Pole Placement Control Design

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Assumptions:

- The system is completely state controllable.
- The state variables are measurable and are available for feedback.
- Control input is unconstrained.

Pole Placement Control Design

Objective:

The closed loop poles should lie at $\mu_1, \ldots, \mu_n$, which are their 'desired locations'.

Difference from classical approach:

Not only the "dominant poles", but "all poles" are forced to lie at specific desired locations.

Necessary and sufficient condition:

The system is completely state controllable.

Closed Loop System Dynamics

$$\dot{X} = AX + BU$$

The control vector U is designed in the following state feedback form

$$U = -KX$$

This leads to the following closed loop system

$$\dot{X} = (A - BK)X = A_{CL}X$$

where $A_{CL} = (A - BK)$

Philosophy of Pole Placement Control Design

The gain matrix K is designed in such a way that

$$\left|sI-(A-BK)\right|=(s-\mu_1)(s-\mu_2)\quad\cdots\quad(s-\mu_n)$$

where $\mu_1,\cdots,\mu_n$ are the desired pole locations.

Pole Placement Design Steps: Method 1 (low order systems, $n \leq 3$)

- Check controllability
- Define $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$
- Substitute this gain in the desired characteristic polynomial equation

$$\left| sI - A + BK \right| = (s - \mu_1) \cdots (s - \mu_n)$$

- Solve for k_1, k_2, k_3 by equating the like powers on both sides

Example: Design state feedback controller equation for the following system:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-7} & \mathbf{-12} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u} \quad \textbf{and } \boldsymbol{y} = [\mathbf{5} \quad \mathbf{0}] \boldsymbol{X}$$

Solution:

$$M = [B\ AB] = \left[\begin{matrix} 0 \\ 1 \end{matrix}\ \begin{bmatrix} 0 & 1 \\ -7 & -12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & -12 \end{bmatrix}$$

Rank(M) is 2, so the system is controllable.

If $\mu_{1,}\ \mu_2$ are the location of desired regulator poles

$$|sI - (A - BK)| = (s - \mu_1)(s - \mu_2)$$
$$= s^2 - (\mu_1 + \mu_2)s + \mu_1\mu_2$$

Let $K = [k_1\ k_2]$ then $BK = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$

$$|sI - (A - BK)| = \left\| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -7 & -12 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right) \right\|$$

$$= \left\| \begin{bmatrix} s & -1 \\ 7 + k_1 & s + 12 + k_2 \end{bmatrix} \right\| = s^2 + (12 + k_2)s + (7 + k_1)$$

$$= s^2 - (\mu_1 + \mu_2)s + \mu_1\mu_2$$

Let $\mu_{1,2} = -7 \pm 0.7j$ then $\mu_1 + \mu_2 = -14$ and $\mu_1\mu_2 = 49.49$

So, k_1=49.49-7 = 42.49 and $k_2 = 14 - 12 = 2$

The state feedback controller equation

$$u = -49.49x_1 - 2x_2$$

Pole Placement Control Design: Method – 2

$$\dot{X} = AX + Bu$$

$$u = -KX, \qquad K = \begin{bmatrix} k_1 & k_2 \cdots k_n \end{bmatrix}$$

Let the system be in first companion (controllable canonical) form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

After applying the control, the closed loop system dynamics is given by

$$\dot{X} = \left(A - BK \right) X = A_{CL} X$$

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ k_1 & k_2 & k_3 & \cdots & k_n \end{bmatrix}$$

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \\ \left(-a_n - k_1\right) & \left(-a_{n-1} - k_2\right) & \cdots & \cdots & \cdots & \left(-a_1 - k_n\right) \end{bmatrix} \cdots\cdots\cdots(1)$$

Pole Placement Control Design: Method – 2

If $\mu_1, \cdots, \mu_n$ are the desired poles. Then the desired characteristic polynomial is given by,

$$(s-\mu_1)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

This characteristic polynomial, will lead to the closed loop system matrix as

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & & \\ & & & & \\ \vdots & & \ddots & & 1 \\ 0 & 0 & & \cdots & \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \cdots\cdots(2)$$

State space form

Pole Placement Control Design: Method – 2

Comparing Equation (1) and (2), we arrive at:

$$\begin{bmatrix} a_n + k_1 = \alpha_n \\ a_{n-1} + k_2 = \alpha_{n-1} \\ \vdots \\ a_1 + k_n = \alpha_1 \end{bmatrix} \Rightarrow \begin{bmatrix} k_1 = (\alpha_n - a_n) \\ k_2 = (\alpha_{n-1} - a_{n-1}) \\ \vdots \\ k_n = (\alpha_1 - a_1) \end{bmatrix}$$

$$K = (\alpha - a) \qquad (\text{Row vector form})$$

Bass-Gura Method

Example: Design state feedback controller equation for the following system:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-7} & \mathbf{-12} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u} \quad \textbf{and } \boldsymbol{y} = [\mathbf{5} \quad \mathbf{0}] \boldsymbol{X}$$

Solution:

$$M = [B\ AB] = \left[\begin{matrix} 0 \\ 1 \end{matrix} \ \begin{bmatrix} 0 & 1 \\ -7 & -12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 1 & -12 \end{bmatrix}$$

Rank(M) is 2, so the system is controllable.

$$|[sI - A]| = \left| \begin{bmatrix} s & -1 \\ 7 & s + 12 \end{bmatrix} \right| = s^2 + 12s + 7 = s^2 + a_1 s + a_2$$

Let, $\mu_{1,2} = -7 \pm 0.7j$

So $(s - \mu_1)(s - \mu_2) = s^2 + 14s + 49.49 = s^2 + \alpha_1 s + \alpha_2$

$k_1 = \alpha_2 - a_2 = 49.49 - 7 = 42.49$ and

$k_2 = \alpha_1 - a_1 = 14 - 12 = 2$

The state feedback controller equation $u = -42.49x_1 - 2x_2$

What if the system is not given in the first companion form?

Define a transformation $X = T\hat{X}$

$$\dot{\hat{X}} = T^{-1}\dot{X}$$

$$\dot{\hat{X}} = T^{-1}\left(AX + Bu\right)$$

$$\dot{\hat{X}} = \left(T^{-1}AT\right)\hat{X} + \left(T^{-1}B\right)u$$

Design a T such that $T^{-1}AT$ will be in first companion form.

Select $T = MW$

where $M \triangleq \left[B \quad AB \cdots\cdots\cdots A^{n-1}B\right]$ is the controllability matrix

Pole Placement Control Design: Method – 2

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & & \iddots & \iddots & 0 \\ & \iddots & \iddots & \cdots & \vdots \\ a_1 & 1 & \cdots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Next, design a controller for the transformed system (using the technique for systems in first companion form).

$$u = -\hat{K}\hat{X} = -(\hat{K}T^{-1})X = -KX$$

Note:* Because of its role in control design as well as the use of** M ***(Controllability Matrix) in the process, the '<u>first companion form</u>' is also known as 'Controllable Canonical form'.

Pole Placement Design Steps: Method 2: Bass-Gura Approach

- Check the controllability condition
- Form the characteristic polynomial for *A*

  $$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

  find a_i's
- Find the Transformation matrix $T=MW$
- Write the desired characteristic polynomial

  $$(s-\mu_1)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

  and determine the α_i's
- The required state feedback gain matrix is

  $$K = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \cdots \quad (\alpha_1 - a_1)]\, T^{-1}$$

Bass-Gura Method

Example: Design state feedback controller equation for the following system:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{-7} & \mathbf{-12} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u} \quad \textbf{and } \boldsymbol{y} = [\mathbf{5} \quad \mathbf{0}]\boldsymbol{X}$$

Solution:

$$M = [B\ AB] = \begin{bmatrix} 0 & \begin{bmatrix} 1 & 3 \\ -7 & -12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & -12 \end{bmatrix}$$

Rank(M) is 2, so the system is controllable.

$$|[sI - A]| = \left| \begin{bmatrix} s-1 & -3 \\ 7 & s+12 \end{bmatrix} \right| = s^2 + 11s + 9 = s^2 + a_1 s + a_2$$

Let, $\mu_{1,2} = -7 \pm 0.7j$ So $(s - \mu_1)(s - \mu_2) = s^2 + 14s + 49.49 = s^2 + \alpha_1 s + \alpha_2$

$W = \begin{bmatrix} 11 & 1 \\ 1 & 0 \end{bmatrix}$ so, $T = MW = \begin{bmatrix} 0 & 3 \\ 1 & -12 \end{bmatrix} \begin{bmatrix} 11 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$ or $T^{-1} = \begin{bmatrix} 0.33 & 0 \\ 0.33 & 1 \end{bmatrix}$

$$K = [\alpha_2 - a_2 \quad \alpha_1 - a_1] * T^{-1} = [40.49 \quad 3] \begin{bmatrix} 0.33 & 0 \\ 0.33 & 1 \end{bmatrix} = [14.49 \quad 3]$$

The state feedback controller equation $u = -14.49x_1 - 3x_2$

Pole Placement Design Steps: Method 3 (Ackermann's formula)

Define $\tilde{A} = A - BK$

desired characteristic equation is

$$\left|sI - \left(A - BK\right)\right| = \left(s - \mu_1\right)\cdots\left(s - \mu_n\right)$$

$$|sI - \tilde{A}| = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

Caley-Hamilton theorem states that every matrix A satisfies its own characteristic equation

$$\phi(\tilde{A}) = \tilde{A}^n + \alpha_1 \tilde{A}^{n-1} + \alpha_2 \tilde{A}^{n-2} + \cdots + \alpha_{n-1} \tilde{A} + \alpha_n = 0$$

For the case $n = 3$ consider the following identities.

$$\mathbf{I} = \mathbf{I}$$

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\tilde{\mathbf{A}}^2 = (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\tilde{\mathbf{A}}$$

$$\tilde{\mathbf{A}}^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2$$

Pole Placement Design Steps: Method 3: (Ackermann's formula)

Multiplying the identities in order by α_3, α_2, α_1 respectively and adding we get

$$\begin{aligned}
&\alpha_3\mathbf{I} + \alpha_2\tilde{\mathbf{A}} + \alpha_1\tilde{\mathbf{A}}^2 + \tilde{\mathbf{A}}^3 \\
&= \alpha_3\mathbf{I} + \alpha_2(\mathbf{A} - \mathbf{BK}) + \alpha_1(\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\tilde{\mathbf{A}}) + \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} \\
&\quad - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 \\
&= \alpha_3\mathbf{I} + \alpha_2\mathbf{A} + \alpha_1\mathbf{A}^2 + \mathbf{A}^3 - \alpha_2\mathbf{BK} - \alpha_1\mathbf{ABK} - \alpha_1\mathbf{BK}\tilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK} \\
&\quad - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 \qquad \cdots\cdots\cdots\cdot\mathbf{(1)}
\end{aligned}$$

From Caley-Hamilton Theorem for $\tilde{\mathbf{A}}$

$$\alpha_3\mathbf{I} + \alpha_2\tilde{\mathbf{A}} + \alpha_1\tilde{\mathbf{A}}^2 + \tilde{\mathbf{A}}^3 = \phi(\tilde{\mathbf{A}}) = \mathbf{0}$$

And also we have for A

$$\alpha_3\mathbf{I} + \alpha_2\mathbf{A} + \alpha_1\mathbf{A}^2 + \mathbf{A}^3 = \phi(\mathbf{A}) \neq \mathbf{0}$$

Pole Placement Design Steps: Method 3 (Ackermann's formula)

Substituting $\varphi(\tilde{\mathbf{A}})$ and $\varphi(\mathbf{A})$ in equation (1) we get

$$\phi(\tilde{\mathbf{A}}) = \phi(\mathbf{A}) - \alpha_2\mathbf{BK} - \alpha_1\mathbf{BK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 - \alpha_1\mathbf{ABK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$

0

$$\phi(\mathbf{A}) = \mathbf{B}(\alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2) + \mathbf{AB}(\alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}) + \mathbf{A}^2\mathbf{BK}$$

$$= [\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}]\begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Since system is completely controllable inverse of the controllability matrix exists we obtain

$$[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = \begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} \quad \text{.........}\mathbf{(2)}$$

Pole Placement Design Steps: Method 3 (Ackermann’s formula)

Pre multiplying both sides of the equation **(2)** with **[0 0 1]**

$$[0 \quad 0 \quad 1][\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = [0 \quad 0 \quad 1]\begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} = \mathbf{K}$$

- For an arbitrary positive integer *n* (number of states) *Ackermann’s formula* for the state feedback gain matrix *K* is given by

$$K = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}\begin{bmatrix} B & AB & A^2B & \cdots\cdots\cdots & A^{n-1}B \end{bmatrix}^{-1}\phi(A)$$

where $\phi(A) = A^n + \alpha_1 A^{n-1} + \cdots\cdots\cdots + \alpha_{n-1}A + \alpha_n I$

α_i's are the coefficients of the desired characteristic polynomial

Ackermann's formula

Example: Design state feedback controller equation for the following system:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-7} & \mathbf{-12} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u} \quad \textbf{and } \boldsymbol{y} = [\mathbf{5} \quad \mathbf{0}]\boldsymbol{X}$$

Solution:

Let, $\mu_{1,2} = -7 \pm 0.7j$

So $(s - \mu_1)(s - \mu_2) = \mathrm{s}^2 + 14s + 49.49 = \mathrm{s}^2 + \alpha_1 s + \alpha_2$

$$\phi(A) = A^2 + \alpha_1 A + \alpha_2 I = \begin{bmatrix} -7 & -12 \\ 84 & 137 \end{bmatrix} + 14 \begin{bmatrix} 0 & 1 \\ -7 & -12 \end{bmatrix} + 49.49 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 42.49 & 2 \\ -14 & 18.49 \end{bmatrix}$$

$$M = [B\ AB] = \left[\begin{matrix} 0 \\ 1 \end{matrix} \ \begin{bmatrix} 0 & 1 \\ -7 & -12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 1 & -12 \end{bmatrix} \text{ or } M^{-1} = \begin{bmatrix} 12 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K = [0 \quad 1] M^{-1} \phi(A) = [0 \quad 1] \begin{bmatrix} 12 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 42.49 & 2 \\ -14 & 18.49 \end{bmatrix} = [42.49 \quad 2]$$

The state feedback controller equation $u = -42.49x_1 - 2x_2$

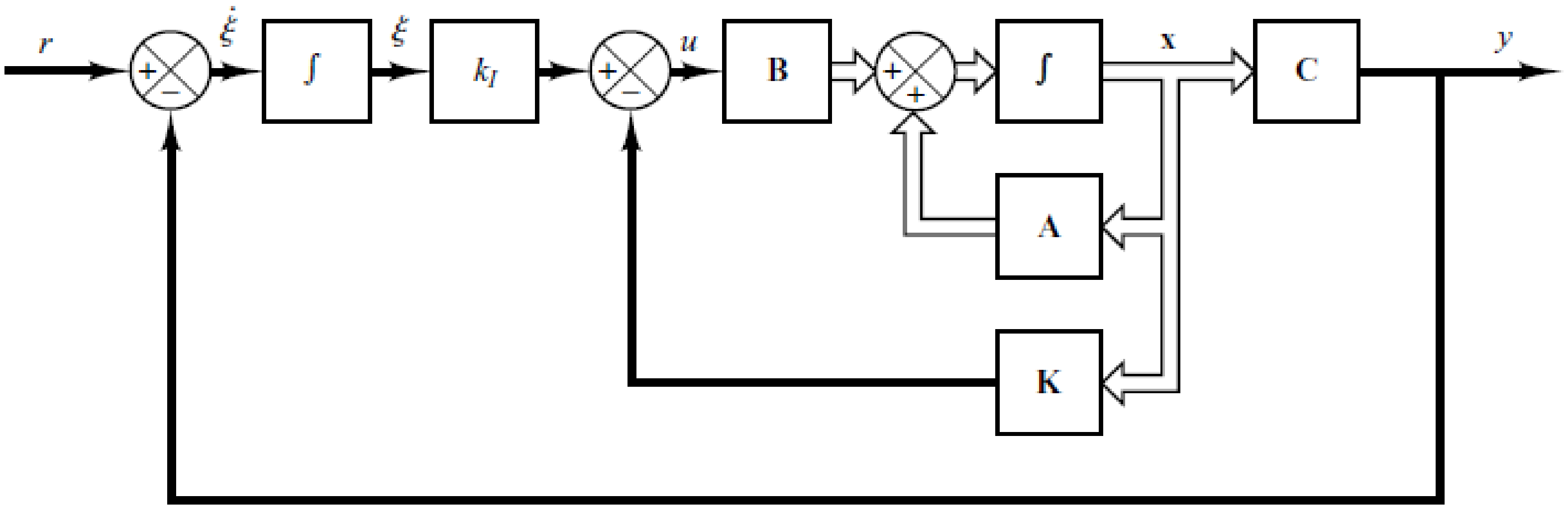
Choice of closed loop poles : Guidelines

- Do not choose the closed loop poles far away from the open loop poles, otherwise it will demand high control effort

- Do not choose the closed loop poles very negative, otherwise the system will be fast reacting (i.e. it will have a small time constant)
  - In frequency domain it leads to large bandwidth, and hence noise gets amplified

Design of State Feedback Controller for Servo situation

System: $$\dot{X} = AX + Bu$$
$$y = CX$$



Design of State Feedback Controller for Servo situation

- From the Block Diagram, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I\xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

- We assume there is no zero at the origin of the plant

Design of State Feedback Controller for Servo situation

- Let a step function is applied on reference input at t =0, then for t > 0, the system dynamics are

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \qquad (1)$$

- At steady state, $\dot{\xi}(t) = 0$ and y(∞)=r
- So, at steady state

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty) \qquad (2)$$

Design of State Feedback Controller for Servo situation

- Since, r(t) = r(∞) = r for t > 0 (for step input)
- Subtracting eqn (2) from eqn (1)

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)]$$

- Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$

$$\xi(t) - \xi(\infty) = \xi_e(t)$$

$$u(t) - u(\infty) = u_e(t)$$

Design of State Feedback Controller for Servo situation

- In terms of new variables,

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t) \qquad (3)$$

- Where,

$$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t) \qquad (4)$$

Define a new $(n + 1)$th-order error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Design of State Feedback Controller for Servo situation

Then equation (3) becomes,

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \qquad (5)$$

Where,

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

Equation (4) becomes

$$u_e = -\hat{\mathbf{K}}\mathbf{e} \qquad (6)$$

Where,

$$\hat{\mathbf{K}} = \left[\mathbf{K} \;\vdots\; -k_I\right]$$

Design of State Feedback Controller for Servo situation

Then Substituting eqn (6) into eqn (5) becomes,

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e}$$

For desired eigenvalues $\mu_1, \mu_2, \ldots \mu_{n+1}$ of matrix $\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}$ the state feedback controller gain **K** and integral gain K_I can be obtained by pole placement technique provided the system defined by equation (5) is state controllable.

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e$$

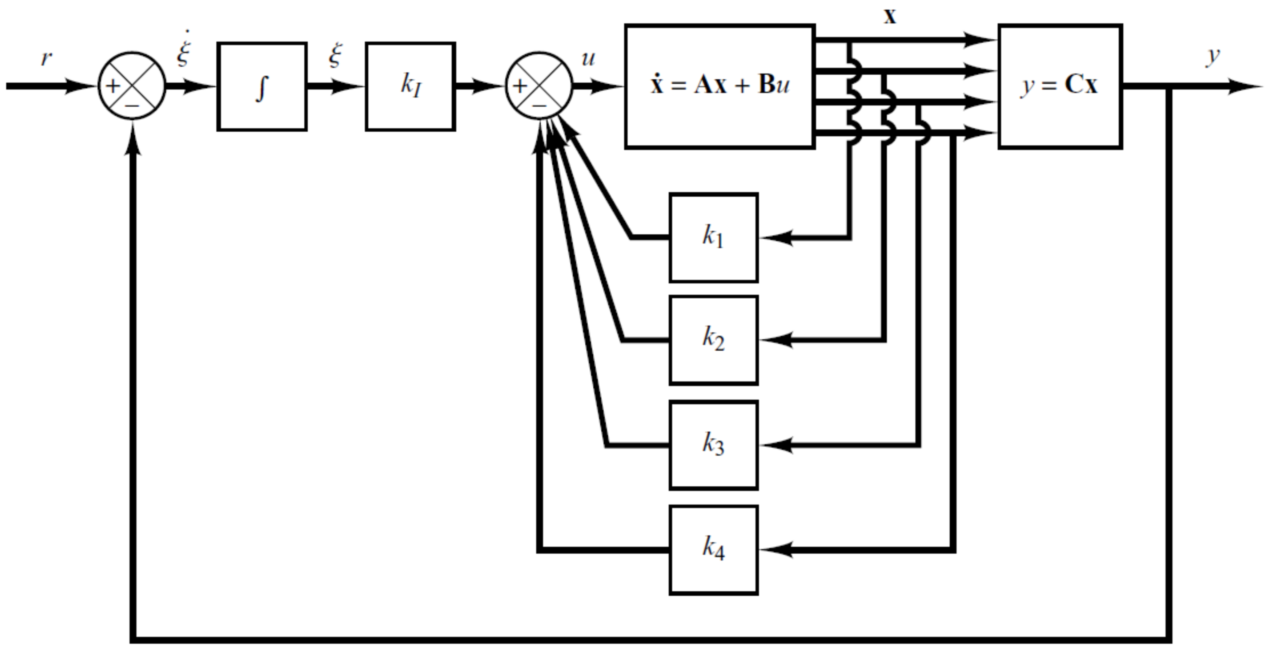
Design of State Feedback Controller for Servo situation

The system $\dot{\mathbf{e}} = \widehat{\mathbf{A}}\mathbf{e} + \widehat{\mathbf{B}}\mathbf{u}_\mathbf{e}$ will be state controllable when Rank of $\mathbf{M} = [\widehat{\mathbf{B}}\ \widehat{\mathbf{A}}\widehat{\mathbf{B}} \quad \ldots \quad \widehat{\mathbf{A}}^{\mathbf{n}}\widehat{\mathbf{B}}]$ = n+1

Further, It can also be shown that the system $\dot{\mathbf{e}} = \widehat{\mathbf{A}}\mathbf{e} + \widehat{\mathbf{B}}\mathbf{u}_\mathbf{e}$ is state controllable if the

Rank of $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}$ = n+1.

Inverted Pendulum on a Cart



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I\xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

Inverted Pendulum on a Cart

- State Error Equation: $\dot{\mathbf{e}} = \widehat{\mathbf{A}}\mathbf{e} + \widehat{\mathbf{B}}\mathbf{u_e}$

where,

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

Control Signal: $\mathbf{u_e} = -\widehat{\mathbf{K}}\mathbf{e}$

where, $\widehat{\mathbf{K}} = [\mathbf{K}|-\mathrm{k_I}] = [k_1 \;\; k_2 \;\; k_3 \;\; k_4 \;\; -k_I]$

Let us choose the desired close loop poles at s=μ_i (i=1,5)

$$\mu_{1,2} = -1 \pm j\sqrt{3}, \qquad \mu_{3,4,5} = -5$$

Inverted Pendulum on a Cart

- Checking the controllability of $\dot{\mathbf{e}} = \widehat{\mathbf{A}}\mathbf{e} + \widehat{\mathbf{B}}\mathbf{u_e}$

$$P = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.6 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.49 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Rank of P is 5. So the system is state controllable and arbitrary pole placement is possible

Inverted Pendulum on a Cart

MATLAB program to calculate $\widehat{K}$

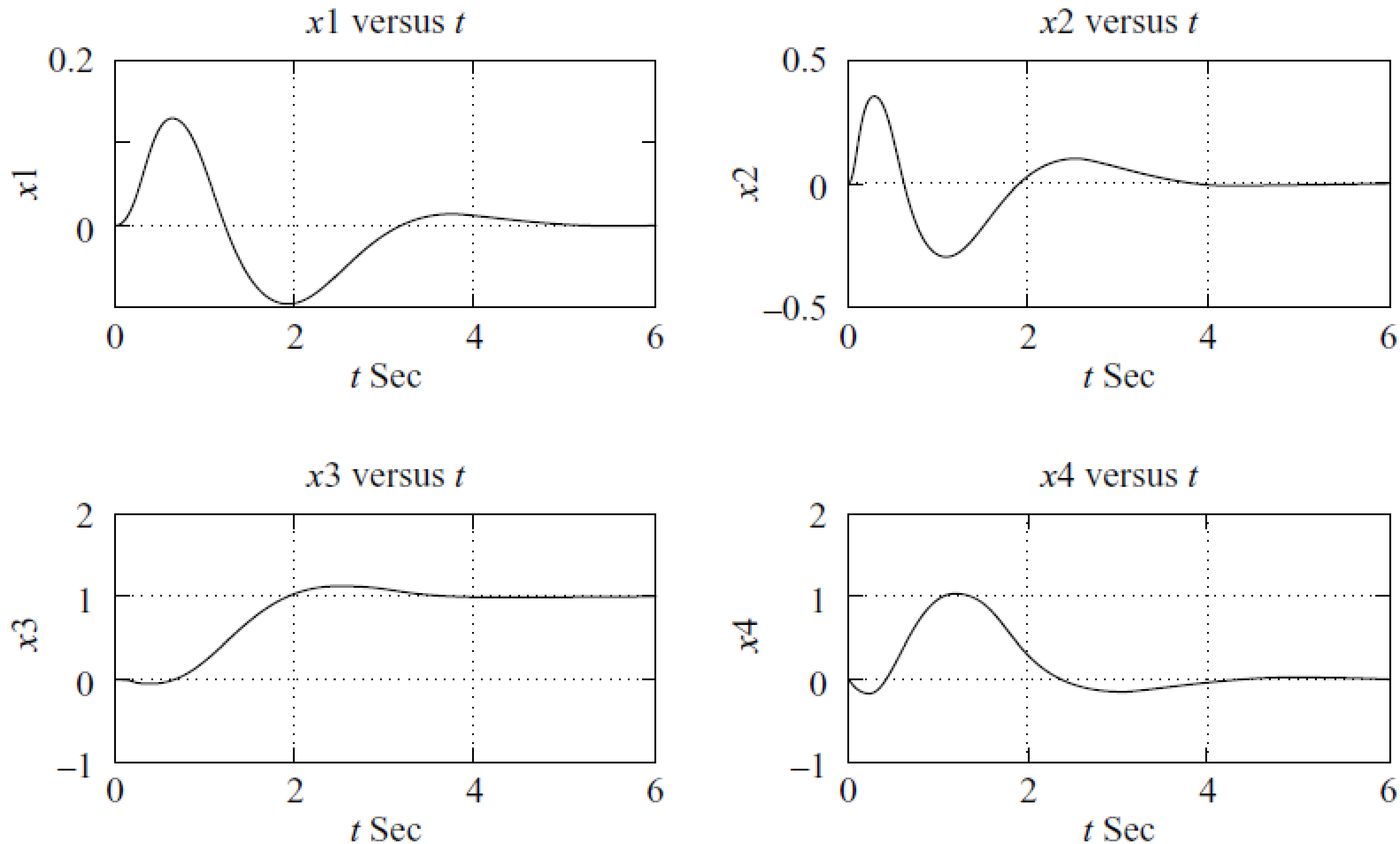
MATLAB Program

```
A = [0  1  0  0; 20.601  0  0  0;  0  0  0  1; -0.4905  0  0  0];
B = [0;-1;0;0.5];
C = [0  0  1  0];
Ahat = [A  zeros(4,1); -C  0];
Bhat = [B;0];
J = [-1+j*sqrt(3)  -1-j*sqrt(3)  -5  -5  -5];
Khat = acker(Ahat,Bhat,J)

Khat =

  -157.6336  -35.3733  -56.0652  -36.7466  50.9684
```

Thus, $K = [-157.63 \quad -35.37 \quad -56.06 \quad -36.75]$

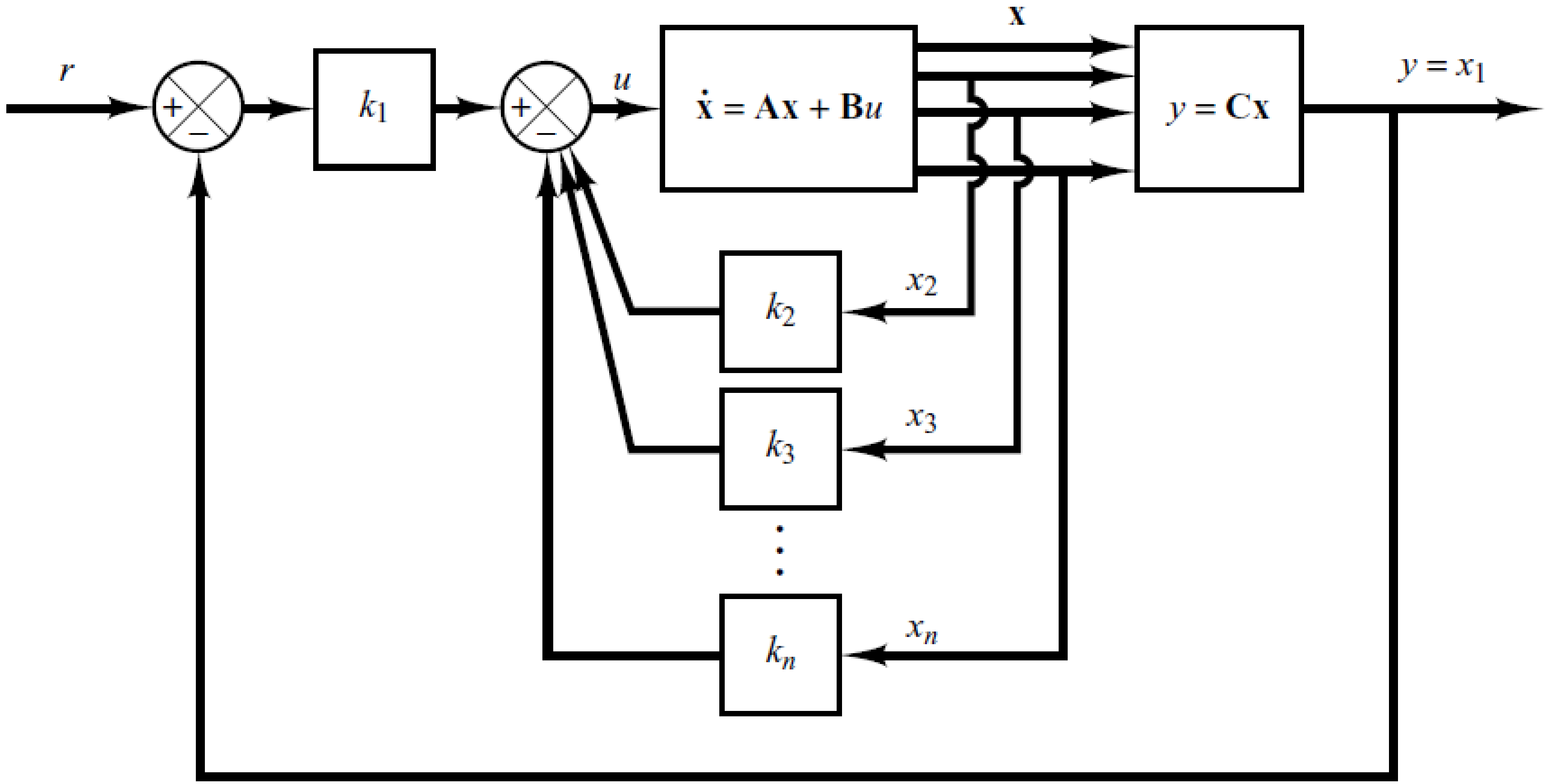
$$k_I = -50.97$$

Inverted Pendulum on a Cart



Plant with Integrator

- System: $\dot{X} = AX + Bu$
  $y = CX$
- Let $y = x_1$; then the block diagram



Plant with Integrator

- In this system we use the following control signal
- $u = -[0\ k_2\ k_3\ \ \ldots k_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + k_1(r - x_1)$

$= -\mathbf{K}\,\mathrm{X}\ + k_1 r$

So, $\dot{X} = AX + B(-KX + k_1 r) = (A - BK)X + Bk_1 r$

At Steady state,

$$\dot{X}(\infty) = (A - BK)X(\infty) + Bk_1 r(\infty)$$

$$\dot{e} = \dot{X}(t) - \dot{X}(\infty) = (A - BK)\ e$$

References

- K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999.
- B. Friedland: *Control System Design*, McGraw Hill, 1986.