In this lesson we shall first present complex form of Fourier integral. We then introduce Fourier sine and cosine integral. The convergence of these integrals with its application to evaluate integrals will be discussed. In this lesson will be very useful to introduce Fourier transforms.

12.1 The Exponential Fourier Integral

It is often convenient to introduce complex form of Fourier integral. In fact, using complex form of Fourier integral we shall introduce Fourier transform, sometimes referred as Fourier exponential transform, in the next lesson. We start with the following Fourier integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos\alpha(u-x)\,\mathrm{d}u\,\mathrm{d}\alpha \tag{12.1}$$

Note that the integral

$$\int_{-\infty}^\infty f(u)\cos\alpha(u-x)\,\mathrm{d}u$$

is an even function of α and therefore the integral (12.1) can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\cos\alpha(u-x)\,\mathrm{d}u\,\mathrm{d}\alpha \tag{12.2}$$

Also, note that the integral

$$\int_{-\infty}^\infty f(u)\sin\alpha(u-x)\,\mathrm{d}u$$

is an odd function of α and therefore we have the following result

$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\sin\alpha(u-x)\,\mathrm{d}u\,\mathrm{d}\alpha = 0 \tag{12.3}$$

Multiplying the equation (12.3) by i and adding into the equation (12.2) we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\left[\cos\alpha(u-x) + i\sin\alpha(u-x)\right]\,\mathrm{d}u\,\mathrm{d}\alpha \tag{12.4}$$

This may be rewritten as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(u-x)}\,\mathrm{d}u\,\mathrm{d}\alpha \tag{12.5}$$

If we subtract the equation (12.3) after multiplying by i from the equation (12.2) we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{-i\alpha(u-x)}\,\mathrm{d}u\,\mathrm{d}\alpha \tag{12.6}$$

Either (12.5) or (12.6) are exponential form of the Fourier integral.

12.1.1 Example

Compute the complex Fourier integral representation of $f(x) = e^{-a|x|}$.

Solution: The complex integral representation of f is given as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\alpha(u-x)} \, \mathrm{d}u \, \mathrm{d}\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \int_{-\infty}^{\infty} f(u) e^{i\alpha u} \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.7}$$

We first compute the inner integral

$$\int_{-\infty}^{\infty} f(u) e^{i\alpha u} \, \mathrm{d}u = \int_{-\infty}^{0} e^{au} e^{i\alpha u} \, \mathrm{d}u + \int_{0}^{\infty} e^{-au} e^{i\alpha u} \, \mathrm{d}u = \left[\frac{e^{(a+i\alpha)u}}{a+i\alpha} \right]_{-\infty}^{0} + \left[-\frac{e^{-(a-i\alpha)u}}{a-i\alpha} \right]_{0}^{\infty}$$

This can be further simplified

$$\int_{-\infty}^{\infty} f(u) e^{-i\alpha u} \, \mathrm{d}u = \left(\frac{1}{a+i\alpha} + \frac{1}{a-i\alpha} \right) = \frac{2a}{a^2+\alpha^2}$$

Then the complex Fourier integral representation of f is

$$f(x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+\alpha^2} e^{-i\alpha x} \, \mathrm{d}\alpha \tag{12.8}$$

In this lesson we describe Fourier transform. We shall connect Fourier series with the Fourier transform through Fourier integral. Several interesting properties of the Fourier Transform such as linearity, shifting, scaling etc. will be discussed.

12.2 Fourier Transform

Consider the Fourier integral defined in earlier lessons as

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\cos\alpha(u-x)du\right]d\alpha$$

Since the inner integral is an even function of α we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(u)\cos\alpha(u-x)du\right]d\alpha \qquad (12.9)$$

Further note that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\sin\alpha(u-x)du\,d\alpha \qquad (12.10)$$

as the integral

$$\int_{-\infty}^\infty f(u)\sin\alpha(u-x)du\,d\alpha$$

is an odd function of α. Multiplying the equation (12.10) by the imaginary unit i and adding to the equation (12.9), we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(u-x)}du\,d\alpha \qquad (12.11)$$

This is the complex Fourier integral representation of f on the real line. Now we split the exponential integrands and the pre-factor $1/(2\pi)$ as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(u)e^{i\alpha u}du\right]e^{-i\alpha x}d\alpha \qquad (12.12)$$

The term in the parentheses is what we will the *Fourier transform* of f. Thus the Fourier transform of f, denoted by $\hat{f}(\alpha)$, is defined as

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(u)e^{i\alpha u}du$$

Now the equation (12.12) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha \tag{12.13}$$

The function $f(x)$ in equation (12.13) is called the inverse Fourier transform of $\hat{f}(\alpha)$. We shall use F for Fourier transformation and F^{-1} for inverse Fourier transformation in this lesson.

Remark: *It should be noted that there are a number of alternative forms for the Fourier transform. Different forms deals with a different pre-factor and power of exponential. For example we can also define Fourier and inverse Fourier transform in the following manner.*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

or

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

or

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

We shall remain with our original form because it is easy to remember because of the same pre-factor in front of both forward and inverse transforms.