Understanding the Need for Optimal Rebalancing

What Is a Portfolio?

A **portfolio** is a collection of financial assets—such as equities, bonds, gold, or mutual funds—held by an investor to achieve a balance between **risk** and **return**.
Each asset contributes differently to overall performance; some offer higher expected returns but greater volatility, while others provide stability.

$$V(t) = V_A(t) + V_B(t)$$

where $V_A(t) = w_A(t) \times V(t)$ and $V_B(t) = w_B(t) \times V(t)$

where $w_A(t)$ and $w_B(t) = 1 - w_A(t)$ represent the fractions of total wealth invested in **Asset A** and **Asset B,** respectively, at time t.

Why Do Portfolio Weights Change?

Even if an investor initially chooses an **ideal mix** (say 60 % in stocks and 40 % in bonds), asset prices evolve at different rates.
When one asset outperforms the other, its share in total wealth increases—this is called **portfolio drift**.

For instance:

Day	**Value of Stocks**	**Value of Bonds**	**Total (₹)**	**Weight in Stocks**
0	60,000	40,000	1,00,000	0.60
1	63,000 (+5%)	39,200 (–2%)	1,02,200	0.616

After one day, the equity proportion rises from 0.60 to 0.616 without any trade.
If this continues, the portfolio becomes **riskier** than intended.

Rebalancing — The Corrective Mechanism

Rebalancing means adjusting portfolio weights back to their target values by buying or selling assets.
It is the only way to realign the portfolio with the investor's desired **risk–return profile**.

However, rebalancing introduces a dilemma:

Decision	Benefit	Drawback
Rebalance frequently	Keeps portfolio close to target; risk well-controlled	Incurs high transaction costs (brokerage, taxes, slippage)
Rebalance rarely	Saves transaction cost	Let's drift grow; higher portfolio risk and lower utility

Thus, investors constantly face a **trade-off** between the *cost of trading* and the *cost of being off target*.

When and how much should we rebalance?

Mathematically, this can be expressed as minimising the **total expected cost** over the investment horizon T:

$$\min_{\text{policy}} \ \mathbb{E}\left[\sum_{t=0}^{T} \Big(\text{Transaction Cost}_t + \text{Utility Shortfall}_t\Big)\right].$$

Traditional (Heuristic) Rebalancing Rules

1. **Calendar-Based Rebalancing**
 – Rebalance every month or quarter regardless of drift magnitude.
 – *Limitation:* ignores market volatility and current deviation.

2. **Tolerance-Band (± x %) Rules**
 – Rebalance only when weight deviates beyond a fixed band (e.g., ± 5 %).
 – *Limitation:* bands are static and arbitrarily chosen.

Both methods lack **adaptivity**: they do not account for current transaction-cost levels, volatility changes, or utility loss due to drift.

Hence, they can be **inefficient in dynamic markets.**

Why Rebalancing Matters in Practice

In real-world investing, **portfolios are never static.**
Institutional investors—mutual funds, pension funds, hedge funds, and wealth managers—manage **billions in assets** across diverse asset classes.
Market fluctuations continuously alter their portfolio weights.
If these weights drift too far from the strategic targets, **risk exposure**, **tracking error**, and **expected returns** deviate significantly from plan.

Thus, rebalancing is not optional—it's a **core risk-control mechanism**.

The Shortcomings of Static Policies

Let us analyse why *calendar* and *tolerance band* policies fail theoretically.

(a) Calendar Rebalancing:

Rebalancing on a fixed schedule (like every month) assumes markets move steadily. But in reality, markets have calm stretches and then sudden stormy periods, which makes this fixed strategy inefficient.

Mathematically, under volatility σ_t that varies with time:

$$E[|w_t - w^*|] \propto \sigma_t$$

Hence, rebalancing should be **more frequent when volatility is high**, less when stable — but calendar rules ignore this.

(b) Fixed Band Rebalancing:

A fixed ±x% band assumes the same cost–benefit trade-off regardless of cost levels or volatility.
But the optimal bandwidth **depends on TC(Transaction cost) and σ^2(volatility)**:

$$\text{Optimal Band Width} \propto \left(\frac{TC}{\sigma^2}\right)^{1/3}.$$

- When transaction costs (TC) are high, **the band gets wider** to avoid paying expensive fees too often.
- When volatility (**σ^2**) is high, **the band gets narrower,** i.e we need to **rebalance more frequently** to quickly control the high risk of drifting too far, too fast.

Hence, a 5% band may be optimal one year, but inefficient in another year

DP considers rebalancing as a **sequential decision problem**. (explained in a later section).

Problem Definition (Conceptual)

Now, after all the analyses above, the central question of our study is:-

> **How can an investor decide, at every point in time, whether to rebalance their portfolio or wait, in a way that minimises the total cost over the investment horizon?**

This is not a one-time optimization; it is a **sequential decision-making problem** that unfolds through time.

The Decision Environment

We consider a simple **two-asset fully invested portfolio** with:

- Asset A — riskier, higher expected return
- Asset B — safer, lower expected return

At any time t:

$$w_t = \text{weight in Asset A}, \quad (1 - w_t) = \text{weight in Asset B}$$

When markets move, the weight w_t drifts away from its desired value w^* (the mean–variance optimal weight).

The State, Action, and Transition

Term	Meaning	Mathematical Expression
State (x_t**)**	current portfolio composition	$x_t = w_t$
Action (a_t**)**	post-trade or "go-to" weight	$a_t \in [0, 1]$
Transition	How the next weight evolves after returns	$w_{t+1} = \dfrac{a_t(1 + r_{A,t+1})}{a_t(1 + r_{A,t+1}) + (1 - a_t)(1 + r_{B,t+1})}$

The Two Types of Costs

Every rebalancing decision leads to **two distinct costs**:

(a) Transaction Cost (TC)

The **cash cost** of trading, proportional to how much you adjust your portfolio.

$$TC_t(a_t; w_t) = C_A|a_t - w_t| + C_B|(1 - a_t) - (1 - w_t)|$$

where
C_A, C_B = per-unit trading cost for each asset (in decimals).

(b) Certainty-Equivalent Cost (CEC)

The **utility penalty** for being off the optimal risk–return mix.

Let $U(w)$ denote expected utility for weight w and $U_{opt} = U(w^*)$.
Then:

$$CEC_t(w_t) = [exp(U_{opt}) - exp(U_{w_t})] \times V_t$$

This measures the *loss in risk-adjusted wealth* by deviating from w^*

Total Cost Function

At each time t and decision a_t :

$$g_t(w_t, a_t) = TC_t(a_t; w_t) + CEC_t(a_t).$$

The investor seeks to choose actions $\{a_t\}_{t=0}^{T}$ that minimise the **total expected cost** over the horizon T:

$$\min_{\{a_t\}} E\left[\sum_{t=0}^{T} g_t(w_t, a_t)\right].$$

This is the **objective function** of our project.

Dynamic Programming Representation

We represent the problem recursively using a **value function** $J_t(w_t)$:

$$J_t(w_t) = \min_{a_t \in [0,1]} \Big[g_t(w_t, a_t) + E[J_{t+1}(w_{t+1})]\Big], \quad J_T(w_T) = CEC_T(w_T)$$

$g_t(w_t, a_t)$is the *immediate* cost of trading today.

$E[J_{t+1}(w_{t+1})]$, is the *expected future* cost of tomorrow's drift

Dynamic programming solves this **backwards in time**, ensuring globally optimal decisions.

The Optimal Policy

The solution to the above **Bellman recursion** gives a **policy function**:

Economically,
$\pi_t(w_t)$ = “If today my portfolio weight is w_t, what weight should I move to so that the sum of trading and risk costs is minimised in the long run?”

This policy has a **threshold (band) structure**, which we will visualise later.

Boundary Conditions

- At the final horizon T: $J_T(w_T) = CEC_T(w_T)$, since no further trades are possible.
- For costless markets (TC = 0) $a_t^* = w^*$, always rebalances perfectly.
- For infinite costs (TC → ∞): never rebalance ,$a_t^* = w_t$.
- A realistic case lies **between** these extremes.

Mathematical Setup and Notations

Model Overview

We consider a **two-asset portfolio** comprising:

- **Asset A:** relatively risky asset (e.g., equity)
- **Asset B:** relatively safe asset (e.g., bond or gold ETF)

At any discrete time $t \in \{0, 1, 2, \ldots, T\}$,
let w_t denote the fraction of total wealth invested in Asset A,
and hence $1 - w_t$ in Asset B.

The total portfolio value is denoted by V_t.

$$V_t = w_t V_t^A + (1 - w_t) V_t^B$$

Since we are fully invested $w_t + (1 - w_t) = 1$.

Asset Returns and Wealth Dynamics

Let:

$$r_{A,t+1} = \frac{P_{A,t+1} - P_{A,t}}{P_{A,t}}, \quad r_{B,t+1} = \frac{P_{B,t+1} - P_{B,t}}{P_{B,t}}$$

be the simple returns on each asset over a period $(t, t+1]$.

Then the portfolio's gross return is:

$$r_{p,t+1}(w_t) = w_t r_{A,t+1} + (1 - w_t) r_{B,t+1}.$$

After one period without trading:

$$V_{t+1} = V_t(1 + r_{p,t+1}).$$

The new **drifted weight** (before rebalancing at $t + 1$) becomes:

$$w_{t+1}^{\text{drift}} = \frac{w_t(1 + r_{A,t+1})}{w_t(1 + r_{A,t+1}) + (1 - w_t)(1 + r_{B,t+1})}.$$

New Stock Weight = (New Dollar Value of Stocks) / (New Total Dollar Value of portfolio)

This non-linear update is the key reason we use **Dynamic Programming** — today's decision affects tomorrow's state in a stochastic way.

Mean–Variance Framework

We summarise return statistics using a **mean vector** and a **covariance matrix**:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix}.$$

The portfolio mean and variance for a given weight w are:

$$\begin{aligned} \mu_p(w) &= [w, (1 - w)]\, \boldsymbol{\mu} = w\mu_A + (1 - w)\mu_B, \\ \sigma_p^2(w) &= [w, (1 - w)]\, \boldsymbol{\Sigma}\, [w, (1 - w)]^\top \\ &= w^2\sigma_A^2 + 2w(1 - w)\sigma_{AB} + (1 - w)^2\sigma_B^2. \end{aligned}$$

Optimal Weight in the Frictionless Case

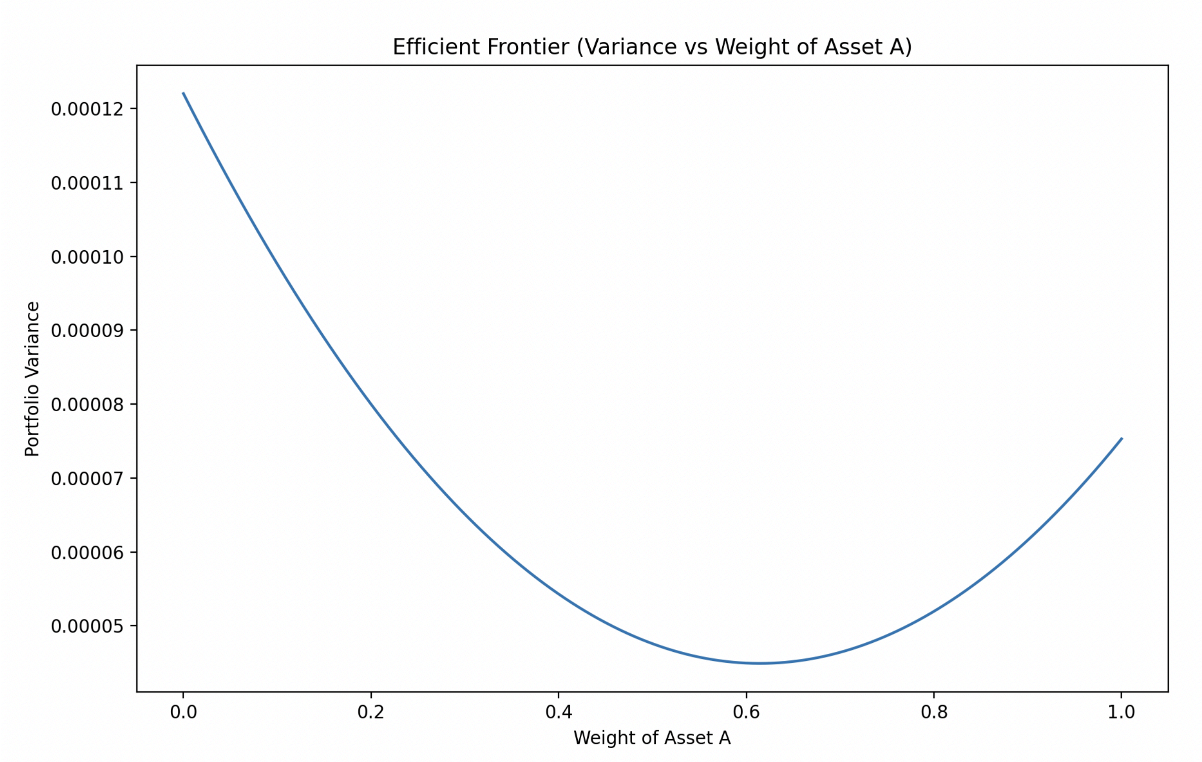
The **Global Minimum Variance (GMV) weight** w^* is obtained by minimising $\sigma_p^2(w)$ with respect to w:

$$\frac{d\sigma_p^2(w)}{dw} = 2w(\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}) - 2(\sigma_B^2 - \sigma_{AB}) = 0$$

$$\Rightarrow w^* = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}.$$

This w^* represents the **target weight** the investor would always hold if there were no trading costs.

Portfolio Variance vs. Weight of Asset A



Risk Preference: Expected Utility Approximation

To measure the investor's satisfaction with risk and return, we employ a **log-utility approximation**, which captures risk aversion in a tractable way.

$$U(\mu, \sigma^2) = \ln(1+\mu) - \frac{\sigma^2}{2(1+\mu)^2}.$$

For each portfolio weight w:

$$U(w) = U(\mu_p(w), \sigma_p^2(w)).$$

The **optimal utility** (at the GMV weight) is:

Certainty-Equivalent (CE) and Utility Gap

The **Certainty-Equivalent (CE)** wealth corresponding to a utility $U(w)$ is:

Hence, the **Certainty-Equivalent Cost (CEC)** for deviating from the optimal weight is:

$$CEC_t(w_t) = [exp(U_{opt}) - exp(U_{w_t})] \times V_t$$

This quantifies the “psychological” or *risk-adjusted* cost of being off-target, measured in monetary terms.

Risk Preferences and Utility: Incorporating Investor Psychology

While MVO captures the mean–variance trade-off, it doesn’t directly measure *how the investor feels about risk*.
In reality, investors maximise **expected utility**, not just minimise variance.

Hence, we move from MVO to **Expected Utility Theory**, using a log-utility function as a tractable, realistic choice for moderately risk-averse investors.

The Log-Utility Function

Utility of wealth W:

$$U(W) = \ln(W)$$

Expected utility over one period:

$$E[U(W_{t+1})] = E[\ln(W_t(1 + r_p))] = \ln(W_t) + E[\ln(1 + r_p)].$$

Ignoring constants and using a Taylor expansion for small r_p:

$$E[\ln(1 + r_p)] \approx \ln(1 + \mu_p) - \frac{\sigma_p^2}{2(1 + \mu_p)^2}.$$

Thus, the **approximate expected log-utility** is:

$$U(\mu_p, \sigma_p^2) = \ln(1 + \mu_p) - \frac{\sigma_p^2}{2(1 + \mu_p)^2}.$$

$$\pi_t(w_t) = \arg\min_{a_t} \Big[g_t(w_t, a_t) + E[J_{t+1}(w_{t+1})]\Big].$$

Portfolio Utility Function

Substitute the mean and variance of the portfolio:

$$U(w) = \ln(1 + [w, 1-w]\mu) - \frac{([w, 1-w]\Sigma[w, 1-w]^T)}{2(1 + [w, 1-w]\mu)^2}.$$

This function assigns a **utility score** to every possible weight w, combining expected return and volatility into a single metric.

Optimal Utility Weight

If there were no transaction costs, the investor would choose:

$$w^* = \arg\max_w U(w).$$

In practice, this w^* is close to the MVO(Mean-Variance Optimisation) solution (especially for small μ and σ), so we use it as our **benchmark utility-optimal weight**.

Quantifying Utility Loss

If the investor holds a suboptimal weight w_t instead of w^*,
The utility gap becomes:

$$\Delta U(w_t) = U(w^*) - U(w_t)$$

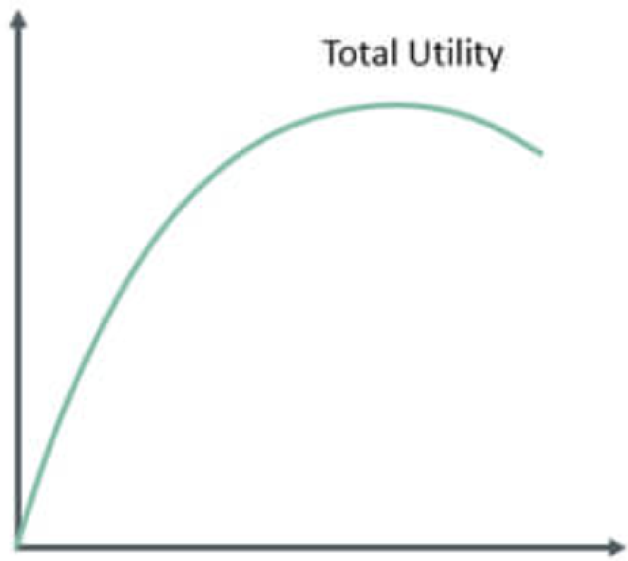
Multiplying by the current portfolio value V_t gives the **Certainty-Equivalent Cost**:

$$CEC_t(w_t) = [exp(U_{opt}) - exp(U_{w_t})] \times V_t$$

This is our **first cost component** — the *penalty for being off-optimal*.

Example: Utility vs Weight Curve

Imagine plotting $U(w)$ against w:



U(w)

w

- The curve is concave with a maximum at w^*.
- Moving away from w^* reduces utility → that drop is the *risk penalty* captured by CEC.

Defining the Optimal Rebalancing Rule

We seek:

$$a_t^* = \arg\min_{a_t}[TC_t(a_t; w_t) + CEC_t(a_t)]$$

In the multi-period (DP) framework, this expands to:

$$a_t^* = \arg\min_{a_t} \Big[TC_t(a_t; w_t) + CEC_t(a_t) + E[J_{t+1}(w_{t+1})]\Big].$$

But even in the static, single-period version, the logic is identical — we rebalance only when the marginal utility gain exceeds the transaction cost.

Dynamic Programming Formulation

Elements of the Dynamic Programming Model

Component	Meaning	Symbolic Representation
State variable	Current portfolio weight	$x_t = w_t$
Action variable	Post-trade (chosen) weight	$a_t \in [0, 1]$
Immediate cost	Trading + risk penalty today	$g_t(x_t, a_t) = TC_t(a_t; x_t) + CEC_t(a_t)$
Transition function	How tomorrow's weight evolves	$w_{t+1} = f(a_t, r_{A,t+1}, r_{B,t+1})$
Objective	Minimise total expected cost	$E[\sum g_t]$

The Bellman Optimality Equation

Let $J_t(w_t)$ represent the *minimum expected future cost* starting from time t.

Then, according to Bellman's principle of optimality:

$$J_t(w_t) = \min_{a_t \in [0,1]} \Big[g_t(w_t, a_t) + E[J_{t+1}(w_{t+1})] \Big],$$

with terminal condition:

$$J_T(w_T) = CEC_T(w_T).$$

- $g_t(w_t, a_t)$: Immediate, observable cost today (trade + utility loss).
- $E[J_{t+1}(w_{t+1})]$: Expected future cost if we follow the optimal policy thereafter.
- $J_t(w_t)$: Total minimal expected cost from now till horizon T.

DP solves this **backwards in time**, starting from T and working to $t = 0$.

Optimal Policy

The optimal post-trade weight (the “go-to” weight) is given by:

$$a_t^*(w_t) = \arg\min_{a_t} \Big[TC_t(a_t; w_t) + CEC_t(a_t) + E[J_{t+1}(w_{t+1})]\Big].$$

This mapping $\pi_t(w_t) = a_t^*(w_t)$ defines the **optimal rebalancing policy**.

Numerical Implementation

We discretise the weight domain $[0, 1]$ into G grid points (e.g., step size = 0.01).
At each grid point w_t:

1. Compute $g_t(w_t, a_t)$ for all candidate actions a_t.
2. Approximate expectation $E[J_{t+1}(w_{t+1})]$ using mean–variance assumptions for returns.
3. Select the a_t one that minimises the total.

Complexity: $O(T \times G^2)$, but can be reduced using convexity or interpolation.

The No-Trade Region and Threshold Policy

Deriving the Thresholds

The thresholds occur where the **marginal benefit of trading** = the **marginal cost of trading**.

Mathematically:

$$\frac{\partial CEC_t(w)}{\partial w}\Big|_{w=L_t} = \frac{\partial TC_t(w; w_t)}{\partial w}\Big|_{w=L_t}$$

Minimum of total cost $g_t(w_t, a_t)$ = **go-to weight** a_t^*.

Analytical Insights

1. **Wider Bands**
   - When transaction cost ($C_A + C_B$) ↑, rebalancing becomes more expensive → **wider no-trade zone**.
2. **Narrower Bands**
   - When volatility (σ) ↑, utility loss increases faster → **narrower band** (trade more frequently).
3. **Asymmetry**

   - If assets are not symmetric (different volatilities or costs), L_t and H_t are asymmetric.

Economic Interpretation

The band policy mimics rational human behaviour:

- You don't rebalance for every small drift.
- You wait until the deviation becomes "painful enough" (utility loss outweighs trading cost).
- When that happens, you trade *just enough* to restore efficiency.

This makes the policy **parsimonious** and **intuitive**, aligning with the behaviour of real fund managers.

Simplified Heuristic Approximation

For small drift variance, theory (Constantinides, 1986) gives approximate proportionality:

$$\text{Band Width} \propto \left(\frac{TC}{\sigma^2}\right)^{1/3}.$$

This relationship elegantly captures intuition:
Higher costs → fewer trades; higher volatility → more trades.

Key Takeaway

The optimal rebalancing strategy derived through DP has a **threshold structure**:

- A *no-trade zone* where inertia is optimal.
- *Rebalancing boundaries* (L, H) is determined endogenously by model parameters.
- *Go-to weight* a_t^* defines where to move when rebalancing.

This structure is the **quantitative backbone** of our project's simulation results.

Numerical Procedure and Algorithm

Step	Action	Output
1	Import historical/simulated prices for Assets A, B	Price series
2	Compute daily returns r_A, r_B	Return vectors
3	Estimate mean (μ) and covariance (Σ)	Risk–return parameters
4	Compute GMV weight w^*	Target weight
5	Simulate drift: evolve w_t without trading	Drift path
6	For each day and weight w∈[0,1]: – Compute $U(w)$ – Compute $CEC_t(w) = [U(w^*) - U(w)]V_t$ – Compute $(TC_t(w; w_t) = (C_A + C_B)$	w-w_t
7	Identify L_t, H_t where $CEC = TC$; find $a_t^* = \arg\min(g_t)$	Band & Go-to weight
8	Apply band policy: – If $w_t \in [L_t, H_t]$: Hold – Else: Rebalance to a_t^*	Policy path
9	Record all trades and transaction costs	Trade log

10	Summarize: – Total TC – Average deviation – Number of trades	Evaluation metrics

Computational Complexity

For:

- T = number of periods
- G = number of grid points for w

→ Time complexity ≈ $O(T \times G^2)$.

Optimizations:

- Use **symmetry** to halve the grid search.
- Use **vectorisation** (as in Python/Numpy).
- Use **interpolation** between grid points for smooth curves.

Outputs Produced

1. **Efficient Frontier Plot** – variance vs. weight curve; identifies w*.
2. **Weight Drift Chart** – shows deviation over time without trading.
3. **Cost Curves (CEC, TC, Total)** – illustrates intersection & optimal bands.
4. **Policy Simulation Plot** – compares “No-Rebalance” vs “DP Policy.”
5. **Trade Log Table** – lists days traded, trade size, TC, cumulative cost.

Evaluation Metrics

Metric	Formula	Interpretation
Total Transaction Cost	$\sum TC_t$	₹ cost of trading
Average Drift	$E[x]$	$w_t - w^*$

Utility Loss (CEC)	$E[U(w^*) - U(w_t)]$	Effectiveness of control
Turnover Ratio	sum	δw_t

Reference

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