04/01/19

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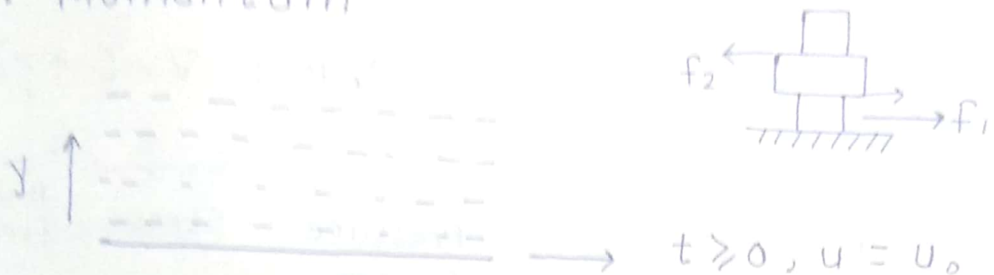
Unsteady / Transient transport of M^2 / Heat / Mass –

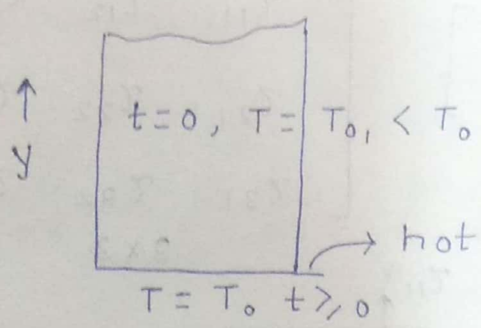
- Momentum



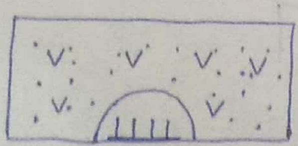
Semi infinite plate

Here, momentum is taking some time to get transported from bottom layer to other layers above – hence "unsteady" state.

- Heat



- Mass



$| \rightarrow$ A (sensor)

$\cdot \rightarrow$ B (Blood)

A is getting consumed / reacting with B. First, V decreases at its surface, then a conc^n gradient occurs in vertical direction, then it develops laterally. Therefore, unsteady state.

- Thermodynamics tells about equilibrium, steady state.
- Heat Transfer deals with kinetics.

Cauchy Momentum Balance:

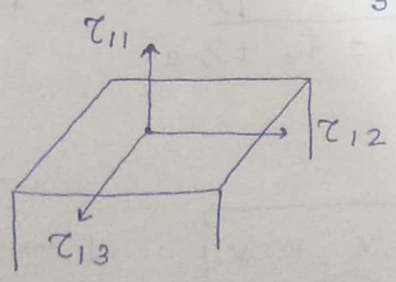
$$\underbrace{\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla v\right)}_{\text{acceleration}} = \underbrace{-\nabla P}_{\text{Pressure forces}} + \underbrace{\nabla\cdot\bar{\bar{\tau}}}_{\text{Stress}} + \underbrace{\rho\vec{g}}_{\text{Body force}}$$

A

* N.S is applicable for Newtonian, incompressible fluids.

Tensor - has no physical meaning
is just a matrix of 9 component

$$\underset{1\times 3}{\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}} = 1\times 3$$



Operators, tensors are introduced to simplify mathematics.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

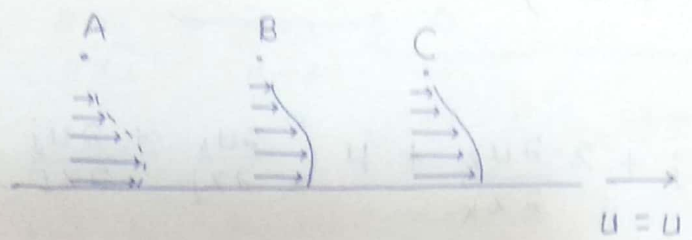
In Ⓐ, we have one having 3 component
" " 9 "

$\nabla v \rightarrow$ tensor (3 X 3 combinations) = 9

. Ⓐ combines all components in 1 eq^n

. Momentum -

A, B and C are hydrodynamically equal. The conditions / environment surrounding them are same. So, why will the velocities be different ?



No pressure gradient. Flow is due to movement of bottom plate

$$\frac{\partial u}{\partial t} + u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v}}_{0} \frac{\partial u}{\partial y} + \underbrace{\cancel{w}}_{0} \frac{\partial u}{\partial z} = -\frac{1}{\rho} \underbrace{\cancel{\frac{\partial p}{\partial x}}}_{0} + \nu \Delta^2 u$$

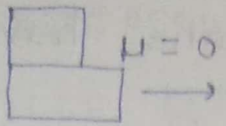
$$\rightarrow \frac{\partial u}{\partial x} = 0 , \quad v = 0 , \quad w = 0$$

$$\therefore \frac{\partial u}{\partial t} = \underbrace{\frac{\mu}{\rho}}_{\nu} \frac{\partial^2 u}{\partial y^2}$$

ν : momentum diffusivity

More momentum gets transferred with a greater (μ).

No slip doesn't tell about static / dynamic but deals with relative velocity.



Slipping occurs

There will be no momentum transport if $\mu = 0$.

I.C : $u(y, t = 0) = 0$

B.C : $u(y = 0, \forall t) = u_0$

$u(y \to \infty, \forall t) = 0$

Scaling Analysis :

To see the relative importance of terms.

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + \vec{U} . \nabla \vec{V} \right) = \underbrace{- \nabla P + \nabla . \overline{\overline{\tau}}}_{} + \rho \vec{g}$$

$$\tau_{ij} = - P \delta_{ij} + \lambda \frac{\partial u_K}{\partial x_K} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} ; \text{ Identity Matrix}$$

$$\frac{\partial u_K}{\partial x_K} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

P here is the thermodynamic pressure.

$$P_{mech} = \frac{(\tau_{11} + \tau_{22} + \tau_{33})}{3}$$

For NV eq^n to work,

$P_{ther} = P_{mech}$.

Stokesian Fluid :-

If the relaxation time, i.e., time required by P_{ther} to become equal to P_{mech}, is very small, fluid is said to be stokesian.

$$-P_{mech} = -P_{therm} + \left(\lambda + \frac{2}{3}\mu\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

For NS eq^n to work,

$$P_{mech} = P_{therm}$$

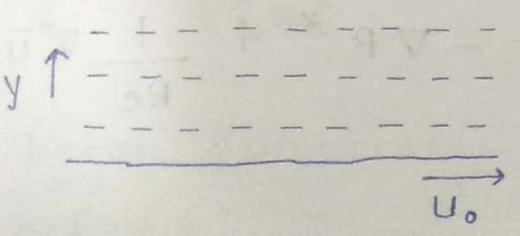
if $\left(\lambda + \frac{2}{3}\mu\right) = 0$, fluids follow Stokes hypothesis.

For incompressible fluid,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Hence, NS eq^n will be applicable.

No hydrodynamic B.L. Throughout viscous forces important.



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

I.C. : $t = 0 \quad u = 0 \quad \forall\, y$

B.C. : $y = 0 \quad u = u_\infty \quad \forall\, t$

$y \to \infty \quad u = 0 \quad \forall\, t$

Scaling Analysis :-

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\cdot\nabla\bar{u}\right) = -\nabla P + \mu\nabla^2\bar{V} + \rho\vec{g}$$

$u_0 = 10^{-3}$ m/s

$y = L = 10^{-4}$ m

$\rho = 10^3$ kg/m^3

$\mu = 10^{-3}$ Pa.s

Order of magnitude analysis :-

$$\sim \frac{U_0}{t_{ref}} \sim \frac{U_0^2}{L} \sim \frac{\rho U_0^2}{L} + \sim \frac{\mu U_0}{L^2}$$

If $\frac{\partial \bar{u}}{\partial t}$ and $\bar{u}.\nabla\bar{u}$ has to survive, order of the two terms should be equal.

$$\frac{U_0}{t_{ref}} = \frac{U_0^2}{L} \Rightarrow t_{ref} = \frac{L}{U_0}$$

$$u^* = \frac{U}{U_0}$$

$$\rho\left(\frac{U_0^2}{L}\frac{\partial \bar{U}^*}{\partial t^*} + \frac{U_0^2}{L}\bar{U}^*.\nabla\bar{U}^*\right) = -\frac{\rho U_0^2}{L}\nabla P + \mu\frac{U_0}{L^2}\nabla^2\bar{U}^*$$

$$\Rightarrow \left(\frac{\partial \bar{u}^*}{\partial t^*} + \bar{u}^*.\nabla\bar{u}^*\right) = -\nabla P^* + \frac{1}{Re}\nabla^2\bar{u}^*$$

where, $Re = \frac{\rho U_0 L}{\mu}$

$$Re = \frac{\rho U_0^2}{\mu . \frac{U_0}{L}} = \frac{\text{Inertial forces}}{\text{Viscous forces}}$$

For higher velocity, inertial forces dominate viscous forces, second term on R.H.S is zero, therefore potential flow regime. For lower velocity, L.H.S = 0 therefore creeping flow regime.

For potential flow, bernoulli's eq^n applicable.

$$Re = \frac{10^3 \times 10^{-3} \times 10^{-4}}{10^{-3}} = 0.1$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\sim \frac{U_0}{t_{ref}} \sim \nu \frac{U_0}{\delta^2}$$; δ varies with time $\delta = \delta(t)$

$t_{ref} = t$ { t : running time }

$$\frac{\cancel{U_0}}{t} = \nu \frac{\cancel{U_0}}{\delta^2} \Rightarrow \delta \sim \sqrt{\nu t}$$

δ is momentum diffusion length scale.

Hydrodynamic B.L. : viscous forces important

$$\delta = \sqrt{2^2 \nu t}$$

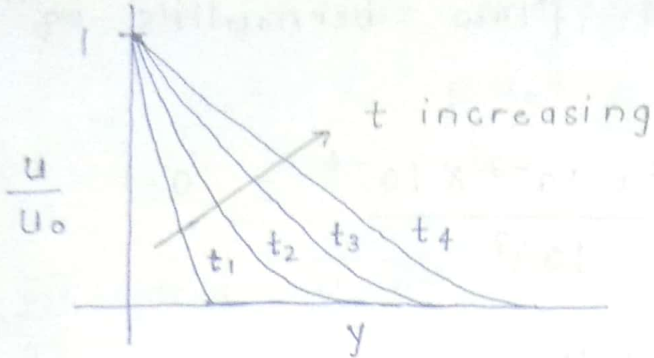
Error function :-

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

Gamma function :-

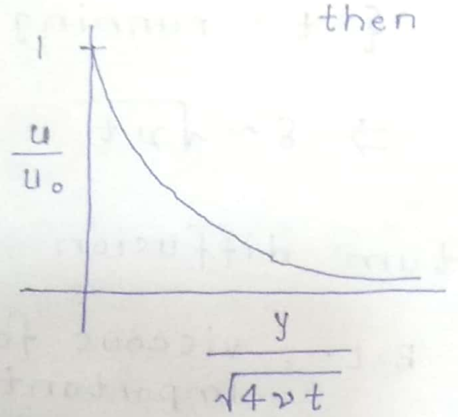
$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Scanned by CamScanner



Self - Similarity :

if the distance from the plate is non-dimensionalised, we get a single curve, then self - similar.



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\bar{u} = \frac{u}{u_0} \qquad \eta = \frac{y}{\delta}$$

$$\frac{\partial \bar{u}}{\partial t} = \nu \frac{\partial^2 \bar{u}}{\partial y^2}$$

B.C. :

$y = 0 \quad \bar{u} = 1$

$y = \infty \quad \bar{u} = 0$

I.C. : $t = 0 \quad \bar{u} = 0 \quad \forall\, y$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

Similarity transformation variable

$$\bar{u} = u/u_0 = f(\eta)$$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -\frac{1}{2}\left(\frac{\partial f}{\partial \eta}\right)\frac{y}{\sqrt{4\nu t}\cdot t}$$

$$= -\frac{\eta}{2t}\frac{\partial f}{\partial \eta}$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial f}{\partial \eta} \cdot \frac{1}{\sqrt{4\nu t}}$$

$$\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial\left(\frac{\partial \bar{u}}{\partial y}\right)}{\partial y} = \frac{\partial\left(\frac{\partial f}{\partial \eta}\frac{1}{\sqrt{4\nu t}}\right)}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{1}{\sqrt{4\nu t}}\left(\frac{\partial^2 f}{\partial \eta^2}\right)\frac{1}{\sqrt{4\nu t}} = \frac{1}{4\nu t}\frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{-\eta}{\cancel{2t}}\frac{\partial f}{\partial \eta} = \frac{\cancel{\nu}}{2\cancel{4}\cancel{\nu t}}\frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{\partial^2 f}{\partial \eta^2} + 2\eta\frac{\partial f}{\partial \eta} = 0$$

$\eta = 0 \longrightarrow f = 1$

$(y = 0,\ t \to \infty)$

$\eta = \infty \longrightarrow f = 0$

Let $\frac{\partial f}{\partial \eta} = \psi$.

$$\Rightarrow \frac{\partial \psi}{\partial \eta} + 2\eta\psi = 0$$

$$\frac{\partial \psi}{\psi} = -2\eta\, d\eta$$

$$\ln \psi = -\eta^2 + c$$

$$\Rightarrow \psi = c\, e^{-\eta^2}$$

$$\frac{\partial f}{\partial \eta} = c\, e^{-\eta^2}$$

$$f_{\eta=\eta} - f_{\eta=0} = \int_0^{\eta} c\, e^{-\eta^2}\, d\eta$$

$$f = 1 + c \int_0^{\eta} e^{-\eta^2}\, d\eta$$

$$\eta \rightarrow \infty \qquad f = 0$$

$$0 = 1 + c \int_0^{\infty} e^{-\eta^2}\, d\eta$$

$$\eta^2 = z \Rightarrow \eta = \sqrt{z} \Rightarrow d\eta = \frac{1}{2\sqrt{z}}\, dz$$

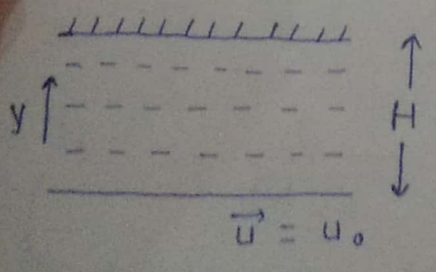
$$0 = 1 + \frac{c}{2} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1}\, dz$$

$$0 = 1 + \frac{c}{2} \Gamma(1/2) = 1 + \frac{c}{2}\sqrt{\pi}$$

$$\Rightarrow c = -\frac{2}{\sqrt{\pi}}$$

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2}\, d\eta = 1 - \text{erf}(\eta)$$

$$\frac{u}{u_o} = 1 - \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right).$$

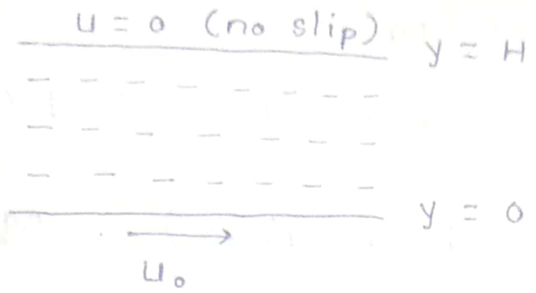


$$u = f(y, t)$$

$$\left.\begin{array}{l} y = 0, \quad u = u_o \\ y = H, \quad u = 0 \end{array}\right\} \text{B.C.}$$

$$t = 0 \quad u = 0 \;\; \forall y \;\} \text{ I.C.}$$

6.02.2019



Unsteady state confined flows

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad u = u(y)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} \frac{U_o}{t_c} = \nu \frac{U_o}{H^2} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

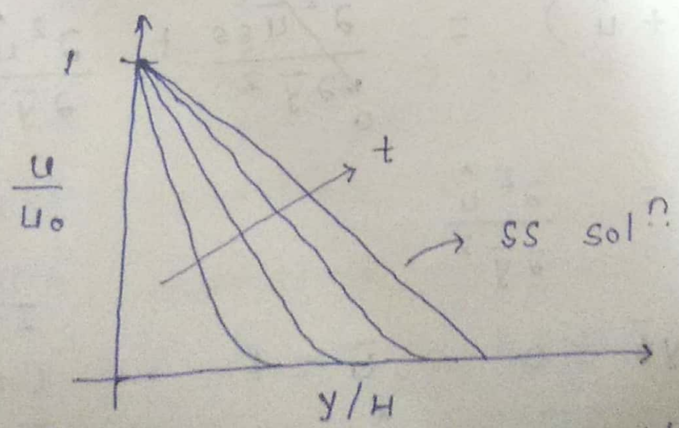
$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c \nu}{H^2} \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right) \Rightarrow$$ All terms have order of mag. equal to 1.

$$\frac{t_c \nu}{H^2} \sim O(1) \Rightarrow t_c = \frac{H^2}{\nu}$$

where t_c is M^2 diffusion time scale

$t << t_c$, it reduces to infinite domain problem and error function sol^n will work.

$t >> t_c$, steady state $u = ay + b$



This profile is not self similar like the one in previous problem.

$$t_c = \frac{(100 \times 10^{-6})^2}{10^{-3}/10^3} = 0.01\ s$$

(for water where $H = 100\ \mu m$)

$\frac{t_c \nu}{H^2} = 1$ (deliberately) , eq^n becomes

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

BC : $\bar{y} = 0$ $\bar{u} = 1$
$\bar{y} = 1$ $\bar{u} = 0$

I.C. : $\bar{t} = 0$ $\forall\ \bar{y}$
$\bar{u} = 0$

$\bar{u} = \bar{u}_{ss} + \hat{u}$ ⟶ unsteady part

↳ ss sol^n

SS solution –

$$0 = \frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2}$$

$\Rightarrow \bar{u}_{ss} = a\bar{y} + b$ $\bar{y} = 0$ $\bar{u}_{ss} = 1$
$\bar{y} = 1$ $\bar{u}_{ss} = 0$

$\Rightarrow \bar{u}_{ss} = 1 - \bar{y}$

Unsteady part –

$$\frac{\partial(\bar{u}_{ss} + \hat{u})}{\partial \bar{t}} = \cancel{\frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2}}_{0} + \frac{\partial^2 \hat{u}}{\partial \bar{y}^2}$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial \bar{t}} = \frac{\partial^2 \hat{u}}{\partial \bar{y}^2}$$

B.C. $\bar{y} = 0$ $\bar{u} = 1$

$\bar{u}_{ss} + \hat{u} = 1$

$\Rightarrow \hat{u} = 0$

I.C. :-

$\bar{t} = 0$ $\bar{u} = 0$

$\bar{u}_{ss} + \hat{u} = 0$

$\hat{u} = -\bar{u}_{ss}$

$\hat{u} = \bar{y} - 1$

$\bar{y} = 1 \quad \hat{u} = 0$

Homogeneous B.C. So separation of variables work.

Separation of variables -

$\hat{u} = f(\bar{y})\, g(\bar{t}) = fg$

$fg' = gf''$

$$\frac{g'}{g} = \frac{f''}{f} = a$$

$$\frac{g'}{g} = a \Rightarrow \frac{dg}{g} = a\, d\bar{t} \Rightarrow g = c e^{a\bar{t}}$$

a should be negative. Let $a = -\lambda^2$

$g = c e^{-\lambda^2 \bar{t}}$

$$\frac{f''}{f} = -\lambda^2 \Rightarrow f'' + \lambda^2 f = 0$$

$f = a \cos \lambda \bar{y} + b \sin \lambda \bar{y}$

$\Rightarrow \forall\, \bar{t} \quad \hat{u} = 0 \Rightarrow fg = 0 \Rightarrow f = 0$
$(\bar{y} = 0)$

when $\bar{y} = 1$, $\hat{u} = 0 \Rightarrow fg = 0 \Rightarrow f = 0$

From 1^{st} condition $a = 0$

From 2^{nd} condition

$f = b \sin \lambda \bar{y} \Rightarrow 0 = b \sin \lambda$

$b \neq 0$, $\therefore \sin \lambda = 0 \Rightarrow \lambda_n = n\pi$

$\Rightarrow \hat{u} = \sum b_n \sin(\lambda_n \bar{y}) c e^{-\lambda_n^2 \bar{t}}$

$\bar{u} = (1-\bar{y}) + \sum b_n (\sin(\lambda_n \bar{y})) c e^{-\lambda_n^2 \bar{t}}$

↳ general sol^n

$\bar{u} = (1-\bar{y}) + \sum A_n \sin(\lambda_n \bar{y})\, e^{-\lambda_n^2 \bar{t}}$

where $\{ A_n = c b_n \}$

At $t = 0$, $\bar{u} = 0 \quad \forall \bar{y}$

$\therefore \bar{y} - 1 = \sum A_n \sin \lambda_n \bar{y} \quad - (*)$

Let $q_n = \sin \lambda_n \bar{y}$

We have, $f'' + \lambda^2 f = 0$

$\Rightarrow \dfrac{d^2 f_n}{d\bar{y}^2} + \lambda_n^2 f_n = 0 \quad \{ f_n = A_n \sin \lambda_n \bar{y}$

For some m,

$$\int_0^1 f_m \frac{d^2 f_n}{d\bar{y}^2} d\bar{y} + \int_0^1 \lambda_n^2 f_n f_m \, d\bar{y} = 0$$

$$\cancel{f_m \frac{df_n}{d\bar{y}} \Big|_0^1}_{\,0} - \int_0^1 \frac{df_m}{d\bar{y}} \frac{df_n}{d\bar{y}} d\bar{y} + \lambda_n^2 \int_0^1 f_m f_n \, d\bar{y} = 0 \quad - (b)$$

Swapping m and n

$$-\int_0^1 \frac{df_n}{d\bar{y}} \frac{df_m}{d\bar{y}} d\bar{y} + \lambda_m^2 \int_0^1 f_m f_n \, d\bar{y} = 0 \quad - (c)$$

$$(\lambda_m^2 - \lambda_n^2)\int_0^1 f_m f_n \, d\bar{y} = 0$$

$$\Rightarrow m \neq n \quad \int_0^1 f_m f_n \, d\bar{y} = 0$$

From (*)

$$(\bar{y} - 1) = \Sigma A_n (\sin \lambda_n \bar{y}) = \Sigma A_n q_n$$

Multiplying by q_m

$$\Rightarrow \int_0^1 (\bar{y}-1) q_m \, d\bar{y} = \int_0^1 \Sigma A_n q_n q_m \, d\bar{y}$$

R.H.S. $= 0$ for $n \neq m$.

For $n = m$

$$\int_0^1 (\bar{y}-1) q_m \, d\bar{y} = \int_0^1 A_n q_n^2 \, d\bar{y}$$

$$\int_0^1 (\bar{y}-1) \sin \lambda_n \bar{y} \, d\bar{y} = \int_0^1 A_n \sin^2 \lambda_n \bar{y} \, d\bar{y}$$

$$\int_0^1 \bar{y} \sin \lambda_n \bar{y} \, d\bar{y} - \int_0^1 \sin \lambda_n \bar{y} \, d\bar{y} = \text{R.H.S}$$

$$\left[-\bar{y} \frac{\cos \lambda_n \bar{y}}{\lambda_n} - \left\{ -\frac{\sin \lambda_n \bar{y}}{\lambda_n^2} \right\} \right]_0^1$$

$$+ \left. \frac{\cos \lambda_n \bar{y}}{\lambda_n} \right|_0^1 = \text{R.H.S}$$

$$-\frac{1 \cdot (-1)^n}{\lambda_n} - 0 - \{0 - 0\} + \frac{(-1)^n}{\lambda_n}$$

$$-\frac{1}{\lambda_n} = \text{R.H.S}$$

$$\text{R.H.S.} = \frac{A_n}{2}$$

$$\Rightarrow -\frac{1}{\lambda_n} = \frac{A_n}{2}$$

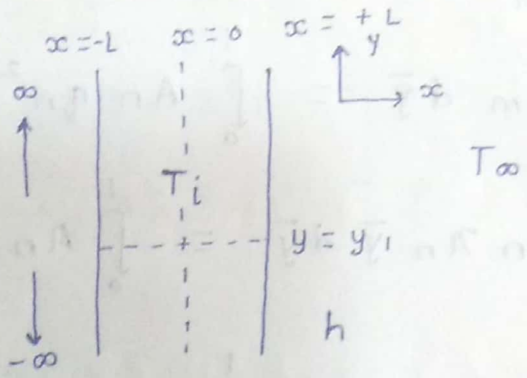
$$\Rightarrow A_n = -\frac{2}{\lambda_n} = -\frac{2}{n\pi}$$

$$\bar{u} = (1-\bar{y}) + \sum \left(A_n \sin \lambda_n \bar{y}\right) e^{-\lambda_n^2 \bar{t}}$$

$$\bar{u} = (1-\bar{y}) - \frac{2}{\pi} \sum \frac{\sin \lambda_n \bar{y}}{n} e^{-\lambda_n^2 \bar{t}}$$

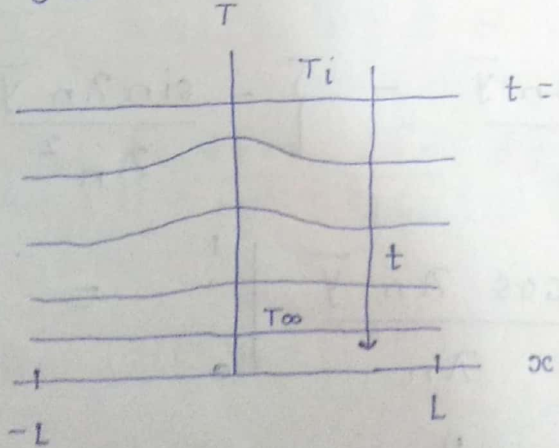
Unsteady Heat Transfer :-

1-D heat conduction



$T_i > T_\infty$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = \frac{k}{\rho C_p}$$



$y = y_1$

τ = thermal diffusion time scale

$$\tau = \frac{L^2}{\alpha}$$

$x = 0 \qquad \frac{\partial T}{\partial x} = 0$

B.C.

$x = \pm L \qquad -k \frac{\partial T}{\partial x}\Big|_{x=L} = h\left(T_{x=L} - T_\infty\right)$

I.C. $\quad t = 0, \; T = T_i \quad \forall x$

Non-dimensionalising the eq^n

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad \bar{x} = \frac{x}{L} \qquad \bar{t} = \frac{t}{t_c}$$

$$\frac{(\cancel{T_i - T_\infty})}{t_c}\frac{\partial \theta}{\partial \bar{t}} = \frac{\alpha(\cancel{T_i - T_\infty})}{L^2}\frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\frac{1}{t_c}\frac{\partial \theta}{\partial \bar{t}} = \frac{\alpha}{L^2}\frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\left(\frac{\partial \theta}{\partial \bar{t}}\right) = \left(\frac{\alpha t_c}{L^2}\right)\left(\frac{\partial^2 \theta}{\partial \bar{x}^2}\right)$$

$$\frac{\alpha t_c}{L^2} \sim 1 \quad \Rightarrow t_c = \frac{L^2}{\alpha} \qquad \boxed{F_o = \frac{t\alpha}{L^2}} = \frac{t}{t_c}$$

$$\frac{\partial \theta}{\partial \bar{t}} = \frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\frac{\partial \theta}{\partial F_o} = \frac{\partial^2 \theta}{\partial \bar{x}^2}$$

B.C. :-

$$\bar{x} = 0 \qquad \frac{\partial \theta}{\partial \bar{x}} = 0$$

$$\bar{x} = 1, \quad -k\frac{(\cancel{T_i - T_\infty})}{L}\frac{\partial \theta}{\partial \bar{x}}\Big|_{\bar{x}=1} = h(\cancel{T_i - T_\infty}) \times \theta|_{x=1}$$

$$\Rightarrow \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = -\frac{hL}{k} \theta|_{x=1}$$

I.C: -

$$\Rightarrow \theta'(1) = -Bi\, \theta(1)$$

$F_o = 0, \theta = 1$

$$T = f(\bar{x})\, g(F_o) = \theta$$

$$fg' = g f''$$

$$\frac{g'}{g} = \frac{f''}{f} = -\lambda^2$$

$$\frac{g'}{g} = -\lambda^2 \Rightarrow g = c_1 e^{-\lambda^2 F_o}$$

$$f'' + \lambda^2 f = 0 \Rightarrow f = a\cos(\lambda\bar{x}) + b\sin(\lambda\bar{x})$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = 0$$

$$f'(1) = -Bi(f(1))$$

$$\Rightarrow f' = -a\lambda \sin(\lambda\bar{x})$$

$$f'_{x=0} = b\lambda = 0$$

$$\Rightarrow b = 0$$

$$f'(1) = -a\lambda\sin\lambda = -Bi\, a\cos\lambda$$

$\Rightarrow \lambda\tan\lambda = Bi \rightarrow$ not a finite no. for every value of λ.

For some λ_n,

$$\lambda_n \tan\lambda_n = Bi$$

$$\theta = \sum A_n \cos(\lambda_n \bar{x})\, e^{-\lambda_n^2 \bar{t}}$$

Applying the initial condition,

$$1 = \sum A_n \cos\lambda_n\bar{x}$$

$$\int_0^1 (\cos\lambda_m\bar{x})\, d\bar{x} = \sum A_n \int_0^1 (\cos\lambda_n\bar{x})(\cos\lambda_m\bar{x})\, d\bar{x}$$

$$\left.\frac{\sin \lambda_m \bar{x}}{\lambda_m}\right|_0^1 = \frac{\sum A_n}{2} \int_0^1 \cos(\lambda_m + \lambda_n)\bar{x}\, d\bar{x} + \frac{\sum A_n}{2} \int_0^1 \cos(\lambda_m - \lambda_n)\bar{x}\, d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2} \left\{ \frac{\sin(\lambda_m + \lambda_n)\bar{x}}{\lambda_m + \lambda_n} + \frac{\sin(\lambda_m - \lambda_n)\bar{x}}{\lambda_m - \lambda_n} \right\}_0^1$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2} \left\{ \frac{\sin(\lambda_m + \lambda_n)}{(\lambda_m + \lambda_n)} + \frac{\sin(\lambda_m - \lambda_n)}{(\lambda_m - \lambda_n)} \right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2} \cdot \frac{1}{\lambda_m^2 - \lambda_n^2} \left\{ (\lambda_m - \lambda_n)\sin(\lambda_m + \lambda_n) + (\lambda_m + \lambda_n)\sin(\lambda_m - \lambda_n) \right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{\sum A_n}{2} \cdot \frac{1}{\lambda_m^2 - \lambda_n^2} \left\{ \lambda_m \{\sin(\lambda_m + \lambda_n) + \sin(\lambda_m - \lambda_n)\} - \lambda_n \{\sin(\lambda_m + \lambda_n) - \sin(\lambda_m - \lambda_n)\} \right\}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \sum A_n \frac{\left(\lambda_m \sin \lambda_m \cos \lambda_n - \lambda_n \cos \lambda_m \sin \lambda_n\right)}{\lambda_m^2 - \lambda_n^2} \quad (*)$$

$\lambda_n \tan \lambda_n = \lambda_m \tan \lambda_m = Bi$

$\Rightarrow \lambda_n \sin \lambda_n \cos \lambda_m = \lambda_m \sin \lambda_m \cos \lambda_m$

R.H.S. of (*) is zero for $m \neq n$

for $m = n$

$$\int_0^1 (\cos \lambda_m \bar{x})\, d\bar{x} = A_m \int_0^1 \cos^2 \lambda_m \bar{x}\, d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \int_0^1 (\cos 2\lambda_m \bar{x} + 1)\, d\bar{x}$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \left[\frac{\sin 2\lambda_m \bar{x}}{2\lambda_m} + \bar{x} \right]_0^1$$

$$\frac{\sin \lambda_m}{\lambda_m} = \frac{A_m}{2} \left[\frac{\sin 2\lambda_m}{2\lambda_m} + 1 \right]$$

$$\Rightarrow A_m = \frac{2 \sin \lambda_m}{\cancel{\lambda_m}} \left(\frac{2 \cancel{\lambda_m}}{2\lambda_m + \sin 2\lambda_m} \right)$$

$$\Rightarrow A_m = \frac{4 \sin \lambda_m}{2\lambda_m + \sin 2\lambda_m}$$

$$\theta = \sum A_m \cos(\lambda_m \bar{x})\, e^{-\lambda_m^2 F_o}$$

where $A_m = \dfrac{4 \sin \lambda_m}{2\lambda_m + \sin 2\lambda_m}$

Heisler Chart :

$\theta_0 = \theta_0 (F_0)$ $\{ \theta_0 = \text{Mid line temp.} \}$

For some $F_0 = a$,

$$\theta_0 = \sum A_n e^{-\lambda_n^2 a} \quad - (1)$$

$$\theta_{F_0 = a} (\bar{x}) = \sum A_n (\cos \lambda_n \bar{x}) e^{-\lambda_n^2 a} \quad - (2)$$

eq^n (2) / (1)

$$\frac{\theta_{(F_0 = a)} (\bar{x})}{\theta_0 (F_0 = a)} = \frac{\sum A_n (\cos \lambda_n \bar{x}) e^{-\lambda_n^2 a}}{\sum A_n e^{-\lambda_n^2 a}}$$