Revision for Midsem

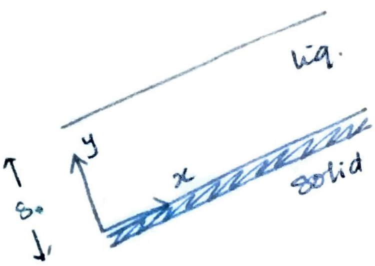
★ $\tau_{yx} = -\mu \frac{dv_x}{dy}$

$q = -k \frac{dT}{dx}$

$N_A = -D_{AB} \frac{dC}{dy}$

★ **Shell Momentum Balance**

No slip conditions, No shear conditions.



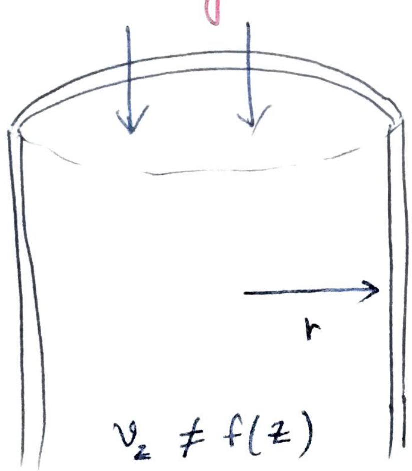
At $y = 0$, $v_x = 0$

At $y = \delta$, $\tau_{yx} = 0$

(Rate of momentum in − out) + $\Sigma F = 0$

Convective, conductive

Flow through a circular pipe



Convective momentum rate

$$\left[\cancel{(2\pi r \Delta r v_z \rho) v_z \Big|_{z=0}} - \cancel{(2\pi r \Delta r v_z \rho) v_z \Big|_{z=L}} \right] \quad v_z \neq f(z)$$

Conductive momentum rate

$$+ \left[2\pi r L \tau_{rz} \Big|_{\substack{z=0 \\ r=r}} - 2\pi(r+\Delta r) \tau_{rz} \Big|_{r+\Delta r} \right]$$

pressure forces

$$+ \left(2\pi r \Delta r\, p_0 \Big|_{z=0} - 2\pi r \Delta r\, p_L \Big|_{z=L} \right)$$

Gravitational force

$$+ 2\pi r \Delta r L \rho g = 0.$$

$$\frac{d}{dr}(r \tau_{rz}) = \left(\frac{P_0 - P_L}{L} \right) r$$

$$\frac{d}{dr}\left(r \cdot -\mu \frac{dv_z}{dr} \right) = \left(\frac{P_0 - P_L}{L} \right) r$$

} v_z and τ_{rz} can be found.

Equation of Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

At constant density (incompressible fluid);

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\left(\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}\right) = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$\boxed{\frac{D\rho}{Dt} = -\rho(\nabla v)} \quad \text{or} \quad \boxed{\frac{\partial \rho}{\partial t} = -\nabla(\rho v)}$$

Substancial Derivative

Equation of Motion

$$\boxed{\rho \frac{Dv}{Dt} = -\nabla p - [\nabla \tau] + \rho g}$$

At constant ρ,

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g \rightarrow \text{Navier Stokes' equation}$$

Neglecting viscous forces;

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho g \rightarrow \text{Euler eq.}$$

In x direction for cartesian system,

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \right. + \rho g_x.$$

In cylindrical coordinates

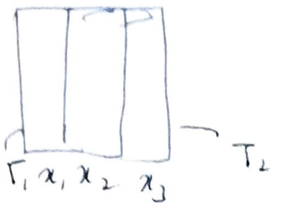
Eq. of continuity -

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

$$q' = k \frac{dT}{dx}$$

$$q = k_1 A \frac{dT_1}{dx_1} + k_2 A \frac{dT_2}{dx_2} + k_3 A \frac{dT_3}{dx_3} \rightarrow \text{series plates.}$$

$$\dot{q} = \frac{T_1 - T_2}{\frac{x_1}{k_1 A} + \frac{x_2}{k_2 A} + \frac{x_3}{k_3 A}}$$



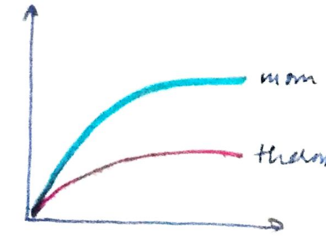
Thermal contact resistance

$$R_c = \frac{T_1 - T_{11}}{\dot{q}}$$

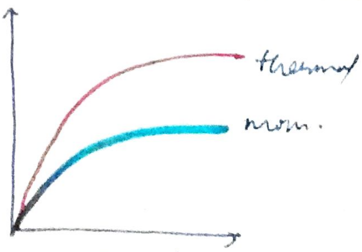
$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

Convection $\dot{q} = h_1 A (T_1 - T_2)$

Thermal boundary layer
Velocity boundary layer } Prandtl number $Pr = \frac{\nu}{\alpha}$ $\nu \rightarrow$ mom. diffusivity, $\alpha \rightarrow$ thermal diffusivity



$Pr > 1$



$Pr < 1$

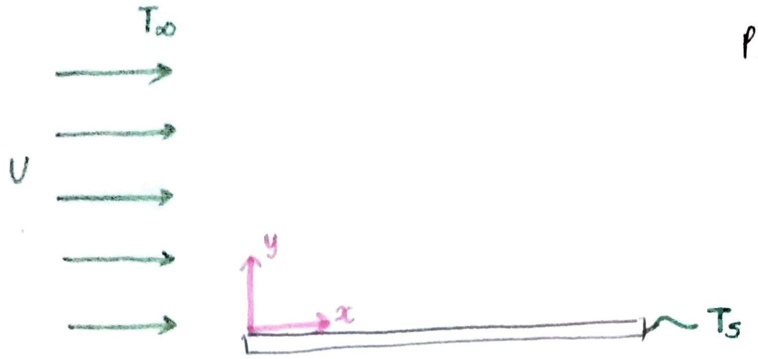
$Pr = \frac{\nu}{\alpha}$ $\nu \uparrow \longrightarrow$ thickness will be $\frac{\nu}{\alpha}$

$\nu >> \alpha$

t_m

13.02.19

Flow of liquid metal over a flat plate



$Pr << 1$

U: Free stream velocity

Outside BL → inviscid flow → no viscosity ($\mu = 0$) → Euler's equation

Within BL → viscous flow → NS equation.

Prandtl number → gives relation b/w thickness of thermal BL and momentum BL.

$Pr = \frac{C_p \mu}{k}$ → very small for liquid metals → Relatively thicker thermal bound[...] layer compared to hydrodyna[...] boundary lay[...]

Simplified energy equation

$$\rho C_p \left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$ → governing eq.

We know that;

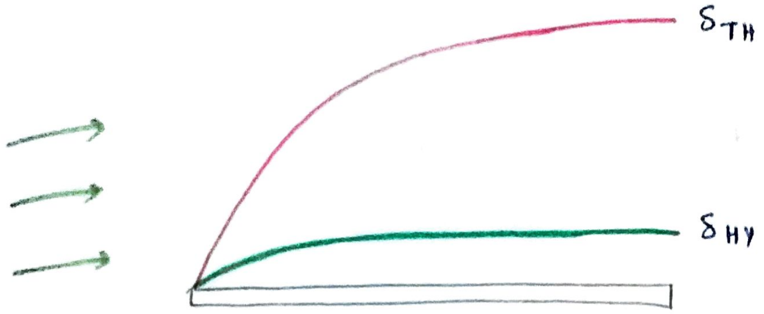
$$v_x >> v_y$$
$$\frac{\partial T}{\partial x} << \frac{\partial T}{\partial y}$$

} So, we can not drop either of $v_x \frac{\partial T}{\partial x}$ or $v_y \frac{\partial T}{\partial y}$

and $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2}$ → we can drop $\frac{\partial^2 T}{\partial x^2}$

$$\therefore \rho C_p \left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2}$$

(or) $v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$ where $\alpha = \frac{k}{\rho C_p}$



The hydrodynamic BL is very thin. So, for most of the region of interest; $v_x \approx U$ or the free stream velocity.

$$\underset{\text{large}}{U}\ \underset{\text{small}}{\frac{\partial T}{\partial x}} + \overset{\text{neglected}}{\underset{\text{small}}{v_y}\ \underset{\text{small}}{\frac{\partial T}{\partial y}}} = \alpha \frac{\partial^2 T}{\partial y^2}$$

↳ because the thickness of thermal BL is large.

$$\therefore \frac{\partial T}{\partial x} = \frac{\alpha}{U} \frac{\partial^2 T}{\partial y^2}$$

define $T^* = \dfrac{T - T_\infty}{T_s - T_\infty}$

governing eq. $\rightarrow \dfrac{\partial T^*}{\partial x} = \dfrac{\alpha}{U} \cdot \dfrac{\partial^2 T^*}{\partial y^2}$

Boundary conditions:

$y = 0$; $T^* = 1$

$y = \infty$; $T^* = 0$

Initial condition:

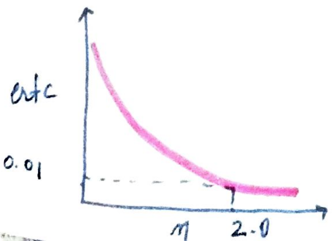
$x = 0$; $T^* = 0$.

Upon solving (refer to MK part for sol^n);

we get;

$$T^* = \frac{T - T_\infty}{T_s - T_\infty} = 1 - \text{erf}\left(\frac{y}{\sqrt{4\frac{\alpha}{U}x}}\right)$$

$$= \text{erfc}\left(\frac{y}{\sqrt{4\frac{\alpha}{U}x}}\right) = \text{erfc}(\eta)$$



For $\eta = 2.0$; $y = \delta$ → physical significance of this equation

Formula: $\frac{d}{dy}\left\{ \operatorname{erf} \frac{y}{\sqrt{4Ax}} \right\}\Big|_{y=0} = \frac{1}{(\pi A x)^{1/2}}$

and WKT

$$T^* = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\frac{\alpha}{U} x}} \right)$$

Find the Nusselt number.

~~WKT~~ $\cancel{-k \frac{\partial T}{\partial y}\Big|_{y=0} = h(T_s - T_\infty)}$

$$-k_s \frac{\partial T}{\partial y}\Big|_{y=0} = -k_{liq} \frac{\partial T}{\partial y}\Big|_{y=0} = h(T_s - T_\infty)$$

temp grad of solid (inside)

temp grad of liquid

$$-k_{liq} \cdot \frac{\partial T^*}{\partial y}\Big|_{y=0} \cancel{(T_s - T_\infty)} = h \cancel{(T_s - T_\infty)}$$

$$+k_{liq} \cdot \frac{\partial}{\partial y}\left(\operatorname{erf}\left(\frac{y}{\sqrt{4Ax}} \right)\right)\Big|_{y=0} = h$$

$$k_{liq} \cdot \frac{1}{\left(\pi \frac{\alpha}{U} . x\right)^{1/2}} = h$$

$$Nu = \frac{hL}{k} = L . \frac{1}{\sqrt{\pi} . \sqrt{\frac{k}{\rho C_p} \cdot \frac{x}{U}}}$$

$Re_x = \frac{\rho x U}{\mu}$

$Pr. = \frac{\mu}{\alpha} = \frac{\mu \rho C_p}{\rho k}$

$\therefore Re_x \cdot Pr = \frac{\rho x U}{\mu} \cdot \frac{\mu}{\rho}$

$$\Rightarrow Nu_x = \frac{1}{\sqrt{\pi}} \cdot (Re_x . Pr)^{1/2}$$

$$Nu_L = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{L} \int_0^L (Re_x Pr)^{1/2}$$

Nusselt no (Nu) = $\frac{\text{Convective heat transfer}}{\text{conductive heat transfer}} = \frac{hL}{k}$

Prandtl no. = $\frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}} = \frac{\nu}{\alpha}$

Reynold's no. = $\frac{\text{Inertial force}}{\text{viscous force}} = \frac{\rho v L}{\mu}$

Stanton no. = $\frac{\text{Heat transferred}}{\text{thermal capacity}} = \frac{h}{u \rho C_p} = \frac{Nu}{Re.Pr}$

Peclet no. = $(Pr. Re) = \frac{\text{advective transport rate}}{\text{diffusive transport rate}}$

Biot no. = $\frac{lh}{k} = \frac{\text{interior resistance}}{\text{exterior resistance}}$

VI

28.02.19

Post Mid Semester

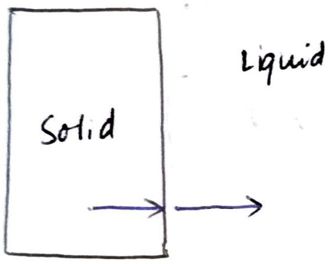
$$N_{Az} = -cD_{AB}\frac{\partial x_A}{\partial z} + x_A(N_{Az} + N_{Bz})$$

$-cD_{AB}\frac{\partial x_A}{\partial z}$ → diffusion (Ficks' law)

$x_A(N_{Az} + N_{Bz})$ → convection (bulk flow)

Boundary conditions are composed of concentrations and molar flux, equilibrium conditions.

- Impermeable surface → equivalent to adiabatic surface in heat transfer
  ↓
  molar flux across is zero

- 

$$-k\left.\frac{dT}{dz}\right|_{z=0} = h(T_{x=0} - T_\infty)$$

equivalent to;

$$-D_{AB}\left.\frac{dC_A}{dz}\right|_0 = h_m(C_{x=0} - C_\infty)$$

$$\frac{hD}{k} = Nu = f(Re, Pr)$$

Similarly, $\frac{h_m D}{D_{AB}} = Sh = f(Re, Sc)$

- Mass in − Mass out + Species generation (due to reaction) = mass accumulation

  Species generation → refers to homogeneous reactions only

  mass accumulation → 0 for constant density

Heterogeneous reactions (on a catalyst surface) do not translate into this equation.

For heterogeneous reactions at the catalyst surface -

Rate at which species reaches surface = Reaction rate

$$-D_{AB}\left.\frac{dC_B}{dz}\right|_{z=0} = \text{reaction rate}$$

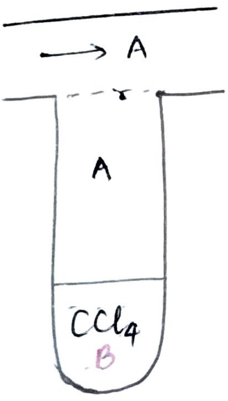
$\downarrow$

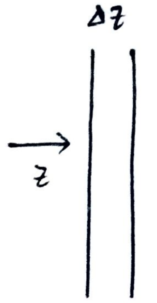
Only diffusion is taken as the mass transfer at the boundary /or on one stagnant surface is due to diffusion only.

$k_1'' \rightarrow$ refers to heterogeneous reaction

$k_1''' \rightarrow$ refers to homogeneous reaction

For a very fast reaction, concentration of the reactant at the catalyst surface is taken to be zero.

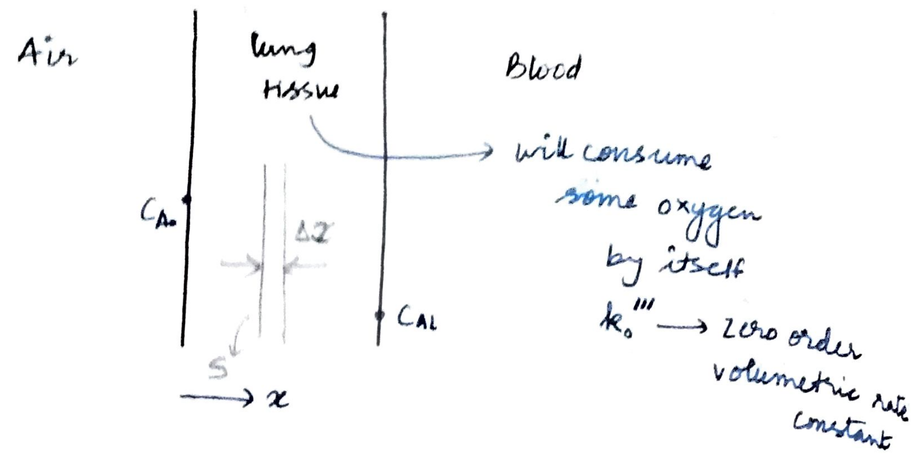




$$N_A\big|_z \cdot S - N_A\big|_{z+\Delta z} \cdot S \pm \text{Reaction} = 0$$

$$\frac{dN_A}{dz} = \text{some form of reaction}$$

Question:



C_{A0}, C_{AL} → concentrations of oxygen

Find -

1) The O_2 conc. profile in the tissue

$$N_A\big|_x \cdot S - N_A\big|_{x+\Delta x} \cdot S - k_o''' \cdot (S \cdot \Delta x) = 0$$

$$-\left[\lim_{\Delta x \to 0} \frac{N_A\big|_{x+\Delta x} - N_A\big|_x}{\Delta x}\right] = k_o''(1)$$

$$k_o'' = -\frac{dN_A}{dx}$$

or $$\boxed{\frac{dN_{Ax}}{dx} + k_o'' = 0}$$

Boundary conditions -

At $x = 0$, $C_A = C_{A0}$

At $x = L$; $C_A = C_{AL}$

$$N_{Ax} = -D_{AB}\frac{dC_A}{dx}$$

$$\therefore -D_{AB} \cdot \frac{d^2C_A}{dx^2} + k_o'' = 0$$

$$\frac{dC_A}{dx} = \frac{k_o''}{D_{AB}} \cdot x + C_1 \Rightarrow C_A = \frac{k_o''}{D_{AB}} \cdot \frac{x^2}{2} + C_1 x + C_2$$

At $x = 0$, $C_A = C_{A0} = C_2$

At $x = L$; $C_{AL} = \frac{k_0''}{D_{AB}} \cdot \frac{L^2}{2} + C_1 L + C_{A0}$

$$C_1 = \frac{C_{AL} - C_{A0}}{L} - \frac{k_0'''}{D_{AB}} \cdot \frac{L}{2}$$

$$\therefore C_A - C_{A0} = \frac{k_0'''}{2D_{AB}}\left(x^2 - xL\right) + \frac{x}{L}\left(C_{AL} - C_{A0}\right)$$

$$\therefore \boxed{C_A - C_{A0} = \frac{k_0'''}{2D_{AB}} x(x - L) + \frac{x}{L}(C_{AL} - C_{A0})}$$

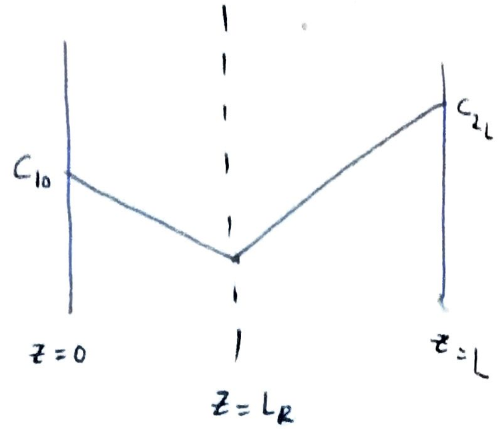
2) O_2 assimilation rate by the blood per unit area

$$\text{Rate of blood per unit area} = -D_{AB} \left.\frac{dC_A}{dx}\right|_{x=L}$$

$$= -\frac{k_0''' L}{2} + \frac{D_{AB}}{L}\left(C_{A0} - C_{AL}\right)$$

If reaction takes place;

We got that at $z = L_R$,
reaction takes place.



From $z=0$ to $z=L_R$,
conc. of only 1 exists with no reaction. 1 meets 2 at $z=L_R$ and disappears.

$$\left.\begin{aligned} \therefore \frac{d^2 c_1}{dz^2} &= 0 \text{ from } z=0 \text{ to } L_R \\ \frac{d^2 c_2}{dz^2} &= 0 \text{ from } z=L \text{ to } L_R \end{aligned}\right\} \text{ linear profiles.}$$

$\therefore c_1 = a_1 + b_1 z \rightarrow$ at $z=0$, $c_1 = c_{10}$
at $z = L_R$, $c_1 = 0$

$$\therefore \boxed{c_1 = c_{10} + \left(\frac{-c_{10}}{L_R}\right) z}$$

$c_2 = a_2 + b_2 z \rightarrow$ at $z = L_R$, $c_2 = 0$
at $z = L$, $c_2 = c_{2L}$

$$c_2 = \frac{-c_{2L} L_R}{L - L_R} + \frac{c_{2L}}{L - L_R} \cdot z$$

$$\boxed{c_2 = \frac{c_{2L}}{L - L_R}(z - L_R)} \text{ for } z > L_R$$

Since 1 mole of 1 reacts with 1 mole of 2,

$$+N_1 = -N_2$$

$$-D_1 \frac{dc_1}{dz} = D_2 \frac{dc_2}{dz}$$

$$-D_1 b_1 = D_2 b_2$$

$$\frac{D_1 C_{10}}{L_R} = \frac{C_{2L} D_2}{L - L_R}$$

$$\Rightarrow \boxed{L_R = \frac{1}{1 + \frac{C_{2L} D_2}{C_{10} D_1}}}$$

and $N_1 = \frac{C_{10} D_1}{L} \left[1 + \frac{C_{2L} D_2}{C_{10} D_1} \right]$

If component 2 was absent; $N_1 = \frac{C_{10} D_1}{L}$

That means, the presence of 2 increased the dissolution of 1.

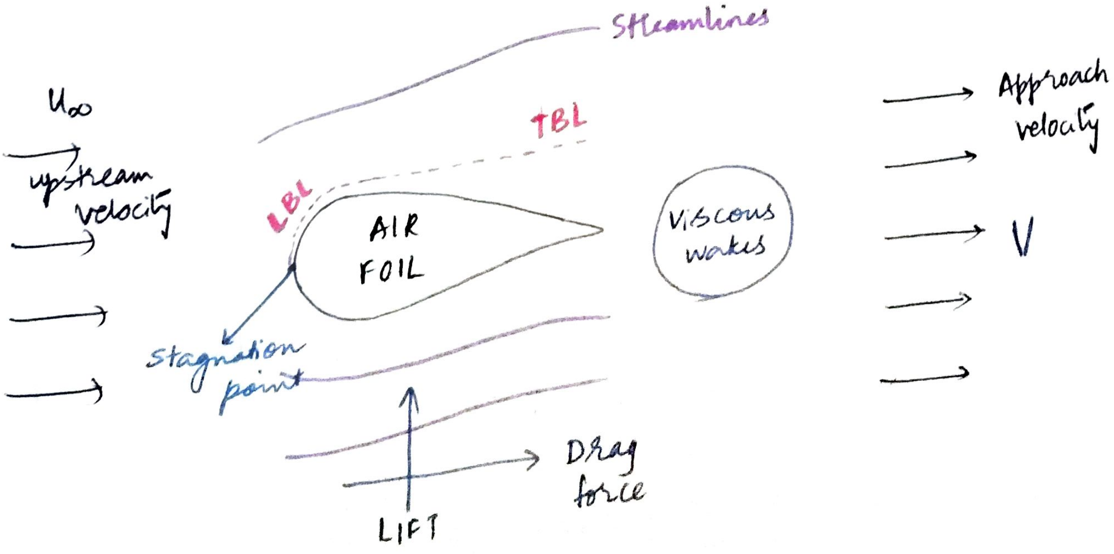
14.03.19

BOUNDARY LAYER CONCEPTS

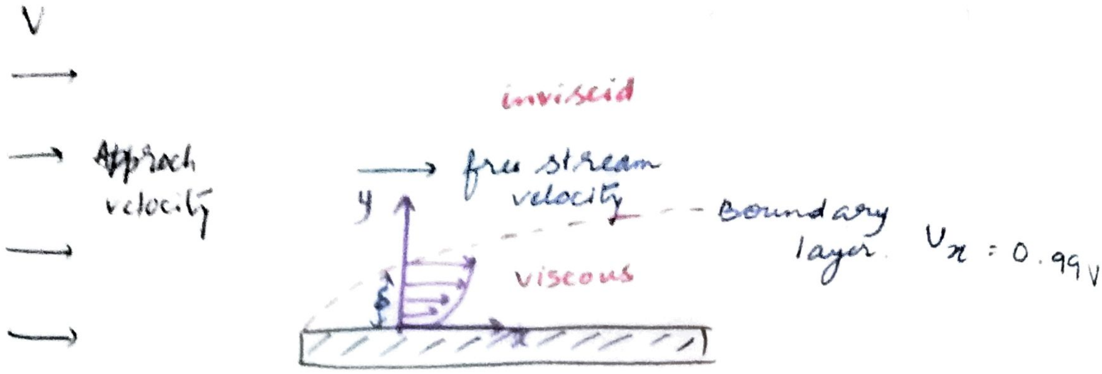
Euler equation – spl. case of NS equation when flow is frictionless.

Viscous wakes – low P region behind a moving object.

Dimples on golf balls – cause turbulence, travel farther than normal ball.



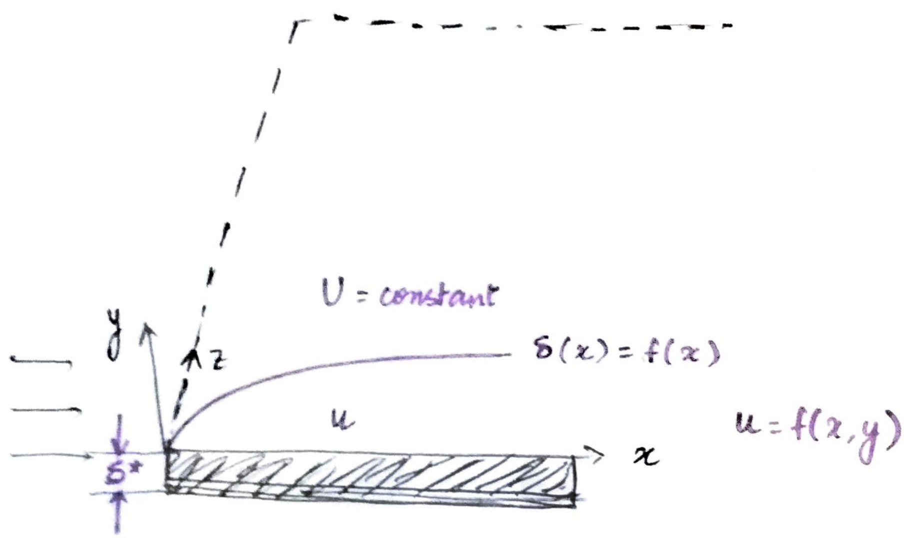
Approach velocity and free stream velocity.



$V_x(x, y)$

$\delta \rightarrow y$ at which $V_x = V(0.99)$

It is impossible to pinpoint the exact place where boundary layer ends.



Without the effect of viscosity for unit z,
the flow rate is $= \rho(U.\delta) \rightarrow$ (inside BL)

With the effect of viscosity for unit z,
the flow rate inside BL is $= \int_0^\delta \rho u \, dy$

Reduction in mass flow rate due to presence of viscosity in boundary layer $= \rho \left[\int_0^\delta (U - u) \, dy \right]$

Loss in mass flow rate due to the barrier alone (in inviscid flow) $= \rho U \delta^*$

when,

$$\rho U \delta^* = \rho \int_0^{\delta} (U-u)\,dy = \rho \int_0^{\infty} (U-u)\,dy$$ as from δ to ∞, $u=U$ anyway.

δ^* → displacement thickness

↓

The thickness of the barrier in inviscid flow that would cause the same loss in mass flow rate as within the BL in a viscous flow.

So, δ^* → integral thickness → gives overall picture as opposed to δ which is a differential thickness.

$$\boxed{\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy}$$

Reduction in mass flow rate of a viscous flow over a flat plate = reduction in mass flow rate of an inviscid flow over a flat plate of thickness δ^*

δ^* → displacement thickness based on mass.

Similarly there should be an entity for momentum as well.

⇒ Mass flow rate in BL $= \int_0^{\delta} \rho u\,dy$

If it was flowing in inviscid conditions, its momentum is given by $= \int_0^{\delta} \rho u\,dy \cdot U$

Its momentum in viscous flow $= \int_0^{\delta} \rho u \cdot u\,dy$

∴ Reduction in momentum $= \int_0^{\delta} \rho u (U-u)\,dy$.

⇒ The loss of momentum in inviscid flow due to a plate of thickness θ is $(\rho U \theta) \cdot U = \rho U^2 \theta$

$$\int u(U-u)\,dy$$

$$\boxed{\theta = \int_0^{\delta} \frac{u}{U}\left[1-\frac{u}{U}\right] dy} \longrightarrow$$ where θ is the momentum thickness.

Boundary layer approximations

* Laminar flow on a flat plate

$$v_x\left[\frac{\partial v_x}{\partial x}\right] + v_y\left[\frac{\partial v_x}{\partial y}\right] = \nu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right]$$

$$\left.\begin{aligned} & v_x \gg v_y \\ & \frac{\partial v_x}{\partial y} \gg \frac{\partial v_x}{\partial x} \end{aligned}\right\}$$ So $v_x \cdot \frac{\partial v_x}{\partial x}$ and $v_y \frac{\partial v_x}{\partial y}$ either can't be eliminated.

But $\frac{\partial^2 v_x}{\partial x^2} \ll \frac{\partial^2 v_x}{\partial y^2}$

$$\therefore\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$y = 0,\ v_x = 0$

$y \to \infty,\ v_x = U$

$x = 0,\ v_x = U$

Order of magnitude analysis

$\delta(x) \to$ small

$$\frac{\partial v_x}{\partial x} \to \sim \frac{U}{l}$$

$$\frac{\partial v_x}{\partial y} \to \sim \frac{U}{\delta}$$

Blasius sol^{n}

↳ attempted to convert pde to ode.

He said if, dimensionless velocity is expressed as a function of dimensionless distance, then all solutions/data sets converge into single form.

$$\boxed{\frac{v_x}{U} = g(\eta)} \quad \text{where } \eta = \frac{y}{\delta(x)} \quad \text{①}$$

↳ use this

$$\rightarrow v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \left[\frac{\partial^2 v_x}{\partial y^2} \right] \quad \text{②}$$

Eq. of continuity is automatically satisfied if we define stream function ψ.

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}$$

$$\therefore \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \Rightarrow \psi \text{ has to be an exact differential for order of diff to not matter.}$$

Near the edge of BL;

$$v_x \sim U, \quad y \sim \delta, \quad \frac{\partial v_x}{\partial y} \approx 0$$

$$\therefore U \cdot \frac{U}{x} \sim \nu \cdot \frac{U}{\delta^2} \quad \text{from ②}$$

$$\delta^2 \sim \frac{\nu x}{U} \Rightarrow \delta \sim \sqrt{\frac{\nu x}{U}}$$

$$\therefore \boxed{\eta = \frac{y}{\delta} = y\sqrt{\frac{U}{\nu x}}}$$

dimensionless stream f^n

$$f(\eta) = \frac{\psi}{\sqrt{\nu x U}}$$ from trial and error

$$\rightarrow v_x = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \sqrt{\nu x U} \cdot \frac{\partial f}{\partial \eta} \cdot \sqrt{\frac{U}{\nu x}}$$

$$\boxed{v_x = U \cdot \frac{\partial f}{\partial \eta}} \longrightarrow (a)$$

$$\rightarrow v_y = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[f \sqrt{\nu x U} \right]$$

$$= -\left[\sqrt{\nu x U} \frac{\partial f}{\partial x} + \frac{1}{2} \sqrt{\frac{\nu U}{x}} \cdot f \right]$$

$$\frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\boxed{v_y = \frac{\sqrt{\nu U}}{2\sqrt{x}} \left[\eta \frac{\partial f}{\partial \eta} - f \right]} \longrightarrow (b)$$

Put (a) and (b) in (2) to get an ODE in terms of f and

$$\therefore \boxed{2 \frac{d^3 f}{d\eta^3} + f \cdot \frac{d^2 f}{d\eta^2} = 0} \rightarrow \text{Blasius sol}^n \text{ ODE}$$

(A)

B.C.s: $\eta = 0 \quad f = \frac{df}{d\eta} = 0$

$\eta = \infty \quad \frac{df}{d\eta} = 1$

→ Numerical Solution given by Howarth.

Boundary layer obtained at $v_x = 0.99\, U = U \frac{df}{d\eta}$

$$\therefore \boxed{\text{BL at } f' = 0.99}$$

We get that $f' = 0.9951$ for $\eta = 5$.

$\therefore \quad \eta = y\sqrt{\dfrac{U}{\nu x}}$ (we know this from before)

At boundary layer, $y = \delta$

$$5 = \delta \sqrt{\frac{U}{\nu x}}$$

some form of Reynolds no.

$$\boxed{\therefore \; \delta = \frac{5x}{\sqrt{Re_x}}}$$

Shear stress → an engg. aspect.

- ↳ gradient of velocity
- ↳ function of f''

$$\therefore \; \tau_w = \mu \left.\frac{\partial V_x}{\partial y}\right|_{y=0}$$

$$= \mu \cdot \frac{\partial}{\partial y}\left[\frac{\partial \psi}{\partial y}\right]_{y=0}$$

$$= \mu \cdot \frac{\partial}{\partial y}\left[U \cdot \frac{\partial f}{\partial \eta}\right]_{\eta=0} = \mu \cdot U \cdot \frac{\partial^2 f}{\partial \eta^2} \cdot \left.\frac{\partial \eta}{\partial y}\right|_{y=0}$$

$$\therefore \; \tau_w = \mu U \sqrt{\frac{U}{\nu x}} \cdot \left.\frac{\partial^2 f}{\partial \eta^2}\right|_{\eta=0}$$

from (A), it is equal to 0.332

$$\therefore \; \boxed{\tau_w = \mu U (0.332)\sqrt{\frac{U}{\nu x}}} \Rightarrow \boxed{\tau_w = \frac{0.332\, \rho U^2}{\sqrt{Re_x}}}$$

Shear stress coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \boxed{\frac{0.664}{\sqrt{Re_x}} = C_f} \; \& \; \boxed{\delta = \frac{5x}{\sqrt{Re_x}}}$$

Also, drag coeff. $C_D = \dfrac{\int_A \tau_w \, dA}{\frac{1}{2}\rho U^2 A}$ can be calculated

$$C_D = \frac{\int_A C_f \cdot \frac{1}{2}\rho U^2 \, dA}{\frac{1}{2}\rho U^2 A}$$

$$\therefore \boxed{C_D = \frac{\int_A C_f \, dA}{A}}$$

drag coefficient

Area of flat plate.

For unit width of the plate, $A = L \times 1$

$$\therefore C_D = \frac{\int_0^L C_f \, dL}{L} = \frac{\int_0^L \frac{0.664}{\sqrt{Re_x}} \, dL}{L}$$

$$\boxed{\therefore C_D = \frac{1}{L}\int_0^L \frac{0.664}{\sqrt{Re_x}} \, dL = 2\,\overline{C_{fL}}}$$

Q1. Use the numerical results of Howarth to evaluate the following quantities (laminar BL flat plate)

i) $\frac{\delta^*}{\delta}$ at $\eta = 5$ and at $\eta \to \infty$

At $\eta = 5$ (Boundary layer), $u = 0.99\,U$

We know that $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\delta (1 - 0.99)\, dy$

$= 0.01\,(\delta)$

$$\therefore \boxed{\frac{\delta^*}{\delta} = 0.01} \text{ at } \eta = 5$$

At $\eta = 5$

$\eta = \dfrac{y}{\sqrt{\dfrac{\gamma x}{U}}} \Rightarrow dy = \sqrt{\dfrac{\gamma x}{U}}\, d\eta$

dimensionless distance

At $\eta = 5$; $y = \boxed{5\sqrt{\dfrac{\gamma x}{U}} = \delta}$ $\left(\text{as at } \eta = 5,\ v_x = U(0.99) \right)$

Boundary value.

(a)

$$\delta^* = \int_0^\delta \left(1 - \frac{v_x}{U}\right) dy$$

We know that at $y = \delta$, $\eta = 5$

and at $y = 0$, $\eta = 0$

$$\therefore \delta^* = \int_0^5 \left(1 - \frac{df}{d\eta}\right) \cdot \sqrt{\frac{\gamma x}{U}}\, d\eta$$

$$= \left(\frac{\delta}{5}\right) \cdot \int_0^5 \left(1 - \frac{df}{d\eta}\right) d\eta$$

from (a)

$$\delta^* = \frac{\delta}{5}\left[(5-0) - \int_0^{3.2833} df\right]$$

At $\eta = 0$; $f = 0$

At $\eta = 5$ (BL);

$f = 3.2833$

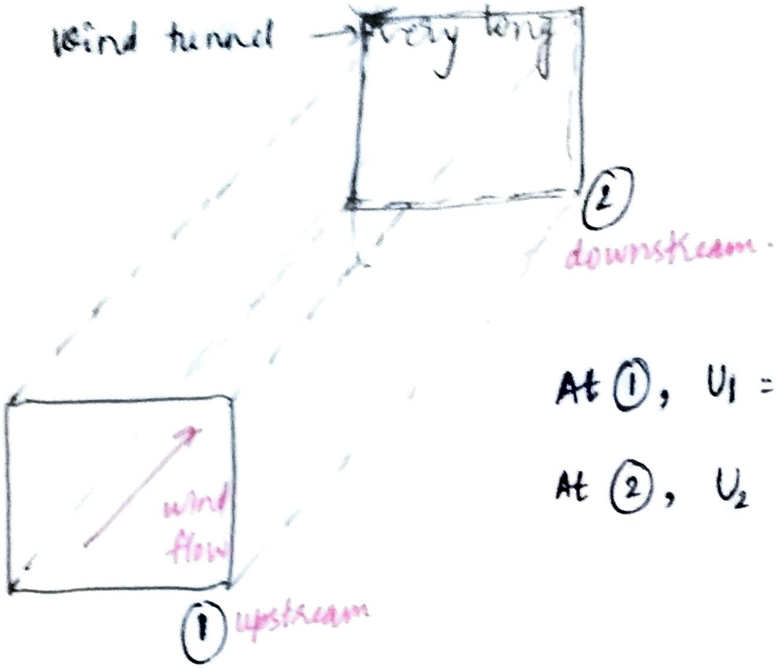
$$\therefore \frac{\delta^*}{\delta} = \frac{1}{5}\left[5 - 3.2833\right]$$

$$\boxed{\frac{\delta^*}{\delta} = 0.34334}$$

At $\eta \to \infty$

$$\delta^* = \int_0^\infty \left(1 - \frac{df}{d\eta}\right) \cdot \sqrt{\frac{\gamma x}{U}}\, d\eta$$

① Wind tunnel



At ①, $U_1 = 26$ m/s , $\delta_1^* = 1.5$

At ②, $U_2 = ?$, $\delta_2^* = 2.1$ mm

Calculate the change in pressure b/w ① and ② as a fraction of the free stream dynamic pressure at ① : $\frac{1}{2}\rho U_1^2$

This is a problem where free stream velocity varies with x. Along the direction ~~the~~ of flow, the four boundary layers along four walls grow and diverge thus reducing the area for inviscid flow which also causes a change in free stream velocity to maintain continuity.

Consider a flow on a flat plate (unbounded flow)

B/w 1 and 2,

B/w 3 and 3'

28.03.19

Integral method, unlike the differential method that we have been using, looks at the overall picture.

It is more practical.

Macroscopic balance :

$\boxed{\text{Fox and Mc. Donald}}$

$$\left.\frac{dN}{dt}\right|_{system} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall}_{\text{transient term}} + \underbrace{\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}}_{\text{some sort of convective term}}$$

$N \rightarrow$ any extensive macroscopic property

$\frac{N}{m} \rightarrow$ corresponding intensive property.

Mass coming in $\rightarrow$ negative, mass flows out $\rightarrow$ positive

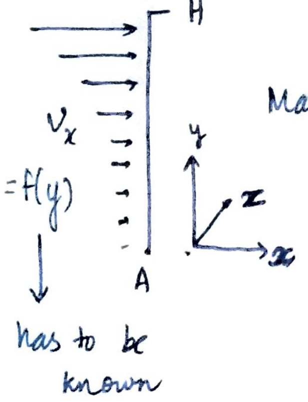
If N is mass, $\eta = 1$ $\left(\eta = \frac{N}{M} \right)$

$d\vec{A} \rightarrow$ area vector, directed outward.

$\int \vec{V} \cdot d\vec{A} \rightarrow$ continuity eq.

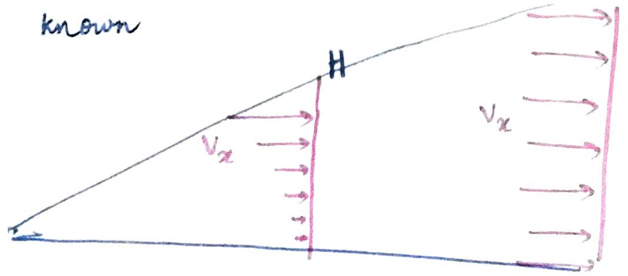
($\Sigma V_1 A_1$ when V_1 is constant over A_1)

Mass flow rate of fluid coming in considering constant density,



Mass flow rate $\Rightarrow$ $\rho \int \left\{ \int_0^H v_x \, dy \right\} dz$

Consider a flat plate.



$V_x = 0$ at $y = 0$

$V_x = U$ at $y = H$

$\frac{\partial V_x}{\partial y} = 0$ at $y = H$

If $N \rightarrow$ momentum, $\vec{P}$

$$\boxed{\vec{F} = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}\rho\,\vec{V}\cdot d\vec{A}}$$

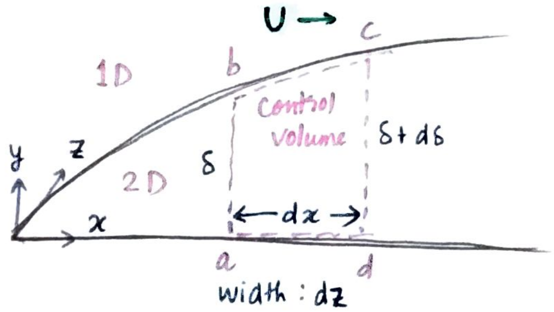
vector equation.
Can be written for F_x, F_y, F_z

$$\vec{F_x} = \vec{F_{Sx}} + \vec{F_{Bx}}$$

$\swarrow$ surface or shear force

$\searrow$ body force.

Velocity varies with x and y.
Body forces are zero within the boundary layer.



$V_x = 0$ at $y = 0$

$V_x = U$ at $y = \delta$

$\frac{\partial V_x}{\partial y} = 0$ at $y = \delta$

$$\vec{F_S} + \vec{F_B} = \cancel{\frac{\partial}{\partial t}}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}\rho\,\vec{V}\cdot d\vec{A}$$

0 (steady state)

Continuity equation:

$$\int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

Speciality of this equation, we have never used if the flow is laminar or turbulent. So this equation is universal.

$$\dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} + \cancel{\dot{m}_{ad}}^{\,0} = 0$$

$$\left(\int_0^\delta \rho V_x\, dy\right) dz = \dot{m}_{ab} \quad \text{and} \quad \dot{m}_{ab} = -\dot{m}_{bc} - \dot{m}_{cd}$$

$$\dot{m}_{x+dx} = \dot{m}_x + \left.\frac{\partial \dot{m}}{\partial x}\right|_x dx \qquad \text{(taylor series expansion)}$$

$$\therefore \dot{m}_{cd} = \dot{m}_{ab} + \left.\frac{\partial \dot{m}_{ab}}{\cancel{\partial y}\,\partial x}\right| \cancel{dy}\, dx$$

$M_{f\,bc}$ (momentum at bc)

$M_{f\,bc} = \dot{m}_{bc} \cdot U$

go through notes sent by SDG via email for detailed notes

Missed class - 29th March (Friday)

$$\theta = \int_0^{\delta} \frac{v_x}{U}\left(1 - \frac{v_x}{U}\right) dy \quad , \quad \delta^* = \int_0^{\delta} \left(1 - \frac{v_x}{U}\right) dy$$

MI eq: $$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\theta\right) + \delta^* U \frac{dU}{dx}$$

Turbulent flow

Emperical relations -

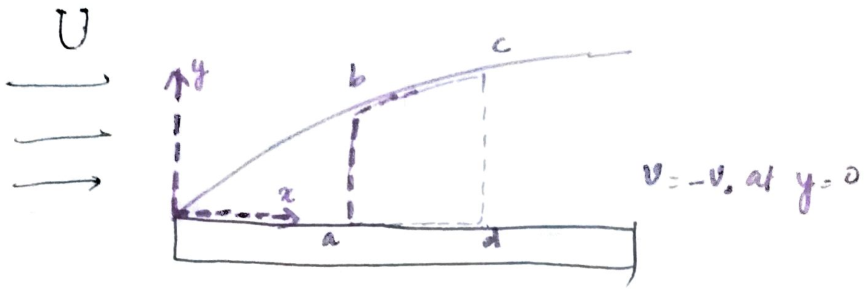
Power law equation :

$$\frac{\bar{v}_z}{U} = \left(\frac{y}{R}\right)^{1/7}$$

distance from wall

radius

Q. Consider horizontal



$$\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V_0} \, d\vec{A}$$

$$\vec{F}_{Sx} + \vec{F}_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho \, d\forall + \int_{CS} u\rho \vec{V_0} \, d\vec{A}$$

04.04.19

Q: The laminar to turbulent transition on a new cricket ball (of diameter 7.2 cm) occurs at a Re_D of about 1.4×10^5 if the flow doesn't encounter the seam. But it can be triggered by the seam e.g., at a R_o as low as 9.5×10^4 (when it is at $30°$ to the airflow). Use the above info to advise a seam bowler about the speed in which he has to bowl to achieve swing of the ball. A very brief (two lines) reason must accompany your suggestion. The kinetic viscosity of air is 1.5×10^{-5} m^2/s.

For $Re = 1.4 \times 10^5$;

$$v = Re\left(\frac{\mu}{\rho}\right) \times \frac{1}{d}$$

$v = 105$ km/hr $\longrightarrow$ onset of turbulence on smooth side

For $Re = 9.5 \times 10^4$;

$$v = Re\left(\frac{\mu}{\rho}\right) \times \frac{1}{d} = 71.25 \text{ km/hr}$$ $\rightarrow$ onset of turbulence on seam side

For a ball to swing, it must encounter laminar flow on one side and turbulent flow on other side. Hence the velocity of ball must be between these two values, where seam side is turbulent and the other side is laminar.

Q: Cricket commentators often talk of late swings referring to balls that swing unpredictably late in flight. Explain this phenomenon based on following facts - Consider a cricket ball of mass 0.156 kg and diameter 7.2 cm being bowled at a speed of V_o (greater than the upper critical speed of the previous problem). If the drag coefficient in this situation is constant at 0.15 estimate the velocity at which the ball must be bowled so that it starts to late swing at a distance of 15 m from the bowling end. From your calcs, do you feel that a swing bowler can plan his delivery for a late swing or whether this delivery is just a matter of chance? Given $\rho_{air} = 1.22$ kg/m^3.

Ball at speed greater than upper critical speed slows down due to drag and gets into the swing range

turbulent on both sides though seam is at 30°

$$C_D = \frac{F_D}{\frac{1}{2}\rho v^2 A_p}$$

Ball to be bowled such that under this drag force its velocity becomes 105 kmph after 15 m.

$$A_p = \pi\left(\frac{7.2\times10^{-2}}{2}\right)^2$$

$$F_D = \frac{1}{2}(0.15)(1.22)\left(\pi\left(\frac{7.2\times10^{-2}}{2}\right)^2\right)v^2$$

$$F_D = (3.7254\times10^{-4})\,v^2$$

$$-m.\frac{dv}{dt} = (3.7254\times10^{-4})\,v^2$$

$$\frac{d^2x}{dt^2} = (2.388\times10^{-3}).\left(\frac{dx}{dt}\right)^2$$

$$\int_{V_0}^{105}\frac{dv}{v^2} = \frac{2.388\times10^{-3}}{1}\int_0^t dt$$

$$-m.\frac{dv}{dx}.\frac{dx}{dt} = (3.7254\times10^{-4})\,v^2$$

$$\int_{V_0}^{105}\frac{dv}{v} = (2.388\times10^{-3})\int_0^{15}dx$$

$$\therefore \ln\left(\frac{105}{V_0}\right) = -0.03582$$

$$\boxed{V_0 = 108.83 \text{ kmph}}$$

A supertanker has a displacement (or mass) of 600,000 metric tons and $L = 300$ m, beam (width) = 80m and draft (depth) $D = 25$ m. The ship moves at 14 knots (7.20 m/s) through seawater ($\nu = 1.9 \times 10^{-6}$ m²/s) and ($\rho = 1.025 \times 10^{3}$ kg/m³). Find,

i) thickness of B.L. at the end (stern) of the ship. 1.7 m

ii) total skin friction drag ~~act~~ acting on the ship. 1.6 MN

iii) 11.4 MW

Boundary layer similarity -

The normalised convection transport equations

Using dimensionless terms;

$$u^* \frac{du^*}{dx^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{\nu}{VL} \frac{\partial^2 u^*}{\partial y^{*2}} \longrightarrow ①$$

$$\left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \rightarrow \text{Conservation eq.} \right\}$$

In T, concentration domain;

$$u^* \frac{dT^*}{dx^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\alpha}{VL} \frac{\partial^2 T^*}{\partial y^{*2}} \longrightarrow ②$$

$\frac{1}{Re.Pr}$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{V}, \quad v^* = \frac{v}{V}, \quad T^* = \frac{T_s - T}{T_s - T_\infty}$$

Boundary conditions:

For eq ① ; $u^*(x^*, 0) = 0$ $\qquad u^*(x^*, \infty) = \frac{u_\infty(x^*)}{V}$

$v^*(x^*, 0) = 0$

Functional form of sol^n's

$$u^* = f_1\left(x^*, y^*, Re_L, \cancel{\frac{dp^*}{dx^*}}\right)$$

• when geometry known

$f_4 \rightarrow Nu$

$f_7 \rightarrow Sh$

Shear stress at the surface ($y^* = 0$)

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

For temp domain; $T^* = f_3\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$

$$q_s = -k_f \left.\frac{dT}{dy}\right|_{y=0} \rightarrow \text{Surface heat flux}$$

If $Pr, Sc, D_{AB} = 1$; Mass, momentum and heat conservation eqs. become identical.

Then, $$\boxed{C_f \cdot \frac{Re_L}{2} = Nu = Sh \Leftrightarrow \frac{C_f}{2} = St = St_m}$$ → Reynold's analogy → for dynamic similarity

Stanton no. (St); Stanton no. on mass (St_m)

↳ due to functional similarity

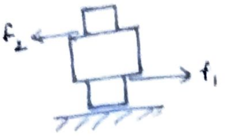
Equation	BCs	Similarity parameters
$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{VL} \frac{\partial^2 u^*}{\partial y^{*2}}$	$u^*(x^*, 0) = 0$; $v^*(x^*, 0) = 0$; $u^*(x^*, \infty) = \frac{u_\infty(x^*)}{V}$	Re_L
$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\alpha}{VL} \frac{\partial^2 T^*}{\partial y^{*2}}$	$T^*(x^*, 0) = 0$; $T^*(x^*, \infty) = 1$	Re_L, Pr
$u^* \frac{\partial C_A^*}{\partial x^*} + v^* \frac{\partial C_A^*}{\partial y^*} = \frac{D_{AB}}{VL} \frac{\partial^2 C_A^*}{\partial y^{*2}}$	$C_A^*(x^*, 0) = 0$; $C_A^*(x^*, \infty) = 1$	Re_L, Sc

04.01.19

Unsteady / Transient transport of momentum / heat / mass

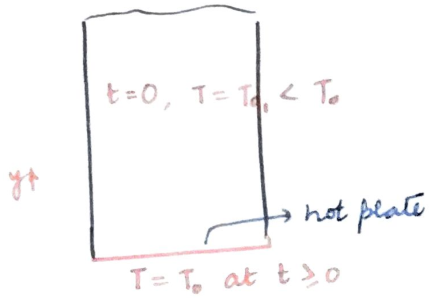
* Momentum

semi-infinite plate

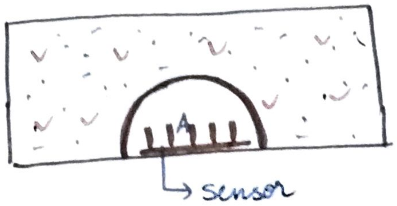


Here, momentum is taking some time to get transported from bottom layer to other layers above - hence unsteady state or 'temporal' state.

* Heat



* Mass



A is consuming / reacting with V. First, V decreases at its surface, then it develops a concentration gradient in vertical direction which is also developed laterally later.
- unsteady state.

→ Thermodynamics talks about equilibrium / steady state

→ Heat transfer deals with kinetics.

→ Navier Stokes eq. is applicable for Newtonian, incompressible fluids.

Cauchy momentum balance

$$\overline{\overline{\tau}} = \mu(\nabla v)$$

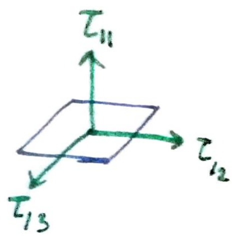
$$\text{(A)} \rightarrow \rho\left(\frac{\partial \bar{v}}{\partial t} + \bar{v}\cdot\nabla v\right) = -\nabla p + \nabla\cdot\overline{\overline{\tau}} + \rho\vec{g} \rightarrow \text{how?}$$

acceleration — Pressure forces — stress — body forces

Tensors - They have no physical meaning. It is just a mat of nine components introduced to simplify mathematic

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

3 X 3

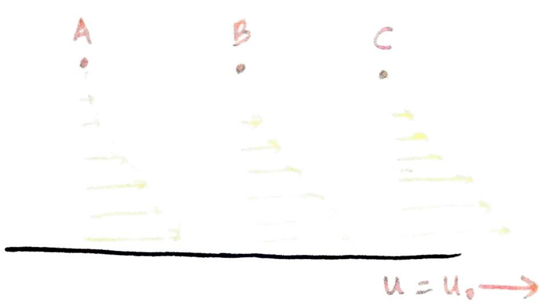


$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

In eq (A), ∇v is a tensor, has 9 components

MOMENTUM

Consider a semi-infinite plate moving at a constant velo $u = u_0$.



Points A, B, C are hydrodynam-ally equal. The conditions surrounding them are also same. Why are their velociti different though?

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial u}{\partial x} = 0, \quad v = w = 0$$

$$\therefore \frac{\partial u}{\partial t} = \nu (\nabla^2 u) \quad \text{where } \nu = \frac{\mu}{\rho}$$

$$\boxed{\frac{\partial u}{\partial t} = \nu . \frac{\partial^2 u}{\partial y^2}}$$

→ momentum diffusivity where more momentum is transfered with more viscosity.

No slip condition - doesn't tell about the static / dynamic state but deals with the relative velocity.

There will be no momentum transport if $\mu = 0$.

I.C : $u(y, t=0) = 0$

B.C : $u(y=0, \forall t) = u_0$

$u(y \to \infty, \forall t) = 0$

Scaling analysis - We see the relative importance of terms

$$\frac{\partial u_k}{\partial x_k} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$ where 1, 2, 3 are the three components.

If $i = j$, $\delta_{ij} = 1$

$i \neq j$, $\delta_{ij} = 0$

because pressure will always be normal.

$$\tau_{ij} = -P\delta_{ij} + \lambda \frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \longrightarrow B$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$ Identity matrix

→ P in equation (B) is $P_{thermodynamic}$ – manifests all degree of freedom

Has –ve sign as it is applied inwards but shear is defined outwards.

→ $$P_{mechanical} = \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3}$$

→ For Navier Stokes' eq. to be applicable,

$$P_{mechanical} = P_{thermodynamic}$$

From eq (B);

$$-P_{mech} = -P_{thermo} + \left(\lambda + \frac{2\mu}{3}\right)\overbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}^{A}$$

∴ For fluids following Stokes' hypothesis,

$\lambda + \frac{2\mu}{3} = 0$ where λ, μ are functions of DOFs.

(or) $A = 0$

Therefore for Navier Stokes' eq. to be applicable;

(i) Fluid is Stokesian $\left[\lambda + \frac{2\mu}{3} = 0\right]$

(ii) Fluid is incompressible $\left[\frac{\partial \mu_1}{\partial x_1} + \frac{\partial \mu_2}{\partial x_2} + \frac{\partial \mu_3}{\partial x_3} = 0\right]$

→ If $A = 0$; it doesn't matter which fluid it is, i.e., it is an incompressible fluid.

Scaling Analysis :

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}.\nabla\bar{u}\right) = -\nabla P + \mu\nabla^2\bar{u} + \rho\bar{g}$$

$$\frac{\rho u_o}{t_{ref}} \qquad \frac{\rho u_o^2}{L} \qquad \frac{\rho u_o^2}{L} \qquad \frac{\mu u_o}{L^2}$$

$\therefore\ t_{ref} = \frac{L}{u_o}$

scaling terms

(For these terms to survive, their order of magnitude should be equal)

$$\rho\left[\frac{u_o^2}{L}\cdot\frac{\partial u^*}{\partial t^*} + \frac{u_o^2}{L}\,u^*\nabla u^*\right] = -\frac{\rho u_o^2}{L}\nabla P^* + \mu\frac{u_o}{L^2}\nabla^2 u^*$$

$$\therefore\ \frac{\partial \bar{u}^*}{\partial t^*} + \bar{u}^*\nabla\bar{u}^* = -\nabla P^* + \frac{\nabla^2\bar{u}^*}{\left(\frac{\rho u_o L}{\mu}\right)} \rightarrow \text{Reynold's no.}$$

$= \dfrac{\text{inertial forces}}{\text{viscous forces}}$

$= \dfrac{\rho u_o^2}{\mu\frac{u_o}{L}} \rightarrow$ stress

Order of magnitude of $u^* = 1$

→ When Re is very high, inertial forces dominate viscous forces. Loss is so small due to viscosity that we can almost ignore it and apply Bernoulli's theorem.
Such a flow is called potential flow.

→ When viscous forces dominate, Re is very low and LHS = 0.
Such a flow is called creeping flow.
To increase dominance of creeping flow, decrease L.
Therefore, in microfluidics, we have creeping flow in microchannels.

$$\text{WKT} \quad \frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \sim \frac{u_o}{t_{ref}} \sim \nu\frac{u_o}{\delta^2}$$

Upon scaling this

with $t_{ref} = \frac{L}{u_o}$; $\cancel{\frac{\partial u^*}{\partial t^*}} = \cancel{\rho\frac{\partial u^*}{}}\ \frac{u_o}{t} = \nu\frac{u_o}{\delta^2}$

$\delta = \sqrt{\nu t}$ → dimensionally

But with appropriate constants,

$$\boxed{\delta = \sqrt{4 \nu t}}$$

Boundary layer

→ For boundary layer, viscous forces are important. Above it, inertial forces are important.

→ 16.01.19

Error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

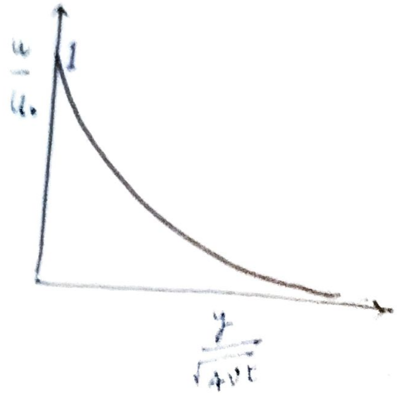
Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Upon thinking qualitatively, for the semi-infinite plate moving with u_0:

$t \to \infty$, $\frac{u}{u_0}$ will touch ∞

Upon scaling,



We had derived that,

$$\frac{\partial u}{\partial t} = \nu . \frac{\partial^2 u}{\partial y^2} \longrightarrow (1)$$

Considering scaling factors,

$$f(\eta) = \bar{u} = \frac{u}{u_o} \quad \text{and} \quad \eta = \frac{y}{\delta} \quad \text{where } \delta = \sqrt{4\nu t}$$

Boundary Conditions:

$y = 0, \ \bar{u} = 1$

$y = \infty, \ \bar{u} = 0$

Initial conditions:

$t = 0, \ \bar{u} = 0 \ \forall y.$

→ Combine space and time to convert PDE to ODE.

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \eta} . \frac{\partial \eta}{\partial t} = \frac{\partial f}{\partial \eta}\left[\frac{-1}{2\sqrt{t}.t} . \frac{y}{4\nu}\right]$$

$$\boxed{\frac{\partial \bar{u}}{\partial t} = -\frac{\eta}{2t} . \frac{\partial f}{\partial \eta}}$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial \eta} . \frac{\partial \eta}{\partial y} = \frac{\partial f}{\partial \eta}\left[\frac{1}{\sqrt{4\nu t}}\right]$$

$$\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial \eta} . \frac{1}{\delta}\right] = \frac{\partial^2 f}{\partial \eta^2}\left[\frac{1}{\delta}\right] . \frac{\partial \eta}{\partial y}$$

$$\boxed{\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial^2 f}{\partial \eta^2} . \frac{1}{4\nu t}}$$

Upon substituting in eq (1);

$$-\frac{\eta}{2t} . \frac{\partial f}{\partial \eta} = \nu . \frac{1}{4\nu t} . \frac{\partial^2 f}{\partial \eta^2}$$

or;

$$\boxed{\frac{\partial^2 f}{\partial \eta^2} + 2\eta \frac{\partial f}{\partial \eta} = 0} \longrightarrow (2)$$

To solve eq (2).

take $\frac{\partial f}{\partial \eta} = \psi$

At $\eta = 0$; $y = 0$, $t \to \infty$, $f \to 1$.

$$\frac{\partial \psi}{\partial \eta} + 2\eta\psi = 0$$

$$\ln \psi = -\eta^2 + \ln c$$

$$\psi = c e^{-\eta^2}$$

$$\frac{\partial f}{\partial \eta} = c e^{-\eta^2}$$

$$\int_{f|_{\eta=0}}^{f|_{\eta}} \partial f = c \int_0^{\eta} e^{-\eta^2} d\eta$$

$$\boxed{f = 1 + c \int_0^{\eta} e^{-\eta^2} d\eta}$$

WKT, as $\eta \to \infty$, $f \to 0$

$$0 = 1 + c \int_0^{\infty} e^{-\eta^2} d\eta$$

take $\eta^2 = z \Rightarrow \eta = \sqrt{z}$

$$d\eta = \frac{1}{2\eta} dz$$

$$\therefore 0 = 1 + \frac{c}{2} \int_0^{\infty} e^{-z} . z^{-1/2} dz$$

$$0 = 1 + \frac{c}{2} \int_0^{\infty} e^{-z} \left(z^{\frac{1}{2}-1} \right) dz$$

$$0 = 1 + \frac{c}{2} \Gamma\left(\frac{1}{2}\right) = 1 + \frac{c\sqrt{\pi}}{2}$$ (gamma function)

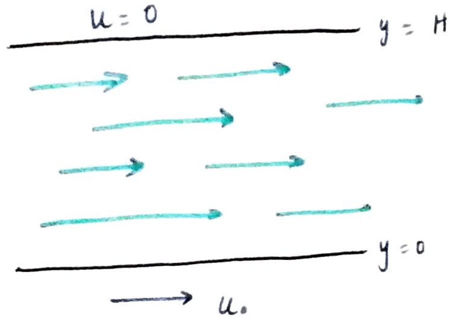
$$\therefore \boxed{c = -\frac{2}{\sqrt{\pi}}}$$

$$\therefore f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

$$\therefore f = 1 - \mathrm{erf}(\eta)$$

(or) $$\boxed{\frac{u}{u_o} = 1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)}$$

06.02.19



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad u = u(y)$$

$$\frac{u_o}{t_c} \cdot \frac{\partial \bar{u}}{\partial \bar{t}} = \nu \cdot \frac{u_o}{H^2} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

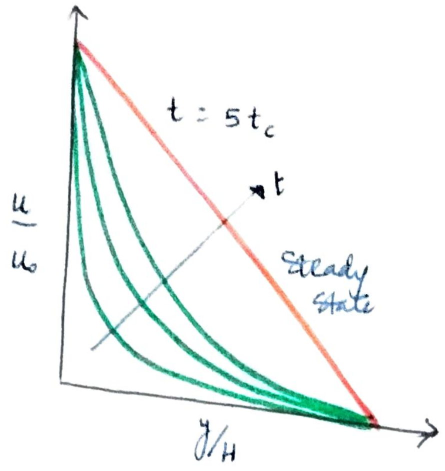
$$\left(\frac{\partial \bar{u}}{\partial \bar{t}}\right) = \left(\frac{t_c \nu}{H^2}\right)\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)$$

$$\frac{t_c \nu}{H^2} \sim 1 \Rightarrow \boxed{t_c = \frac{H^2}{\nu}}$$ $\rightarrow$ momentum diffusion time scale

$t << t_c$ or $\bar{t} << 1$ $\rightarrow$ errors function will work as the profile looks like that of an infinite plate as the Boundary layer hasn't been established yet $\rightarrow$ a sol^n as found earlier.

$t >> t_c$ or $\bar{t} >> 1$ $\rightarrow$ steady state solution $\rightarrow$ already known. $\hookrightarrow$ $u = ay + b$

How would it be for $\bar{t} \sim 1$?



$$\therefore \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$(\text{or}) \quad \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = \frac{\partial \bar{u}}{\partial t}$$

$$\bar{u} = \bar{u}_{ss} + \hat{u}$$

$\bar{u}_{ss}$ → Complementary sol^n → steady state

$\hat{u}$ → Particular solution → unsteady state

$$\frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2} = 0$$

$$\bar{u}_{ss} = a\bar{y} + b$$

$\bar{y} = 0 \rightarrow \bar{u}_{ss} = 1$

$\bar{y} = 1 \rightarrow \bar{u}_{ss} = 0$

$$\therefore \boxed{\bar{u}_{ss} = 1 - \bar{y}} \rightarrow \text{steady state sol}^n$$

For unsteady part;

$$\frac{\partial \hat{u}}{\partial \bar{t}} = \frac{\partial^2 \hat{u}}{\partial \bar{y}^2} \longrightarrow (1)$$

Boundary conditions: for $\bar{y} = 0$,

$\bar{u} = 1$

$\bar{u}_{ss} + \hat{u} = 1$

$\therefore \hat{u} = 0$

For $\bar{y} = 1$

$\bar{u} = 0$

$\bar{u}_{ss} + \hat{u} = 0$

$\therefore \hat{u} = 0$

For $\bar{y} = 0, 1$

$\hat{u} = 0$

↓

homogeneous

The equation is homogeneous → separation of variables is possible.

Initial conditions:

$$\bar{t} = 0 \rightarrow \bar{u} = 0$$

$$\bar{u}_{ss} + \hat{u} = 0$$

$$1 - \bar{y} + \hat{u} = 0$$

$$\boxed{\hat{u} = \bar{y} - 1}$$

Separation of variables:

$$\hat{u} = f(\bar{y}) \cdot g(\bar{t}) = fg$$

Putting this in eq (1);

$$\frac{\partial \hat{u}}{\partial \bar{t}} = \frac{\partial^2 \hat{u}}{\partial \bar{y}^2}$$

$$fg' = gf''$$

$$\frac{g'}{g} = \frac{f''}{f} = a$$

→ $\frac{g'}{g} = a \Rightarrow \frac{dg}{g} = a\,dt \Rightarrow \boxed{g = c_1 e^{at}}$

This has to reach steady state; so a has to be a negative real no.

Therefore, replacing a with $-\lambda^2$;

$$\boxed{g = c_1 e^{-\lambda^2 t}} \longrightarrow (i)$$

→ $\frac{f''}{f} = -\lambda^2$

$$f'' + \lambda^2 f = 0$$

$$\boxed{f = b \sin \lambda \bar{y} + a \cos \lambda \bar{y}} \longrightarrow (ii)$$

From BC; we got that $\forall t$, $\bar{y} = 0, 1$; $\hat{u} = 0$; i.e.; $fg = 0$ for $\forall t$

g is a function of t and can't be 0 at all t. ∴ $f = 0$ for $\bar{y} = 0, 1$

Substituting this in (ii) ;

At $\bar{y} = 0$; $0 = a + 0 \Rightarrow a = 0$

At $\bar{y} = 1$; $0 = b\sin\lambda$

$\lambda_n = n\pi$

$\therefore \hat{u} = \sum b_n \sin(\lambda_n \bar{y}) . c e^{-\lambda_n^2 \bar{t}}$

$$\boxed{\bar{u} = \underbrace{(1 - \bar{y})}_{\bar{u}_{ss}} + \underbrace{\sum A_n \sin(\lambda_n \bar{y}) . e^{-\lambda_n^2 \bar{t}}}_{\hat{u}}} \longrightarrow (a)$$

07.02.19

From eq (a); we get that at $\bar{t} = 0$;

$$\hat{u}\Big|_{\bar{t}=0} = \sum A_n \sin(\lambda_n \bar{y})$$

And from initial conditions, wkt,

$$\hat{u}\Big|_{\bar{t}=0} = \bar{y} - 1$$

$$\therefore \boxed{\bar{y} - 1 = \sum A_n \sin(\lambda_n \bar{y})} \longrightarrow (b)$$

$\longrightarrow$ Consider $f_n = A_n \sin \lambda_n \bar{y}$ and $f_m = A_m \sin \lambda_m \bar{y}$

$$f'' + \lambda^2 f = 0$$

$$f_m \left(\frac{d^2 f_n}{d\bar{y}^2} + \lambda_n^2 f_n \right) = 0 \times f_m$$

$$\int_0^1 f_m \frac{d^2 f_n}{d\bar{y}^2} d\bar{y} + \int_0^1 \lambda_n^2 f_n f_m \, d\bar{y} = 0$$

$$\Rightarrow \underbrace{\cancel{f_m \frac{d f_n}{d\bar{y}}}\Big|_0^1}_{0 \text{ as } f_m = 0 \text{ at } 0 \text{ and } 1} - \int_0^1 \frac{d f_m}{d\bar{y}} . \frac{d f_n}{d\bar{y}} . d\bar{y} + \int_0^1 \lambda_n^2 f_n f_m \, d\bar{y} = 0$$

Similarly, integrating $f_n f_m'' + \lambda^2 f_n f_m = 0$; we get;

$$-\int_0^1 \frac{df_m}{d\bar{y}} \cdot \frac{df_n}{d\bar{y}} \cdot d\bar{y} + \int_0^1 \lambda_m^2 f_n f_m \, d\bar{y} = 0$$

upon subtracting these two eqs., we get,

$$\int_0^1 (\lambda_n^2 - \lambda_m^2) f_n f_m \, d\bar{y} = 0$$

$$\therefore \boxed{\int_0^1 f_n f_m \, d\bar{y} = 0 \quad \text{as} \quad \lambda_n^2 - \lambda_m^2 \neq 0 \text{ for } m \neq n}$$

From eq. (b);

$$(\bar{y} - 1) = \sum A_n \sin(\lambda_n \bar{y})$$

Take $q_n = \sin(\lambda_n \bar{y})$

$$\therefore (\bar{y} - 1) q_m = \sum A_n q_n q_m$$

$$\int_0^1 (\bar{y} - 1) q_m \, d\bar{y} = \int_0^1 \sum A_n q_n q_m \, d\bar{y}$$

for $m = n$;

$$\int_0^1 (\bar{y} - 1) q_m \, d\bar{y} = 0 \text{ for } m \neq n.$$

$$\int_0^1 (\bar{y} - 1) q_m \, d\bar{y} = \int_0^1 \sum A_m q_m^2 \, d\bar{y}$$

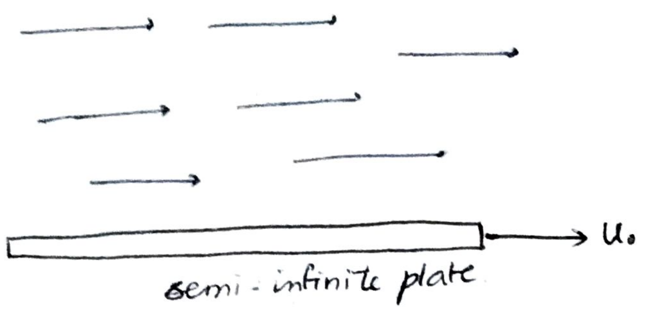
$$\int_0^1 (\bar{y} - 1) \sin(\lambda_n \bar{y}) \, d\bar{y} = \int_0^1 \sum A_n \sin^2(\lambda_n \bar{y}) \, d\bar{y}$$

upon taking by-parts and integrating,

$$\frac{-1}{\lambda_n} = \frac{A_n}{2} \Rightarrow \boxed{A_n = \frac{-2}{\lambda_n} = \frac{-2}{n\pi}}$$

$$\boxed{\therefore \bar{u} = (1 - \bar{y}) + \sum A_n \sin(\lambda_n \bar{y}) \, e^{-\lambda_n^2 t} \quad \text{where } A_n = \frac{-2}{n\pi}}$$

→ Momentum Transfer



NS equation :

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \cdot \cancel{\frac{\partial u_x}{\partial x}} + \cancel{u_y \cdot \frac{\partial u_x}{\partial y}} + u_z \cancel{\frac{\partial u_x}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2} + \cancel{\frac{\partial^2 u_x}{\partial z^2}}\right)$$

$$\frac{\partial u_x}{\partial t} = +\frac{\mu}{\rho}\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial u_x}{\partial t} + \nu \cdot \frac{-\partial^2 u_x}{\partial y^2} = 0$$

(or) $\boxed{\dfrac{\partial u_x}{\partial t} = \nu \cdot \dfrac{\partial^2 u_x}{\partial y^2}}$ ⟶ $\dfrac{\cancel{u_0}}{t_{ref}} = \nu \cdot \dfrac{\cancel{u_0}}{y_{ref}^2}$

$y_{ref} = \sqrt{\nu t_{ref}}$

Taking $y_{ref} = \sqrt{4 \nu t_{ref}}$ for mathematical ease

* Non dimensionalisation

$$\bar{u} = \frac{u_x}{u_0} \; ; \; \bar{y} = \frac{y}{y_{ref}}$$

$$\cancel{u_0}\frac{\partial \bar{u}}{\partial t} = \nu \cdot \frac{\cancel{u_0}}{y_{ref}^2} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

Put $y_{ref} = \sqrt{4 \nu t_{ref}} = \sqrt{4 \nu t}$

$$\frac{\partial \bar{u}}{\partial t} = \nu \cdot \frac{\partial^2 \bar{u}}{\partial y^2}$$

$$\cancel{\frac{\partial \bar{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial y^2} \quad \text{where } y}$$

$$\frac{\partial \bar{u}}{\partial t} = \nu . \frac{\partial^2 \bar{u}}{\partial y^2} \qquad \text{where } \eta = \frac{y}{\sqrt{4\nu t}}$$

- $$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{y}{\sqrt{4\nu t}} \cdot \frac{1}{t}\left(\frac{-1}{2}\right)$$
$$= \frac{-\eta}{2t} \cdot \frac{\partial \bar{u}}{\partial \eta}$$

- $$\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial \eta} \cdot \frac{1}{\sqrt{4\nu t}}\right)$$
$$= \frac{\partial}{\partial \eta}\left(\frac{\partial \bar{u}}{\partial \eta} \cdot \frac{1}{\sqrt{4\nu t}}\right)\frac{\partial \eta}{\partial y}$$
$$= \frac{\partial^2 \bar{u}}{\partial \eta^2}\left(\frac{1}{\sqrt{4\nu t}}\right)^2$$

$$\therefore \frac{-\eta}{2\cancel{t}} \cdot \frac{\partial \bar{u}}{\partial \eta} = \cancel{\nu}\frac{\partial^2 \bar{u}}{\partial \eta^2} \cdot \frac{1}{4\cancel{\nu}\cancel{t}}$$

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} = -\overset{2}{\cancel{4}}\cancel{\nu}\eta \cdot \frac{\partial \bar{u}}{\partial \eta}$$

$$\boxed{\therefore \frac{\partial^2 \bar{u}}{\partial \eta^2} + 2\eta\frac{\partial \bar{u}}{\partial \eta} = 0}$$

Take $\frac{\partial \bar{u}}{\partial \eta} = \psi$

$$\frac{\partial \psi}{\partial \eta} + 2\eta\psi = 0$$

Initial and Boundary conditions:

At $t=0$, $u_x = 0$, $\bar{u} = 0 \quad \forall\, y$

At $t > 0$, $u_x = u_0$ at $y = 0 \quad \rightarrow \quad \bar{u} = 1$ at $y=0 \,/\, \eta = 0$

At $t \geq 0$, $u_x = 0$ at $y \to \infty \quad \rightarrow \quad \bar{u} = 0$ at $\eta = \infty$

$$\frac{\partial \psi}{\partial \eta} + 2\eta\psi = 0$$

$$\cancel{IF \rightarrow e^{\int 2\psi\, d\eta} = e^{2\eta}}$$

$$\int \frac{\partial \psi}{\psi} = \int -2\eta\, \partial \eta$$

$$\ln \psi = -\eta^2 + c$$

$$\psi = c e^{-\eta^2}$$

~~At $\eta = 0$, $\psi =$~~

$$\psi = \frac{\partial \bar{u}}{\partial \eta} = c e^{-\eta^2}$$

$$\int_1^{\bar{u}} \bar{u} = \int_0^{\eta} c e^{-\eta^2} d\eta$$

$$\bar{u} - 1 = \int_0^{\eta} c e^{-\eta^2} d\eta$$

$$\boxed{\bar{u} = 1 + c \int_0^{\eta} e^{-\eta^2} d\eta}$$

At $\eta \to \infty$; $\quad 0 = 1 + c \left[\int_0^{\infty} e^{-\eta^2} d\eta \right]$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx, \quad \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

~~$\therefore \bar{u} = 1 + c$~~

$$\eta^2 = z$$

$$2\eta\, d\eta = dz$$

$$\therefore \int_0^\infty e^{-\eta^2}\, d\eta = \int_0^\infty e^{-z} \cdot \frac{dz}{2\sqrt{z}}$$

$$= \frac{1}{2}\int_0^\infty z^{\frac{1}{2}-1} \cdot e^{-z}\, dz$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\pi} \qquad \text{as } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

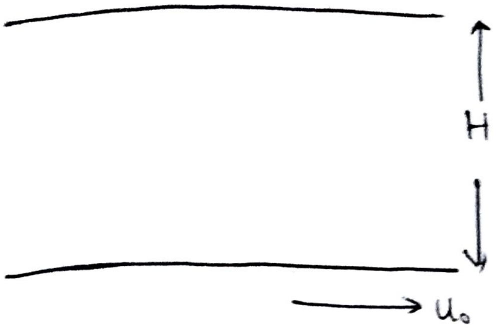
$$\therefore 0 = 1 + c \cdot \frac{\sqrt{\pi}}{2}$$

$$c = -\frac{2}{\sqrt{\pi}}$$

and $\bar{u} = 1 - \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \operatorname{erf}(\eta) \cdot \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$

$$\boxed{\bar{u} = 1 - \operatorname{erf}(\eta)}$$

(or) $\boxed{\bar{u} = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)}$ → see how plot look



$$\frac{\partial u}{\partial t} = \nu \cdot \frac{\partial^2 u}{\partial y^2}$$

Here, $u = \bar{u}\, u_0$, $y = \bar{y} H$

and $\bar{t} = t/t_c$

$$\therefore \frac{u_0}{t_c}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\nu \cdot u_0}{H^2}\frac{\partial^2 \bar{u}}{\partial y^2}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\nu t_c}{H^2} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

Take $t_c = \frac{H^2}{\nu}$.

$$\therefore \frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

Conditions;

At $\bar{y} = 0$; $\bar{u} = 1$ at $\forall t \geq 0$

At $t = 0$, $\bar{u} = 0$

At $\bar{y} = 1$, $\bar{u} = 0$

$$\bar{u} = \underbrace{\bar{u}_{ss}}_{\text{Steady State}} + \underbrace{\hat{u}}_{\text{unsteady sol}^n}$$

At steady state; $\frac{\partial \bar{u}_{ss}}{\partial \bar{t}} = 0 \Rightarrow \frac{\partial^2 \bar{u}_{ss}}{\partial \bar{y}^2} = 0$

$$\bar{u}_{ss} = a\bar{y} + b$$

$b = 1$, $a = -1$

$$\bar{u}_{ss} = 1 - \bar{y}$$

$$\bar{u} = (1-\bar{y}) + \hat{u}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \Rightarrow \frac{\partial \hat{u}}{\partial t} = \cdot \left[\frac{\partial^2 \hat{u}}{\partial \bar{y}^2}\right]$$

Conditions for $\hat{u}$

At $t = 0$; $\bar{u} = 0$

$$(1-\bar{y}) + \hat{u} = 0 \text{ at } t = 0$$

At $t \geq 0$ at $\bar{y} = 0$; $\bar{u} = 1$

$$(1-\cancel{\bar{y}}) + \hat{u} = 1$$

$$\bar{y} = 0, \hat{u} = 0$$

$$\bar{y} = 1; \bar{u} = 0$$

$$\cancel{(1-\bar{y})} + \hat{u} = 0$$

$$\bar{y} = 1, \hat{u} = 0$$

$$\boxed{\begin{array}{l} \text{At } t = 0;\ \hat{u} = \bar{y} - 1 \\ \text{At } \bar{y} = 0, 1;\ \hat{u} = 0 \end{array}}$$

Homogeneous B.C $\rightarrow$ separation of variables

$$\hat{u} = f(\bar{y}) \cdot g(\bar{t})$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$f \cdot \frac{\partial g}{\partial \bar{t}} = g \cdot \frac{\partial^2 f}{\partial \bar{y}^2}$$

$$\frac{1}{f} \frac{\partial^2 f}{\partial \bar{y}^2} = \frac{1}{g} \cdot \frac{\partial g}{\partial \bar{t}} = k$$

$$\frac{\partial^2 f}{\partial \bar{y}^2} - kf = 0$$

$$g = c_1 e^{k\bar{t}}$$

At $\bar{t} \to \infty$, $\hat{u} = 0$ (s.s.)

$\therefore k = -\lambda^2$

$$\boxed{g = c_1 e^{-\lambda^2 \bar{t}}}$$

~~$D^2 - k = 0$~~

~~$D = \pm\sqrt{k}$~~

~~$f = c$~~

$$D^2 + \lambda^2 = 0$$

$$D = \pm \lambda i \rightarrow f = a\cos\lambda\bar{y} + b\sin\lambda\bar{y}$$

$$\therefore \hat{u} = c_1 e^{-\lambda^2 \bar{t}} \left[a \cos \lambda \bar{y} + b \sin \lambda \bar{y} \right]$$

At $\bar{y} = 0$;

$$\hat{u} = c_1 e^{-\lambda^2 \bar{t}} [a] = 0$$

$$\therefore \boxed{a = 0}$$

At $\bar{y} = 1$;

$$0 = c_1 e^{-\lambda^2 \bar{t}} [b \sin \lambda]$$

$$\boxed{\lambda = n\pi}$$

$$\therefore \hat{u} = c_1 e^{-\lambda_n^2 \bar{t}} \left[\sum b_n \sin (\bar{y} n \pi) \right]$$

$$\boxed{\therefore \bar{u} = (1 - \bar{y}) + \left(\sum A_n \sin(\lambda_n \bar{y}) \right) e^{-\lambda_n^2 \bar{t}}}$$

At $t = 0$; $\bar{u} = 0$

$$\boxed{\bar{y} = 1 + \sum A_n \sin(\lambda_n \bar{y})}$$

Let $f_n = A_n \sin(\lambda_n \bar{y})$

$$f_n'' + \lambda_n^2 f_n = 0$$

Multiply with f_m.

$$\int_0^1 f_m \left(\frac{\partial^2 f_n}{\partial \bar{y}^2} \right) + \int_0^1 \lambda_n^2 f_n f_m = 0$$

$$\cancel{f_m \cdot \frac{\partial f_n}{\partial \bar{y}} \Big|_0^1} - \int_0^1 \frac{\partial f_m}{\partial \bar{y}} \cdot \frac{\partial f_n}{\partial \bar{y}} \, d\bar{y} + \lambda_n^2 \int_0^1 f_n f_m d\bar{y} = 0.$$

0

Similarly, $$- \int_0^1 \frac{\partial f_n}{\partial \bar{y}} \bullet \frac{\partial f_m}{\partial \bar{y}} \cdot d\bar{y} + \lambda_m^2 \int_0^1 f_n f_m \, d\bar{y} = 0$$

(−) (−)

$$\Rightarrow \left(\lambda_m^2 - \lambda_n^2 \right) \int_0^1 f_n f_m \, d\bar{y} = 0$$

$$(\lambda_m^2 - \lambda_n^2)\int_0^1 f_n f_m \, d\bar{y} = 0$$

for $m \neq n$; $\int_0^1 f_n f_m \, d\bar{y} = 0.$

$$\bar{y} - 1 = \sum A_n \sin(\lambda_n \bar{y})$$

$$\int_0^1 (\bar{y}-1) f_m \, d\bar{y} = \int_0^1 \sum f_n f_m \, d\bar{y}$$

For $n \neq m$;

$$\int_0^1 (\bar{y}-1) f_m \, d\bar{y} = 0$$

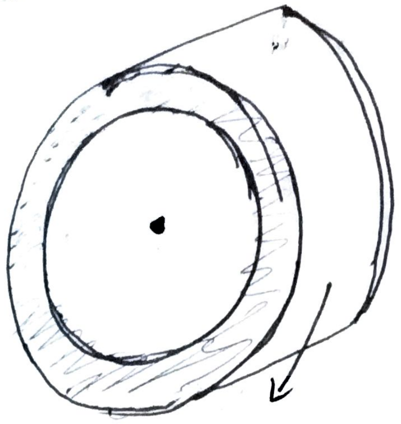
For $n = m$;

$$\int_0^1 (\bar{y}-1) f_m \, d\bar{y} = \int_0^1 (\sum f_m) f$$

$$\int_0^1 (\bar{y}-1) \cancel{A_n} \sin\lambda_n \bar{y} \, d\bar{y} = \int_0^1 \sum A_n^2 \sin^2\lambda_n \bar{y} \, d\bar{y}$$

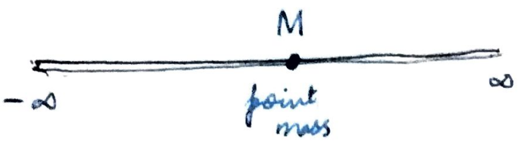
$$\int_0^1 \bar{y} \sin\lambda_n \bar{y} \, d\bar{y} - \int_0^1 \sin\lambda_n \bar{y} \, d\bar{y} = \cancel{\sum A_n} \sum A_n \int_0^1 \frac{1 - \sin 2\lambda_n}{2}$$

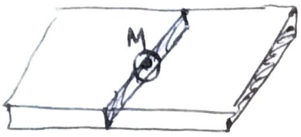
I'm fucked!!



Post Midsem syllabus

1D diffusion from instantaneous point source





Taking the 1D crosssection of area A
Mass M is confined in this space.

$$dV = \lim_{\Delta x \to 0} A . \Delta x$$

$$c(t \to 0, x = 0) = \lim_{\Delta x \to 0} \frac{M}{A \Delta x}$$

$$c(t = 0) = \frac{M}{A} . \delta(x)$$

↳ Some inverse form of Δx.

We know the property of Δx but not exactly what it is.
Mass is finite, A is finite but c is infinite due to $\delta(x)$ at $x =$

Initial conditions:

$$c(t = 0) = \frac{M}{A} \delta(x)$$

Boundary conditions:

$$c(t, x = \pm\infty) = 0$$

$$F(\alpha, t) = \int_{-\infty}^{\infty} F(x,t)\, e^{-i\alpha x}\, dx$$

$$\frac{\partial F}{\partial t} = F_t(\alpha, t) = \frac{\partial \hat{F}}{\partial t}$$

$$\frac{\partial^2 F}{\partial x^2} = -\alpha^2 F$$

$\hat{c}(\alpha, t) = F(\alpha)\exp(-D\alpha^2 t)$ as $\dfrac{d\hat{c}}{dt} + D\alpha^2 \hat{c} = 0$

where $\hat{c}(\alpha, t) = \displaystyle\int_{-\infty}^{\infty} c(x,t)\, e^{-i\alpha x}\, dx$

$$\frac{\partial \hat{c}(\alpha,t)}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial c(x,t)}{\partial t}\, e^{-i\alpha x}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\partial c(x,t)}{\partial x}\, e^{-i\alpha x}\, dx = i\alpha\, \hat{c}(\alpha, t)$$

$$\int_{-\infty}^{\infty} \frac{\partial^2 c(x,t)}{\partial x^2}\, e^{-i\alpha x}\, dx = -\alpha^2\, \hat{c}(\alpha, t)$$

$$\therefore \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial c}{\partial t}\, e^{-i\alpha x}\, dx = D \int_{-\infty}^{\infty} \frac{\partial^2 c}{\partial x^2}\, e^{-i\alpha x}\, dx$$

$$\frac{d\hat{c}}{dt} = -D\alpha^2 \hat{c}$$

(or) $\dfrac{d\hat{c}}{dt} + D\alpha^2 \hat{c} = 0$

and $\hat{c}(\alpha, t) = \underbrace{F(\alpha)}_{\text{Some constant expression}} \cdot \exp(-D\alpha^2 t)$

At $t = 0$, $\hat{c}(\alpha, 0) = F(\alpha) = \displaystyle\int_{-\infty}^{\infty} c(x,0)\, e^{-i\alpha x}\, dx$

$\therefore \hat{c}(\alpha, 0) = F(\alpha) = \displaystyle\int_{-\infty}^{\infty} \frac{M}{A}\, \delta(x)\, e^{-i\alpha x}\, dx = \frac{M}{}$ (dirac delta

$\therefore \hat{c}(x,t) = \frac{M}{A} e^{-D\alpha^2 t}$

$\downarrow$

fourier transform of $c(x,t)$

Taking inverse fourier transform,

Let $\alpha = \frac{y}{\sqrt{Dt}}$

$d\alpha = \frac{dy}{\sqrt{Dt}}$

$$c(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M}{A}\exp(-D\alpha^2 t).\, e^{i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M}{A}\exp(-y^2).\frac{1}{\sqrt{Dt}}.\cos\left(\frac{yx}{\sqrt{Dt}}\right).\, dy$$

$$\boxed{c(x,t) = \frac{2}{2\pi}\int_{0}^{\infty} \frac{M}{A} e^{-y^2}.\left(\cos\frac{yx}{\sqrt{Dt}}\right).\frac{1}{\sqrt{Dt}}\, dy}$$

$$I(\eta) = \int_0^\infty e^{-y^2}\cos(\eta y)\, dy \quad \text{where } \eta = \frac{x}{\sqrt{Dt}}$$

$$\cancel{\frac{dI}{d\eta} = \int_0^\infty (-2y)\, e^{-y^2} \sin \eta y\, dy}$$

$$\frac{dI(\eta)}{d\eta} = \int_0^\infty e^{-y^2}.\left[(\sin \eta y).y\right] dy.$$

$$= \frac{-1}{2}\left[\int_0^\infty -2y.\, e^{-y^2}.\, \sin\eta y\, dy\right]$$

$$= \frac{-1}{2}\left[\int_0^\infty \sin\eta y\; d(e^{-y^2})\right]$$

$$= \underbrace{\frac{e^{-y^2}}{2}\cancel{\sin \eta y}\Big|_0^\infty}_{0} - \frac{\eta}{2}\int_0^\infty e^{-y^2}\cos(\eta y)\, dy$$

$$I(\eta) = -\frac{\eta}{2}\left[\int_0^\infty e^{-y^2}\cos\eta y\, dy\right] \longrightarrow (*)$$

$$\frac{dI(\eta)}{d\eta} + \frac{\eta}{2}I(\eta) = 0 \quad \Rightarrow \quad I(\eta) = I_0 \exp\left(-\frac{\eta^2}{4}\right)$$

Putting $\eta = 0$ in (*) ;

$$I_0 = \frac{\sqrt{\pi}}{2} \quad \text{(from gamma function)}$$

$$I(\eta) = \frac{\sqrt{\pi}}{2} \exp\left(\frac{-\eta^2}{4}\right)$$

$$\boxed{c(x,t) = \frac{M}{A} \cdot \left(\frac{1}{2\sqrt{\pi}}\right) \cdot \frac{1}{\sqrt{Dt}} \exp\left(\frac{-x^2}{4Dt}\right) = \frac{M}{2A\sqrt{\pi D t}} \exp\left(\frac{-x^2}{4Dt}\right)}$$

$M = \int_{\forall} c(x,t)\, d\forall$

$$= \int \frac{M}{2A\sqrt{\pi D t}} \exp\left(\frac{-x^2}{4Dt}\right) d\forall$$

Introducing non dimensional terms,

$$\pi_1 = \frac{C}{M/A\sqrt{Dt}} \quad , \quad \pi_2 = \frac{x}{\sqrt{Dt}}$$

$$\pi_1 = f(\pi_2)$$

We had defined, $\eta = \frac{x}{\sqrt{Dt}}$

$$\frac{\partial \eta}{\partial t} = \frac{-1}{2\sqrt{Dt}} \cdot \frac{x}{t} = \frac{-\eta}{2t}$$

13.03.19 – **Non dimensional approach to the same question.**

Dependent variables $\rightarrow C$

Independent variables $\rightarrow \left(\frac{M}{A}\right), D, x, t$

Fundamental dimensions/ variables $\rightarrow M, L, T \rightarrow (m = 3)$

Right choice: T depends on C, $\frac{M}{A}$, D as

$$T = C^{x} \cdot \left(\frac{M}{A}\right)^{y} \cdot D^{z}$$

$$T = \left(\frac{M}{L^3}\right)^{x} \cdot \left(\frac{M}{L^2}\right)^{y} \cdot \left(\frac{L^2}{T}\right)^{z}$$

$$z = -\cancel{2}1$$

$$-3x - 2y + 2z = 0$$

$$\rightarrow 2z = -\cancel{2} \Rightarrow +3x + 2y = -2$$

$$x + y = 0 \Rightarrow x = -y$$

$$x = -\cancel{2}, \quad y = \cancel{2}$$

$$\boxed{z = -1, \ y = +2, \ x = -2}$$

Non dimensional time

$$\text{Let } \pi_1' = \frac{t}{T} = \frac{t}{C^{-2}\left(\frac{M}{A}\right)^2 D^{-1}} = \frac{C^2}{\left(\frac{M}{A}\right)^2 \cdot \frac{1}{Dt}}$$

$$\pi_1 = \frac{C}{\frac{M}{A} \cdot \frac{1}{\sqrt{Dt}}}$$

$$\pi_1 = f(\pi_2)$$

Next possible ~~π~~ is $\pi_2 = \frac{x}{\sqrt{Dt}}$; Characteristic length $= \sqrt{Dt}$

If any other π_2 obtained, then it is some form of repetition and not unique.

$$c(x,0) = \frac{M}{A}\,\delta(x)$$

$$\frac{\partial c}{\partial t} = \frac{\partial\left(\frac{M}{A}\cdot\frac{1}{\sqrt{Dt}}\cdot f(\eta)\right)}{\partial t}$$

$$= \frac{M}{A}\cdot\frac{1}{\sqrt{Dt}}\cdot\frac{\partial f}{\partial t} + \frac{M}{A}f(\eta)\cdot\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{Dt}}\right)$$

$$= \frac{-M}{2A\sqrt{Dt}}\cdot\frac{1}{t}\left[\eta\frac{\partial f}{\partial\eta} + f\right]$$

$$\frac{\partial c}{\partial x} = \frac{M}{A\sqrt{Dt}}\cdot\frac{\partial f}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}$$

$$= \frac{M}{A\sqrt{Dt}}\cdot\frac{\partial f}{\partial\eta}\cdot\frac{1}{\sqrt{Dt}}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{M}{ADt}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\eta}\right) = \frac{M}{ADt}\cdot\frac{\partial}{\partial\eta}\left(\frac{\partial f}{\partial\eta}\right)\cdot\frac{\partial\eta}{\partial x}$$

$$= \frac{M}{A(Dt)^{3/2}}\frac{\partial^2 f}{\partial\eta^2}$$

$$\frac{-M}{2At\sqrt{Dt}}\left[f + \eta\frac{\partial f}{\partial\eta}\right] = \left(\frac{M}{A}\cdot\frac{D}{(Dt)\sqrt{Dt}}\right)\frac{\partial^2 f}{\partial\eta^2}$$

$$\frac{\partial^2 f}{\partial\eta^2} + \frac{1}{2}\cdot\frac{\partial(f\eta)}{\partial\eta} = 0$$

$$\frac{\partial}{\partial\eta}\left[\frac{\partial f}{\partial\eta} + \frac{f\eta}{2}\right] = 0$$

$$\frac{\partial f}{\partial\eta} + \frac{f\eta}{2} = \text{Constant} \longrightarrow \text{(i)}$$

For all t, we must have,

$$M = \int_{-\infty}^{\infty} cA\,(dx)$$

$$M = \int_{-\infty}^{\infty} \frac{M}{A\sqrt{Dt}} \cdot f\left(\frac{x}{\sqrt{Dt}}\right) \cdot A\, dx$$

$\frac{x}{\sqrt{Dt}} \downarrow \eta$ $\qquad d\eta = \frac{dx}{\sqrt{Dt}}$

$$M = \int_{-\infty}^{\infty} M f(\eta)\, d\eta \Rightarrow \boxed{\int_{-\infty}^{\infty} f(\eta)\, d\eta = 1} \longrightarrow \text{(ii)}$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

$$c = \frac{M}{A\sqrt{Dt}} f(\eta) \quad \text{where } \eta = \frac{x}{\sqrt{Dt}}$$

$$c(\mp\infty, t) = 0$$

$$f(\mp\infty) = 0$$

$$c(x, 0) = \frac{M}{A} \delta(x)$$

$$f\left(\frac{x}{\sqrt{Dt}}\right) = \delta(x) \cdot \sqrt{Dt}$$

$$f(\infty) = 0$$

From (i);

$$\frac{\partial f}{\partial \eta} + \frac{f\eta}{2} = C_0$$

Let's take -
$C_0 = 0$

$$\frac{\partial f}{\partial \eta} + \frac{f\eta}{2} = 0$$

$$\frac{\partial f}{f} = -\left(\eta \frac{\partial \eta}{2}\right)$$

$$\ln f = -\frac{\eta^2}{4} + C_1 \Rightarrow f = c_1 \exp\left(-\frac{\eta^2}{4}\right)$$

From (ii)

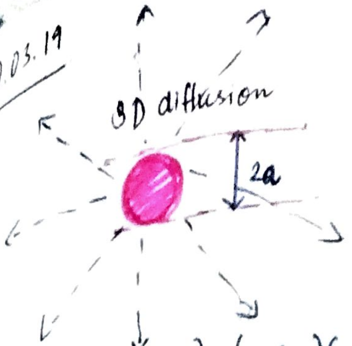
$$2\int_0^{\infty} c_1 \exp\left(-\frac{\eta^2}{4}\right) d\eta = 1$$

$$\Rightarrow 4\underbrace{\int_0^{\infty} c_1 \exp(-\varepsilon^2)\, d\varepsilon}_{} = 1 \qquad \text{taking } \varepsilon = \frac{\eta}{2}$$

error func

$$\Rightarrow 4C_1 \frac{\sqrt{\pi}}{2} = 1 \Rightarrow C_1 = 1/2\sqrt{\pi}$$

20.03.19



$\underline{t=0}$ $\quad C_0 = \dfrac{M}{\frac{4}{3}\pi a^3}$ from 0 to a

beyond that, $C_0 = 0$.

$$\frac{\partial C}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial C}{\partial r}\right)$$

$$C = \frac{C_0}{2r(\pi D t)^{1/2}} \int_0^a r' \left\{ \exp\left(\frac{-(r-r')^2}{4Dt}\right) - \exp\left(\frac{-(r+r')^2}{4Dt}\right) \right\} dr'$$

for $a \to 0$ $\quad C = \left(\dfrac{M}{\frac{4}{3}\pi a^3}\right) \dfrac{1}{8(\pi k t)^{1/2}} e^{-r^2/4Dt}$

$$\Rightarrow \boxed{C = \frac{C_0}{8(\pi k t)^{1/2}} e^{-r^2/4Dt}}$$

- Put a spherical dye in a finite reservoir at $t=0$

Solving: $\dfrac{\partial C}{\partial t} = \dfrac{D}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial C}{\partial r}\right)$

Take $C = \dfrac{u}{r}$

$$\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial u}{\partial t} \quad ; \quad \frac{\partial C}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial r} + u\left(\frac{-1}{r^2}\right)$$

$$\therefore \frac{1}{\cancel{r}}\frac{\partial u}{\partial t} = \frac{D}{r^{\cancel{2}}} \cdot \frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r} - \frac{u}{1}\right]$$

$$\frac{\partial u}{\partial t} = \frac{D}{r}\left[\cancel{\frac{\partial u}{\partial r}} + r\frac{\partial^2 u}{\partial r^2} - \cancel{\frac{\partial u}{\partial r}}\right]$$

$$\therefore \boxed{\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial r^2}} \longrightarrow A$$

$$\boxed{\text{Particular solution: } t^{-1/2} \exp\left(-\frac{(r-r')^2}{4Dt}\right)} \quad r' = \text{any constant}$$

check if this particular solution satisfies the DE.

$$\frac{\partial u}{\partial t} = \exp\left(-\frac{(r-r')^2}{4Dt}\right)\left[\frac{(r-r')^2}{4Dt^{5/2}} + \left(-\frac{1}{2}\right)t^{-3/2}\right] \rightarrow (i)$$

$$\frac{\partial u}{\partial r} = t^{-1/2}\left[\exp\left(-\frac{(r-r')^2}{4Dt}\right)\right] \cdot \left(-\frac{2(r-r')}{4Dt}\right)$$

$$= -\frac{(r-r')}{2Dt^{3/2}} \exp\left(-\frac{(r-r')^2}{4Dt}\right)$$

$$\frac{\partial^2 u}{\partial r^2} = \left(\exp\left[-\frac{(r-r')^2}{4Dt}\right]\right)\left(\frac{-1}{2Dt^{3/2}} + \left(\frac{-(r-r')}{2Dt^{3/2}}\right)\cdot\left(\frac{-\cancel{2}(r-r')}{\cancel{2}4Dt}\right)\right.$$

$$\frac{\partial^2 u}{\partial r^2} = \left[\exp\left(\frac{-(r-r')^2}{4Dt}\right)\right]\left[-\frac{1}{2Dt^{3/2}} + \frac{(r-r')^2}{4D^2t^{5/2}}\right] \rightarrow (ii)$$

$(i) = (ii) \rightarrow$ hence it satisfies.

$\rightarrow$ general solⁿ of (A)

Note: $f = \sum_{n=0}^{\infty} a_n \cos \lambda_n x \quad , \quad \lambda_n = (\quad)$

$$\therefore \quad u = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(r') \exp\left(-\frac{(r-r')^2}{4Dt}\right) dr'$$

Let $r' = r + 2\sqrt{Dt}\,\varepsilon$

$dr' = 2\sqrt{Dt}\,\varepsilon$

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(r + 2\sqrt{Dt}\,\varepsilon)\, e^{-\varepsilon^2}\, d\varepsilon$$