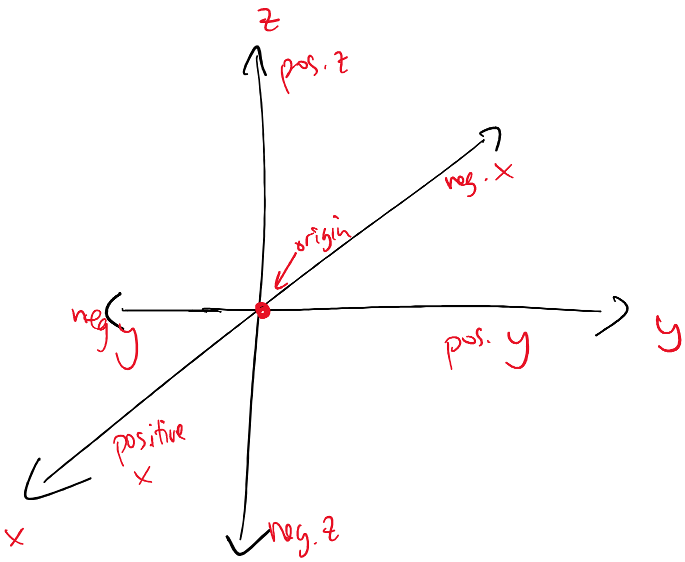
August 26 - 11.1

Monday, August 26, 2024 09:25

Ch 11 : Vectors, Lines, Planes

(no calculus in Ch.11)

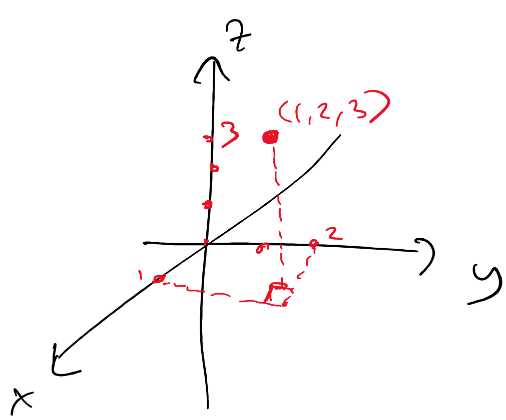


x-axis, y-axis, z-axis

each point in space is described by Cartesian coordinates

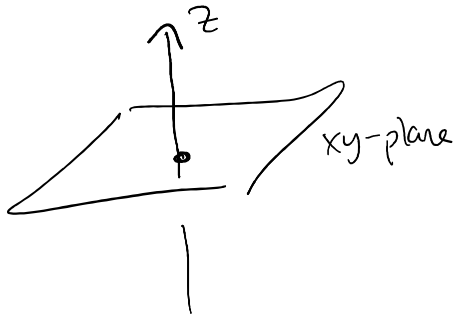
(x, y, z)

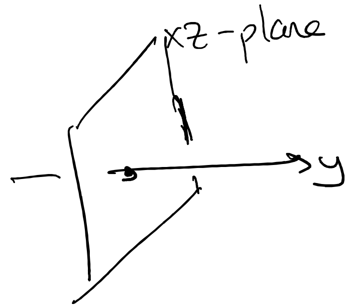
eg. $(1,2,3)$

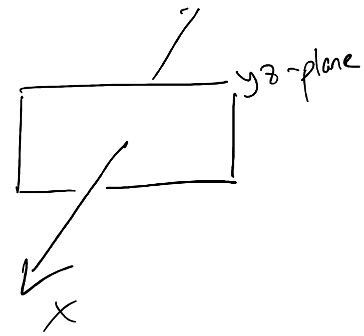


xy-plane = points where $z=0$, ie. $(x,y,0)$

Similarly: xz-plane & yz-plane



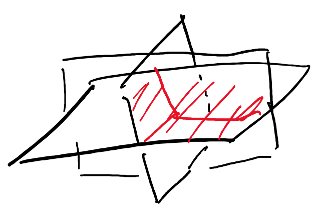




these 3 are called <u>coordinate planes</u>.

These planes divide space into eight regions called <u>octants</u>.

(cf. quadrants in 2D). The <u>first octant</u> is where $x, y, z > 0$.



<u>Distance</u>: between 2 points

if $P = (x, y, z)$ and $Q = (x', y', z')$

then $\boxed{|PQ| = \text{distance from } P \text{ to } Q = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.}$

eg. dist. from $P = (1, 0, -1)$ to $Q = (2, -3, -2)$ is eg. dist. from $P = (1,0,-1)$ to $Q = (2,-3,-2)$ is

$$|PQ| = \sqrt{(1-2)^2 + (0-(-3))^2 + ((-1)-(-2))^2}$$

$$= \sqrt{1 + 9 + 1} = \sqrt{11}.$$

eg. dist. from $O = (0,0,0)$ to $P = (x,y,z)$ is

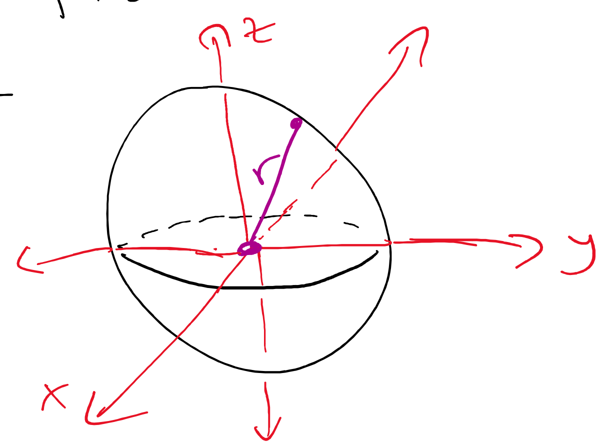
$$|OP| = \sqrt{(0-x)^2 + (0-y)^2 + (0-z)^2}$$

$$= \sqrt{x^2+y^2+z^2}$$

The sphere of radius r centered at O is all points distance r away from O, i.e. all $P = (x,y,z)$ satisfying

$$|OP| = r, \text{ i.e. } \sqrt{x^2+y^2+z^2} = r$$

i.e. $x^2+y^2+z^2 = r^2$



Similarly, sphere centered at $Q = (x_0, y_0, z_0)$ is all

$P = (x, y, z)$ dist. r from Q, so $|PQ| = r$

i.e. $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r$

i.e. $$\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.}$$

eg For what values of C is $x^2+y^2+z^2 = 2x-4y+4z+C$ a sphere? What is the center & radius?

Sol'n: Combine variables by completing the square:

$$(x^2-2x) + (y^2+4y) + (z^2-4z) = C$$

$$\to (x^2-2x+1) + (y^2+4y+4) + (z^2-4z+4) = C+1+4+4$$

$$\to (x-1)^2 + (y+2)^2 + (z-2)^2 = C+9$$

center $= (1, -2, 2)$

radius $= \sqrt{C+9}$ only a sphere when $C > -9$.

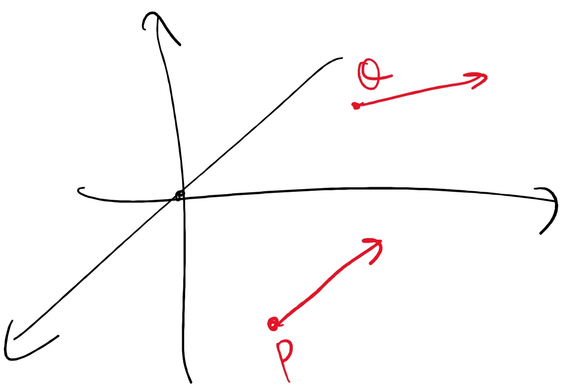
$c > -9$.

August 28 - 11.2

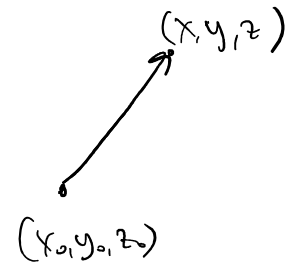
Wednesday, August 28, 2024 11:22

Vectors :

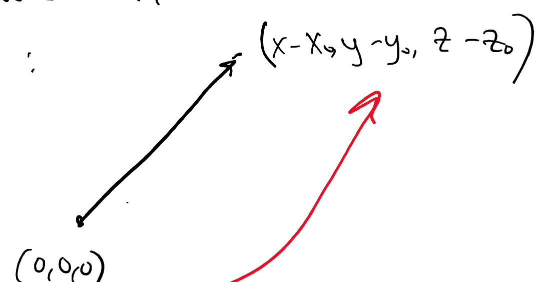
Vectors describe a direction and a magnitude from a point in space.



If we have



then we shift this vector to the origin :



use this to describe the vector.

⚠ we identify vectors that have different starting points.

Hence, a vector is described by an ordered triple (a,b,c), called the components of the vector.

$\overrightarrow{PQ}$ = vector starting at P and ending at Q

$\vec{PQ}$ = vector starting at P and ending at Q

if $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, then

$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

We write vectors by hand as $\vec{v}$ or $\overrightarrow{v}$ or $\bar{v}$

The <u>magnitude</u> or <u>norm</u> of a vector $\bar{v} = (a, b, c)$ is

$$\boxed{\|\bar{v}\| = \sqrt{a^2 + b^2 + c^2}.}$$

A <u>unit vector</u> has norm $= 1$.

A <u>scalar</u> is a real number.

The <u>zero vector</u> is $\bar{0} = (0, 0, 0)$.

<u>Arithmetic with vectors:</u>

If $\bar{v} = (v_1, v_2, v_3)$, $\bar{w} = (w_1, w_2, w_3)$, c = scalar,

then $\bar{v} \pm \bar{w} = (v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3)$

$c\bar{v} = (cv_1, cv_2, cv_3).$

<u>Properties</u>:

$\bar{v} + \bar{w} = \bar{w} + \bar{v}$

$\bar{0} + \bar{v} = \bar{v}$

$\bar{v} + (-1)\bar{w} = \bar{v} - \bar{w}$

$0 \cdot \bar{v} = \bar{0}$

$1 \cdot \bar{v} = \bar{v}$

$c(\bar{v} + \bar{w}) = c\bar{v} + c\bar{w}$

We have special vectors:

$$\boxed{\begin{aligned} \bar{i} &= (1,0,0) \\ \bar{j} &= (0,1,0) \\ \bar{k} &= (0,0,1) \end{aligned}}$$

all unit vectors.

if $\bar{v} = (v_1, v_2, v_3)$, then $\bar{v} = \underbrace{v_1\bar{i}}_{\bar{i}\text{-component}} + \underbrace{v_2\bar{j}}_{\bar{j}\text{-component}} + \underbrace{v_3\bar{k}}_{\bar{k}\text{-component}}$

<u>eg</u>. $P = (1,0,-1)$, $Q = (2,-3,-2)$

then $\overrightarrow{PQ} = ((2)-(1), (-3)-(0), (-2)-(-1))$

$= (1, -3, -1)$

$$= \bar{i} - 3\bar{j} - \bar{k}$$

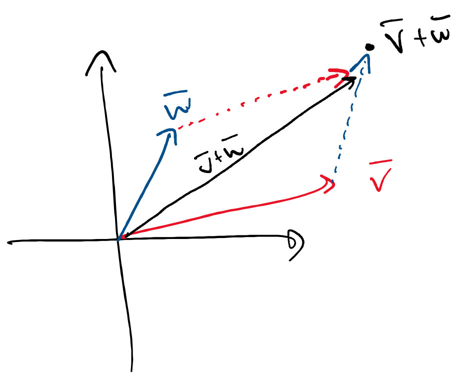
then $\|\overrightarrow{PQ}\| = \sqrt{(1)^2 + (-3)^2 + (-1)^2} = \sqrt{11}$.

eg. $\bar{v} = (3,4,0)$

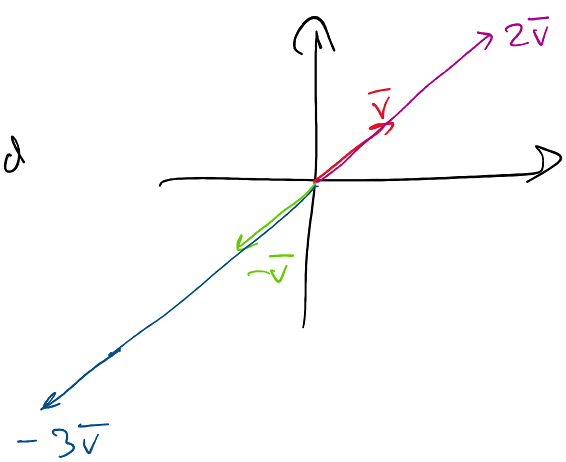
$$\|c\bar{v}\| = \|(3c, 4c, 0)\|$$
$$= \sqrt{(3c)^2 + (4c)^2} = \sqrt{25c^2} = 5|c|.$$

In general, $\boxed{\|c\bar{v}\| = |c| \cdot \|\bar{v}\|}$.

<u>Geometry of vectors</u>: (easier to see in 2D)



and



We say non-zero vectors $\bar{v}$ and $\bar{w}$ are <u>parallel</u> if

$\bar{v} = c\bar{w}$ (or $\bar{w} = c\bar{v}$) for some scalar c.

<u>eg.</u> Are $\bar{v} = (1, 2, -1)$ and $\bar{w} = (2, 4, 2)$ parallel?

<u>Sol'n</u>: If $\bar{w} = c\bar{v}$, then $(2,4,2) = (c, 2c, -c)$

$$\Rightarrow \begin{cases} 2 = c & \rightarrow c = 2 \\ 4 = 2c & \\ 2 = -c & \rightarrow c = -2 \end{cases} \quad \text{Contradiction.}$$

So they're not parallel.

The unit vectors parallel to any non-zero $\bar{v}$

are $\pm \dfrac{\bar{v}}{\|\bar{v}\|}$, since

$\dfrac{\bar{v}}{c}$ means $\frac{1}{c}\cdot\bar{v}$

$$\left\| \pm \frac{\bar{v}}{\|\bar{v}\|} \right\| = \left| \pm \frac{1}{\|\bar{v}\|} \right| \cdot \|\bar{v}\| = \frac{1}{\|\bar{v}\|} \cdot \|\bar{v}\| = 1$$

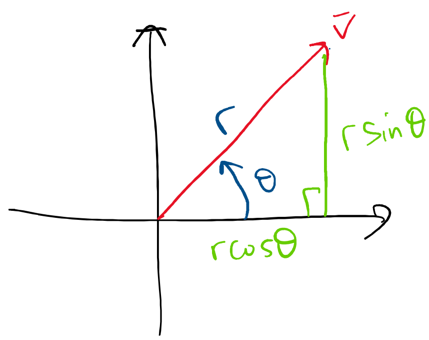
$$\boxed{\text{The } \underline{\text{unit vector in the direction of } \bar{v}} \text{ is } \frac{\bar{v}}{\|\bar{v}\|}.}$$

e.g. The unit vector in the direction of $\bar{v} = \bar{i} - 3\bar{j} - \bar{k}$

eg. The unit vector in the direction of $\bar{v} = \bar{i} - 3\bar{j} - \bar{k}$

is $\dfrac{\bar{v}}{\|\bar{v}\|} = \dfrac{\bar{v}}{\sqrt{11}} = \dfrac{1}{\sqrt{11}}\bar{i} - \dfrac{3}{\sqrt{11}}\bar{j} - \dfrac{1}{\sqrt{11}}\bar{k}$.

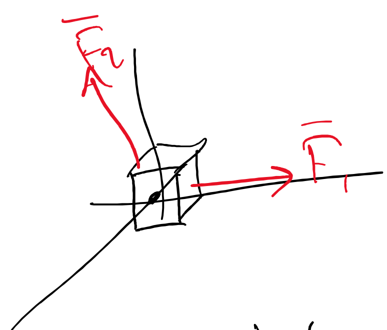
2D vectors: can be described by magnitude and angle (cf. polar coordinates)



so $\bar{v} = r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j}$

and $\|\bar{v}\| = r$.

Forces acting on a body:



$\bar{F}_1 = (0,3,0)$

$\bar{F}_2 = (1,0,2)$

want to apply a force $\bar{F}_3$ so that the object doesn't move, ie. forces cancel out.

Sol'n: want $\bar{F}_1 + \bar{F}_2 + \bar{F}_3 = \bar{0}$

$$\Rightarrow \bar{F}_3 = -\bar{F}_1 - \bar{F}_2$$

$$= -(0,3,0) - (1,0,2) = (-1,-3,-2).$$

August 30 - 11.3

Friday, August 30, 2024 11:59

11.3 : Dot Product

We saw $\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

The dot product of vectors $\bar{v}$ and $\bar{w}$ is

$$\boxed{\bar{v} \cdot \bar{w} = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

Then $\bar{v} \cdot \bar{v} = v_1^2 + v_2^2 + v_3^2$, so $\boxed{\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}}}$

eg. $\bar{v} = (1, 0, -1)$, $\bar{w} = (2, -3, -2)$, then

$$\begin{aligned} \bar{v} \cdot \bar{w} &= (1)(2) + (0)(-3) + (-1)(-2) \\ &= 2 + 0 + 2 \\ &= 4. \end{aligned}$$

Properties :

Properties:

$$\bar{v} \cdot \bar{w} = \bar{w} \cdot \bar{v}.$$

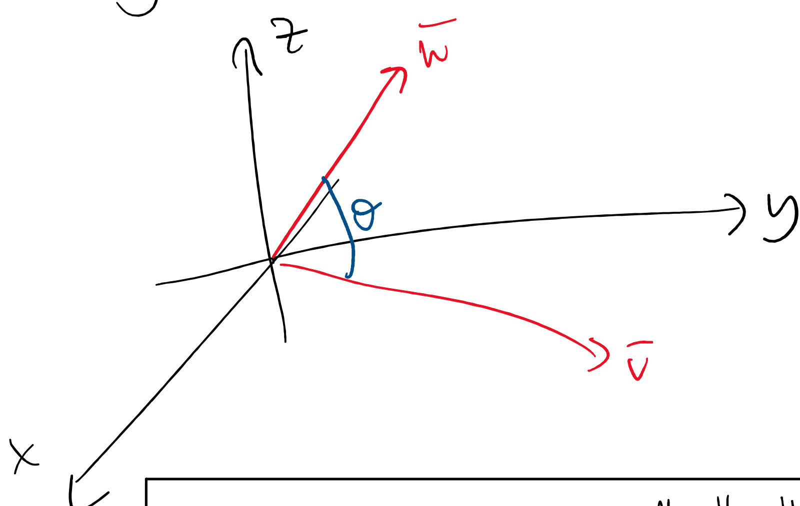
$$\bar{i} \cdot \bar{i} = 1, \quad \bar{j} \cdot \bar{j} = 1, \quad \bar{k} \cdot \bar{k} = 1$$

$$\bar{i} \cdot \bar{j} = 0, \quad \bar{i} \cdot \bar{k} = 0, \quad \bar{j} \cdot \bar{k} = 0.$$

$$\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$$

$$c\,(\bar{v} \cdot \bar{w}) = (c\bar{v}) \cdot \bar{w} = \bar{v} \cdot (c\bar{w})$$

Angles:



θ = angle between $\bar{v}$ and $\bar{w}$.

$$\boxed{\text{Theorem: } \quad \bar{v} \cdot \bar{w} = \|\bar{v}\| \cdot \|\bar{w}\| \cdot \cos\theta.}$$

We say $\bar{v}, \bar{w}$ are <u>perpendicular</u> or <u>orthogonal</u> if $\theta = \frac{\pi}{2}$.

We say $\bar{v}, \bar{w}$ are <u>perpendicular</u> or <u>orthogonal</u> if $\theta = \frac{\pi}{2}$.

<u>Theorem</u>: $\bar{v}, \bar{w}$ are orthogonal $\iff \bar{v} \cdot \bar{w} = 0$.

eg. Find the value of α for which

$$\bar{v} = 3\bar{i} - \alpha\bar{j} + \bar{k}$$

is orthogonal to $\bar{w} = \bar{i} + 2\bar{j} + 0\bar{k}$

<u>Sol'n</u>: want $\bar{v} \cdot \bar{w} = 0$

calculate: $\bar{v} \cdot \bar{w} = (3)(1) + (-\alpha)(2) + (1)(0)$

$$= 3 - 2\alpha$$

so set $3 - 2\alpha = 0 \rightsquigarrow \alpha = \frac{3}{2}$.

eg. Find the angle between $\bar{v} = (2,3,2)$ and $\bar{w} = (1,2,-1)$

<u>Sol'n</u>: $\bar{v} \cdot \bar{w} = \|\bar{v}\| \|\bar{w}\| \cos\theta \longrightarrow \boxed{\cos\theta = \frac{\bar{v} \cdot \bar{w}}{\|\bar{v}\| \|\bar{w}\|}}$

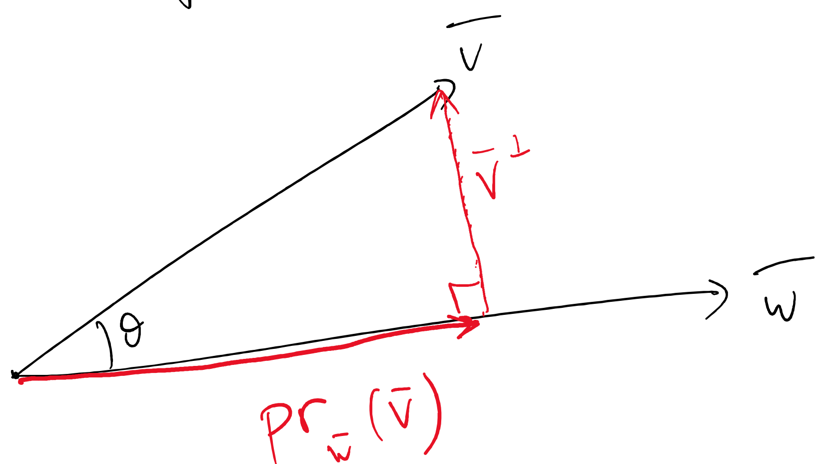
Here: $\bar{v} \cdot \bar{w} = 2 + 6 - 2 = 6$

$$\|\bar{v}\| = \sqrt{4+9+4} = \sqrt{17}$$

$$\|\bar{w}\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\Rightarrow \cos\theta = \frac{6}{\sqrt{17}\sqrt{6}} \Rightarrow \theta = \arccos\left(\frac{6}{\sqrt{17}\sqrt{6}}\right) \approx 54°.$$

Vector Projections:



$\hookrightarrow$ projection of $\bar{v}$ onto $\bar{w}$.

set up so that $\bar{v} = pr_{\bar{w}}(\bar{v}) + \bar{v}^{\perp}$

and $\bar{v}^{\perp} \cdot \bar{w} = 0$

and $pr_{\bar{w}}(\bar{v})$ is parallel to $\bar{w}$.

So $pr_{\bar{w}}(\bar{v}) = c\bar{w}$ for some c.

So $\quad pr_{\bar{w}}(\bar{v}) = c\,\bar{w}$ for some c.

and $\quad \|pr_{\bar{w}}(\bar{v})\| = c\|\bar{w}\| = \underbrace{\|\bar{v}\|\cos\theta}$

by trig. on the right triangle above.

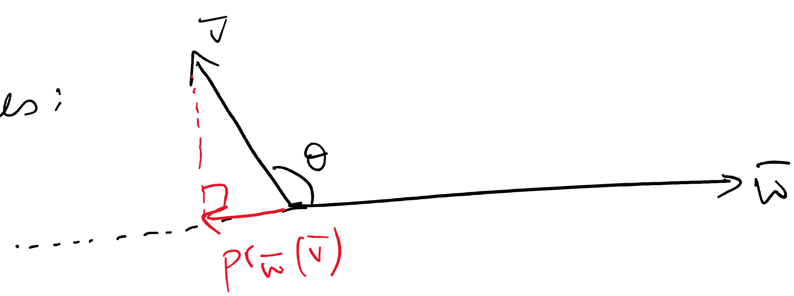
We know $\quad \cos\theta = \dfrac{\bar{v}\cdot\bar{w}}{\|\bar{v}\|\,\|\bar{w}\|}$.

$$\Rightarrow \quad c\|\bar{w}\| = \|\bar{v}\|\cdot\left(\frac{\bar{v}\cdot\bar{w}}{\|\bar{v}\|\,\|\bar{w}\|}\right).$$

$$\Rightarrow \quad c = \frac{\bar{v}\cdot\bar{w}}{\|w\|^2}.$$

$$\Rightarrow \quad \boxed{pr_{\bar{w}}(\bar{v}) = \left(\frac{\bar{v}\cdot\bar{w}}{\|\bar{w}\|^2}\right)\bar{w}.}$$

With obtuse angles:



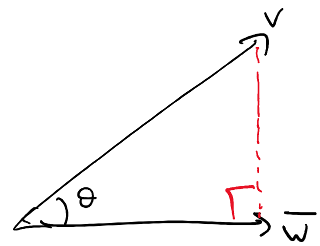
Same formula!

eg. Find the projection of $\bar{v} = (2,3,2)$ onto $\bar{w} = (1,2,-1)$.

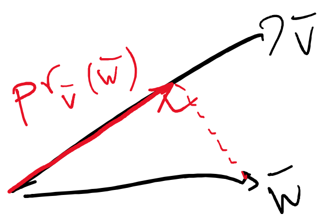
Sol'n: We already saw $\bar{v} \cdot \bar{w} = 6$, $\|\bar{w}\| = \sqrt{6}$

$$\Rightarrow \quad pr_{\bar{w}}(\bar{v}) = \left(\frac{\bar{v} \cdot \bar{w}}{\|\bar{w}\|^2}\right)\bar{w} = \left(\frac{6}{\sqrt{6}^2}\right)\bar{w} = \bar{w}$$

this corresponds to:



But $pr_{\bar{v}}(\bar{w}) = \left(\frac{\bar{v} \cdot \bar{w}}{\|\bar{v}\|^2}\right)\bar{v} = \left(\frac{6}{\sqrt{17}^2}\right)\bar{v} = \frac{6}{17}\bar{v}$

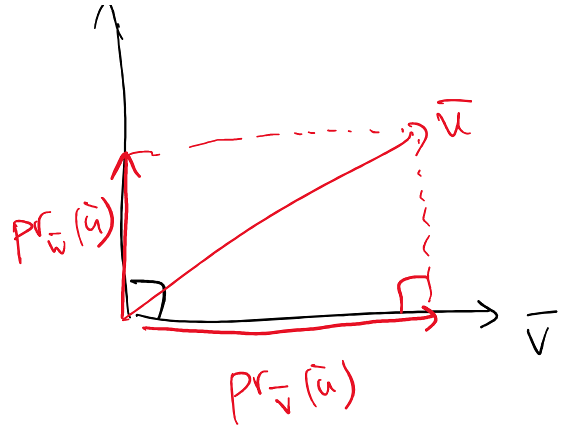


$$= \left(\frac{12}{17}, \frac{18}{17}, \frac{12}{17}\right)$$

eg. Rewrite $\bar{u} = 3\bar{i} + 3\bar{k}$ as $\alpha\bar{v} + \beta\bar{w}$

where $\bar{v} = \bar{i} + 2\bar{k}$, $\bar{w} = 2\bar{i} - \bar{k}$

Sol'n: Notice $\bar{v} \cdot \bar{w} = 0$.



Then $\bar{u} = pr_{\bar{v}}(\bar{u}) + pr_{\bar{w}}(\bar{u})$.

$$\left(\frac{9}{5}, 0, \frac{18}{5}\right) \qquad \left(\frac{6}{5}, 0, \frac{-3}{5}\right)$$

This is called <u>resolving $\bar{u}$ into vectors parallel to $\bar{v}$ & $\bar{w}$</u>.

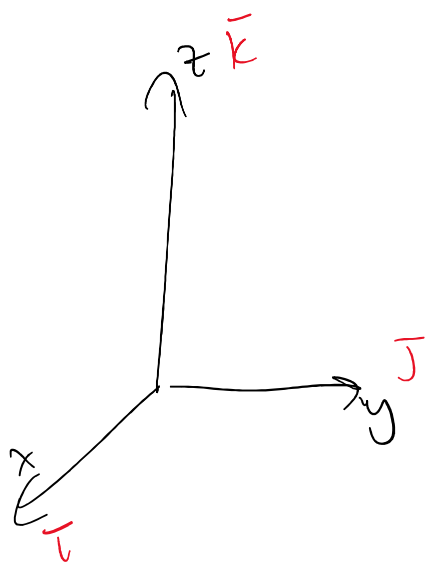
September 4 - 11.4

Wednesday, September 4, 2024 11:57

Quiz 1 tomorrow - on 11.1 - 11.3

11.4 : Cross Product

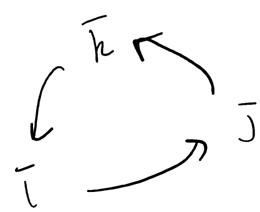
The cross product of two vectors is a vector.



$\bar{i} \times \bar{i} = \bar{0}$, $\bar{j} \times \bar{j} = \bar{0}$, $\bar{k} \times \bar{k} = \bar{0}$

and $\bar{i} \times \bar{j} = \bar{k}$, $\bar{j} \times \bar{k} = \bar{i}$, $\bar{k} \times \bar{i} = \bar{j}$

and $\bar{j} \times \bar{i} = -\bar{k}$, $\bar{k} \times \bar{j} = -\bar{i}$, $\bar{i} \times \bar{k} = -\bar{j}$



CCW $\rightarrow$ + sign

CW $\rightarrow$ - sign

eg. $\bar{v} = \bar{i} - \bar{j}$, $\bar{w} = \bar{i} + \bar{k}$, then

$$\bar{v} \times \bar{w} = (\bar{i} - \bar{j}) \times (\bar{i} + \bar{k})$$

$$= \bar{i} \times \bar{i} + \bar{i} \times \bar{k} - \bar{j} \times \bar{i} - \bar{j} \times \bar{k}$$

$$= (\bar{0}) + (-\bar{j}) - (-\bar{k}) - (\bar{i})$$

$$= -\bar{i} - \bar{j} + \bar{k}.$$

<u>Property</u> : $c(\bar{v} \times \bar{w}) = (c\bar{v}) \times \bar{w} = \bar{v} \times (c\bar{w})$

<u>Faster, using linear algebra</u>:

$$\boxed{\begin{aligned} \bar{v} \times \bar{w} &= (v_2 w_3 - v_3 w_2)\bar{i} + (v_3 w_1 - v_1 w_3)\bar{j} + (v_1 w_2 - v_2 w_1)\bar{k}. \\ &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \longleftarrow \text{determinant of } 3 \times 3 \text{ matrix} \end{aligned}}$$

Crash course on determinants :

of 2×2 : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

of 3×3 : $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

eg $(\bar{i} - \bar{j}) \times (\bar{i} + \bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \bar{i}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= \bar{i}\big((-1)(1) - (0)(0)\big) - \bar{j}\big((1)(1) - (0)(1)\big) + \bar{k}\big((1)(0) - (-1)(1)\big)$$

$$= -\bar{i} - \bar{j} + \bar{k}$$

<u>Properties</u> : $c(\bar{v} \times \bar{w}) = (c\bar{v}) \times \bar{w} = \bar{v} \times (c\bar{w})$

$$\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}$$

$$\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times \bar{w}$$

$$\bar{v} \times \bar{v} = \bar{0}$$

⚠ In general, $\bar{u} \times (\bar{v} \times \bar{w}) \neq (\bar{u} \times \bar{v}) \times \bar{w}$

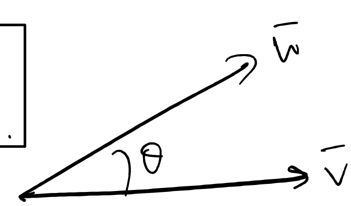
<u>eg</u>. $\bar{i} \times (\underline{\bar{i} \times \bar{j}}) = \bar{i} \times \bar{k} = -\bar{j}$

but $(\bar{i} \times \bar{i}) \times \bar{j} = \bar{0} \times \bar{j} = \bar{0}$

<u>Theorem</u> : • $\bar{v} \times \bar{w}$ is perpendicular to both $\bar{v}$ and $\bar{w}$.

so $\bar{v} \cdot (\bar{v} \times \bar{w}) = 0$, $\bar{w} \cdot (\bar{v} \times \bar{w}) = 0$.

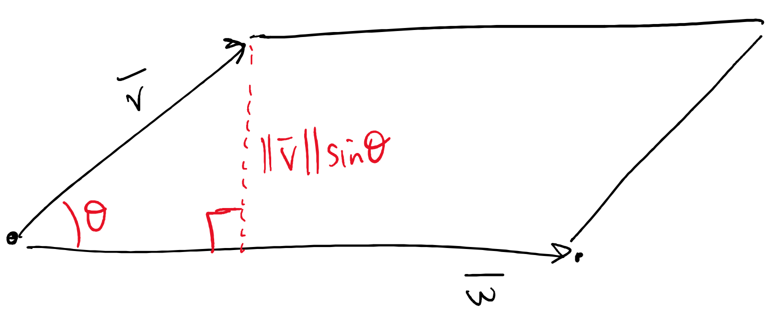
• $\boxed{\|\bar{v} \times \bar{w}\| = \|\bar{v}\| \cdot \|\bar{w}\| \cdot \sin\theta}$.



Hence: $\boxed{\bar{v} \,\&\, \bar{w} \text{ are parallel} \iff \theta = 0 \text{ or } \pi \iff \bar{v} \times \bar{w} = \bar{0}.}$

since $\sin(0) = \sin(\pi) = 0$

Parallelograms:

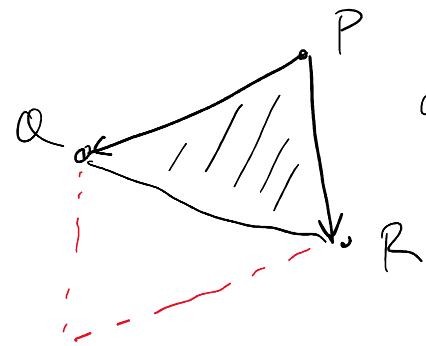


$$\text{Area} = \underbrace{\text{base}}_{\|\bar{w}\|} \times \underbrace{\text{height}}_{\|\bar{v}\|\sin\theta} = \|\bar{v}\|\,\|\bar{w}\| \sin\theta = \|\bar{v} \times \bar{w}\|.$$

eg. Find area of the triangle with vertices $P = (1,0,1)$, $Q = (2,-1,1)$, $R = (2,0,2)$.

Sol'n:

$\overrightarrow{PQ} = (2,-1,1) - (1,0,1)$
$= (1,-1,0)$



area(Δ) $= \frac{1}{2}$ area($\square$)

$\overrightarrow{PR} = (1,0,1)$

$\vec{PR} = (1,0,1)$

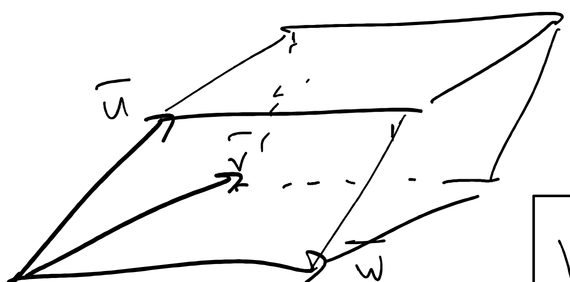
$\Rightarrow$ area $(\Delta) = \frac{1}{2}$ area (parallelogram made with sides $\vec{PQ}, \vec{PR}$)

$$= \frac{1}{2} \left\| \vec{PQ} \times \vec{PR} \right\| = \frac{1}{2} \left\| (\bar{i} - \bar{j}) \times (\bar{i} + \bar{k}) \right\|$$

$$= \frac{1}{2} \left\| -\bar{i} - \bar{j} + \bar{k} \right\| = \frac{1}{2}(\sqrt{3}) = \frac{\sqrt{3}}{2}.$$

as before

Triple Products:



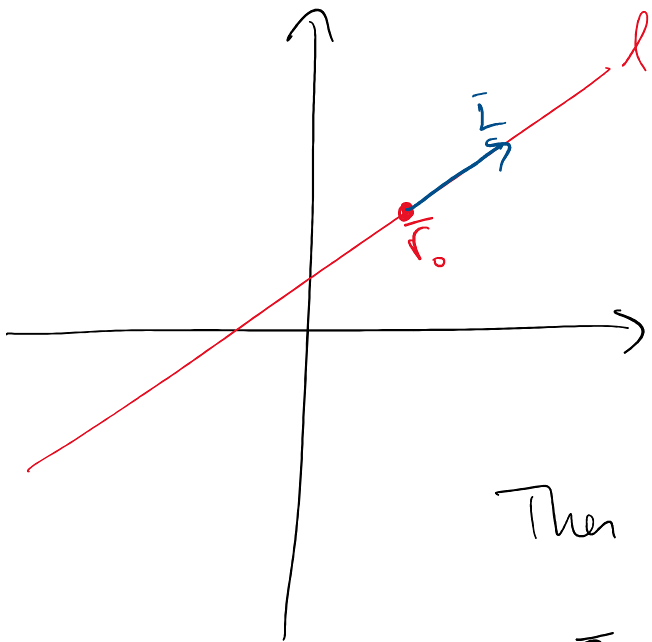
parallelepiped with sides $\bar{u}, \bar{v}, \bar{w}$.

$$\boxed{\text{Volume} = \left| \bar{u} \cdot (\bar{v} \times \bar{w}) \right|.}$$

September 6 - 11.5

Friday, September 6, 2024 11:58

11.5 : Lines in Space

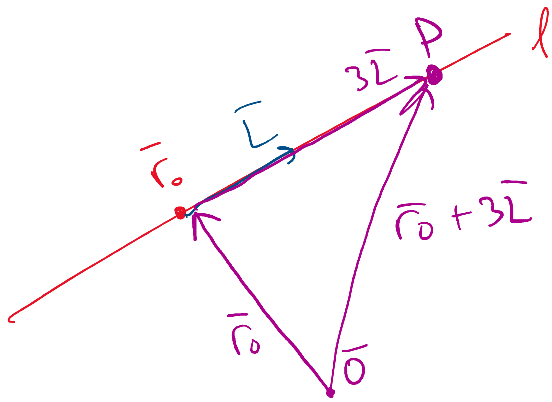


lines (in 2D/3D) are determined by a point $\bar{r}_0$ on the line and a direction vector $\bar{L}$ (a vector parallel to the line)

Then any point on the line is

$\bar{r}_0 + t\bar{L}$, for some real number t.

e.g.



The equation of a line in space is

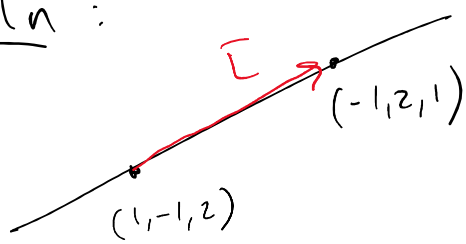
$$\bar{r}(t) = \bar{r}_0 + t\bar{L}.$$

As t varies through all real numbers, $\bar{r}(t)$ traces out all points on the line.

⚠ the choice of $\bar{r}_0$ and $\bar{L}$ is not unique.

eg. Find an equation of the line through $(1,-1,2)$ and $(-1,2,1)$.

Sol'n:



choose $\bar{r}_0 = (1,-1,2)$ (or $(-1,2,1)$)

choose $\bar{L}$ to be the vector between the two points.

$$\Rightarrow \bar{L} = (-1,2,1) - (1,-1,2) = (-2,3,-1)$$

So $\bar{r}(t) = \bar{r}_0 + t\bar{L} = (1,-1,2) + t(-2,3,-1)$

$$= (1,-1,2) + (-2t, 3t, -t)$$

$$= (1-2t, -1+3t, 2-t)$$

this is a <u>vector equation</u> of the line

Alternatively, we can write:

$$\begin{cases} x = 1-2t \\ y = -1+3t \\ z = 2-t \end{cases}$$

called parametric equations of the line.

t = parameter.

Alternatively, can solve for t and eliminate it:

$$x = 1-2t \rightsquigarrow t = \frac{x-1}{-2}$$

$$y = -1+3t \rightsquigarrow t = \frac{y+1}{3}$$

$$z = 2-t \rightsquigarrow t = \frac{z-2}{-1}$$

$$\Rightarrow \quad \frac{x-1}{-2} = \frac{y+1}{3} = \frac{z-2}{-1}$$

called symmetric equations for the line.

Recap : (1) $\bar{r}(t) = (x_0, y_0, z_0) + t\,(a, b, c)$ — vector equation

$$= (x_0 + at)\bar{i} + (y_0 + bt)\bar{j} + (z_0 + ct)\bar{k}$$

(2) $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ — parametric equations

(3) $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ — symmetric eq'ns.

eg. Find symmetric eq'ns of the line parallel to our previous example and passing through $(1,1,0)$.

Sol'n : parallel means same direction vector.

$\Rightarrow \bar{L} = (-2, 3, -1)$

Our point is $(1,1,0)$

$$\Rightarrow \quad \frac{x-1}{-2} = \frac{y-1}{3} = \frac{z-0}{-1}$$

or $\dfrac{z}{-1}$

eg Find Symmetric eq'ns of $\bar{r}(t) = (1,2,3) + t(1,0,2)$

Sol'n: $\dfrac{x-1}{1} = \dfrac{y-2}{0} = \dfrac{z-3}{2}$

not good! can't divide by 0

look at parametric eq'ns:

$$x = 1 + t \longrightarrow t = \frac{x-1}{1}$$
$$y = 2 + 0t = 2 \longrightarrow \text{can't solve for } t$$
$$z = 3 + 2t \longrightarrow t = \frac{z-3}{2}$$

so our symmetric eq'ns:

$$\frac{x-1}{1} = \frac{z-3}{3}, \quad y = 2$$

eg. Identify the line $x=1, y=2$

Sol'n: No restriction on z, so z can be anything.

Introduce a parameter t, with $z = t$

$$\Rightarrow \begin{cases} x = 1 + 0t \\ y = 2 + 0t \\ z = 0 + 1 \cdot t \end{cases}$$

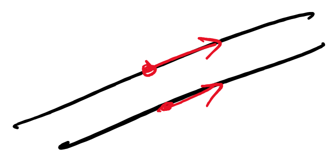
$\Rightarrow \bar{r}(t) = (1,2,0) + t\,(0,0,1)$

ie. the line parallel to z axis through the point $(1,2,0)$.

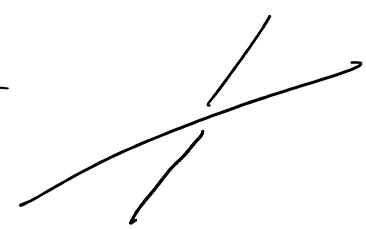
Two lines:

Lines that don't intersect are called:

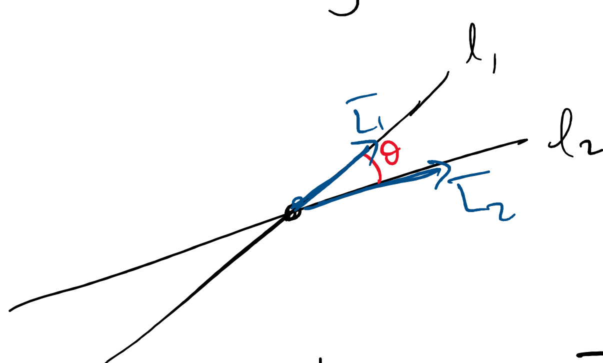
- **parallel** if they have parallel direction vectors.



- **Skew** if they don't



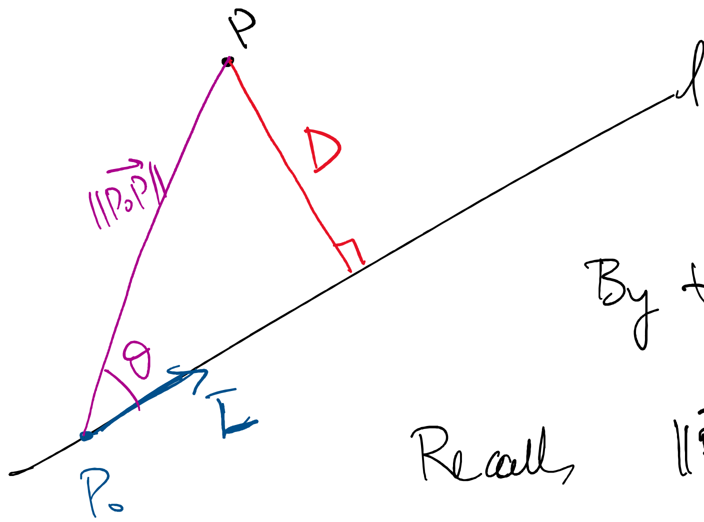
Angles between intersecting lines:



Sol'n: find angle between $\bar{L}_1$ and $\bar{L}_2$.

Sol'n: find angle between L_1 and L_2.

Distance from a point to the line:



By trig, $D = \|\overrightarrow{P_0P}\| \sin\theta$

Recall $\|\overrightarrow{P_0P} \times \bar{L}\| = \|\overrightarrow{P_0P}\| \cdot \|\bar{L}\| \cdot \sin\theta$

$$\Rightarrow \boxed{D = \frac{\|\overrightarrow{P_0P} \times \bar{L}\|}{\|\bar{L}\|}}$$

eg. Find the distance from $P = (1,0,2)$ to the line through $P_0 = (1,-1,1)$ parallel to $\bar{i} - 2\bar{j} - 2\bar{k}$

Sol'n: $\bar{L} = (1,-2,-2)$

$\overrightarrow{P_0P} = (1,0,2) - (1,-1,1) = (0,1,1)$

$$\overrightarrow{P_0P} \times \bar{L} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = \cdots\cdots = \bar{j} - \bar{k}$$

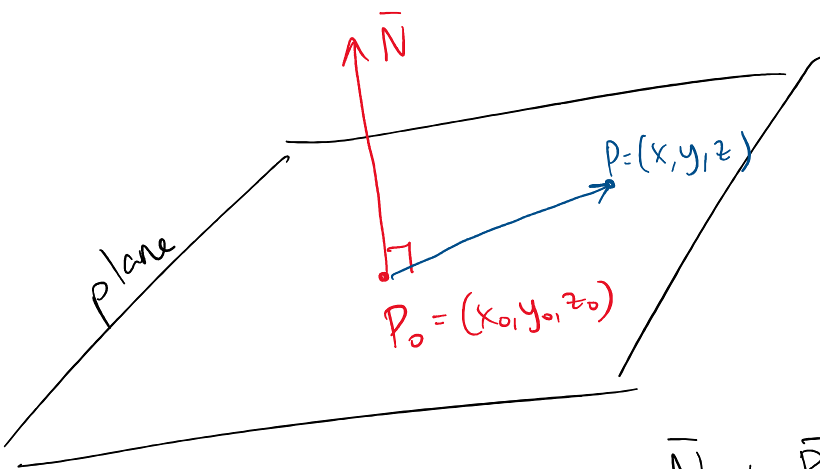
so $$D = \frac{\|\bar{j} - \bar{k}\|}{\|\bar{i} - 2\bar{j} - 2\bar{k}\|} = \frac{\sqrt{2}}{\sqrt{9}} = \frac{\sqrt{2}}{3}.$$

September 9 - 11.6

Monday, September 9, 2024 11:58

Quiz 2 on Thurs — on 11.4, 11.5

11.6 : Planes in Space



$\bar{N}$ = normal vector to the plane

= vector perpendicular to the plane

P_0 = point on the plane

$\bar{N} \perp \overrightarrow{P_0P}$ for any $P=(x,y,z)$ on the plane.

$\Longrightarrow$ $\boxed{\bar{N} \cdot \overrightarrow{P_0P} = 0}$ describes the plane.

if $N=(a,b,c)$, $P_0=(x_0,y_0,z_0)$, $P=(x,y,z)$

then $\bar{N}\cdot\overrightarrow{P_0P}=0$ becomes

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

or simplify to $\boxed{ax + by + cz = d}$

$= ax_0 + by_0 + cz_0.$

The plane consists of all points (x, y, z) that satisfy the equations above.

eg. Find some points on $2x + 3y - z = 7$

Soln: pick values for 2 variables, calculate the third.

eg $x = 1, y = 2 \Rightarrow 2(1) + 3(2) - z = 7$

$\Rightarrow 2 + 6 - z = 7 \Rightarrow z = 1$

$\Rightarrow (1, 2, 1)$ is on the plane.

eg. $x = 0, y = 0. \Rightarrow -z = 7 \Rightarrow z = -7$

$\Rightarrow (0, 0, -7)$ is on the plane

eg $x = 0, z = 0 \Rightarrow 3y = 7 \Rightarrow y = 7/3$

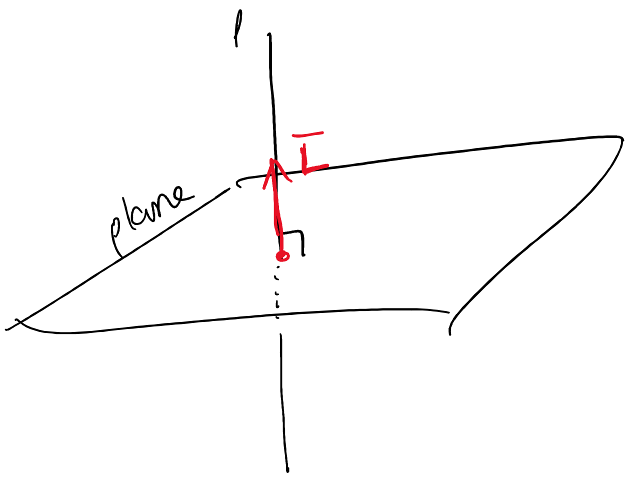
$\Rightarrow (0, 7/3, 0)$ is on the plane

eg $y = 0, z = 0 \Rightarrow 2x = 7 \Rightarrow x = 7/2$

$\Rightarrow (7/2, 0, 0)$ is on the plane.

eg Find an equation of the plane through $P_0 = (-2,3,5)$ perpendicular to the line
$$\begin{cases} x = -2 + t \\ y = 1 + 2t \\ z = 4 + 0t \end{cases}$$

Soln:



use direction vector $\bar{L}$ of the line as our normal vector $\bar{N}$ to the plane.

here: $\bar{L} = (1,2,0)$

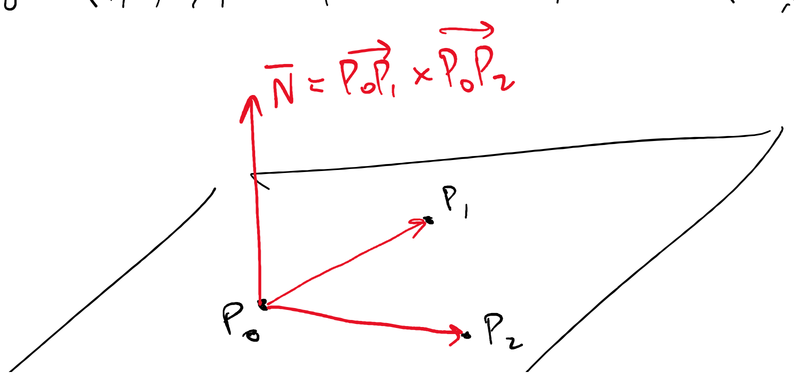
so use $\bar{N} = (1,2,0)$

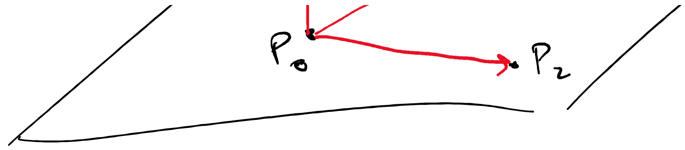
so the eq'n is $1\cdot(x-(-2)) + 2\cdot(y-3) + 0\cdot(z-5) = 0$

simplify $\Rightarrow$ $x + 2y = 4$.

eg. Find an eq'n of the plane passing through $P_0 = (1,0,2)$, $P_1 = (1,1,0)$, $P_2 = (2,0,2)$.

Soln:





Calculate $\overrightarrow{P_0P_1} = (0,1,-2)$, $\overrightarrow{P_0P_2} = (1,0,0)$

then set $\bar{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = (\bar{j} - 2\bar{k}) \times \bar{i}$

since cross prod. is perpendicular to both vectors

$$= \bar{j} \times \bar{i} - 2\bar{k} \times \bar{i}$$

$$= (-\bar{k}) - 2(\bar{j})$$

$$= -2\bar{j} - \bar{k} = (0,-2,-1)$$

$\Rightarrow$ eq'n of the plane is (using P_0 as the point):

$$0(x-1) + (-2)(y-0) + (-1)(z-2) = 0$$

$$\Rightarrow \quad 2y + z = 2.$$

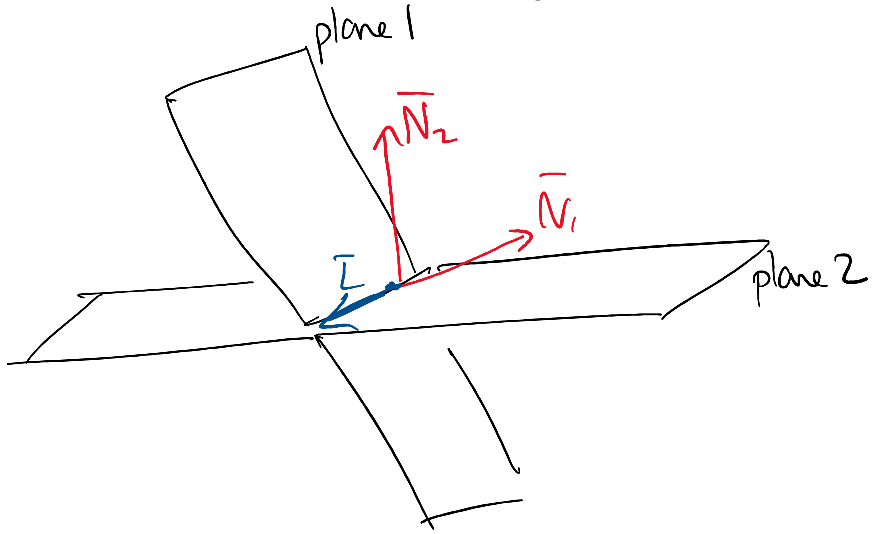
Intersecting Planes:

If planes are parallel, their normals are parallel

eg. $x+y+z=3$ is parallel to $-3x-3y-3z=4$

as $(1,1,1)$ is parallel to $(-3,-3,-3)$.

If they're not parallel, they intersect in a line.



$\overline{L}$ is in plane 1, so $\overline{L} \perp \overline{N}_1$

$\overline{L}$ is in plane 2, so $\overline{L} \perp \overline{N}_2$

$$\boxed{\Rightarrow \overline{L} \text{ is parallel to } \overline{N}_1 \times \overline{N}_2}$$

eg. Find the equation of the intersection of

$$2x - 3y + 2z = 9$$

and $$x + 2y - z = -4$$

Sol'n: $\overline{N}_1 = (2, -3, 2)$, $\overline{N}_2 = (1, 2, -1)$

these aren't parallel, so the planes intersect in a line.

Choose $\overline{L} = \overline{N}_1 \times \overline{N}_2 = \cdots = -\bar{i} + 4\bar{j} + 7\bar{k}$.

Now we need a point on the line, ie. a point on

Now we need a point on the line, ie. a point on both planes. So solve:

$$\begin{cases} 2x - 3y + 2z = 9 \\ x + 2y - z = -4 \end{cases}$$

To simplify, try setting $x = 0$.

$$\Rightarrow \begin{cases} -3y + 2z = 9 & (1) \\ 2y - z = -4 & (2) \end{cases}$$

$(1) + 2(2) \rightsquigarrow (-3y + 2z) + 2(2y - z) = 9 + 2(-4)$

$$\Rightarrow y = 1$$

$(2) \rightsquigarrow 2(1) - z = -4 \Rightarrow z = 6.$

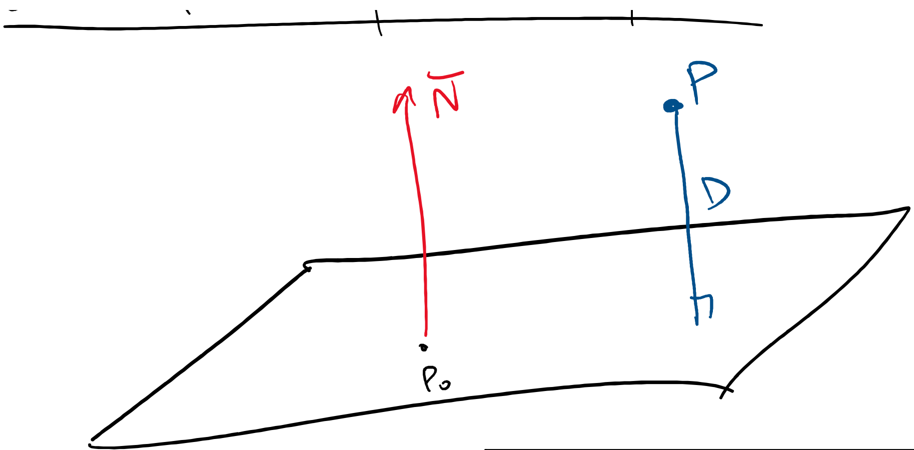
So $x = 0, y = 1, z = 6$ is a point on the line.

ie. $(0, 1, 6)$

Recall: $\vec{l} = (-1, 4, 7)$

So parametric eq'ns for the line: $\begin{cases} x = 0 - t \\ y = 1 + 4t \\ z = 6 + 7t \end{cases}$

Distance from a point to a plane:



works out that $\boxed{D = \left| \dfrac{\overrightarrow{P_0P} \cdot \bar{N}}{\|\bar{N}\|} \right|}$

eg. Find the distance from $P = (0,0,4)$ to $x + y + 2z = 2$.

Soln: Pick a pt on the plane. $P_0 = (0,0,1)$

$$\overrightarrow{P_0P} = (0,0,3)$$

$$\bar{N} = (1,1,2)$$

$$\overrightarrow{P_0P} \cdot \bar{N} = 6$$

$$\Rightarrow \quad D = \left| \frac{6}{\|\bar{N}\|} \right| = \frac{6}{\sqrt{1^2+1^2+2^2}} = \frac{6}{\sqrt{6}}.$$

September 11 - 12.1, 12.2

Wednesday, September 11, 2024 11:58

Quiz - on last week's material

Ch 12 : vector-valued functions

12.1 : Definitions and Examples

We've already seen lines $\bar{r}(t) = (x_0 + at)\bar{i} + (y_0 + bt)\bar{j} + (z_0 + ct)\bar{k}$

Generalize : $\bar{F}(t) = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$

for any real-valued functions f_1, f_2, f_3.

Called a vector-valued function.

eg. $\bar{F}(t) = (\cos t)\,\bar{i} + (\sin t)\,\bar{j} + t\,\bar{k}$

domain of $\bar{F}(t)$ = all t values you can plug in

f_1, f_2, f_3 = component functions of $\bar{F}(t)$

Can also use parametric equations:

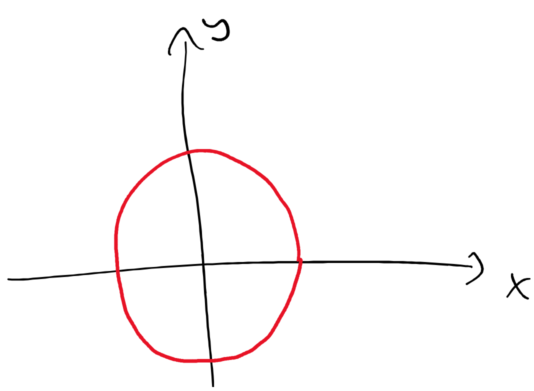
$$x = f_1(t)$$
$$y = f_2(t)$$
$$z = f_3(t)$$

eg $x = \cos t$, $y = \sin t$, $z = t$

If we plot all the points $\bar{F}(t)$ in space as t ranges over all real values (in the domain), we trace out a curve in space.

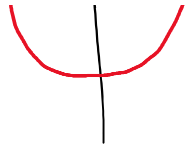
eg $\bar{F}(t) = \cos t\,\bar{i} + \sin t\,\bar{j}$ $(+0\bar{k})$

traces out a circle in xy-plane, counterclockwise.



$\bar{F}(0) = \cos 0\,\bar{i} + \sin 0\,\bar{j}$
$= \bar{i} + 0\bar{j} = \bar{i}$

$\bar{F}(\pi/2) = \bar{j}$, $\bar{F}(\pi) = -\bar{i}$



$\bar{F}(\pi/2) = \bar{j}$, $\bar{F}(\pi) = -\bar{i}$

$\bar{F}(\frac{3\pi}{2}) = -\bar{j}$, $\bar{F}(2\pi) = \bar{i}$, etc.

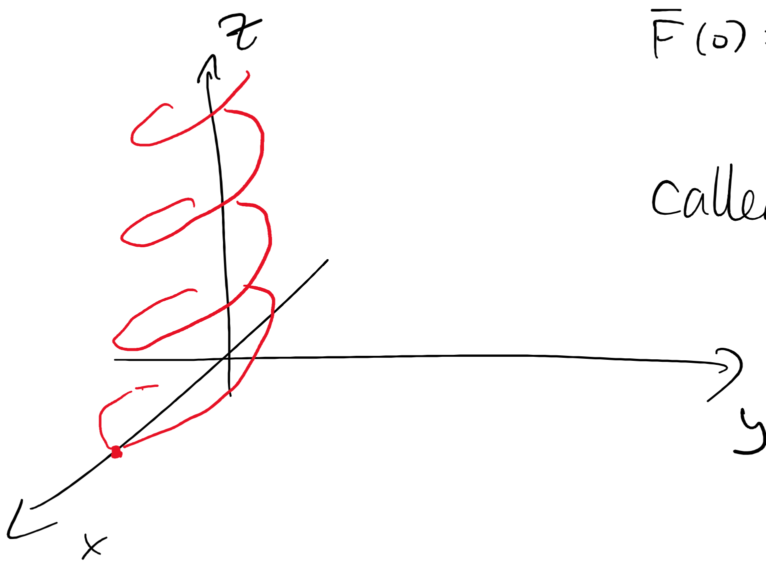
(think: $x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$.
and $x^2 + y^2 = 1$ gives a circle.)

(look at §6.9 in textbook)

eg. $\bar{F}(t) = \cos t \, \bar{i} + \sin t \, \bar{j} + t\bar{k}$

the xy-coords trace out a circle, but as t increases,
the z-coord. is increasing.

$\bar{F}(0) = \bar{i}$, $\bar{F}(\frac{\pi}{2}) = \bar{j} + \frac{\pi}{2}\bar{k}$



called a <u>circular helix</u>

<u>We can define:</u>

$(\bar{F} \pm \bar{G})(t) = \bar{F}(t) \pm \bar{G}(t)$ ⟵ vector-valued

$(\bar{F} \cdot \bar{G})(t) = \bar{F}(t) \cdot \bar{G}(t)$ ⟵ real-valued

$(\bar{F} \times \bar{G})(t) = \bar{F}(t) \times \bar{G}(t)$ ⟵ vector-valued

$(f\bar{F})(t) = f(t)\,\bar{F}(t)$ ⟵ vector-valued

$(\bar{F} \circ g)(t) = \bar{F}(g(t))$ ⟵ vector-valued

12.2 : Limits & Continuity

Want to understand $\lim_{t \to t_0} \bar{F}(t)$.

Thm: If $\bar{F}(t) = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$, then

$\lim_{t \to t_0} \bar{F}(t)$ exists $\Longleftrightarrow$ $\lim_{t \to t_0} f_i(t)$ exists for all $i = 1, 2, 3$

and $\lim_{t \to t_0} \bar{F}(t) = \left(\lim_{t \to t_0} f_1(t)\right)\bar{i} + \left(\lim_{t \to t_0} f_2(t)\right)\bar{j} + \left(\lim_{t \to t_0} f_3(t)\right)\bar{k}$.

eg. Find $\lim\limits_{t\to 0} \left(\cos(t+\pi)\,\bar{i} + \sin(t+\pi)\,\bar{j} + e^{-t^2}\,\bar{k} \right)$

Sol'n: $= \left(\lim\limits_{t\to 0} (\cos(t+\pi))\right)\bar{i} + \text{---}\ \bar{j} + \text{---}\ \bar{k}$

$$= \cos(0+\pi)\,\bar{i} + \sin(0+\pi)\,\bar{j} + e^{-(0)^2}\,\bar{k}$$

$$= -\bar{i} + \bar{k}$$

Properties: (write $\lim$ for $\lim\limits_{t\to t_0}$)

$$\lim\left(\bar{F}(t) \pm \bar{G}(t)\right) = \lim \bar{F}(t) \pm \lim \bar{G}(t)$$

$$\lim\,(f\bar{F})(t) = \left(\lim f(t)\right)\left(\lim \bar{F}(t)\right)$$

$$\lim\,(\bar{F}\cdot\bar{G})(t) = \lim \bar{F}(t) \cdot \lim \bar{G}(t)$$

$$\lim\,(\bar{F}\times\bar{G})(t) = \lim \bar{F}(t) \times \lim \bar{G}(t)$$

$$\lim_{t\to t_0} (\bar{F}\circ g)(t) = \lim_{s\to s_0} \bar{F}(s), \quad \text{if } \lim_{t\to t_0} g(t) = s_0$$

eg. $\bar{F}(t) = \sin t\,\bar{i} + \cos t\,\bar{j} + \tan t\,\bar{k}$

$\bar{G}(t) = e^t\,\bar{i} + te^t\,\bar{j} + t^2e^t\,\bar{k}$.

Find $\lim_{t \to 0} (\bar{F} \cdot \bar{G})(t)$ and $\lim_{t \to 0} (\bar{F} \times \bar{G})(t)$

Sol'n: $\lim_{t \to 0} \bar{F}(t) = \sin(0)\,\bar{i} + \cos(0)\,\bar{j} + \tan(0)\,\bar{k} = \bar{j}$

$\lim_{t \to 0} \bar{G}(t) = e^0\,\bar{i} + 0 \cdot e^0\,\bar{j} + 0^2 e^0\,\bar{k} = \bar{i}$

$\Rightarrow \lim_{t \to 0} (\bar{F} \cdot \bar{G})(t) = \bar{j} \cdot \bar{i} = 0$

$\lim_{t \to 0} (\bar{F} \times \bar{G})(t) = \bar{j} \times \bar{i} = -\bar{k}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \bar{i} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - \bar{j} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \bar{k} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= \bar{i}\,(1 \cdot 0 - 0 \cdot 0) - \bar{j}\,(0 \cdot 0 - 0 \cdot 1) + \bar{k}\,(0 \cdot 0 - 1 \cdot 1)$$

$$= -\bar{k}$$

Defn: A vector-valued function $\bar{F}(t)$ is <u>continuous</u> at $t = t_0$ if $\lim_{t \to t_0} \bar{F}(t) = \bar{F}(t_0)$.

Thm: $\bar{F}(t)$ is continuous $\Leftrightarrow$ each component function is continuous.

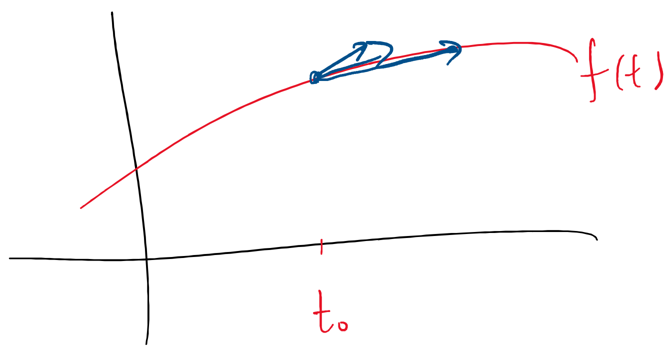
September 13 - 12.3

Friday, September 13, 2024 11:59

12.3 : Derivatives/Integrals

Recall : $f(t)$ is a real-valued function, then

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$



Now : If $\bar{F}(t)$ is a vector-valued function, we define

$$\bar{F}'(t_0) = \lim_{h \to 0} \frac{\bar{F}(t_0+h) - \bar{F}(t_0)}{h} = \lim_{t \to t_0} \frac{\bar{F}(t) - \bar{F}(t_0)}{t - t_0}.$$

The derivative $\bar{F}'(t_0)$ is a vector, pointing tangent to the curve $\bar{F}(t)$ in the same direction as $\bar{F}(t)$ is traced as t increases.

<u>Interpretation</u>: If $\bar{F}(t)$ is the location of a particle at time t, then $\bar{F}'(t_0)$ is the direction the particle is moving at time t_0.

If $\bar{F}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, then

$$\frac{\bar{F}(t) - \bar{F}(t_0)}{t - t_0} = \frac{(f_1(t) - f_1(t_0))}{t - t_0}\bar{i} + \frac{(f_2(t) - f_2(t_0))}{t - t_0}\bar{j} + \frac{(f_3(t) - f_3(t_0))}{t - t_0}\bar{k}$$

taking limits:

$$\boxed{\bar{F}'(t_0) = f_1'(t_0)\bar{i} + f_2'(t_0)\bar{j} + f_3'(t_0)\bar{k}}$$

eg $\bar{F}(t) = a\bar{i} + b\bar{j} + c\bar{k}$ for constants a, b, c.

then $\bar{F}'(t) = \frac{d}{dt}(a)\bar{i} + \frac{d}{dt}(b)\bar{j} + \frac{d}{dt}(c)\bar{k} = \bar{0}$.

eg. $\bar{F}(t) = t\bar{i} + \sqrt{t}\bar{j} - e^t\bar{k}$

then $\bar{F}'(t) = \bar{i} + \frac{1}{2\sqrt{t}}\bar{j} - e^t\bar{k}$.

eg. $\bar{F}(t) = f(t)\,\bar{v}$, where $\bar{v}$ is a constant vector.

like $\bar{F}(t) = \sin t\,(\bar{i} - \bar{j})$

then $\bar{F}'(t) = f'(t)\,\bar{v}$

so $\bar{F}'(t) = \cos t \cdot (\bar{i} - \bar{j})$

<u>Properties</u> : (omitting t)

$$(\bar{F} \pm \bar{G})' = \bar{F}' \pm \bar{G}'$$

$$(f\bar{F})' = f'\bar{F} + f\bar{F}'$$

product rule

$$(\bar{F} \cdot \bar{G})' = \bar{F}' \cdot \bar{G} + \bar{F} \cdot \bar{G}'$$

$$(\bar{F} \times \bar{G})' = \bar{F}' \times \bar{G} + \bar{F} \times \bar{G}'$$

$$(\bar{F} \circ g)' = g'\,(\bar{F}' \circ g) \quad \leftarrow \text{chain rule}$$

eg. $\bar{F}(t) = \sin t\,\bar{i} + \cos t\,\bar{j} + \tan t\,\bar{k}$

$\bar{G}(t) = e^t\,\bar{i} + te^t\,\bar{j} + t^2e^t\,\bar{k}$

Find $(\bar{F}\cdot\bar{G})'(0)$ and $(\bar{F}\times\bar{G})'(0)$.

Sol'n : long way : calculate $\bar{F}(t)\cdot\bar{G}(t)$ or $\bar{F}(t)\times\bar{G}(t)$
take deriv.
plug in $t=0$.

Instead : $\bar{F}(0)=\bar{j}$, $\bar{G}(0)=\bar{i}$

$\bar{F}'(t) = \cos t\,\bar{i} - \sin t\,\bar{j} + \sec^2 t\,\bar{k}$

$\bar{G}'(t) = e^t\,\bar{i} + (e^t + te^t)\bar{j} + (2te^t + t^2e^t)\bar{k}$

$\Rightarrow \bar{F}'(0) = \bar{i} + \bar{k}$

$\bar{G}'(0) = \bar{i} + \bar{j}$

then $(\bar{F}\cdot\bar{G})'(0) = \bar{F}'(0)\cdot\bar{G}(0) + \bar{F}(0)\cdot\bar{G}'(0)$

$= (\bar{i}+\bar{k})\cdot(\bar{i}) + (\bar{j})\cdot(\bar{i}+\bar{j}) = 1+1=2$

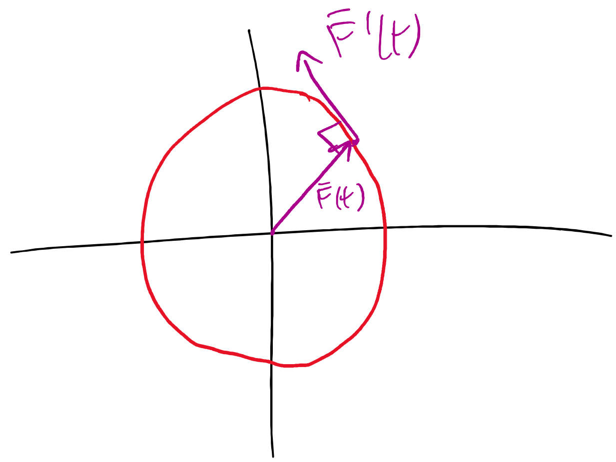
and $(\bar{F}\times\bar{G})'(0) = \bar{F}'(0)\times\bar{G}(0) + \bar{F}(0)\times\bar{G}'(0)$

$= (\bar{i}+\bar{k})\times(\bar{i}) + (\bar{j})\times(\bar{i}+\bar{j})$

$= \underbrace{\cancel{\bar{i}\times\bar{i}}}_{\bar{0}} + \bar{k}\times\bar{i} + \bar{j}\times\bar{i} + \underbrace{\cancel{\bar{j}\times\bar{j}}}_{\bar{0}}$

$= (\bar{j}) + (-\bar{k}) = \bar{j} - \bar{k}$.

<u>Fact</u> : $\boxed{\text{If } \|\bar{F}(t)\| = \text{constant, then } \bar{F}(t)\cdot\bar{F}'(t) = 0.}$



<u>Why?</u> $\|\bar{F}(t)\|^2 = \sqrt{\bar{F}(t)\cdot\bar{F}(t)}^{\,2} = \text{constant}^2$

$\Rightarrow \bar{F}(t)\cdot\bar{F}(t) = \text{const.}$

take derivs: $\underbrace{\bar{F}'(t)\cdot\bar{F}(t)} + \underbrace{\bar{F}(t)\cdot\bar{F}'(t)} = 0.$

same thing

$\Rightarrow \bar{F}(t)\cdot\bar{F}'(t) = 0$

<u>Physics interpretation</u>:

$\bar{F}(t)$ = <u>position</u> of a particle at time t

$\bar{v}(t) = \bar{F}'(t)$ = <u>velocity</u>.

$\|\bar{v}(t)\| = \underline{\text{speed}}$

$\bar{a}(t) = \bar{v}'(t) = \bar{F}''(t) = \underline{\text{acceleration}}$

Integrals :

$$\boxed{\int \bar{F}(t)\,dt = \left(\int f_1(t)\,dt\right)\bar{i} + \left(\int f_2(t)\,dt\right)\bar{j} + \left(\int f_3(t)\,dt\right)\bar{k}}$$

and same for definite integrals.

eg. $\int_0^1 t\bar{i} + \sqrt{t+1}\,\bar{j} - e^t\bar{k}\,dt$

$$= \left(\int_0^1 t\,dt\right)\bar{i} + \left(\int_0^1 \sqrt{t+1}\,dt\right)\bar{j} - \left(\int_0^1 e^t\,dt\right)\bar{k}$$

$$= \left[\frac{t^2}{2}\right]_0^1 \bar{i} + \left[\frac{2}{3}(t+1)^{3/2}\right]_0^1 \bar{j} - \left[e^t\right]_0^1 \bar{k}$$

$$= \frac{1}{2}\bar{i} + \frac{2}{3}\left(\sqrt{2}-1\right)\bar{j} + (1-e)\bar{k}.$$

Note: $\boxed{\text{If } \quad \bar{F}'(t) = \bar{G}(t), \quad \text{then} \qquad \int \bar{G}(t)\,dt = \bar{F}(t) + \bar{C}}$

constant vector.

eg. Newton's law says $\overline{Force}(t) = m \cdot \bar{a}(t)$ (*)

mass accel.

Find $\bar{r}(t)$ = position vector if $\overline{Force} = -mg\bar{k}$.

Sol'n: (*) $m \cdot \bar{r}''(t) = -mg\bar{k}$

$$\Rightarrow \bar{r}''(t) = -g\bar{k}$$

$$\Rightarrow \bar{r}'(t) = \int -g\bar{k}\,dt = -gt\bar{k} + \bar{C}_1$$

$$\Rightarrow \bar{r}(t) = -\tfrac{1}{2}gt^2\bar{k} + t\bar{C}_1 + \bar{C}_2.$$

$\bar{V}(0)$ initial velocity

$\bar{r}(0)$ initial position.

Table of Contents


```
%Person A, Person B, Person C
%Section 101
%Matlab Project 1
```

Problem 1

```
3+4
disp('We just computed a sum')
```

```
ans =

     7

We just computed a sum
```

Problem 2

```
A=[1 2; 3 4; 5 6]

A  % Shows the matrix again
A; % Suppressed;  Matrix A is not shown again
```

```
A =

     1     2
     3     4
     5     6


A =

     1     2
     3     4
     5     6
```

Problem 3

```
syms x
f(x)=sin(x)
```

```
f(pi)
disp('This shows that if we evaluate the sine function at pi, the value is 0')
```

```
f(x) =
sin(x)

ans =
0
This shows that if we evaluate the sine function at pi, the value is 0
```

Published with MATLAB® R2022b

Matlab Guide

Contents

This guide was prepared by Jamie Conway for Fall 2024, updating an older guide originally written by Justin Wyss-Gallifent.

1 Preliminary Stuff

1.1 Installing Matlab

The best way to use Matlab in this course is to download install it on your own computer. Every UMD student has free acccess to the latest version of Matlab to download and install. To do so, go to `https://terpware.umd.edu/`, and search for "matlab". Find "MATLAB for Students", and choose your operating system (Windows, Mac, or Linux). Follow the instructions on that page to download and install Matlab on your computer (you may need to make an account with Mathworks; use your UMD email address to do so).

1.2 I've opened Matlab; now what?

When you run Matlab you will see a bunch of windows. The important one will have a prompt in it which looks like `>>` (the Command Window). This is where we will tell Matlab what to do.

Go into this window and type "$2 + 3$" and press Enter. You will see:

```
>> 2+3

ans =

     5
```

Oh yeah, you know Matlab! Let's do something more relevant to calculus like take a derivative. Before we do that it's important to note that Matlab works a lot with what are called symbolic expressions. If we want to take a derivative we use the `diff` command. It's tempting to do `diff(x^2)` but this gives an error. Try it and see! Why does this happen? The reason is that Matlab doesn't know what `x` is and we must tell it that `x` is a symbol to work with. We do so with the `syms` command. So we can do the following two commands. The first line tells Matlab that `x` is symoblic and will be symbolic until we tell it otherwise.

```
>> syms x
>> diff(x^2)

ans =

     2*x
```

Matlab is very good at doing complicated derivatives (with `diff`) and integrals (with `int`), even with standard functions. The result is not always simplified, so you can have Matlab try to simplify it for you:

```
>> diff(2*x^2*sin(x)/cos(x))

ans =

2*x^2 + (2*x^2*sin(x)^2)/cos(x)^2 + (4*x*sin(x))/cos(x)


>> simplify(diff(2*x^2*sin(x)/cos(x)))
```

```
ans =

(2*x*(x + sin(2*x)))/cos(x)^2
```

Notice that you didn't have to type `syms x` again. As long as this session is open, Matlab will remember what you've told it about `x`.

You can factor polynomials:

```
>> factor(x^4-3*x^3-8*x^2+21*x+9)

ans =

(x^2 + 3*x + 1)*(x - 3)^2
```

And you can solve equations. Make sure to use `==` with two equal signs.

```
>> solve(x^2+x==2)

ans =

-2
 1
```

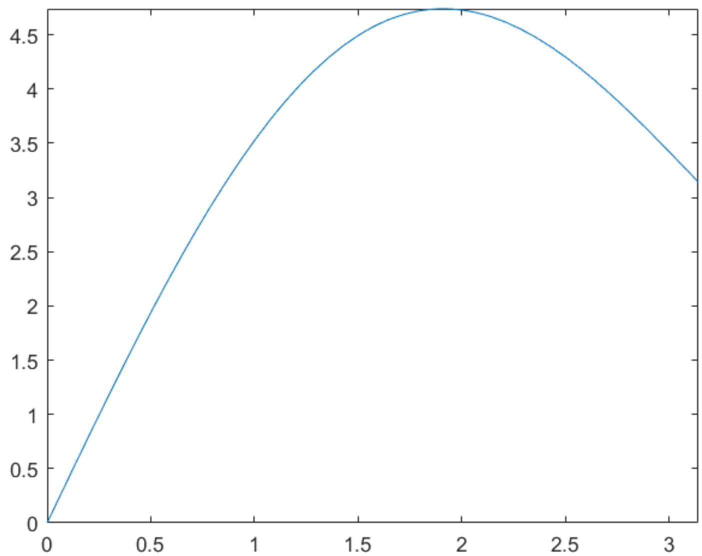
You can substitute values in for a variable:

```
>> subs(x^3+4/x+1,-1)

ans =

-4
```

1.3 Plotting

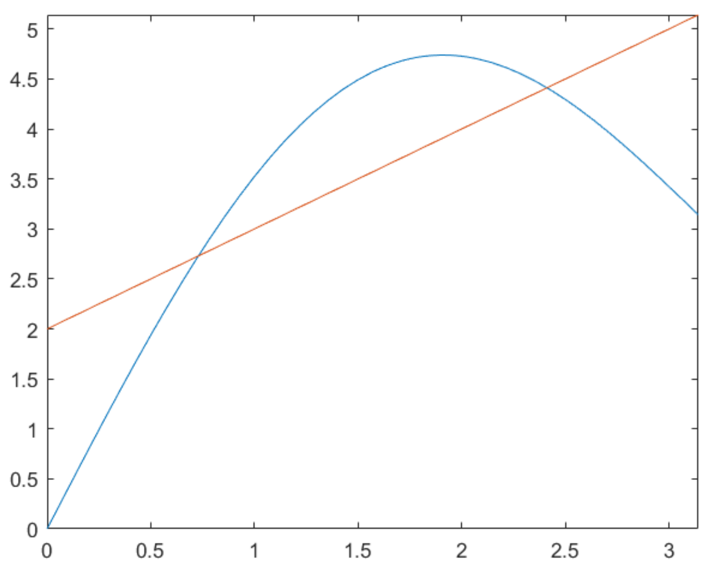
And you can plot functions. The first argument of `fplot` is the function, and the second argument is the x-interval for your plot, in the form `[a b]`. The plot will appear as a separate window.

```
>> fplot(x+3*sin(x),[0 pi])
```



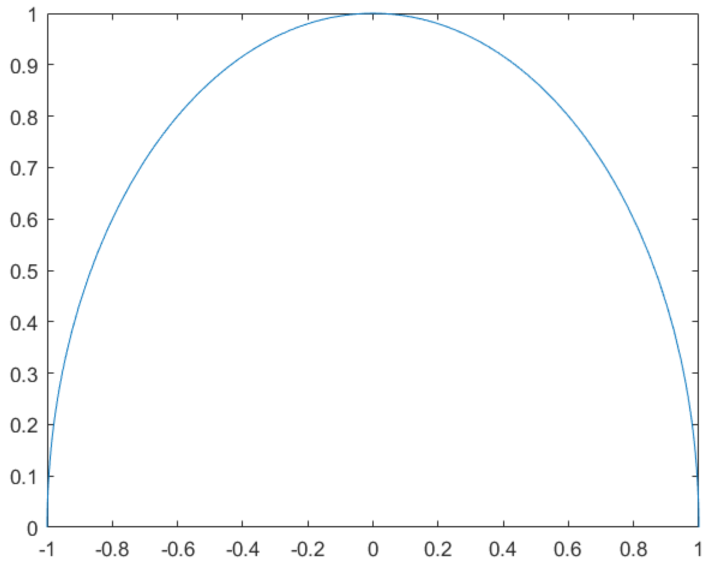
You may want to plot two functions on the same graph. To do this, type `hold on` in between your `fplot` commands. Don't forget to use `hold off` when you want a fresh graph!

```
>> fplot(x+3*sin(x),[0 pi])
>> hold on
>> fplot(x+2,[0 pi])
>> hold off
```



You can plot parametric curves using `fplot`.

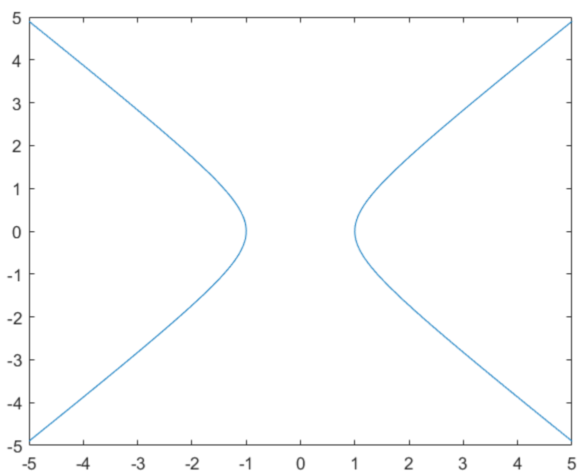
```
>> syms t
>> fplot(cos(t),sin(t),[0 2*pi])
```



You can see other interesting uses of `fplot` by looking at the help center online. In general, the help center is a great way to get a feel for how Matlab commands work, and to find the Matlab command you're looking for.

If we want to graph an inplicitly defined function, like $x^2 - y^2 = 1$, we first need to write it as $\ldots = 0$ (so $x^2 - y^2 - 1 = 0$), and then use the `fimplicit` command:

```
>> syms x y
>> fimplicit(x^2-y^2-1,[-5 5,-5 5])
```



1.4 Scripts and Publishing

The Command Window is great for doing some quick work, but for longer projects that you want to save and then publish to PDF, scripts are the best way. In this class, I prefer you use a script and *not* a live script. Your scripts will save as a .m file, and you can publish them to PDF by using the Publish ribbon at the top of the program. Before publishing, make sure to set your Publishing Options (by clicking the arrow underneath Publish). You want to choose Output file format to be pdf, and under Code Settings, make sure that both Include code and Evaluate code are set to true. Please look at Example_Matlab_File.m on Canvas to see how to format your script so that it breaks the output into sections, one for each problem. You can also see what the output should look like, to make sure that you're publishing settings are correct.

While you're working on your code, you can use Run (or Run Section) to see what the output would be. The output will appear in the Command Window (which might be a tab on the right of your window). If you put a semicolon after a line in your code, it will still evaluate, but not output anything. Use this only for output that you don't need graded (like a table of thousands of x- and y-values). If you want to have text written in your output, use `disp('type your text here')`.

2 Calculus III Material

2.1 Matlab Project 1 Material

2.1.1 Vectors

Define a vector. Vectors in Matlab are treated like in linear algebra so this may be new to you. The vector `2 i + 3 j + 7 k` can be entered either as a horizontal vector or a vertical vector. Whatever you choose to use, be consistent. I will use horizontal vectors in this guide because that's how we think of them in Math 241. As a vertical vector we use semicolons to separate rows:

```
u=[2;3;7]

u =

     2
     3
     7
```

As a horizontal vector (like I'll do) we use spaces or commas. I'll use spaces out of habit.

```
u=[2 3 7]

u =
```

```
     2     3     7
```

We can put variables inside our vectors to get vector-valued functions, and then take derivatives/integrals. The %% signs denote a comment, and everything on after them on the line is ignored by Matlab.

```
syms t
a=[t^2 1/t 2*t]
diff(a)
int(a,1,3) %% this integrates from 1 to 3

a =

[t^2, 1/t, 2*t]

ans =

[2*t, -1/t^2, 2]

ans =

[26/3, log(3), 8]
```

Here are various combinations of vectors. Notice also here the semicolons at the end of the first two lines to suppress the output, meaning we're telling Matlab "assign the vectors and keep quiet about it". Also note the use of `norm` for the length (magnitude) of a vector. Don't use the Matlab command `length`, because `length` just tells you how many elements are in the vector.

```
u=[6 9 12];
v=[-1 0 3];
norm(u)
w=u+v
dot(u,v)
cross(u,v)
dot(u,cross(u,v))
dot(u,v)/dot(v,v)*v

ans =

   16.1555

w =

     5     9    15
```

```
ans =

    30

ans =

    27   -30     9

ans =

     0

ans =

    -3     0     9
```

Warning: There are certain things which behave differently in Matlab than you might expect because Matlab knows a bit more than you might about some things. For example the dot product of two vectors has a slightly different definition than you might expect if the entries are complex numbers. In the above example this was never an issue since neither **u** nor **v** is complex, but if we try to use a variable look at what happens. Notice I've put `clear all` first which completely clears out Matlab so we know we're starting anew.

```
clear all;
syms t;
a=[t 2*t 5];
dot(a,a)
```

```
ans =

5*t*conj(t) + 25
```

What happened? Matlab doesn't know that `t` is a real number and so it does the more general dot product which we are not familiar with (involving a complex conjugate). If we want Matlab to know that `t` is a real number, we can tell it:

```
clear all;
syms t;
assume(t,'real');
a=[t 2*t 5];
dot(a,a)
```

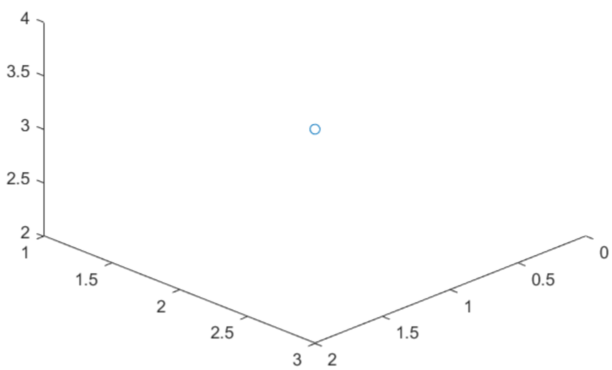
```
ans =
```

```
5*t^2 + 25
```

2.1.2 Plotting in 3D

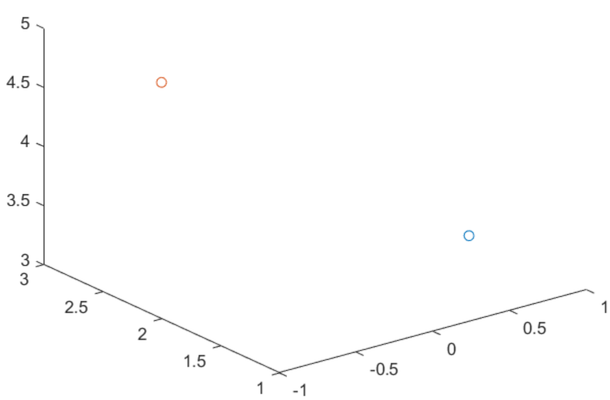
Plotting a single point can be done with `plot3`. The first three arguments are the coordinates of the point, and the fourth argument tells Matlab what to draw there (there are limited options).

```
plot3(1,2,3,'o')
```



We can plot two points in a similar way:

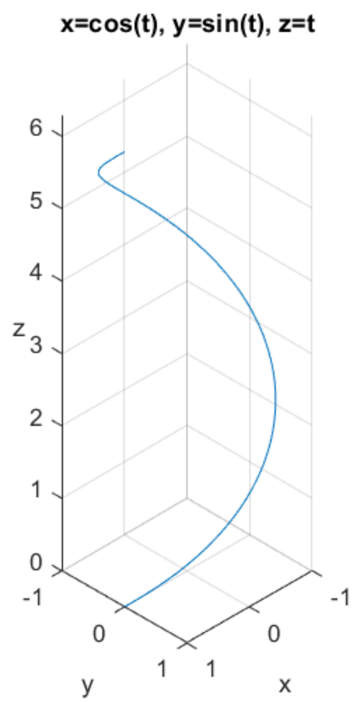
```
plot3(1,2,3,'o',-1,2,5,'o')
```



You can drag on the graph to rotate your point of view. To reset your point of view to the usual, type `view([10 10 10])` into the command window, to put our point-of-view at the point $(10, 10, 10)$.

To plot parametric curves, we use `fplot3`. Here we plot the curve $\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ for $0 \leq t \leq 2\pi$. The last command (`daspect`) tells Matlab to use the same scale for all the axes (why isn't this default behavior!?). For the `zlabel`, if you don't set Rotation to be 0, then 'z' will be sideways.

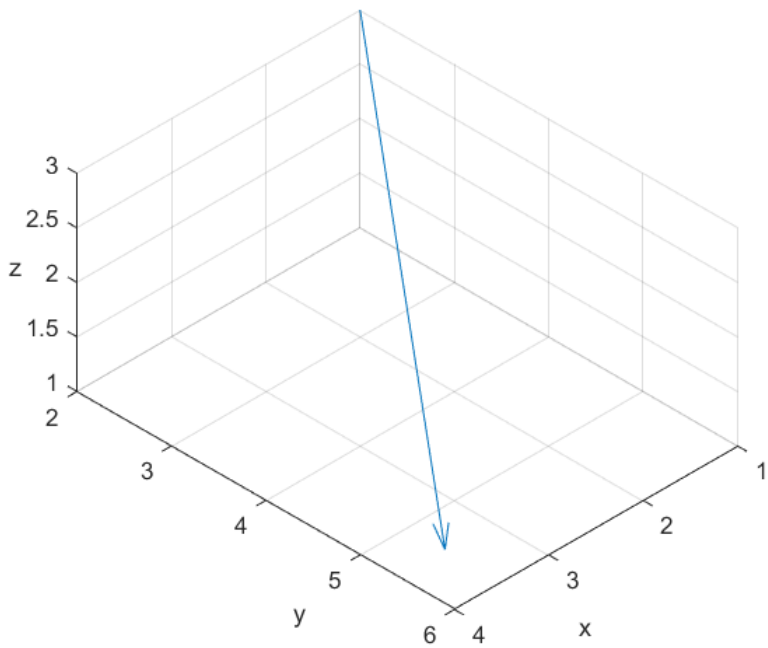
```
syms t
fplot3(cos(t), sin(t), t, [0 2*pi])
view([10 10 10])
title('x=cos(t), y=sin(t), z=t')
xlabel('x');
ylabel('y');
zlabel('z','Rotation', 0);
daspect([1 1 1])
```



We can also plot arrows representing vectors in 3D, using `quiver3`. The first three arguments are the coordinates of the base of the arrow, and the last three arguments are the vector, so the code below will plot $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ from $(1, 2, 3)$.

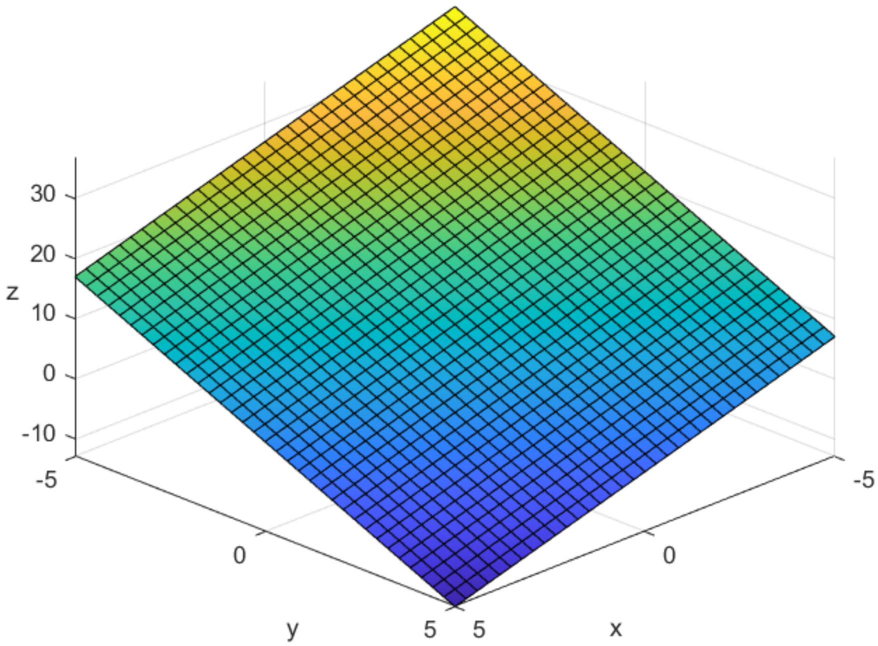
```
quiver3(1,2,3,3,4,-2)
view([10 10 10])
xlabel('x');
ylabel('y');
```

```
zlabel('z','Rotation', 0);
daspect([1 1 1])
```



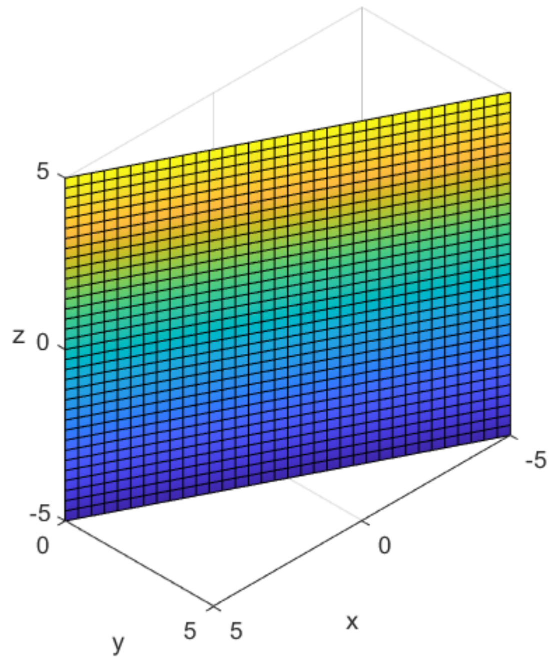
To plot a plane, we can't just plug in the plane equation; instead, we first need to solve for z to get $z = \ldots$. For example, the plane $2x+3y+2x = 12$ is equivalent to $z = 12-2x-3y$. Here we plot the function $12-2x-3y$ for x and y bounds -5 to 5.

```
syms x y;
fsurf(x,y,12-2*x-3*y,[-5 5,-5 5])
view([10 10 10])
xlabel('x');
ylabel('y');
zlabel('z','Rotation', 0);
```



However, if our plane is $2x+4y=10$, then instead of solving for z (which is impossible), we solve for another variable (say, y). We get $y=(10-2x)/4$, and then plot this as follows:

```
syms x z;
fsurf(x,(10-2*x)/4,z,[-5 5,-5 5])
view([10 10 10])
xlabel('x');
ylabel('y');
zlabel('z','Rotation', 0);
daspect([1 1 1])
```



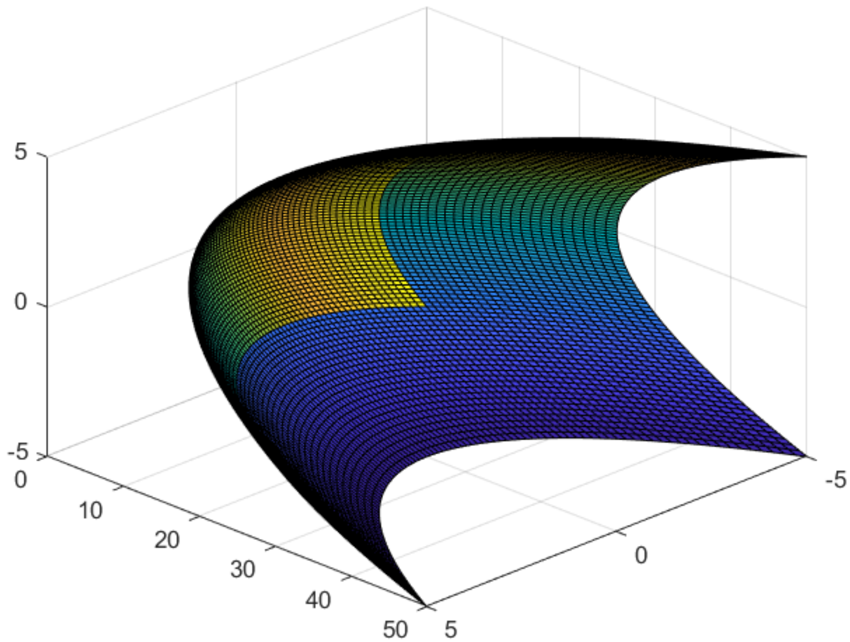
If for some reason you need to be able to see through your plane, use `fmesh` instead of `fsurf` (with the same syntax).

2.2 Matlab Project 2 Material

2.2.1 More plotting

Another way to get 3D graphs that's very useful when we don't have $z = \ldots$ is using `meshgrid` and `surf`. Say we want to plot $y = x^2 + z^2$. We will first use `meshgrid` to generate a bunch of (x, z) coordinates (in this case, values between -5 and 5 in steps of 0.1), and then use them to figure out the corresponding y coordinate. Rather than explain everything in detail, use this example as a model to do your own. Note: please use a semicolon after the coordinate calculations, otherwise you will be outputting massive matrices that no-one wants to see.

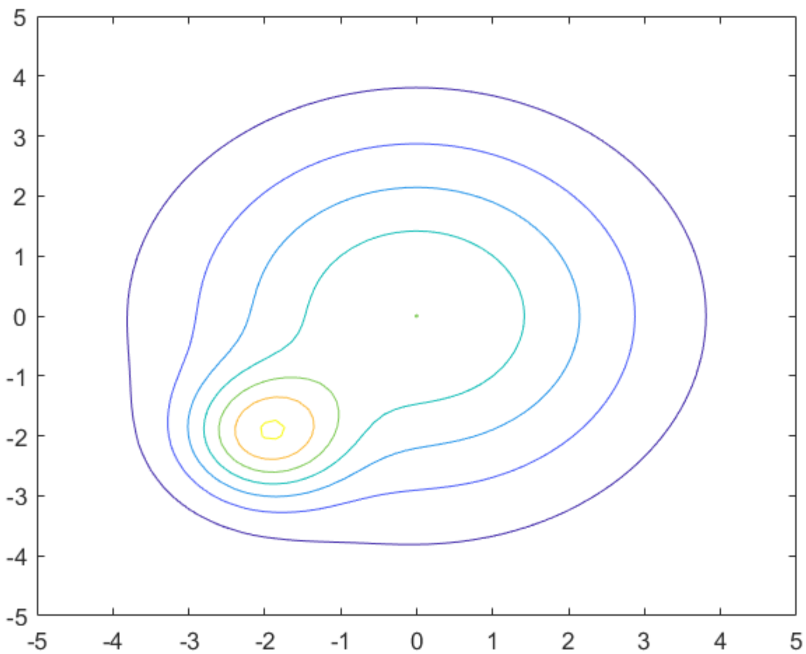
```
[x,z] = meshgrid(-5:0.1:5, -5:0.1:5);
y = x.^2 + z.^2;
surf(x,y,z)
view([10 10 10])
```



It is important that when working with `meshgrid`, multiplication/division/exponents need to have a dot before them, so `.*`, `./`, and `.^` instead of `*`, `/`, and `^`.

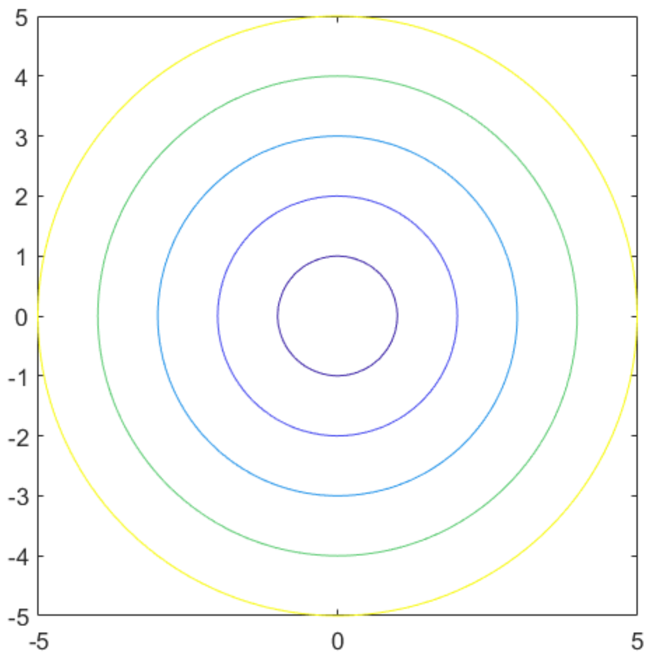
If we want to plot level curves of $f(x, y)$, we can use the `fcontour` command. For example, to plot level curves of $f(x, y) = e^{-(x/3)^2-(y/3)^2} + e^{-(x+2)^2-(y+2)^2}$, we do:

```
syms x y;
fcontour(exp(-(x/3)^2-(y/3)^2) + exp(-(x+2)^2-(y+2)^2), [-5
   5,-5 5])
```



The first argument of `fcontour` is the function, and the second is the range of x and y values to plot. If we want to choose particular values of c for level curves $f(x, y) = c$, we use the `LevelList` property. The following gives level curves for $f(x, y) = x^2 + y^2$ for $c = 1, 4, 9, 16, 25$:

```
syms x y;
fcontour(x^2+y^2, [-5 5,-5 5],'LevelList',[1 4 9 16 25])
daspect([1 1 1])
```



2.2.2 Partial derivatives

Since we can tell Matlab which variable to take the derivative with respect to, finding partial derivatives is as easy as regular derivatives. Here's an example of a partial derivative with respect to y:

```
syms x y;
diff(x^2*y+y^2/x,y)

ans =

(2*y)/x + x^2
```

For second partial derivatives, we have to nest our commands:

```
diff(diff(x^2*exp(x*y^2),y),x)

ans =

2*x^3*y^3*exp(x*y^2) + 6*x^2*y*exp(x*y^2)
```

For finding the gradient ∇f, we use the `jacobian` command.

```
jacobian(x^2*y^3,[x y])
```

```
ans =

[2*x*y^3, 3*x^2*y^2]
```

This command takes the partial derivative of the first argument with respect to the variables in the second argument, and then puts the outputs in a vector. To evaluate this vector at partial (x, y)-value, we use `subs`:

```
gradf = jacobian(x^2*y^3,[x y])
subs(gradf,{x,y},{-1,3})

ans =

[-54, 27]
```

With three variables, this all works the same:

```
syms x y z;
gradf = jacobian(x*exp(x*y)/z,[x y z])
subs(gradf,{x,y,z},{1,0,1})

ans =

[exp(x*y)/z + (x*y*exp(x*y))/z, (x^2*exp(x*y))/z, -(x*exp(x*y))/z^2]

ans =

[1, 1, -1]
```

2.2.3 Finding critical points

We first need to understand how to solve systems of equations using `solve`. We input a system of equations as a vector, where each equation is a component of the vector. For example, to solve $xy = 3$ and $x - y = 2$, we run `solve([x*y==3 x-y==2])`. However, if you try this, it won't immediately output the answers. To get our answers, we need to assign Matlab's output to variables:

```
syms x y;
[xsoln,ysoln] = solve([x*y==3 x-y==2])

xsoln =

-1
 3
```

```
ysoln =

-3
 1
```

Notice that there are two answers each for x and y. This means that there are two solutions, and the solution $x = -1$ corresponds to the solution $y = -3$, and the solution $x = 3$ corresponds to the solution $y = 1$. Note that Matlab will order the variables in alphabetical order (and x comes before y), so make sure to order your `xsoln` and `ysoln` appropriately. Note also that you want to use names like `xsoln` instead of `x`, otherwise you won't be able to use `x` as a variable anymore.

A useful shorthand for `solve` is that when solving equations $\ldots = 0$, you can omit the `==0` part of the equation. This comes in handy when working with gradients, because the `jacobian` command just outputs the partial derivatives, and we don't need to edit that before plugging it into solve to find critical points (which are when the partial derivatives equal 0).

```
syms x y;
f=x^2-2*x*y+1/3*y^3-3*y;
[xsoln,ysoln]=solve(jacobian(f,[x y]))
```

```
xsoln =

-1
 3


ysoln =

-1
 3
```

We see that the critical points of $f(x, y) = x^2 - 2xy + \frac{1}{3}y^3 - 3y$ are $(-1, -1)$ and $(3, 3)$.

2.2.4 Lagrange multiplier problems

When we solve a problem using Lagrange multipliers what we're doing is solving $\nabla f = L \cdot \nabla g$ along with the constraint (we'll use L instead of λ here). The first of these is the same as solving $\nabla f - L\nabla g = 0$ and the constraint can be rewritten to equal 0 too. The tricky thing is solving them all at once. This is how it's done for finding the extreme values for $f(x, y) = 3x^2 + 2y^2 - 4y + 1$ subject to $x^2 + y^2 = 16$. First here we'll just solve the equations:

```
syms x y L;
f=3*x^2+2*y^2-4*y+1;
g=x^2+y^2-16;
```

```
firstpart=jacobian(f,[x y])-L*jacobian(g,[x y]);
[Lsoln,xsoln,ysoln] = solve([firstpart,g])
```

```
Lsoln =

3/2
5/2
  3
  3


xsoln =

        0
        0
-2*3^(1/2)
 2*3^(1/2)


ysoln =

 4
-4
-2
-2
```

Note that L is alphabetically before x and y, so we need to put `Lsoln` first. We don't actually care about the values of L, but Matlab outputs solutions for all three variables, and we can't grab the x and y solutions without grabbing the L solutions. We see then that the system of equations has solutions $(0, 4)$, $(0, -4)$, $(-2\sqrt{3}, -2)$, $(2\sqrt{3}, -2)$. To find which gives us the min/max, we can plug each one into f using `subs`. Even better, we can do them all at once:

```
subs(f,{x,y},{xsoln,ysoln})
```

```
ans =

17
49
53
53
```

The minimum value 17 comes from the first solution, which was $(0, 4)$, and the maximum value 53 from the last two solutions, which were $(\pm 2\sqrt{3}, -2)$.

2.3 Matlab Project 3 Material

2.3.1 Multiple Integrals

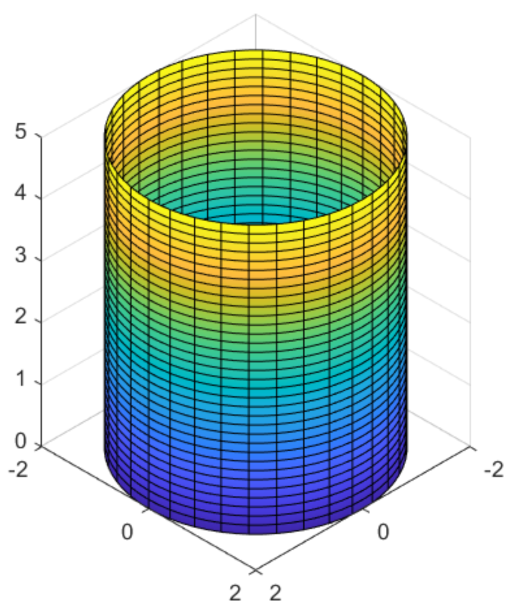
Since Matlab does integrals so well, this is easy and we just nest the integrals. For example consider the following. Read it carefully from the innermost `int` outwards. When we do `int(f,x,a,b)`, we integrate f with respect to x from a to b. Here a and b may also contain other variables.

```
syms x y z;
int(int(int(x^2*y+z,z,x+y,x),y,0,2*x),x,-1,2)

ans =

-81/2
```

2.3.2 Plotting parametric surfaces

Say we have the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$. We can parametrize this using θ and z as $\mathbf{r}(\theta, z) = 2\cos(\theta)\mathbf{i} + 2\sin(\theta)\mathbf{j} + z\mathbf{k}$, where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 5$. To plot this, we use `fsurf`:

```
syms theta z;
fsurf(2*cos(theta),2*sin(theta),z,[0 2*pi,0 5])
view([10 10 10])
daspect([1 1 1])
```



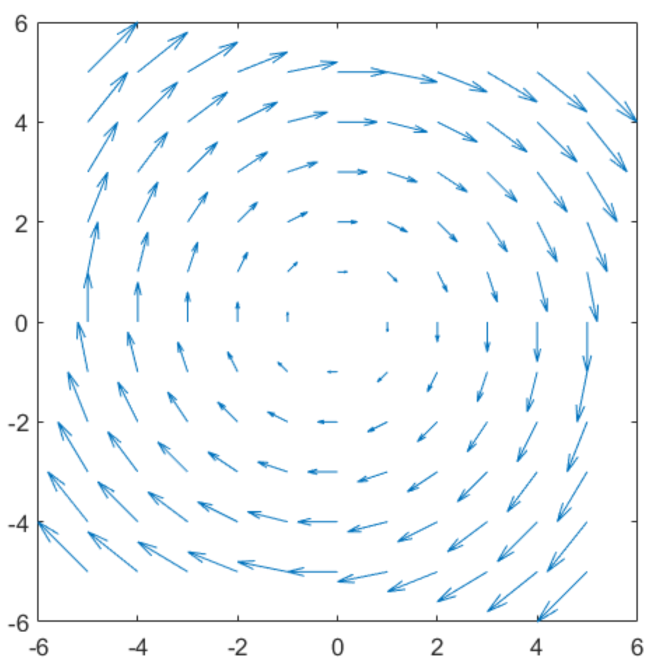
Here, theta comes alphabetically before z, so the first set of bounds (0 2*pi) applies to theta, and the second (0 5) applies to z. If you already have your parametrization in a vector, we can reference the components of the vector to get the same output:

```
syms theta z;
r = [2*cos(theta) 2*sin(theta) z]
fsurf(r(1),r(2),r(3),[0 2*pi,0 5])
view([10 10 10])
daspect([1 1 1])
```

2.3.3 Vector fields

We've already drawn arrows in 3D using quiver3, and now we will draw arrows in 2D using quiver. Say we wish to plot the vector field $\mathbf{F}(x,y) = \frac{y}{5}\mathbf{i} - \frac{x}{5}\mathbf{j}$. To do this, we first need to generate a matrix of (x,y)-coordinates using meshgrid, and then plug these into quiver. Don't make your meshgrid too fine (here we use a step size of 1), or you won't be able to see individual arrows.

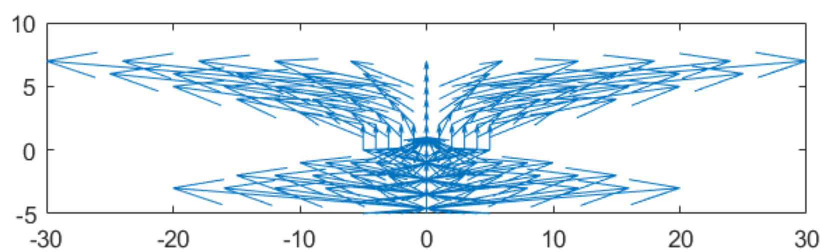
```
[x,y] = meshgrid(-5:1:5,-5:1:5);
quiver(x,y,y/5,-x/5,0)
daspect([1 1 1])
```



Remember, the first two arguments of quiver tells Matlab where the base of the arrow should be, and the next two are the vector. If you omit the 0 at the end, then Matlab will automatically rescale arrows that overlap, and we don't want that.

If we wanted to plot the vector field $\mathbf{F}(x, y) = xy\mathbf{i} + 2\mathbf{j}$, then since we'll be doing arithmetic with meshgrid data, we have to use `.*` instead of `*`. Additionally, `quiver` will throw an error if you just use `2` as the second component of the vector (since `2` is a constant and not a matrix like the other arguments). To get around this, we create a matrix (11×11, since that is the size of the `meshgrid` outputs) called `two` and assign `2` to all its entries.

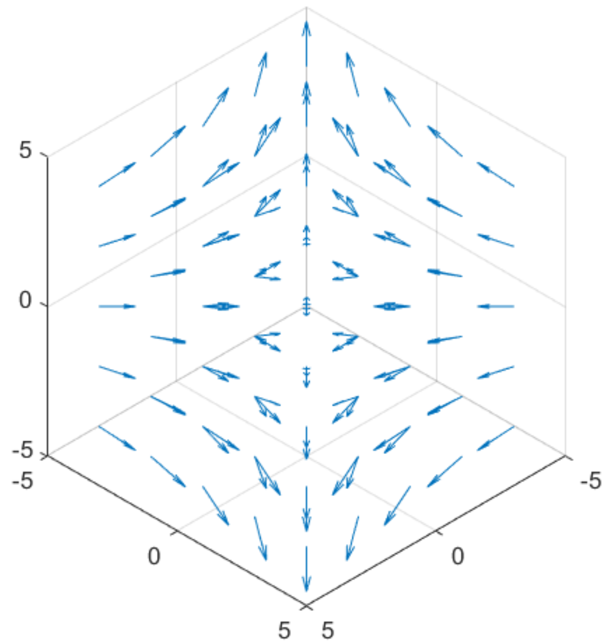
```
[x,y] = meshgrid(-5:1:5,-5:1:5);
two(1:11,1:11) = 2;
quiver(x,y,x.*y,two,0)
daspect([1 1 1])
```



See what the plot would look like if you omitted the `0` in `quiver`.

The same idea is used to generate plots of 3D vector fields. For example, to plot $\mathbf{F}(x, y, z) = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j} + \frac{z}{5}\mathbf{k}$, we do:

```
[x,y,z] = meshgrid(-4:2:4,-4:2:4,-4:2:4);
quiver3(x,y,z,y./sqrt(x.^2+y.^2),x./sqrt(x.^2+y.^2),z./5,0)
view([10 10 10])
daspect([1 1 1])
```



Try rotating this plot in Matlab — it's very cool!

2.3.4 Line integrals, surface integrals

For these, you just have to break it up into pieces and tell Matlab what to do using the `diff` and `int` functions you already know.

For example, to find the line integral of $f(x, y, z) = x + y^2 - 2z$ along the curve $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + 2t\mathbf{k}$ from $0 \le t \le 1$, we do the following. Make sure you can follow along.

```
syms t x y z;
r=[t,-3*t,2*t];
f=x+y^2-2*z;
normrprime = simplify(norm(diff(r,t))); %% this is |r'(t)|
fsubbed = subs(f,[x,y,z],r); %% this is f(r(t))
int(fsubbed*normrprime,t,0,1) %% integrate f(r(t))*|r'(t)|
```

```
ans =

(3*14^(1/2))/2
```

Here is an example of integrating $f(x, y, z) = x^2+y^2$ over the surface $\mathbf{r}(\phi, \theta) = 2\sin(\phi)\cos(\theta)\mathbf{i}+ 2\sin(\phi)\sin(\theta)\mathbf{j}+2\cos(\phi)\mathbf{k}$ for $0 \le \phi \le \pi/2$ and $0 \le \theta \le 3\pi/2$ (do you know what this surface is?).

```
syms phi theta x y z;
assume(phi,'Real');
assume(theta,'Real');
r=[2*sin(phi)*cos(theta),2*sin(phi)*sin(theta),2*cos(phi)];
f=x^2+y^2;
N = cross(diff(r,phi),diff(r,theta)); %% this is N
normN = simplify(norm(N)); %% this is |N|
fsubbed = subs(f,[x,y,z],r); %% this is f(r(phi,theta))
int(int(fsubbed*normN,phi,0,pi/2),theta,0,3*pi/2) %% integrate
   f(r(phi,theta))*|N|

ans =

16*pi
```

If we didn't assume that `phi` and `theta` are real, then Matlab wouldn't be able to sufficiently simplify the expressions to get a result (since $|x|^2 = x^2$ is true only for real x values).

Similarly, here is the surface integral of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface $\mathbf{r}(\theta, y) = \cos(\theta)\mathbf{i} + y\mathbf{j} + \sin(\theta)\mathbf{k}$, for $0 \leq \theta \leq 2\pi$ and $-2 \leq y \leq 1$:

```
syms theta x y z;
r=[cos(theta),y,sin(theta)];
F=[x y z];
N = cross(diff(r,theta),diff(r,y)); %% this is N
Fsubbed = subs(F,[x,y,z],r); %% this is F(r(theta,y))
int(int(dot(Fsubbed,N),y,-2,1),theta,0,2*pi) %% integrate F(r(
   phi,theta)) dot N

ans =

-6*pi
```

Here we didn't need to assume that `theta` and `y` were real. It depends on the problem, and you only need to add it if Matlab isn't simplifying/calculating how you expect.

MATH 241 – MATLAB Project 1 – due September 30, 11:59pm

- **What to turn in:** You will need to write a script (.m file) and then "publish" it in MATLAB as a PDF in order to turn it in on Gradescope.
- **What is a script?** A script is just a sequence of MATLAB commands, with each command on its own line. If you click "Run", MATLAB will execute all of the commands in the script, in order. The output will be produced in the Command Window, as if you entered each command individually. Your script should include all the necessary commands to solve the problems in the assignment. Scripts are useful because you can easily make edits to the commands if you make a mistake, or if you want to change something. To open a new script, click on 'New Script' in the top menu bar.
- **What is publishing?** When you "publish" a script, MATLAB will produce a nice-looking document that displays all of the code in your script **as well as** the output produced by the commands after they were executed.
  - You may need to go into the options in MATLAB to change the format of the published document from HTML to PDF. Please be sure to submit a PDF document. This can be done by going to Publish → Edit Publishing Options → Output File Format.
  - In the published document, MATLAB displays ALL of the commands and then ALL of the output afterwards. For a long script, this results in something that is very difficult to read. Here is how you separate your code problem-by-problem in order to make it better: Before each numbered problem, type %% in the script to make a heading. This signifies to MATLAB that it is a new portion of code, and MATLAB separates it accordingly when published. Any text after the %% reads as a heading; remember to put a space between it and any text!

Do this	Not this
%% Problem 1	%%Problem 1

    After you publish, it should look like
    * Code for Problem 1
    * Output for Problem 1
    * Code for Problem 2
    * Output for Problem 2
    * ⋮
  - You will need to include some answers to questions that are not MATLAB commands (e.g. you may have to explain why something happened). These questions are marked with a ⋆, and they require you to type an answer that is not just a MATLAB command. You should have MATLAB print these answers using the `disp` or `fprintf` commands in your script. You may need to manually add line breaks if your explanation is long. If the grader cannot read what you wrote because the text was chopped off, you won't get full credit!
  - You can see an example .m file and its output in the Files section on Canvas.
- **Rules for working in groups:** You are encouraged to collaborate on this project, if it is a true collaboration: no slackers allowed. You can't learn a programming language without typing it yourself. Projects may be turned in by groups of no more than three students, and in Gradescope for group submissions, there will be an option to include both group member's names. **If you don't include the names of all group members, not everyone will get credit.** It is possible to add names to the group even after submission.
- **But first:** Before you begin the project, go through the guide to Matlab posted in Canvas, if you haven't done so already. Be sure to type along in MATLAB as you read. That should be enough to get you through this project. Don't turn in anything from what you did in the guide. If you want more, there is a getting started course that you can find by Googling "MATLAB onramp tutorial".

0. Reread all directions on the previous page that are typed in boldface, and read the guide to Matlab, available on the main course page.

1. (a) Use the `diff` command to differentiate the functions $f(x) = \tan(x^2{+}2x)$ and $g(x) = \sqrt{e^{\sin x} + \ln(2x)}$. Then use the `subs` command to evaluate the derivatives at the point $x = \pi$. [Note: In MATLAB, the exponential is denoted `exp(x)` and the natural log is `log(x)`.]

   (b) Use the `int` command to evaluate the following integrals:

   i. $\int x \sin(x^2)\, \mathrm{d}x$

   ii. $\int_0^1 \sin(3x)\sqrt{1 - \cos(3x)}\, \mathrm{d}x$

   For the second integral, display the answer in decimal form. This can be done by using the command `double(ans)` where `ans` stands for 'answer', and is the name of the variable assigned to a command output if not otherwise specified by the user.

2. (a) Compute the projection of $\mathbf{v} = (1, 4, 2)$ onto the vector $\mathbf{w} = (1, 1, 1)$.

   (b) Compute the distance from $(1, -2, 5)$ to the line with parametric equations

$$x = 1 + 3t, y = -2 - 4t, z = 12t.$$

3. (a) Plot the function $f(x) = x^2 \sin(10x)$.

   (b) Plot the 2D curve given by the parametric equations $x = \sin(2t)$, $y = \cos(2t)$, with t in the range $-10 \le t \le 10$.

   (c) Plot the 3D curve given by the parametric equations $x = \sqrt{t}\cos(4t)$, $y = \sqrt{t}\sin(4t)$, $z = \sqrt{t}$, with t in the range $0 \le t \le 15$.

4. (a) Compute the distance from the point $P = (1, 1, 1)$ to the plane $x + y + z = 1$.

   (b) Let Q be the point on the plane closest to P. Find Q, then plot the vector $\overrightarrow{QP}$, and the plane $x + y + z = 1$, on the same graph. Use the interval ranges $0 \le x \le 1$, $0 \le y \le 1$.

   Note: This can done using the `hold on` and `hold off` commands. More specifically, part of your code will have the following form:

```
fmesh(...)
hold on
quiver3(...)
daspect([1 1 1])
hold off
```

   where you need to insert the appropriate arguments in the parentheses. `daspect` just makes the plot look better by choosing axes with equal spacings. The effect of `hold on` is to keep the current figure window open so we can add additional plots to it without overwriting, and the `hold off` turns this feature off.

5. (a) Plot $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$ for $-1 \le t \le 1$.

   (b) Find the length of the curve from $-1 \le t \le 1$. Display your answer in decimal form, as in 1(b).

   (c) Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ for $\mathbf{r}(t)$. Confirm that $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal.

   In order to avoid issues, define your variable as real (and not complex) by writing `assume(t,'real')`. Then, use the `simplify` command to simplify your expressions at each stage and to confirm that the dot product gives you 0.

MATH 241 – MATLAB Project 2 – due October 28, 11:59pm

- **What to turn in:** You will need to write a script (.m file) and then "publish" it in MATLAB as a PDF in order to turn it in on Gradescope.
- **What is a script?** A script is just a sequence of MATLAB commands, with each command on its own line. If you click "Run", MATLAB will execute all of the commands in the script, in order. The output will be produced in the Command Window, as if you entered each command individually. Your script should include all the necessary commands to solve the problems in the assignment. Scripts are useful because you can easily make edits to the commands if you make a mistake, or if you want to change something. To open a new script, click on 'New Script' in the top menu bar.
- **What is publishing?** When you "publish" a script, MATLAB will produce a nice-looking document that displays all of the code in your script **as well as** the output produced by the commands after they were executed.
  - You may need to go into the options in MATLAB to change the format of the published document from HTML to PDF. Please be sure to submit a PDF document. This can be done by going to Publish → Edit Publishing Options → Output File Format.
  - In the published document, MATLAB displays ALL of the commands and then ALL of the output afterwards. For a long script, this results in something that is very difficult to read. Here is how you separate your code problem-by-problem in order to make it better: Before each numbered problem, type %% in the script to make a heading. This signifies to MATLAB that it is a new portion of code, and MATLAB separates it accordingly when published. Any text after the %% reads as a heading; remember to put a space between it and any text!

Do this	Not this
%% Problem 1	%%Problem 1

    After you publish, it should look like
    - Code for Problem 1
    - Output for Problem 1
    - Code for Problem 2
    - Output for Problem 2
    - ⋮
  - You will need to include some answers to questions that are not MATLAB commands (e.g. you may have to explain why something happened). These questions are marked with a ⋆, and they require you to type an answer that is not just a MATLAB command. You should have MATLAB print these answers using the `disp` or `fprintf` commands in your script. You may need to manually add line breaks if your explanation is long. If the grader cannot read what you wrote because the text was chopped off, you won't get full credit!
  - You can see an example .m file and its output in the Files section on Canvas.
- **Rules for working in groups:** You are encouraged to collaborate on this project, if it is a true collaboration: no slackers allowed. You can't learn a programming language without typing it yourself. Projects may be turned in by groups of no more than three students, and in Gradescope for group submissions, there will be an option to include both group member's names. **If you don't include the names of all group members, not everyone will get credit.** It is possible to add names to the group even after submission.
- **But first:** Before you begin the project, go through the guide to Matlab posted in Canvas, if you haven't done so already. Be sure to type along in MATLAB as you read. That should be enough to get you through this project. Don't turn in anything from what you did in the guide. If you want more, there is a getting started course that you can find by Googling "MATLAB onramp tutorial".

1. (a) Plot the function $f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$ over the domain $-1 \le x \le 1$, $-1 \le y \le 1$.
   (b) In a single graph, plot some level curves. (A good choice of levels might be $c = 0, 0.5, 1, ..., 3$.) (Note: Instead of using a separate plot command for each curve, you can use the `fcontour` command.)
   (c) Suppose we are walking along the surface described by f. If we walk along a single level curve, how does our height change? If we cross level curves, how does our height change? You can either answer this question generally or use specific examples from part (b). (There is no Matlab calculation for this qusetion, you need to write out your answer in text that will display in the output.)
   (d) (unrelated to the above) Use the `meshgrid` and `surf` commands (*not* `fsurf` or `fmesh`) to plot $y = \sin(x \cos(z))$.
2. (a) Plot the function $f(x, y) = x \sin(xy) - y \sin(5y)$.
   (b) Compute the gradient of f in Matlab using the `jacobian` function.
   (c) If we are on the surface of f when $(x, y) = (1, 1)$, in which direction should we walk so that our vertical height increases most rapidly? Justify your answer (referring to class results or textbook results).
3. (a) Plot the function $f(x, y) = 2x^3y - y^2 - 3xy$. Based on the graph, how many maximum and minimum points do you expect f has, and how many saddle points?
   (b) Find f_x and f_y using the `diff` command. Setting these to zero, solve the resulting system of equations using the `solve` command. How many critical points do you obtain?
   (c) Compute the matrix of second partial derivatives and use it to determine whether f has a relative minimum, relative maximum, or saddle point at each critical point. If the test is inconclusive, resort to looking at the graph to make a conclusion about the nature of the critical points.
4. Use Lagrange multipliers to solve the following problem. Do **all** the computations in Matlab.

   Suppose a new planet has been discovered and we'd like to land a spacecraft on the planet where the magnetic field is weakest. The planet is perfectly spherically shaped with radius 3 (in some units that don't matter), and we take the center of the planet as the origin of our coordinate system. From satellite measurements, the strength of the magnetic field is approximated by $M(x, y, z) = xz - y^2 + 3x + 3$. Find the point(s) on the surface of the planet where M is minimal (i.e. the magnetic field is weakest).

MATH 241 – MATLAB Project 3 – due December 9, 11:59pm

- **What to turn in:** You will need to write a script (.m file) and then "publish" it in MATLAB as a PDF in order to turn it in on Gradescope.
- **What is a script?** A script is just a sequence of MATLAB commands, with each command on its own line. If you click "Run", MATLAB will execute all of the commands in the script, in order. The output will be produced in the Command Window, as if you entered each command individually. Your script should include all the necessary commands to solve the problems in the assignment. Scripts are useful because you can easily make edits to the commands if you make a mistake, or if you want to change something. To open a new script, click on 'New Script' in the top menu bar.
- **What is publishing?** When you "publish" a script, MATLAB will produce a nice-looking document that displays all of the code in your script **as well as** the output produced by the commands after they were executed.
  - You may need to go into the options in MATLAB to change the format of the published document from HTML to PDF. Please be sure to submit a PDF document. This can be done by going to Publish → Edit Publishing Options → Output File Format.
  - In the published document, MATLAB displays ALL of the commands and then ALL of the output afterwards. For a long script, this results in something that is very difficult to read. Here is how you separate your code problem-by-problem in order to make it better: Before each numbered problem, type %% in the script to make a heading. This signifies to MATLAB that it is a new portion of code, and MATLAB separates it accordingly when published. Any text after the %% reads as a heading; remember to put a space between it and any text!

Do this	Not this
%% Problem 1	%%Problem 1

    After you publish, it should look like
    - Code for Problem 1
    - Output for Problem 1
    - Code for Problem 2
    - Output for Problem 2
    - ⋮
  - You will need to include some answers to questions that are not MATLAB commands (e.g. you may have to explain why something happened). These questions are marked with a $\star$, and they require you to type an answer that is not just a MATLAB command. These answers will need to be typed as *comments* in your script, which is done by typing a single % sign, followed by your comment, like this:

    `% This is a comment!`

    This % tells MATLAB to not treat what follows as code. A detailed example and instructions are here: `https://www.mathworks.com/help/matlab/matlab_prog/publishing-matlab-code.html`.
  - An example of a Published PDF using these guidelines can be found on the main class page. See the file 'Publishing Example Code' and 'Publishing Example Output'. You can download the m-file and use it as a template.
- **Rules for working in groups:** You are encouraged to collaborate on this project, if it is a true collaboration: no slackers allowed. You can't learn a programming language without typing it yourself. Group submissions can be no more than two students. If you do work with a group, be sure to list your group members on the project, along with your name.

- **But first:** Before you begin the project, go through the guide to Matlab posted in Canvas, if you haven't done so already. Be sure to type along in MATLAB as you read. That should be enough to get you through this project. Don't turn in anything from what you did in the guide. If you want more, there is a getting started course that you can find by Googling "MATLAB onramp tutorial".

1. (a) Evaluate the integral $\iiint_D \mathrm{d}V$ for the volume of the region D bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the elliptic paraboloid $z = 4 - x^2 - 3y^2$. Give your answer in decimal form.
   (b) Plot the 2D vector field $\mathbf{F} = (x, -y)$, for $-5 \leq x, y \leq 5$ (you can use step size 1).
   (c) Plot the 3D vector field $\mathbf{F} = (y, -x, \sin z)$, for $-5 \leq x, y, z \leq 5$ (you can use step size 1).
   (Note: In 3D, vector fields can be confusing to look at. Rotate the plot using the 'rotate' tool in the graphing window to get an idea of what this vector field looks like. For submission, it doesn't matter the perspective of the graph.)

2. (a) Compute the work done (i.e. compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$) by the force field $\mathbf{F}(x, y, z) = (\sin z, \cos \sqrt{y}, x^3)$ along the curve C, parametrized by $\mathbf{r}(t) = (\sin t, t^2, t)$ for $0 \leq t \leq 2\pi$.
   (b) Plot the curve C from part (a).

3. (a) Plot the sphere $x^2 + y^2 + z^2 = 10$.
   (b) Suppose a fluid having constant density 50 g/m^3 flows with velocity $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a spherical filter, defined by $x^2 + y^2 + z^2 = 10$. Determine the rate of mass flow through the sphere in the direction of the outward normal. Convert your answer to decimal form.

4. (a) Plot the part of the cone $x^2 + z^2 = y^2$ that lies above the xy-plane and between the planes $y = 0$ and $y = 1$.
   (b) Suppose $T(x, y, z)$ represents the temperature at a point (x, y, z). According to Fourier's Law, heat 'flows' with the vector field $\mathbf{F} = -k\nabla T$, where k is a positive constant describing the rate of flow. The total rate of heat flow across a surface S is given by $\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$, where $\mathbf{n}$ is the outward unit normal.

   If $T(x, y, z) = x^2 + y^2 + z^2$ and S is the unit sphere $x^2 + y^2 + z^2 = 1$, find the total rate of heat flow across the surface S. (Assume $k = 1$.)