)$

$a_2 = \cancel{q}$ $\Rightarrow a_0 v(b) u'(b) - a_0 v'(b) u(b) - a_0 v(a) u'(a) + a_0 v'(a) u(a)$

$$\Rightarrow a_0(b) * (-u(b)) \left[\frac{\beta_2}{\beta_1} v(b) + v'(b) \right]$$

$$\Rightarrow a_0(a) * \left\{ \frac{\alpha_2}{\alpha_1} v(a) + v'(a) \right\} u(a)$$

$$\Rightarrow \left.\begin{aligned} &\text{at } x = a \\ &\alpha_1 \frac{dv}{dx} + \alpha_2 v = 0 \\ &\text{at } x = b \\ &\beta_1 \frac{dv}{dx} + \beta_2 v = 0 \end{aligned}\right\} B^*$$

$\Rightarrow \boxed{B = B^*}$

$\boxed{L = L^*}$

$\rightarrow L$ is self adjoint.

$\boxed{\text{Theorem \# 1:}}$ For Sturm-Louiville operater (S-L)
$\hookrightarrow$ Infinite eigen values.

Definition of orthogonal function:

$\phi_m(x)$ & $\phi_n(x)$ are orthogonal to each other w.r.t weight function $r(x)$. if $\langle \phi_m, \phi_n \rangle = 0$.

$$\Rightarrow \int_X \phi_m \phi_n r(x)\, dx = 0$$

if functi of (x,y)

$$\int_{(X,Y)} \phi_m \phi_n r(x,y)\, dx\, dy = 0$$

similarly for (x,y,z)

Theorem 2 : If $\lambda_m \neq \lambda_n$ ... are corresponding eigen functions.

y_m & y_n are orthogonal function w.r.t $r(x)$.

$$\Rightarrow \quad Ly = -\lambda r y \quad \Rightarrow \quad Ly_n = -\lambda_n r y_n \quad --- (1)$$

Eigen value problem.

$$Ly_m = -\lambda_m r y_m \quad --- (2)$$

Take inner product of eq(1) w.r.t y_m

" " " eq 2 w.r.t y_n.

& Subtract.

$$\boxed{\langle v, Lu\rangle = \langle u, L^* v\rangle + J(u,v)}$$

$$\Rightarrow \int y_m (Ly_n)dx - \int y_n (Ly_m)dx = (\lambda_m - \lambda_n)\int y_m y_n r dx$$

$$\int y_n L^* y_m dx + J(y_m, y_n) - \int y_n (L y_m) dx = (\lambda_m - \lambda_n)\int y_m y_n r dx$$

$L = L^*$ & $J(u,v) = 0$

$$\Rightarrow \boxed{\underbrace{(\lambda_m - \lambda_n)}_{\neq 0}\underbrace{\int y_m y_n r dx}_{=0} = 0} \qquad \boxed{\int y_m y_n r dx = 0}$$

Summary: * Characteristics of PDEs
* Various BC's
* Definition of well & ill-posed problem.
  ↳ convert into well posed problem.
* Special ODEs ($\Rightarrow$ corresponding eigenvalue problem).
* Finding out adjoint operator & BCs of adjoint problem.
* Definition of S-L operator
* Definition / Theorems.

Solving PDE's using separation of variables

→ Cartesian co-ord / Parabolic PDE.

Well posed Problem

Ex: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

at $t=0$, $u = u_0$

at $x=0$,
$x=1$,

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

at $t=0$, $T = T_0$

$x = a$, $T = T_1$

$x = b$, $T = T_1$

$$\Rightarrow X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{Const} = -\lambda$$

+ve   -ve   zero

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda X = 0$$

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \quad \text{at } x=0, =1 \; X=0$$

$$\boxed{X_n = C_n \sin(n\pi x)}$$

$\alpha_n = n\pi$, $n = 1, 2, \ldots \infty$

$$X = \sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$$\Rightarrow \frac{1}{T_n}\frac{dT_n}{dt} = -n^2\pi^2$$

$$T_n = C_1 \exp(-n^2\pi^2 t)$$

$$U_n = T_n X_n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n = \sum C_n' \exp(-n^2\pi^2 t)\sin(n\pi x)}$$

at $t=0$, $u = u_0$

$$u_0 = \sum C_n' \sin(n\pi x)$$

Multiply both sides by $\sin(m\pi x)\,dx$

$$u_0 \sin(m\pi x)\,dx = \sum_{n=1}^{\infty} C_n \int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx$$

$$= C_1 \int \sin(m\pi x)\sin(\pi x)\,dx + C_2 \int \sin(m\pi x)\sin(2\pi x)\,dx$$

$$= C_n \int_0^1 \sin^2(n\pi x)\,dx$$

$$\Rightarrow u_0 \int_0^1 \sin(m\pi x)\,dx = C_n' \underbrace{\int_0^1 \sin^2(n\pi x)\,dx}_{1/2}$$

$\Downarrow$
$m \to n$

$$\Rightarrow u_0 \int_0^1 \sin(n\pi x)\,dx = C_n'/2 \quad \Rightarrow u_0\left(\frac{1-\cos n\pi}{n\pi}\right) = C_n'/2$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n = 2u_0 \sum \frac{(1-\cos n\pi)}{n\pi} \exp(-n^2\pi^2 t)\sin(n\pi x)}$$

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Date : 31/08/2021 Prof. Sirshendu De

2 Dimensional Parabolic well posed problem.

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$$

at $t=0$, $u=u_0$

$x=0$, $\frac{\partial u}{\partial x}=0$

$x=1$, $u=0$

$u = X(x)T(t)$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\alpha_n^2$$

$$\frac{d^2X}{dx^2} + \alpha_n^2 X = 0$$

at $x=0 \Rightarrow \frac{dX}{dx}=0$
$=1 \Rightarrow X=0$

$\alpha_n = (2n-1)\pi/2$

$X_n = C_1 \cos(\alpha_n x)$

$$\Rightarrow \frac{1}{T_n}\frac{dT_n}{dt} = -\alpha_n^2$$

$$T_n = C_2 \exp(-\alpha_n^2 T)$$

$$u_n = X_n T_n$$
$$= C_n \exp(-\alpha_n^2 t) \cdot \cos(\alpha_n x)$$

$$\boxed{u = \sum_{n=1}^{\infty} u_n = \sum C_n \exp(-\alpha_n^2 t)\cos(\alpha_n x)}$$

$u_0 = \sum C_n \cos(\alpha_n x)$

multiply both sides by $\cos(\alpha_m x)dx$ & integrate over x

$$\int_0^1 u_0 \cos(\underbrace{\alpha_m}_{m \to n} x)\,dx = \sum_{n=1}^{\infty} \int_0^1 C_n \cos(\alpha_n x)\cos(\alpha_n x)\,dx$$

$$= C_n \underbrace{\int_0^1 \cos^2(\alpha_n x)\,dx}_{1/2}$$

$$u_0 \int_0^1 \cos(\alpha_n x)\,dX = \frac{C_n}{2}.$$

$$\boxed{C_n = 2u_0 \frac{\sin\alpha_n}{\alpha_n}}$$

$$u(x,t) = 2u_0 \sum_{n=1} \frac{\sin\alpha_n}{\alpha_n} \exp(-\alpha_n^2 t) \cos(\alpha_n x)$$ $\alpha_n = (2n-1)\pi/2$, $n = 1, 2, 3 \ldots \infty$

* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, at $t=0$, $u = u_0$

$x = 0$, $u = 0$

$x = 1$, $\frac{\partial u}{\partial x} + \beta u = 0.$

$u = X(x)\,T(t)$; $\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$

at $x = 0$, $X = 0$

$x = 1$, $\frac{dX}{dx} + \beta X = 0$

$X_n = C_1 \sin(\alpha_n x) \Rightarrow \alpha_n + \beta \tan\alpha_n = 0$

↳ Eigen values are roots of this equ.

$T_n = C_2 \exp(-\alpha_n^2 t)$; $u_n = T_n X_n$

$= C_n \exp(-\alpha_n^2 t) \sin(\alpha_n x)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp(-\alpha_n^2 t) \sin(\alpha_n x)$$

$$u_0 = \sum_{n=1}^{\infty} C_n \sin(\alpha_n x)$$

$$u_0 \int_0^1 \sin(\alpha_n x)\,dx = C_n \int_0^1 \sin^2(\alpha_n x)\,dx$$

$$u_0 \left(\frac{1-\cos\alpha_n}{\alpha_n}\right) = \frac{C_n}{2} \int_0^1 2\sin^2(\alpha_n x)\,dx$$

$$= \frac{C_n}{2} \int_0^1 \left[1 - \cos(2\alpha_n x)\right] dx$$

$$= \frac{C_n}{2}\left[1 - \int \cos(2\alpha_n x)\,dx\right]$$

$$= \frac{C_n}{2}\left[1 - \left\{\frac{\sin(2\alpha_n x)}{2\alpha_n}\right\}_0^1\right] \Rightarrow \frac{C_n}{2}\left[1 - \left\{\frac{\sin(2\alpha_n)}{2\alpha_n}\right\}\right]$$

$$u_0\left(\frac{1-\cos\alpha_n}{\alpha_n}\right) = \frac{C_n}{2}\left[1 - \frac{2\tan\alpha_n}{1+\tan^2\alpha_n}\left(\frac{1}{2\alpha_n}\right)\right]$$

$$= \frac{C_n}{2}\left[1 - \frac{\tan\alpha_n}{\alpha_n(1+\tan^2\alpha_n)}\right] = \frac{C_n}{2}\left[1 - \frac{1}{\alpha_n}\frac{(-\alpha_n/\beta)}{\left(1+\frac{\alpha_n^2}{\beta^2}\right)}\right]$$

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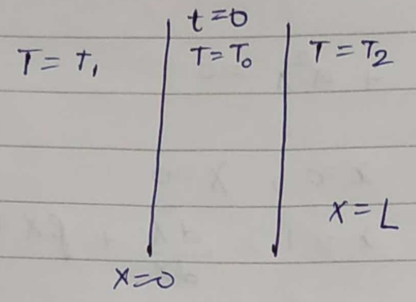
$$\frac{C_n}{2}\left[1+\frac{1}{\beta}\cdot\frac{\beta}{\beta^2+\alpha_n^2}\right] = \frac{C_n}{2}\left[\frac{\beta+\beta^2+\alpha_n^2}{\beta^2+\alpha_n^2}\right]$$

$$\boxed{a_n = 2u_0\left(\frac{1-\cos\alpha_n}{\alpha_n}\right)\frac{\beta^2+\alpha_n^2}{\beta^2+\beta+\alpha_n^2}}$$

$$\Rightarrow u(x,t) = 2u_0\sum_{n=1}^{\infty}\left(\frac{1-\cos\alpha_n}{\alpha_n}\right)\left(\frac{\beta^2+\alpha_n^2}{\beta^2+\beta+\alpha_n^2}\right)\sin(\alpha_n x)\,(\exp(-\alpha_n^2 t))$$

$\Rightarrow$ <u>2D heat conduction problem.</u>



Biot no. $(Bi) = \frac{hL}{K} < 0.1$ (lumped system analysis)

if K is high, Gov eqⁿ would be ODE.

K = Therml Conductivity.

$$\Rightarrow \rho C_p \frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}$$

at $t=0$, $T=T_0$

$x=0$, $T=T_1$

$x=L$, $T=T_2$

$\theta = \frac{T-T_1}{T_1}$; $x^* = x/L$

idea is BC should be homogen.

$$\Rightarrow \rho C_p\frac{\partial\theta}{\partial t} = \frac{K}{L^2}\frac{\partial^2\theta}{\partial x^{*2}}$$

$$\Rightarrow L^2\left(\frac{\rho C_p}{K}\right)\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial(x^*)^2} \Rightarrow \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^{*2}}$$

α → Therml diffusivity.

$$\boxed{\frac{t\alpha}{L^2} = \tau}$$

Dimensionless no.

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial x^{*2}}$$

at $\tau=0$, $\theta = \frac{T_0-T_1}{T_1} = \theta_0$

at $x^*=0$, $\theta=0$

$x^*=1$, $\theta = \frac{T_2-T_1}{T_1} = \theta_1$

ILL POSED PROBLEM.

$\theta = \theta_1 + \theta_2$

$\theta_1 : \frac{\partial\theta_1}{\partial\tau} = \frac{\partial^2\theta_1}{\partial x^{*2}}$

at $\tau=0$, $\theta_1=\theta_0$

$x^*=0$, $\theta_1=0$

$x^*=1$, $\theta_1=0$

$\theta_2 : \frac{\partial\theta_2}{\partial\tau} = \frac{\partial^2\theta_2}{\partial x^{*2}}$

$\theta_1 = 2\theta_0 \sum^{\infty} \left(\frac{1-\cos n\pi}{n\pi}\right) (e^{\ldots})(\sin n\pi x)$

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$\theta_2 = \theta_2^s(x^*) + \theta_2^t(x^*, \tau)$; $\theta_2^s : \frac{d^2\theta_2^s}{dx^{*2}} = 0$ | $\theta_2^t : \frac{\partial\theta_2^t}{\partial\tau} = \frac{\partial^2\theta_2^t}{\partial x^{*2}}$

at $x^* = 0,\ \theta_2^s = 0$

$x^* = 1,\ \theta_2^s = \theta_{10}$

at $t=0,\ \theta_2^t = -\theta_{10}x^*$ ~~$\theta_2(x^*)$~~

at $\left.\begin{matrix} x^* = 0 \\ = 1 \end{matrix}\right\}\ \theta_2^t = 0$

$\theta_2^s = c_1 x^*$

$\theta_{10} = c_1$

$\boxed{\theta_2^s = \theta_{10} x^*}$

$\theta_2^t(x^*, \tau)$

$= \sum C_n \exp(-n^2\pi^2\tau)\sin(n\pi x^*)$

$C_n = -2\theta_{10}\int_0^1 x^* \sin(n\pi x^*)\,dx$

$C_n = -2\theta_{10}\Big[$

$\theta_2(x^*, \tau) = \theta_{10}x^* + \sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2\tau)\sin(n\pi x^*)$

$\boxed{\underset{(x^*, \tau)}{\theta} = \theta_1 + \theta_2}$

01/09/202

Tomorrow: 9:00 clock.
class test - I
Syllabus Till today.
1 hr