13.1 Properties of Fourier Transform

We now list a number of properties of the Fourier transform that are useful in their manipulation.

1. Linearity: Let f and g are piecewise continuous and absolutely integrable functions. Then for constants a and b we have

$$F(af + bg) = aF(f) + bF(g)$$

Proof: Similar to the Fourier sine and cosine transform this property is obvious and can be proved just using linearity of the Fourier integral.

2. Change of Scale Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right), \quad a \neq 0$$

Proof: By the definition of Fourier transform we get

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{i\alpha x}dx$$

Substituting $ax = t$ so that $adx = dt$, we have

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\alpha\frac{t}{a}}\frac{dt}{a} = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right).$$

3. Shifting Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[f(x-a)] = e^{i\alpha a}F[f(x)]$$

Proof: By definition, we have

$$\begin{aligned} F[f(x-a)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x-a)e^{i\alpha x}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(t)e^{i\alpha(t+a)}dt = e^{i\alpha a}\hat{f}(\alpha) \end{aligned}$$

3. Duality Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[\hat{f}(x)] = f(-\alpha)$$

Proof: By definition of the inverse Fourier transform, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$$

Renaming x to α and α to x, we have

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-i\alpha x} dx$$

Replacing α to $-\alpha$, we obtain

$$f(-\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i\alpha x} dx = F[\hat{f}(x)].$$

13.2 Fourier Transforms of Derivatives

13.2.1 Theorem

If $f(x)$ is continuously differential and $f(x) \to 0$ as $|x| \to \infty$, then

$$F[f'(x)] = (-i\alpha)F[f(x)] = (-i\alpha)\hat{f}(\alpha).$$

Proof: By the definition of Fourier transform we have

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{i\alpha x} dx$$

Integrating by parts we obtain

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \left\{ \left[f(x) e^{i\alpha x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) e^{i\alpha x} (i\alpha) dx \right\}.$$

Since $f(x) \to 0$ as $|x| \to \infty$, we get

$$F[f'(x)] = -i\alpha \hat{f}(\alpha).$$

This proves the result. ∎

Note that the above result can be generalized. If $f(x)$ is continuously n-times differentiable and $f^k(x) \to 0$ as $|x| \to \infty$ for $k = 1, 2, ..., n-1$, then the Fourier transform of nth derivative is

$$F[f^n(x)] = (-i\alpha)^n \hat{f}(\alpha).$$

13.3 Convolution for Fourier Transforms

13.3.1 Theorem

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is $\sqrt{2\pi}$ times the product of the Fourier transforms of $f(x)$ and $g(x)$, i.e.,

$$F[f * g] = \sqrt{2\pi} F(f)F(g).$$

Proof: By definition, we have

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)g(x - y)\,\mathrm{d}y \right) e^{i\alpha x} \mathrm{d}x$$

Changing the order of integration we obtain

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x - y)e^{i\alpha x} \mathrm{d}x\,\mathrm{d}y$$

By substituting $x - y = t \Rightarrow dx = dt$ we get

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(t)e^{i\alpha(y+t)} \mathrm{d}t\,\mathrm{d}y$$

Splitting the integrals we get

$$F[f * g] = \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{i\alpha y}\,\mathrm{d}y \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{i\alpha t}\,\mathrm{d}t \right)$$

Finally we have the following result

$$F[f * g] = \sqrt{2\pi}\, F[f]F[g] = \sqrt{2\pi}\, \hat{f}(\alpha)\hat{g}(\alpha)$$

This proves the result. ∎

The above result is sometimes written by taking the inverse transform on both the sides as

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

or

$$\int_{-\infty}^{\infty} f(y)g(x - y)\mathrm{d}y = \int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

13.4 Parseval's Identity for Fourier Transforms

13.4.1 Theorem

If $\hat{f}(\alpha)$ *and* $\hat{g}(\alpha)$ *are the Fourier transforms of the* $f(x)$ *and* $g(x)$ *respectively, then*

$$(i) \int_{-\infty}^{\infty} \hat{f}(\alpha)\overline{\hat{g}(\alpha)}\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x \quad (ii) \int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} |f(\alpha)|^2\,\mathrm{d}\alpha.$$

Proof: (i) Use of the inversion formula for Fourier transform gives

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)}e^{i\alpha x}\,\mathrm{d}\alpha\right)\,\mathrm{d}x$$

Changing the order of integration we have

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\hat{g}(\alpha)}e^{i\alpha x}\,\mathrm{d}x\,\mathrm{d}\alpha$$

Using the definition of Fourier transform we get

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)}\hat{f}(\alpha)\,\mathrm{d}\alpha.$$

(ii) Taking $f(x) = g(x)$ we get,

$$\int_{-\infty}^{\infty} \hat{f}(\alpha)\overline{\hat{f}(\alpha)}\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} f(x)\overline{f(x)}\,\mathrm{d}x$$

This implies
$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |f(\hat{\alpha})|^2\,\mathrm{d}\alpha.$$

13.5 Example Problems

13.5.1 Problem 1

Find the Fourier transform of the following function

$$X_{[-a,a]}(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a. \end{cases} \tag{13.1}$$

Solution: By the definition of Fourier transform, we have

$$F\left[X_{[-a,a]}(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_{[-a,a]}(x) e^{i\alpha x} dx$$

Using the given value of given function we get

$$\begin{aligned} F\left[X_{[-a,a]}(x)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} (e^{i\alpha a} - e^{-i\alpha a}) \\ &= \frac{2}{\sqrt{2\pi}} \left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i\alpha} \right) = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin(\alpha a)}{\alpha} \right). \end{aligned}$$

13.5.2 Problem 2

Find the Fourier transform of e^{-ax^2}.

Solution: Using the definition of the Fourier Transform

$$F(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{i\alpha x} dx$$

Further simplifications leads to

$$\begin{aligned} F\left[e^{-ax^2}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{[-a(x-\frac{i\alpha}{2a})^2 - \frac{\alpha^2}{4a}]} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{4a}} \int_{-\infty}^{\infty} e^{-ay^2} dy = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}} \end{aligned}$$

If $a = 1/2$ then $F\left[e^{-\frac{1}{2}x^2}\right] = e^{-\frac{\alpha^2}{2}}$. This shows $F[f(x)] = f(\alpha)$ such function is said to be self-reciprocal under the Fourier transformation.

13.5.3 Problem 3

Find the inverse Fourier transform of $\hat{f}(\alpha) = e^{-|\alpha|y}$, where $y \in (0, \infty)$.

Solution: By the definition of inverse Fourier transform

$$\begin{aligned} F^{-1}\left[\hat{f}(\alpha)\right] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\alpha|y} e^{-i\alpha x} d\alpha \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\alpha y} e^{-i\alpha x} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha y} e^{-i\alpha x} d\alpha \end{aligned}$$

Combining the two exponentials in the integrands

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(y-ix)\alpha} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(y+ix)\alpha} d\alpha$$

Now we can integrate the above two integrals to get

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(y-ix)\alpha}}{(y-ix)}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(y+ix)\alpha}}{-(y+ix)}\right]_{0}^{\infty}$$

Noting $\lim_{\alpha\to-\infty} e^{(y-ix)\alpha} = 0$ and $\lim_{\alpha\to\infty} e^{-(y+ix)\alpha} = 0$, we obtain

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{y-ix} + \frac{1}{\sqrt{2\pi}} \frac{1}{y+ix}$$

This can be further simplified to give

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \frac{y+ix+y-ix}{(y-ix)(y+ix)}$$

Hence we get

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{y}{(x^2+y^2)}.$$