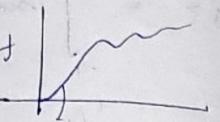


$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin nx + \sum_{n=1}^{\infty} b_n \cos nx \rightarrow \text{Fourier series}$$

→ why is Fourier required?
(or) any other expansion



You
have to write
the function
in terms of cos, sin
as weight function.

→ Adj.

Adjoint operator

$$\underline{A} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = 14$$

$$\langle \underline{u}, \underline{A}\underline{v} \rangle = -2$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = \langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle$$

$$= \left\langle \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= a_{11}u_1v_1 + a_{12}u_2v_1 + a_{21}u_1v_2 + a_{22}u_2v_2$$

$$= u_1(a_{11}v_1 + a_{12}v_2) + u_2(a_{21}v_1 + a_{22}v_2)$$

$$\begin{aligned} \langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} \rangle &= \langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11}u_1 & a_{12}u_1 \\ a_{21}u_1 & a_{22}u_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle \\ &= \langle \underline{u}, \underline{A}\underline{v} \rangle \\ \langle \underline{A}\underline{u}, \underline{v} \rangle &= \langle \underline{u}, \underline{A}^*\underline{v} \rangle \end{aligned}$$

$$\langle \underline{u}, \underline{A}^*\underline{v} \rangle = 14$$

A* is adjoint operator of A

Every matrix can be an operator

→ $\oplus: \mathbb{C}^{2 \times 1} \rightarrow \mathbb{C}^{2 \times 1}$ → generate new vector within the same vector space
(operator vector maps the vector)

$\oplus: \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{n \times p}$
 $2 \times 3 \xrightarrow{3 \times 1}$ original vector space

$$\text{Q) } \underline{A} = \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix}$$

$$\underline{u} = [u_1 \ u_2]^T$$

$$\underline{v} = [v_1 \ v_2]^T$$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = \langle \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle$$

$$= \langle \begin{bmatrix} u_1 - 2iu_2 \\ 3u_1 + iu_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle$$

$$= (u_1 - 2iu_2)v_1 + (3u_1 + iu_2)v_2$$

$$= u_1(v_1 + 3v_2) + u_2(-2iv_1 + iv_2)$$

$$= \langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 + 3v_2 \\ -2iv_1 + iv_2 \end{bmatrix} \rangle$$

$$\textcircled{X} \quad = \langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -2i & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle$$

complex
number
↓
inner product
is different
take complex
conjugate

$$\underline{\underline{A^*}} = \begin{bmatrix} 1 & 3 \\ +2i & -i \end{bmatrix}$$

a) The eigen values of a self adjoint operator are real

$$\langle \underline{\underline{A}} \underline{\underline{u}}, \underline{\underline{v}} \rangle = \langle \underline{\underline{u}}, \underline{\underline{A}} \underline{\underline{v}} \rangle$$

let $\underline{\underline{u}}$ be an eigenvector of $\underline{\underline{A}}$ with the corresponding eigenvalue as λ

$$\underline{\underline{A}} \underline{\underline{u}} = \lambda \underline{\underline{u}} \rightarrow \text{scaling}$$

$$\langle \underline{\underline{A}} \underline{\underline{u}}, \underline{\underline{u}} \rangle = \langle \lambda \underline{\underline{u}}, \underline{\underline{u}} \rangle = \lambda \langle \underline{\underline{u}}, \underline{\underline{u}} \rangle \rightarrow \textcircled{1}$$

$$\langle \underline{\underline{u}}, \underline{\underline{A}} \underline{\underline{u}} \rangle = \langle \underline{\underline{u}}, \lambda \underline{\underline{u}} \rangle = \bar{\lambda} \langle \underline{\underline{u}}, \underline{\underline{u}} \rangle \rightarrow \textcircled{2}$$

$$\lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \in \mathbb{R}$$

$$\textcircled{1} = \textcircled{2}$$

→ for self adjoint

→ The eigen vectors of a self adjoint operator corresponding to distinct eigen values are always orthogonal.

$$\underline{\underline{A}} \underline{\underline{u}} = \lambda \underline{\underline{u}}$$

$$\underline{\underline{A}} \underline{\underline{v}} = \lambda \underline{\underline{v}}$$

$$\langle \underline{\underline{A}} \underline{\underline{u}}, \underline{\underline{v}} \rangle = \langle \lambda \underline{\underline{u}}, \underline{\underline{v}} \rangle = \lambda \langle \underline{\underline{u}}, \underline{\underline{v}} \rangle \rightarrow \textcircled{1}$$

$$\langle \underline{\underline{u}}, \underline{\underline{A}} \underline{\underline{v}} \rangle = \lambda \underline{\underline{v}} \langle \underline{\underline{u}}, \underline{\underline{v}} \rangle \rightarrow \textcircled{2}$$

$$\textcircled{1} = \textcircled{2}$$

$$\text{if } \langle \underline{\underline{u}}, \underline{\underline{v}} \rangle = 0$$

$$\rightarrow \langle \underline{\underline{u}}, \underline{\underline{v}} \rangle = 0$$

$$\rightarrow \underline{\underline{A}} = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix} \xrightarrow{\text{S.P.}} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\underline{\underline{u}} = \begin{bmatrix} a \\ 3a \end{bmatrix}$$

$\underline{\underline{A}} \underline{\underline{u}} = \underline{\underline{v}}$

if $\underline{\underline{u}}$ is eigen vector $\rightarrow \underline{\underline{A}} \underline{\underline{u}} = \lambda \underline{\underline{u}}$

eigen vector
↓
special
vector
such that
direction
is same
now changes

eigenvalue ✓ eigenvalue ✗

eigenvalue

$$a) \frac{d^2 f}{dx^2} + \alpha f = 0$$

$\hat{L} = \frac{d^2}{dx^2} + \alpha$ → it is a self adjoint operator

$$\int_0^1 (\frac{d^2 f}{dx^2} + \alpha f) g dx = \int_0^1 f'' g dx + \int_0^1 \alpha f g dx$$

$$= \left[f \frac{d f}{dx} - f \frac{d g}{dx} \right]_0^1 + \alpha \int_0^1 f g dx$$

force
if $f(0) = f(1) = 0$
 $g'(0) = g'(1) = 0$

a) Identify the solvability condition & determine the range space of the following equation.

$$x_1 + x_2 + x_3 = b_1$$

$$2x_1 - x_2 + x_3 = b_2$$

$$x_1 - 2x_2 = b_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & -1 & 1 & b_2 \\ 1 & -2 & 0 & b_3 \end{array} \right] \xrightarrow{\begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -3 & -1 & b_2 - 2b_1 \\ 0 & -3 & -1 & b_3 - b_1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & -3 & -1 & b_2 - 2b_1 \\ 0 & -3 & -1 & b_3 - b_1 \end{array} \right] \xrightarrow{\begin{matrix} R_2 \rightarrow R_2 - R_3 \\ R_3 \rightarrow R_3 - R_2 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - b_1 - (b_2 - 2b_1) \\ 0 & 0 & 0 & b_3 - b_1 - (b_2 - 2b_1) \end{array} \right]$$

$$b_3 - b_1 - b_2 + 2b_1 = 0$$

$$b_3 - b_2 + b_1 = 0$$

$$\underline{b_3 = b_2 - b_1}$$

$$\left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_2 - b_1 \end{array} \right] \Rightarrow$$

$$b_1 = \alpha, b_2 = \beta$$

$$\left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right] = \alpha \left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right] + \beta \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]$$

dimension = 2
basis
 $\left[\begin{array}{c} 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$

Frobenius alternative method (range space)

$A\bar{x} = b$ is solvable if $\langle b, y \rangle = 0 \iff A^T y = 0$

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{array} \right]$$

$$A^T = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{array} \right]$$

$$A^T y = 0 \iff \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$-3y_2 - 3y_3 = 0$$

$$y_1 + 2y_2 + y_3 = 0$$

$$y_3 = -y_2$$

$$y_1 + 2y_2 - y_2 = 0 \Rightarrow y_1 = -y_2$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$DOF = 2$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\langle b, y \rangle = \langle \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rangle = b_1(-1) + b_2(1) + b_3(-1) = 0$$

$$-b_1 + b_2 - b_3 = 0 \quad DOF = 2$$

$$b_3 = b_1 - b_2$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvalue Problems

$$\frac{dx}{dt} = 2x \Rightarrow x = e^{2t} x_1$$

$$\frac{dx}{dt} = -3x \Rightarrow x = e^{-3t} x_2$$

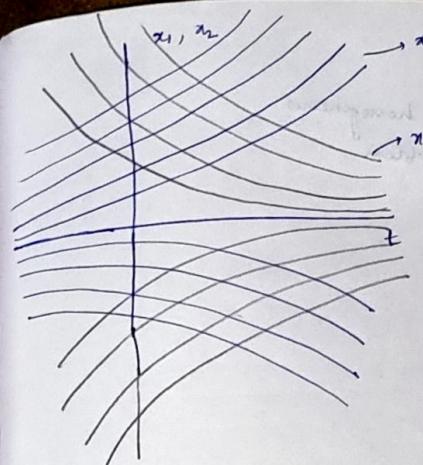
$$\rightarrow \frac{dx_1}{dt} = 2x_1$$

$$\frac{dx_2}{dt} = -3x_2$$

simultaneous equations

$$\{x_1, x_2\} = \{e^{2t}, e^{-3t}\}$$

$$\textcircled{X}$$



points of intersection
are the solution of
the simultaneous equations
(This will make a
linear vector space)

$$\frac{dx_1}{dt} = 2x_1 + 3x_2$$

$$\frac{dx_2}{dt} = -2x_1 - 3x_2$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d}{dt} \underline{x} = A \underline{x}$$

$$\frac{dx}{dt} = a \underline{x} \quad \text{eigenvalue} \rightarrow L \underline{x} = a \underline{x}$$

$$\frac{dx}{dt} = \underline{\lambda} \underline{x} \quad a \rightarrow \underline{\lambda} \quad (\text{similar equations})$$

at

$$x = c e^{\underline{\lambda} t}$$

$$\underline{x} = c e^{\underline{\lambda} t} \underline{v} \quad \text{eigenvector}$$

$$\underline{x} = c e^{\underline{\lambda} t} \underline{v} \quad \text{eigen value}$$

$$\underline{x} = e^{\underline{\lambda} t} \underline{v} \quad \text{not } f(t)$$

$$\underline{x} = e^{\underline{\lambda} t} \underline{v} \quad \text{corresponding eigenvector}$$

$$\frac{d}{dt}(\underline{x}) = \frac{d}{dt}(e^{\underline{\lambda} t} \underline{v}) = e^{\underline{\lambda} t} \underline{v} \quad \text{const}$$

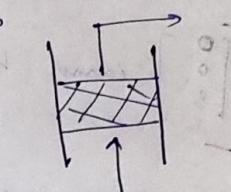
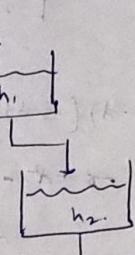
$$\underline{x} = (e^{\underline{\lambda} t} \underline{v}) \underline{A} = e^{\underline{\lambda} t} \underline{A} \underline{v} = \underline{\lambda} e^{\underline{\lambda} t} \underline{v}$$

$$\underline{x} = f(x, p, t)$$

$$p(x, t)$$

$$t(x, p)$$

$$\underline{\lambda} \underline{v} = \underline{\lambda} \underline{v}$$



$$xi$$

$$p$$

$$T$$

$$x = f(x, p, t)$$

$$p(x, t)$$

$$t(x, p)$$

$$\frac{dx}{dt} = Ax$$

$$x = \sum_{i=1}^n c_i e^{\lambda_i t} v_i$$

only for homogeneous problem.

$$A = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ -2 & -3-\lambda \end{vmatrix}$$

$$(2-\lambda)(-3-\lambda) + 6 = 0$$

$$-6 + \lambda^2 - 2\lambda + 3\lambda + 6 = 0$$

$$\lambda = 0 \quad \lambda = -1$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

cannot be an eigenvector

eigenvector
only gives one direction

$$A v_1 = 0 v_1$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2v_1 + 3v_2 = 0$$

$$\begin{aligned} v_2 &= x \\ v_1 &= -\frac{3}{2}x \end{aligned}$$

$$v_1 = \alpha \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \alpha' \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A v_2 = (-1) v_2$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = - \begin{bmatrix} v \\ u \end{bmatrix}$$

$$2v + 3u = -v \quad 2v + 3u = 0$$

$$-2v - 3u = -u$$

$$3v + 3u = 0$$

$$-2v - 2u = 0$$

$$v + u = 0$$

$$v_2 = \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{0t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = -3C_1 + C_2 e^{-t} \\ x_2 = 2C_1 + C_2 e^{-t} \end{cases}$$

all the solutions lie in the vector space

Q) $\frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 5x = 0$

$$x(0) = C_1, \quad \frac{dx}{dt}|_0 = C_2 \rightarrow \text{IVP}$$

$$x(0) = C_1, \quad x(1) = S \rightarrow \text{BVP}$$

$$D^2 + 6D + 5 = 0$$

$$\text{Solve } (D = -1, -5)$$

$$x(t) = C_1 e^{-t} + C_2 e^{-5t}$$

1st order ODE
↳ always IVP

$$\text{let } \frac{dy}{dt} = y$$

$$\frac{dy}{dt} + cy + 5x = 0$$

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 \\ -5 & -c-\lambda \end{bmatrix} = 0$$

$$+\lambda(6+\lambda) + 5 = 0$$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = -1, -5$$

$$\begin{bmatrix} 0 & 1 \\ -5 & -c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_2 = -v_1$$

$$-5v_1 - cv_2 = -5v_1$$

$$v_1 + v_2 = 0$$

$$v_1 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 \\ -5 & -c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -5 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (-5)\alpha$$

$$v_2 = -5v_1$$

$$-5v_1 - cv_2 = -5v_2$$

$$\underline{v}_2 = \beta \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

(Method)

Advantages → 2nd method why?

Find derivative directly
n-th order equation

$$\Rightarrow Q) \frac{d^2y}{dx^2} - y = 0 \Rightarrow y(x) = C_1 e^x + C_2 e^{-x}$$

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow y(x) = C_1 \cos x + C_2 \sin x$$

$$Q) \frac{dy}{dx} = 0y + 1z$$

$$\frac{dz}{dx} = y + 0z.$$

$$\frac{d}{dx} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

$$\lambda = 1, -1$$

$$\underline{v}_1 = 1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\underline{v}_2 = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\underline{v}_1 = \underline{v}_2$$

$$\rightarrow \underline{v}_1 = \underline{v}_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_1 = 1 \rightarrow \underline{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_2 = -1 \rightarrow \underline{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{\lambda_1 x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 x} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a) \frac{dy}{dx} = y$$

$$\frac{dy}{dx} = -y$$

$$\frac{d}{dx} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} iv_1 \\ iv_2 \end{bmatrix}$$

$$\underline{v_2 = iv_1} \quad \underline{-v_1 = iv_2}$$

$$\underline{v_1 = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} -iv_1 \\ -iv_2 \end{bmatrix}$$

$$\underline{v_2 = -iv_1}$$

$$\underline{v_1 = \beta \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{ix} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-ix} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$y = c_1' \cos x + i c_2' \sin x$$

$$\rightarrow y = z_1 + iz_2$$

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + ay = 0$$

$$\text{with } a_2 = \frac{d^2}{dx^2}(z_1 + iz_2) + a_1 \frac{d}{dx}(z_1 + iz_2) + a_0(z_1 + iz_2) = 0.$$

$$\begin{aligned} & a_2 \frac{d^2 z_1}{dx^2} + a_1 \frac{dz_1}{dx} + a_0 z_1 + i \left(a_2 \frac{d^2 z_2}{dx^2} + a_1 \frac{dz_2}{dx} + a_0 z_2 \right) = 0. \\ & + (i \frac{d^2 z_1}{dx^2} + i \frac{dz_1}{dx}) + (a_2 - (i \frac{d^2 z_2}{dx^2} + i \frac{dz_2}{dx})) = 0 \end{aligned}$$

\therefore both z_1 & z_2 are original solution.

$$\rightarrow \operatorname{Re} y = c_1 \cos x$$

$$\operatorname{Im} y = c_2 \sin x$$

$$y(x) = d_1 \cos x + d_2 \sin x$$

\Rightarrow Frobenius method

$$\frac{d^2 y}{dx^2} - y = 0$$

$$\frac{dy}{dx} + y = 0$$

$$\text{assume } y = \sum_{i=0}^{\infty} c_i x^{i+k}$$

$$\frac{dy}{dx} = \sum_{i=0}^{\infty} c_i (i+k) x^{i+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2} \rightarrow \text{substitute this in ODE}$$

$$\sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2} - \sum_{i=0}^{\infty} c_i x^{i+k} = 0$$

$$c_0 k(k-1)x^{k-2} + c_1 (k+1)kx^{k-1} + c_2 (2+k)(k+1)x^k +$$

$$c_3 (k+3)(k+2)x^{k+1} + c_4 (k+4)(k+3)x^{k+2} + \dots = 0$$

$$-c_0 x^k - c_2 x^{k+2} - c_4 x^{k+4} - \dots = 0$$

$\therefore c_0 k(k-1) = 0 \quad \& \quad c_4 (k+1)(k) = 0 \rightarrow \text{bcuz on the RHS no terms are there for power } k-2, k+1$

$$\rightarrow x^k (c_2(2+k)(k+1) - c_0) + x^{k+1} (c_3(k+3)(k+2) - c_1) +$$

$$x^{k+2} (c_4(k+4)(k+3) - c_2) = \dots = 0$$

$$c_2(2+k)(k+1) - c_0 = 0 \quad \rightarrow \text{Individual coefft} = 0$$

$$c_4(k+4)(k+3) - c_2 = 0$$

$$c_3(k+3)(k+2) - c_1 = 0$$

$$c_4 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} c_0$$

so,

$$n(n-1)c_n - c_{n-2} = 0$$

$$c_n = \frac{c_0}{n!} \rightarrow n = \text{even}$$

$$\frac{n=0 \text{ odd}}{c_n = \frac{c_1}{n!}}$$

$$c_3 = \frac{1}{3!} c_0$$

$$c_4 = \frac{1}{4!} c_0$$

$$c_6 = \frac{1}{6!} c_0$$

$$c_3 = \frac{1}{3!} c_1$$

$$c_4 = \frac{1}{4!} c_1$$

$$c_7 = \frac{1}{7!} c_1$$

$$\rightarrow y = \sum_{i=0}^{\infty} c_i x^i$$

$$y = c_0 + c_1 x + \frac{1}{2!} c_2 x^2 + \frac{1}{3!} c_3 x^3 + \dots$$

$$y = c_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) + c_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right)$$

$$y = c_0 \cosh x + c_1 \sinh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}$$

initial value conditions will not work

$$y = (c_0 + c_1) e^x + \left(\frac{c_0 - c_1}{2} \right) e^{-x} \quad \boxed{y = d_1 e^x + d_2 e^{-x}}$$

basis & orthogonal

why this method?

$$\frac{dy}{dx} + p(x)y = 0 \quad (\text{linear homogeneous ODE})$$

$$\frac{dy}{dx} - q(x)y = 0 \quad (\text{non-homogeneous ODE})$$

won't be able to solve using other methods
use frobenius method

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_1, a_2, a_0 \in C^{1,2,1} \quad (\text{continuous})$$

$a_i \in f(x) \rightarrow$ use frobenius method can be used.

$a_i \in f(y) \rightarrow$ non linear case

STURM - LIOUVILLE THEORY

$\frac{dy}{dx} + y = 0 \rightarrow$ solved using 3 methods - (1)

$$① Ad^2 + bD + c = 0$$

② Converting this to simultaneous DE then into matrix eqn (eigenvalues & then eigenvectors)

③ Frobenius method

$\frac{dy}{dx} + \lambda^2 y = 0 \rightarrow \lambda$ is a parameter

$\frac{dy}{dx} + \lambda^2 y = 0 \rightarrow \lambda = \lambda(z) \rightarrow \lambda$ is a parameter
not a variable
(as it is not a function of x, y)

$\frac{dy}{dx} + \lambda^2 y = 0 \rightarrow \lambda = \lambda(z) \rightarrow \lambda$ is a variable
so use frobenius method

if this is dependent variable then the equation is nonlinear

④) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad n > 0 \quad$ - (2) (2nd order ODE)

⑤) $(1-x^2) \frac{dy}{dx} = 2x \frac{dy}{dx} + ny = 0 \quad -1 < x < 1$

(linear or non-linear systems are defined)
(by model equation not solution
($cy_1 - y = \sin x / \cos x$)

$$y = e^{\int c dx}$$

take $y = y_1 + y_2$
 \rightarrow y_1 will satisfy the ODE
then take $y = cy_2$

⑥) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \rightarrow$ Bessel equation

$$y = \sum_{i=0}^{\infty} c_i x^{i+k}, \quad \frac{dy}{dx} = \sum_{i=0}^{\infty} c_i (i+k) x^{i+k-1},$$

$$\frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2}$$

$$\rightarrow \sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k} + \sum_{i=0}^{\infty} c_i (i+k) x^{i+k} + \sum_{i=0}^{\infty} c_i x^{i+k+2} = 0$$

$$\sum_{i=0}^{\infty} c_i x^{i+k+2} = 0$$

$$c_0(k)(k-1) + c_0(k) - n^2 c_0 = 0 \rightarrow \text{coeff of } x^k.$$

$$c_0 k^2 - n^2 c_0 = 0 \rightarrow \text{coeff of } x^k.$$

$$c_0(k^2 - n^2) = 0.$$

$$c_0 = 0 \quad (\text{or}) \quad \boxed{k = \pm n}$$

k is known

Two solutions \rightarrow one involving n & one involving $-n$.

$$\boxed{k = +n}$$

$$\sum_{i=0}^{\infty} c_i (i+n)(i+n-1) x^{i+n} + \sum_{i=0}^{\infty} c_i (i+n) x^{i+n} - n^2 \sum_{i=0}^{\infty} c_i x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$$

coeff of x^{n+1} $\cancel{c_1(n+1)(n)} \quad \cancel{c_1(1+n)} - c_1 n^2 = 0$

$$c_1 ((n+1)^2 + 1 + n - n^2) = 0$$

$$\boxed{c_1 = 0}$$

(or) $n+1+n = 0$
 $n = -1/2$ \times not possible.

$\therefore c_1$ has to be 0.

$$\sum_{i=0}^{\infty} c_i (i^2 + i + n^2 - n) x^{i+n} + \sum_{i=0}^{\infty} c_i (i+n) x^{i+n} - n \sum_{i=0}^{\infty} c_i x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$$

$$\sum_{i=0}^{\infty} c_i (i^2 + 2in - i - n + n^2 + (i+n) - n^2) x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$$

$$\sum_{i=0}^{\infty} c_i (i^2 + 2in - i - n + n^2 + (i+n) - n^2) x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$$

coeff of x^{n+2} $\cancel{c_1(n+1)} + \cancel{c_2(n+2)} = 0$

$$c_2(4 + 4n) + c_1 = 0.$$

$$c_2 = \frac{-c_1}{4(n+1)}$$

Coeff of x^{n+3}

$$c_3(9 + 6n)x^{n+3} + c_2 x^{n+2}$$

$$\boxed{c_3(9 + 6n) + 4 = 0}$$

$$c_3 = \frac{-4}{3(2n+3)}$$

$$4 = 0 \rightarrow c_3 = 0$$

$$G_1 = G_2 = G_3 = G_4 = 0 \quad \dots = 0$$

$$\rightarrow C_4(16 + \epsilon n) x^{n+4} + C_2 x^{n+4}$$

$$C_4 = \frac{C_2}{8(n+2)}$$

$$C_2 = \frac{C_0}{4(n+1)(n+2)}$$

$$C_0 = \frac{C_0}{32(n+1)(n+2)}$$

$$C_0 = f(a) + \alpha$$

$$y(r) = (C_0 + C_2 r^2 + C_4 r^4 + \dots) x^n$$

Bessel function
for specific value of C_0

(a) A metallic rod of infinite length at temp T_0 is dipped in an infinite reservoir at temperature T_∞ . The radius of the rod is R . Determine the spatio-temporal evolution of the temperature of the rod if the governing equation is given by

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

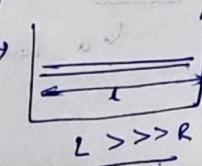
temp will be limited to T_0, T_∞

$\int_{T_0 \rightarrow \text{cold}}^{T_\infty \rightarrow \text{hot}} \dots$
 $\infty \text{ length}$
(large)

$\infty (n_0 + P) \dots$

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

cylindrically symmetric



$T_0 \rightarrow T_\infty \rightarrow L$
order of magnitude analysis

Compared to radial.

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)$$

$$T = X(t) Y(r)$$

homogeneous

$$Y \frac{dX}{dt} = \alpha \left(X \frac{d^2 Y}{dr^2} + \frac{1}{r} \frac{dY}{dr} X \right)$$

$$\frac{1}{X} \frac{dX}{dt} = \alpha \left(\frac{1}{Y} \frac{d^2 Y}{dr^2} + \frac{1}{rY} \frac{dY}{dr} \right) = \text{const}$$

$$\frac{\alpha}{Y} \left(\frac{d^2 Y}{dr^2} + \frac{1}{r} \frac{dY}{dr} \right) = -\beta \quad \text{as } \beta = \text{const}$$

$$\frac{d^2 Y}{dr^2} + \frac{1}{r} \frac{dY}{dr} - \frac{\beta}{\alpha} Y = 0$$

$$\frac{r^2}{\alpha} \frac{d^2 Y}{dr^2} + r \frac{dY}{dr} - \left(\frac{\beta r^2}{\alpha} \right) Y = 0$$

solution to this equation is bessel function.

$$Y = C_1 J_n(r) + C_2 J_{-n}(r)$$

from problem

zeros of $\cos \omega r$ have one zero of $\sin \omega r$ (Same thing happens with bessel function)



→ spheres $T = f(\theta, \phi)$

$$\text{we will get } (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + dy = 0 \quad -1 < x < 1$$

Legendre differential equation

solutions = Legendre polynomials

Instead of infinite series we get finite polynomial

bcz series get truncated

$$L_{n+1} = (-)^n L_n \rightarrow \text{recursion relation}$$

self adjoint operator

(λ) we get orthogonal polynomials.

(eigenvalues are real)

$$(x \cdot \frac{dy}{dx} + \frac{dy}{dx})$$

⇒ Any equation of the above kind can be written

$$\text{as } \hat{L} = \frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x) \rightarrow \text{Sturm-Liouville operator}$$

$$\hat{L} y + q(x) y = 0 \rightarrow \text{Sturm-Liouville problem}$$

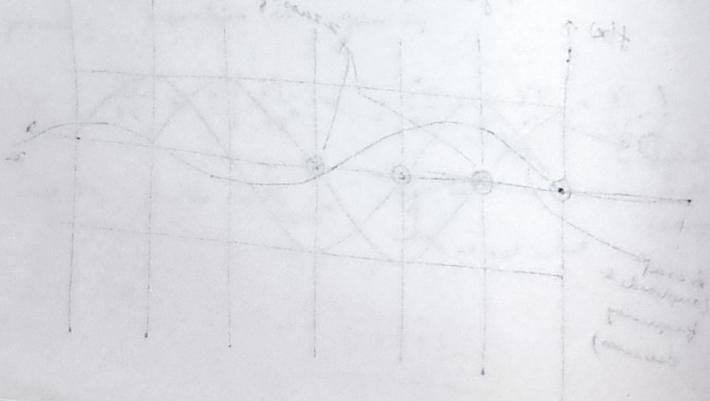
→ 4 independent variables in any problem $\rightarrow x, y, z, t$

(other things are parameters)

$$p(x) = \frac{dy}{dx} + \frac{q(x)}{y}$$

independently of each other

$$(x^2 \frac{d^2y}{dx^2} + 15) \frac{dy}{dx} + y = 0$$



$$p(x) = 11x^2 y^2 + \left[\frac{2}{x^2} y^2 \right] y^2$$

$$\left[\frac{2}{x^2} y^2 \right]^2 y^2 = \left[\frac{2}{x^2} y^2 \right] y^2 = 11x^2 y^2$$

$$11x^2 y^2 + \left[\frac{2}{x^2} y^2 \right] y^2 = \left[\frac{2}{x^2} y^2 \right] y^2$$

$$\left[\frac{2}{x^2} y^2 \right]^2 y^2 = \left[\frac{2}{x^2} y^2 \right] y^2$$

$$\left[\frac{2}{x^2} y^2 \right] y^2 = \left[\frac{2}{x^2} y^2 \right] y^2$$

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