

## MATH 246: Chapter 0 Section 0: Course Introduction and Overview

### Justin Wyss-Gallifent

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Main Topics:

- What is a differential equation and what does it mean to solve one?
  - Ordinary vs. Partial DEs.
  - Order of a DE.
  - Linear vs. Nonlinear DEs.
  - A system of DEs.
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1. What is a differential equation and what does it mean to solve one?

- (a) The most straightforward definition of a *differential equation* (a DE) is that it's an equation involving some or all of the following: An unknown function of one or more variables such as  $y(t)$ , derivatives of that function such as  $y'$ ,  $y''$ , and so on, and other functions of the same variable(s) such as  $\sin(t)$  and  $t^2$ .

**Example:**  $f'(t) + f(t) = 10$  in which  $f$  is our unknown function of  $t$ .

**Example:**  $y'' + 3y' - xy = 6$  in which  $y$  is our unknown function of  $x$ .

**Example:**  $t^2 f''(t) = 5 - f'(t)(\sin t)$  in which  $f$  is our unknown function of  $t$ .

**Example:**  $17 \frac{dy}{dx} - x \frac{d^2 y}{dx^2} = xy$  in which  $y$  is our unknown function of  $x$ .

**Example:**  $\partial_x u + \sin(x) \partial_y u = y^3 \partial_{xy} u$  in which  $u$  is our unknown function of both  $x$  and  $y$ .

- (b) *Solving a DE* means finding a function which makes the DE true when you plug that function in.

**Example:**  $f(t) = e^t$  is a solution to the DE  $f(t) - f'(t) = 0$ .

**Example:**  $y(t) = \sin(t)$  is a solution to the DE  $y + y'' = 0$ .

**Example:**  $f(t) = t + e^{2t}$  is a solution to the DE  $f''(t) + 4t = 4f(t)$ .

**Example:**  $f(x) = x^2$  is not a solution to the DE  $xf'(x) = f(x)$ .

Just as regular equations can have more than one solution ( $x^2 - 9 = 0$  has two solutions) so can a DE. In fact usually a DE will have infinitely many solutions.

**Example:**  $f(t) = 487e^t$  is another solution to the DE  $f(t) - f'(t) = 0$ . You can probably see lots more now.

2. Associated definitions

- (a) A DE is called *ordinary* (so an ODE) if the unknown function is just a function of one variable. Otherwise it's *partial* (so a PDE). Generally in this course when we talk about a DE we mean an ODE.

**Example:**  $f'(t) + 3tf''(t) = e^t$  is an ODE.

**Example:**  $u_x(x, y) + u_{yx}(x, y) + y = 3$  is a PDE. If you've not seen partial derivatives before don't worry.

- (b) The *order* of a DE is the highest derivative that appears in it. We say things like *first-order* and *second-order* and so on.

**Example:**  $x^7 f'(x) + (\cos x)f(x) + x = e^x$  is first-order.

**Example:**  $tf(t) + e^t f''(t) = 1 - f'(t)$  is second-order.

(c) A DE is *linear* if it can be written as a sum of some or all of:

- i. An unknown  $f$  multiplied by a coefficient.
- ii. Derivatives of the unknown  $f$  multiplied by coefficients.
- iii. Coefficients.

By *coefficients* we mean they can be other functions of the same variables that  $f$  is, including just constants, including 0.

**Example:** The DE  $5tf(t) + (\ln t)f'(t) = 5$  is linear.  
**Example:** The DE  $(\tan t)y(t) - t^3y'(t) + 7y''(t) = 1$  is linear.  
**Example:** The DE  $f(x)\sqrt{x} + (1-x)f'''(x) = f'(x)$  is linear.  
**Example:** The DE  $f(t)^2 + f'(t) = 7$  is nonlinear because the  $f(t)^2$  is not permitted.  
**Example:** The DE  $\sin(y') + y' - y = x$  is nonlinear because the  $\sin(y')$  is not permitted.  
**Example:** The DE  $y'y + y = xy$  is nonlinear because the  $y'y$  is not permitted.

Clarification perhaps:

A *first-order linear differential equation* using the variable  $t$  and the unknown function  $y$  will have the form

$$a_1(t)y' + a_0(t)y = c(t)$$

A *second-order linear differential equation* with the function  $y(t)$  will have the form:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = c(t)$$

An  $n^{\text{th}}$  order linear DE with the function  $y(t)$  will have the form:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = c(t)$$

(d) A *system* of DEs is just that, a collection of more than one DE where the goal is to find a single function that makes them all true. The *order* of such a system is the highest derivative that appears in any of the DEs.

**Example:** A first-order system of two linear DEs:

$$\begin{aligned} ty + t^2y' &= e^t \\ 3y + 5y' &= \sin(t) \end{aligned}$$

### 3. Moving onwards.

At this point you can probably start to wrap your head around which DEs look like they might be easier to handle. The following is a list of DEs of increasing complexity. Even though you don't really know how to solve any of these just yet (that's not true, you can do the first one!) you can almost certainly look at them in order and get an appreciation for the fact that they start pretty nice and get more convoluted! Don't worry that some of the words on the right might not make sense.

$y' = t^2$	Explicit first order linear ODE
$5y' - 4y = 0$	Homogeneous first order linear ODE with constant coefficients
$2y'' + 5y' - 4y = 0$	Homogeneous second order linear ODE with constant coefficients
$7y' - 2y = t$	Nonhomogeneous first order linear ODE with constant coefficients
$t^2y' + e^ty = 1 + t$	Nonhomogeneous first order linear ODE

With some quirky exceptions our approach will pretty much be like that in that we'll first tackle the easier types. This will help us develop some theory which will then support us as we move to the more complicated types, and then to systems of these.

## MATH 246: Chapter 1 Section 1: Introduction to First-Order DEs

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Main Topics:

- Overview of first-order (O)DEs.
  - Explicit first-order DEs.
  - General solutions, initial value problems, specific (particular) solutions.
  - Underlying theory regarding existence of solutions on an interval.
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### 1. Introductory overview of first-order ODEs.

- (a) A first-order ODE (not necessarily linear) is permitted to have an unknown function  $y$  (of a single variable, say  $t$ ) its derivative  $y'$  and then some other functions of  $t$ .

**Example:**  $t(y')^2 + y = \sin t$

**Example:**  $y' - ty = e^{2t}$

**Example:**  $\sin(y') + e^{y'} = t$

- (b) In general these can be very hard! The first step is always algebra though, basically we first solve for  $y'$  and then proceed from there. Thus for the next few sections we'll assume that we've solved for  $y'$  in terms of  $t$  and  $y$  and we'll focus on DEs that have the form  $y' = f(t, y)$ .

That  $f$  might be confusing, it's not the unknown function but rather it just represents the fact that we can have a bunch of  $y$  and  $t$  on the right hand side. In other words things like this:

**Example:**  $y' = ty$

**Example:**  $y' = 4t - 8y$

**Example:**  $y' = \frac{y}{t}$ .

### 2. Explicit first-order DEs.

- (a) Because solving even first-order ODEs is hard we'll go down even further and look at *explicit* first-order ODEs that have the form  $y' = f(t)$ .

**Example:**  $y' = t^2$ .

**Example:**  $y' = 4t + \sin t$ .

- (b) At this point you might have an epiphany and realize that often you can solve these because solving these is as easy as integrating the right side.

**Example:**  $y' = t^2$ . To solve this we integrate to get  $y = \frac{1}{3}t^2 + C$  for any constant  $C$ .

**Example:**  $y' = 4t + \sin t$ . To solve this we integrate to get  $y = 2t^2 - \cos t + C$  for any constant  $C$ .

### 3. General solutions, initial value problems, specific (particular) solutions

- (a) We've started to notice that we can have many solutions to a DE. In the explicit DEs above get a constant  $C$  which can be anything.
- (b) A *general solution* to a DE is a solution involving constants and for which different constants will give all solutions.

- (c) A *specific solution* or a *particular solution* is a solution in which a specific (particular) choice of constant(s) has been made.

**Example:** The general solution to  $y' = t^2$  is  $y = \frac{1}{3}t^3 + C$ . Some specific solutions are  $y = \frac{1}{3}t + 1$ ,  $y = \frac{1}{3}t - 107$  and  $y = \frac{1}{3}t + \pi$ .

- (d) Often when we encounter a DE it comes pre-packaged with an *initial value*, or IV. In our simple exact case (and in many future cases) this will be an instance that  $y(t_I) = y_I$  for specific  $t_I$  and  $y_I$ . The DE and the IV together form an *initial value problem* or IVP. It's very common that  $t_I = 0$  but this isn't always the case!

**Example:**  $y' = 2t$  with  $y(0) = 3$  is an IVP.

**Example:**  $y' = 2t$  with  $y(0) = 5$  is an IVP with the same DE but different IV.

**Example:**  $y' = 2t$  with  $y(1) = 3$  is an IVP with again the same DE but different IV.

- (e) When we're given an IVP the idea will be to first solve the DE to get the general solution and then use the IV to get the specific solution.

**Example:**  $y' = 2t$  with  $y(0) = 3$ . First we find  $y = t^2 + C$ , the general solution, and then  $y(0) = 0^2 + C = 3$  so  $C = 3$  and the specific solution is  $y = t^2 + 3$ .

**Example:**  $y' = 2t$  with  $y(1) = 3$ . First we find  $y = t^2 + C$ , the general solution, and then  $y(1) = 1^2 + C = 3$  so  $C = 2$  and the specific solution is  $y = t^2 + 2$ .

#### 4. Intervals of Existence and Theory for Explicit IVPs:

We now know that solving the explicit DE given by  $y' = f(t)$  is as easy (or hard) as finding an antiderivative for  $f(t)$ . However the Fundamental Theorem of Calculus tells us something interesting. It states that if a function is continuous on an open interval then it has an antiderivative on that open interval. This means that even if we can't actually find the antiderivative of  $f(t)$  using techniques that we know, we still know that it exists, and therefore that there is a solution on an open interval as long as  $f(t)$  is continuous on that open interval.

**Example:** Consider the explicit DE given by  $y' = t$ . Since the function  $t$  is continuous on  $(-\infty, \infty)$  we know it has an antiderivative on  $(-\infty, \infty)$  and therefore the DE has a solution there. In this case the general solution is  $y = \frac{1}{2}t^2 + C$ .

**Example:** Consider the explicit DE given by  $y' = \frac{1}{t^2}$ . The function  $\frac{1}{t^2}$  is continuous on  $(-\infty, 0)$  and on  $(0, \infty)$ . What this means is that it has solutions on each of those intervals.

When it comes to an explicit IVP we start with  $y = f(t)$  and  $y(t_I) = y_I$ . We say that the *interval of existence* is the largest open interval containing  $t_I$  on which a solution exists. This is found by finding the largest open interval containing  $t_I$  on which  $f(t)$  is continuous.

**Example:**  $y' = \frac{1}{t^2}$  with  $y(1) = 5$ . We notice the largest open interval containing  $t_I = 1$  on which  $\frac{1}{t^2}$  is defined is  $(0, \infty)$  and so this is the IE. Notice that we don't need to solve it, but we could, since the general solution is  $y = -\frac{1}{t} + C$  and then  $y(1) = -1 + C = 5$  so  $C = 6$  and the specific solution is  $y = -\frac{1}{t} + 6$ .

**Example:**  $y' = \frac{t}{(t-3)(t+6)}$  with  $t(0) = 17$ . We notice the largest open interval containing  $t_I = 0$  on which  $\frac{t}{(t-3)(t+6)}$  is defined is  $(-6, 3)$  so this is the IE. We could possibly solve this with some messy partial fractions but we won't. However we do know for sure that there is a solution on this interval.

## MATH 246: Chapter 1 Section 2: Linear First-Order DEs

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Main Topics:

- Linear First-Order DEs and Linear Normal Form.
  - General approach.
  - Initial Value Problems.
  - Theory
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### 1. Linear first-order ODEs.

Recall that these will all have the form  $a_1(t)y' + a_0(t)y = c(t)$  where  $a_1, a_0, c$  can be any functions of  $t$ .

**Example:**  $4y' + 5y = 0$

**Example:**  $4ty' + e^t y = \sin t$

### 2. Linear Normal Form.

#### (a) Introduction:

We will usually divide through by  $a_1(t)$  and re-label a bit to get what is known as the *linear normal form*:

$$y' + a(t)y = f(t) \text{ for functions } a(t) \text{ and } f(t)$$

#### (b) General Solution.

These we can actually handle, and most of you did in Calculus II though it may be rusty.

Method: If we let  $A(t)$  be an antiderivative (any antiderivative, meaning use  $+0$  for the constant) of  $a(t)$  so that  $A'(t) = a(t)$  then observe:

$$\begin{aligned} y' + a(t)y &= f(t) \\ e^{A(t)}y' + e^{A(t)}a(t)y &= f(t)e^{A(t)} \\ \frac{d}{dt} \left( e^{A(t)}y \right) &= f(t)e^{A(t)} \\ e^{A(t)}y &= \int f(t)e^{A(t)} dt \\ y &= e^{-A(t)} \int f(t)e^{A(t)} dt \end{aligned}$$

The only step that might concern you here is from line 2 to line 3. This is just the reverse of the product rule with a bit of chain rule thrown in. Reading it from line 3 to line 2 might be easier. We'll see this sort of thing happen again with exact DEs later.

This process can either be repeated for each problem or treated simply as a recipe, meaning find  $A(t)$  and then use the formula at the end of the calculation.

Be careful though, the  $e^{-A(t)}$  is multiplied by the entire integral, meaning the  $+C$  too when you integrate.

We'll call this final expression the *integral-form solution*:

$$y = e^{-A(t)} \int f(t)e^{A(t)} dt$$

**Example:** Consider  $y' + 5y = 2$ . We see that  $a(t) = 5$  so  $A(t) = 5t$  and the solution is

$$\begin{aligned} y &= e^{-5t} \int 2e^{5t} dt \\ &= e^{-5t} \left( \frac{2}{5} e^{5t} + C \right) \leftarrow \text{Note the parentheses!!!} \\ &= \frac{2}{5} + Ce^{-5t} \end{aligned}$$

If you'd have forgotten the parentheses you'd have got an incorrect answer which I won't even write here!

**Example:** Consider  $ty' + 2y = t^4$  with  $t > 0$ . This is not in linear normal form so we divide by  $t$  to get  $y' + \frac{2}{t}y = t^3$ . Then  $a(t) = \frac{2}{t}$  so  $A(t) = 2 \ln t$  and the solution is

$$\begin{aligned} y &= e^{-2 \ln t} \int t^3 e^{2 \ln t} dt \\ y &= e^{-2 \ln t} \int t^3 e^{2 \ln t} dt \\ &= t^{-2} \int t^5 dt \\ &= t^{-2} \left( \frac{1}{6} t^6 + C \right) \\ &= \frac{1}{6} t^4 + \frac{C}{t^2} \end{aligned}$$

Here's one with an IVP:

**Example:** Consider  $y' - 6y = e^t$  with  $y(0) = 2$ . We see that  $a(t) = -6$  so  $A(t) = -6t$  and the general solution is

$$\begin{aligned} y &= e^{-(-6t)} \int e^t e^{-6t} dt \\ &= e^{6t} \int e^{-5t} dt \\ &= e^{6t} \left( -\frac{1}{5} e^{-5t} + C \right) \\ &= -\frac{1}{5} e^t + Ce^{6t} \end{aligned}$$

At this point  $y(0) = -\frac{1}{5} e^0 + Ce^0 = -\frac{1}{5} + C = 2$  so that  $C = \frac{11}{5}$  so the specific solution is

$$y = -\frac{1}{5} e^t + \frac{11}{5} e^{6t}$$

At this point you can probably see that solving a first-order linear ODE is as easy (or as hard) as first finding  $A(t)$  and then finding  $\int f(t)e^{A(t)} dt$ .

- (c) Note about the choice of  $A(t)$ . You might wonder what happened if you didn't choose  $+0$  as your constant when choosing  $A(t)$ . In fact it makes no difference. Suppose we took  $A(t)$  and adjusted it by adding some number like  $+7$ . The solution would then be:

$$y = e^{-(A(t)+7)} \int f(t)e^{A(t)+7} dt = e^{-7}e^{-A(t)} \int f(t)e^7e^{A(t)} dt = e^{-A(t)} \int f(t)e^{A(t)} dt$$

which is exactly the same.

### 3. Theory!

The Second Fundamental Theorem of Calculus states that if a function is continuous on an open interval then it has an antiderivative on that interval and that antiderivative will be continuous. What this means is that if  $a(t)$  is continuous then  $A(t)$  will exist and therefore so will  $e^{-A(t)}$  and then provided that  $f(t)$  is continuous then so will  $\int f(t)e^{A(t)} dt$ .

Warning! This doesn't mean that these things are easy to calculate, just that they exist!

What this means is that if we have an initial value  $y(t_I) = y_I$  then the interval of existence of the solution will be the largest open interval containing  $t_I$  on which both  $f(t)$  and  $a(t)$  are continuous. As before this lets us find the IE even when we can't solve the IVP.

**Example:** Consider  $y' + \frac{1}{t^2}y = \frac{1}{t-5}$  with  $y(2) = 17$ . Here  $a(t) = \frac{1}{t^2}$  and  $f(t) = \frac{1}{t-5}$ . The largest open interval containing  $t_I = 2$  on which both are continuous is  $(0, 5)$  so this is the IE of the solution. Finding the solution is a different matter entirely but it exists on  $(0, 5)$ !

### 4. Integration Comment.

As a final note observe that there are two antiderivatives involved in the problem, finding  $A(t)$  and finding  $\int f(t)e^{A(t)} dt$ . This latter one will often involve simplification involving  $e$  and  $\ln$  as well as substitution and integration by parts.

## MATH 246: Chapter 1 Section 3: Separable DEs

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Main Topics:

- Separable DE and Method of Solution.
  - Implicit vs. Explicit Solutions.
  - Constant Solutions.
  - Autonomous DEs.
  - Effect of Initial Values on Solution Choice.
  - Non-uniqueness of Solutions.
  - Method Notation Note.
- 

#### 1. Separable ODEs.

A DE is *separable* if it can be written in the form  $y' = f(t)g(y)$ . The word separable comes from the fact that the right side is separated into a product of a function of  $t$  and a function of  $y$ .

**Example:**  $y' = ty$  is separable - it is already separated!

**Example:**  $ty' + y' = y^2$  is separable because it can be separated, first by factoring  $y'(t+1) = y^2$  and then dividing  $y' = \frac{y^2}{t+1}$  and thinking of it as  $y' = \left(\frac{1}{t+1}\right)y^2$ .

**Example:**  $y' = t + y$  is not separable. There is no way to write the right side as a product of a function of  $t$  and a function of  $y$ .

#### 2. Method: Solving for Non-Constant Solutions.

The method of solution for non-constant solutions is really slick:

$$\begin{aligned}y' &= f(t)g(y) \\ \frac{dy}{dt} &= f(t)g(y) \\ \frac{1}{g(y)}dy &= f(t)dt \\ \int \frac{1}{g(y)}dy &= \int f(t)dt\end{aligned}$$

Where the integral on the left is with respect to  $y$  and the integral on the right is with respect to  $t$ . Since both indefinite integrals should get their own constant, instead we just put a single  $+C$  on the right.

The  $\frac{1}{g(y)}$  looks really icky to integrate but in our examples it generally works out pretty nicely because it's often not a quotient at all.

Side note: We're really taking a stick and beating the notation into a pulp here. If this weird approach bothers you I've attached a note at the end explaining why it's simply a shorthand notation for something more rigorous.



**Example:** Consider  $y' = \frac{t}{y^2}$ .

Note that you could think of this as  $y' = t \left( \frac{1}{y^2} \right)$ . The key is to get all the  $y$ s together multiplied by the  $y'$  on the left and leave all the  $t$ s together on the right. We work as follows, multiplying by  $y^2$  first, or if you prefer to think of it that way, dividing by  $1/y^2$ :

$$\begin{aligned}y' &= \frac{t}{y^2} \\y^2 \frac{dy}{dt} &= t \\y^2 dy &= t dt \\\int y^2 dy &= \int t dt \\\frac{1}{3}y^3 &= \frac{1}{2}t^2 + C \\y &= \sqrt[3]{\frac{3}{2}t^2 + 3C}\end{aligned}$$

There's an argument to be made at this point that since  $3C$  is just a constant we could write  $C$  instead. However this results in a problem having two different  $C$ 's meaning two different things and given that we will often solve for this  $C$  it's far safer to just leave it as is.

### 3. Observation: Constant Solutions:

When we solve a separable DE we do often actually divide by some  $g(y)$  and in that case we have to independently look at the possibility that  $g(y) = 0$ . This may lead to constant functions which are additional solutions to the separable DE.

**Example:** Consider  $y' = e^t \sqrt{1 - y^2}$ .

The method of solution above goes as follows:

$$\begin{aligned}\frac{dy}{dx} &= e^t \sqrt{1 - y^2} \\\frac{1}{\sqrt{1 - y^2}} dy &= e^t dt \\\int \frac{1}{\sqrt{1 - y^2}} dy &= \int e^t dt \\\sin^{-1} y &= e^t + C \\y &= \sin(e^t + C)\end{aligned}$$

There is nothing wrong with this provided  $\sqrt{1 - y^2} \neq 0$ . So what if  $\sqrt{1 - y^2} = 0$ ? This would arise if  $y = \pm 1$  and in fact these are completely valid solutions (functions!) to the DE. Thus overall the DE has two constant solutions as well as the nonconstant solutions!

All Solution:  $y = -1$ ,  $y = 1$ ,  $y = \sin(e^t + C)$

Conclusion: When we divide by some  $g(y)$  the functions arising from when  $g(y) = 0$  are valid *constant solutions* to the DE and must be included in our final list.

#### 4. Implicit versus Explicit Solutions:

It's entirely possible that when we solve a separable DE we are unable to solve for  $y$  at the end, or it may be very difficult.

**Example:** Consider  $y' = \frac{t}{e^y + 1}$ .

Notice there are no constant solutions here because to solve it we do not divide by something which can be zero.

We work as follows:

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{e^y + 1} \\ (e^y + 1)dy &= t dt \\ \int e^y + 1 dy &= \int t dt \\ e^y + y &= \frac{1}{2}t^2 + C\end{aligned}$$

In this case it's reasonable to stop here and say that we have *implicitly* defined the solutions. An *implicit solution* is a solution in which we have not actually achieved  $y =$ .

Ideally of course we would be able to solve for  $y$ , this would yield an *explicit solution*.

#### 5. Autonomous ODEs:

There is a special kind of separable ODE called *autonomous*. This occurs when  $f(t) = 1$  and so instead we have  $y' = g(y)$ . This can be solved like any other separable ODE. We only mention it because these will arise repeatedly over the course in various places.

**Example:** Consider  $y' = (y - 4)^2$ .

Here  $g(y) = (y - 4)^2$  which equals 0 when  $y = 4$  so this is the constant solution. The nonconstant solutions we obtain as follows:

$$\begin{aligned}\frac{dy}{dt} &= (y - 4)^2 \\ (y - 4)^{-2} dy &= 1 dt \\ \int (y - 4)^{-2} dy &= \int 1 dt \\ -(y - 4)^{-1} &= t + C \\ (y - 4)^{-1} &= -(t + C) \\ (y - 4) &= \frac{-1}{t + C} \\ y &= \frac{-1}{t + C} + 4\end{aligned}$$

## 6. Two Small Initial Value Notes:

### (a) Choosing Solutions:

When we solve a separable ODE and get an implicit solution for which there seems to be more than one explicit solution, an initial value usually tells us which one it is:

**Example:** Consider  $y' = \frac{t}{y}$  with  $y(1) = -3$ .  
First we solve the DE:

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{y} \\ y \, dy &= t \, dt \\ \int y \, dy &= \int t \, dt \\ \frac{1}{2}y^2 &= \frac{1}{2}t^2 + C \\ y^2 &= t^2 + 2C \\ y &= \pm\sqrt{t^2 + 2C}\end{aligned}$$

We see that there are two explicit solutions to the DE.

When we consider the initial value we have  $y(1) = \pm\sqrt{1^2 + 2C} = -3$  so we are forced to use the negative in front of the square root. Thus  $-\sqrt{1^2 + 2C} = -3$  so  $1 + 2C = 9$  and  $C = 4$ . Then the explicit solution is  $y = -\sqrt{t^2 + 8}$ .  
Note: There are no constant solutions here since  $g(y) = \frac{1}{y}$  is never 0.

### (b) Uniqueness (?) of Solutions:

The existence of constant solutions often leads to non-unique solutions to IVPs. This can happen when a constant solution satisfies the DE but also the procedural method gives another solution. The way to manage this is to not forget to find your constant solutions and check if they satisfy the IV.

**Example:** Consider  $y' = y^{2/3}$  with  $y(0) = 0$ . Notice that  $y = 0$  is a constant solution which also satisfies the DE. However the DE is separable:

$$\begin{aligned}\frac{dy}{dt} &= y^{2/3} \\ y^{-2/3} \, dy &= 1 \, dt \\ \int y^{-2/3} \, dy &= \int 1 \, dt \\ 3y^{1/3} &= t + C \\ y &= \left(\frac{1}{3}t + \frac{1}{3}C\right)^3\end{aligned}$$

Then  $y(0) = \left(\frac{1}{3}C\right)^3 = 0$  so  $C = 0$ . This gives the additional solution  $y = \left(\frac{1}{3}t\right)^3 = \frac{1}{27}t^3$ .

## 7. Overlap

At this juncture it might be helpful to notice that an ODE doesn't need to be just one of the categories we've looked at - explicit, first-order linear, and separable - it could fall into more than one category.

**Example:**  $y' = ty$  is both separable and first-order linear.

**Example:**  $y' = t^2$  is all of explicit, separable and first-order linear.

8. Justification for the strange approach.

It may bother you that we can rip apart the DE like we do and simply stick an integral sign on both sides. Really this is just shorthand notation for a more rigorous approach. More rigorously once our DE is rewritten as:

$$G(y(t))y'(t) = f(t)$$

Since they are equal we can integrate both sides with respect to  $t$ :

$$\int G(y(t))y'(t) dt = \int f(t) dt$$

On the left we make the substitution  $u = y(t)$  which yields  $\frac{du}{dt} = y'(t)$  which we then substitute in:

$$\int G(u) \frac{du}{dt} dt = \int f(t) dt$$

This then simplifies to:

$$\int G(u) du = \int f(t) dt$$

## MATH 246: Chapter 1 Section 5: Graphical Methods

Justin Wyss-Gallifent

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Main Topics:

- Quantitative versus Qualitative Solutions.
  - Phase-Line Portraits and Sketching Autonomous Systems.
  - Contour Plots for Implicit Solutions.
  - Slope Fields.
- 

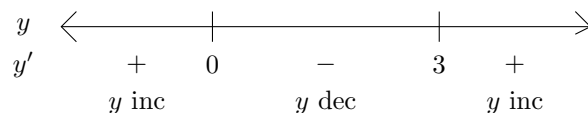
### 1. Introduction:

The overarching goal of this section is to find things out about solutions to DEs without actually solving them explicitly. Instead we attack them graphically.

### 2. Phase Line Portraits for Autonomous Differential Equations.

#### (a) Introductory Example.

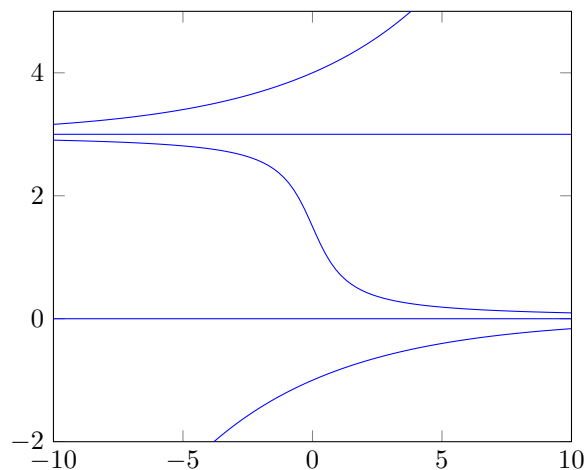
Consider the autonomous DE  $y' = y(y-3)$ . This has constant solutions  $y = 0$  and  $y = 3$ . But what about other solutions? Well the DE tells us that if we know  $y$  then we know  $y'$ . A sign chart giving information about  $y'$  can give us a wealth of information:



Consider now that:

- A solution below  $y = 0$  will be increasing. Moreover for larger positive  $t$ -values it will get closer to  $y = 0$  where it will level off and for larger negative  $t$ -values it will get closer to  $\infty$ .
- A solution between  $y = 0$  and  $y = 3$  will be decreasing. Moreover for larger positive  $t$ -values it will get closer to  $y = 0$  where it will level off and for larger negative  $t$ -values it will get closer to  $y = 3$  where it will level off.
- A solution above  $y = 3$  will be increasing. Moreover for larger positive  $t$ -values it will get closer to  $\infty$  and for larger negative  $t$ -values it will get closer to  $y = 3$  where it will level off.

Basically all possible solutions will look like these:



(b) General Understanding:

An autonomous DE will generally have constant solutions. Between the constant solutions is where interesting things happen. To see what's going on between the constant solutions draw a number line which gives information about whether solutions are increasing or decreasing.

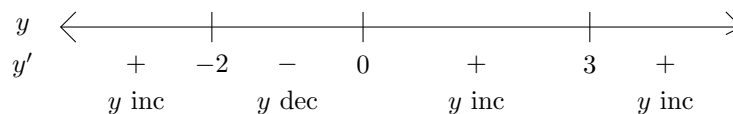
This number line is the *phase line portrait* of the autonomous differential equation.

Solutions approaching constant solutions will do so asymptotically. This is true both for larger positive  $t$ -values and larger negative  $t$ -values.

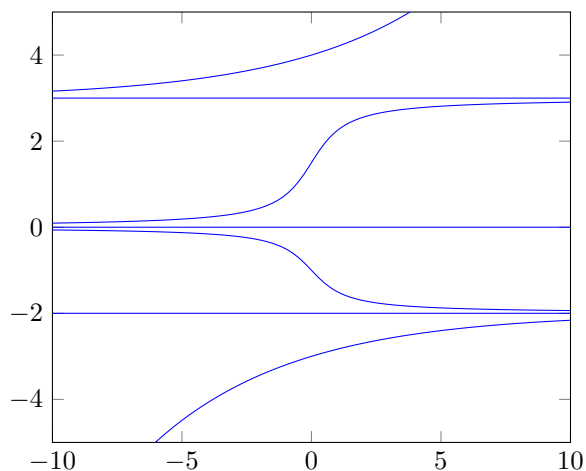
Moreover this gives some information about those constant solutions:

- If nearby solutions move away from the constant solution on both sides then the constant solution is unstable.
- If nearby solutions move toward the constant solution on both sides then the constant solution is stable.
- If there is different behavior on each side then the constant solution is *semistable*.

**Example:** Consider  $y' = y(y - 3)(y + 2)^2$ . This has the following sign chart:



Basically all possible solutions will look like these:



From these families of solutions we can draw all sorts of conclusions:

- The constant solution  $y = -2$  is stable, the constant solution  $y = 0$  is unstable, and the constant solution  $y = 3$  is semistable.
- The particular solution  $y(t)$  satisfying  $y(0) = \alpha$  has  $\lim_{t \rightarrow \infty} y(t) = 2$  when  $-\infty < \alpha < 0$ .
- The particular solution  $y(t)$  satisfying  $y(0) = \alpha$  is decreasing when  $-2 < \alpha < 0$ .
- $y = -2$  is stable.
- $y = 0$  is unstable.
- $y = 3$  is semistable.

### 3. Contour Plots of Implicit Solutions

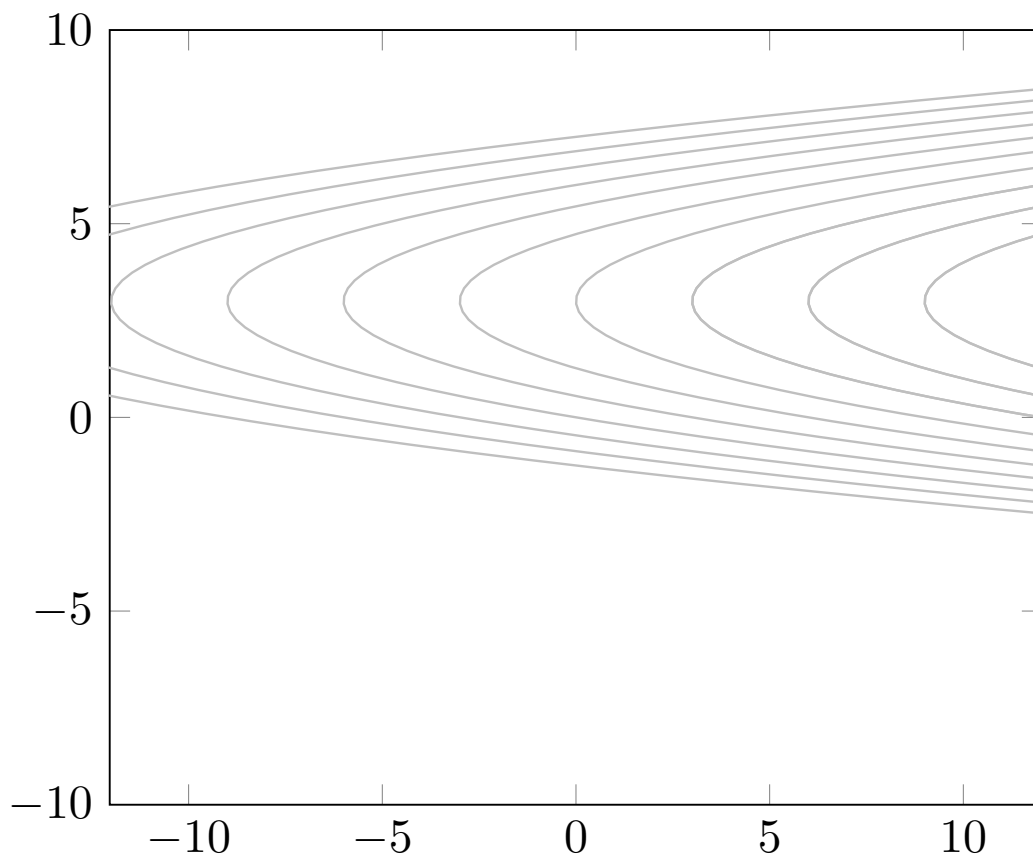
- (a) When we solve a separable DE we often get an implicit solution with a  $C$  in it. This implicit solution is an *equation*. If we pick various values of  $C$  and plot the resulting equations we get a *contour plot*.

What's useful about these contour plots is that the parts of the curves that form functions ( $y$  in terms of  $t$ ) that satisfy the implicit solution. This means that we can pick a point on a curve and follow it as far left and right as possible and the result is the graph of an explicit solution to the DE.

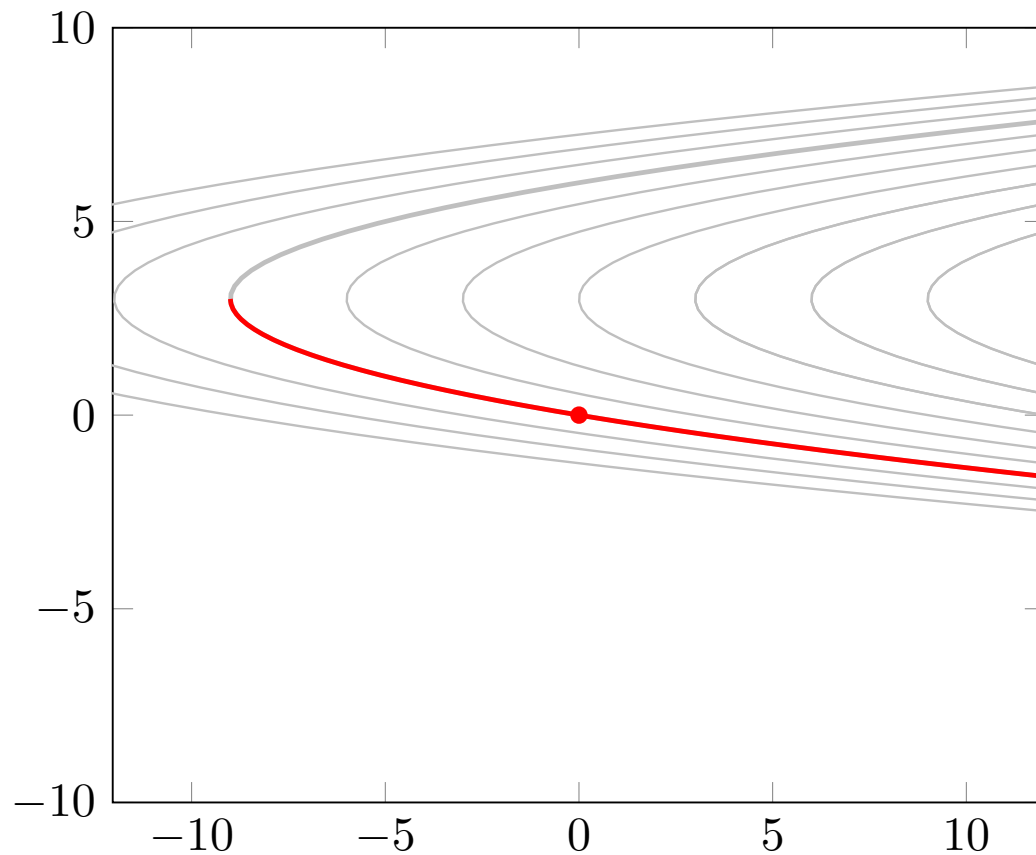
**Example:** Consider  $\frac{dy}{dx} = \frac{1}{2y-6}$ . This is separable with general solution  $y^2 - 6y = x + C$ . This is not as bad as it looks:

$$\begin{aligned}y^2 - 6y &= x + C \\y^2 - 6y + 9 &= x + C + 9 \\(y - 3)^2 &= x + C + 9\end{aligned}$$

These are all parabolas opening right with their vertices at  $y = 3$ . If we sketch a few of these (note that they extend out forever, this is just a subset):



Solutions to the DE are functions (must pass the vertical line test!) which lie along these parabolic curves. For example the specific solution satisfying  $y(0) = 0$  corresponds to  $C = 0$  and looks like this:



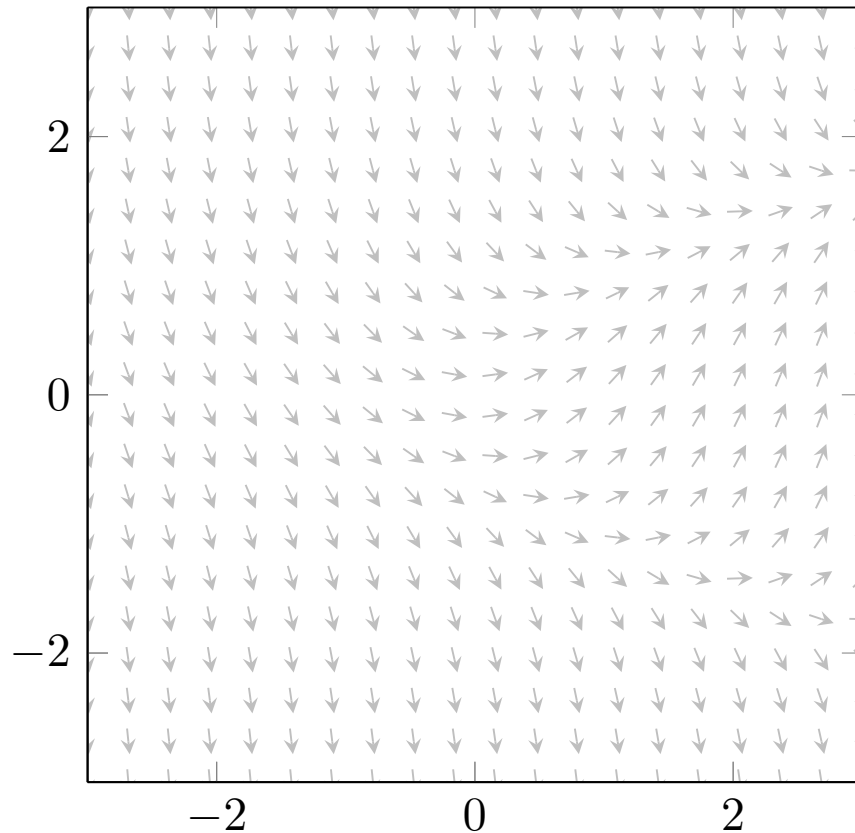
From this contour plot we can draw all sorts of conclusions:

- Solutions extend infinitely far to the right but not the left.
- Solutions are either increasing or decreasing but not both.
- The specific solution  $y(t)$  with  $y(0) = 0$  is a decreasing function with  $\lim_{t \rightarrow \infty} y(t) = -\infty$ .
- The specific solution with  $y(0) = 5$  is an increasing function.
- Solutions are either always increasing or always decreasing.

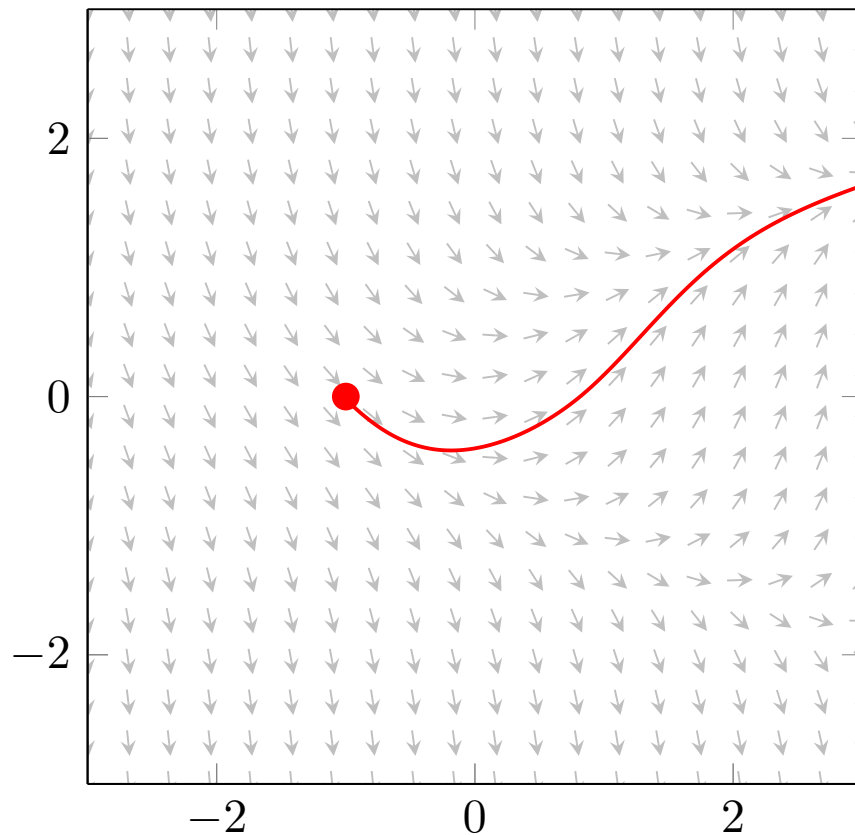


#### 4. Direction (Slope) Fields

- (a) As a last-ditch effort any  $\frac{dy}{dt} = f(t, y)$  (any first-order) is essentially telling us the slope of a solution at a point. Consequently we can plug in lots of  $t$  and  $y$  and indicate what the slope would be of a solution passing through that point. The result is a *direction field* or *slope field*. Then we can trace functions which follow the field and draw conclusions.
- (b) **Example:** Here is the direction field for  $\frac{dy}{dt} = t - y^2$ , a hard DE to solve:



Solutions to the DE are functions (must pass the vertical line test!) which follow these arrows. For example the specific solution satisfying  $y(-1) = 0$  looks like this:



From this direction field we can draw all sorts of conclusions:

- The specific DE solution satisfying  $y(-1) = 0$  has a relative minimum approximately  $(0, -0.5)$ .
- We can observe categories, for example not all solutions have relative minima.

Note however that we're somewhat restricted by the range we drew!

## MATH 246: Chapter 1 Section 6: Applications

### Justin Wyss-Gallifent

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Main Topics:

- Population Dynamics.
  - Water Tanks.
  - Motion.
- 

#### 1. Population dynamics:

- (a) Introduction: In precalculus you probably learned that if a population grows at rate 5% then it obeys the formula

$$P = Ae^{0.05t}$$

But why? The answer is that to say “a population grows at rate of 5%” means that the instantaneous change in population at any time equals 5% of the actual population, meaning:

$$p' = 0.05p$$

This is a first order linear differential equation (it's also separable). If we rewrite it as  $p' - 0.05p = 0$  then  $a(t) = -0.05$  so  $A(t) = -0.05t$  and the solution is:

$$p(t) = e^{-(-0.05t)} \int 0 \, dt = Ce^{0.05t}$$

That's why!

- (b) General Approach: Our general formula will involve a population with a certain growth rate  $R$  but in addition some new amount may arrive or depart every time period, maybe by being introduced, removed, etc. So in general we have

$$p' = Rp + a(t)$$

Our rate will always be constant but the amount that are introduced or subtracted may vary.

- (c) Examples:

**Example:** A population of monkeys starts with 100. It has a growth rate of 4% per year but an additional 8 monkeys join each year from a neighboring troop. Find the number of monkeys after  $t$  years.

**Solution:** Here we have  $p' = 0.04p + 8$  with  $p(0) = 100$ .

The solution (work omitted) is  $p = 300e^{0.04t} - 200$ .

**Example:** In a certain neighborhood there is a mosquito problem. The population starts at 10M and has a growth rate of 20% monthly. Traps are put out and these traps kill 3M monthly. Find the number of mosquitos after  $t$  months and determine when the mosquitos will be wiped out.

**Solution:** Here we have  $p' = 0.2p - 3$  with  $p(0) = 10$ .

The solution (work omitted) is  $p = -5e^{0.2t} + 15$  and if we solve  $-5e^{0.2t} + 15 = 0$  we get  $t = 5 \ln(3) \approx 5.49$  months.

## 2. Tanks.

- (a) Introduction: We have a tank that contains a saltwater mixture. As time goes by, saltwater is being pumped in and out. Our goal is to know how much salt there is at any time  $t$ .
- (b) General Approach: If  $Q$  is the amount of salt at time  $t$  then we'll have

$$Q' = \text{Rate In} - \text{Rate Out}$$

The only confusing thing about these problems is we usually have to do some work with quantities to figure out the rates.

- (c) Examples:

**Example:** A tank initially contains 500L of saltwater with a concentration of 0.2kg/L. Saltwater with a concentration of 0.3kg/L is being pumped in at 10L/min while the tank is being emptied of the mixture at the same rate. Find the amount of salt in the tank at time  $t$ .

**Solution:** We're interested in the quantity of salt, not saltwater.

- Initially there is  $(500 \text{ L})(0.2 \text{ kg/L})=100 \text{ kg}$  of salt so  $Q(0) = 100$ .
- Salt is entering at  $(10 \text{ L/min})(0.3 \text{ kg/L})=3 \text{ kg/min}$ .
- Salt is leaving at  $(10 \text{ L/min})(Q \text{ kg} / 500 \text{ L})=0.02Q \text{ kg/min}$ .

Note this is because at any instant there is  $y$  kg of salt in the tank and the tank always has 500 L of mixture in it because the rate in equals the rate out.

Therefore we have  $Q' = 3 - 0.02Q$  with  $Q(0) = 100$ .

This is first-order linear rewritten as  $Q' + 0.02Q = 3$  so we set  $a(t) = 0.02$  and so  $A(t) = 0.02t$  and then the solution is:

$$\begin{aligned} Q &= e^{-0.02t} \int 3e^{0.02t} dt \\ &= e^{-0.02t} [150e^{0.02t} + C] \\ &= 150 + Ce^{-0.02t} \end{aligned}$$

Then the initial value  $Q(0) = 100$  gives us  $C = -50$  and so the solution is  $Q = 150 - 50e^{-0.2t}$ .

**Example:** A 300 gal tank initially contains 200 gal of saltwater with a concentration of 0.15 lb/gal. Saltwater with a concentration of 0.2 lb/gal is being pumped in at 6 gal/min while the tank is being emptied of the mixture at 4 gal/min. How much salt will be in the tank when it is full?

**Solution:** Observe that the tank does not start out full but gains 2 gal/min, meaning after time  $t$  it will have  $200 + 2t$  gal in it. This will be important in the DE!

Again note:

- Initially there is  $(200 \text{ gal})(0.15 \text{ lb/gal})=30$  gal of salt so  $Q(0) = 30$ .
- Salt is entering at  $(6 \text{ gal/min})(0.2 \text{ lb/gal})=1.2 \text{ lb/min}$ .
- Salt is leaving at  $(4 \text{ gal/min})(Q \text{ lb} / 200 + 2t \text{ gal}) = \frac{4Q}{200+2t} \text{ lb/min}$ .  
Note this is because at any instant there is  $y$  lb of salt in the tank and the tank has  $200 + 2t$  gal in it.

Therefore we have  $Q' = 1.2 - \frac{4Q}{200+2t}$  with  $Q(0) = 30$ .

This is first-order linear rewritten as:

$$Q' + \left[ \frac{2}{100+t} \right] Q = 1.2$$

so we set  $a(t) = \frac{2}{100+t}$  and so  $A(t) = 2\ln(t+100)$  and then the solution is:

$$\begin{aligned} Q &= e^{-2\ln(t+100)} \int 1.2e^{2\ln(t+100)} dt \\ &= e^{\ln(t+100)^{-2}} \int 1.2e^{\ln(t+100)^2} dt \\ &= (t+100)^{-2} \int 1.2(t+100)^2 dt \\ &= (t+100)^{-2} [0.4(t+100)^3 + C] \\ &= 0.4(t+100) + \frac{C}{(t+100)^2} \end{aligned}$$

Using the initial value  $Q(0) = 30$ :

$$\begin{aligned} 30 &= 0.4(100) + \frac{C}{100^2} \\ 30 &= 40 + \frac{C}{10000} \\ -10 &= \frac{C}{10000} \\ C &= -100000 \end{aligned}$$

and so the solution is:

$$Q = 0.4(t+100) - \frac{100000}{(t+100)^2}$$

The tank is full when  $200 + 2t = 300$  so  $t = 50$ . The amount of salt is then

$$Q(50) = 0.4(50+100) - \frac{100000}{(50+100)^2}$$

### 3. Motion:

- (a) Introduction: In calculus you probably learned that a falling object with no air resistance has

$$a(t) = -9.8$$

But why? The answer comes from equating two forces. if the object has acceleration  $a(t)$  and mass  $m$  then the force on it is  $ma(t)$ . The force from gravity is  $-9.8m$ . When we equate these we get

$$ma(t) = -9.8m$$

and then we cancel the  $m$ .

- (b) Adding some air: When we add air resistance there are now two forces. First there is gravity pulling down and then drag (air resistance for example) pushing up. These two forces combine to form the total force. We know the total force is  $ma(t) = mv'(t)$ . Therefore

$$ma(t) = \text{force of gravity} + \text{drag force}$$

The force of gravity is  $-9.8m$ . The drag force is harder, it's  $mkv^2$  where  $k$  is the drag coefficient. Thus we have

$$ma(t) = -9.8m + mkv^2$$

or, cancelling the  $m$  again:

$$a(t) = -9.8 + kv^2$$

Finally we replace  $a(t)$  by  $v'(t)$  to get:

$$\frac{dv}{dt} = -9.8 + kv^2$$

- (c) General Approach: We'll generally just use the IVP

$$\frac{dv}{dt} = -9.8 + kv^2 \text{ with } v(0) = 0 \text{ (usually)}$$

to find  $v$  at time  $t$  and answer questions from that. Things to note:

- $v(0)$  might not be 0 if there is some initial velocity.
- In the Metric system  $k$  will be in  $m^{-1}$ , mass will be in kg and 9.8 stays as-is.
- In the English system  $k$  will be in  $ft^{-1}$ , mass will be in slugs and we use 32.2 (instead of 9.8).
- Terminal velocity occurs when  $v' = 0$  so this is when  $v = \sqrt{\frac{9.8}{k}}$ .
- If we know  $v$  then we can also find out distance travelled since  $v = h'$  and so  $h(t_E) - h(t_I) = \int_{t_I}^{t_E} v(t) dt$
- We can certainly change from air to some other substance or from earth gravity to some other standard. Information would have to be given.
- $\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \tanh^{-1} \frac{x}{a} + C$
- $\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}$
- $\tanh^{-1}(z) = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right)$
- $\int \tanh(z) = \ln(\cosh(z)) + C$
- $\cosh(z) = \frac{e^z + e^{-z}}{2}$

(d) Examples:

**Example:** A skydiver leaps out of a plane at 3000m. The drag coefficient is  $0.002m^{-1}$ . What is the IVP here? What is her terminal velocity? Find her velocity at time  $t$ .

**Solution:** We have  $\frac{dv}{dt} = -9.8 + 0.002v^2$  with  $v(0) = 0$ .

Her terminal velocity is  $v = \sqrt{\frac{9.8}{0.002}} = \sqrt{4900} = 70$  m/s.

The solution to the IVP is shown here:

$$\begin{aligned}\frac{dv}{dt} &= -9.8 + 0.002v^2 \\ \frac{dv}{dt} &= 0.002(-4900 + v^2) \\ \frac{1}{v^2 - 4900} dv &= 0.002 dt \\ \int \frac{1}{v^2 - 4900} dv &= \int 0.002 dt \\ -\frac{1}{70} \tanh^{-1}\left(\frac{v}{70}\right) &= 0.002t + C \\ \tanh^{-1}\left(\frac{v}{70}\right) &= -0.14t + C \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ v &= 70 \tanh(-0.14t + C)\end{aligned}$$

Then  $v(0) = 70 \tanh(C) = 0$  so  $C = \tanh^{-1} 0 = 0$  and our final answer is

$$v = -70 \tanh(0.14t)$$

At this point if we wish to know the height we can integrate:

$$h(t) = \int v(t) dt = \int -70 \tanh(0.14t) dt = -\frac{70}{0.14} \ln(\cosh(0.14t)) + C$$

Then we can use  $h(0) = 3000$  to find  $C$ .

**MATH 246: Chapter 1 Section 7: Approximation Methods**  
**Justin Wyss-Gallifent**

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Main Topics:

- Euler's Method (The Left-Sum Method).
  - The Runge-Trapezoid Method.
  - The Runge-Midpoint Method.
- 

1. Euler's Method

(a) Introduction Suppose we're dealing with the IVP given by:

$$\frac{dy}{dt} = t + y \text{ with } y(1) = 2$$

Suppose we'd really like to know  $y(2)$ .

The DE tells us that at the point  $(1, 2)$  the slope of the solution is  $\frac{dy}{dt}(1, 2) = 3$ . Of course the solution is not a straight line, meaning if we move right 1 we won't go up exactly 3, but if things aren't too bad then we would go up approximately 3. Thus we can conclude that  $y(1 + 1) \approx 2 + 3$  or  $y(2) \approx 5$ .

This approximately probably stinks, so what we can do instead is go to the right just 0.5 and up  $0.5(3)$ , then do the process again, now anchored at the new point. That is:

At  $(1, 2)$  the slope is  $\frac{dy}{dt}(1, 2) = 3$  so we go over 0.5 and up  $0.5(3)$  and now we're at  $(1 + 0.5, 2 + 0.5(3)) = (1.5, 3.5)$ .

At  $(1.5, 3.5)$  the slope is  $\frac{dy}{dt}(1.5, 3.5) = 5$  so we go over 0.5 and up  $0.5(5)$  and now we're at  $(1.5 + 0.5, 3.5 + 0.5(5)) = (2, 6)$

Then we conclude  $y(2) \approx 6$ . This approximation is probably better.



(b) Euler's Method.

This process is known as Euler's Method. We start with an IVP given by  $\frac{dy}{dt} = f(t, y)$  with  $y(t_0) = y_0$ . and we choose a small  $h$ . We did  $h = 1$  and then  $h = 0.5$ . We then proceed as follows:

$$\begin{aligned}(t_1, y_1) &= (t_0 + h, y_0 + hf(t_0, y_0)) \\ (t_2, y_2) &= (t_1 + h, y_1 + hf(t_1, y_1))\end{aligned}$$

Or, more generally:

Euler's Method

$$\begin{aligned}t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + hf(t_{i-1}, y_{i-1})\end{aligned}$$

**Example:** Again with  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . Let's approximate  $y(2)$  using  $n = 10$  steps of size  $h = 0.1$ .

This can all be put more nicely into a table as follows:

0	1	2	y(1)=2
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$1 + 0.1 = 1.1$	$2 + 0.3 = 2.3$	$y(1.1) \approx 2.3$
2	$1.1 + 0.1 = 1.2$	$2.3 + 0.34 = 2.64$	$y(1.2) \approx 2.64$
3	$1.2 + 0.1 = 1.3$	$2.64 + 0.384 = 3.024$	$y(1.3) \approx 3.024$
4	$1.3 + 0.1 = 1.4$	$3.024 + 0.4324 = 3.4564$	$y(1.4) \approx 3.4564$
5	$1.4 + 0.1 = 1.5$	$3.4564 + 0.48564 = 3.94204$	$y(1.5) \approx 3.94204$
6	$1.5 + 0.1 = 1.6$	$3.94204 + 0.544204 = 4.48624$	$y(1.6) \approx 4.48624$
7	$1.6 + 0.1 = 1.7$	$4.48624 + 0.608624 = 5.09487$	$y(1.7) \approx 5.09487$
8	$1.7 + 0.1 = 1.8$	$5.09487 + 0.679487 = 5.77436$	$y(1.8) \approx 5.77436$
9	$1.8 + 0.1 = 1.9$	$5.77436 + 0.757436 = 6.53179$	$y(1.9) \approx 6.53179$
10	$1.9 + 0.1 = 2$	$6.53179 + 0.843179 = 7.37497$	$y(2) \approx 7.37497$

Of course the further we go the less accurate we get but if the DE is not so bad then maybe we're good. The solution to the above DE (first-order linear) is  $y(t) = 4e^{t-1} - t - 1$  and so  $y(2) = 4e - 2 - 1 \approx 7.8731273138361809414411498854106$  so our approximation is not terrible.

**Example:** Same IVP but we could do better by reducing  $h$  and increasing the number of steps. Just for fun, compare to 1000 steps of size  $h = 0.001$  each and see how close the approximation is at the end!

Note: This was generated in Python and some approximation and truncation is taking place.

0	1	2	y(1)=2
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$1 + 0.001 = 1.001$	$2 + 0.003 = 2.003$	$y(1.001) \approx 2.003$
2	$1.001 + 0.001 = 1.002$	$2.003 + 0.003004 = 2.006$	$y(1.002) \approx 2.006$
3	$1.002 + 0.001 = 1.003$	$2.006 + 0.003008 = 2.00901$	$y(1.003) \approx 2.00901$
...	...	...	...
998	$1.997 + 0.001 = 1.998$	$7.83816 + 0.00983516 = 7.84799$	$y(1.998) \approx 7.84799$
999	$1.998 + 0.001 = 1.999$	$7.84799 + 0.00984599 = 7.85784$	$y(1.999) \approx 7.85784$
1000	$1.999 + 0.001 = 2$	$7.85784 + 0.00985684 = 7.8677$	$y(2) \approx 7.8677$

## 2. Improving:

First off recall that for a continuous function  $y(t)$  the Fundamental Theorem of Calculus tells us that:

$$\int_a^b \left[ \frac{dy}{dt} \right] dt = y(b) - y(a)$$

With our differential equation given that we're looking for some  $y(t)$  satisfying  $\frac{dy}{dt} = f(t, y(t))$  this translates to:

$$\int_a^b f(t, y(t)) dt = y(b) - y(a)$$

Given that we started this whole process knowing  $y_0$  and wanting  $y_1$  we can write:

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t)) dt$$

which can then be rewritten as our *Basic Formula*:

$$y_1 = y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt$$

So the real question is how to tackle the integral.

Let's revisit integrals. Suppose you wanted to know  $\int_a^b g(x) dx$  but couldn't do it. One really bad approximation is just a left rectangle. That is

$$\int_a^b g(x) dx \approx (b - a)g(a)$$

Using this in the *Basic Formula* yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + (t_1 - t_0)f(t_0, y(t_0)) \\ y_1 &\approx y_0 + (t_1 - t_0)f(t_0, y_0) \\ y_1 &\approx y_0 + hf(t_0, y_0) \end{aligned}$$

Well then, we've just got Euler's Method!

What this suggests is that better methods of approximating the integral yield better approximations for our IVP.

### 3. The Runge-Trapezoid Method:

A second way to approximate the integral would be to construct a trapezoid using the endpoints:

$$\int_a^b g(x) dx \approx \frac{1}{2}(b-a)(g(a) + g(b))$$

Using this in the *Basic Formula* yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + \frac{1}{2}(t_1 - t_0)(f(t_0, y(t_0)) + f(t_1, y(t_1))) \\ y_1 &\approx y_0 + \frac{1}{2}h(f(t_0, y_0) + f(t_0 + h, y(t_1))) \end{aligned}$$

Which is all fun and games until we notice the right side has an  $y(t_1)$  in it, and this is what we want. How can we resolve this? We do something slick and we plug in the result of Euler's Method into this:

$$y_1 \approx y_0 + \frac{1}{2}h(f(t_0, y_0) + f(t_0 + h, \underbrace{y_0 + hf(t_0, y_0)}_{\text{Euler}}))$$

Haha what fun. What we're really doing is using one approximation of  $y(t_1)$  to get what we think will be a better one.

Runge-Trapezoidal Method

$$\begin{aligned} t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + \frac{1}{2}h \left( f(t_{i-1}, y_{i-1}) + f(t_{i-1} + h, y_{i-1} + hf(t_{i-1}, y_{i-1})) \right) \end{aligned}$$

Back to our first IVP  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . If  $h = 0.1$  then proceeding one step gives us  $t_1 = 1.1$  and:

$$\begin{aligned} y_1 &\approx y_0 + \frac{1}{2}h (f(t_0, y_0) + f(t_0 + h, y_0 + hf(t_0, y_0))) \\ &\approx 2 + \frac{1}{2}(0.1) (f(1, 2) + f(1 + 0.1, 2 + 0.1f(1, 2))) \\ &\approx 2 + \frac{1}{2}(0.1) (1 + 2 + f(1.1, 2 + 0.1(1 + 2))) \\ &\approx 2 + \frac{1}{2}(0.1) (1 + 2 + 1.1 + 2 + 0.1(1 + 2)) = 2.32 \end{aligned}$$

Here's the Runge-Trapezoidal Method applied to our first IVP with 10 steps of size 0.1:

0	1	2	y(1)=2
$i$	$t_i$	$y_i$	So
1	$1 + 0.1 = 1.1$	2.32	$y(1.1) \approx 2.32$
2	$1.1 + 0.1 = 1.2$	2.6841	$y(1.2) \approx 2.6841$
3	$1.2 + 0.1 = 1.3$	3.09693	$y(1.3) \approx 3.09693$
4	$1.3 + 0.1 = 1.4$	3.56361	$y(1.4) \approx 3.56361$
5	$1.4 + 0.1 = 1.5$	4.08979	$y(1.5) \approx 4.08979$
6	$1.5 + 0.1 = 1.6$	4.68171	$y(1.6) \approx 4.68171$
7	$1.6 + 0.1 = 1.7$	5.34629	$y(1.7) \approx 5.34629$
8	$1.7 + 0.1 = 1.8$	6.09116	$y(1.8) \approx 6.09116$
9	$1.8 + 0.1 = 1.9$	6.92473	$y(1.9) \approx 6.92473$
10	$1.9 + 0.1 = 2$	7.85632	$y(2) \approx 7.85632$

Remember the exact value of  $y(2) = 4e - 2 - 1 \approx 7.8731273138361809414411498854106$ .

#### 4. The Runge-Midpoint Method:

A third way to approximate the integral is a midpoint rectangle:

$$\int_a^b g(x) dx \approx (b-a)g\left(\frac{a+b}{2}\right)$$

Using this in the *Basic Formula* and using the fact that our midpoint is  $t_0 + \frac{1}{2}h$  yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + (t_1 - t_0)f\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right) \\ y_1 &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right) \end{aligned}$$

Which again is all fun and games until we realize we don't know  $y(t_0 + \frac{1}{2}h)$  so we swap in Euler's Method again using a half-step, that is  $y_0 + \frac{1}{2}hf(t_0, y_0)$  and so

$$y_1 \approx y_0 + hf\left(t_0 + \frac{1}{2}h, \underbrace{y_0 + \frac{1}{2}hf(t_0, y_0)}_{\text{Euler}}\right)$$

#### Runge-Midpoint Method

$$\begin{aligned} t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + hf\left(t_{i-1} + \frac{1}{2}h, y_{i-1} + \frac{1}{2}hf(t_{i-1}, y_{i-1})\right) \end{aligned}$$

Back to our first IVP  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . If  $h = 0.1$  then proceeding one step gives us  $t_1 = 1.1$  and:

$$\begin{aligned} y_i &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(t_0, y_0)\right) \\ &\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)f(1, 2)\right) \\ &\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)(1 + 2)\right) \\ &\approx 2 + 0.1\left(1 + \frac{1}{2}(0.1) + 2 + \frac{1}{2}(0.1)(1 + 2)\right) = 2.32 \end{aligned}$$

This is actually the same as the Runge-Trapezoidal Method and in fact for this particular IVP the Runge-Midpoint Method applied to our first IVP actually gives the same result as the Runge-Trapezoidal Method, so we omit the full table.

5. Everything together:

Let  $y(t)$  be the solution to  $\frac{dy}{dt} = ty + t$  with  $y(0) = 1$ . Approximate  $y(1)$  using  $n = 10$  steps of size  $h = 0.1$ :

Euler			
0	0	1	y(0)=1
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$0 + 0.1 = 0.1$	$1 + 0 = 1$	$y(0.1) \approx 1$
2	$0.1 + 0.1 = 0.2$	$1 + 0.02 = 1.02$	$y(0.2) \approx 1.02$
3	$0.2 + 0.1 = 0.3$	$1.02 + 0.0404 = 1.0604$	$y(0.3) \approx 1.0604$
4	$0.3 + 0.1 = 0.4$	$1.0604 + 0.061812 = 1.12221$	$y(0.4) \approx 1.12221$
5	$0.4 + 0.1 = 0.5$	$1.12221 + 0.0848885 = 1.2071$	$y(0.5) \approx 1.2071$
6	$0.5 + 0.1 = 0.6$	$1.2071 + 0.110355 = 1.31746$	$y(0.6) \approx 1.31746$
7	$0.6 + 0.1 = 0.7$	$1.31746 + 0.139047 = 1.4565$	$y(0.7) \approx 1.4565$
8	$0.7 + 0.1 = 0.8$	$1.4565 + 0.171955 = 1.62846$	$y(0.8) \approx 1.62846$
9	$0.8 + 0.1 = 0.9$	$1.62846 + 0.210277 = 1.83873$	$y(0.9) \approx 1.83873$
10	$0.9 + 0.1 = 1$	$1.83873 + 0.255486 = 2.09422$	$y(1) \approx 2.09422$

Runge-Trapezoidal			
0	0	1	y(0)=1
$i$	$t_i$	$y_i$	So
1	$0 + 0.1 = 0.1$	1.01	$y(0.1) \approx 1.01$
2	$0.1 + 0.1 = 0.2$	1.04035	$y(0.2) \approx 1.04035$
3	$0.2 + 0.1 = 0.3$	1.09197	$y(0.3) \approx 1.09197$
4	$0.3 + 0.1 = 0.4$	1.16645	$y(0.4) \approx 1.16645$
5	$0.4 + 0.1 = 0.5$	1.2661	$y(0.5) \approx 1.2661$
6	$0.5 + 0.1 = 0.6$	1.39414	$y(0.6) \approx 1.39414$
7	$0.6 + 0.1 = 0.7$	1.55478	$y(0.7) \approx 1.55478$
8	$0.7 + 0.1 = 0.8$	1.75355	$y(0.8) \approx 1.75355$
9	$0.8 + 0.1 = 0.9$	1.99751	$y(0.9) \approx 1.99751$
10	$0.9 + 0.1 = 1$	2.29576	$y(1) \approx 2.29576$

Runge-Midpoint			
0	0	1	y(0)=1
$i$	$t_i$	$y_i$	So
1	$0 + 0.1 = 0.1$	1.01	$y(0.1) \approx 1.01$
2	$0.1 + 0.1 = 0.2$	1.0403	$y(0.2) \approx 1.0403$
3	$0.2 + 0.1 = 0.3$	1.09182	$y(0.3) \approx 1.09182$
4	$0.3 + 0.1 = 0.4$	1.16613	$y(0.4) \approx 1.16613$
5	$0.4 + 0.1 = 0.5$	1.26556	$y(0.5) \approx 1.26556$
6	$0.5 + 0.1 = 0.6$	1.39328	$y(0.6) \approx 1.39328$
7	$0.6 + 0.1 = 0.7$	1.55351	$y(0.7) \approx 1.55351$
8	$0.7 + 0.1 = 0.8$	1.75172	$y(0.8) \approx 1.75172$
9	$0.8 + 0.1 = 0.9$	1.99497	$y(0.9) \approx 1.99497$
10	$0.9 + 0.1 = 1$	2.2923	$y(1) \approx 2.2923$

For reference the actual answer is  $2e^{0.5} - 1 \approx 2.2974425414002562936973015756283$ .

**MATH 246: Chapter 1 Section 8: Exact Differential Equations**  
**Justin Wyss-Gallifent**

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Main Topics:

- Exact Differential Equations.
  - Method of Solution.
  - Integrating Factors.
- 

1. **A Bit of History and Introduction:** Suppose  $H(x, y)$  is a function and  $y$  is a function of  $x$ . Then by the chain rule we know  $\frac{d}{dx}H(x, y) = H_x(x, y) + H_y(x, y)\frac{dy}{dx}$ .

So now consider the following differential equation:

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

You may notice that the left side looks like the result of the chain rule and is actually so, when  $H(x, y) = x^3y^2$ . Don't worry about if there's a formal method for where  $H(x, y)$  comes from for now, just notice that  $H_x(x, y) = 3x^2y^2$  and  $H_y(x, y) = 2x^3y$ . What this means is that the differential equation may be rewritten by undoing the chain rule on the left:

$$\underbrace{3x^2y^2 + 2x^3y\frac{dy}{dx}}_{\frac{d}{dx}[x^3y^2]} = 0$$

So then when the derivative of something is zero, that thing is a constant:

$$\begin{aligned}\frac{d}{dx}[x^3y^2] &= 0 \\ x^3y^2 &= C\end{aligned}$$

and we've solved it, at least implicitly!

2. **Definition and Method:** A differential equation is *exact* if it has the form:

$$H_x(x, y) + H_y(x, y)\frac{dy}{dx} = 0$$

for some function  $H(x, y)$ . When a differential equation is exact, solving implicitly is as easy as finding  $H(x, y)$  and setting  $H(x, y) = C$  for any constant.

Here are a few exact differential equations. For each,  $H(x, y)$  is written in the middle and the implicit solution to the right.

Exact DE	$H(x, y)$	Solution to DE
$y + x\frac{dy}{dx} = 0$	$H(x, y) = xy$	$xy = C$
$y + (x + 2y)\frac{dy}{dx} = 0$	$H(x, y) = xy + y^2$	$xy + y^2 = C$
$\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$	$H(x, y) = \frac{x}{y}$	$\frac{x}{y} = C$
$y \cos(xy) + x \cos(xy)\frac{dy}{dx} = 0$	$H(x, y) = \sin(xy)$	$\sin(xy) = C$

3. **Detecting Exactness and Finding H:** There is a trick to detecting whether a differential equation is exact. If the differential equation has the form:

$$M + N \frac{dy}{dx} = 0$$

then it is exact if and only if  $M_y = N_x$ . You can test all the ones above. Then you can check that this next one is not exact:

$$xy + y \frac{dy}{dx} = 0$$

In this case  $M_y = x$  and  $N_x = 0$ . Not equal, not exact.

Once you know that your differential equation is exact, often you can guess at  $H(x, y)$ . However if you're struggling, there's a systematic method for finding it. Here's an example from above:

$$y + (x + 2y) \frac{dy}{dx} = 0$$

We want  $H(x, y)$  with (A)  $H_x(x, y) = y$  and (B)  $H_y(x, y) = x + 2y$ . Observe:

We want (A):	$H_x(x, y) = y$
This tells us that:	$H(x, y) = xy + h(y)$
From this line:	$H_y(x, y) = x + h'(y)$
But from (B):	$H_y(x, y) = x + 2y$
Set these equal:	$x + h'(y) = x + 2y$
Solve for $h'(y)$ :	$h'(y) = 2y$
Find $h(y)$ :	$h(y) = y^2 + D$
Put back into second line:	$H(x, y) = xy + y^2 + D$

We can choose any  $D$  so choose  $D = 0$  to get  $H(x, y) = xy + y^2$ .

**Example:** Find  $H(x, y)$  to solve  $x + 1 + \frac{1}{y} - \frac{x}{y^2} \frac{dy}{dx} = 0$ . Follow the exact procedure above, here we want (A)  $H_x(x, y) = x + 1 + \frac{1}{y}$  and (B)  $H_y(x, y) = -\frac{x}{y^2}$ :

We want (A):	$H_x(x, y) = x + 1 + \frac{1}{y}$
This tells us that:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + h(y)$
From this line:	$H_y(x, y) = -\frac{x}{y^2} + h'(y)$
But from (B):	$H_y(x, y) = -\frac{x}{y^2}$
Set these equal:	$-\frac{x}{y^2} + h'(y) = -\frac{x}{y^2}$
Solve for $h'(y)$ :	$h'(y) = 0$
Find $h(y)$ :	$h(y) = D$
Put back into second line:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + D$

Then choose  $D = 0$  to get  $H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y}$  and the solution to our DE is  $\frac{1}{2}x^2 + x + \frac{x}{y} = C$ .

4. **Integrating Factors:** It's not uncommon to have a differential equation which is not quite exact but can be made exact by multiplying through by some function called an *integrating factor*. For example the differential equation

$$2y + x \frac{dy}{dx} = 0$$

is not exact because  $M_y = 2$  and  $N_x = 1$  so  $M_y \neq N_x$ . But if we multiply through by  $x$  we get the new differential equation

$$2xy + x^2 \frac{dy}{dx} = 0$$

which is exact because  $M_y = 2x$  and  $N_x = 2x$ . Now  $H(x, y) = x^2y$  and the solution is  $x^2y = C$ .

The question is how to come up with this integrating factor. This is very challenging so we'll look at two specific cases, either the integrating factor is a function  $f(x)$  of only  $x$  or the integrating factor is a function  $g(y)$  of only  $y$ .



## 5. Examples:

**Example 1:** Consider the differential equation we've seen before:

$$2y + x \frac{dy}{dx} = 0$$

Here  $M = 2y$  and  $N = x$ , these are different so it's not exact. Let's look for some  $f(x)$  so that when we multiply through the result is exact:

$$2yf(x) + xf(x) \frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned} [xf(x)]_x &= [2yf(x)]_y \\ 1f(x) + xf'(x) &= 2f(x) + 2y(0) \\ xf'(x) &= f(x) \\ f'(x) &= \frac{f(x)}{x} \end{aligned}$$

We can see that  $f(x) = x$  does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$2xy + x^2 \frac{dy}{dx} = 0$$

which has  $H(x, y) = x^2y$  and hence solution  $x^2y = C$ .

**Addendum:** If we tried  $g(y)$  we'd want this to be exact:

$$2yg(y) + xg(y) \frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned} [xg(y)]_x &= [2yg(y)]_y \\ 1g(y) + x(0) &= 2g(y) + 2yg'(y) \\ g'(y) &= -\frac{g(y)}{2y} \end{aligned}$$

It's much harder to see what might work here. Interestingly  $g(y) = y^{-1/2}$  will work, yielding the exact:

$$2y^{1/2} + xy^{-1/2} \frac{dy}{dx}$$

which has  $H(x, y) = 2xy^{1/2}$  and hence solution  $2xy^{1/2} = C$ .

**Example 2:** Consider the differential equation

$$y + (x + xy) \frac{dy}{dx} = 0$$

Here  $M = y$  and  $N = x + xy$ , these are different so it's not exact. Let's look for some  $g(y)$  so that when we multiply through the result is exact:

$$yg(y) + (x + xy)g(y) \frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned} [(x + xy)g(y)]_x &= [yg(y)]_y \\ (1 + y)g(y) &= 1g(y) + yg'(y) \\ g(y) + yg(y) &= g(y) + yg'(y) \\ g'(y) &= g(y) \end{aligned}$$

We can see that  $g(y) = e^y$  does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$ye^y + (x + xy)e^y \frac{dy}{dx} = 0$$

which has  $H(x, y) = xye^y$  and hence solution  $xye^y = C$ .

**Addendum:** If we tried  $f(x)$  we'd want this to be exact:

$$yf(x) + (x + xy)f(x) \frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned} [(x + xy)f(x)]_x &= [yf(x)]_y \\ (1 + y)f(x) + (x + xy)f'(x) &= 1f(x) + y(0) \\ f(x) + yf(x) + (x + xy)f'(x) &= f(x) \\ yf(x) + (x + xy)f'(x) &= 0 \\ f'(x) &= -\frac{yf(x)}{(x + xy)} \end{aligned}$$

It's not at all obvious if anything this works.

## MATH 246: Chapter 2 Section 1: Intro to Higher Order Linear

Justin Wyss-Gallifent

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Main Topics:

- Reminder and Notation.
  - Interval of Existence.
- 

1. **Introduction** Since higher order DEs are difficult we're going to focus on linear higher order DEs. We'll narrow it down even more but for now that's where we are. Just a reminder that these look like, all in *linear normal form*:

First-Order	$y' + a(t)y = f(t)$ (We can solve these)
Second-Order	$y'' + a(t)y' + b(t)y = f(t)$
Third-Order	$y''' + a(t)y'' + b(t)y' + c(t)y = f(t)$
Etc.	Etc.

### 2. Notation Note

It's not uncommon to see an alternate notation for the derivative from here on. We often write  $Dy$  instead of  $y'$ ,  $D^2y$  instead of  $y''$  and so on.

### 3. Existence Theory

The theory is similar to what we've seen for first-order but the initial value needs a bit more:

- A first order linear IVP requires knowing  $y(t_I) = y_0$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $f(t)$  are differentiable.
- A second order linear IVP requires knowing  $y(t_I) = y_0$  and  $y'(t_I) = y_1$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$ ,  $b(t)$  and  $f(t)$  are differentiable.
- A third order linear IVP requires knowing  $y(t_I) = y_0$  and  $y'(t_I) = y_1$  and  $y''(t_I) = y_2$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$ ,  $b(t)$ ,  $c(t)$  and  $f(t)$  are differentiable.
- From here you can certainly see the pattern.

Note: The proof (of existence and uniqueness) is difficult. The special case for first order linear is easy and we saw it because we explicitly constructed the solution.

**Example:**  $y'' + \frac{1}{t}y' - \frac{1}{t-3}y = t$  with  $y(1) = 17$  and  $y'(1) = 2$  has a unique solution on  $(0, 3)$ . If instead we have  $y(4) = 17$  then this has a unique solution on  $(3, \infty)$ .

**Example:** For  $t^{-1/2}y'' + e^t y' - \sin(t)y = \frac{t}{6-t}$  with  $y(1) = 8$  and  $y'(1) = 3$  we have to first rewrite in linear normal form as  $y'' + e^t \sqrt{t}y' - \sqrt{t}\sin(t)y = \frac{t^{3/2}}{6-t}$  which then has a unique solution on  $(0, 6)$ .

**Example:** The IVP  $y''' - \frac{1}{t}y'' + e^t y' - \sin(t)y = \frac{t}{10-t}$  with  $y(3) = 8$  and  $y'(3) = 3$  and  $y''(3) = 5$  has a unique solution on  $(0, 10)$ .

**Example:** The IVP  $D^2y - Dy - 2y = 0$  with  $y(0) = 1$  and  $y'(0) = -3$  has a unique solution on  $(-\infty, \infty)$ . If we notice that  $y = e^{2x}$  is a solution then we know it's the only solution.

## MATH 246: Chapter 2 Section 2: Homogeneous Equations - Method and Theory

Justin Wyss-Gallifent

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Main Topics:

- Definition of Homogeneous.
  - Motivational Example for Second Order.
  - Theory for Second and Third Order.
- 

1. **Introduction:** Since even linear higher-order DEs are difficult we are going to simplify even more. For today we're going to look at *homogeneous* higher-order linear DEs, in which the forcing function  $f(t)$  is equal to 0. That is:

First-Order	$y' + a(t)y = 0$
Second-Order	$y'' + a(t)y' + b(t)y = 0$
Third-Order	$y''' + a(t)y'' + b(t)y' + c(t)y = 0$
$\vdots$	$\vdots$

2. **A Motivational Example:** Consider the second-order homogeneous linear DE:

$$y'' - y' - 2y = 0$$

Next look at the following two functions, don't worry about where they came from:

$$Y_1(t) = e^{2t} \text{ and } Y_2(t) = e^{-t}$$

We can easily see that these are both solutions to the DE by plugging them (and their derivatives) in and checking.

- (a) **Observation 1 - Getting More Solutions:**

Notice that if we take a *linear combination* of these two, meaning

$$Y(t) = C_1 e^{2t} + C_2 e^{-t}$$

where  $C_1$  and  $C_2$  are constants. Then we can easily see that this is also a solution to the DE by plugging it (and its derivatives) in and checking.

- (b) **Observation 2 - Getting All Solutions:**

We can build new solutions from these two but can we build all solutions this way? Well suppose that we had some solution to the DE, call it  $Y(t)$ . What we want to know is if we can find  $C_1$  and  $C_2$  so that  $Y(t) = C_1 e^{2t} + C_2 e^{-t}$  for this  $Y(t)$ ?

Well, suppose we find that  $Y(0) = y_0$  and  $Y'(0) = y_1$ . Since  $Y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$  we would need

$$\begin{aligned} y_0 &= Y(0) = C_1 + C_2 \\ y_1 &= Y'(0) = 2C_1 - C_2 \end{aligned}$$

Can we find such values? Since  $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \neq 0$  there is a unique solution.

Notice now that since this is a solution to the IVP and since there is only one solution to the IVP this must be the solution we were looking for.

(c) **Observation 3 - Anything Special About Those Two?**

We can't just start with any two solutions. To see this observe that if we'd started with  $Y_1(t) = e^{2t}$  and  $Y_2(t) = 17e^{2t}$  that both of these are solutions. Again any linear combination  $Y(t) = C_1e^{2t} + C_217e^{2t}$  is a solution. However is every solution to the DE a linear combination? Again, suppose  $Y(t)$  is a solution and  $Y(0) = y_0$  and  $Y'(0) = y_1$ . Then  $Y'(t) = 2C_1e^{2t} + 34C_2e^{2t}$  and we would need

$$\begin{aligned}y_0 &= Y(0) = C_1 + 17C_2 \\y_1 &= Y'(0) = 2C_1 + 34C_2\end{aligned}$$

Since  $\begin{vmatrix} 1 & 17 \\ 2 & 34 \end{vmatrix} = 0$  there may be no solution. That is, we can't guarantee a solution.

3. **Theory:**

(a) **Theory for Second-Order Homogeneous:**  $y'' + a(t)y' + b(t)y = 0$

- For a second-order homogeneous linear DE we need to find two solutions  $Y_1(t)$  and  $Y_2(t)$  with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2] = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}$$

Alternately the two solutions cannot be multiples of each other. They form a *fundamental set* or *fundamental pair* of solutions  $\{Y_1(t), Y_2(t)\}$ .

- Every solution is then a linear combination of the fundamental pair. This means the general solution is  $Y(t) = C_1Y_1(t) + C_2Y_2(t)$ .
- A second-order IVP must provide  $y(t_I)$  and  $y'(t_I)$  in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $b(t)$  are differentiable.

(b) **Theory for Third-Order Homogeneous:**  $y''' + a(t)y'' + b(t)y' + c(t)y = 0$

- For a third-order homogeneous linear DE we need to find three solutions  $Y_1(t)$ ,  $Y_2(t)$ , and  $Y_3(t)$  with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} Y_1 & Y_2 & Y_3 \\ Y_1' & Y_2' & Y_3' \\ Y_1'' & Y_2'' & Y_3'' \end{vmatrix}$$

Alternately it must be impossible to write one of the solutions as a linear combination of the others. They form a *fundamental set* of solutions  $\{Y_1(t), Y_2(t), Y_3(t)\}$ .

- Every solution is then a linear combination of the fundamental set. This means the general solution is  $Y(t) = C_1Y_1(t) + C_2Y_2(t) + C_3Y_3(t)$ .
- A third-order IVP must provide  $y(t_I)$ ,  $y'(t_I)$ , and  $y''(t_I)$  in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $b(t)$  and  $c(t)$  are differentiable.

(c) **Theory for Higher-Order:**

You can probably see the pattern.

- (d) **Critical Note:** Don't worry about where these fundamental sets are coming from right now, just realize that we (somehow) need obtain them!

#### 4. Practice for Both:

Here are some examples:

**Example:** Consider  $y'' + 4y = 0$ . First we'll show that  $Y_1(t) = \sin(2t)$  and  $Y_2(t) = \cos(2t)$  form a fundamental pair. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} = -2\sin^2(2t) - 2\cos^2(2t) = -2 \neq 0$$

This tells us that  $Y_1(t)$  and  $Y_2(t)$  form a fundamental pair and that the general solution is:

$$Y(t) = C_1 \sin(2t) + C_2 \cos(2t)$$

So now if we have the IVP with  $Y(0) = 4$  and  $Y'(0) = 2$  we can find the specific solution by first finding:

$$Y'(t) = 2C_1 \cos(2t) - 2C_2 \sin(2t)$$

and then solving the system:

$$\begin{aligned} 4 &= Y(0) = C_1 \sin(2(0)) + C_2 \cos(2(0)) = C_2 \\ 2 &= Y'(0) = 2C_1 \cos(2(0)) - 2C_2 \sin(2(0)) = 2C_1 \end{aligned}$$

So that  $C_1 = 1$  and  $C_2 = 4$  and the specific solution is:

$$Y(t) = \sin(2t) + 4\cos(2t)$$

**Example:** Consider  $(1 + t^2)y'' - 2ty' + 2y = 0$ . First we'll show that  $Y_1(t) = t$  and  $Y_2(t) = t^2 - 1$  form a fundamental pair. Notice that it doesn't matter whether we divide by  $1 + t^2$  or not when we check these. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2 - 1 \\ 1 & 2t \end{vmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \neq 0$$

This tells us that  $Y_1(t)$  and  $Y_2(t)$  form a fundamental pair and that the general solution is:

$$Y(t) = C_1 t + C_2(t^2 - 1)$$

So now if we have the IVP with  $Y(2) = -5$  and  $Y'(2) = 7$  we can find the specific solution by first finding:

$$Y'(t) = C_1 + 2C_2 t$$

and then solving the system:

$$\begin{aligned} -5 &= Y(2) = C_1(2) + C_2(2^2 - 1) = 2C_1 + 3C_2 \\ 7 &= Y'(2) = C_1 + 2C_2(2) = C_1 + 4C_2 \end{aligned}$$

So that  $C_1 = -\frac{41}{5}$  and  $C_2 = -\frac{19}{5}$  and the specific solution is:

$$Y(t) = -\frac{41}{5}t + \frac{19}{5}(t^2 - 1)$$

**Example:** Consider  $D^3y - 2D^2y = 0$  First we'll show that  $Y_1(t) = 1$ ,  $Y_2(t) = t$  and  $Y_3(t) = e^{2t}$  form a fundamental set. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} 1 & t & e^{2t} \\ 0 & 1 & 2e^{2t} \\ 0 & 0 & 4e^{2t} \end{vmatrix} = 4e^{2t} \neq 0$$

This tells us that  $Y_1(t)$ ,  $Y_2(t)$  and  $Y_3(t)$  form a fundamental set and that the general solution is:

$$Y(t) = C_1 + C_2t + C_3e^{2t}$$

So now if we have the IVP with  $Y(0) = 1$ ,  $Y'(0) = 0$  and  $Y''(0) = -4$  we can find the specific solution by first finding:

$$Y'(t) = C_2 + 2C_3e^{2t}$$

$$Y''(t) = 4C_3e^{2t}$$

and then solving the system:

$$1 = Y(0) = C_1 + C_3$$

$$0 = Y'(0) = C_2 + 2C_3$$

$$-4 = Y''(0) = 4C_3$$

So that  $C_3 = -1$ ,  $C_2 = 2$  and  $C_1 = 2$  and the specific solution is:

$$Y(t) = 2 + 2t - e^{2t}$$

## 5. More about Fundamental Sets:

### (a) Natural Fundamental Sets

There's more than just one fundamental set, and one that comes up a lot is called the *natural fundamental set*.

In the second-order case this is the set  $\{Y_1, Y_2\}$  with  $Y_1$  having  $Y_1(t_I) = 1$  and  $Y_1'(t_I) = 0$  and with  $Y_2$  having  $Y_2(t_I) = 0$  and  $Y_2'(t_I) = 1$ .

In the third-order case this is the set  $\{Y_1, Y_2, Y_3\}$  with  $Y_1$  having  $Y_1(t_I) = 1$ ,  $Y_1'(t_I) = 0$ , and  $Y_1''(t_I) = 0$ , with  $Y_2$  having  $Y_2(t_I) = 0$ ,  $Y_2'(t_I) = 1$ , and  $Y_2''(t_I) = 0$ , and with  $Y_3$  having  $Y_3(t_I) = 1$ ,  $Y_3'(t_I) = 0$ , and  $Y_3''(t_I) = 1$ ,

Beyond there you can probably see the pattern.

### (b) Reduction of Order (OMITTED)

The big question of course is where the fundamental set comes from. We'll address that a bit later but for now we have one helper.

If we have one solution  $Y_1(t)$  then the second one is very often a multiple of the first. So we can set  $Y_2(t) = uY_1(t)$  and when we plug this into the DE and use the fact that  $Y_1(t)$  is a solution we end up in a situation where we can find a first-order DE (hence the name) that we can use to find  $u$ .

**Example:** You can check that  $Y_1(t) = e^{5t}$  is a solution to  $y'' - 3y' - 10y = 0$ . To find the other by reduction of order we put  $Y_2(t) = ue^{5t}$ . We then find

$$Y_2'(t) = u'e^{5t} + 5ue^{5t} \text{ and } Y_2''(t) = u''e^{5t} + 5u'e^{5t} + 5u'e^{5t} + 25ue^{5t} = u''e^{5t} + 10u'e^{5t} + 25ue^{5t}$$

and plug these into the DE:

$$\begin{aligned} y'' - 3y' - 10y &= 0 \\ (u''e^{5t} + 10u'e^{5t} + 25ue^{5t}) - 3(u'e^{5t} + 5ue^{5t}) - 10(ue^{5t}) &= 0 \\ u'' + 10u' + 25u - 3u' - 15u - 10u &= 0 \\ u'' + 7u' &= 0 \end{aligned}$$

If we let  $w = u'$  then this gives us  $w' + 7w = 0$  which has solution  $w = Ce^{-7t}$  and so  $u' = Ce^{-7t}$  and so  $u = -\frac{1}{7}Ce^{-7t} + D$  and another solution is

$$Y_2(t) = \left(-\frac{1}{7}Ce^{-7t} + D\right)e^{5t} = -\frac{1}{7}Ce^{-2t} + De^{5t}$$

Since this is true for any  $C$  and  $D$  we can pick the solution

$$Y_2(t) = e^{-2t}$$

for which  $W[Y_1, Y_2] \neq 0$  and we have our fundamental pair.



## MATH 246: Chapter 2 Section 3: Matrices and Determinants

Justin Wyss-Gallifent

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Main Topics:

- Matrices
  - Determinants
  - Relationship to Linear Systems
- 

### 1. Introduction

As the course progresses we'll run into matrices and we'll need some basic facts. For now we simply need to know what a matrix is, what a determinant is, and what they can be used for.

### 2. Matrices

A matrix is basically a rectangular array of numbers. In this course pretty much all the matrices we'll work with will be square and either  $2 \times 2$  or  $3 \times 3$ .

**Examples:**

$$A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{bmatrix}$$

### 3. Determinants

The determinant of a matrix is a single number associated with the matrix which tells us certain properties of that matrix. It is the single most important number associated with a matrix. It can be denoted either by putting  $\det$  in front of the matrix or by putting the matrix values (not the brackets) inside vertical bars like absolute values.

It can be defined recursively but we'll only need it for  $2 \times 2$  and  $3 \times 3$  so here are the rules:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

**Example:**  $\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = (3)(1) - (-2)(0) = 3$

**Example:**  $\begin{vmatrix} 1 & 3 \\ -5 & 7 \end{vmatrix} = (1)(7) - (3)(-5) = 22$

**Example:**

$$\begin{aligned} \begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} &= 4 \begin{vmatrix} 5 & 0 \\ 8 & -3 \end{vmatrix} - 3 \begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} + 1 \begin{vmatrix} -2 & 5 \\ 0 & 8 \end{vmatrix} \\ &= 4(-15) - 3(6) + 1(-16) \\ &= -94 \end{aligned}$$

#### 4. Relationship to Systems of Equations

In linear algebra matrices are used to solve linear systems of equations. We don't need to do that but we do need to know that determinants of matrices can tell us information about the solutions.

##### (a) All Systems

If we put the coefficients of the variables into a matrix and find the determinant then this determinant will be nonzero if and only if there is a unique solution to the system. If we do get zero then there will be either no solutions or infinitely many solutions. For now we need not distinguish between these outcomes.

**Example:** The system

$$2x + 3y = 7$$

$$5x - 7y = 23$$

Since  $\begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix} = -29 \neq 0$  there is only one solution.

**Example:** The system

$$4x + 8y = 3$$

$$6x + 12y = -8$$

Since  $\begin{vmatrix} 4 & 8 \\ 6 & 12 \end{vmatrix} = 0$  there are either no solutions or infinitely many solutions.

**Example:** The system

$$4x + 3y + 1z = 7$$

$$-2x + 5y + 0z = -17$$

$$0x + 8y - 3z = 2$$

Since  $\begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} = -94 \neq 0$  there is only one solution.

(b) **Homogenous Systems**

A homogeneous linear system is when all the constant terms are 0. Setting all the variables to be zero always gives a solution, called the *trivial solution*. In this case if the determinant is zero then there must be infinitely many solutions, meaning there are *nontrivial solutions*, and if the determinant is nonzero then there is only the trivial solution, so there are no nontrivial solutions.

**Example:** The system

$$2x + 3y = 0$$

$$5x - 7y = 0$$

has the trivial solution  $x = y = 0$ . In addition since  $\begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix} = -29 \neq 0$  this is the only solution.

**Example:** The system

$$4x + 8y = 0$$

$$6x + 12y = 0$$

has the trivial solution  $x = y = 0$ . In addition since  $\begin{vmatrix} 4 & 8 \\ 6 & 12 \end{vmatrix} = 0$  there are (infinitely many) other nontrivial solutions.

**Example:** The system

$$4x + 3y + 1z = 0$$

$$-2x + 5y + 0z = 0$$

$$0x + 8y - 3z = 0$$

has the trivial solution  $x = y = z = 0$ . In addition since  $\begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} = -94 \neq 0$  this is the only solution.

## MATH 246: Chapter 2 Section 4: Homogeneous Equations with Constant Coefficients

Justin Wyss-Gallifent

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Main Topics:

- The Characteristic Polynomial
  - Real Simple Roots
  - Real Multiple Roots
  - Complex Simple Roots
  - Complex Multiple Roots
- 

### 1. Introduction:

We've established the fact that for an  $n^{\text{th}}$  order homogeneous linear differential equation that we need to find a fundamental set of  $n$  solutions denoted  $Y_1, \dots, Y_n$ . Once we do this we have the general solution

$$Y(t) = C_1 Y_1 + C_2 Y_2 + \dots C_n Y_n$$

The next question is how to get that fundamental set.

This is hard, so for now we'll focus on the simpler situation where the DE has constant coefficients. Examples:

$$\begin{aligned}y'' + 2y' - 3y &= 0 \\ 2y''' - 5y &= 0 \\ D^4 y + D^3 y - 2D^2 y + Dy + y &= 0\end{aligned}$$

2. **Inspirational Example:** Consider  $y'' + 2y' - 3y = 0$ . Here because we've got constant multiples of derivatives added to get zero, we think that perhaps solutions might be functions whose derivatives are constant multiples of themselves. The primary example is  $e^{zt}$  for various  $z$ . So let's try - if  $y = e^{zt}$  is a solution to our equation then  $y' = ze^{zt}$  and  $y'' = z^2 e^{zt}$  and we have:

$$\begin{aligned}y'' + 2y' - 3y &= 0 \\ z^2 e^{zt} + 2ze^{zt} - 3e^{zt} &= 0 \\ (z^2 + 2z - 3)e^{zt} &= 0\end{aligned}$$

Since  $e^{zt}$  is always positive we must have

$$\begin{aligned}z^2 + 2z - 3 &= 0 \\ (z + 3)(z - 1) &= 0\end{aligned}$$

and so either  $z = -3$  or  $z = 1$ .

Lo and behold we've actually found two solutions, and we only needed two. We've found

$$Y_1(t) = e^{-3t} \text{ and } Y_2(t) = e^{1t}$$

Thus the general solution is

$$Y(t) = C_1 e^{-3t} + C_2 e^t$$

3. **General Idea:** We see that the DE  $y'' + 2y' - 3y = 0$  gave us a polynomial  $z^2 + 2z - 3 = 0$ . This will happen in every case, that polynomial is called the *characteristic polynomial*. The roots of the polynomial will give us solutions and we will get enough to complete our fundamental set. However there are nuances, primarily that we have to deal with both real and complex roots and each of these has two subcategories.

In addition, though we won't prove this, the process is guaranteed to result in a fundamental set, meaning the Wronskian would be nonzero if we checked it.

4. **Real Simple Roots:** A *real simple root* is a root which only appears once when we factor the characteristic polynomial. For a simple real root  $z = r$  we get a solution  $e^{rt}$ . If there are  $n$  distinct simple real roots then we get  $n$  solutions and we're done.

**Example:** The DE  $y'' - 3y' - 10y = 0$  has characteristic polynomial  $z^2 - 3z - 10$  or  $(z-5)(z+2)$  with roots  $z = 5$  and  $z = -2$ . So we have two simple real roots and therefore the fundamental set is  $\{e^{5t}, e^{-2t}\}$  and the general solution is  $Y(t) = C_1e^{5t} + C_2e^{-2t}$ .

**Example:** The DE  $y''' - 5y'' + 6y' = 0$  has characteristic polynomial  $z^3 - 5z^2 + 6z$  or  $z(z-3)(z-2)$  with roots  $z = 0$ ,  $z = 3$  and  $z = 2$ . So we have three simple real roots and therefore the fundamental set is  $\{1, e^{3t}, e^{2t}\}$  (notice that  $e^{0t} = 1$ ) and the general solution is  $Y(t) = C_1 + C_2e^{3t} + C_3e^{2t}$ .

5. **Real Multiple Roots** A *real multiple root* is a root which appears more than once when we factor the characteristic polynomial.

**Example:** The DE  $D^3y - 15D^2y + 75Dy - 125y = 0$  has characteristic polynomial  $z^3 - 15z^2 + 75z - 125$  or  $(z-5)^3$ . there is only the root  $z = 5$  with multiplicity  $m = 3$ .

**Example:** The DE  $y''' - 4y'' + 4y' = 0$  has characteristic polynomial  $z^3 - 4z^2 + 4z$  or  $z(z-2)^2$ . There is the root  $z = 0$  which is a simple root (multiplicity 1) and the root  $z = 2$  with multiplicity  $m = 2$ .

For a real multiple root  $z = r$  with multiplicity  $m$  we get  $m$  solutions:

$$e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$$

We'll discuss where these come from in a later section.

**Example:** The DE  $D^3y - 15D^2y + 75Dy - 125y = 0$  has characteristic polynomial  $z^3 - 15z^2 + 75z - 125$  or  $(z-5)^3$ . there is only the root  $z = 5$  with multiplicity  $m = 3$ . So we have the fundamental set  $\{e^{5t}, te^{5t}, t^2e^{5t}\}$  and the general solution  $Y(t) = C_1e^{5t} + C_2te^{5t} + C_3t^2e^{5t}$ .

**Example:** The DE  $y''' - 4y'' + 4y' = 0$  has characteristic polynomial  $z^3 - 4z^2 + 4z$  or  $z(z-2)^2$ . There is the root  $z = 0$  which is a simple root (multiplicity 1) and the root  $z = 2$  with multiplicity  $m = 2$ . So we have the fundamental set  $\{1, e^{2t}, te^{2t}\}$  and the general solution  $Y(t) = C_1 + C_2e^{2t} + C_3te^{2t}$ .

6. **Complex Simple Roots:** Complex roots always come in pairs. This means that if  $r + si$  is a root then so is  $r - si$ . A *complex simple root pair* is a pair of roots  $r \pm si$  which appears only once when we factor the characteristic polynomial. We may have to actually solve via the quadratic formula to really see what's up.

**Example:** The DE  $y'' + y' + 2y = 0$  has characteristic polynomial  $z^2 + z + 2$ . This doesn't obviously factor. To find the roots we set it equal to 0 and use the quadratic formula to get  $z = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{-7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$ .

**Example:** The DE  $y'' + 4y = 0$  has characteristic polynomial  $z^2 + 4$  and setting  $z^2 + 4 = 0$  gives  $z = \pm \sqrt{-4} = 0 \pm 2i$ . We've made the 0 clear for a reason as you'll see.

To see what's going to happen, let's just charge ahead for a minute using the previous approach. If  $r + si$  is a root then  $e^{(r+si)t}$  is a solution. But note that we can rewrite this:

$$e^{(r+si)t} = e^{rt}e^{(st)i} = e^{rt}(\cos(st) + i\sin(st))$$

Since  $r - si$  is also a root then  $e^{(r-si)t}$  is also a solution and we can rewrite this too:

$$e^{(r-si)t} = e^{rt}e^{(-st)i} = e^{rt}(\cos(-st) + i\sin(-st)) = e^{rt}(\cos(st) - i\sin(st))$$

But these are complex solutions which don't fill out our fundamental set. But since linear combinations of solutions are solutions, we can be sneaky and observe:

- $\frac{1}{2}(\text{sum of complex solutions}) = e^{rt}\cos(st)$  is a solution.
- $\frac{1}{2i}(\text{difference of complex solutions}) = e^{rt}\sin(st)$  is a solution.

Thus for each complex simple root pair  $z = r \pm si$  we get a pair of solutions  $e^{rt}\cos(st)$  and  $e^{rt}\sin(st)$ .

**Example:** The DE  $y'' + y' + 2y = 0$  has characteristic polynomial  $z^2 + z + 2$  with roots we saw as  $-\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$ . The fundamental set is then  $\left\{e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right), e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)\right\}$  and the general solution is  $Y(t) = C_1e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + C_2e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)$ .

**Example:** The DE  $y'' + 4y = 0$  has characteristic polynomial  $z^2 + 4$  with roots  $z = 0 \pm 2i$ . The fundamental set is then  $\{\cos(2t), \sin(2t)\}$  and the general solution is  $Y(t) = C_1\cos(2t) + C_2\sin(2t)$ .

**Example:** The DE  $y''' - y'' - 4y' - 6y = 0$  has characteristic polynomial  $z^3 - z^2 - 4z - 6$  or  $(z - 3)(z^2 + 2z + 2)$ . We see that  $z = 3$  is a root but for the rest we set  $z^2 + 2z + 2 = 0$  and get  $z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = -1 \pm i$ . The fundamental set is then  $\{e^{3t}, e^{-t}\cos(t), e^{-t}\sin(t)\}$  and the general solution is  $Y(t) = C_1e^{3t} + C_2e^{-t}\cos(t) + C_3e^{-t}\sin(t)$ .

7. **Complex Multiple Roots:** This expands like with real roots. For each complex multiple root pair  $z = r \pm si$  with multiplicity  $m$  we get  $m$  pairs of solutions

$$e^{rt} \cos(st), te^{rt} \cos(st), \dots, t^{m-1} e^{rt} \cos(st)$$

$$e^{rt} \sin(st), te^{rt} \sin(st), \dots, t^{m-1} e^{rt} \sin(st)$$

**Example:** The DE  $D^6y + 8D^5y + 65D^4y + 232D^3y + 904D^2y + 1440Dy + 3600y = 0$  has characteristic polynomial  $z^6 + 8z^5 + 65z^4 + 232z^3 + 904z^2 + 1440z + 3600$  or  $(z^2 + 9)(z^2 + 4z + 20)^2$ . The first part gives us  $z = 0 \pm 3i$  and the second part gives us  $z = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2} = -2 \pm 4i$  with multiplicity 2. The fundamental set is then  $\{\cos(3t), \sin(3t), e^{-2t} \cos(4t), e^{-2t} \sin(4t), te^{-2t} \cos(4t), te^{-2t} \sin(4t)\}$  and the general solution is  $Y(t) = C_1 \cos(3t) + C_2 \sin(3t) + C_3 e^{-2t} \cos(4t) + C_4 e^{-2t} \sin(4t) + C_5 te^{-2t} \cos(4t) + C_6 te^{-2t} \sin(4t)$ .

8. **Summary:** In summary we construct the characteristic polynomial and find the roots. The roots tell us how to construct the fundamental set as follows:

- (a) If  $r$  is a real simple root then put in:

$$e^{rt}$$

- (b) If  $r$  is a multiple simple root with multiplicity  $m$  then put in:

$$e^{rt}, te^{rt}, \dots, t^{m-1} e^{rt}$$

- (c) If  $r \pm si$  is a complex simple root pair then put in:

$$e^{rt} \cos(st), e^{rt} \sin(st)$$

- (d) If  $r \pm si$  is a complex multiple root pair with multiplicity  $m$  then put in:

$$e^{rt} \cos(st), te^{rt} \cos(st), \dots, t^{m-1} e^{rt} \cos(st)$$

and

$$e^{rt} \sin(st), te^{rt} \sin(st), \dots, t^{m-1} e^{rt} \sin(st)$$

## MATH 246: Chapter 2 Section 5: Nonhomogeneous Equations - Method and Theory

Justin Wyss-Gallifent

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Main Topics:

- Inspirational Example
  - General Method
- 

### 1. Introduction:

Now that we know how to handle homogeneous linear differential equations with constant coefficients we'd like to see what happens if we make one small change. We'll look at nonhomogeneous linear differential equations. For now we'll momentarily even remove the requirement that we have constant coefficients. Examples:

$$\begin{aligned}y'' + 2y' - 3y &= e^t \\ 2y''' - 5ty &= \cos(2t) \\ y'''' + e^t y''' - 2ty'' + Dy + t^2y &= t\end{aligned}$$

### 2. An Inspirational Example:

The general idea is actually quite simple. To see, let's look at the example  $y'' + 2y' - 3y = f(t)$  where  $f(t)$  is unknown but nonzero.

Suppose that somehow we obtained just one solution. Call that solution  $Y_p(t)$  where the  $p$  stands for *particular*. Suppose that  $Y(t)$  is any other solution. Then look at what happens when we plug in  $Y(t) - Y_p(t)$ :

$$\begin{aligned}(Y(t) - Y_p(t))'' + 2(Y(t) - Y_p(t))' - 3(Y(t) - Y_p(t)) \\&= [Y''(t) + 2Y'(t) - 3Y(t)] - [Y_p''(t) + 2Y_p'(t) - 3Y_p(t)] \\&= f(t) - f(t) \\&= 0\end{aligned}$$

What this is telling us is that  $Y(t) - Y_p(t)$  is a solution to the *homogeneous version* of the DE. Since we know that all the solutions to the homogeneous version look like  $C_1 e^{-3t} + C_2 e^t$  this then tells us that

$$\begin{aligned}Y(t) - Y_p(t) &= C_1 e^{-3t} + C_2 e^t \\ Y(t) &= Y_p(t) + C_1 e^{-3t} + C_2 e^t\end{aligned}$$

### 3. General Method:

What this tells us is actually pretty fantastic. It says that when we're confronted by an  $n^{\text{th}}$  order nonhomogeneous linear differential equation, all we need to do is two things:

- Find the fundamental set  $\{Y_1, \dots, Y_n\}$  for the homogeneous version.
- Find one single solution  $Y_p$  for the original differential equation.

Then the general solution to the original differential equation is

$$Y(t) = Y_p(t) + C_1 Y_1(t) + \dots + C_n Y_n(t)$$

Of course these two things may not be easy. If the homogeneous version has constant coefficients at least part (a) is easy. Let's gloss over these for now to see things in practice.

- Warning** Keep in mind that the nonhomogeneous version doesn't have a fundamental set, it's the homogeneous version which does. Then the fundamental set for the homogeneous version helps us construct the general solution to the nonhomogeneous version.



## 5. Examples:

Here are some examples of both DEs and IVPs:

**Example:** Consider the differential equation  $y'' + 4y = 4t$ .

(a) The homogeneous version  $y'' + 4y = 0$  has fundamental set  $\{\cos(2t), \sin(2t)\}$ .

(b) The function  $Y_p(t) = t$  is a solution to  $y'' + 4y = 4t$ .

Thus the general solution to  $y'' + 4y = 4t$  is:

$$Y(t) = t + C_1 \cos(2t) + C_2 \sin(2t)$$

**Example:** Consider the differential equation  $D^3y - 2D^2y = 9e^{3t}$ .

(a) The homogeneous version  $D^3y - 2D^2y = 0$  has fundamental set  $\{1, t, e^{2t}\}$ .

(b) The function  $Y_p(t) = e^{3t}$  is a solution to  $D^3y - 2D^2y = 9e^{3t}$ .

Thus the general solution to  $D^3y - 2D^2y = 9e^{3t}$  is:

$$Y(t) = e^{3t} + C_1 + C_2t + C_3e^{2t}$$

**Example:** Consider the initial value problem  $(1 + t^2)y'' - 2ty' + 2y = 6$  with  $Y(0) = 2$  and  $Y'(0) = 1$ .

(a) The homogeneous version  $(1 + t^2)y'' - 2ty' + 2y = 0$  has fundamental set  $\{t, t^2 - 1\}$ .

(b) The function  $Y_p(t) = 3$  is a solution to  $(1 + t^2)y'' - 2ty' + 2y = 6$ .

Thus the general solution to  $(1 + t^2)y'' - 2ty' + 2y = 6$  is:

$$Y(t) = 3 + C_1t + C_2(t^2 - 1)$$

To solve the IVP we find

$$Y'(t) = C_1 + 2tC_2$$

and we solve:

$$2 = Y(0) = 3 + C_1(0) + C_2(0^2 - 1)$$

$$1 = Y'(0) = C_1 + 2(0)C_2$$

This tells us that  $C_2 = 1$  and  $C_1 = 1$  and so the specific solution is

$$Y(t) = 6 + t + (t^2 - 1)$$

## MATH 246: Chapter 2 Section 6: Undetermined Coefficients

Justin Wyss-Gallifent

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Main Topics:

- General Idea
  - Solution Building
- 

### 1. Introduction

Remember where we are: We have a non-homogenous linear differential equation with constant coefficients. We know how to deal with the homogeneous version and we know that all we need to do is get ahold of a single solution to the non-homogeneous version denoted  $Y_p(t)$  and then we can construct all solutions to the non-homogeneous version.

### 2. General Idea

The Method of Undetermined Coefficients will only work if the right side of the differential equation (the forcing function  $f(t)$ ) is one of our nice forms like  $e^{2t}$  or  $t^2$  or  $e^{2t} \cos(5t)$ . It is based on the premise that we know what the answer looks like and we only need to work out some coefficients.

Just to warm up before we get all formal:

If $f(t) = e^{5t}$	then $Y_p(t)$ probably looks like $Ce^{5t}$
If $f(t) = \cos(2t)$	then $Y_p(t)$ probably looks like $C_1 \cos(2t) + C_2 \sin(2t)$
If $f(t) = te^{5t}$	then $Y_p(t)$ probably looks like $C_1 e^{5t} + C_2 t e^{5t}$ or $(C_1 + C_2 t)e^{5t}$

The Method of Undetermined Coefficients will work as follows - we'll suggest what the solution looks like, with unknown constants, then we'll plug it into the DE and find the constants that make it work.

**Example:** If we have  $y'' - y = 2e^{5t}$  and we suggest that  $Y_p(t) = Ce^{5t}$ , then  $Y_p'(t) = 5Ce^{5t}$  and  $Y_p''(t) = 25Ce^{5t}$ . If we plug these into the DE we get  $25Ce^{5t} - Ce^{5t} = 2e^{5t}$  and so  $24Ce^{5t} = 2e^{5t}$  and so  $C = \frac{1}{12}$  and the solution is  $Y_p(t) = \frac{1}{12}e^{5t}$ . Wow, that was easy!

### 3. Building A Solution

This is very procedural and works as follows. Here  $Q_n(t)$  means a known polynomial of degree  $n$  and  $GP_n$  means an unknown generic polynomial of degree  $n$  with undetermined coefficients for which we fill in  $A$ ,  $B$ ,  $C$ , etc. For example  $GP_2 = At^2 + Bt + C$ . This is actually far easier in practice than it looks, the most common mistake is forgetting to check the multiplicity.

- If  $f(t)$  has the form  $Q_n(t)e^{rt}$ , first find  $m$  the multiplicity of  $z$  as a root of the characteristic polynomial. Often  $m = 0$ .  
 $\implies Y_p(t) = t^m [GP_n] e^{rt}$
- If  $f(t)$  has the form  $Q_n(t)e^{rt} \cos(st)$  or  $Q_n(t)e^{rt} \sin(st)$ , first find  $m$  the multiplicity of  $r + si$  as a root of the characteristic polynomial. Often  $m = 0$ .  
 $\implies Y_p(t) = t^m [GP_n] e^{rt} \cos(st) + t^m [GP_n] e^{rt} \sin(st)$   
Here the  $GP_n$  are different with different coefficients!
- If  $f(t)$  is the sum of such forms, we add the resulting forms together. Make sure to never, ever, repeat undetermined coefficients.

#### 4. What to do with the Solution

Once we have our guess we plug it into the DE and simplify like mad. The result will have similar functions on the left and right but the coefficients on the left will be the unknowns. We then equate the coefficients on each side and solve for those unknowns.

Theoretical Note: What allows us to do this last step is that the functions in the solution are linearly independent according to our construction of our solution.

#### 5. Examples:

**Example:** Consider  $y'' + 3y' + 2y = 2e^{3t}$ .

Forcing:  $f(t) = 2e^{3t}$ .

CP:  $p(z) = z^2 + 3z + 2 = (z + 2)(z + 1)$ .

We have  $r = 3$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial 2 is degree 0, we have:

$$Y_p(t) = t^m [GP_0] e^{3t} = t^0(A)e^{3t} = Ae^{3t}$$

Then  $Y_p' = 3Ae^{3t}$  and  $Y_p'' = 9Ae^{3t}$ . We plug these into the DE to get

$$9Ae^{3t} + 3(3Ae^{3t}) + 2Ae^{3t} = 2e^{3t}$$

$$20Ae^{3t} = 2e^{3t}$$

$$A = \frac{1}{10}$$

Then  $Y_p(t) = \frac{1}{10}e^{3t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{-2t}, e^{-t}\}$  the general solution is  $Y(t) = \frac{1}{10}e^{3t} + C_1e^{-2t} + C_2e^{-t}$ .

**Example:** Consider  $y'' - 3y' + 2y = 7e^{2t}$ .

Forcing:  $f(t) = 7e^{2t}$ .

CP:  $p(z) = z^2 - 3z + 2 = (z - 2)(z - 1)$ .

We have  $r = 2$  which is a root of  $p(z)$  of multiplicity  $m = 1$ . Because the coefficient polynomial 7 is degree 0, we have:

$$Y_p(t) = t^m [GP_0] e^{2t} = t^1(A)e^{2t} = Ate^{2t}$$

Then  $Y_p' = Ae^{2t} + 2Ate^{2t}$  and  $Y_p'' = 4Ae^{2t} + 4Ate^{2t}$ . We plug these into the DE to get

$$4Ae^{2t} + 4Ate^{2t} - 3(Ae^{2t} + 2Ate^{2t}) + 2(Ate^{2t}) = 7e^{2t}$$

$$Ae^{2t} = 7e^{2t}$$

$$A = 7$$

Then  $Y_p(t) = 7te^{2t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{2t}, e^t\}$  the general solution is  $Y(t) = 7te^{2t} + C_1e^{2t} + C_2e^t$ .

**Example:** Consider  $y''' - y'' = t^2$ .

Forcing:  $f(t) = t^2 = t^2 e^{0t}$ .

CP:  $p(z) = z^3 - z^2 = z^2(z - 1)$ .

We have  $r = 0$  which is a root of  $p(z)$  of multiplicity  $m = 2$ . Because the coefficient polynomial  $t^2$  is degree 2, we have:

$$Y_p(t) = t^m [GP_2] e^{0t} = t^2(At^2 + Bt + C)e^{0t} = At^4 + Bt^3 + Ct^2$$

Then  $Y_p' = 4At^3 + 3Bt^2 + 2Ct$ ,  $Y_p'' = 12At^2 + 6Bt + 2C$  and  $Y_p''' = 24At + 6B$ . We plug these into the DE to get

$$\begin{aligned} 24At + 6B - (12At^2 + 6Bt + 2C) &= t^2 \\ -12At^2 + (24A - 6B)t + (6B - 2C) &= t^2 \end{aligned}$$

Therefore  $-12A = 1$ ,  $24A - 6B = 0$  and  $6B - 2C = 0$ , giving  $A = -\frac{1}{12}$ ,  $B = -\frac{1}{4}$  and  $C = -\frac{3}{4}$ .

Then  $Y_p(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2$ .

Note: Because the fundamental set for the homogeneous version is  $\{1, t, e^t\}$  the general solution is  $Y(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2 + C_1 + C_2t + C_3e^t$ .

**Example:** Consider  $y'' - 3y' + 2y = 5te^{4t}$ .

Forcing:  $f(t) = 5te^{4t}$ .

CP:  $p(z) = z^2 - 3z + 2 = (z - 1)(z - 2)$

We have  $r = 4$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $5t$  is degree 1, we have:

$$Y_p(t) = t^m [GP_1] e^{4t} = t^0(At + B)e^{4t} = (At + B)e^{4t}$$

Then  $Y_p' = Ae^{4t} + (4At + 4B)e^{4t}$  and  $Y_p'' = 8Ae^{4t} + (16At + 16B)e^{4t}$ . We plug these into the DE to get

$$\begin{aligned} 8Ae^{4t} + (16At + 16B)e^{4t} - 3(Ae^{4t} + (4At + 4B)e^{4t}) + 2((At + B)e^{4t}) &= 5te^{4t} \\ (5A + 6B)e^{4t} + (6A)t e^{4t} &= 5te^{4t} \\ (5A + 6B) + (6A)t &= 5t \end{aligned}$$

Therefore  $5A + 6B = 0$  and  $6A = 5$ , giving  $A = \frac{5}{6}$  and  $B = -\frac{25}{36}$ .

Then  $Y_p(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{2t}, e^t\}$  the general solution is  $Y(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t} + C_1e^{2t} + C_2e^t$ .

**Example:** Consider  $y'' + y' = t + 3e^t$ .

Forcing:  $f(t) = t + 3e^t = te^{0t} + 3e^{1t}$  which has two parts.

CP:  $p(z) = z^2 + z = z(z + 1)$ .

For the  $te^{0t}$  part we have  $r = 0$  which is a root of  $p(z) = z^2 + z = z(z + 1)$  of multiplicity  $m = 1$ . Because the coefficient polynomial  $t$  is degree 1, we have:

$$\text{First Part: } Y_p(t) = t^1 [GP_1] e^{0t} = t(At + B)e^{0t} = At^2 + Bt$$

For the  $3e^{1t}$  part we have  $r = 1$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $3$  is degree 0, we have:

$$\text{Second Part: } Y_p(t) = t^m [GP_0] e^t = t^0(C)e^t = Ce^t$$

Combining these we have:

$$Y_p(t) = At^2 + Bt + Ce^t$$

Then  $Y'_p = 2At + B + Ce^t$  and  $Y''_p = 2A + Ce^t$ . We plug these into the DE to get

$$2A + Ce^t + 2At + B + Ce^t = t + 3e^t$$

$$2At + (2A + B) + 2Ce^t = t + 3e^t$$

Therefore  $A = \frac{1}{2}$ ,  $B = -1$  and  $C = \frac{3}{2}$ .

Then  $Y_p(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t$ .

Note: Because the fundamental set for the homogeneous version is  $\{1, e^{-t}\}$  the general solution is  $Y(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t + C_1 + C_2e^{-t}$ .

**Example:** Consider  $y'' + 2y' + 2y = 17\cos(3t)$ .

Forcing:  $f(t) = 17\cos(3t) = 17e^{0t}\cos(3t)$ .

CP:  $p(z) = z^2 + 2z + 2$  with roots  $z = 1 \pm 1i$ .

We have  $r + si = 0 + 3i$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $17$  is degree 0, we have:

$$\begin{aligned} Y_p(t) &= t^m [GP_0] \cos(3t) + t^m [GP_0] \sin(3t) \\ &= t^0(A) \cos(3t) + t^0(B) \sin(3t) = A \cos(3t) + B \sin(3t) \end{aligned}$$

Then  $Y'_p = -3A \sin(3t) + 3B \cos(3t)$  and  $Y''_p = -9A \cos(3t) - 9B \sin(3t)$ . We plug these into the DE to get

$$\begin{aligned} -9A \cos(3t) - 9B \sin(3t) + 2(-3A \sin(3t) + 3B \cos(3t)) + 2(A \cos(3t) + B \sin(3t)) &= 17 \cos(3t) \\ (-7A + 6B) \cos(3t) + (-6A - 7B) \sin(3t) &= 17 \cos(3t) \end{aligned}$$

Therefore  $-7A + 6B = 17$  and  $-6A - 7B = 0$ , giving  $A = -\frac{7}{5}$  and  $B = \frac{6}{5}$ .

Then  $Y_p(t) = -\frac{7}{5} \cos(3t) + \frac{6}{5} \sin(3t)$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{-t} \cos(t), e^{-t} \sin(t)\}$  the general solution is  $Y(t) = -\frac{7}{5} \cos(3t) + \frac{6}{5} \sin(3t) + C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$

## 6. Unfinished Examples:

**Example:** Consider  $y'' + 2y' + y = t^2e^{-t} + 17te^t \cos(3t)$ .

Forcing:  $f(t) = t^2e^{-t} + 17te^t \cos(3t)$  which has two parts.

CP:  $p(z) = z^2 + 2z + 1 = (z + 1)^2$ .

For the  $t^2e^{-t}$  part we have  $r = -1$  which is a root of  $p(z)$  of multiplicity 2. Because the coefficient polynomial  $t^2$  is degree 2 we have:

$$\text{First Part: } Y_p(t) = t^m [GP_2] e^{-t} = t^2(At^2 + Bt + C)e^{-t} = (At^4 + Bt^3 + Ct^2)e^{-t}$$

For the  $17te^t \cos(3t)$  part we have  $r + si = 1 + 3i$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $17t$  is degree 1 we have:

$$\begin{aligned} \text{Second Part: } Y_p(t) &= t^m [GP_1] e^t \cos(3t) + t^m [GP_1] e^t \sin(3t) \\ &= t^0(Dt + E)e^t \cos(3t) + t^0(Ft + G)e^t \sin(3t) \\ &= (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t) \end{aligned}$$

Combining these we have:

$$Y_p(t) = (At^4 + Bt^3 + Ct^2)e^{-t} + (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t)$$

## MATH 246: Chapter 2 Section 7: Variation of Parameters

Justin Wyss-Gallifent

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Main Topics:

- Restrictions.
  - General Idea and Method.
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### 1. Introduction

For the last chapter we've been focusing on finding a single solution  $Y_P(t)$  to a non-homogeneous linear differential equation with constant coefficients where  $f(t)$  is a familiar form.

What we're going to do now is remove both the restriction that the coefficients be constant and the restriction that  $f(t)$  is of our familiar form. We will restrict to second-order though, and we'll make sure the coefficient of  $y''$  is 1 (linear normal form), which can easily be attained through division. The goal will be the same, to find some  $Y_P(t)$ , because again the general solution will be  $Y(t) = Y_p(t) + c_1 Y_1(t) + c_2 Y_2(t)$  where  $\{Y_1(t), Y_2(t)\}$  is the fundamental set for the homogeneous version.

It may seem like if we could do this then why would we need the previous chapter? The answer is that the method of this section gets extremely complicated for third and higher order and can be computationally intensive even for second order. However it has the main advantage of providing a formulaic solution.

### 2. General Idea

The general idea is to start with the fundamental set  $\{Y_1, Y_2\}$  for the homogeneous version and ask a simple question - is it possible to find two function  $u_1(t)$  and  $u_2(t)$  such that  $y = u_1 Y_1 + u_2 Y_2$  is a solution to the nonhomogeneous version?

It turns out that simply plugging this  $y$  into the DE leaves us with quite a mess:

$$\begin{aligned} y'' + a(t)y' + b(t)y &= f(t) \\ (u_1 Y_1 + u_2 Y_2)'' + a(t)(u_1 Y_1 + u_2 Y_2)' + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ \underbrace{(u_1' Y_1 + u_1 Y_1' + u_2' Y_2 + u_2 Y_2')' + a(t)(u_1' Y_1 + u_1 Y_1' + u_2' Y_2 + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2)}_{\text{Quite a Mess!}} &= f(t) \end{aligned}$$

However if we look at this mess we notice that  $u_1' Y_1 + u_2' Y_2$  shows up in two places. It turns out that if  $u_1' Y_1 + u_2' Y_2 = 0$  then the above "Quite a Mess" tidies up:

$$\begin{aligned} (u_1 Y_1' + u_2 Y_2')' + a(t)(u_1 Y_1' + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ u_1' Y_1' + u_1 Y_1'' + u_2' Y_2' + u_2 Y_2'' + a(t)(u_1 Y_1' + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ u_1 \underbrace{(Y_1'' + a(t)Y_1' + b(t)Y_1)}_{= 0 \text{ bc homog soln}} + u_2 \underbrace{(Y_2'' + a(t)Y_2' + b(t)Y_2)}_{= 0 \text{ bc homog soln}} + u_1' Y_1' + u_2' Y_2' &= f(t) \\ u_1' Y_1' + u_2' Y_2' &= f(t) \end{aligned}$$

The practical upshot of all this is that if we can find  $u_1$  and  $u_2$  satisfying the system

$$\begin{aligned}u_1'Y_1 + u_2'Y_2 &= 0 \\u_1'Y_1' + u_2'Y_2' &= f(t)\end{aligned}$$

then  $Y_p = u_1Y_1 + u_2Y_2$  will be a solution to the nonhomogeneous DE.

Conveniently this is easy to solve because it's a system of two equations and two unknowns where the unknowns are  $u_1'$  and  $u_2'$ .

Linear algebra gives us a generic formula because this is a matrix equation:

$$\begin{aligned}\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} Y_2' & -Y_2 \\ -Y_1' & Y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} -Y_2f(t) \\ Y_1f(t) \end{bmatrix}\end{aligned}$$

Thus:

$$u_1' = -\frac{Y_2f(t)}{W[Y_1, Y_2]} \text{ and } u_2' = \frac{Y_1f(t)}{W[Y_1, Y_2]}$$

Then

$$u_1 = -\int \frac{Y_2f(t)}{W[Y_1, Y_2]} dt \text{ and } u_2 = \int \frac{Y_1f(t)}{W[Y_1, Y_2]} dt$$

and the final result is:

$$Y_p = u_1Y_1 + u_2Y_2$$

It's worth noting that it's sometimes messy but we can directly write down a formula for  $Y_p(t)$ :

$$Y_p(t) = -Y_1 \int \frac{Y_2f(t)}{W[Y_1, Y_2]} dt + Y_2 \int \frac{Y_1f(t)}{W[Y_1, Y_2]} dt$$



### 3. Examples

**Example:** Consider  $y'' + y = \sec t$ .

Since the characteristic polynomial is  $z^2 + 1$  with roots  $0 \pm 1i$  the fundamental set for the homogeneous version is  $\{\cos t, \sin t\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and then we simply evaluate:

$$\begin{aligned} u_1 &= -\int \frac{(\sin t)(\sec t)}{1} dt = -\int \tan t dt = \ln |\cos t| + C && \text{Choose } u_1 = \ln |\cos t| \\ u_2 &= \int \frac{(\cos t)(\sec t)}{1} dt = \int 1 dt = t + C && \text{Choose } u_2 = t \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = (\ln |\cos t|) \cos t + t \sin t$$

and the general solution to the nonhomogeneous version is

$$Y_p(t) = (\ln |\cos t|) \cos t + t \sin t + c_1 \cos t + c_2 \sin t$$

**Example:** Consider  $y'' - 3y' + 2y = t$ .

Since the characteristic polynomial is  $z^2 - 3z + 2 = (z - 1)(z - 2)$  with roots 1, 2 the fundamental set for the homogeneous version is  $\{e^t, e^{2t}\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

and then we simply evaluate (some IBP here):

$$\begin{aligned} u_1 &= -\int \frac{e^{2t}t}{e^{3t}} dt = -\int te^{-t} dt = te^{-t} - e^{-t} + C && \text{Choose } u_1 = te^{-t} - e^{-t} \\ u_2 &= \int \frac{e^t t}{e^{3t}} dt = \int te^{-2t} = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C && \text{Choose } u_2 = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = (te^{-t} - e^{-t}) e^t + \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) e^{2t} = \frac{1}{2}t - \frac{3}{4}$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{2}t - \frac{3}{4} + c_1 e^t + c_2 e^{3t}$$

Side Note: The Method of Undetermined Coefficients is much nicer for this problem.

**Example:** Consider  $(t^2 + 1)y'' - 2ty' + 2y = (t^2 + 1)^2$ .

First we rewrite as  $y'' - \frac{2t}{t^2+1}y' + \frac{2}{t^2+1}y = t^2 + 1$ . It's worth noting that even though this looks uglier the only thing it affects that we need is the right side. We have no technique for finding the fundamental set for the homogeneous version so I'll just give it to you, it's  $\{t, t^2 - 1\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2 - 1 \\ 1 & 2t \end{vmatrix} = t^2 + 1$$

and then we simply evaluate:

$$\begin{aligned} u_1 &= -\int \frac{(t^2-1)(t^2+1)}{t^2+1} dt = -\int t^2 - 1 dt = -\frac{1}{3}t^3 + t + C && \text{Choose } u_1 = -\frac{1}{3}t^3 + t \\ u_2 &= \int \frac{(t)(t^2+1)}{t^2+1} dt = \int t dt = \frac{1}{2}t^2 + C && \text{Choose } u_2 = \frac{1}{2}t^2 \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = \left(-\frac{1}{3}t^3 + t\right)t + \left(\frac{1}{2}t^2\right)(t^2 - 1) = \frac{1}{6}t^4 + \frac{1}{2}t^2$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 + c_1 t + c_2(t^2 - 1)$$

We can make this a pretty nice IVP by adding the condition  $Y(1) = 0$  and  $Y'(1) = 1$ . Since  $Y'(t) = \frac{2}{3}t^3 + t + c_1 + 2c_2 t$  we then have

$$\begin{aligned} Y(1) &= \frac{1}{6} + \frac{1}{2} + c_1 = 0 \\ Y'(1) &= \frac{2}{3} + 1 + c_1 + 2c_2 = 1 \end{aligned}$$

so then  $c_1 = -\frac{2}{3}$  and  $c_2 = 0$  so the specific solution to the IVP is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 - \frac{2}{3}t$$

## MATH 246: Chapter 2 Section 8: Mechanical Vibration

### Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Notes
  - Unforced and Undamped Motion
  - Unforced and Damped Motion
  - Forced Motion
- 

#### 1. Introduction

Important: Positive is up and negative is down.

Imagine a spring hanging with no weight on it. We then attach a mass  $m$  which stretches the spring a distance of  $y_R < 0$ . We are now at the rest point. At this point the force of gravity is  $mg$  (negative since  $g < 0$ , think  $g = -9.8$  if it helps) and the force of the spring by Hooke's Law is  $-ky_R$  (the spring force is upwards so we negate against  $y_R < 0$ ). Consequently because we are at rest we have  $mg + (-ky_R) = 0$  and so  $mg = ky_R$ .

Now then, imagine the object and spring system is in motion and at any time  $t$  the displacement from the rest point is given by  $y(t)$ . At any instant now there could be multiple forces acting on the object:

Gravity	$mg$	Acting downwards with $g < 0$ .
Spring	$-k(y + y_R)$	Acting against the displacement.
Damping	$-\gamma y'$	Acting against and proportional to velocity, here $\gamma > 0$ .
External	$f(t)$	Some other external force.

When we put these all together we get:

$$\begin{aligned}F_{Tot} &= F_{Grav} + F_{Spring} + F_{Damping} + F_{External} \\my'' &= mg - k(y + y_R) - \gamma y' + f(t) \\my'' &= ky_r - ky - ky_R - \gamma y' + f(t) \\my'' &= -ky - \gamma y' + f(t)\end{aligned}$$

We finally rewrite this as:

$$my'' + \gamma y' + ky = f(t)$$

If this doesn't look familiar then you've been asleep!

#### 2. A Few Notes

- In the Metric system we may either have length, time, mass and force in meters, seconds, kilograms and newtons ( $\text{newton} = \text{kg} \cdot \text{meter}/\text{s}^2$ ) respectively, or in centimeters, seconds, grams and dynes ( $\text{dyne} = \text{g} \cdot \text{cm}/\text{s}^2$ ) respectively. In the British system we may have feet, seconds, slugs and pounds ( $\text{lb} = \text{slug} \cdot \text{ft}/\text{s}^2$ ). Note also in the British system that weight is also in pounds with  $\text{lb} = \text{slug} \cdot \text{gravity}$ .
- If  $k$  is not given we may need to find it using  $mg = ky_R$ . We would be given the mass  $m$  of the object and the displacement  $y_R$ . We can then find  $k$ . For example if an object of mass 2 kilograms displace a spring 0.5 meters downwards then  $(2)(-9.8) = k(-0.5)$ .
- If  $\gamma$  is not given we may need to find it using  $F_{Damping} = \gamma y'$ . We would be given the damping force for a certain velocity. For example if a mass traveling at  $0.1\text{m/s}$  upwards invokes a damping force of  $0.3\text{N}$  downwards then  $-0.3 = -\gamma(0.1)$ .

### 3. Unforced and Undamped

The simplest situation is when there is no external force and no damping. In this case we have  $my'' + ky = 0$ . The characteristic polynomial has roots  $0 \pm i\sqrt{\frac{k}{m}}$  and so the solution is given by

$$y(t) = c_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m}}\right)$$

This can be rewritten using the Subtraction Formula for Cosine as

$$y(t) = A \cos\left(t\sqrt{\frac{k}{m}} - \delta\right)$$

where  $A = \sqrt{c_1^2 + c_2^2}$  is the amplitude and  $\delta$  satisfies  $\cos \delta = \frac{c_1}{A}$  and  $\sin \delta = \frac{c_2}{A}$ . The graph of this makes good sense for a spring that's bouncing up and down forever.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$ . It is pulled down  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards.

We have  $0.4y'' + 0.1y = 0$  or  $y'' + 0.25y = 0$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

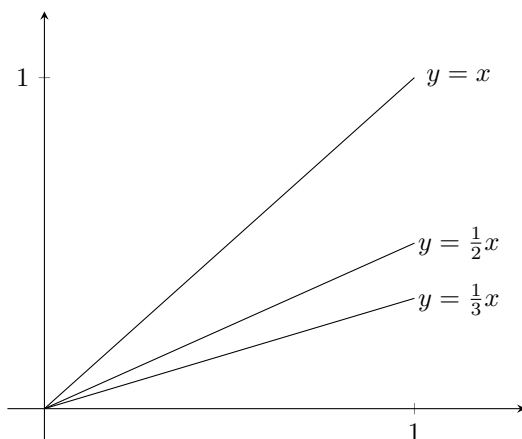
The characteristic polynomial has roots  $0 \pm i\sqrt{\frac{0.1}{0.4}} = 0 \pm 0.5i$ . The general solution is then

$$y(t) = c_1 \cos 0.5t + c_2 \sin 0.5t$$

For the initial value observe  $y'(t) = -0.5c_1 \sin 0.5t + 0.5c_2 \cos 0.5t$  and so  $y(0) = c_1 = -0.2$  and  $y'(0) = 0.5c_2 = -0.3$  so  $c_2 = -0.6$ . This gives us the specific solution

$$y(t) = -0.2 \cos 0.5t - 0.6 \sin 0.5t$$

The amplitude is  $A = \sqrt{(-0.2)^2 + (0.6)^2} = \sqrt{0.4} \approx 0.63$ . We can even draw a nice sketch.



Sketch omitted, but this starts at  $(0, -0.2)$  with a slope of  $-0.3$  and settles into a regular oscillation.

#### 4. Unforced with Damping

Now we have  $my'' + \gamma y' + ky = 0$ . The characteristic polynomial has roots  $-\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$ . the behavior of this depends strongly on  $\gamma^2 - 4mk$ .

- (a) **Underdamped:** When  $\gamma^2 - 4mk < 0$  (meaning the damping coefficient is small) we have complex roots and our solution has both exponential and trigonometric components. The function starts out oscillating but then the amplitude drops, limiting to zero.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.15$ . It is pulled down  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards.

We have  $0.4y'' + 0.15y' + 0.1y = 0$  or  $y'' + 0.375y' + 0.25y = 0$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

The characteristic polynomial has roots  $z = \frac{-0.15 \pm \sqrt{(0.15)^2 - 4(0.4)(0.1)}}{2(0.4)} = -0.1875 \pm \frac{\sqrt{0.1375}}{0.8}i$  The general solution is then

$$y(t) = c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{0.8}t\right) + c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{0.8}t\right)$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, -0.2)$  with a slope of  $-0.3$  and settles into an oscillation which reduces over time and limits to zero.

- (b) **Critically Damped:** When  $\gamma^2 - 4mk = 0$  we have a real root of multiplicity two. This is the special critically damped case. It corresponds to the smallest possible  $\gamma$  for which the oscillation stops.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.15625$  in a fluid with damping coefficient  $\gamma = 0.5$ . It is pulled down  $0.7m$  and released with zero velocity.

We have  $0.4y'' + 0.5y' + 0.15625y = 0$  or  $y'' + 1.25y' + 0.390625y = 0$  with  $y(0) = -0.7$  and  $y'(0) = 0$ .

The characteristic polynomial  $z^2 + 1.25z + 0.390625$  has a single root of multiplicity 2 as it factors as  $(z + 0.625)^2$ . The general solution is then

$$y(t) = c_1e^{-0.625t} + c_2te^{-0.625t}$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, -0.7)$  with a slope of 0 and heads directly but asymptotically to the  $t$ -axis.

- (c) **Overdamped:** When  $\gamma^2 - 4mk > 0$  we have two real roots and the system is overdamped.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.5$ . It is pushed up  $0.6m$  and released with zero velocity.

We have  $0.4y'' + 0.5y' + 0.1y = 0$  or  $y'' + 1.25y' + 0.25y = 0$  with  $y(0) = 0.6$  and  $y'(0) = 0$ .

The characteristic polynomial  $z^2 + 1.25z + 0.25$  factors as  $(z + 1)(z + 0.25)$  with roots  $-1, -0.25$  and hence the general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{-0.25t}$$

For the initial value observe  $y'(t) = -c_1 e^{-t} - 0.25c_2 e^{-0.25t}$  and so  $y(0) = c_1 + c_2 = 0.6$  and  $y'(0) = -c_1 - 0.25c_2$ . This yields  $c_2 = 0.8$  and  $c_1 = -0.2$ . This gives us the specific solution

$$y(t) = -0.2e^{-t} + 0.8e^{-0.25t}$$

We can even draw a nice sketch.

Sketch omitted, but this starts at  $(0, 0.6)$  with a slope of 0 and heads directly but asymptotically to the  $t$ -axis.

- (d) **A Note on Critically Damped vs. Overdamped:** These two functions look very similar. The critical thing to note is that a damped system oscillates, an overdamped system doesn't, and a critically damped system doesn't either but lies right on the edge of the other two.

## 5. Forced

With forced motions  $f(t) \neq 0$  and all bets are off. We know we need to find a particular solution  $Y_p$  and then add the general solution to the homogeneous system. This makes sense because the system is governed by both that forcing function and the usual spring motion stuff. In general the behavior will look springy at the start, although the damping might suppress this a bit, and in the long term (assuming damping) will look as if only the forcing function is acting on it.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.15$ . It is pulled up  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards. An additional external force  $f(t) = 0.3$  acts downwards on it.

We have  $0.4y'' + 0.15y' + 0.1y = -0.3$  or  $y'' + 0.375y' + 0.25y = -0.75$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

The Method of Undetermined Coefficients gives us one solution  $Y_p(t) = 3$ . The homogeneous version is an earlier problem with general solution

$$c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2}t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2}t\right)$$

and hence the general solution to our forced problem is

$$Y(t) = -3 + c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2}t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2}t\right)$$

Notice that in the long term the exponentials take complicated part to zero so that  $\lim_{t \rightarrow \infty} Y(t) = 3$ . So in the long term only the forcing function remains acting on it.

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, 0.2)$  with slope  $-0.3$  and oscillates with reducing amplitude as it settles down and limits to  $y = -3$ .



## MATH 246: Chapter 2 Section 9: Laplace Transforms

Justin Wyss-Gallifent

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Main Topics:

- Formal Definition
  - Rules
  - Reversing the Rules
  - Derivative Rules
  - Solving IVPs
  - Step Functions with Laplace Transforms
- 

1. **Introduction:** Laplace transforms are a way of changing one function into another function. Basically we start with a function of  $t$  and change it to a function of  $s$ . We can also do the reverse. The Laplace transform has some really useful properties which will help us solve initial value problems.
2. **Formal Definition:** If  $f(t)$  is a function then the Laplace transform of this function is formally defined by:

$$\mathcal{L}[y(t)](s) = \int_0^{\infty} y(t)e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b y(t)e^{-st} dt$$

For an unknown  $y(t)$  we will often write  $\mathcal{L}[y(t)]$  or just  $\mathcal{L}[y]$  for readability. This formal definition is used to build a set of rules and the rules are what we'll use.

**Example:** If  $y(t) = 1$  then we get:

$$\begin{aligned}\mathcal{L}[1] &= \int_0^{\infty} 1e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{s}e^{-st} \right|_0^b \\ &= \lim_{b \rightarrow \infty} -\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s(0)} \\ &= \frac{1}{s}\end{aligned}$$

Thus  $\mathcal{L}[1] = \frac{1}{s}$ .

**Example:** If  $y(t) = t$  then we get:

$$\begin{aligned}
\mathcal{L}[t] &= \int_0^{\infty} t e^{-st} dt \\
&= \lim_{b \rightarrow \infty} \int_0^b t e^{-st} dt \\
&= \lim_{b \rightarrow \infty} t \left( -\frac{1}{s} \right) e^{-st} \Big|_0^b - \int_0^b \left( -\frac{1}{s} \right) e^{-st} dt \\
&= \lim_{b \rightarrow \infty} t \left( -\frac{1}{s} \right) e^{-st} \Big|_0^b - \frac{1}{s^2} e^{-st} \Big|_0^b \\
&= \lim_{b \rightarrow \infty} \left[ -\frac{b}{s} e^{-sb} - 0 \right] - \left[ \frac{1}{s^2} e^{-sb} - \frac{1}{s^2} \right] \\
&= \frac{1}{s^2}
\end{aligned}$$

Thus  $\mathcal{L}[1] = \frac{1}{s^2}$ .

### 3. Function Rules

Using this same approach we can prove the following rules for common functions:

Function	Example
$\mathcal{L}[0] = 0$	n/a
$\mathcal{L}[c] = \frac{c}{s}$	$\mathcal{L}[42] = \frac{42}{s}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}[t^3] = \frac{3!}{s^4}$
$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}[e^{5t}] = \frac{1}{s-5}$
$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$	$\mathcal{L}[\cos(7t)] = \frac{s}{s^2+49}$
$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$	$\mathcal{L}[\sin(7t)] = \frac{7}{s^2+49}$
$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}[e^{2t}t^4] = \frac{4!}{(s-2)^5}$
$\mathcal{L}[e^{at} \cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t} \cos(3t)] = \frac{s-5}{(s-5)^2+9}$
$\mathcal{L}[e^{at} \sin(bt)] = \frac{b}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t} \sin(3t)] = \frac{3}{(s-5)^2+9}$
$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$	$\mathcal{L}[2 + 5t] = \mathcal{L}[2] + 5\mathcal{L}[t] = \frac{2}{s} + 5\left(\frac{1}{s^2}\right)$

**Notation:** Sometimes if a function is denoted  $y(t)$  then some sources use  $Y(s)$  instead of  $\mathcal{L}[y(t)]$  for convenience. I will tend to not do this much.

4. **Reversing** If we start with  $\mathcal{L}[y]$  we can also work in reverse. Here are some examples with comments because sometimes we need to manipulate the function first. This can also be written with inverse notation.

**Example:** If  $\mathcal{L}[y] = \frac{2}{s}$  then  $y(t) = 2$ . aka  $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = s$ .

**Example:** If  $\mathcal{L}[y] = \frac{1}{s^5}$  then first we rewrite  $\mathcal{L}[y] = \frac{1}{4!} \left(\frac{4!}{s^5}\right)$  and then we see  $y(t) = \frac{1}{24}t^4$ . aka  $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \frac{1}{24}t^4$ .

**Example:** If  $\mathcal{L}[y] = \frac{1}{s+4}$  we think of it as  $\mathcal{L}[y] = \frac{1}{s-(-4)}$  then  $y(t) = e^{-4t}$ .

**Example:** If  $\mathcal{L}[y] = \frac{-18}{s^2+9}$  then first rewrite to get  $\mathcal{L}[y] = -6 \left(\frac{3}{s^2+9}\right)$  and then we see  $y(t) = -6 \sin(3t)$ .

**Example:** If  $\mathcal{L}[y] = \frac{2}{s^2+s}$  then first we see  $\mathcal{L}[y] = \frac{2}{s(s+1)}$  and then we need to rewrite with partial fractions first and then following this we need a bit more rewriting to fit the formulas so  $\mathcal{L}[y] = \frac{2}{s} - \frac{2}{s+1} = \frac{2}{s} - 2 \left(\frac{1}{s-(-1)}\right)$ , and then we see  $y(t) = 2 - 2e^{-t}$ .

**Example:** If  $\mathcal{L}[y] = \frac{4s+3}{s^2+25}$  then we need to break it up and rewrite a little to fit the formulas:  
 $\mathcal{L}[y] = \frac{4s+3}{s^2+25} = \frac{4s}{s^2+25} + \frac{3}{s^2+25} = 4 \left(\frac{s}{s^2+25}\right) + \frac{3}{5} \left(\frac{5}{s^2+25}\right)$  and then we see that  
 $y(t) = 4 \cos(5t) + \frac{3}{5} \sin(5t)$ .

**Example:** If  $\mathcal{L}[y] = \frac{s+1}{s^2-4s+5}$  then the denominator doesn't factor so instead we complete the square and then do a bit more rewriting to get  $\mathcal{L}[y] = \frac{s+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$  and then we see that  $y(t) = e^{2t} \cos(t) + 3e^{2t} \sin(t)$ .

## 5. Derivative Rules

It turns out that the Laplace transfer is nice with derivatives of functions too.

**Example:** Observe that:

$$\begin{aligned}
 \mathcal{L}[y'(t)] &= \lim_{b \rightarrow \infty} \int_0^b y'(t) e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} \left. y(t) e^{-st} \right|_0^b + s \int_0^b y(t) e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} [y(b) e^{-sb} - y(0)] + s \mathcal{L}[y(t)] \\
 &= -y(0) + s \mathcal{L}[y(t)] \\
 &= s \mathcal{L}[y(t)] - y(0)
 \end{aligned}$$

In general we have the following pattern for an unknown  $y(t)$ :

$$\begin{array}{ll}
 \mathcal{L}[y'] = s \mathcal{L}[y(t)] - y(0) & = s \mathcal{L}[y] - y(0) \\
 \mathcal{L}[y''] = s^2 \mathcal{L}[y(t)] - sy(0) - y'(0) & = s^2 \mathcal{L}[y] - sy(0) - y'(0) \\
 \mathcal{L}[y'''] = s^3 \mathcal{L}[y(t)] - s^2 y(0) - sy'(0) - y''(0) & = s^3 \mathcal{L}[y] - s^2 y(0) - sy'(0) - y''(0) \\
 \vdots & \vdots
 \end{array}$$

For example if  $y(t)$  is unknown but we know  $y(0) = 7$  and  $y'(0) = -3$  then the second rule tells us that

$$\begin{aligned}
 \mathcal{L}[y''] &= s^2 \mathcal{L}[y] - sy(0) - y'(0) \\
 &= s^2 \mathcal{L}[y] - s(7) - (-3) \\
 &= s^2 \mathcal{L}[y] - 7s + 3
 \end{aligned}$$

We'll see very soon why this is significant.

## 6. Solving Initial Value Problems

Laplace Transforms are incredibly useful for dealing with IVPs when  $t_I = 0$ . Other values of  $t_I$  can be dealt with using a function shift but we won't deal with those here.

When dealing with such an initial value problem our approach will be the following:

- (a) Take the Laplace transform of each side.
- (b) Apply the rules for functions and for derivatives to eliminate all the  $t$ , all the derivatives and substitute all the initial values.
- (c) Solve the result for  $\mathcal{L}[y]$ .
- (d) Reverse the Laplace transform to get the solution  $y(t)$ .

**Example:** Suppose we have  $y' = 3$  with  $y(0) = 1$ . We do the following:

$$\begin{aligned}y' &= 3 \\ \mathcal{L}[y'] &= \mathcal{L}[3] \\ s\mathcal{L}[y] - y(0) &= \frac{3}{s} \\ s\mathcal{L}[y] - 1 &= \frac{3}{s} \\ s\mathcal{L}[y] &= \frac{3}{s} + 1 \\ \mathcal{L}[y] &= \frac{3}{s^2} + \frac{1}{s} \\ y(t) &= 3t + 1\end{aligned}$$

And we've solved it! Notice that you really need to understand how the various tables are being used here. The Laplace transform table is used at the beginning and end and the derivative rules are also used early on.

**Example:** Suppose we have  $y'' - 2y' - 3y = 0$  with  $y(0) = 1$  and  $y'(0) = 4$ . We do the following:

$$\begin{aligned}
 y'' - 2y' - 3y &= 0 \\
 \mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[0] \\
 (s^2\mathcal{L}[y] - sy(0) - y'(0)) - 2(s\mathcal{L}[y] - y(0)) - 3\mathcal{L}[y] &= 0 \\
 s^2\mathcal{L}[y] - s - 4 - 2s\mathcal{L}[y] + 2 - 3\mathcal{L}[y] &= 0 \\
 \mathcal{L}[y](s^2 - 2s - 3) - s - 2 &= 0 \\
 \mathcal{L}[y](s^2 - 2s - 3) &= s + 2 \\
 \mathcal{L}[y] &= \frac{s + 2}{s^2 - 2s - 3} \\
 \mathcal{L}[y] &= \frac{s + 2}{(s - 3)(s + 1)}
 \end{aligned}$$

Now we need to do some manipulation with partial fractions:

$$\begin{aligned}
 \frac{s + 2}{(s - 3)(s + 1)} &= \frac{A}{s - 3} + \frac{B}{s + 1} \\
 s + 2 &= A(s + 1) + B(s - 3)
 \end{aligned}$$

At this point  $s = -1$  gives us  $B = -1/4$  and  $s = 3$  gives us  $A = 5/4$ . Back to our problem with the most recent line rewritten:

$$\begin{aligned}
 \mathcal{L}[y] &= \frac{s + 2}{(s - 3)(s + 1)} \\
 \mathcal{L}[y] &= \frac{5/4}{s - 3} + \frac{-1/4}{s + 1} \\
 \mathcal{L}[y] &= \frac{5}{4} \left( \frac{1}{s - 3} \right) - \frac{1}{4} \left( \frac{1}{s - (-1)} \right) \\
 y(t) &= \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t}
 \end{aligned}$$

**Example:** Suppose we have  $y'' + 4y = 2t$  with  $y(0) = 1$  and  $y'(0) = 0$ . We do the following:

$$\begin{aligned}
 y'' + 4y &= 2t \\
 \mathcal{L}[y''] + 4\mathcal{L}[y] &= \mathcal{L}[2t] \\
 s^2\mathcal{L}[y] - sy(0) - y'(0) + 4\mathcal{L}[y] &= \frac{2}{s^2} \\
 s^2\mathcal{L}[y] - s - 0 + 4\mathcal{L}[y] &= \frac{2}{s^2} \\
 \mathcal{L}[y](s^2 + 4) &= \frac{2}{s^2} + s \\
 \mathcal{L}[y] &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4}
 \end{aligned}$$

This doesn't look so nice. The second part is okay (it's from  $\cos$ ) but the first part is not in our table. Instead we need to break it up with partial fractions:

$$\begin{aligned}
 \frac{2}{s^2(s^2 + 4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4} \\
 2 &= As(s^2 + 4) + B(s^2 + 4) + (Cs + D)s^2 \\
 2 &= (A + C)s^3 + (B + D)s^2 + 4As + 4B
 \end{aligned}$$

Comparing coefficients gives us  $A + C = 0$ ,  $B + D = 0$ ,  $4A = 0$  and  $4B = 2$  so that  $B = 1/2$ ,  $A = 0$ ,  $D = -1/2$  and  $C = 0$  and so back our process with the most recent line rewritten:

$$\begin{aligned}
 \mathcal{L}[y] &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4} \\
 \mathcal{L}[y] &= \frac{1/2}{s^2} + \frac{-1/2}{s^2 + 4} + \frac{s}{s^2 + 4} \\
 \mathcal{L}[y] &= \frac{1}{2} \left( \frac{1}{s^2} \right) - \frac{1}{4} \left( \frac{2}{s^2 + 4} \right) + \frac{s}{s^2 + 4} \\
 y(t) &= \frac{1}{2}t - \frac{1}{4}\sin(2t) + \cos(2t)
 \end{aligned}$$

Compare this to before where we'd need to find the general solution to the homogeneous version of the differential equation, also find a specific solution to the nonhomogeneous version, add them, then use the initial values to find the constants. This way is significantly faster.

Sometimes it's good practice just to do the first part of a problem:

**Example:** Find the Laplace Transform of the solution to the initial value problem  $y'' + y' - 3y = t + e^{2t} \cos(3t)$  with  $y(0) = -1$  and  $y'(0) = 2$ .

Here all we need to get to is  $\mathcal{L}[y]$ . We do the following:

$$\begin{aligned}
 y'' + y' - 3y &= t + e^{2t} \cos(3t) \\
 \mathcal{L}[y''] + \mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[t] + \mathcal{L}[e^{2t} \cos(3t)] \\
 s^2 \mathcal{L}[y] - sy(0) - y'(0) + s\mathcal{L}[y] - y(0) - 3\mathcal{L}[y] &= \frac{s-2}{(s-2)^2+9} \\
 s^2 \mathcal{L}[y] + s - 2 + s\mathcal{L}[y] + 1 - 3\mathcal{L}[y] &= \frac{2-s}{(s-2)^2+9} \\
 \mathcal{L}[y](s^2 + s - 3) + s - 1 &= \frac{2-s}{(s-2)^2+9} \\
 \mathcal{L}[y](s^2 + s - 3) &= \frac{2-s}{(s-2)^2+9} + 1 - s \\
 \mathcal{L}[y] &= \frac{2-s}{((s-2)^2+9)(s^2+s-3)} + \frac{1-s}{s^2+s-3}
 \end{aligned}$$

To finish, this would need to undergo a partial fractions decomposition and then the rules would need to be applied.



7. **Step Functions** The most basic step function is the function which returns 0 up until (but not including)  $t = 0$  and then 1 after that. More specifically we have

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

There are other options. If we want to use a value other than 0 we denote it  $u_c(t)$ :

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Step functions are useful because they turn other functions on and off. For example the product function  $u_\pi(t) \sin(t - \pi)$  is 0 for  $t < \pi$  and  $\sin(t - \pi)$  for  $t \geq \pi$ .

It may seem odd that we have  $\sin(t - \pi)$  here rather than just  $\sin(t)$  but there's a reason why this will usually happen. When a function "kicks in" at a certain  $t$ -value this usually means that before that  $t$ -value the function is 0 and then at that  $t$ -value the function begins as though 0 were plugged into it. So for example  $u_\pi(t) \sin(t - \pi)$  equals 0 until  $t = \pi$  at which point the  $\sin(t - \pi)$  part starts behaving as if 0 were plugged in (because of the  $t - \pi$  in there).

**Example:** Suppose a function equals 0 until  $t = \pi/4$  and then starts behaving like the sine function, meaning like the sine function does at  $t = 0$ . This new function would be  $u_{\pi/4}(t) \sin(t - \pi/4)$ .

**Example:** Suppose a function equals 0 until  $t = 3$  and then starts behaving like the exponential function  $e^t$ , meaning like the exponential function does at  $t = 0$ . This new function would be  $u_3(t)e^{t-3}$ .

## 8. Laplace Transforms and Step Functions

Step functions have the following Laplace transform related behavior:

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs}\mathcal{L}[f(t)]$$

This is a bit confusing so for the forward and backwards directions think:

**Forward:** Pull out the  $u_c(t)$  which changes to  $e^{-cs}$  and change all  $t - c$  to  $t$  then continue.

**Example:**  $\mathcal{L}[u_3(t)(t - 3)^5] = e^{-3s}\mathcal{L}[t^5] = e^{-3s}\left(\frac{5!}{s^6}\right)$

**Example:**  $\mathcal{L}[u_\pi(t)\sin(t - \pi)] = e^{-\pi s}\mathcal{L}[\sin(t)] = e^{-\pi s}\left(\frac{1}{s^2 + 1}\right)$

**Example:**  $\mathcal{L}[u_2(t)e^{4(t-2)}(t - 2)^5] = e^{-2s}\mathcal{L}[e^{4t}t^5] = e^{-2s}\left(\frac{5!}{(s-4)^6}\right)$

**Backward:** For  $\mathcal{L}[y] = e^{-cs}J(s)$  first find  $j(t)$  with  $\mathcal{L}[j(t)] = J(s)$ , replace the  $t$  by  $t - c$  and put a  $u_c(t)$  in front.

**Example:** If  $\mathcal{L}[y] = e^{-5s}\left(\frac{s}{s^2 + 49}\right)$  then we note  $\mathcal{L}[\cos(7t)] = \frac{s}{s^2 + 49}$ , replace the  $t$  by  $t - 5$  and put  $u_5(t)$  in front, yielding  $y(t) = u_5(t)\cos(7(t - 5))$ .

**Example:** If  $\mathcal{L}[y] = e^{3s}\left(\frac{5!}{s^6}\right)$  then we note  $\mathcal{L}[t^5] = \frac{5!}{s^6}$ , replace the  $t$  by  $t - (-3)$  and put  $u_{(-3)}(t)$  in front, yielding  $y(t) = u_{(-3)}(t)(t + 3)^5$ .

**Example:** If  $\mathcal{L}[y] = e^{-5s}\left(\frac{6!}{(s-3)^7}\right)$  then we note  $\mathcal{L}[e^{3t}t^6] = \frac{6!}{(s-3)^7}$ , replace the  $t$  by  $t - 5$  and put  $u_5(t)$  in front, yielding  $y(t) = u_5(t)e^{3(t-5)}(t - 5)^6$ .

This can then be tied into initial value problems.

**Example:** Suppose  $y' - 2y = f(t)$  where

$$f(t) = \begin{cases} 0 & t < 3 \\ t - 3 & t \geq 3 \end{cases}$$

and where  $y(0) = 0$ .

We first note that  $f(t) = u_3(t)(t - 3)$  and then proceed:

$$\begin{aligned} y' - 2y &= u_3(t)(t - 3) \\ \mathcal{L}[y'] - 2\mathcal{L}[y] &= \mathcal{L}[u_3(t)(t - 3)] \\ s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] &= e^{-3s}\mathcal{L}[t] \\ s\mathcal{L}[y] - 2\mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2}\right) \\ \mathcal{L}[y](s - 2) &= e^{-3s}\left(\frac{1}{s^2}\right) \\ \mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2(s - 2)}\right) \\ &\dots\text{Partial Fractions Not Shown}\dots \\ \mathcal{L}[y] &= e^{-3s}\underbrace{\left(\frac{-1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s - 2}\right)}_{\text{Yields: } -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}} \\ y(t) &= u_3(t)\left(-\frac{1}{4} - \frac{1}{2}(t - 3) + \frac{1}{4}e^{2(t-3)}\right) \end{aligned}$$

**Example:** Suppose  $y'' - y' - 2y = f(t)$  where

$$f(t) = \begin{cases} 0 & t < 3 \\ 7 & t \geq 3 \end{cases}$$

and where  $y(0) = 0$  and  $y'(0) = -2$ .

We first note that  $f(t) = 7u_3(t)$  and then proceed:

$$\begin{aligned} y'' - y' - 2y &= 7u_3(t) \\ \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] &= 7\mathcal{L}[u_3(t)] \\ s^2\mathcal{L}[y] - sy(0) - y'(0) - (s\mathcal{L}[y] - y(0)) - 2\mathcal{L}[y] &= 7e^{-3t} \\ s^2\mathcal{L}[y] - s + 2 - s\mathcal{L}[y] - 2\mathcal{L}[y] &= 7e^{-3t} \\ \mathcal{L}[y](s^2 - s - 2) - s + 2 &= 7e^{-3t} \\ \mathcal{L}[y](s^2 - s - 2) &= 7e^{-3t} + s - 2 \\ \mathcal{L}[y] &= 7e^{-3t} \left( \frac{1}{s^2 - s - 2} \right) + \frac{s - 2}{s^2 - s - 2} \\ \mathcal{L}[y] &= 7e^{-3t} \left( \frac{1}{(s - 2)(s + 1)} \right) + \frac{1}{s + 1} \\ &\dots \text{Partial Fractions Not Shown} \dots \\ \mathcal{L}[y] &= 7e^{-3t} \underbrace{\left( \frac{1/3}{s - 2} - \frac{1/3}{s + 1} \right)}_{\text{Yields: } \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}} + \frac{1}{s + 1} \\ y(t) &= 7u_3(t) \left( \frac{1}{3}e^{2(t-3)} - \frac{1}{3}e^{-1(t-3)} \right) + e^{-t} \end{aligned}$$

**Example:** Suppose  $y'' - y' = f(t)$  where

$$f(t) = \begin{cases} 0 & t < \pi/4 \\ \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$$

and where  $y(0) = 0$  and  $y'(0) = 0$ . We first note that  $f(t) = u_{\pi/4}(t) \cos(t - \pi/4)$  and then proceed:

$$\begin{aligned} y'' - y' &= u_{\pi/4} \cos(t - \pi/4) \\ \mathcal{L}[y''] - \mathcal{L}[y'] &= \mathcal{L}[u_{\pi/4} \cos(t - \pi/4)] \\ (s^2 \mathcal{L}[y] - sy(0) - y'(0)) - (s \mathcal{L}[y] - y(0)) &= e^{-(\pi/4)s} \mathcal{L}[\cos(t)] \\ s^2 \mathcal{L}[y] - s \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{s^2 + 1} \right) \\ \mathcal{L}[y] (s^2 - s) &= e^{-(\pi/4)s} \left( \frac{s}{s^2 + 1} \right) \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{(s^2 + 1)(s^2 - s)} \right) \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{(s^2 + 1)s(s - 1)} \right) \\ &\dots \text{Partial Fractions Not Shown} \dots \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{2}}{s - 1} \right) \\ \mathcal{L}[y] &= -\frac{1}{2} e^{-(\pi/4)s} \left( \frac{s + 1}{s^2 + 1} - \frac{1}{s - 1} \right) \\ \mathcal{L}[y] &= -\frac{1}{2} e^{-(\pi/4)s} \underbrace{\left( \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{1}{s - 1} \right)}_{\text{Yields: } \cos(t) + \sin(t) - e^t} \\ y(t) &= -\frac{1}{2} u_{\pi/4}(t) \left( \cos(t - \pi/4) + \sin(t - \pi/4) - e^{(t - \pi/4)} \right) \end{aligned}$$

## MATH 246: Chapter 3 Section 1: Intro to First Order Systems

Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Rewriting Single Higher Order as Systems
  - Multiple Tank Problems
- 

### 1. Introduction

All that we've studied so far are DEs involving a single function  $y$  depending on a single variable  $t$ . In the real world things can get far more complicated. As a classic example consider a predator-prey situation. The rate of growth of the prey depends on both the number of prey and the number of predators, possibly as well as time, and similarly for the rate of growth of the predators.

What we'll study now are systems of first-order DEs. In the most basic case we'll have two functions  $x_1(t)$  and  $x_2(t)$  in which their derivatives  $x'_1$  and  $x'_2$  depend on both  $x_1$  and  $x_2$  themselves and maybe on other functions of  $t$ .

The goal will be to find two functions  $x_1(t)$  and  $x_2(t)$  which simultaneously satisfy the system.

**Example:**

$$\begin{aligned}x'_1 &= x_1 - x_2 \\x'_2 &= -3x_1 - x_2\end{aligned}$$

In this example the pair  $x_1(t) = 2e^{-2t}$  and  $x_2(t) = 6e^{-2t}$  form a solution.

**Example:**

$$\begin{aligned}x'_1 &= 2t^2x_1 - 3x_2 + \cos t \\x'_2 &= x_1 + 4tx_2 - 2t\end{aligned}$$

In this example a solution pair is not easy at all.

We can also have more functions and equations.

**Example:**

$$\begin{aligned}x'_1 &= 2x_1 - 3x_2 - x_3 + \cos t \\x'_2 &= x_1 + 4x_2 - 2t \\x'_3 &= -x_1 + 20x_2 - 7x_3 + e^t\end{aligned}$$

In addition we could have an initial value, which would mean an initial value for each of the functions.

**Example:** Here is an IVP.

$$\begin{aligned}x'_1 &= tx_1 - 3x_2 \\x'_2 &= x_1 + 4t^2x_2\end{aligned}$$

With  $x_1(0) = 2$  and  $x_2(0) = -3$ .

## 2. Rewriting Single Higher-Order as Systems

Single higher-order DEs can be rewritten as systems of first-order DEs. This may be useful as we go on to develop methods of solving systems. The general idea for an  $n^{\text{th}}$  order DE will be to rewrite it as a system of  $n$  first-order DEs.

(a) When dealing with just the DE part it's simple. For an  $n^{\text{th}}$  order system we assign:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy \\x_3 &= D^2y \\&\vdots \\x_{n-1} &= D^{n-2}y \\x_n &= D^{n-1}y\end{aligned}$$

The first  $n - 1$  then give us:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= x_4 \\&\vdots \\x'_{n-1} &= x_n\end{aligned}$$

We get one more  $x'_n = \dots$  from the DE because  $x'_n = y^{(n)}$ , which we can find, and replacing all the other derivatives by their respective  $x_i$ .

**Example:** Consider  $D^2y + tDy - 3y = t$ . We assign:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy\end{aligned}$$

The first then gives us:

$$x'_1 = y' = Dy = x_2$$

The differential equation gives us:

$$x'_2 = D^2y = t - tDy - 3y = t - tx_2 - 3x_1$$

Thus our final system is:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= -3x_1 - tx_2 + t\end{aligned}$$

**Example:** Consider  $D^3y - 2D^2y + tDy - e^t y = \sin t$ . We assign and get:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy \\x_3 &= D^2y\end{aligned}$$

The first two of these give us:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3\end{aligned}$$

The third  $x'_3 = \dots$  comes from the DE and is:

$$x'_3 = D^3y = 2D^2y - tDy + e^t y + \sin t = 2x_3 - tx_2 + e^t x_1 + \sin t$$

All together we have the system:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= 2x_3 - tx_2 + e^t x_1 + \sin t\end{aligned}$$

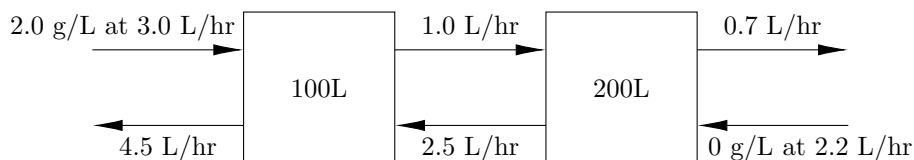
(b) With Initial Values:

Initial values are easy to rewrite. Since we know  $y(t_I), Dy(t_I), \dots, D^{n-1}y(t_I)$  these just become  $x_1(t_I), x_2(t_I), \dots, x_n(t_I)$ .

**Example:** Consider  $D^2y + tDy - 3y = t$  again. Suppose we know that  $y(1) = 2$  and  $y'(1) = -3$ . We set  $x_1 = y$  and  $x_2 = Dy$ . Then we know  $x_1(1) = 2$  and  $x_2(1) = y'(1) = -3$ .

### 3. Tank Problems

A classic example of these are tank problems. Imagine two tanks containing salt water. Water is being pumped into and out of these tanks in a variety of ways. For example in the following scenario there are two tanks. The one on the left contains 100L and the one on the right 200L. These quantities do not change in this example because the liters into each equals the gallons out.



The quantity 2.0 g/L at 3.0 L/hr indicates that salt water with that density is flowing into the left tank at that rate, and the quantity 0 g/L at 2.2 L/hr indicates the same for the right tank.

The other quantities do not have densities because they are assumed to be mixtures from the tank.

Imagine now that  $x_1$  is the amount of salt in the left tank and  $x_2$  is the amount of salt in the right tank, each at time  $t$ . Then the density of salt in the left tank is  $\frac{x_1}{100}$  and in the right tank is  $\frac{x_2}{200}$ . The entire scenario is then modeled by the system:

$$\begin{aligned}x_1' &= [\text{Rate of Salt In}] - [\text{Rate of Salt Out}] \\x_2' &= [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]\end{aligned}$$

which is:

$$\begin{aligned}x_1' &= +(2)(3) + (x_2/200)(2.5) - (x_1/100)(4.5) - (x_1/100)(1) \\x_2' &= +(x_1/100)(1) + (0)(2.2) - (x_2/200)(0.7) - (x_2/200)(2.5)\end{aligned}$$

This simplifies to:

$$\begin{aligned}x_1' &= -0.055x_1 + 0.0125x_2 + 6 \\x_2' &= 0.01x_1 - 0.0475x_2\end{aligned}$$

Suppose in addition we know that at time  $t = 0$  there is 10g of salt in the left tank and 20g of salt in the right tank. Then we can add in the initial value  $x_1(0) = 10$  and  $x_2(0) = 20$ .



## MATH 246: Chapter 3 Section 10: Population Dynamics

Justin Wyss-Gallifent

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Main Topics:

- Predator-Prey Models
  - Competing Species Models
  - Cooperating Species Models
- 

### 1. Predator-Prey Models

Consider a interaction between predators and prey. Suppose the number of prey is  $x(t)$  while the number of predators is  $y(t)$ . A simple but reasonable system of differential equations modeling these could be:

$$\begin{aligned}x' &= (r - ax - by)x \\ y' &= (-s + cx - dy)y\end{aligned}$$

To understand the meaning of these constants, consider that  $r - ax - by$  is the growth rate for prey while  $-s + cx - dy$  is the growth rate for predators. This is the reason they're multiplied by  $x$  and  $y$  respectively to get  $x'$  and  $y'$ . Moreover:

- The constant  $r > 0$  gives the intrinsic growth rate of the prey. This is positive because by default (in absense of predators) the prey will reproduce.
- The growth rate of prey may decline as the number of prey grows due to competetiveness. This is managed by the constant  $a \geq 0$ .
- The growth rate of the prey will decline as the number of predators grows. This is managed by the constant  $d > 0$ .
- The constant  $s > 0$  gives the intrinsic growth rate of the predators. We have  $-s$  because by default (in absense of prey) the predators will die out.
- The growth rate of predators will increase with the number of prey. This is managed by the constant  $c > 0$ .
- The growth rate of predators may decline as the number of predators grows due to competetiveness. This is managed by the constant  $d \geq 0$ .

Our goal will be to analyze such systems and understand what happens to the populations in the long term.

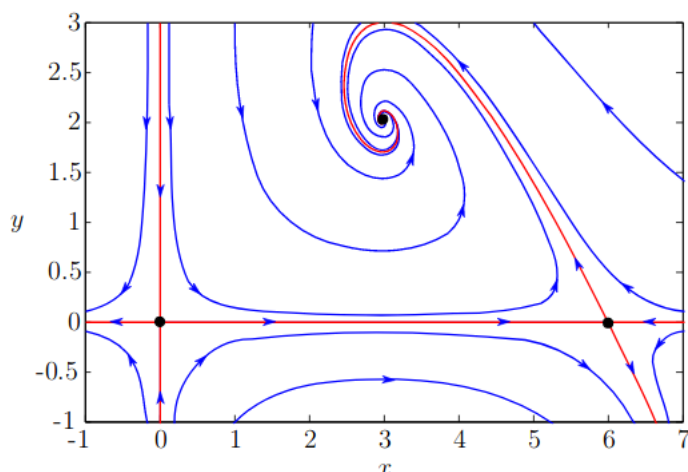
**Example:** Consider the model:

$$\begin{aligned}x' &= (12 - 2x - 3y)x \\ y' &= (-15 + 5x)y\end{aligned}$$

There are three stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$  with eigenpairs  $\left(12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a saddle.
- $(6, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -12 & -18 \\ 0 & 15 \end{bmatrix}$  with eigenpairs  $\left(-12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(15, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(3, 2)$  has  $\partial \bar{F} = \begin{bmatrix} -6 & -9 \\ 10 & 0 \end{bmatrix}$  with eigenvalues  $-3 \pm 9i$ .  
It follows that this is a counterclockwise spiral sink.

The following picture was pilfered from Levermore's notes:



So now an initial population of  $(0.1, 2)$  will undergo a decrease in predators, resulting in an increase in prey, resulting in an increase in predators, resulting in a decrease in prey, and so on, and will eventually spiral into the stable point  $(3, 2)$ .

**Example:** Consider the model:

$$\begin{aligned}x' &= (6 - 3y)x \\ y' &= (-15 + 5x)y\end{aligned}$$

There are two stationary points which we analyze as follows:

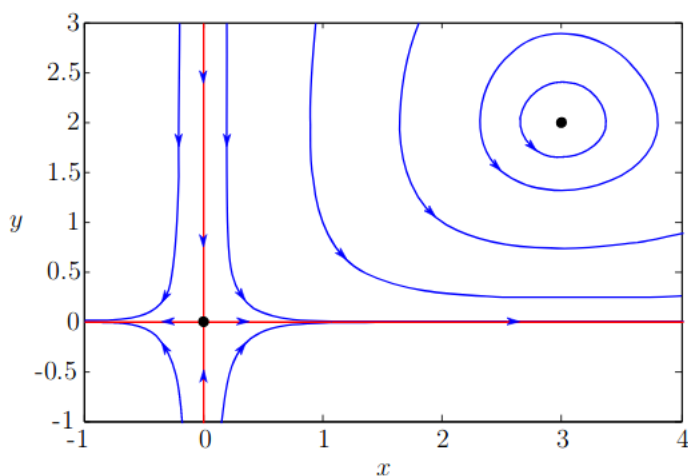
- $(0,0)$  has  $\partial\bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$  with eigenpairs  $\left(6, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .

It follows that this is a saddle.

- $(3,2)$  has  $\partial\bar{F} = \begin{bmatrix} 0 & -9 \\ 10 & 0 \end{bmatrix}$  with eigenvalues  $0 \pm 90i$ .

It follows that this is a counterclockwise circle.

The following picture was pilfered from Levermore's notes:



So now an initial population in the first quadrant will tend to circle around  $(3,2)$  but it will not approach it in a spiral sense.

## 2. Competing Species Models

Competing species models look like this:

$$\begin{aligned}x' &= (r - ax - by)x \\ y' &= (s - cx - dy)y\end{aligned}$$

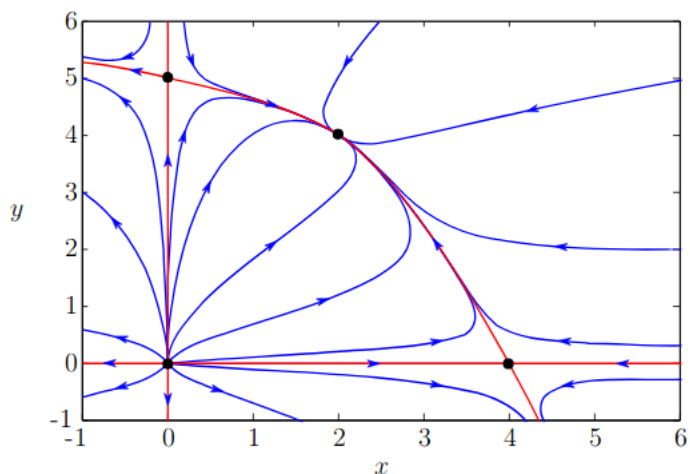
**Example:** Consider the model:

$$\begin{aligned}x' &= (16 - 4x - 2y)x \\ y' &= (10 - x - 2y)y\end{aligned}$$

There are four stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 16 & 0 \\ 0 & 10 \end{bmatrix}$  with eigenpairs  $\left(16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a nodal source..
- $(0, 5)$  has  $\partial \bar{F} = \begin{bmatrix} 6 & 0 \\ -5 & -10 \end{bmatrix}$  with eigenpairs  $\left(6, \begin{bmatrix} 16 \\ -5 \end{bmatrix}\right), \left(-10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(4, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -16 & -8 \\ 0 & 6 \end{bmatrix}$  with eigenpairs  $\left(-16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(6, \begin{bmatrix} -4 \\ 11 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(2, 4)$  has  $\partial \bar{F} = \begin{bmatrix} -8 & -4 \\ -4 & -8 \end{bmatrix}$  with eigenpairs  $\left(-12, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(-4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$   
It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both  $x$  and  $y$  positive then it will tend towards  $(2, 4)$  but this can happen in a variety of ways.

For example if we start at  $(10, 0.1)$  this means there are lots of species  $x$  and few of species  $y$ . Because there are a lot of species  $x$  they are constrained by resources and hence their population drops. When it gets close to  $x = 4$  however the rate of drop decreases and at that point resources are not so constraining it tends to start to stabilize. However at that point since  $x$  and  $y$  are competing  $y$  can grow now, since there aren't so many  $x$ . And so it does, and this causes  $x$  to drop more. In the long term (infinity) the pair heads to  $(2, 4)$ .

### 3. Cooperating Species Models

Cooperating species models look like this;

$$\begin{aligned}x' &= (r - ax + by)x \\ y' &= (s + cx - dy)y\end{aligned}$$

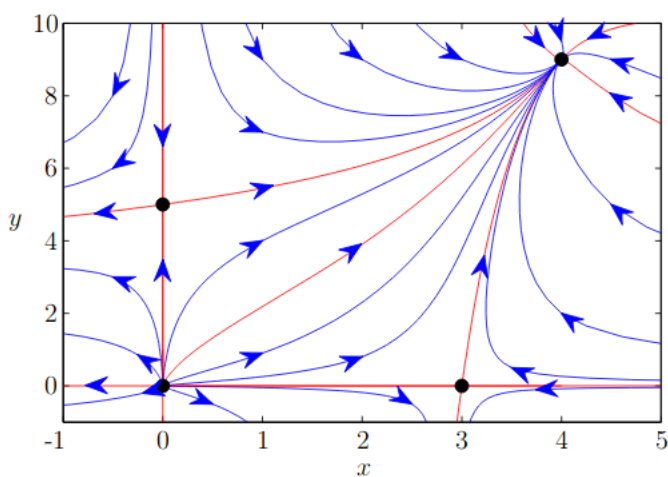
**Example:** Consider the model:

$$\begin{aligned}x' &= (27 - 9x + y)x \\ y' &= (20 + 4x - 4y)y\end{aligned}$$

There are four stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 27 & 0 \\ 0 & 20 \end{bmatrix}$  with eigenpairs  $\left(27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a nodal source..
- $(0, 5)$  has  $\partial \bar{F} = \begin{bmatrix} 32 & 0 \\ 20 & -20 \end{bmatrix}$  with eigenpairs  $\left(32, \begin{bmatrix} 13 \\ 5 \end{bmatrix}\right), \left(-20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(3, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -27 & 3 \\ 0 & 32 \end{bmatrix}$  with eigenpairs  $\left(-27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(32, \begin{bmatrix} 3 \\ 59 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(4, 9)$  has  $\partial \bar{F} = \begin{bmatrix} -36 & 9 \\ 16 & -36 \end{bmatrix}$  with eigenpairs  $\left(-24, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right), \left(-48, \begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$   
It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both  $x$  and  $y$  positive then it will tend towards  $(4, 9)$  but this can happen in a variety of ways.

**MATH 246: Chapter 3 Section 2: Basic Theory and Notation for Systems**  
**Justin Wyss-Gallifent**

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Main Topics:

- Linear Algebra Notation
  - Theory for Homogeneous
  - The Fundamental Matrix
  - Comment on Nonhomogeneous
- 

**1. Matrix and Vector Notation for Systems of First Order Linear DEs**

Consider the following:

- A first order linear system of differential equations may be written:

$$\bar{x}' = A(t)\bar{x} + \bar{f}(t)$$

- This system is homogeneous when  $\bar{f}(t) = \bar{0}$  and we say the system has constant coefficients when the matrix  $A(t)$  is all constants.
- An initial value can then be written as  $\bar{x}(t_I) = \bar{x}_I$ ,
- The solution can then be given as a single  $\bar{x}$ .

**Example:** Consider the following initial value problem:

$$\begin{aligned}x_1' &= 3x_1 + 2tx_2 + t & x_1(0) &= 1 \\x_2' &= t^2x_1 + 3x_2 & x_2(0) &= -2\end{aligned}$$

This can be rewritten as:

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\bar{x}'} = \underbrace{\begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} t \\ 0 \end{bmatrix}}_{\bar{f}(t)} \text{ with } \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\bar{x}(0)} = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\bar{x}_I}$$

Thus more simply:

$$\bar{x}' = \begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

**Example:** Consider the following solution:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

One solution to this is:

$$\bar{x} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$$

This can be checked with a matrix calculation:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 3e^{5t} + 2e^{5t} \\ 2e^{5t} + 3e^{5t} \end{bmatrix} = \begin{bmatrix} 5e^{5t} \\ 5e^{5t} \end{bmatrix} = \bar{x}'$$

## 2. Theory for Homogeneous - Fundamental Sets and Fundamental Matrices

Note: In what follows I've written  $n = 2$  to mean that I'm giving a specific example that generalizes. You could substitute  $n = 3, 4, \dots$  and the theory would still be good. In cases where it's not clear what happens for  $3, 4, \dots$  I've said more.

Theory: A homogeneous system of  $n = 2$  DEs has a fundamental pair/set consisting of  $n = 2$  solutions  $\bar{x}_1$  and  $\bar{x}_2$  (more if  $n \geq 3$ )

A fundamental set has nonzero Wronskian where

$$W[\bar{x}_1, \bar{x}_2] = |\bar{x}_1 \ \bar{x}_2|$$

That determinant is just found by dumping the vectors  $\bar{x}_1$  and  $\bar{x}_2$  together in a matrix and going from there.

When we have a fundamental pair/set the matrix used to determine the Wronskian is called the *fundamental matrix* and is usually denoted  $\Psi$  or  $\Psi(t)$ .

The general solution to the system then consists of all linear combinations of those  $n = 2$  solutions, this can be written several ways:

$$\bar{x}(t) = c_1 \bar{x}_1 + c_2 \bar{x}_2 = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t) \bar{c}$$

**Example:** Consider the system

$$\bar{x}' = \begin{bmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{bmatrix} \bar{x}$$

This has solutions (calculation omitted)  $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} 1 + t^3 \\ t \end{bmatrix}, \begin{bmatrix} t^2 \\ 1 \end{bmatrix} \right\}$ .

These form a fundamental pair because  $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} 1 + t^3 & t^2 \\ t & 1 \end{vmatrix} = 1 \neq 0$ .

Consequently the general solution to the system is:

$$\bar{x}(t) = c_1 \begin{bmatrix} 1 + t^3 \\ t^2 \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t^2 + c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + t^3 & t^2 \\ t^2 & 1 \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

It's worth noting that if we go back and think of the original problem as:

$$\begin{aligned} x_1' &= t^2 x_1 + (2t - t^4) x_2 \\ x_2' &= x_2 - t^2 x_2 \end{aligned}$$

Then the general solution is:

$$\begin{aligned} x_1 &= c_1(1 + t^3) + c_2 t^2 \\ x_2 &= c_1 t + c_2 \end{aligned}$$

**Example 1:** Consider the system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has solutions (calculation omitted)  $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}, \begin{bmatrix} e^t \\ -e^t \end{bmatrix} \right\}$ .

These form a fundamental pair because  $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{vmatrix} = -2e^{6t} \neq 0$ .

Consequently the general solution to the system is:

$$\bar{x}(t) = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{bmatrix} = \underbrace{\begin{bmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

If we turn this into an initial value problem with the initial condition:

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can find the specific solution using a simple matrix calculation:

$$\begin{aligned} \bar{x}(0) &= \Psi(0)\bar{c} \\ \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{c} \\ \bar{c} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= -\frac{1}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ \bar{c} &= \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Hence the specific solution can be written a few ways:

$$\bar{x}(t) = \Psi(t)\bar{c} = \begin{bmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{bmatrix} \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

It's worth noting that if we go back and think of the original problem as the IVP:

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 & x_1(0) &= 1 \\ x_2' &= 2x_1 + 3x_2 & x_2(0) &= 2 \end{aligned}$$

Then the specific solution is:

$$\begin{aligned} x_1 &= \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ x_2 &= \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{aligned}$$



### 3. Natural Fundamental Sets and Matrices

The natural fundamental matrix associated to a specific  $t_I$  is a specific fundamental matrix which is incredibly useful.

Suppose we solve the two initial value problems:

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{call this solution } x_1$$

and

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \text{call this solution } x_2$$

we get what are known together as the *natural fundamental set associated to  $t_I$*  and if we put these together in a matrix we get the *natural fundamental matrix associated to  $t_I$*  which is denoted  $\Phi(t)$  or just  $\Phi$ . There is only one of these for a given  $t_I$ .

**Example:** Consider the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has natural fundamental matrix associated to  $t_I = 0$  of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

This matrix  $\Phi(t)$  is incredibly useful. To see this, suppose we have a system given by

$$\bar{x}' = A(t)\bar{x}$$

If we have the natural fundamental matrix  $\Phi(t)$  associated to some  $t_I$  and we wish to solve the initial value problem with:

$$\bar{x}(t_I) = \bar{x}_I = \begin{bmatrix} a \\ b \end{bmatrix}$$

Consider the vector:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

Observe that by definition of matrix/vector multiplication we have:

$$\bar{x}(t) = \Phi(t)\bar{x}_I = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1 + b\bar{x}_2$$

Since  $\bar{x}$  is a linear combination of  $\bar{x}_1$  and  $\bar{x}_2$  it is a solution to the DE. Moreover observe that:

$$\bar{x}(t_I) = \Phi(t_I)\bar{x}_I = \Phi(t_I) \begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1(t_I) + b\bar{x}_2(t_I) = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \bar{x}_I$$

It follows that the solution to the IVP is simply given by:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

This is handy when we need to solve repeated initial value problems with the same  $t_I$ .

**Example:**

Revisit the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

We saw this has natural fundamental matrix associated to  $t_I = 0$  of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

So now we can throw out solutions to IVPs easily:

- If we have:  $\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Then the solution is:

$$\bar{x} = \Phi(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

- If we have:  $\bar{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

Then the solution is:

$$\bar{x} = \Phi(t) \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{5t} + 3e^t \\ e^{5t} - 3e^t \end{bmatrix}$$

What's even better is that if we have any fundamental matrix  $\Psi(t)$  then we can find the natural fundamental matrix for any  $t_I$  by calculating:

$$\Phi(t) = \Psi(t)\Psi(t_I)^{-1}$$

**Example:**

The system:

$$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$$

has fundamental matrix:

$$\Psi(t) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$$

Suppose we wish to solve the IVP with  $\bar{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ .

We first find  $\Phi$  associated to  $t_I = 0$  by doing the following:

$$\begin{aligned} \Phi &= \Psi(t)\Psi(0)^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \end{aligned}$$

Then the solution to the IVP is given by:

$$\bar{x} = \Phi \bar{x}_I = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cos x + 7 \sin x \\ -2 \sin x + 7 \cos x \end{bmatrix}$$

## MATH 246: Chapter 3 Section 3: Matrix and Vector Essentials

Justin Wyss-Gallifent

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Main Topics:

- Basic Definitions
  - Combining Matrices and Vectors
  - Inverses
  - Determinants
  - Transpose and Complex Conjugate
  - Eigenstuff
- 

### 1. Introduction:

It's far easier to manage systems of differential equations when we can rephrase them in the language of matrices and vectors. To that end, here are the essentials.

### 2. Basic Definitions:

- (a) A *matrix* is a rectangular array of numbers. It has size  $m \times n$  if there are  $m$  rows and  $n$  columns. Matrices are typically denoted using capital letters:

**Example:** Here is a  $3 \times 4$  matrix:

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 5 & 0 & 17 \\ 2 & 2 & -7 & 3 \end{bmatrix}$$

- (b) Most of the matrices in this class will be *square*, meaning they have the same number of rows and columns. Mostly we'll deal with  $2 \times 2$  and  $3 \times 3$  matrices.
- (c) The *identity matrix*  $I_n$  is the square  $n \times n$  matrix with 1s on the *main diagonal* (upper-left to lower right) and 0s elsewhere. When the size is clear from context we just write  $I$ .

**Example:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (d) The *zero matrix* is matrix of all zeros.
- (e) A *vector* is a matrix which is a single column. Vectors are usually denoted in lower-case with a bar over the letter.

**Example:**  $\bar{a} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

### 3. Combining Matrices and Vectors:

- (a) We can add matrices and vectors by adding matching entries provided they both have the same size.

**Example:** For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 9 & 8 \end{bmatrix}$$

- (b) We can multiply an  $n \times n$  matrix  $A$  by a vector  $\bar{x}$  with  $n$  entries to get a new vector with  $n$  entries. The formal definition of this is that we take the linear combination of the columns of  $A$  using the weights in  $\bar{x}$ . More informally we do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

**Example:** We have:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} &= 5 \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} (1)(5) + (2)(3) + (-3)(2) \\ (0)(5) + (4)(3) + (7)(2) \\ (8)(5) + (-1)(3) + (5)(2) \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 26 \\ 47 \end{bmatrix} \end{aligned}$$

- (c) We can multiply an  $n \times n$  matrix by another  $n \times n$  matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors and putting them together in a new matrix.

**Example:** Here it is with lots of brackets to help you see what's going on:

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 9 \end{bmatrix} &= \left[ \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \right] \\ &= \left[ \begin{bmatrix} (2)(-1) + (1)(5) \\ (4)(-1) + (3)(5) \end{bmatrix} \quad \begin{bmatrix} (2)(3) + (1)(9) \\ (4)(3) + (3)(9) \end{bmatrix} \right] \\ &= \begin{bmatrix} 3 & 15 \\ 9 & 39 \end{bmatrix} \end{aligned}$$

- (d) It's almost always the case that for matrices  $A$  and  $B$  that  $AB \neq BA$ .
- (e) The identity matrix acts like the number 1 in that for any matrix  $A$  we have:

$$AI = IA = A$$

#### 4. Determinants:

- (a) The *determinant* of a matrix, denoted  $\det A$  or by putting the matrix in vertical bars instead of brackets, is a number calculated from the matrix. We've seen this for  $2 \times 2$  and  $3 \times 3$  matrices.
- (b) Properties include:
- A matrix  $A$  has an inverse if and only if  $\det A \neq 0$ .
  - For a  $2 \times 2$  case  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ .

#### 5. Inverses:

- (a) The *inverse* of an  $n \times n$  matrix  $A$  is another matrix denoted  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = I$ . It's like a "reciprocal" for matrices.
- (b) For the  $2 \times 2$  size there is a formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Example:** For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}$$

- (c) Properties include:
- $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$
  - We have  $\det(A^{-1}) = 1/(\det A)$ .

#### 6. Transpose and Complex Conjugate:

- (a) The *transpose* of an  $n \times n$  matrix  $A$ , denoted  $A^T$ , is found by reflecting the matrix in its main diagonal.

**Example:** We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

- (b) A matrix may have complex numbers in it, in which case its *complex conjugate* denoted either  $\bar{A}$  or  $A^c$ , is found by taking the complex conjugate of each entry.

**Example:** We have:

$$\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^c = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}$$

## 7. Eigenstuff:

If we have a matrix, the determinant is the most important number associated to it. After the determinant the next most important items are eigenvalues and eigenvectors.

- (a) If  $A$  is an  $n \times n$  matrix, an *eigenvalue* of  $A$  is a number  $\lambda$  with the property that there is some  $\bar{v} \neq \bar{0}$  such that  $A\bar{v} = \lambda\bar{v}$ . The vector  $\bar{v}$  is then an *eigenvector* associated to  $\lambda$  and we say that  $(\lambda, \bar{v})$  is an *eigenpair* of  $A$ .

**Example:** Observe that:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so we would say that  $\lambda = 3$  is an eigenvalue,  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector and  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  is an eigenpair for the matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ .

- (b) Any nonzero multiple of an eigenvector is also an eigenvector, so in the above example  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,

$\begin{bmatrix} 17 \\ 17 \end{bmatrix}$  and  $\begin{bmatrix} -7 \\ -7 \end{bmatrix}$  are all eigenvectors for the same eigenvalue.

- (c) If we have a complex eigenpair  $(\lambda, \bar{v})$  then  $(\bar{\lambda}, \bar{\bar{v}})$  is also an eigenpair.

- (d) Given an  $n \times n$  matrix  $A$ , the value  $\lambda$  will be an eigenvalue if and only if there is some  $\bar{v} \neq \bar{0}$  such that

$$A\bar{v} = \lambda\bar{v}$$

We can manipulate this:

$$\begin{aligned} \lambda\bar{v} - A\bar{v} &= \bar{0} \\ \lambda I\bar{v} - A\bar{v} &= \bar{0} \\ (\lambda I - A)\bar{v} &= \bar{0} \end{aligned}$$

This will have a nontrivial solution precisely when:

$$\det(\lambda I - A) = 0$$

So what we do is we define the characteristic polynomial of  $A$  as:

$$p(z) = \det(zI - A)$$

Then we know that the eigenvalues of  $A$  are the roots of this characteristic polynomial.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det \left( z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ -2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) - 4 \\ &= z^2 - 6z + 5 \\ &= (z-5)(z-1) \end{aligned}$$

The eigenvalues are then the roots so  $\lambda_1 = 5$  and  $\lambda_2 = 1$ .

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det \left( z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-4 & 1 \\ -1 & z-2 \end{bmatrix} \\ &= (z-4)(z-2) + 1 \\ &= z^2 - 6z + 9 \\ &= (z-3)^2 \end{aligned}$$

The only eigenvalue is the root  $\lambda = 3$ . However this multiplicity counts, so we can think  $\lambda_1 = 3$  and  $\lambda_2 = 3$ .



Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ 2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) + 4 \\ &= z^2 - 6z + 13 \end{aligned}$$

This does not factor so we use the quadratic formula:

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2} = 3 \pm 2i$$

The eigenvalues are then  $\lambda_1 = 3 + 2i$  and  $\lambda_2 = 3 - 2i$ .

- (e) Once we find the eigenvalues we take each eigenvalue  $z = \lambda$  and solve the matrix equation  $A\bar{v} = \lambda\bar{v}$ , or  $A\bar{v} - \lambda\bar{v} = \bar{0}$ , or  $(A - \lambda I)\bar{v} = \bar{0}$ . This can be fairly intensive for large cases. For the  $2 \times 2$  case there is a trick, though, which is really useful:

For  $\lambda_1$  choose any nonzero column of  $A - \lambda_2 I$ .

For  $\lambda_2$  choose any nonzero column of  $A - \lambda_1 I$ .

**Example:** We saw that the eigenvalues for  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$  are  $\lambda_1 = 5$  and  $\lambda_2 = 1$ .

Then:

- For  $\lambda_1 = 5$  choose any nonzero column of

$$A - \lambda_2 I = A - 1I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

so  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  will do. Since any multiple of this works, pick the nicer  $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

- For  $\lambda_2 = 1$  choose any nonzero column of

$$A - \lambda_1 I = A - 5I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

so  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  will do. Since any multiple of this works, pick the nicer  $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

We thus have eigenpairs  $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  and  $\left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ .

**Example:** We saw that the eigenvalue for  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$  is  $\lambda_1 = \lambda_2 = 3$ . Then:

- For  $\lambda_1 = 3$  choose any nonzero column of  $A - \lambda_2 I = A - 3I = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  so  $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  will do.
- Notice that  $\lambda_2 = \lambda_1$  so we get nothing new.

We thus have the single eigenpair  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ .

**Example:** We saw that the eigenvalues for  $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$  are  $\lambda_1 = 3 + 2i$  and  $\lambda_2 = 3 - 2i$ . Then:

- For  $\lambda_1 = 3 + 2i$  choose any nonzero column of

$$A - \lambda_2 I = A - (3 - 2i)I = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 + 2i & 0 \\ 0 & 3 + 2i \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ -2 & -2i \end{bmatrix}$$

so  $\begin{bmatrix} 2i \\ -2 \end{bmatrix}$  will do. Since any multiple of this works, we pick the nicer  $\bar{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ .

- We know from earlier that for  $\lambda_2 = 3 - 2i$  we can use the conjugate so  $\bar{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

We thus have eigenpairs  $\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$  and  $\left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ .

## MATH 246: Chapter 3 Section 5: Using Eigenpairs to Construct Solutions

Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Real Eigenvalues with Multiplicity 1
  - Real Eigenvalues with Multiplicity 2
  - Pairs of Complex Eigenvalues
- 

### 1. Using Eigenpairs to Construct Solutions:

If we go back to  $\bar{x}' = A\bar{x}$  observe that if  $(\lambda, \bar{v})$  is an eigenpair then it turns out that  $\bar{x} = e^{\lambda t}\bar{v}$  is a solution:

$$\bar{x}' = \frac{d}{dt}(e^{\lambda t}\bar{v}) = e^{\lambda t}\lambda\bar{v} = e^{\lambda t}A\bar{v} = Ae^{\lambda t}\bar{v} = A\bar{x}$$

This tells that an eigenpair yields a solution.

However there are some nuances. The process gets significantly harder at the  $3 \times 3$  case and above as we have to start to consider things like eigenspace dimensions and obtaining linearly independent sets of eigenvalues. Consequently we will stay with the  $2 \times 2$  case.

### 2. Two Real Eigenvalues

If we have two real eigenpairs with distinct eigenvectors:

$$(\lambda_1, \bar{v}_1) \text{ and } (\lambda_2, \bar{v}_2) \text{ with } \lambda_1 \neq \lambda_2$$

So our two solutions are:

$$\{\bar{x}_1, \bar{x}_2\} = \{e^{\lambda_1 t}\bar{v}_1, e^{\lambda_2 t}\bar{v}_2\}$$

and they will form a fundamental set.

**Example:** The system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

has a matrix with eigenpairs

$$\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

Therefore we have two solutions

$$\{\bar{x}_1, \bar{x}_2\} = \left\{e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

So the general solution is:

$$\bar{x} = C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

### 3. One Real Eigenvalue

If we have just one eigenpair  $(\lambda, \bar{v})$  with multiplicity 2 then the situation is trickier. This eigenpair will give us one solution

$$\bar{x}_1 = \bar{v}e^{\lambda t}$$

It turns out that a second solution can be obtained by choosing  $\bar{w}$  to be a nonzero multiple of  $\bar{v}$  and then assigning:

$$\bar{x}_2 = e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w})$$

The proof of why this works is not at all obvious.

So our two solutions are:

$$\{\bar{x}_1, \bar{x}_2\} = \{e^{\lambda t} \bar{v}, e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w})\}$$

**Example:**

$$\bar{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpair:

$$\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

This gives us one solution

$$\bar{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find another choose  $\bar{w}$  to be any non-multiple of  $\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , for example  $\bar{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Then a second solution is:

$$\begin{aligned} x_2 &= e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w}) \\ &= e^{3t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= e^{3t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ &= e^{3t} \begin{bmatrix} 1+t \\ t \end{bmatrix} \end{aligned}$$

So the general solution is

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1+t \\ t \end{bmatrix}$$

#### 4. Two Complex Conjugate Eigenvalues

If we have two complex conjugate eigenpairs then we do get two solutions but they are not real solutions. We've seen this issue before.

To understand the method we will work through an example the long way and then point out that there's a short method. Then we will work through another example with the short method.

**Example:**

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) \text{ and } \left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$$

**Long Way:** The first gives us the solution:

$$\begin{aligned} \bar{x} &= e^{(3+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{3t} (\cos(2t) + i \sin(2t)) \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ i \cos(2t) - \sin(2t) \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{bmatrix} \quad \textcircled{1} \end{aligned}$$

The second gives us the solution:

$$\begin{aligned} \bar{x} &= e^{(3-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{3t} (\cos(-2t) + i \sin(-2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= (e^{3t} \cos(2t) - i e^{3t} \sin(2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) - i \sin(2t) \\ -i \cos(2t) - \sin(2t) \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) - i \sin(2t) \\ -\sin(2t) - i \cos(2t) \end{bmatrix} \quad \textcircled{2} \end{aligned}$$

Since linear combinations of solutions are solutions if we take half of the sum of these we get the solution:

$$\bar{x}_1 = \frac{1}{2} [\textcircled{1} + \textcircled{2}] = e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

and if we take  $\frac{1}{2i}$  times the difference we get the solution:

$$\bar{x}_1 = \frac{1}{2i} [\textcircled{1} - \textcircled{2}] = e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

**Short Way:** If we look back at ① and break it into real and imaginary parts then the two solutions can be extracted from the real and imaginary parts:

$$\begin{aligned}\bar{x} &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{bmatrix} \\ &= e^{3t} \underbrace{\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}_{\bar{x}_1} + i e^{3t} \underbrace{\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}_{\bar{x}_2}\end{aligned}$$

So the general solution is:

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

We can see now that the short method would be to take one eigenpair:

$$(r + si, \bar{v})$$

Use this to calculate ① and extract the real and imaginary parts. But where did ① come from? It came from doing the calculation:

$$e^{rt} (\cos(st) + i \sin(st)) \bar{v}$$

So the key is to do this calculation and extract the real and imaginary parts and those will be  $\bar{x}_1$  and  $\bar{x}_2$ .

**Example:**

$$\bar{x}' = \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(2 + i\sqrt{5}, \begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix}\right) \text{ and } \left(2 - i\sqrt{5}, \begin{bmatrix} -i\sqrt{5} \\ -5 \end{bmatrix}\right)$$

We then calculate:

$$\begin{aligned} & e^{2t} \left( \cos(\sqrt{5}t) + i \sin(\sqrt{5}t) \right) \begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) + i\sqrt{5} \cos(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) - 5i \sin(\sqrt{5}t) \end{bmatrix} \end{aligned}$$

From here we extract the real and imaginary parts:

$$\begin{aligned} \bar{x}_1 &= e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) \end{bmatrix} \\ \bar{x}_2 &= e^{2t} \begin{bmatrix} -\sqrt{5} \cos(\sqrt{5}t) \\ -5 \sin(\sqrt{5}t) \end{bmatrix} \end{aligned}$$

So the general solution is

$$\bar{x} = C_1 e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\sqrt{5} \cos(\sqrt{5}t) \\ -5 \sin(\sqrt{5}t) \end{bmatrix}$$

Note: Other valid answers can look quite different from this since any multiple of an eigenvector is an eigenvector and since complex multiples can look quite surprising.

5. **An Initial Value Problem:** Since we haven't done one from start to finish, here is an initial value problem:

**Example:** Solve

$$\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- (a) Find the eigenvalues:

$$\begin{aligned} p(z) &= \det \begin{bmatrix} z-5 & -1 \\ 3 & z-1 \end{bmatrix} = (z-5)(z-1) - (-1)(3) \\ &= z^2 - 6z + 8 = (z-2)(z-4) \end{aligned}$$

So the eigenvalues are  $\lambda_1 = 2$  and  $\lambda_2 = 4$ .

- (b) Find the eigenvectors:

For  $\lambda_1 = 2$  choose a nonzero column of  $A - \lambda_2 I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$  so

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For  $\lambda_2 = 4$  choose a nonzero column of  $A - \lambda_1 I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$  so

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- (c) Write down the general solution:

We have

$$\bar{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- (d) Plug in the initial value and solve:

$$\bar{x}(0) = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \bar{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so that:

$$\bar{c} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

- (e) Write down the answer:

$$\bar{x} = \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



## MATH 246: Chapter 3 Section 6: Graphs of Solutions

Justin Wyss-Gallifent

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Main Topics:

- Basic Idea
  - Gallery of Possibilities
- 

### 1. Introduction:

The goal of this section is to understand what the solutions of the system:

$$\bar{x}' = A\bar{x}$$

look like graphically when we are in the  $n = 2$  case.

To make this a little clearer instead of thinking of  $\bar{x}'$  as  $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  we'll think of  $\bar{x}'$  as  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  because this way a solution can be thought of as a curve which moves around in the  $xy$ -plane as a function of time  $t$ .

In addition we'll think of the matrix  $A$  as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

What we'll do first is go through one example thoroughly and then the rest will be categorized without too much explanation.

### 2. Broad Strokes:

The general idea is that the solutions can always be graphed using just the eigenvalues and sometimes (but not always) the eigenvectors and sometimes (but not always) the matrix. The solutions will not be perfect but they'll give us a lot of insight.

Here the types of solutions have been broken down into five categories to make them easier to remember. Each category has subcategories.

While this seems like a lot there are many similarities and you'll find that patterns repeat over and over and make a lot of sense, so it's really not that terrible!

### 3. First Example:

Consider the system:

$$\bar{x}' = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix} \bar{x}$$

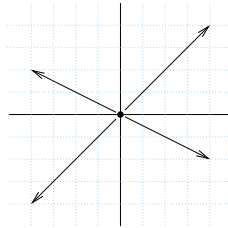
The eigenpairs of  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$  are  $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$  and  $\left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . The general solution is then

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let's analyze a few solutions:

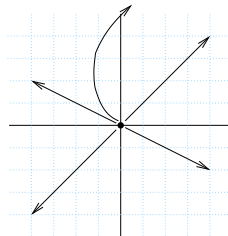
- If  $C_1 = 0$  and  $C_2 = 0$  then we get  $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  which is a constant solution which sits at the origin for all  $t$ .
- If  $C_1 = 0$  and  $C_2 = 1$  then we get  $\bar{x} = e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}$ . Notice that  $x(t) = e^{9t}$  and  $y = e^{9t}$  are always positive and equal. As  $t \rightarrow \infty$  this point moves away from the origin and as  $t \rightarrow -\infty$  this point moves toward but never touches (slows down as it goes) the origin.
- If  $C_1 = 0$  and  $C_2 = -1$  we get a similar thing, the only difference being that both  $x(t)$  and  $y(t)$  are negative.
- If  $C_1 = 1$  and  $C_2 = 0$  then we get  $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{3t} \\ e^{3t} \end{bmatrix}$ . This solution always has  $x = -2y$  but otherwise has the same behavior.
- If  $C_1 = -1$  and  $C_2 = 0$  we get the opposite of the previous.

All together we get the following five solutions:



One more solution:

- If  $C_1 = 1$  and  $C_2 = 1$  then we get  $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . For large negative  $t$  the  $e^{9t}$  is closer to zero than the  $e^{3t}$  and so the function behaves like  $e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ . On the other hand for large positive  $t$  the  $e^{3t}$  still exists and contributes but the  $e^{9t}$  is much more significant and so the function turns out to be approaching parallel to  $e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The result is:



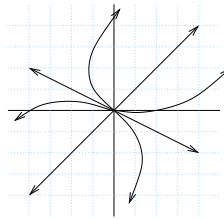
#### 4. Categories of Solutions:

(a) The eigenvalues  $\lambda_1, \lambda_2$  are both real, nonzero and different.

i. Both eigenvalues are positive: **Nodal Source - Unstable**

In this case there are four straight-line solutions moving away from the origin along the vectors  $\pm \bar{v}_1$  and  $\pm \bar{v}_2$ . The other solutions move away from the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

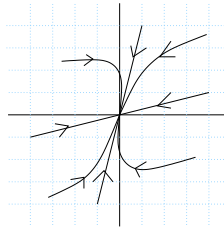
**Example 1:** If  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$  then the epairs are  $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



ii. Both eigenvalues are negative: **Nodal Sink - Stable**

In this case there are four straight-line solutions moving toward the origin along the vectors  $\pm \bar{v}_1$  and  $\pm \bar{v}_2$ . The other solutions move toward the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

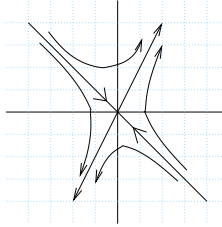
**Example 2:** If  $A = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$  then the epairs are  $\left(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right), \left(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$ . Picture:



iii. One eigenvalue is positive and the other is negative: **Saddle - Unstable**

In this case there are four straight-line solutions. The two corresponding to the positive eigenvalue move away from the origin (along the eigenvector and its opposite) and the two corresponding to the negative eigenvalue move toward the origin (along the eigenvector and its opposite). The other solutions move toward the origin initially parallel to the straight-line solutions moving toward the origin but then curve and move away parallel to the other straight-line solution.

**Example 3:** If  $A = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$  then the epairs are  $\left(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .  
Picture:

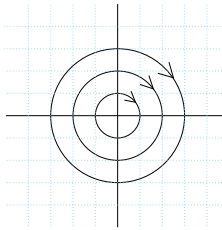


(b) The eigenvalues are complex conjugates.

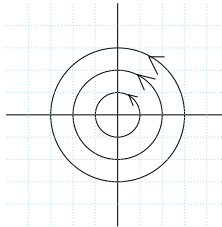
i. They have the form  $0 \pm si$ : **Circle - Stable**

In this case the solutions are circles around the origin. They are clockwise if  $a_{12} > 0$  and counterclockwise if  $a_{12} < 0$ .

**Example 4:** If  $A = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$  then the evals are  $\pm 2i$ . Picture:



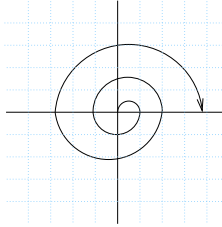
**Example 5:** If  $A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$  then the evals are  $\pm 6i$ . Picture:



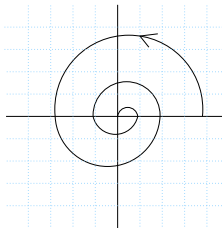
ii. They have the form  $r \pm si$ : **Spiral Source - Unstable (if out) or Sink - Stable (if in)**

In this case the solutions are spirals around the origin. They are clockwise if  $a_{12} > 0$  and counterclockwise if  $a_{12} < 0$  and they spiral in if  $r < 0$  and out if  $r > 0$ .

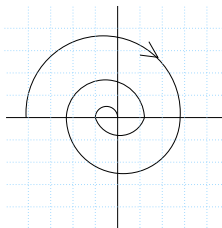
**Example 6:** If  $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$  then the evals are  $3 \pm 2i$ . Picture:



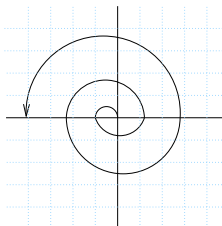
**Example 7:** If  $A = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$  then the evals are  $-3 \pm 2i$ . Picture:



**Example 8:** If  $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$  then the evals are  $-2 \pm 3i$ . Picture:



**Example 9:** If  $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$  then the evals are  $2 \pm 6i$ . Picture:

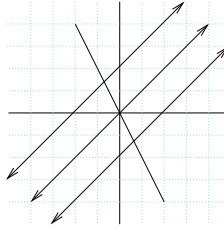


(c) One eigenvalue is 0, the other  $\lambda$  is real and not zero.

i. The other eigenvalue is positive: **Linear Source - Unstable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move away from that line and are parallel to the eigenvector corresponding to  $\lambda$ .

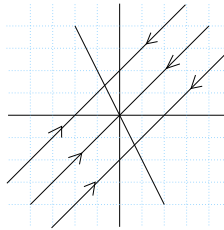
**Example 10:** If  $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$  then the epairs are  $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



ii. The other eigenvalue is negative: **Linear Sink - Stable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move toward that line and are parallel to the eigenvector corresponding to  $\lambda$ .

**Example 11:** If  $A = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$  then the epairs are  $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:

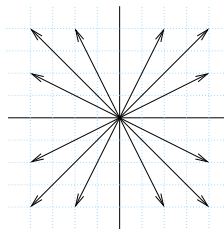


(d) There is a single nonzero eigenvalue  $\lambda$  and  $A$  looks like  $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ :

i. If the eigenvalue is positive: **Radial Source - Unstable**

In this case all the solutions are straight lines moving away from the origin.

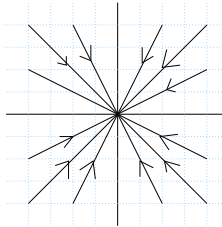
**Example 12:** If  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  then the eval is 2. Picture:



ii. If the eigenvalue is negative: **Radial Sink - Stable**

In this case all the solutions are straight lines moving toward the origin.

**Example 13:** If  $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$  then the eval is  $-3$ . Picture:

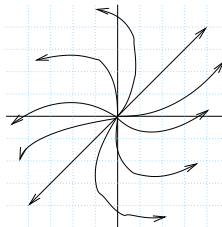


(e) There is a single nonzero eigenvalue  $\lambda$  and  $A$  does not look like that:

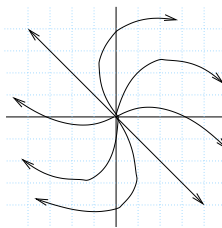
i. If the eigenvalue is positive: **Twist Source - Unstable**

In this case there are two straight-line solutions moving away from the origin along the eigenvector corresponding to  $\lambda$ . The other solutions are all curved solutions which move away from the origin in a clockwise direction if  $a_{12} > 0$  and in a counterclockwise direction if  $a_{12} < 0$ .

**Example 14:** If  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$  then the epair is  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



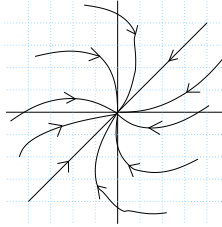
**Example 15:** If  $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$  then the epair is  $\left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ . Picture:



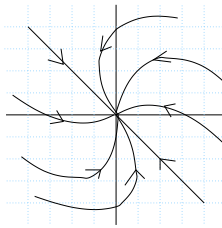
ii. If the eigenvalue is negative: **Twist Sink - Stable**

In this case there are two straight-line solutions moving toward the origin along the eigenvector corresponding to  $\lambda$ . The other solutions are all curved solutions which move toward the origin in a clockwise direction if  $a_{12} > 0$  and in a counterclockwise direction if  $a_{12} < 0$ .

**Example 16:** If  $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$  then the epair is  $\left(-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



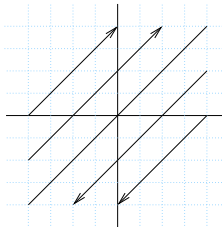
**Example 17:** If  $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$  then the epair is  $\left(-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ . Picture:



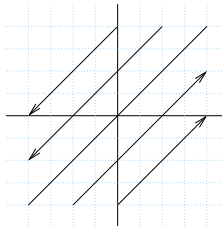
iii. If the eigenvalue is zero: **Parallel Shear - Unstable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions. The other solutions are straight lines parallel to that one, “clockwise” if  $a_{12} > 0$  and “counterclockwise” if  $a_{12} < 0$ .

**Example 18:** If  $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$  then the epair is  $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



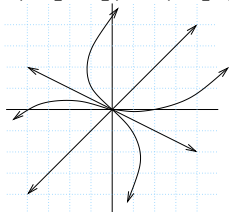
**Example 19:** If  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  then the epair is  $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



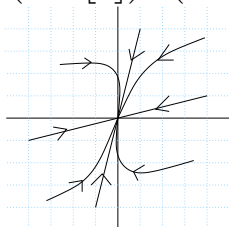


# MATH 246: Chapter 3 Section 6: Graphs of Solutions Cheatsheet

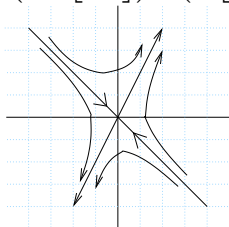
1.  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$ :  
 $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



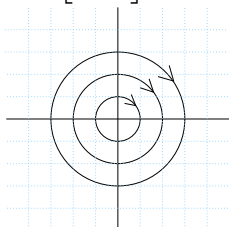
2.  $A = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$ :  
 $\left(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right), \left(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$



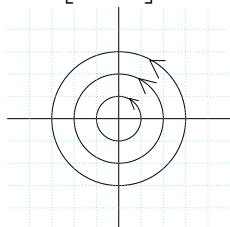
3.  $A = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$ :  
 $\left(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$



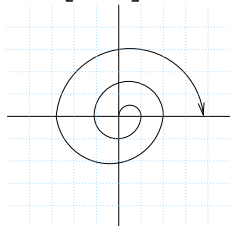
4.  $A = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$ :  $\lambda = 0 \pm 2i$



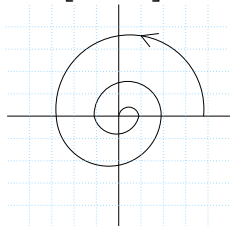
5.  $A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$ :  $\lambda = 0 \pm 6i$



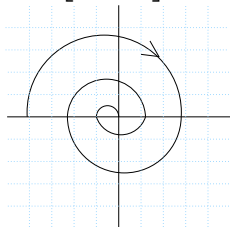
6.  $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$ :  $\lambda = 3 \pm 2i$



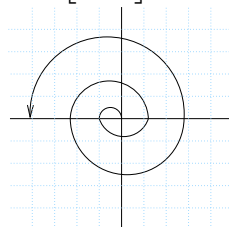
7.  $A = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$ :  $\lambda = -3 \pm 2i$



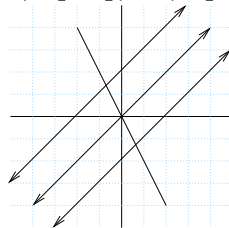
8.  $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$ :  $\lambda = -2 \pm 3i$



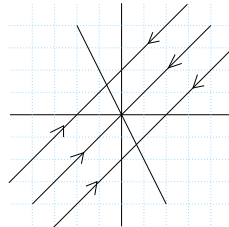
9.  $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ :  $\lambda = 2 \pm 6i$



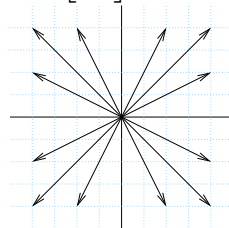
10.  $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ :  
 $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



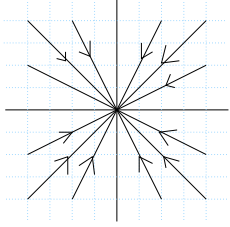
11.  $A = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$ :  
 $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



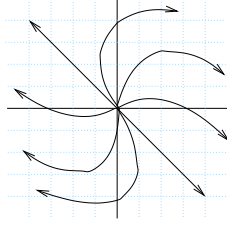
12.  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ :  $\lambda = 2$



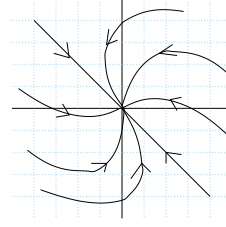
13.  $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} : \lambda = -3$



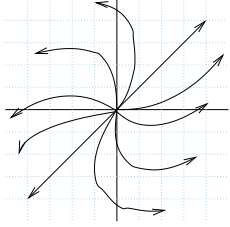
15.  $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} : \left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$



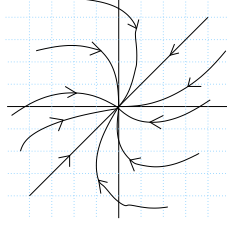
17.  $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix} : \left(-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$



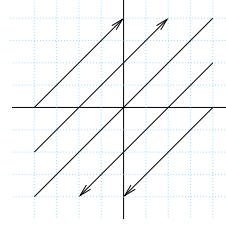
14.  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} : \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



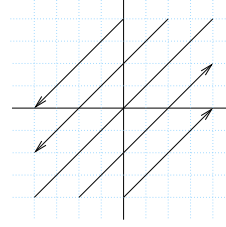
16.  $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} : \left(-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



18.  $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} : \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



19.  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} : \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



## MATH 246: Chapter 3 Section 7: Hamiltonian Systems

Justin Wyss-Gallifent

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Main Topics:

- Preliminaries
  - Hamiltonian Systems
  - Stationary Solutions/Points
  - Analysis with the Hessian
- 

### 1. Preliminaries

The goal of this section is to look at a couple of more specific things related to systems of two differential equations. We're going to first throw out the requirement that the system is linear, meaning we can't assume it looks like  $\bar{x}' = A\bar{x}$ . Instead we'll think of these as:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

**Example:** One example would be something like:

$$\begin{aligned}x' &= x^2 - y^2 \\ y' &= y + 2\end{aligned}$$

2. **Definition of Hamiltonian** One very special type of system of differential equations is a Hamiltonian system. A Hamiltonian system is a system in which there is some function  $H(x, y)$  such that:

$$\begin{aligned}x' &= H_y(x, y) \\ y' &= -H_x(x, y)\end{aligned}$$

The reason that these are nice is that for a Hamiltonian system we have:

$$\begin{aligned}-H_x(x, y) \frac{dx}{dt} &= H_y(x, y) \frac{dy}{dt} \\ H_x(x, y) \frac{dy}{dt} + H_y(x, y) \frac{dx}{dt} &= 0 \\ \frac{d}{dt} H(x, y) &= 0 \\ H(x, y) &= C\end{aligned}$$

For some/any constant  $C$ . This means that solutions to the system of differential equations are level curves for  $H$ .

**Example:** The system:

$$\begin{aligned}x' &= 2y \\ y' &= -2x\end{aligned}$$

is Hamiltonian with  $H(x, y) = x^2 + y^2$  because  $x' = H_y = 2y$  and  $y' = -H_x = -2x$ . The solutions then satisfy  $x^2 + y^2 = C$  and so they're circles.

Notice that we could also have seen this using methods from the previous section. Here  $\bar{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \bar{x}$  so the eigenvalues for  $A$  are  $0 \pm 2i$  and since  $a_{12} > 0$  the solutions are (clockwise) circles.

### 3. Determining if a System is Hamiltonian

It turns out that a system of the form:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

is Hamiltonian if  $f_x + g_y = 0$  and if it is then we can find  $H$  using a process we used before with exact differential equations.

**Example:** Show the following system is Hamiltonian and find  $H(x, y)$ :

$$\begin{aligned}x' &= x^2 + 2y \\ y' &= -2xy\end{aligned}$$

First note that  $f_x + g_y = 2x - 2x = 0$  so the system is Hamiltonian. We wish to find  $H(x, y)$  with  $H_y(x, y) = x^2 + 2y$  and  $-H_x(x, y) = -2xy$ . The latter tells us  $H_x(x, y) = 2xy$  and so  $H(x, y) = x^2y + g(y)$ . From here  $H_y(x, y) = x^2 + g'(y) = x^2 + 2y$  so  $g'(y) = 2y$  and  $g(y) = y^2 + C$ . Then  $H(x, y) = x^2y + y^2 + C$ . Since we can choose any constant we let  $H(x, y) = x^2y + y^2$ . Thus the solutions, when plotted, satisfy the equation  $x^2y + y^2 = C$ , whatever this looks like!

### 4. Analysis of Stationary Solutions/Points

A stationary solution is a solution for which  $x' = 0$  and  $y' = 0$ . These are easy to solve for.

**Example:** Find the stationary solutions to:

$$\begin{aligned}x' &= x^2 - y^2 \\ y' &= y + 2\end{aligned}$$

We set  $x^2 - y^2 = 0$  and  $y + 2 = 0$ . The latter gives us  $y = -2$  and then the former gives us  $x^2 - 4 = 0$  so  $x = \pm 2$ . Therefore there are two stationary solutions,  $(2, -2)$  and  $(-2, -2)$ .

Hamiltonian systems can be analyzed further by looking at the Hessian at each stationary point. The Hessian is the following matrix:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}$$

We'll only look at those for which the determinant of the Hessian is nonzero.

If  $\det \partial^2 H < 0$  then the stationary point is a saddle.

If  $\det \partial^2 H > 0$  then the stationary point is a circle.

Directions can be figured out by testing points.

**Example:** Consider the system

$$\begin{aligned}x' &= 4y - y^3 \\y' &= x\end{aligned}$$

To find the stationary solutions we set  $4y - y^3 = y(4 - y^2) = 0$  and  $x = 0$ . The former gives us  $y = 0, \pm 2$  so there are three stationary solutions at  $(0, 0)$ ,  $(0, 2)$  and  $(0, -2)$ . Noting that  $H_y = 4y - y^3$  and  $-H_x = x$ , so  $H_x = -x$ , we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 - 3y^2 \end{bmatrix}$$

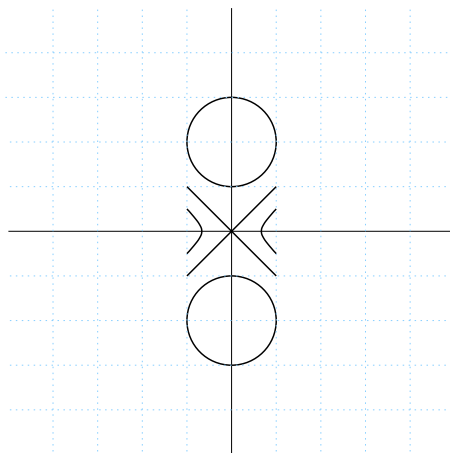
Then at each point:

$$\det \partial^2 H(0, 0) = \det \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = -4 \text{ so } (0, 0) \text{ is a saddle}$$

$$\det \partial^2 H(0, 2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0, 2) \text{ is a circle}$$

$$\det \partial^2 H(0, -2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0, -2) \text{ is a circle.}$$

A preliminary picture is:



Notice how the circles fit nicely with the saddle shape!

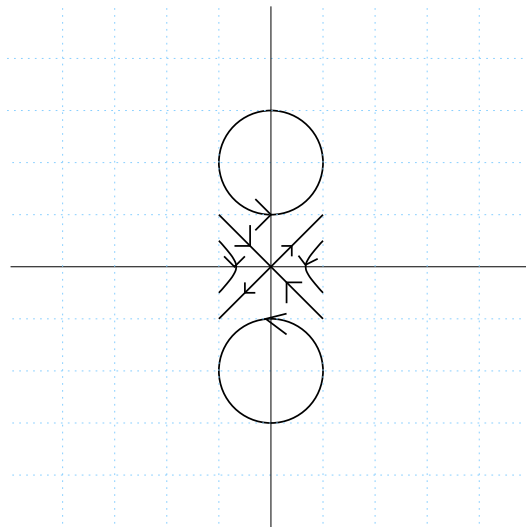
**Example (Continued):**

We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point  $(0, 1)$  we have:

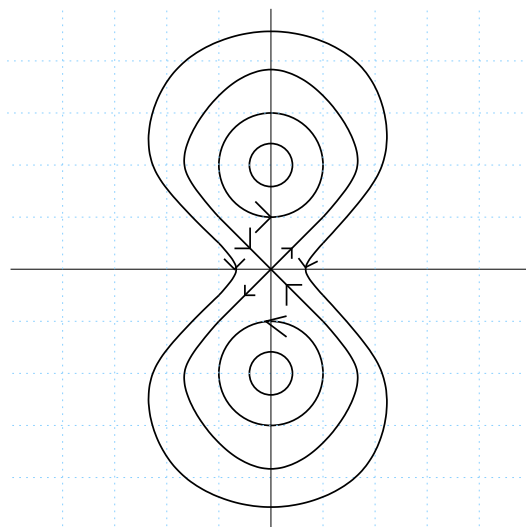
$$x'(0, 1) = 3$$

$$y'(0, 1) = 0$$

Meaning at  $(0, 1)$  the solution is moving to the right. Everything else is filled in according to rules of compatibility.



If we were to fill in more solutions it starts to look pretty:



**Example:** Consider the system

$$\begin{aligned}x' &= -x + y + x^2 \\y' &= y - 2xy\end{aligned}$$

To find the stationary points we set  $-x + y + x^2 = 0$  and  $y - 2xy = y(1 - 2x) = 0$ . The latter gives us  $y = 0$  or  $x = \frac{1}{2}$ . If  $y = 0$  then the former gives us  $x = 0, 1$  and if  $x = \frac{1}{2}$  then the former gives us  $y = \frac{1}{4}$ . So there are three stationary points at  $(0, 0)$ ,  $(1, 0)$  and  $(\frac{1}{2}, \frac{1}{4})$ .

Noting that  $H_y = -x + y + x^2$  and  $-H_x = y - 2xy$ , so  $H_x = 2xy - y$ , we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 1 \\ 2x - 1 & 1 \end{bmatrix}$$

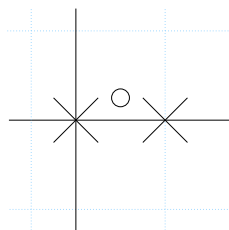
Then at each point:

$$\det \partial^2 H(0, 0) = \det \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = -1 \text{ so } (0, 0) \text{ is a saddle.}$$

$$\det \partial^2 H(1, 0) = \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 \text{ so } (1, 0) \text{ is a saddle.}$$

$$\det \partial^2 H\left(\frac{1}{2}, \frac{1}{4}\right) = \det \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \text{ so } \left(\frac{1}{2}, \frac{1}{4}\right) \text{ is a center.}$$

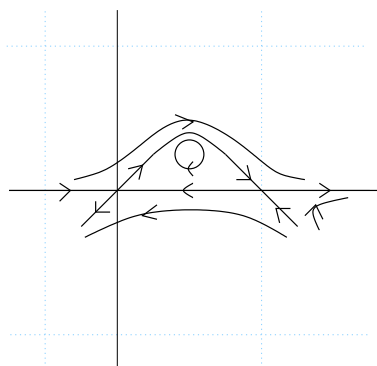
A preliminary picture is:



We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point  $(\frac{1}{2}, 0)$  we have:

$$\begin{aligned}x' \left( \frac{1}{2}, 0 \right) &= - \\y' \left( \frac{1}{2}, 0 \right) &= 0\end{aligned}$$

Meaning at  $(\frac{1}{2}, 0)$  the solution is moving to the left. Everything else is filled in according to rules of compatibility. Some adjustment of the saddles is also needed!



## MATH 246: Chapter 3 Section 9: Linearization of Nonlinear Systems

Justin Wyss-Gallifent

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Main Topics:

- Linearization
  - Stationary Point Analysis
- 

### 1. Introduction

The goal of this section is to do a bit of analysis of nonlinear systems which are not necessarily Hamiltonian. The approach is similar though - find stationary solutions, find what they look like, fill them in, figure out what the remaining solutions look like.

### 2. Linearization at the Stationary Solutions

This sounds far more complicated than it sounds. In Calculus suppose you know that  $f(x) = x^2 - 9$  and you're investigating this function. You might notice that the  $x$ -intercepts are  $x = \pm 3$  and you might want to know what happens at those points. You might notice that  $f'(x) = 2x$  so  $f'(-3) = -6$  and  $f'(3) = 6$  and so the function is decreasing at  $x = -3$  and increasing at  $x = 3$ . What you just did was that you *linearized* the function at  $x = \pm 3$ , meaning you sort of made it a line with slope  $\pm 6$  at those points.

What we'll do is precisely the same thing but with a system of nonlinear differential equations.

Given a system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

First a notation point - sometimes we write

$$\bar{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Using this notation the linearization matrix for this system is the matrix:

$$\partial \bar{F} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

**Example:** Consider the system:

$$\begin{aligned}x' &= y \\ y' &= 4x - x^3\end{aligned}$$

Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$



### 3. Stationary Point Analysis

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that  $f'(3) = 6$  you know the function is increasing at that point but if you discovered that  $f'(3) = 0$  then the function could be increasing or decreasing or neither at that point.

#### 4. Stationary Point Analysis

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that  $f'(3) = 6$  you know the function is increasing at that point but if you discovered that  $f'(3) = 0$  then the function could be increasing or decreasing or neither at that point.

**Example:**

Consider the system:

$$\begin{aligned}x' &= 1 - y \\ y' &= x^2 - y^2\end{aligned}$$

Since  $f(x, y) = 1 - y$  and  $g(x, y) = x^2 - y^2$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$$

The stationary solutions are  $(1, 1)$  and  $(-1, 1)$ . We check the linearization matrix at those points:

- At  $(1, 1)$ :  $\partial \bar{F}(1, 1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$ .

The eigenvalues are  $\lambda = -1 \pm i$  so the point is a counterclockwise spiral sink.

- At  $(-1, 1)$ :  $\partial \bar{F}(-1, 1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ . The eigenpairs are

$$\left( -1 + \sqrt{3}, \begin{bmatrix} 1 + \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} 2.7 \\ -2 \end{bmatrix} \right) \quad \text{and} \quad \left( -1 - \sqrt{3}, \begin{bmatrix} 1 - \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} -1.7 \\ -2 \end{bmatrix} \right).$$

This is a saddle.

**Example:** Consider the system:

$$\begin{aligned}x' &= y \\y' &= 4x - x^3\end{aligned}$$

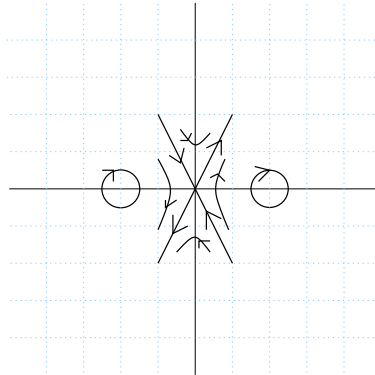
Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

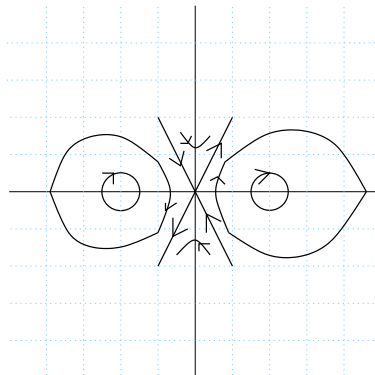
The stationary solutions are  $(0, 0)$ ,  $(2, 0)$  and  $(-2, 0)$ . We check the linearization matrix at those points:

- At  $(0, 0)$ :  $\partial \bar{F}(0, 0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$  and  $\left(2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ . This is a saddle.
- At  $(2, 0)$ :  $\partial \bar{F}(2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .
- At  $(-2, 0)$ :  $\partial \bar{F}(-2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .

Together we get the picture:



From here we can fill in a nice family of solutions:



**Example:** Consider the system:

$$\begin{aligned}x' &= (y - x)(x - 1) \\ y' &= (3 + 2x - x^2)y\end{aligned}$$

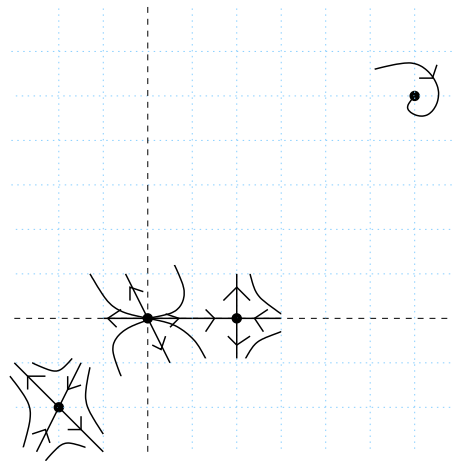
Since  $f(x, y) = xy - x^2 - y + x$  and  $g(x, y) = 3y + 2xy - x^2y$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} y - 2x + 1 & x - 1 \\ 2y - 2xy & 3 + 2x - x^2 \end{bmatrix}$$

The stationary solutions are  $(0, 0)$ ,  $(-1, -1)$ ,  $(1, 0)$  and  $(3, 3)$ . We check the linearization at those point:

- At  $(0, 0)$ :  $\partial \bar{F}(0, 0) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ . The eigenpairs are  $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(3, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ . This is a source. Solutions close to  $(0, 0)$  are tangent to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .
- At  $(-1, -1)$ :  $\partial \bar{F}(-1, -1) = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ . This is a saddle.
- At  $(1, 0)$ :  $\partial \bar{F}(1, 0) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ . The eigenpairs are  $\left(-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ . This is a saddle.
- At  $(3, 3)$ :  $\partial \bar{F}(3, 3) = \begin{bmatrix} -2 & 2 \\ 12 & 0 \end{bmatrix}$ . The eigenvalues are  $-1 \pm i\sqrt{23}$ . Since  $a_{12} > 0$  this is a clockwise spiral sink.

Together we get the picture:



## MATH 246: Chapter 0 Section 0: Course Introduction and Overview

### Justin Wyss-Gallifent

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Main Topics:

- What is a differential equation and what does it mean to solve one?
  - Ordinary vs. Partial DEs.
  - Order of a DE.
  - Linear vs. Nonlinear DEs.
  - A system of DEs.
- 

1. What is a differential equation and what does it mean to solve one?

- (a) The most straightforward definition of a *differential equation* (a DE) is that it's an equation involving some or all of the following: An unknown function of one or more variables such as  $y(t)$ , derivatives of that function such as  $y'$ ,  $y''$ , and so on, and other functions of the same variable(s) such as  $\sin(t)$  and  $t^2$ .

**Example:**  $f'(t) + f(t) = 10$  in which  $f$  is our unknown function of  $t$ .

**Example:**  $y'' + 3y' - xy = 6$  in which  $y$  is our unknown function of  $x$ .

**Example:**  $t^2 f''(t) = 5 - f'(t)(\sin t)$  in which  $f$  is our unknown function of  $t$ .

**Example:**  $17 \frac{dy}{dx} - x \frac{d^2 y}{dx^2} = xy$  in which  $y$  is our unknown function of  $x$ .

**Example:**  $\partial_x u + \sin(x) \partial_y u = y^3 \partial_{xy} u$  in which  $u$  is our unknown function of both  $x$  and  $y$ .

- (b) *Solving a DE* means finding a function which makes the DE true when you plug that function in.

**Example:**  $f(t) = e^t$  is a solution to the DE  $f(t) - f'(t) = 0$ .

**Example:**  $y(t) = \sin(t)$  is a solution to the DE  $y + y'' = 0$ .

**Example:**  $f(t) = t + e^{2t}$  is a solution to the DE  $f''(t) + 4t = 4f(t)$ .

**Example:**  $f(x) = x^2$  is not a solution to the DE  $xf'(x) = f(x)$ .

Just as regular equations can have more than one solution ( $x^2 - 9 = 0$  has two solutions) so can a DE. In fact usually a DE will have infinitely many solutions.

**Example:**  $f(t) = 487e^t$  is another solution to the DE  $f(t) - f'(t) = 0$ . You can probably see lots more now.

2. Associated definitions

- (a) A DE is called *ordinary* (so an ODE) if the unknown function is just a function of one variable. Otherwise it's *partial* (so a PDE). Generally in this course when we talk about a DE we mean an ODE.

**Example:**  $f'(t) + 3tf''(t) = e^t$  is an ODE.

**Example:**  $u_x(x, y) + u_{yx}(x, y) + y = 3$  is a PDE. If you've not seen partial derivatives before don't worry.

- (b) The *order* of a DE is the highest derivative that appears in it. We say things like *first-order* and *second-order* and so on.

**Example:**  $x^7 f'(x) + (\cos x)f(x) + x = e^x$  is first-order.

**Example:**  $tf(t) + e^t f''(t) = 1 - f'(t)$  is second-order.

(c) A DE is *linear* if it can be written as a sum of some or all of:

- i. An unknown  $f$  multiplied by a coefficient.
- ii. Derivatives of the unknown  $f$  multiplied by coefficients.
- iii. Coefficients.

By *coefficients* we mean they can be other functions of the same variables that  $f$  is, including just constants, including 0.

**Example:** The DE  $5tf(t) + (\ln t)f'(t) = 5$  is linear.  
**Example:** The DE  $(\tan t)y(t) - t^3y'(t) + 7y''(t) = 1$  is linear.  
**Example:** The DE  $f(x)\sqrt{x} + (1-x)f'''(x) = f'(x)$  is linear.  
**Example:** The DE  $f(t)^2 + f'(t) = 7$  is nonlinear because the  $f(t)^2$  is not permitted.  
**Example:** The DE  $\sin(y') + y' - y = x$  is nonlinear because the  $\sin(y')$  is not permitted.  
**Example:** The DE  $y'y + y = xy$  is nonlinear because the  $y'y$  is not permitted.

Clarification perhaps:

A *first-order linear differential equation* using the variable  $t$  and the unknown function  $y$  will have the form

$$a_1(t)y' + a_0(t)y = c(t)$$

A *second-order linear differential equation* with the function  $y(t)$  will have the form:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = c(t)$$

An  $n^{\text{th}}$  order linear DE with the function  $y(t)$  will have the form:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = c(t)$$

(d) A *system* of DEs is just that, a collection of more than one DE where the goal is to find a single function that makes them all true. The *order* of such a system is the highest derivative that appears in any of the DEs.

**Example:** A first-order system of two linear DEs:

$$\begin{aligned} ty + t^2y' &= e^t \\ 3y + 5y' &= \sin(t) \end{aligned}$$

### 3. Moving onwards.

At this point you can probably start to wrap your head around which DEs look like they might be easier to handle. The following is a list of DEs of increasing complexity. Even though you don't really know how to solve any of these just yet (that's not true, you can do the first one!) you can almost certainly look at them in order and get an appreciation for the fact that they start pretty nice and get more convoluted! Don't worry that some of the words on the right might not make sense.

$y' = t^2$	Explicit first order linear ODE
$5y' - 4y = 0$	Homogeneous first order linear ODE with constant coefficients
$2y'' + 5y' - 4y = 0$	Homogeneous second order linear ODE with constant coefficients
$7y' - 2y = t$	Nonhomogeneous first order linear ODE with constant coefficients
$t^2y' + e^ty = 1 + t$	Nonhomogeneous first order linear ODE

With some quirky exceptions our approach will pretty much be like that in that we'll first tackle the easier types. This will help us develop some theory which will then support us as we move to the more complicated types, and then to systems of these.

## MATH 246: Chapter 1 Section 1: Introduction to First-Order DEs

Justin Wyss-Gallifent

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Main Topics:

- Overview of first-order (O)DEs.
  - Explicit first-order DEs.
  - General solutions, initial value problems, specific (particular) solutions.
  - Underlying theory regarding existence of solutions on an interval.
- 

### 1. Introductory overview of first-order ODEs.

- (a) A first-order ODE (not necessarily linear) is permitted to have an unknown function  $y$  (of a single variable, say  $t$ ) its derivative  $y'$  and then some other functions of  $t$ .

**Example:**  $t(y')^2 + y = \sin t$

**Example:**  $y' - ty = e^{2t}$

**Example:**  $\sin(y') + e^{y'} = t$

- (b) In general these can be very hard! The first step is always algebra though, basically we first solve for  $y'$  and then proceed from there. Thus for the next few sections we'll assume that we've solved for  $y'$  in terms of  $t$  and  $y$  and we'll focus on DEs that have the form  $y' = f(t, y)$ .

That  $f$  might be confusing, it's not the unknown function but rather it just represents the fact that we can have a bunch of  $y$  and  $t$  on the right hand side. In other words things like this:

**Example:**  $y' = ty$

**Example:**  $y' = 4t - 8y$

**Example:**  $y' = \frac{y}{t}$ .

### 2. Explicit first-order DEs.

- (a) Because solving even first-order ODEs is hard we'll go down even further and look at *explicit* first-order ODEs that have the form  $y' = f(t)$ .

**Example:**  $y' = t^2$ .

**Example:**  $y' = 4t + \sin t$ .

- (b) At this point you might have an epiphany and realize that often you can solve these because solving these is as easy as integrating the right side.

**Example:**  $y' = t^2$ . To solve this we integrate to get  $y = \frac{1}{3}t^3 + C$  for any constant  $C$ .

**Example:**  $y' = 4t + \sin t$ . To solve this we integrate to get  $y = 2t^2 - \cos t + C$  for any constant  $C$ .

### 3. General solutions, initial value problems, specific (particular) solutions

- (a) We've started to notice that we can have many solutions to a DE. In the explicit DEs above get a constant  $C$  which can be anything.
- (b) A *general solution* to a DE is a solution involving constants and for which different constants will give all solutions.

- (c) A *specific solution* or a *particular solution* is a solution in which a specific (particular) choice of constant(s) has been made.

**Example:** The general solution to  $y' = t^2$  is  $y = \frac{1}{3}t^3 + C$ . Some specific solutions are  $y = \frac{1}{3}t + 1$ ,  $y = \frac{1}{3}t - 107$  and  $y = \frac{1}{3}t + \pi$ .

- (d) Often when we encounter a DE it comes pre-packaged with an *initial value*, or IV. In our simple exact case (and in many future cases) this will be an instance that  $y(t_I) = y_I$  for specific  $t_I$  and  $y_I$ . The DE and the IV together form an *initial value problem* or IVP. It's very common that  $t_I = 0$  but this isn't always the case!

**Example:**  $y' = 2t$  with  $y(0) = 3$  is an IVP.

**Example:**  $y' = 2t$  with  $y(0) = 5$  is an IVP with the same DE but different IV.

**Example:**  $y' = 2t$  with  $y(1) = 3$  is an IVP with again the same DE but different IV.

- (e) When we're given an IVP the idea will be to first solve the DE to get the general solution and then use the IV to get the specific solution.

**Example:**  $y' = 2t$  with  $y(0) = 3$ . First we find  $y = t^2 + C$ , the general solution, and then  $y(0) = 0^2 + C = 3$  so  $C = 3$  and the specific solution is  $y = t^2 + 3$ .

**Example:**  $y' = 2t$  with  $y(1) = 3$ . First we find  $y = t^2 + C$ , the general solution, and then  $y(1) = 1^2 + C = 3$  so  $C = 2$  and the specific solution is  $y = t^2 + 2$ .

#### 4. Intervals of Existence and Theory for Explicit IVPs:

We now know that solving the explicit DE given by  $y' = f(t)$  is as easy (or hard) as finding an antiderivative for  $f(t)$ . However the Fundamental Theorem of Calculus tells us something interesting. It states that if a function is continuous on an open interval then it has an antiderivative on that open interval. This means that even if we can't actually find the antiderivative of  $f(t)$  using techniques that we know, we still know that it exists, and therefore that there is a solution on an open interval as long as  $f(t)$  is continuous on that open interval.

**Example:** Consider the explicit DE given by  $y' = t$ . Since the function  $t$  is continuous on  $(-\infty, \infty)$  we know it has an antiderivative on  $(-\infty, \infty)$  and therefore the DE has a solution there. In this case the general solution is  $y = \frac{1}{2}t^2 + C$ .

**Example:** Consider the explicit DE given by  $y' = \frac{1}{t^2}$ . The function  $\frac{1}{t^2}$  is continuous on  $(-\infty, 0)$  and on  $(0, \infty)$ . What this means is that it has solutions on each of those intervals.

When it comes to an explicit IVP we start with  $y = f(t)$  and  $y(t_I) = y_I$ . We say that the *interval of existence* is the largest open interval containing  $t_I$  on which a solution exists. This is found by finding the largest open interval containing  $t_I$  on which  $f(t)$  is continuous.

**Example:**  $y' = \frac{1}{t^2}$  with  $y(1) = 5$ . We notice the largest open interval containing  $t_I = 1$  on which  $\frac{1}{t^2}$  is defined is  $(0, \infty)$  and so this is the IE. Notice that we don't need to solve it, but we could, since the general solution is  $y = -\frac{1}{t} + C$  and then  $y(1) = -1 + C = 5$  so  $C = 6$  and the specific solution is  $y = -\frac{1}{t} + 6$ .

**Example:**  $y' = \frac{t}{(t-3)(t+6)}$  with  $t(0) = 17$ . We notice the largest open interval containing  $t_I = 0$  on which  $\frac{t}{(t-3)(t+6)}$  is defined is  $(-6, 3)$  so this is the IE. We could possibly solve this with some messy partial fractions but we won't. However we do know for sure that there is a solution on this interval.



## MATH 246: Chapter 1 Section 2: Linear First-Order DEs

Justin Wyss-Gallifent

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Main Topics:

- Linear First-Order DEs and Linear Normal Form.
  - General approach.
  - Initial Value Problems.
  - Theory
- 

### 1. Linear first-order ODEs.

Recall that these will all have the form  $a_1(t)y' + a_0(t)y = c(t)$  where  $a_1, a_0, c$  can be any functions of  $t$ .

**Example:**  $4y' + 5y = 0$

**Example:**  $4ty' + e^t y = \sin t$

### 2. Linear Normal Form.

#### (a) Introduction:

We will usually divide through by  $a_1(t)$  and re-label a bit to get what is known as the *linear normal form*:

$$y' + a(t)y = f(t) \text{ for functions } a(t) \text{ and } f(t)$$

#### (b) General Solution.

These we can actually handle, and most of you did in Calculus II though it may be rusty.

Method: If we let  $A(t)$  be an antiderivative (any antiderivative, meaning use  $+0$  for the constant) of  $a(t)$  so that  $A'(t) = a(t)$  then observe:

$$\begin{aligned} y' + a(t)y &= f(t) \\ e^{A(t)}y' + e^{A(t)}a(t)y &= f(t)e^{A(t)} \\ \frac{d}{dt} \left( e^{A(t)}y \right) &= f(t)e^{A(t)} \\ e^{A(t)}y &= \int f(t)e^{A(t)} dt \\ y &= e^{-A(t)} \int f(t)e^{A(t)} dt \end{aligned}$$

The only step that might concern you here is from line 2 to line 3. This is just the reverse of the product rule with a bit of chain rule thrown in. Reading it from line 3 to line 2 might be easier. We'll see this sort of thing happen again with exact DEs later.

This process can either be repeated for each problem or treated simply as a recipe, meaning find  $A(t)$  and then use the formula at the end of the calculation.

Be careful though, the  $e^{-A(t)}$  is multiplied by the entire integral, meaning the  $+C$  too when you integrate.

We'll call this final expression the *integral-form solution*:

$$y = e^{-A(t)} \int f(t)e^{A(t)} dt$$

**Example:** Consider  $y' + 5y = 2$ . We see that  $a(t) = 5$  so  $A(t) = 5t$  and the solution is

$$\begin{aligned} y &= e^{-5t} \int 2e^{5t} dt \\ &= e^{-5t} \left( \frac{2}{5} e^{5t} + C \right) \leftarrow \text{Note the parentheses!!!} \\ &= \frac{2}{5} + Ce^{-5t} \end{aligned}$$

If you'd have forgotten the parentheses you'd have got an incorrect answer which I won't even write here!

**Example:** Consider  $ty' + 2y = t^4$  with  $t > 0$ . This is not in linear normal form so we divide by  $t$  to get  $y' + \frac{2}{t}y = t^3$ . Then  $a(t) = \frac{2}{t}$  so  $A(t) = 2 \ln t$  and the solution is

$$\begin{aligned} y &= e^{-2 \ln t} \int t^3 e^{2 \ln t} dt \\ y &= e^{-2 \ln t} \int t^3 e^{2 \ln t} dt \\ &= t^{-2} \int t^5 dt \\ &= t^{-2} \left( \frac{1}{6} t^6 + C \right) \\ &= \frac{1}{6} t^4 + \frac{C}{t^2} \end{aligned}$$

Here's one with an IVP:

**Example:** Consider  $y' - 6y = e^t$  with  $y(0) = 2$ . We see that  $a(t) = -6$  so  $A(t) = -6t$  and the general solution is

$$\begin{aligned} y &= e^{-(-6t)} \int e^t e^{-6t} dt \\ &= e^{6t} \int e^{-5t} dt \\ &= e^{6t} \left( -\frac{1}{5} e^{-5t} + C \right) \\ &= -\frac{1}{5} e^t + Ce^{6t} \end{aligned}$$

At this point  $y(0) = -\frac{1}{5} e^0 + Ce^0 = -\frac{1}{5} + C = 2$  so that  $C = \frac{11}{5}$  so the specific solution is

$$y = -\frac{1}{5} e^t + \frac{11}{5} e^{6t}$$

At this point you can probably see that solving a first-order linear ODE is as easy (or as hard) as first finding  $A(t)$  and then finding  $\int f(t)e^{A(t)} dt$ .

- (c) Note about the choice of  $A(t)$ . You might wonder what happened if you didn't choose  $+0$  as your constant when choosing  $A(t)$ . In fact it makes no difference. Suppose we took  $A(t)$  and adjusted it by adding some number like  $+7$ . The solution would then be:

$$y = e^{-(A(t)+7)} \int f(t)e^{A(t)+7} dt = e^{-7}e^{-A(t)} \int f(t)e^7e^{A(t)} dt = e^{-A(t)} \int f(t)e^{A(t)} dt$$

which is exactly the same.

### 3. Theory!

The Second Fundamental Theorem of Calculus states that if a function is continuous on an open interval then it has an antiderivative on that interval and that antiderivative will be continuous. What this means is that if  $a(t)$  is continuous then  $A(t)$  will exist and therefore so will  $e^{-A(t)}$  and then provided that  $f(t)$  is continuous then so will  $\int f(t)e^{A(t)} dt$ .

Warning! This doesn't mean that these things are easy to calculate, just that they exist!

What this means is that if we have an initial value  $y(t_I) = y_I$  then the interval of existence of the solution will be the largest open interval containing  $t_I$  on which both  $f(t)$  and  $a(t)$  are continuous. As before this lets us find the IE even when we can't solve the IVP.

**Example:** Consider  $y' + \frac{1}{t^2}y = \frac{1}{t-5}$  with  $y(2) = 17$ . Here  $a(t) = \frac{1}{t^2}$  and  $f(t) = \frac{1}{t-5}$ . The largest open interval containing  $t_I = 2$  on which both are continuous is  $(0, 5)$  so this is the IE of the solution. Finding the solution is a different matter entirely but it exists on  $(0, 5)$ !

### 4. Integration Comment.

As a final note observe that there are two antiderivatives involved in the problem, finding  $A(t)$  and finding  $\int f(t)e^{A(t)} dt$ . This latter one will often involve simplification involving  $e$  and  $\ln$  as well as substitution and integration by parts.

## MATH 246: Chapter 1 Section 3: Separable DEs

### Justin Wyss-Gallifent

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Main Topics:

- Separable DE and Method of Solution.
  - Implicit vs. Explicit Solutions.
  - Constant Solutions.
  - Autonomous DEs.
  - Effect of Initial Values on Solution Choice.
  - Non-uniqueness of Solutions.
  - Method Notation Note.
- 

#### 1. Separable ODEs.

A DE is *separable* if it can be written in the form  $y' = f(t)g(y)$ . The word separable comes from the fact that the right side is separated into a product of a function of  $t$  and a function of  $y$ .

**Example:**  $y' = ty$  is separable - it is already separated!

**Example:**  $ty' + y' = y^2$  is separable because it can be separated, first by factoring  $y'(t+1) = y^2$  and then dividing  $y' = \frac{y^2}{t+1}$  and thinking of it as  $y' = \left(\frac{1}{t+1}\right)y^2$ .

**Example:**  $y' = t + y$  is not separable. There is no way to write the right side as a product of a function of  $t$  and a function of  $y$ .

#### 2. Method: Solving for Non-Constant Solutions.

The method of solution for non-constant solutions is really slick:

$$\begin{aligned}y' &= f(t)g(y) \\ \frac{dy}{dt} &= f(t)g(y) \\ \frac{1}{g(y)}dy &= f(t)dt \\ \int \frac{1}{g(y)}dy &= \int f(t)dt\end{aligned}$$

Where the integral on the left is with respect to  $y$  and the integral on the right is with respect to  $t$ . Since both indefinite integrals should get their own constant, instead we just put a single  $+C$  on the right.

The  $\frac{1}{g(y)}$  looks really icky to integrate but in our examples it generally works out pretty nicely because it's often not a quotient at all.

Side note: We're really taking a stick and beating the notation into a pulp here. If this weird approach bothers you I've attached a note at the end explaining why it's simply a shorthand notation for something more rigorous.

**Example:** Consider  $y' = \frac{t}{y^2}$ .

Note that you could think of this as  $y' = t \left( \frac{1}{y^2} \right)$ . The key is to get all the  $y$ s together multiplied by the  $y'$  on the left and leave all the  $t$ s together on the right. We work as follows, multiplying by  $y^2$  first, or if you prefer to think of it that way, dividing by  $1/y^2$ :

$$\begin{aligned}y' &= \frac{t}{y^2} \\y^2 \frac{dy}{dt} &= t \\y^2 dy &= t dt \\\int y^2 dy &= \int t dt \\\frac{1}{3}y^3 &= \frac{1}{2}t^2 + C \\y &= \sqrt[3]{\frac{3}{2}t^2 + 3C}\end{aligned}$$

There's an argument to be made at this point that since  $3C$  is just a constant we could write  $C$  instead. However this results in a problem having two different  $C$ 's meaning two different things and given that we will often solve for this  $C$  it's far safer to just leave it as is.

### 3. Observation: Constant Solutions:

When we solve a separable DE we do often actually divide by some  $g(y)$  and in that case we have to independently look at the possibility that  $g(y) = 0$ . This may lead to constant functions which are additional solutions to the separable DE.

**Example:** Consider  $y' = e^t \sqrt{1 - y^2}$ .

The method of solution above goes as follows:

$$\begin{aligned}\frac{dy}{dx} &= e^t \sqrt{1 - y^2} \\\frac{1}{\sqrt{1 - y^2}} dy &= e^t dt \\\int \frac{1}{\sqrt{1 - y^2}} dy &= \int e^t dt \\\sin^{-1} y &= e^t + C \\y &= \sin(e^t + C)\end{aligned}$$

There is nothing wrong with this provided  $\sqrt{1 - y^2} \neq 0$ . So what if  $\sqrt{1 - y^2} = 0$ ? This would arise if  $y = \pm 1$  and in fact these are completely valid solutions (functions!) to the DE. Thus overall the DE has two constant solutions as well as the nonconstant solutions!

All Solution:  $y = -1, y = 1, y = \sin(e^t + C)$

Conclusion: When we divide by some  $g(y)$  the functions arising from when  $g(y) = 0$  are valid *constant solutions* to the DE and must be included in our final list.

#### 4. Implicit versus Explicit Solutions:

It's entirely possible that when we solve a separable DE we are unable to solve for  $y$  at the end, or it may be very difficult.

**Example:** Consider  $y' = \frac{t}{e^y + 1}$ .

Notice there are no constant solutions here because to solve it we do not divide by something which can be zero.

We work as follows:

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{e^y + 1} \\ (e^y + 1)dy &= t dt \\ \int e^y + 1 dy &= \int t dt \\ e^y + y &= \frac{1}{2}t^2 + C\end{aligned}$$

In this case it's reasonable to stop here and say that we have *implicitly* defined the solutions. An *implicit solution* is a solution in which we have not actually achieved  $y =$ .

Ideally of course we would be able to solve for  $y$ , this would yield an *explicit solution*.

#### 5. Autonomous ODEs:

There is a special kind of separable ODE called *autonomous*. This occurs when  $f(t) = 1$  and so instead we have  $y' = g(y)$ . This can be solved like any other separable ODE. We only mention it because these will arise repeatedly over the course in various places.

**Example:** Consider  $y' = (y - 4)^2$ .

Here  $g(y) = (y - 4)^2$  which equals 0 when  $y = 4$  so this is the constant solution. The nonconstant solutions we obtain as follows:

$$\begin{aligned}\frac{dy}{dt} &= (y - 4)^2 \\ (y - 4)^{-2} dy &= 1 dt \\ \int (y - 4)^{-2} dy &= \int 1 dt \\ -(y - 4)^{-1} &= t + C \\ (y - 4)^{-1} &= -(t + C) \\ (y - 4) &= \frac{-1}{t + C} \\ y &= \frac{-1}{t + C} + 4\end{aligned}$$

## 6. Two Small Initial Value Notes:

### (a) Choosing Solutions:

When we solve a separable ODE and get an implicit solution for which there seems to be more than one explicit solution, an initial value usually tells us which one it is:

**Example:** Consider  $y' = \frac{t}{y}$  with  $y(1) = -3$ .  
First we solve the DE:

$$\begin{aligned}\frac{dy}{dt} &= \frac{t}{y} \\ y \, dy &= t \, dt \\ \int y \, dy &= \int t \, dt \\ \frac{1}{2}y^2 &= \frac{1}{2}t^2 + C \\ y^2 &= t^2 + 2C \\ y &= \pm\sqrt{t^2 + 2C}\end{aligned}$$

We see that there are two explicit solutions to the DE.

When we consider the initial value we have  $y(1) = \pm\sqrt{1^2 + 2C} = -3$  so we are forced to use the negative in front of the square root. Thus  $-\sqrt{1^2 + 2C} = -3$  so  $1 + 2C = 9$  and  $C = 4$ . Then the explicit solution is  $y = -\sqrt{t^2 + 8}$ .  
Note: There are no constant solutions here since  $g(y) = \frac{1}{y}$  is never 0.

### (b) Uniqueness (?) of Solutions:

The existence of constant solutions often leads to non-unique solutions to IVPs. This can happen when a constant solution satisfies the DE but also the procedural method gives another solution. The way to manage this is to not forget to find your constant solutions and check if they satisfy the IV.

**Example:** Consider  $y' = y^{2/3}$  with  $y(0) = 0$ . Notice that  $y = 0$  is a constant solution which also satisfies the DE. However the DE is separable:

$$\begin{aligned}\frac{dy}{dt} &= y^{2/3} \\ y^{-2/3} \, dy &= 1 \, dt \\ \int y^{-2/3} \, dy &= \int 1 \, dt \\ 3y^{1/3} &= t + C \\ y &= \left(\frac{1}{3}t + \frac{1}{3}C\right)^3\end{aligned}$$

Then  $y(0) = \left(\frac{1}{3}C\right)^3 = 0$  so  $C = 0$ . This gives the additional solution  $y = \left(\frac{1}{3}t\right)^3 = \frac{1}{27}t^3$ .

## 7. Overlap

At this juncture it might be helpful to notice that an ODE doesn't need to be just one of the categories we've looked at - explicit, first-order linear, and separable - it could fall into more than one category.

**Example:**  $y' = ty$  is both separable and first-order linear.

**Example:**  $y' = t^2$  is all of explicit, separable and first-order linear.

8. Justification for the strange approach.

It may bother you that we can rip apart the DE like we do and simply stick an integral sign on both sides. Really this is just shorthand notation for a more rigorous approach. More rigorously once our DE is rewritten as:

$$G(y(t))y'(t) = f(t)$$

Since they are equal we can integrate both sides with respect to  $t$ :

$$\int G(y(t))y'(t) dt = \int f(t) dt$$

On the left we make the substitution  $u = y(t)$  which yields  $\frac{du}{dt} = y'(t)$  which we then substitute in:

$$\int G(u) \frac{du}{dt} dt = \int f(t) dt$$

This then simplifies to:

$$\int G(u) du = \int f(t) dt$$



## MATH 246: Chapter 1 Section 5: Graphical Methods

Justin Wyss-Gallifent

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Main Topics:

- Quantitative versus Qualitative Solutions.
  - Phase-Line Portraits and Sketching Autonomous Systems.
  - Contour Plots for Implicit Solutions.
  - Slope Fields.
- 

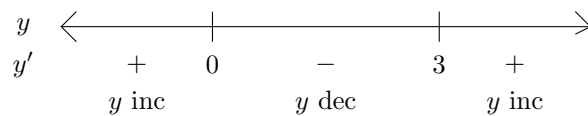
### 1. Introduction:

The overarching goal of this section is to find things out about solutions to DEs without actually solving them explicitly. Instead we attack them graphically.

### 2. Phase Line Portraits for Autonomous Differential Equations.

#### (a) Introductory Example.

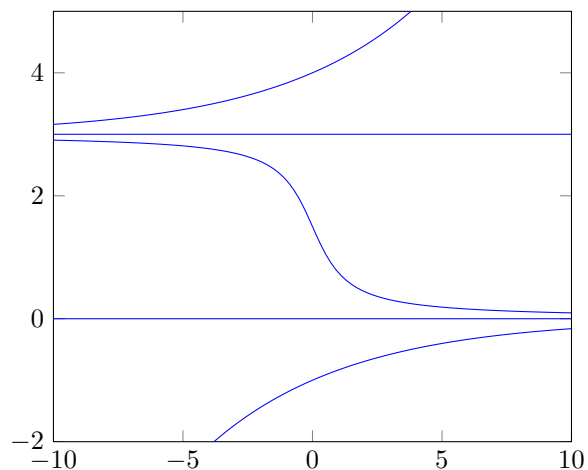
Consider the autonomous DE  $y' = y(y-3)$ . This has constant solutions  $y = 0$  and  $y = 3$ . But what about other solutions? Well the DE tells us that if we know  $y$  then we know  $y'$ . A sign chart giving information about  $y'$  can give us a wealth of information:



Consider now that:

- A solution below  $y = 0$  will be increasing. Moreover for larger positive  $t$ -values it will get closer to  $y = 0$  where it will level off and for larger negative  $t$ -values it will get closer to  $\infty$ .
- A solution between  $y = 0$  and  $y = 3$  will be decreasing. Moreover for larger positive  $t$ -values it will get closer to  $y = 0$  where it will level off and for larger negative  $t$ -values it will get closer to  $y = 3$  where it will level off.
- A solution above  $y = 3$  will be increasing. Moreover for larger positive  $t$ -values it will get closer to  $\infty$  and for larger negative  $t$ -values it will get closer to  $y = 3$  where it will level off.

Basically all possible solutions will look like these:



(b) General Understanding:

An autonomous DE will generally have constant solutions. Between the constant solutions is where interesting things happen. To see what's going on between the constant solutions draw a number line which gives information about whether solutions are increasing or decreasing.

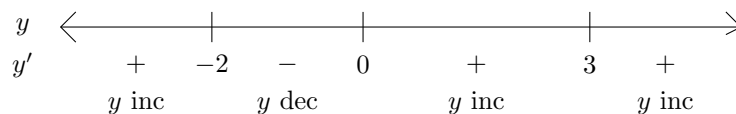
This number line is the *phase line portrait* of the autonomous differential equation.

Solutions approaching constant solutions will do so asymptotically. This is true both for larger positive  $t$ -values and larger negative  $t$ -values.

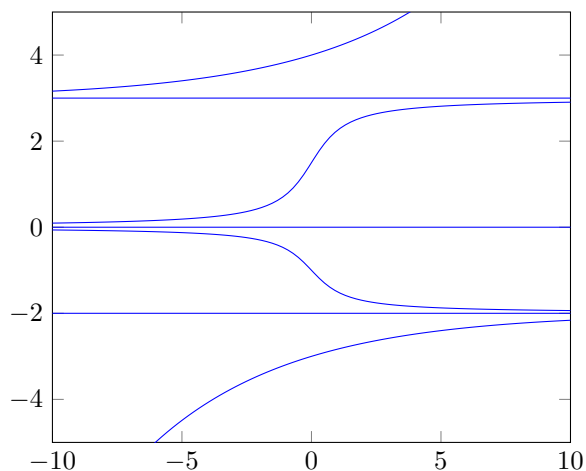
Moreover this gives some information about those constant solutions:

- If nearby solutions move away from the constant solution on both sides then the constant solution is unstable.
- If nearby solutions move toward the constant solution on both sides then the constant solution is stable.
- If there is different behavior on each side then the constant solution is *semistable*.

**Example:** Consider  $y' = y(y - 3)(y + 2)^2$ . This has the following sign chart:



Basically all possible solutions will look like these:



From these families of solutions we can draw all sorts of conclusions:

- The constant solution  $y = -2$  is stable, the constant solution  $y = 0$  is unstable, and the constant solution  $y = 3$  is semistable.
- The particular solution  $y(t)$  satisfying  $y(0) = \alpha$  has  $\lim_{t \rightarrow \infty} y(t) = 2$  when  $-\infty < \alpha < 0$ .
- The particular solution  $y(t)$  satisfying  $y(0) = \alpha$  is decreasing when  $-2 < \alpha < 0$ .
- $y = -2$  is stable.
- $y = 0$  is unstable.
- $y = 3$  is semistable.

### 3. Contour Plots of Implicit Solutions

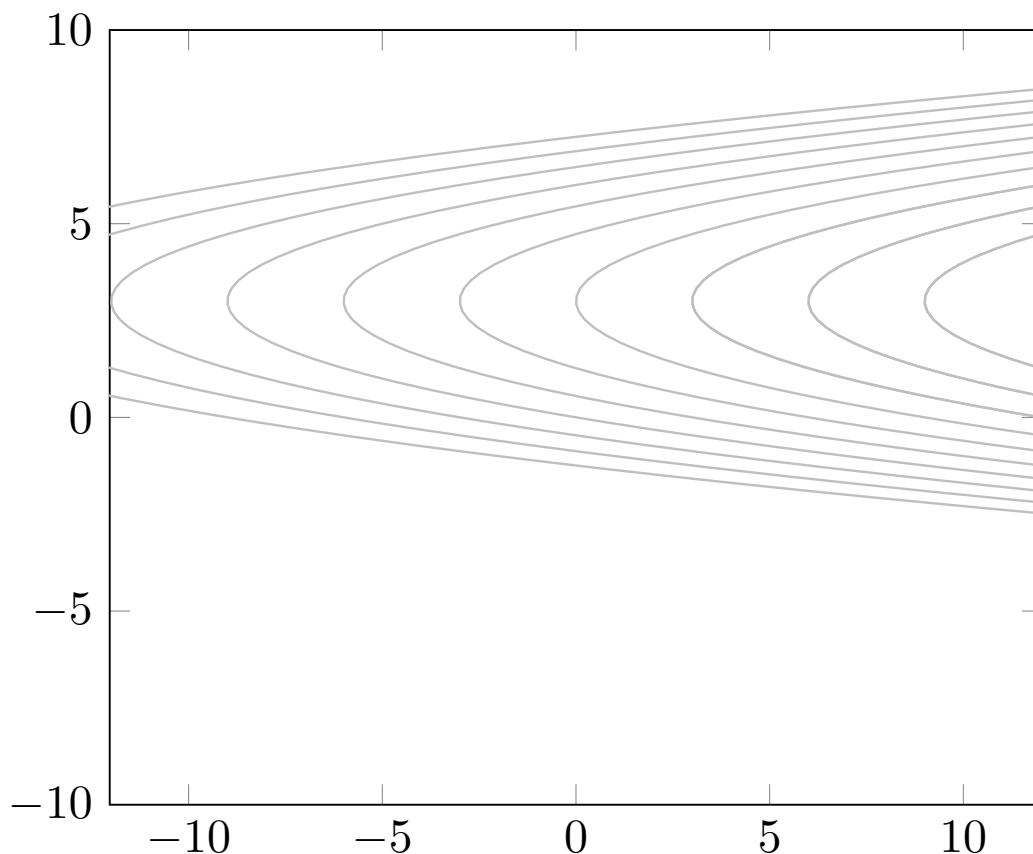
- (a) When we solve a separable DE we often get an implicit solution with a  $C$  in it. This implicit solution is an *equation*. If we pick various values of  $C$  and plot the resulting equations we get a *contour plot*.

What's useful about these contour plots is that the parts of the curves that form functions ( $y$  in terms of  $t$ ) that satisfy the implicit solution. This means that we can pick a point on a curve and follow it as far left and right as possible and the result is the graph of an explicit solution to the DE.

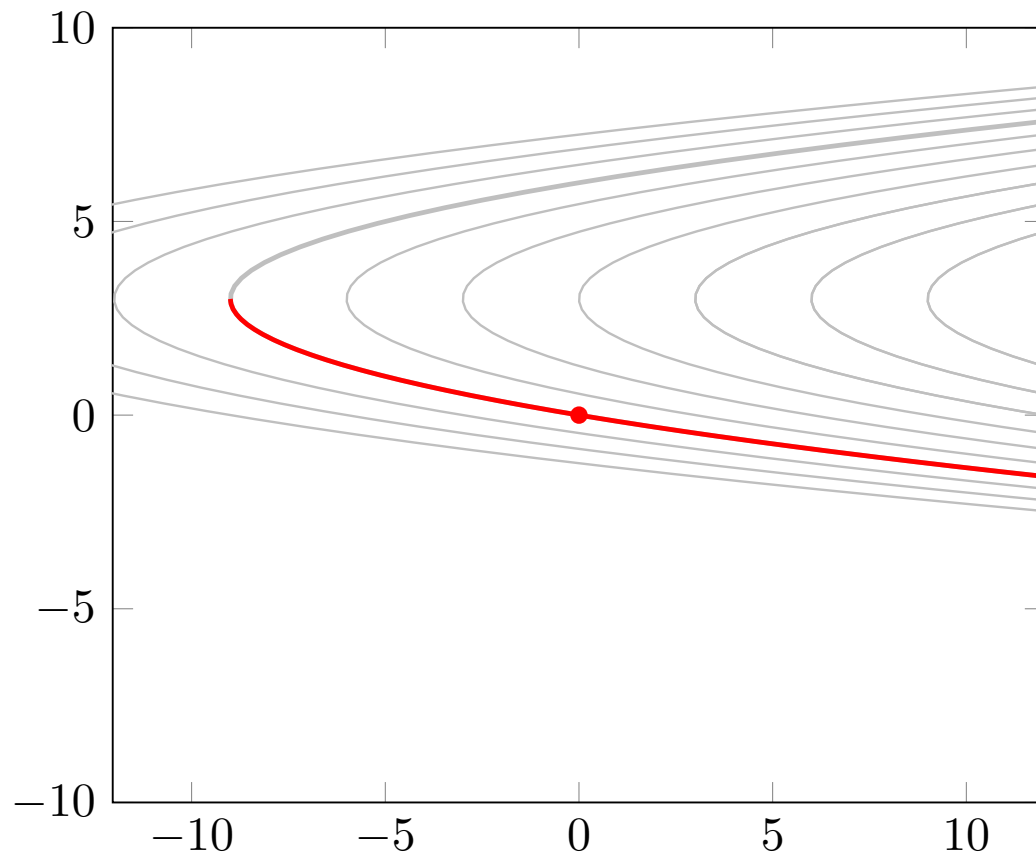
**Example:** Consider  $\frac{dy}{dx} = \frac{1}{2y-6}$ . This is separable with general solution  $y^2 - 6y = x + C$ . This is not as bad as it looks:

$$\begin{aligned}y^2 - 6y &= x + C \\y^2 - 6y + 9 &= x + C + 9 \\(y - 3)^2 &= x + C + 9\end{aligned}$$

These are all parabolas opening right with their vertices at  $y = 3$ . If we sketch a few of these (note that they extend out forever, this is just a subset):



Solutions to the DE are functions (must pass the vertical line test!) which lie along these parabolic curves. For example the specific solution satisfying  $y(0) = 0$  corresponds to  $C = 0$  and looks like this:

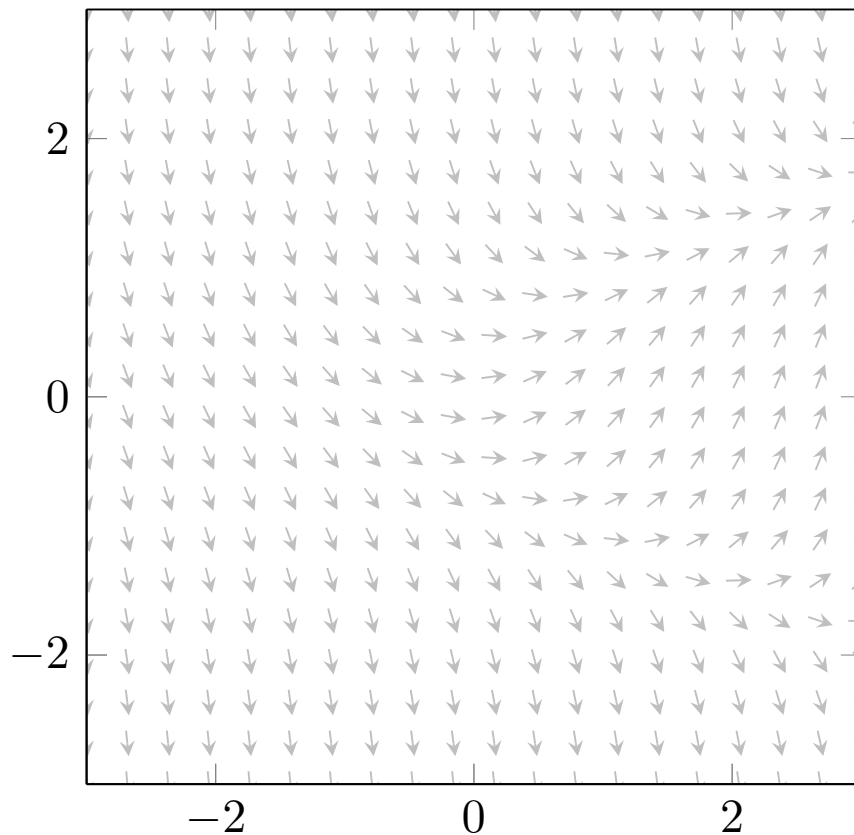


From this contour plot we can draw all sorts of conclusions:

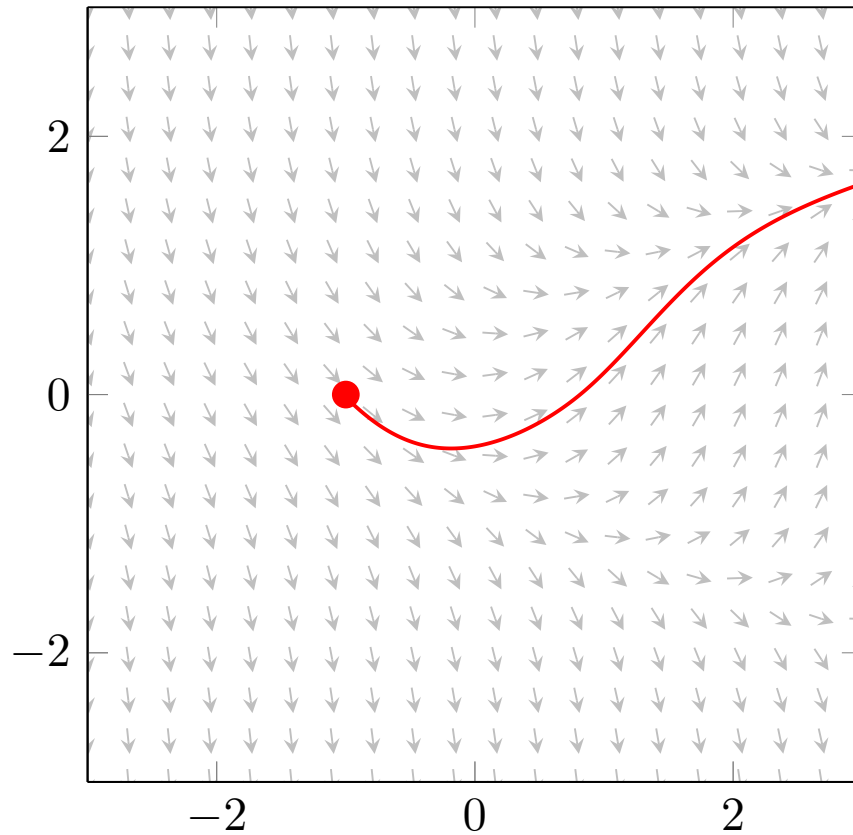
- Solutions extend infinitely far to the right but not the left.
- Solutions are either increasing or decreasing but not both.
- The specific solution  $y(t)$  with  $y(0) = 0$  is a decreasing function with  $\lim_{t \rightarrow \infty} y(t) = -\infty$ .
- The specific solution with  $y(0) = 5$  is an increasing function.
- Solutions are either always increasing or always decreasing.

#### 4. Direction (Slope) Fields

- (a) As a last-ditch effort any  $\frac{dy}{dt} = f(t, y)$  (any first-order) is essentially telling us the slope of a solution at a point. Consequently we can plug in lots of  $t$  and  $y$  and indicate what the slope would be of a solution passing through that point. The result is a *direction field* or *slope field*. Then we can trace functions which follow the field and draw conclusions.
- (b) **Example:** Here is the direction field for  $\frac{dy}{dt} = t - y^2$ , a hard DE to solve:



Solutions to the DE are functions (must pass the vertical line test!) which follow these arrows. For example the specific solution satisfying  $y(-1) = 0$  looks like this:



From this direction field we can draw all sorts of conclusions:

- The specific DE solution satisfying  $y(-1) = 0$  has a relative minimum approximately  $(0, -0.5)$ .
- We can observe categories, for example not all solutions have relative minima.

Note however that we're somewhat restricted by the range we drew!

## MATH 246: Chapter 1 Section 6: Applications

### Justin Wyss-Gallifent

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Main Topics:

- Population Dynamics.
  - Water Tanks.
  - Motion.
- 

#### 1. Population dynamics:

- (a) Introduction: In precalculus you probably learned that if a population grows at rate 5% then it obeys the formula

$$P = Ae^{0.05t}$$

But why? The answer is that to say “a population grows at rate of 5%” means that the instantaneous change in population at any time equals 5% of the actual population, meaning:

$$p' = 0.05p$$

This is a first order linear differential equation (it's also separable). If we rewrite it as  $p' - 0.05p = 0$  then  $a(t) = -0.05$  so  $A(t) = -0.05t$  and the solution is:

$$p(t) = e^{-(-0.05t)} \int 0 \, dt = Ce^{0.05t}$$

That's why!

- (b) General Approach: Our general formula will involve a population with a certain growth rate  $R$  but in addition some new amount may arrive or depart every time period, maybe by being introduced, removed, etc. So in general we have

$$p' = Rp + a(t)$$

Our rate will always be constant but the amount that are introduced or subtracted may vary.

- (c) Examples:

**Example:** A population of monkeys starts with 100. It has a growth rate of 4% per year but an additional 8 monkeys join each year from a neighboring troop. Find the number of monkeys after  $t$  years.

**Solution:** Here we have  $p' = 0.04p + 8$  with  $p(0) = 100$ .

The solution (work omitted) is  $p = 300e^{0.04t} - 200$ .

**Example:** In a certain neighborhood there is a mosquito problem. The population starts at 10M and has a growth rate of 20% monthly. Traps are put out and these traps kill 3M monthly. Find the number of mosquitos after  $t$  months and determine when the mosquitos will be wiped out.

**Solution:** Here we have  $p' = 0.2p - 3$  with  $p(0) = 10$ .

The solution (work omitted) is  $p = -5e^{0.2t} + 15$  and if we solve  $-5e^{0.2t} + 15 = 0$  we get  $t = 5 \ln(3) \approx 5.49$  months.

## 2. Tanks.

- (a) Introduction: We have a tank that contains a saltwater mixture. As time goes by, saltwater is being pumped in and out. Our goal is to know how much salt there is at any time  $t$ .
- (b) General Approach: If  $Q$  is the amount of salt at time  $t$  then we'll have

$$Q' = \text{Rate In} - \text{Rate Out}$$

The only confusing thing about these problems is we usually have to do some work with quantities to figure out the rates.

- (c) Examples:

**Example:** A tank initially contains 500L of saltwater with a concentration of 0.2kg/L. Saltwater with a concentration of 0.3kg/L is being pumped in at 10L/min while the tank is being emptied of the mixture at the same rate. Find the amount of salt in the tank at time  $t$ .

**Solution:** We're interested in the quantity of salt, not saltwater.

- Initially there is  $(500 \text{ L})(0.2 \text{ kg/L})=100 \text{ kg}$  of salt so  $Q(0) = 100$ .
- Salt is entering at  $(10 \text{ L/min})(0.3 \text{ kg/L})=3 \text{ kg/min}$ .
- Salt is leaving at  $(10 \text{ L/min})(Q \text{ kg} / 500 \text{ L})=0.02Q \text{ kg/min}$ .

Note this is because at any instant there is  $y$  kg of salt in the tank and the tank always has 500 L of mixture in it because the rate in equals the rate out.

Therefore we have  $Q' = 3 - 0.02Q$  with  $Q(0) = 100$ .

This is first-order linear rewritten as  $Q' + 0.02Q = 3$  so we set  $a(t) = 0.02$  and so  $A(t) = 0.02t$  and then the solution is:

$$\begin{aligned} Q &= e^{-0.02t} \int 3e^{0.02t} dt \\ &= e^{-0.02t} [150e^{0.02t} + C] \\ &= 150 + Ce^{-0.02t} \end{aligned}$$

Then the initial value  $Q(0) = 100$  gives us  $C = -50$  and so the solution is  $Q = 150 - 50e^{-0.2t}$ .



**Example:** A 300 gal tank initially contains 200 gal of saltwater with a concentration of 0.15 lb/gal. Saltwater with a concentration of 0.2 lb/gal is being pumped in at 6 gal/min while the tank is being emptied of the mixture at 4 gal/min. How much salt will be in the tank when it is full?

**Solution:** Observe that the tank does not start out full but gains 2 gal/min, meaning after time  $t$  it will have  $200 + 2t$  gal in it. This will be important in the DE!

Again note:

- Initially there is  $(200 \text{ gal})(0.15 \text{ lb/gal})=30$  gal of salt so  $Q(0) = 30$ .
- Salt is entering at  $(6 \text{ gal/min})(0.2 \text{ lb/gal})=1.2 \text{ lb/min}$ .
- Salt is leaving at  $(4 \text{ gal/min})(Q \text{ lb} / 200 + 2t \text{ gal}) = \frac{4Q}{200+2t} \text{ lb/min}$ .  
Note this is because at any instant there is  $y$  lb of salt in the tank and the tank has  $200 + 2t$  gal in it.

Therefore we have  $Q' = 1.2 - \frac{4Q}{200+2t}$  with  $Q(0) = 30$ .

This is first-order linear rewritten as:

$$Q' + \left[ \frac{2}{100+t} \right] Q = 1.2$$

so we set  $a(t) = \frac{2}{100+t}$  and so  $A(t) = 2\ln(t+100)$  and then the solution is:

$$\begin{aligned} Q &= e^{-2\ln(t+100)} \int 1.2e^{2\ln(t+100)} dt \\ &= e^{\ln(t+100)^{-2}} \int 1.2e^{\ln(t+100)^2} dt \\ &= (t+100)^{-2} \int 1.2(t+100)^2 dt \\ &= (t+100)^{-2} [0.4(t+100)^3 + C] \\ &= 0.4(t+100) + \frac{C}{(t+100)^2} \end{aligned}$$

Using the initial value  $Q(0) = 30$ :

$$\begin{aligned} 30 &= 0.4(100) + \frac{C}{100^2} \\ 30 &= 40 + \frac{C}{10000} \\ -10 &= \frac{C}{10000} \\ C &= -100000 \end{aligned}$$

and so the solution is:

$$Q = 0.4(t+100) - \frac{100000}{(t+100)^2}$$

The tank is full when  $200 + 2t = 300$  so  $t = 50$ . The amount of salt is then

$$Q(50) = 0.4(50+100) - \frac{100000}{(50+100)^2}$$

### 3. Motion:

- (a) Introduction: In calculus you probably learned that a falling object with no air resistance has

$$a(t) = -9.8$$

But why? The answer comes from equating two forces. if the object has acceleration  $a(t)$  and mass  $m$  then the force on it is  $ma(t)$ . The force from gravity is  $-9.8m$ . When we equate these we get

$$ma(t) = -9.8m$$

and then we cancel the  $m$ .

- (b) Adding some air: When we add air resistance there are now two forces. First there is gravity pulling down and then drag (air resistance for example) pushing up. These two forces combine to form the total force. We know the total force is  $ma(t) = mv'(t)$ . Therefore

$$ma(t) = \text{force of gravity} + \text{drag force}$$

The force of gravity is  $-9.8m$ . The drag force is harder, it's  $mkv^2$  where  $k$  is the drag coefficient. Thus we have

$$ma(t) = -9.8m + mkv^2$$

or, cancelling the  $m$  again:

$$a(t) = -9.8 + kv^2$$

Finally we replace  $a(t)$  by  $v'(t)$  to get:

$$\frac{dv}{dt} = -9.8 + kv^2$$

- (c) General Approach: We'll generally just use the IVP

$$\frac{dv}{dt} = -9.8 + kv^2 \text{ with } v(0) = 0 \text{ (usually)}$$

to find  $v$  at time  $t$  and answer questions from that. Things to note:

- $v(0)$  might not be 0 if there is some initial velocity.
- In the Metric system  $k$  will be in  $m^{-1}$ , mass will be in kg and 9.8 stays as-is.
- In the English system  $k$  will be in  $ft^{-1}$ , mass will be in slugs and we use 32.2 (instead of 9.8).
- Terminal velocity occurs when  $v' = 0$  so this is when  $v = \sqrt{\frac{9.8}{k}}$ .
- If we know  $v$  then we can also find out distance travelled since  $v = h'$  and so  $h(t_E) - h(t_I) = \int_{t_I}^{t_E} v(t) dt$
- We can certainly change from air to some other substance or from earth gravity to some other standard. Information would have to be given.
- $\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \tanh^{-1} \frac{x}{a} + C$
- $\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}$
- $\tanh^{-1}(z) = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right)$
- $\int \tanh(z) = \ln(\cosh(z)) + C$
- $\cosh(z) = \frac{e^z + e^{-z}}{2}$

(d) Examples:

**Example:** A skydiver leaps out of a plane at 3000m. The drag coefficient is  $0.002m^{-1}$ . What is the IVP here? What is her terminal velocity? Find her velocity at time  $t$ .

**Solution:** We have  $\frac{dv}{dt} = -9.8 + 0.002v^2$  with  $v(0) = 0$ .

Her terminal velocity is  $v = \sqrt{\frac{9.8}{0.002}} = \sqrt{4900} = 70$  m/s.

The solution to the IVP is shown here:

$$\begin{aligned}\frac{dv}{dt} &= -9.8 + 0.002v^2 \\ \frac{dv}{dt} &= 0.002(-4900 + v^2) \\ \frac{1}{v^2 - 4900} dv &= 0.002 dt \\ \int \frac{1}{v^2 - 4900} dv &= \int 0.002 dt \\ -\frac{1}{70} \tanh^{-1}\left(\frac{v}{70}\right) &= 0.002t + C \\ \tanh^{-1}\left(\frac{v}{70}\right) &= -0.14t + C \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ \frac{v}{70} &= \tanh(-0.14t + C) \\ v &= 70 \tanh(-0.14t + C)\end{aligned}$$

Then  $v(0) = 70 \tanh(C) = 0$  so  $C = \tanh^{-1} 0 = 0$  and our final answer is

$$v = -70 \tanh(0.14t)$$

At this point if we wish to know the height we can integrate:

$$h(t) = \int v(t) dt = \int -70 \tanh(0.14t) dt = -\frac{70}{0.14} \ln(\cosh(0.14t)) + C$$

Then we can use  $h(0) = 3000$  to find  $C$ .

**MATH 246: Chapter 1 Section 7: Approximation Methods**  
**Justin Wyss-Gallifent**

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Main Topics:

- Euler's Method (The Left-Sum Method).
  - The Runge-Trapezoid Method.
  - The Runge-Midpoint Method.
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1. Euler's Method

(a) Introduction Suppose we're dealing with the IVP given by:

$$\frac{dy}{dt} = t + y \text{ with } y(1) = 2$$

Suppose we'd really like to know  $y(2)$ .

The DE tells us that at the point  $(1, 2)$  the slope of the solution is  $\frac{dy}{dt}(1, 2) = 3$ . Of course the solution is not a straight line, meaning if we move right 1 we won't go up exactly 3, but if things aren't too bad then we would go up approximately 3. Thus we can conclude that  $y(1 + 1) \approx 2 + 3$  or  $y(2) \approx 5$ .

This approximately probably stinks, so what we can do instead is go to the right just 0.5 and up  $0.5(3)$ , then do the process again, now anchored at the new point. That is:

At  $(1, 2)$  the slope is  $\frac{dy}{dt}(1, 2) = 3$  so we go over 0.5 and up  $0.5(3)$  and now we're at  $(1 + 0.5, 2 + 0.5(3)) = (1.5, 3.5)$ .

At  $(1.5, 3.5)$  the slope is  $\frac{dy}{dt}(1.5, 3.5) = 5$  so we go over 0.5 and up  $0.5(5)$  and now we're at  $(1.5 + 0.5, 3.5 + 0.5(5)) = (2, 6)$

Then we conclude  $y(2) \approx 6$ . This approximation is probably better.

(b) Euler's Method.

This process is known as Euler's Method. We start with an IVP given by  $\frac{dy}{dt} = f(t, y)$  with  $y(t_0) = y_0$ . and we choose a small  $h$ . We did  $h = 1$  and then  $h = 0.5$ . We then proceed as follows:

$$\begin{aligned}(t_1, y_1) &= (t_0 + h, y_0 + hf(t_0, y_0)) \\ (t_2, y_2) &= (t_1 + h, y_1 + hf(t_1, y_1))\end{aligned}$$

Or, more generally:

Euler's Method

$$\begin{aligned}t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + hf(t_{i-1}, y_{i-1})\end{aligned}$$

**Example:** Again with  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . Let's approximate  $y(2)$  using  $n = 10$  steps of size  $h = 0.1$ .

This can all be put more nicely into a table as follows:

0	1	2	y(1)=2
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$1 + 0.1 = 1.1$	$2 + 0.3 = 2.3$	$y(1.1) \approx 2.3$
2	$1.1 + 0.1 = 1.2$	$2.3 + 0.34 = 2.64$	$y(1.2) \approx 2.64$
3	$1.2 + 0.1 = 1.3$	$2.64 + 0.384 = 3.024$	$y(1.3) \approx 3.024$
4	$1.3 + 0.1 = 1.4$	$3.024 + 0.4324 = 3.4564$	$y(1.4) \approx 3.4564$
5	$1.4 + 0.1 = 1.5$	$3.4564 + 0.48564 = 3.94204$	$y(1.5) \approx 3.94204$
6	$1.5 + 0.1 = 1.6$	$3.94204 + 0.544204 = 4.48624$	$y(1.6) \approx 4.48624$
7	$1.6 + 0.1 = 1.7$	$4.48624 + 0.608624 = 5.09487$	$y(1.7) \approx 5.09487$
8	$1.7 + 0.1 = 1.8$	$5.09487 + 0.679487 = 5.77436$	$y(1.8) \approx 5.77436$
9	$1.8 + 0.1 = 1.9$	$5.77436 + 0.757436 = 6.53179$	$y(1.9) \approx 6.53179$
10	$1.9 + 0.1 = 2$	$6.53179 + 0.843179 = 7.37497$	$y(2) \approx 7.37497$

Of course the further we go the less accurate we get but if the DE is not so bad then maybe we're good. The solution to the above DE (first-order linear) is  $y(t) = 4e^{t-1} - t - 1$  and so  $y(2) = 4e - 2 - 1 \approx 7.8731273138361809414411498854106$  so our approximation is not terrible.

**Example:** Same IVP but we could do better by reducing  $h$  and increasing the number of steps. Just for fun, compare to 1000 steps of size  $h = 0.001$  each and see how close the approximation is at the end!

Note: This was generated in Python and some approximation and truncation is taking place.

0	1	2	y(1)=2
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$1 + 0.001 = 1.001$	$2 + 0.003 = 2.003$	$y(1.001) \approx 2.003$
2	$1.001 + 0.001 = 1.002$	$2.003 + 0.003004 = 2.006$	$y(1.002) \approx 2.006$
3	$1.002 + 0.001 = 1.003$	$2.006 + 0.003008 = 2.00901$	$y(1.003) \approx 2.00901$
...	...	...	...
998	$1.997 + 0.001 = 1.998$	$7.83816 + 0.00983516 = 7.84799$	$y(1.998) \approx 7.84799$
999	$1.998 + 0.001 = 1.999$	$7.84799 + 0.00984599 = 7.85784$	$y(1.999) \approx 7.85784$
1000	$1.999 + 0.001 = 2$	$7.85784 + 0.00985684 = 7.8677$	$y(2) \approx 7.8677$

## 2. Improving:

First off recall that for a continuous function  $y(t)$  the Fundamental Theorem of Calculus tells us that:

$$\int_a^b \left[ \frac{dy}{dt} \right] dt = y(b) - y(a)$$

With our differential equation given that we're looking for some  $y(t)$  satisfying  $\frac{dy}{dt} = f(t, y(t))$  this translates to:

$$\int_a^b f(t, y(t)) dt = y(b) - y(a)$$

Given that we started this whole process knowing  $y_0$  and wanting  $y_1$  we can write:

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t)) dt$$

which can then be rewritten as our *Basic Formula*:

$$y_1 = y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt$$

So the real question is how to tackle the integral.

Let's revisit integrals. Suppose you wanted to know  $\int_a^b g(x) dx$  but couldn't do it. One really bad approximation is just a left rectangle. That is

$$\int_a^b g(x) dx \approx (b - a)g(a)$$

Using this in the *Basic Formula* yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + (t_1 - t_0)f(t_0, y(t_0)) \\ y_1 &\approx y_0 + (t_1 - t_0)f(t_0, y_0) \\ y_1 &\approx y_0 + hf(t_0, y_0) \end{aligned}$$

Well then, we've just got Euler's Method!

What this suggests is that better methods of approximating the integral yield better approximations for our IVP.

### 3. The Runge-Trapezoid Method:

A second way to approximate the integral would be to construct a trapezoid using the endpoints:

$$\int_a^b g(x) dx \approx \frac{1}{2}(b-a)(g(a) + g(b))$$

Using this in the *Basic Formula* yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + \frac{1}{2}(t_1 - t_0)(f(t_0, y(t_0)) + f(t_1, y(t_1))) \\ y_1 &\approx y_0 + \frac{1}{2}h(f(t_0, y_0) + f(t_0 + h, y(t_1))) \end{aligned}$$

Which is all fun and games until we notice the right side has an  $y(t_1)$  in it, and this is what we want. How can we resolve this? We do something slick and we plug in the result of Euler's Method into this:

$$y_1 \approx y_0 + \frac{1}{2}h(f(t_0, y_0) + f(t_0 + h, \underbrace{y_0 + hf(t_0, y_0))}_{\text{Euler}}))$$

Haha what fun. What we're really doing is using one approximation of  $y(t_1)$  to get what we think will be a better one.

Runge-Trapezoidal Method

$$\begin{aligned} t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + \frac{1}{2}h \left( f(t_{i-1}, y_{i-1}) + f(t_{i-1} + h, y_{i-1} + hf(t_{i-1}, y_{i-1})) \right) \end{aligned}$$

Back to our first IVP  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . If  $h = 0.1$  then proceeding one step gives us  $t_1 = 1.1$  and:

$$\begin{aligned} y_1 &\approx y_0 + \frac{1}{2}h (f(t_0, y_0) + f(t_0 + h, y_0 + hf(t_0, y_0))) \\ &\approx 2 + \frac{1}{2}(0.1) (f(1, 2) + f(1 + 0.1, 2 + 0.1f(1, 2))) \\ &\approx 2 + \frac{1}{2}(0.1) (1 + 2 + f(1.1, 2 + 0.1(1 + 2))) \\ &\approx 2 + \frac{1}{2}(0.1) (1 + 2 + 1.1 + 2 + 0.1(1 + 2))) = 2.32 \end{aligned}$$

Here's the Runge-Trapezoidal Method applied to our first IVP with 10 steps of size 0.1:

0	1	2	y(1)=2
$i$	$t_i$	$y_i$	So
1	$1 + 0.1 = 1.1$	2.32	$y(1.1) \approx 2.32$
2	$1.1 + 0.1 = 1.2$	2.6841	$y(1.2) \approx 2.6841$
3	$1.2 + 0.1 = 1.3$	3.09693	$y(1.3) \approx 3.09693$
4	$1.3 + 0.1 = 1.4$	3.56361	$y(1.4) \approx 3.56361$
5	$1.4 + 0.1 = 1.5$	4.08979	$y(1.5) \approx 4.08979$
6	$1.5 + 0.1 = 1.6$	4.68171	$y(1.6) \approx 4.68171$
7	$1.6 + 0.1 = 1.7$	5.34629	$y(1.7) \approx 5.34629$
8	$1.7 + 0.1 = 1.8$	6.09116	$y(1.8) \approx 6.09116$
9	$1.8 + 0.1 = 1.9$	6.92473	$y(1.9) \approx 6.92473$
10	$1.9 + 0.1 = 2$	7.85632	$y(2) \approx 7.85632$

Remember the exact value of  $y(2) = 4e - 2 - 1 \approx 7.8731273138361809414411498854106$ .

#### 4. The Runge-Midpoint Method:

A third way to approximate the integral is a midpoint rectangle:

$$\int_a^b g(x) dx \approx (b-a)g\left(\frac{a+b}{2}\right)$$

Using this in the *Basic Formula* and using the fact that our midpoint is  $t_0 + \frac{1}{2}h$  yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t)) dt \\ y_1 &\approx y_0 + (t_1 - t_0)f\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right) \\ y_1 &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right) \end{aligned}$$

Which again is all fun and games until we realize we don't know  $y(t_0 + \frac{1}{2}h)$  so we swap in Euler's Method again using a half-step, that is  $y_0 + \frac{1}{2}hf(t_0, y_0)$  and so

$$y_1 \approx y_0 + hf\left(t_0 + \frac{1}{2}h, \underbrace{y_0 + \frac{1}{2}hf(t_0, y_0)}_{\text{Euler}}\right)$$

#### Runge-Midpoint Method

$$\begin{aligned} t_i &= t_{i-1} + h \\ y_i &\approx y_{i-1} + hf\left(t_{i-1} + \frac{1}{2}h, y_{i-1} + \frac{1}{2}hf(t_{i-1}, y_{i-1})\right) \end{aligned}$$

Back to our first IVP  $\frac{dy}{dt} = t + y$  with  $y(1) = 2$ . If  $h = 0.1$  then proceeding one step gives us  $t_1 = 1.1$  and:

$$\begin{aligned} y_i &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(t_0, y_0)\right) \\ &\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)f(1, 2)\right) \\ &\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)(1 + 2)\right) \\ &\approx 2 + 0.1\left(1 + \frac{1}{2}(0.1) + 2 + \frac{1}{2}(0.1)(1 + 2)\right) = 2.32 \end{aligned}$$

This is actually the same as the Runge-Trapezoidal Method and in fact for this particular IVP the Runge-Midpoint Method applied to our first IVP actually gives the same result as the Runge-Trapezoidal Method, so we omit the full table.



5. Everything together:

Let  $y(t)$  be the solution to  $\frac{dy}{dt} = ty + t$  with  $y(0) = 1$ . Approximate  $y(1)$  using  $n = 10$  steps of size  $h = 0.1$ :

Euler			
0	0	1	y(0)=1
$i$	$t_i$	$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$	So
1	$0 + 0.1 = 0.1$	$1 + 0 = 1$	$y(0.1) \approx 1$
2	$0.1 + 0.1 = 0.2$	$1 + 0.02 = 1.02$	$y(0.2) \approx 1.02$
3	$0.2 + 0.1 = 0.3$	$1.02 + 0.0404 = 1.0604$	$y(0.3) \approx 1.0604$
4	$0.3 + 0.1 = 0.4$	$1.0604 + 0.061812 = 1.12221$	$y(0.4) \approx 1.12221$
5	$0.4 + 0.1 = 0.5$	$1.12221 + 0.0848885 = 1.2071$	$y(0.5) \approx 1.2071$
6	$0.5 + 0.1 = 0.6$	$1.2071 + 0.110355 = 1.31746$	$y(0.6) \approx 1.31746$
7	$0.6 + 0.1 = 0.7$	$1.31746 + 0.139047 = 1.4565$	$y(0.7) \approx 1.4565$
8	$0.7 + 0.1 = 0.8$	$1.4565 + 0.171955 = 1.62846$	$y(0.8) \approx 1.62846$
9	$0.8 + 0.1 = 0.9$	$1.62846 + 0.210277 = 1.83873$	$y(0.9) \approx 1.83873$
10	$0.9 + 0.1 = 1$	$1.83873 + 0.255486 = 2.09422$	$y(1) \approx 2.09422$

Runge-Trapezoidal			
0	0	1	y(0)=1
$i$	$t_i$	$y_i$	So
1	$0 + 0.1 = 0.1$	1.01	$y(0.1) \approx 1.01$
2	$0.1 + 0.1 = 0.2$	1.04035	$y(0.2) \approx 1.04035$
3	$0.2 + 0.1 = 0.3$	1.09197	$y(0.3) \approx 1.09197$
4	$0.3 + 0.1 = 0.4$	1.16645	$y(0.4) \approx 1.16645$
5	$0.4 + 0.1 = 0.5$	1.2661	$y(0.5) \approx 1.2661$
6	$0.5 + 0.1 = 0.6$	1.39414	$y(0.6) \approx 1.39414$
7	$0.6 + 0.1 = 0.7$	1.55478	$y(0.7) \approx 1.55478$
8	$0.7 + 0.1 = 0.8$	1.75355	$y(0.8) \approx 1.75355$
9	$0.8 + 0.1 = 0.9$	1.99751	$y(0.9) \approx 1.99751$
10	$0.9 + 0.1 = 1$	2.29576	$y(1) \approx 2.29576$

Runge-Midpoint			
0	0	1	y(0)=1
$i$	$t_i$	$y_i$	So
1	$0 + 0.1 = 0.1$	1.01	$y(0.1) \approx 1.01$
2	$0.1 + 0.1 = 0.2$	1.0403	$y(0.2) \approx 1.0403$
3	$0.2 + 0.1 = 0.3$	1.09182	$y(0.3) \approx 1.09182$
4	$0.3 + 0.1 = 0.4$	1.16613	$y(0.4) \approx 1.16613$
5	$0.4 + 0.1 = 0.5$	1.26556	$y(0.5) \approx 1.26556$
6	$0.5 + 0.1 = 0.6$	1.39328	$y(0.6) \approx 1.39328$
7	$0.6 + 0.1 = 0.7$	1.55351	$y(0.7) \approx 1.55351$
8	$0.7 + 0.1 = 0.8$	1.75172	$y(0.8) \approx 1.75172$
9	$0.8 + 0.1 = 0.9$	1.99497	$y(0.9) \approx 1.99497$
10	$0.9 + 0.1 = 1$	2.2923	$y(1) \approx 2.2923$

For reference the actual answer is  $2e^{0.5} - 1 \approx 2.2974425414002562936973015756283$ .

**MATH 246: Chapter 1 Section 8: Exact Differential Equations**  
**Justin Wyss-Gallifent**

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Main Topics:

- Exact Differential Equations.
  - Method of Solution.
  - Integrating Factors.
- 

1. **A Bit of History and Introduction:** Suppose  $H(x, y)$  is a function and  $y$  is a function of  $x$ . Then by the chain rule we know  $\frac{d}{dx}H(x, y) = H_x(x, y) + H_y(x, y)\frac{dy}{dx}$ .

So now consider the following differential equation:

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

You may notice that the left side looks like the result of the chain rule and is actually so, when  $H(x, y) = x^3y^2$ . Don't worry about if there's a formal method for where  $H(x, y)$  comes from for now, just notice that  $H_x(x, y) = 3x^2y^2$  and  $H_y(x, y) = 2x^3y$ . What this means is that the differential equation may be rewritten by undoing the chain rule on the left:

$$\underbrace{3x^2y^2 + 2x^3y\frac{dy}{dx}}_{\frac{d}{dx}[x^3y^2]} = 0$$

So then when the derivative of something is zero, that thing is a constant:

$$\begin{aligned}\frac{d}{dx}[x^3y^2] &= 0 \\ x^3y^2 &= C\end{aligned}$$

and we've solved it, at least implicitly!

2. **Definition and Method:** A differential equation is *exact* if it has the form:

$$H_x(x, y) + H_y(x, y)\frac{dy}{dx} = 0$$

for some function  $H(x, y)$ . When a differential equation is exact, solving implicitly is as easy as finding  $H(x, y)$  and setting  $H(x, y) = C$  for any constant.

Here are a few exact differential equations. For each,  $H(x, y)$  is written in the middle and the implicit solution to the right.

Exact DE	$H(x, y)$	Solution to DE
$y + x\frac{dy}{dx} = 0$	$H(x, y) = xy$	$xy = C$
$y + (x + 2y)\frac{dy}{dx} = 0$	$H(x, y) = xy + y^2$	$xy + y^2 = C$
$\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0$	$H(x, y) = \frac{x}{y}$	$\frac{x}{y} = C$
$y \cos(xy) + x \cos(xy)\frac{dy}{dx} = 0$	$H(x, y) = \sin(xy)$	$\sin(xy) = C$

3. **Detecting Exactness and Finding H:** There is a trick to detecting whether a differential equation is exact. If the differential equation has the form:

$$M + N \frac{dy}{dx} = 0$$

then it is exact if and only if  $M_y = N_x$ . You can test all the ones above. Then you can check that this next one is not exact:

$$xy + y \frac{dy}{dx} = 0$$

In this case  $M_y = x$  and  $N_x = 0$ . Not equal, not exact.

Once you know that your differential equation is exact, often you can guess at  $H(x, y)$ . However if you're struggling, there's a systematic method for finding it. Here's an example from above:

$$y + (x + 2y) \frac{dy}{dx} = 0$$

We want  $H(x, y)$  with (A)  $H_x(x, y) = y$  and (B)  $H_y(x, y) = x + 2y$ . Observe:

We want (A):	$H_x(x, y) = y$
This tells us that:	$H(x, y) = xy + h(y)$
From this line:	$H_y(x, y) = x + h'(y)$
But from (B):	$H_y(x, y) = x + 2y$
Set these equal:	$x + h'(y) = x + 2y$
Solve for $h'(y)$ :	$h'(y) = 2y$
Find $h(y)$ :	$h(y) = y^2 + D$
Put back into second line:	$H(x, y) = xy + y^2 + D$

We can choose any  $D$  so choose  $D = 0$  to get  $H(x, y) = xy + y^2$ .

**Example:** Find  $H(x, y)$  to solve  $x + 1 + \frac{1}{y} - \frac{x}{y^2} \frac{dy}{dx} = 0$ . Follow the exact procedure above, here we want (A)  $H_x(x, y) = x + 1 + \frac{1}{y}$  and (B)  $H_y(x, y) = -\frac{x}{y^2}$ :

We want (A):	$H_x(x, y) = x + 1 + \frac{1}{y}$
This tells us that:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + h(y)$
From this line:	$H_y(x, y) = -\frac{x}{y^2} + h'(y)$
But from (B):	$H_y(x, y) = -\frac{x}{y^2}$
Set these equal:	$-\frac{x}{y^2} + h'(y) = -\frac{x}{y^2}$
Solve for $h'(y)$ :	$h'(y) = 0$
Find $h(y)$ :	$h(y) = D$
Put back into second line:	$H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y} + D$

Then choose  $D = 0$  to get  $H(x, y) = \frac{1}{2}x^2 + x + \frac{x}{y}$  and the solution to our DE is  $\frac{1}{2}x^2 + x + \frac{x}{y} = C$ .

4. **Integrating Factors:** It's not uncommon to have a differential equation which is not quite exact but can be made exact by multiplying through by some function called an *integrating factor*. For example the differential equation

$$2y + x \frac{dy}{dx} = 0$$

is not exact because  $M_y = 2$  and  $N_x = 1$  so  $M_y \neq N_x$ . But if we multiply through by  $x$  we get the new differential equation

$$2xy + x^2 \frac{dy}{dx} = 0$$

which is exact because  $M_y = 2x$  and  $N_x = 2x$ . Now  $H(x, y) = x^2y$  and the solution is  $x^2y = C$ .

The question is how to come up with this integrating factor. This is very challenging so we'll look at two specific cases, either the integrating factor is a function  $f(x)$  of only  $x$  or the integrating factor is a function  $g(y)$  of only  $y$ .

## 5. Examples:

**Example 1:** Consider the differential equation we've seen before:

$$2y + x \frac{dy}{dx} = 0$$

Here  $M = 2y$  and  $N = x$ , these are different so it's not exact. Let's look for some  $f(x)$  so that when we multiply through the result is exact:

$$2yf(x) + xf(x) \frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned} [xf(x)]_x &= [2yf(x)]_y \\ 1f(x) + xf'(x) &= 2f(x) + 2y(0) \\ xf'(x) &= f(x) \\ f'(x) &= \frac{f(x)}{x} \end{aligned}$$

We can see that  $f(x) = x$  does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$2xy + x^2 \frac{dy}{dx} = 0$$

which has  $H(x, y) = x^2y$  and hence solution  $x^2y = C$ .

**Addendum:** If we tried  $g(y)$  we'd want this to be exact:

$$2yg(y) + xg(y) \frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned} [xg(y)]_x &= [2yg(y)]_y \\ 1g(y) + x(0) &= 2g(y) + 2yg'(y) \\ g'(y) &= -\frac{g(y)}{2y} \end{aligned}$$

It's much harder to see what might work here. Interestingly  $g(y) = y^{-1/2}$  will work, yielding the exact:

$$2y^{1/2} + xy^{-1/2} \frac{dy}{dx}$$

which has  $H(x, y) = 2xy^{1/2}$  and hence solution  $2xy^{1/2} = C$ .

**Example 2:** Consider the differential equation

$$y + (x + xy) \frac{dy}{dx} = 0$$

Here  $M = y$  and  $N = x + xy$ , these are different so it's not exact. Let's look for some  $g(y)$  so that when we multiply through the result is exact:

$$yg(y) + (x + xy)g(y) \frac{dy}{dx} = 0$$

For this to be exact we'd need:

$$\begin{aligned} [(x + xy)g(y)]_x &= [yg(y)]_y \\ (1 + y)g(y) &= 1g(y) + yg'(y) \\ g(y) + yg(y) &= g(y) + yg'(y) \\ g'(y) &= g(y) \end{aligned}$$

We can see that  $g(y) = e^y$  does the job. This is then our integrating factor and we multiply our original differential equation through by it to get the exact differential equation

$$ye^y + (x + xy)e^y \frac{dy}{dx} = 0$$

which has  $H(x, y) = xye^y$  and hence solution  $xye^y = C$ .

**Addendum:** If we tried  $f(x)$  we'd want this to be exact:

$$yf(x) + (x + xy)f(x) \frac{dy}{dx} = 0$$

This would mean:

$$\begin{aligned} [(x + xy)f(x)]_x &= [yf(x)]_y \\ (1 + y)f(x) + (x + xy)f'(x) &= 1f(x) + y(0) \\ f(x) + yf(x) + (x + xy)f'(x) &= f(x) \\ yf(x) + (x + xy)f'(x) &= 0 \\ f'(x) &= -\frac{yf(x)}{(x + xy)} \end{aligned}$$

It's not at all obvious if anything this works.

## MATH 246: Chapter 2 Section 1: Intro to Higher Order Linear

Justin Wyss-Gallifent

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Main Topics:

- Reminder and Notation.
  - Interval of Existence.
- 

1. **Introduction** Since higher order DEs are difficult we're going to focus on linear higher order DEs. We'll narrow it down even more but for now that's where we are. Just a reminder that these look like, all in *linear normal form*:

First-Order	$y' + a(t)y = f(t)$ (We can solve these)
Second-Order	$y'' + a(t)y' + b(t)y = f(t)$
Third-Order	$y''' + a(t)y'' + b(t)y' + c(t)y = f(t)$
Etc.	Etc.

### 2. Notation Note

It's not uncommon to see an alternate notation for the derivative from here on. We often write  $Dy$  instead of  $y'$ ,  $D^2y$  instead of  $y''$  and so on.

### 3. Existence Theory

The theory is similar to what we've seen for first-order but the initial value needs a bit more:

- A first order linear IVP requires knowing  $y(t_I) = y_0$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $f(t)$  are differentiable.
- A second order linear IVP requires knowing  $y(t_I) = y_0$  and  $y'(t_I) = y_1$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$ ,  $b(t)$  and  $f(t)$  are differentiable.
- A third order linear IVP requires knowing  $y(t_I) = y_0$  and  $y'(t_I) = y_1$  and  $y''(t_I) = y_2$ . There is a unique solution on the *interval of existence* which is the largest open interval containing  $t_I$  on which  $a(t)$ ,  $b(t)$ ,  $c(t)$  and  $f(t)$  are differentiable.
- From here you can certainly see the pattern.

Note: The proof (of existence and uniqueness) is difficult. The special case for first order linear is easy and we saw it because we explicitly constructed the solution.

**Example:**  $y'' + \frac{1}{t}y' - \frac{1}{t-3}y = t$  with  $y(1) = 17$  and  $y'(1) = 2$  has a unique solution on  $(0, 3)$ . If instead we have  $y(4) = 17$  then this has a unique solution on  $(3, \infty)$ .

**Example:** For  $t^{-1/2}y'' + e^t y' - \sin(t)y = \frac{t}{6-t}$  with  $y(1) = 8$  and  $y'(1) = 3$  we have to first rewrite in linear normal form as  $y'' + e^t \sqrt{t}y' - \sqrt{t} \sin(t)y = \frac{t^{3/2}}{6-t}$  which then has a unique solution on  $(0, 6)$ .

**Example:** The IVP  $y''' - \frac{1}{t}y'' + e^t y' - \sin(t)y = \frac{t}{10-t}$  with  $y(3) = 8$  and  $y'(3) = 3$  and  $y''(3) = 5$  has a unique solution on  $(0, 10)$ .

**Example:** The IVP  $D^2y - Dy - 2y = 0$  with  $y(0) = 1$  and  $y'(0) = -3$  has a unique solution on  $(-\infty, \infty)$ . If we notice that  $y = e^{2x}$  is a solution then we know it's the only solution.

## MATH 246: Chapter 2 Section 2: Homogeneous Equations - Method and Theory

Justin Wyss-Gallifent

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Main Topics:

- Definition of Homogeneous.
  - Motivational Example for Second Order.
  - Theory for Second and Third Order.
- 

1. **Introduction:** Since even linear higher-order DEs are difficult we are going to simplify even more. For today we're going to look at *homogeneous* higher-order linear DEs, in which the forcing function  $f(t)$  is equal to 0. That is:

First-Order	$y' + a(t)y = 0$
Second-Order	$y'' + a(t)y' + b(t)y = 0$
Third-Order	$y''' + a(t)y'' + b(t)y' + c(t)y = 0$
$\vdots$	$\vdots$

2. **A Motivational Example:** Consider the second-order homogeneous linear DE:

$$y'' - y' - 2y = 0$$

Next look at the following two functions, don't worry about where they came from:

$$Y_1(t) = e^{2t} \text{ and } Y_2(t) = e^{-t}$$

We can easily see that these are both solutions to the DE by plugging them (and their derivatives) in and checking.

- (a) **Observation 1 - Getting More Solutions:**

Notice that if we take a *linear combination* of these two, meaning

$$Y(t) = C_1 e^{2t} + C_2 e^{-t}$$

where  $C_1$  and  $C_2$  are constants. Then we can easily see that this is also a solution to the DE by plugging it (and its derivatives) in and checking.

- (b) **Observation 2 - Getting All Solutions:**

We can build new solutions from these two but can we build all solutions this way? Well suppose that we had some solution to the DE, call it  $Y(t)$ . What we want to know is if we can find  $C_1$  and  $C_2$  so that  $Y(t) = C_1 e^{2t} + C_2 e^{-t}$  for this  $Y(t)$ ?

Well, suppose we find that  $Y(0) = y_0$  and  $Y'(0) = y_1$ . Since  $Y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$  we would need

$$\begin{aligned} y_0 &= Y(0) = C_1 + C_2 \\ y_1 &= Y'(0) = 2C_1 - C_2 \end{aligned}$$

Can we find such values? Since  $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \neq 0$  there is a unique solution.

Notice now that since this is a solution to the IVP and since there is only one solution to the IVP this must be the solution we were looking for.



(c) **Observation 3 - Anything Special About Those Two?**

We can't just start with any two solutions. To see this observe that if we'd started with  $Y_1(t) = e^{2t}$  and  $Y_2(t) = 17e^{2t}$  that both of these are solutions. Again any linear combination  $Y(t) = C_1e^{2t} + C_217e^{2t}$  is a solution. However is every solution to the DE a linear combination? Again, suppose  $Y(t)$  is a solution and  $Y(0) = y_0$  and  $Y'(0) = y_1$ . Then  $Y'(t) = 2C_1e^{2t} + 34C_2e^{2t}$  and we would need

$$\begin{aligned}y_0 &= Y(0) = C_1 + 17C_2 \\y_1 &= Y'(0) = 2C_1 + 34C_2\end{aligned}$$

Since  $\begin{vmatrix} 1 & 17 \\ 2 & 34 \end{vmatrix} = 0$  there may be no solution. That is, we can't guarantee a solution.

3. **Theory:**

(a) **Theory for Second-Order Homogeneous:**  $y'' + a(t)y' + b(t)y = 0$

- For a second-order homogeneous linear DE we need to find two solutions  $Y_1(t)$  and  $Y_2(t)$  with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2] = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}$$

Alternately the two solutions cannot be multiples of each other. They form a *fundamental set* or *fundamental pair* of solutions  $\{Y_1(t), Y_2(t)\}$ .

- Every solution is then a linear combination of the fundamental pair. This means the general solution is  $Y(t) = C_1Y_1(t) + C_2Y_2(t)$ .
- A second-order IVP must provide  $y(t_I)$  and  $y'(t_I)$  in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $b(t)$  are differentiable.

(b) **Theory for Third-Order Homogeneous:**  $y''' + a(t)y'' + b(t)y' + c(t)y = 0$

- For a third-order homogeneous linear DE we need to find three solutions  $Y_1(t)$ ,  $Y_2(t)$ , and  $Y_3(t)$  with a special relationship. That relationship is that their *Wronskian* does not equal the zero function, where:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} Y_1 & Y_2 & Y_3 \\ Y_1' & Y_2' & Y_3' \\ Y_1'' & Y_2'' & Y_3'' \end{vmatrix}$$

Alternately it must be impossible to write one of the solutions as a linear combination of the others. They form a *fundamental set* of solutions  $\{Y_1(t), Y_2(t), Y_3(t)\}$ .

- Every solution is then a linear combination of the fundamental set. This means the general solution is  $Y(t) = C_1Y_1(t) + C_2Y_2(t) + C_3Y_3(t)$ .
- A third-order IVP must provide  $y(t_I)$ ,  $y'(t_I)$ , and  $y''(t_I)$  in order to find the specific solution.
- This solution is unique on the *interval of existence and uniqueness* which is the largest open interval containing  $t_I$  on which  $a(t)$  and  $b(t)$  and  $c(t)$  are differentiable.

(c) **Theory for Higher-Order:**

You can probably see the pattern.

- (d) **Critical Note:** Don't worry about where these fundamental sets are coming from right now, just realize that we (somehow) need obtain them!

#### 4. Practice for Both:

Here are some examples:

**Example:** Consider  $y'' + 4y = 0$ . First we'll show that  $Y_1(t) = \sin(2t)$  and  $Y_2(t) = \cos(2t)$  form a fundamental pair. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{vmatrix} = -2\sin^2(2t) - 2\cos^2(2t) = -2 \neq 0$$

This tells us that  $Y_1(t)$  and  $Y_2(t)$  form a fundamental pair and that the general solution is:

$$Y(t) = C_1 \sin(2t) + C_2 \cos(2t)$$

So now if we have the IVP with  $Y(0) = 4$  and  $Y'(0) = 2$  we can find the specific solution by first finding:

$$Y'(t) = 2C_1 \cos(2t) - 2C_2 \sin(2t)$$

and then solving the system:

$$\begin{aligned} 4 &= Y(0) = C_1 \sin(2(0)) + C_2 \cos(2(0)) = C_2 \\ 2 &= Y'(0) = 2C_1 \cos(2(0)) - 2C_2 \sin(2(0)) = 2C_1 \end{aligned}$$

So that  $C_1 = 1$  and  $C_2 = 4$  and the specific solution is:

$$Y(t) = \sin(2t) + 4\cos(2t)$$

**Example:** Consider  $(1 + t^2)y'' - 2ty' + 2y = 0$ . First we'll show that  $Y_1(t) = t$  and  $Y_2(t) = t^2 - 1$  form a fundamental pair. Notice that it doesn't matter whether we divide by  $1 + t^2$  or not when we check these. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2 - 1 \\ 1 & 2t \end{vmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1 \neq 0$$

This tells us that  $Y_1(t)$  and  $Y_2(t)$  form a fundamental pair and that the general solution is:

$$Y(t) = C_1 t + C_2(t^2 - 1)$$

So now if we have the IVP with  $Y(2) = -5$  and  $Y'(2) = 7$  we can find the specific solution by first finding:

$$Y'(t) = C_1 + 2C_2 t$$

and then solving the system:

$$\begin{aligned} -5 &= Y(2) = C_1(2) + C_2(2^2 - 1) = 2C_1 + 3C_2 \\ 7 &= Y'(2) = C_1 + 2C_2(2) = C_1 + 4C_2 \end{aligned}$$

So that  $C_1 = -\frac{41}{5}$  and  $C_2 = -\frac{19}{5}$  and the specific solution is:

$$Y(t) = -\frac{41}{5}t + \frac{19}{5}(t^2 - 1)$$

**Example:** Consider  $D^3y - 2D^2y = 0$  First we'll show that  $Y_1(t) = 1$ ,  $Y_2(t) = t$  and  $Y_3(t) = e^{2t}$  form a fundamental set. We check they are solutions (omitted) and we check:

$$W[Y_1, Y_2, Y_3] = \begin{vmatrix} 1 & t & e^{2t} \\ 0 & 1 & 2e^{2t} \\ 0 & 0 & 4e^{2t} \end{vmatrix} = 4e^{2t} \neq 0$$

This tells us that  $Y_1(t)$ ,  $Y_2(t)$  and  $Y_3(t)$  form a fundamental set and that the general solution is:

$$Y(t) = C_1 + C_2t + C_3e^{2t}$$

So now if we have the IVP with  $Y(0) = 1$ ,  $Y'(0) = 0$  and  $Y''(0) = -4$  we can find the specific solution by first finding:

$$Y'(t) = C_2 + 2C_3e^{2t}$$

$$Y''(t) = 4C_3e^{2t}$$

and then solving the system:

$$1 = Y(0) = C_1 + C_3$$

$$0 = Y'(0) = C_2 + 2C_3$$

$$-4 = Y''(0) = 4C_3$$

So that  $C_3 = -1$ ,  $C_2 = 2$  and  $C_1 = 2$  and the specific solution is:

$$Y(t) = 2 + 2t - e^{2t}$$

## 5. More about Fundamental Sets:

### (a) Natural Fundamental Sets

There's more than just one fundamental set, and one that comes up a lot is called the *natural fundamental set*.

In the second-order case this is the set  $\{Y_1, Y_2\}$  with  $Y_1$  having  $Y_1(t_I) = 1$  and  $Y_1'(t_I) = 0$  and with  $Y_2$  having  $Y_2(t_I) = 0$  and  $Y_2'(t_I) = 1$ .

In the third-order case this is the set  $\{Y_1, Y_2, Y_3\}$  with  $Y_1$  having  $Y_1(t_I) = 1$ ,  $Y_1'(t_I) = 0$ , and  $Y_1''(t_I) = 0$ , with  $Y_2$  having  $Y_2(t_I) = 0$ ,  $Y_2'(t_I) = 1$ , and  $Y_2''(t_I) = 0$ , and with  $Y_3$  having  $Y_3(t_I) = 1$ ,  $Y_3'(t_I) = 0$ , and  $Y_3''(t_I) = 1$ ,

Beyond there you can probably see the pattern.

### (b) Reduction of Order (OMITTED)

The big question of course is where the fundamental set comes from. We'll address that a bit later but for now we have one helper.

If we have one solution  $Y_1(t)$  then the second one is very often a multiple of the first. So we can set  $Y_2(t) = uY_1(t)$  and when we plug this into the DE and use the fact that  $Y_1(t)$  is a solution we end up in a situation where we can find a first-order DE (hence the name) that we can use to find  $u$ .

**Example:** You can check that  $Y_1(t) = e^{5t}$  is a solution to  $y'' - 3y' - 10y = 0$ . To find the other by reduction of order we put  $Y_2(t) = ue^{5t}$ . We then find

$$Y_2'(t) = u'e^{5t} + 5ue^{5t} \text{ and } Y_2''(t) = u''e^{5t} + 5u'e^{5t} + 5u'e^{5t} + 25ue^{5t} = u''e^{5t} + 10u'e^{5t} + 25ue^{5t}$$

and plug these into the DE:

$$\begin{aligned} y'' - 3y' - 10y &= 0 \\ (u''e^{5t} + 10u'e^{5t} + 25ue^{5t}) - 3(u'e^{5t} + 5ue^{5t}) - 10(ue^{5t}) &= 0 \\ u'' + 10u' + 25u - 3u' - 15u - 10u &= 0 \\ u'' + 7u' &= 0 \end{aligned}$$

If we let  $w = u'$  then this gives us  $w' + 7w = 0$  which has solution  $w = Ce^{-7t}$  and so  $u' = Ce^{-7t}$  and so  $u = -\frac{1}{7}Ce^{-7t} + D$  and another solution is

$$Y_2(t) = \left(-\frac{1}{7}Ce^{-7t} + D\right)e^{5t} = -\frac{1}{7}Ce^{-2t} + De^{5t}$$

Since this is true for any  $C$  and  $D$  we can pick the solution

$$Y_2(t) = e^{-2t}$$

for which  $W[Y_1, Y_2] \neq 0$  and we have our fundamental pair.

## MATH 246: Chapter 2 Section 3: Matrices and Determinants

Justin Wyss-Gallifent

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Main Topics:

- Matrices
  - Determinants
  - Relationship to Linear Systems
- 

### 1. Introduction

As the course progresses we'll run into matrices and we'll need some basic facts. For now we simply need to know what a matrix is, what a determinant is, and what they can be used for.

### 2. Matrices

A matrix is basically a rectangular array of numbers. In this course pretty much all the matrices we'll work with will be square and either  $2 \times 2$  or  $3 \times 3$ .

**Examples:**

$$A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{bmatrix}$$

### 3. Determinants

The determinant of a matrix is a single number associated with the matrix which tells us certain properties of that matrix. It is the single most important number associated with a matrix. It can be denoted either by putting  $\det$  in front of the matrix or by putting the matrix values (not the brackets) inside vertical bars like absolute values.

It can be defined recursively but we'll only need it for  $2 \times 2$  and  $3 \times 3$  so here are the rules:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

**Example:**  $\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = (3)(1) - (-2)(0) = 3$

**Example:**  $\begin{vmatrix} 1 & 3 \\ -5 & 7 \end{vmatrix} = (1)(7) - (3)(-5) = 22$

**Example:**

$$\begin{aligned} \begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} &= 4 \begin{vmatrix} 5 & 0 \\ 8 & -3 \end{vmatrix} - 3 \begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} + 1 \begin{vmatrix} -2 & 5 \\ 0 & 8 \end{vmatrix} \\ &= 4(-15) - 3(6) + 1(-16) \\ &= -94 \end{aligned}$$

#### 4. Relationship to Systems of Equations

In linear algebra matrices are used to solve linear systems of equations. We don't need to do that but we do need to know that determinants of matrices can tell us information about the solutions.

##### (a) All Systems

If we put the coefficients of the variables into a matrix and find the determinant then this determinant will be nonzero if and only if there is a unique solution to the system. If we do get zero then there will be either no solutions or infinitely many solutions. For now we need not distinguish between these outcomes.

**Example:** The system

$$2x + 3y = 7$$

$$5x - 7y = 23$$

Since  $\begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix} = -29 \neq 0$  there is only one solution.

**Example:** The system

$$4x + 8y = 3$$

$$6x + 12y = -8$$

Since  $\begin{vmatrix} 4 & 8 \\ 6 & 12 \end{vmatrix} = 0$  there are either no solutions or infinitely many solutions.

**Example:** The system

$$4x + 3y + 1z = 7$$

$$-2x + 5y + 0z = -17$$

$$0x + 8y - 3z = 2$$

Since  $\begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} = -94 \neq 0$  there is only one solution.

(b) **Homogenous Systems**

A homogeneous linear system is when all the constant terms are 0. Setting all the variables to be zero always gives a solution, called the *trivial solution*. In this case if the determinant is zero then there must be infinitely many solutions, meaning there are *nontrivial solutions*, and if the determinant is nonzero then there is only the trivial solution, so there are no nontrivial solutions.

**Example:** The system

$$2x + 3y = 0$$

$$5x - 7y = 0$$

has the trivial solution  $x = y = 0$ . In addition since  $\begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix} = -29 \neq 0$  this is the only solution.

**Example:** The system

$$4x + 8y = 0$$

$$6x + 12y = 0$$

has the trivial solution  $x = y = 0$ . In addition since  $\begin{vmatrix} 4 & 8 \\ 6 & 12 \end{vmatrix} = 0$  there are (infinitely many) other nontrivial solutions.

**Example:** The system

$$4x + 3y + 1z = 0$$

$$-2x + 5y + 0z = 0$$

$$0x + 8y - 3z = 0$$

has the trivial solution  $x = y = z = 0$ . In addition since  $\begin{vmatrix} 4 & 3 & 1 \\ -2 & 5 & 0 \\ 0 & 8 & -3 \end{vmatrix} = -94 \neq 0$  this is the only solution.

## MATH 246: Chapter 2 Section 4: Homogeneous Equations with Constant Coefficients

Justin Wyss-Gallifent

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Main Topics:

- The Characteristic Polynomial
  - Real Simple Roots
  - Real Multiple Roots
  - Complex Simple Roots
  - Complex Multiple Roots
- 

### 1. Introduction:

We've established the fact that for an  $n^{\text{th}}$  order homogeneous linear differential equation that we need to find a fundamental set of  $n$  solutions denoted  $Y_1, \dots, Y_n$ . Once we do this we have the general solution

$$Y(t) = C_1 Y_1 + C_2 Y_2 + \dots C_n Y_n$$

The next question is how to get that fundamental set.

This is hard, so for now we'll focus on the simpler situation where the DE has constant coefficients. Examples:

$$\begin{aligned} y'' + 2y' - 3y &= 0 \\ 2y''' - 5y &= 0 \\ D^4 y + D^3 y - 2D^2 y + Dy + y &= 0 \end{aligned}$$

2. **Inspirational Example:** Consider  $y'' + 2y' - 3y = 0$ . Here because we've got constant multiples of derivatives added to get zero, we think that perhaps solutions might be functions whose derivatives are constant multiples of themselves. The primary example is  $e^{zt}$  for various  $z$ . So let's try - if  $y = e^{zt}$  is a solution to our equation then  $y' = ze^{zt}$  and  $y'' = z^2 e^{zt}$  and we have:

$$\begin{aligned} y'' + 2y' - 3y &= 0 \\ z^2 e^{zt} + 2ze^{zt} - 3e^{zt} &= 0 \\ (z^2 + 2z - 3)e^{zt} &= 0 \end{aligned}$$

Since  $e^{zt}$  is always positive we must have

$$\begin{aligned} z^2 + 2z - 3 &= 0 \\ (z + 3)(z - 1) &= 0 \end{aligned}$$

and so either  $z = -3$  or  $z = 1$ .

Lo and behold we've actually found two solutions, and we only needed two. We've found

$$Y_1(t) = e^{-3t} \text{ and } Y_2(t) = e^{1t}$$

Thus the general solution is

$$Y(t) = C_1 e^{-3t} + C_2 e^t$$

3. **General Idea:** We see that the DE  $y'' + 2y' - 3y = 0$  gave us a polynomial  $z^2 + 2z - 3 = 0$ . This will happen in every case, that polynomial is called the *characteristic polynomial*. The roots of the polynomial will give us solutions and we will get enough to complete our fundamental set. However there are nuances, primarily that we have to deal with both real and complex roots and each of these has two subcategories.

In addition, though we won't prove this, the process is guaranteed to result in a fundamental set, meaning the Wronskian would be nonzero if we checked it.



4. **Real Simple Roots:** A *real simple root* is a root which only appears once when we factor the characteristic polynomial. For a simple real root  $z = r$  we get a solution  $e^{rt}$ . If there are  $n$  distinct simple real roots then we get  $n$  solutions and we're done.

**Example:** The DE  $y'' - 3y' - 10y = 0$  has characteristic polynomial  $z^2 - 3z - 10$  or  $(z-5)(z+2)$  with roots  $z = 5$  and  $z = -2$ . So we have two simple real roots and therefore the fundamental set is  $\{e^{5t}, e^{-2t}\}$  and the general solution is  $Y(t) = C_1e^{5t} + C_2e^{-2t}$ .

**Example:** The DE  $y''' - 5y'' + 6y' = 0$  has characteristic polynomial  $z^3 - 5z^2 + 6z$  or  $z(z-3)(z-2)$  with roots  $z = 0$ ,  $z = 3$  and  $z = 2$ . So we have three simple real roots and therefore the fundamental set is  $\{1, e^{3t}, e^{2t}\}$  (notice that  $e^{0t} = 1$ ) and the general solution is  $Y(t) = C_1 + C_2e^{3t} + C_3e^{2t}$ .

5. **Real Multiple Roots** A *real multiple root* is a root which appears more than once when we factor the characteristic polynomial.

**Example:** The DE  $D^3y - 15D^2y + 75Dy - 125y = 0$  has characteristic polynomial  $z^3 - 15z^2 + 75z - 125$  or  $(z-5)^3$ . there is only the root  $z = 5$  with multiplicity  $m = 3$ .

**Example:** The DE  $y''' - 4y'' + 4y' = 0$  has characteristic polynomial  $z^3 - 4z^2 + 4z$  or  $z(z-2)^2$ . There is the root  $z = 0$  which is a simple root (multiplicity 1) and the root  $z = 2$  with multiplicity  $m = 2$ .

For a real multiple root  $z = r$  with multiplicity  $m$  we get  $m$  solutions:

$$e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$$

We'll discuss where these come from in a later section.

**Example:** The DE  $D^3y - 15D^2y + 75Dy - 125y = 0$  has characteristic polynomial  $z^3 - 15z^2 + 75z - 125$  or  $(z-5)^3$ . there is only the root  $z = 5$  with multiplicity  $m = 3$ . So we have the fundamental set  $\{e^{5t}, te^{5t}, t^2e^{5t}\}$  and the general solution  $Y(t) = C_1e^{5t} + C_2te^{5t} + C_3t^2e^{5t}$ .

**Example:** The DE  $y''' - 4y'' + 4y' = 0$  has characteristic polynomial  $z^3 - 4z^2 + 4z$  or  $z(z-2)^2$ . There is the root  $z = 0$  which is a simple root (multiplicity 1) and the root  $z = 2$  with multiplicity  $m = 2$ . So we have the fundamental set  $\{1, e^{2t}, te^{2t}\}$  and the general solution  $Y(t) = C_1 + C_2e^{2t} + C_3te^{2t}$ .

6. **Complex Simple Roots:** Complex roots always come in pairs. This means that if  $r + si$  is a root then so is  $r - si$ . A *complex simple root pair* is a pair of roots  $r \pm si$  which appears only once when we factor the characteristic polynomial. We may have to actually solve via the quadratic formula to really see what's up.

**Example:** The DE  $y'' + y' + 2y = 0$  has characteristic polynomial  $z^2 + z + 2$ . This doesn't obviously factor. To find the roots we set it equal to 0 and use the quadratic formula to get  $z = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{-7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$ .

**Example:** The DE  $y'' + 4y = 0$  has characteristic polynomial  $z^2 + 4$  and setting  $z^2 + 4 = 0$  gives  $z = \pm\sqrt{-4} = 0 \pm 2i$ . We've made the 0 clear for a reason as you'll see.

To see what's going to happen, let's just charge ahead for a minute using the previous approach. If  $r + si$  is a root then  $e^{(r+si)t}$  is a solution. But note that we can rewrite this:

$$e^{(r+si)t} = e^{rt}e^{(st)i} = e^{rt}(\cos(st) + i\sin(st))$$

Since  $r - si$  is also a root then  $e^{(r-si)t}$  is also a solution and we can rewrite this too:

$$e^{(r-si)t} = e^{rt}e^{(-st)i} = e^{rt}(\cos(-st) + i\sin(-st)) = e^{rt}(\cos(st) - i\sin(st))$$

But these are complex solutions which don't fill out our fundamental set. But since linear combinations of solutions are solutions, we can be sneaky and observe:

- $\frac{1}{2}(\text{sum of complex solutions}) = e^{rt}\cos(st)$  is a solution.
- $\frac{1}{2i}(\text{difference of complex solutions}) = e^{rt}\sin(st)$  is a solution.

Thus for each complex simple root pair  $z = r \pm si$  we get a pair of solutions  $e^{rt}\cos(st)$  and  $e^{rt}\sin(st)$ .

**Example:** The DE  $y'' + y' + 2y = 0$  has characteristic polynomial  $z^2 + z + 2$  with roots we saw as  $-\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$ . The fundamental set is then  $\left\{e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right), e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)\right\}$  and the general solution is  $Y(t) = C_1e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + C_2e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)$ .

**Example:** The DE  $y'' + 4y = 0$  has characteristic polynomial  $z^2 + 4$  with roots  $z = 0 \pm 2i$ . The fundamental set is then  $\{\cos(2t), \sin(2t)\}$  and the general solution is  $Y(t) = C_1\cos(2t) + C_2\sin(2t)$ .

**Example:** The DE  $y''' - y'' - 4y' - 6y = 0$  has characteristic polynomial  $z^3 - z^2 - 4z - 6$  or  $(z - 3)(z^2 + 2z + 2)$ . We see that  $z = 3$  is a root but for the rest we set  $z^2 + 2z + 2 = 0$  and get  $z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = -1 \pm i$ . The fundamental set is then  $\{e^{3t}, e^{-t}\cos(t), e^{-t}\sin(t)\}$  and the general solution is  $Y(t) = C_1e^{3t} + C_2e^{-t}\cos(t) + C_3e^{-t}\sin(t)$ .

7. **Complex Multiple Roots:** This expands like with real roots. For each complex multiple root pair  $z = r \pm si$  with multiplicity  $m$  we get  $m$  pairs of solutions

$$e^{rt} \cos(st), te^{rt} \cos(st), \dots, t^{m-1} e^{rt} \cos(st)$$

$$e^{rt} \sin(st), te^{rt} \sin(st), \dots, t^{m-1} e^{rt} \sin(st)$$

**Example:** The DE  $D^6y + 8D^5y + 65D^4y + 232D^3y + 904D^2y + 1440Dy + 3600y = 0$  has characteristic polynomial  $z^6 + 8z^5 + 65z^4 + 232z^3 + 904z^2 + 1440z + 3600$  or  $(z^2 + 9)(z^2 + 4z + 20)^2$ . The first part gives us  $z = 0 \pm 3i$  and the second part gives us  $z = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2} = -2 \pm 4i$  with multiplicity 2. The fundamental set is then  $\{\cos(3t), \sin(3t), e^{-2t} \cos(4t), e^{-2t} \sin(4t), te^{-2t} \cos(4t), te^{-2t} \sin(4t)\}$  and the general solution is  $Y(t) = C_1 \cos(3t) + C_2 \sin(3t) + C_3 e^{-2t} \cos(4t) + C_4 e^{-2t} \sin(4t) + C_5 te^{-2t} \cos(4t) + C_6 te^{-2t} \sin(4t)$ .

8. **Summary:** In summary we construct the characteristic polynomial and find the roots. The roots tell us how to construct the fundamental set as follows:

- (a) If  $r$  is a real simple root then put in:

$$e^{rt}$$

- (b) If  $r$  is a multiple simple root with multiplicity  $m$  then put in:

$$e^{rt}, te^{rt}, \dots, t^{m-1} e^{rt}$$

- (c) If  $r \pm si$  is a complex simple root pair then put in:

$$e^{rt} \cos(st), e^{rt} \sin(st)$$

- (d) If  $r \pm si$  is a complex multiple root pair with multiplicity  $m$  then put in:

$$e^{rt} \cos(st), te^{rt} \cos(st), \dots, t^{m-1} e^{rt} \cos(st)$$

and

$$e^{rt} \sin(st), te^{rt} \sin(st), \dots, t^{m-1} e^{rt} \sin(st)$$

## MATH 246: Chapter 2 Section 5: Nonhomogeneous Equations - Method and Theory

Justin Wyss-Gallifent

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Main Topics:

- Inspirational Example
  - General Method
- 

### 1. Introduction:

Now that we know how to handle homogeneous linear differential equations with constant coefficients we'd like to see what happens if we make one small change. We'll look at nonhomogeneous linear differential equations. For now we'll momentarily even remove the requirement that we have constant coefficients. Examples:

$$\begin{aligned}y'' + 2y' - 3y &= e^t \\ 2y''' - 5ty &= \cos(2t) \\ y'''' + e^t y''' - 2ty'' + Dy + t^2y &= t\end{aligned}$$

### 2. An Inspirational Example:

The general idea is actually quite simple. To see, let's look at the example  $y'' + 2y' - 3y = f(t)$  where  $f(t)$  is unknown but nonzero.

Suppose that somehow we obtained just one solution. Call that solution  $Y_p(t)$  where the  $p$  stands for *particular*. Suppose that  $Y(t)$  is any other solution. Then look at what happens when we plug in  $Y(t) - Y_p(t)$ :

$$\begin{aligned}(Y(t) - Y_p(t))'' + 2(Y(t) - Y_p(t))' - 3(Y(t) - Y_p(t)) \\&= [Y''(t) + 2Y'(t) - 3Y(t)] - [Y_p''(t) + 2Y_p'(t) - 3Y_p(t)] \\&= f(t) - f(t) \\&= 0\end{aligned}$$

What this is telling us is that  $Y(t) - Y_p(t)$  is a solution to the *homogeneous version* of the DE. Since we know that all the solutions to the homogeneous version look like  $C_1 e^{-3t} + C_2 e^t$  this then tells us that

$$\begin{aligned}Y(t) - Y_p(t) &= C_1 e^{-3t} + C_2 e^t \\ Y(t) &= Y_p(t) + C_1 e^{-3t} + C_2 e^t\end{aligned}$$

### 3. General Method:

What this tells us is actually pretty fantastic. It says that when we're confronted by an  $n^{\text{th}}$  order nonhomogeneous linear differential equation, all we need to do is two things:

- Find the fundamental set  $\{Y_1, \dots, Y_n\}$  for the homogeneous version.
- Find one single solution  $Y_p$  for the original differential equation.

Then the general solution to the original differential equation is

$$Y(t) = Y_p(t) + C_1 Y_1(t) + \dots + C_n Y_n(t)$$

Of course these two things may not be easy. If the homogeneous version has constant coefficients at least part (a) is easy. Let's gloss over these for now to see things in practice.

- Warning** Keep in mind that the nonhomogeneous version doesn't have a fundamental set, it's the homogeneous version which does. Then the fundamental set for the homogeneous version helps us construct the general solution to the nonhomogeneous version.

## 5. Examples:

Here are some examples of both DEs and IVPs:

**Example:** Consider the differential equation  $y'' + 4y = 4t$ .

(a) The homogeneous version  $y'' + 4y = 0$  has fundamental set  $\{\cos(2t), \sin(2t)\}$ .

(b) The function  $Y_p(t) = t$  is a solution to  $y'' + 4y = 4t$ .

Thus the general solution to  $y'' + 4y = 4t$  is:

$$Y(t) = t + C_1 \cos(2t) + C_2 \sin(2t)$$

**Example:** Consider the differential equation  $D^3y - 2D^2y = 9e^{3t}$ .

(a) The homogeneous version  $D^3y - 2D^2y = 0$  has fundamental set  $\{1, t, e^{2t}\}$ .

(b) The function  $Y_p(t) = e^{3t}$  is a solution to  $D^3y - 2D^2y = 9e^{3t}$ .

Thus the general solution to  $D^3y - 2D^2y = 9e^{3t}$  is:

$$Y(t) = e^{3t} + C_1 + C_2t + C_3e^{2t}$$

**Example:** Consider the initial value problem  $(1 + t^2)y'' - 2ty' + 2y = 6$  with  $Y(0) = 2$  and  $Y'(0) = 1$ .

(a) The homogeneous version  $(1 + t^2)y'' - 2ty' + 2y = 0$  has fundamental set  $\{t, t^2 - 1\}$ .

(b) The function  $Y_p(t) = 3$  is a solution to  $(1 + t^2)y'' - 2ty' + 2y = 6$ .

Thus the general solution to  $(1 + t^2)y'' - 2ty' + 2y = 6$  is:

$$Y(t) = 3 + C_1t + C_2(t^2 - 1)$$

To solve the IVP we find

$$Y'(t) = C_1 + 2tC_2$$

and we solve:

$$2 = Y(0) = 3 + C_1(0) + C_2(0^2 - 1)$$

$$1 = Y'(0) = C_1 + 2(0)C_2$$

This tells us that  $C_2 = 1$  and  $C_1 = 1$  and so the specific solution is

$$Y(t) = 6 + t + (t^2 - 1)$$

## MATH 246: Chapter 2 Section 6: Undetermined Coefficients

Justin Wyss-Gallifent

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Main Topics:

- General Idea
  - Solution Building
- 

### 1. Introduction

Remember where we are: We have a non-homogenous linear differential equation with constant coefficients. We know how to deal with the homogeneous version and we know that all we need to do is get ahold of a single solution to the non-homogeneous version denoted  $Y_p(t)$  and then we can construct all solutions to the non-homogeneous version.

### 2. General Idea

The Method of Undetermined Coefficients will only work if the right side of the differential equation (the forcing function  $f(t)$ ) is one of our nice forms like  $e^{2t}$  or  $t^2$  or  $e^{2t} \cos(5t)$ . It is based on the premise that we know what the answer looks like and we only need to work out some coefficients.

Just to warm up before we get all formal:

If $f(t) = e^{5t}$	then $Y_p(t)$ probably looks like $Ce^{5t}$
If $f(t) = \cos(2t)$	then $Y_p(t)$ probably looks like $C_1 \cos(2t) + C_2 \sin(2t)$
If $f(t) = te^{5t}$	then $Y_p(t)$ probably looks like $C_1 e^{5t} + C_2 t e^{5t}$ or $(C_1 + C_2 t)e^{5t}$

The Method of Undetermined Coefficients will work as follows - we'll suggest what the solution looks like, with unknown constants, then we'll plug it into the DE and find the constants that make it work.

**Example:** If we have  $y'' - y = 2e^{5t}$  and we suggest that  $Y_p(t) = Ce^{5t}$ , then  $Y_p'(t) = 5Ce^{5t}$  and  $Y_p''(t) = 25Ce^{5t}$ . If we plug these into the DE we get  $25Ce^{5t} - Ce^{5t} = 2e^{5t}$  and so  $24Ce^{5t} = 2e^{5t}$  and so  $C = \frac{1}{12}$  and the solution is  $Y_p(t) = \frac{1}{12}e^{5t}$ . Wow, that was easy!

### 3. Building A Solution

This is very procedural and works as follows. Here  $Q_n(t)$  means a known polynomial of degree  $n$  and  $GP_n$  means an unknown generic polynomial of degree  $n$  with undetermined coefficients for which we fill in  $A$ ,  $B$ ,  $C$ , etc. For example  $GP_2 = At^2 + Bt + C$ . This is actually far easier in practice than it looks, the most common mistake is forgetting to check the multiplicity.

- If  $f(t)$  has the form  $Q_n(t)e^{rt}$ , first find  $m$  the multiplicity of  $z$  as a root of the characteristic polynomial. Often  $m = 0$ .  
 $\implies Y_p(t) = t^m [GP_n] e^{rt}$
- If  $f(t)$  has the form  $Q_n(t)e^{rt} \cos(st)$  or  $Q_n(t)e^{rt} \sin(st)$ , first find  $m$  the multiplicity of  $r + si$  as a root of the characteristic polynomial. Often  $m = 0$ .  
 $\implies Y_p(t) = t^m [GP_n] e^{rt} \cos(st) + t^m [GP_n] e^{rt} \sin(st)$   
Here the  $GP_n$  are different with different coefficients!
- If  $f(t)$  is the sum of such forms, we add the resulting forms together. Make sure to never, ever, repeat undetermined coefficients.

#### 4. What to do with the Solution

Once we have our guess we plug it into the DE and simplify like mad. The result will have similar functions on the left and right but the coefficients on the left will be the unknowns. We then equate the coefficients on each side and solve for those unknowns.

Theoretical Note: What allows us to do this last step is that the functions in the solution are linearly independent according to our construction of our solution.

#### 5. Examples:

**Example:** Consider  $y'' + 3y' + 2y = 2e^{3t}$ .

Forcing:  $f(t) = 2e^{3t}$ .

CP:  $p(z) = z^2 + 3z + 2 = (z + 2)(z + 1)$ .

We have  $r = 3$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial 2 is degree 0, we have:

$$Y_p(t) = t^m [GP_0] e^{3t} = t^0(A)e^{3t} = Ae^{3t}$$

Then  $Y_p' = 3Ae^{3t}$  and  $Y_p'' = 9Ae^{3t}$ . We plug these into the DE to get

$$9Ae^{3t} + 3(3Ae^{3t}) + 2Ae^{3t} = 2e^{3t}$$

$$20Ae^{3t} = 2e^{3t}$$

$$A = \frac{1}{10}$$

Then  $Y_p(t) = \frac{1}{10}e^{3t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{-2t}, e^{-t}\}$  the general solution is  $Y(t) = \frac{1}{10}e^{3t} + C_1e^{-2t} + C_2e^{-t}$ .

**Example:** Consider  $y'' - 3y' + 2y = 7e^{2t}$ .

Forcing:  $f(t) = 7e^{2t}$ .

CP:  $p(z) = z^2 - 3z + 2 = (z - 2)(z - 1)$ .

We have  $r = 2$  which is a root of  $p(z)$  of multiplicity  $m = 1$ . Because the coefficient polynomial 7 is degree 0, we have:

$$Y_p(t) = t^m [GP_0] e^{2t} = t^1(A)e^{2t} = Ate^{2t}$$

Then  $Y_p' = Ae^{2t} + 2Ate^{2t}$  and  $Y_p'' = 4Ae^{2t} + 4Ate^{2t}$ . We plug these into the DE to get

$$4Ae^{2t} + 4Ate^{2t} - 3(Ae^{2t} + 2Ate^{2t}) + 2(Ate^{2t}) = 7e^{2t}$$

$$Ae^{2t} = 7e^{2t}$$

$$A = 7$$

Then  $Y_p(t) = 7te^{2t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{2t}, e^t\}$  the general solution is  $Y(t) = 7te^{2t} + C_1e^{2t} + C_2e^t$ .

**Example:** Consider  $y''' - y'' = t^2$ .

Forcing:  $f(t) = t^2 = t^2 e^{0t}$ .

CP:  $p(z) = z^3 - z^2 = z^2(z - 1)$ .

We have  $r = 0$  which is a root of  $p(z)$  of multiplicity  $m = 2$ . Because the coefficient polynomial  $t^2$  is degree 2, we have:

$$Y_p(t) = t^m [GP_2] e^{0t} = t^2(At^2 + Bt + C)e^{0t} = At^4 + Bt^3 + Ct^2$$

Then  $Y_p' = 4At^3 + 3Bt^2 + 2Ct$ ,  $Y_p'' = 12At^2 + 6Bt + 2C$  and  $Y_p''' = 24At + 6B$ . We plug these into the DE to get

$$\begin{aligned} 24At + 6B - (12At^2 + 6Bt + 2C) &= t^2 \\ -12At^2 + (24A - 6B)t + (6B - 2C) &= t^2 \end{aligned}$$

Therefore  $-12A = 1$ ,  $24A - 6B = 0$  and  $6B - 2C = 0$ , giving  $A = -\frac{1}{12}$ ,  $B = -\frac{1}{4}$  and  $C = -\frac{3}{4}$ .

Then  $Y_p(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2$ .

Note: Because the fundamental set for the homogeneous version is  $\{1, t, e^t\}$  the general solution is  $Y(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2 + C_1 + C_2t + C_3e^t$ .

**Example:** Consider  $y'' - 3y' + 2y = 5te^{4t}$ .

Forcing:  $f(t) = 5te^{4t}$ .

CP:  $p(z) = z^2 - 3z + 2 = (z - 1)(z - 2)$

We have  $r = 4$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $5t$  is degree 1, we have:

$$Y_p(t) = t^m [GP_1] e^{4t} = t^0(At + B)e^{4t} = (At + B)e^{4t}$$

Then  $Y_p' = Ae^{4t} + (4At + 4B)e^{4t}$  and  $Y_p'' = 8Ae^{4t} + (16At + 16B)e^{4t}$ . We plug these into the DE to get

$$\begin{aligned} 8Ae^{4t} + (16At + 16B)e^{4t} - 3(Ae^{4t} + (4At + 4B)e^{4t}) + 2((At + B)e^{4t}) &= 5te^{4t} \\ (5A + 6B)e^{4t} + (6A)t e^{4t} &= 5te^{4t} \\ (5A + 6B) + (6A)t &= 5t \end{aligned}$$

Therefore  $5A + 6B = 0$  and  $6A = 5$ , giving  $A = \frac{5}{6}$  and  $B = -\frac{25}{36}$ .

Then  $Y_p(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t}$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{2t}, e^t\}$  the general solution is  $Y(t) = \left(\frac{5}{6}t - \frac{25}{36}\right)e^{4t} + C_1e^{2t} + C_2e^t$ .



**Example:** Consider  $y'' + y' = t + 3e^t$ .

Forcing:  $f(t) = t + 3e^t = te^{0t} + 3e^{1t}$  which has two parts.

CP:  $p(z) = z^2 + z = z(z + 1)$ .

For the  $te^{0t}$  part we have  $r = 0$  which is a root of  $p(z) = z^2 + z = z(z + 1)$  of multiplicity  $m = 1$ . Because the coefficient polynomial  $t$  is degree 1, we have:

$$\text{First Part: } Y_p(t) = t^1 [GP_1] e^{0t} = t(At + B)e^{0t} = At^2 + Bt$$

For the  $3e^{1t}$  part we have  $r = 1$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $3$  is degree 0, we have:

$$\text{Second Part: } Y_p(t) = t^m [GP_0] e^t = t^0(C)e^t = Ce^t$$

Combining these we have:

$$Y_p(t) = At^2 + Bt + Ce^t$$

Then  $Y'_p = 2At + B + Ce^t$  and  $Y''_p = 2A + Ce^t$ . We plug these into the DE to get

$$2A + Ce^t + 2At + B + Ce^t = t + 3e^t$$

$$2At + (2A + B) + 2Ce^t = t + 3e^t$$

Therefore  $A = \frac{1}{2}$ ,  $B = -1$  and  $C = \frac{3}{2}$ .

Then  $Y_p(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t$ .

Note: Because the fundamental set for the homogeneous version is  $\{1, e^{-t}\}$  the general solution is  $Y(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t + C_1 + C_2e^{-t}$ .

**Example:** Consider  $y'' + 2y' + 2y = 17\cos(3t)$ .

Forcing:  $f(t) = 17\cos(3t) = 17e^{0t}\cos(3t)$ .

CP:  $p(z) = z^2 + 2z + 2$  with roots  $z = 1 \pm 1i$ .

We have  $r + si = 0 + 3i$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $17$  is degree 0, we have:

$$\begin{aligned} Y_p(t) &= t^m [GP_0] \cos(3t) + t^m [GP_0] \sin(3t) \\ &= t^0(A) \cos(3t) + t^0(B) \sin(3t) = A \cos(3t) + B \sin(3t) \end{aligned}$$

Then  $Y'_p = -3A \sin(3t) + 3B \cos(3t)$  and  $Y''_p = -9A \cos(3t) - 9B \sin(3t)$ . We plug these into the DE to get

$$\begin{aligned} -9A \cos(3t) - 9B \sin(3t) + 2(-3A \sin(3t) + 3B \cos(3t)) + 2(A \cos(3t) + B \sin(3t)) &= 17 \cos(3t) \\ (-7A + 6B) \cos(3t) + (-6A - 7B) \sin(3t) &= 17 \cos(3t) \end{aligned}$$

Therefore  $-7A + 6B = 17$  and  $-6A - 7B = 0$ , giving  $A = -\frac{7}{5}$  and  $B = \frac{6}{5}$ .

Then  $Y_p(t) = -\frac{7}{5} \cos(3t) + \frac{6}{5} \sin(3t)$ .

Note: Because the fundamental set for the homogeneous version is  $\{e^{-t} \cos(t), e^{-t} \sin(t)\}$  the general solution is  $Y(t) = -\frac{7}{5} \cos(3t) + \frac{6}{5} \sin(3t) + C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$

## 6. Unfinished Examples:

**Example:** Consider  $y'' + 2y' + y = t^2e^{-t} + 17te^t \cos(3t)$ .

Forcing:  $f(t) = t^2e^{-t} + 17te^t \cos(3t)$  which has two parts.

CP:  $p(z) = z^2 + 2z + 1 = (z + 1)^2$ .

For the  $t^2e^{-t}$  part we have  $r = -1$  which is a root of  $p(z)$  of multiplicity 2. Because the coefficient polynomial  $t^2$  is degree 2 we have:

$$\text{First Part: } Y_p(t) = t^m [GP_2] e^{-t} = t^2(At^2 + Bt + C)e^{-t} = (At^4 + Bt^3 + Ct^2)e^{-t}$$

For the  $17te^t \cos(3t)$  part we have  $r + si = 1 + 3i$  which is not a root of  $p(z)$  so  $m = 0$ . Because the coefficient polynomial  $17t$  is degree 1 we have:

$$\begin{aligned} \text{Second Part: } Y_p(t) &= t^m [GP_1] e^t \cos(3t) + t^m [GP_1] e^t \sin(3t) \\ &= t^0(Dt + E)e^t \cos(3t) + t^0(Ft + G)e^t \sin(3t) \\ &= (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t) \end{aligned}$$

Combining these we have:

$$Y_p(t) = (At^4 + Bt^3 + Ct^2)e^{-t} + (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t)$$

## MATH 246: Chapter 2 Section 7: Variation of Parameters

Justin Wyss-Gallifent

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Main Topics:

- Restrictions.
  - General Idea and Method.
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### 1. Introduction

For the last chapter we've been focusing on finding a single solution  $Y_P(t)$  to a non-homogeneous linear differential equation with constant coefficients where  $f(t)$  is a familiar form.

What we're going to do now is remove both the restriction that the coefficients be constant and the restriction that  $f(t)$  is of our familiar form. We will restrict to second-order though, and we'll make sure the coefficient of  $y''$  is 1 (linear normal form), which can easily be attained through division. The goal will be the same, to find some  $Y_P(t)$ , because again the general solution will be  $Y(t) = Y_p(t) + c_1 Y_1(t) + c_2 Y_2(t)$  where  $\{Y_1(t), Y_2(t)\}$  is the fundamental set for the homogeneous version.

It may seem like if we could do this then why would we need the previous chapter? The answer is that the method of this section gets extremely complicated for third and higher order and can be computationally intensive even for second order. However it has the main advantage of providing a formulaic solution.

### 2. General Idea

The general idea is to start with the fundamental set  $\{Y_1, Y_2\}$  for the homogeneous version and ask a simple question - is it possible to find two function  $u_1(t)$  and  $u_2(t)$  such that  $y = u_1 Y_1 + u_2 Y_2$  is a solution to the nonhomogeneous version?

It turns out that simply plugging this  $y$  into the DE leaves us with quite a mess:

$$\begin{aligned} y'' + a(t)y' + b(t)y &= f(t) \\ (u_1 Y_1 + u_2 Y_2)'' + a(t)(u_1 Y_1 + u_2 Y_2)' + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ \underbrace{(u_1' Y_1 + u_1 Y_1' + u_2' Y_2 + u_2 Y_2')' + a(t)(u_1' Y_1 + u_1 Y_1' + u_2' Y_2 + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2)}_{\text{Quite a Mess!}} &= f(t) \end{aligned}$$

However if we look at this mess we notice that  $u_1' Y_1 + u_2' Y_2$  shows up in two places. It turns out that if  $u_1' Y_1 + u_2' Y_2 = 0$  then the above "Quite a Mess" tidies up:

$$\begin{aligned} (u_1 Y_1' + u_2 Y_2')' + a(t)(u_1 Y_1' + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ u_1' Y_1' + u_1 Y_1'' + u_2' Y_2' + u_2 Y_2'' + a(t)(u_1 Y_1' + u_2 Y_2') + b(t)(u_1 Y_1 + u_2 Y_2) &= f(t) \\ u_1 \underbrace{(Y_1'' + a(t)Y_1' + b(t)Y_1)}_{= 0 \text{ bc homog soln}} + u_2 \underbrace{(Y_2'' + a(t)Y_2' + b(t)Y_2)}_{= 0 \text{ bc homog soln}} + u_1' Y_1' + u_2' Y_2' &= f(t) \\ u_1' Y_1' + u_2' Y_2' &= f(t) \end{aligned}$$

The practical upshot of all this is that if we can find  $u_1$  and  $u_2$  satisfying the system

$$\begin{aligned}u_1' Y_1 + u_2' Y_2 &= 0 \\u_1' Y_1' + u_2' Y_2' &= f(t)\end{aligned}$$

then  $Y_p = u_1 Y_1 + u_2 Y_2$  will be a solution to the nonhomogeneous DE.

Conveniently this is easy to solve because it's a system of two equations and two unknowns where the unknowns are  $u_1'$  and  $u_2'$ .

Linear algebra gives us a generic formula because this is a matrix equation:

$$\begin{aligned}\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} Y_2' & -Y_2 \\ -Y_1' & Y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} -Y_2 f(t) \\ Y_1 f(t) \end{bmatrix}\end{aligned}$$

Thus:

$$u_1' = -\frac{Y_2 f(t)}{W[Y_1, Y_2]} \text{ and } u_2' = \frac{Y_1 f(t)}{W[Y_1, Y_2]}$$

Then

$$u_1 = -\int \frac{Y_2 f(t)}{W[Y_1, Y_2]} dt \text{ and } u_2 = \int \frac{Y_1 f(t)}{W[Y_1, Y_2]} dt$$

and the final result is:

$$Y_p = u_1 Y_1 + u_2 Y_2$$

It's worth noting that it's sometimes messy but we can directly write down a formula for  $Y_p(t)$ :

$$Y_p(t) = -Y_1 \int \frac{Y_2 f(t)}{W[Y_1, Y_2]} dt + Y_2 \int \frac{Y_1 f(t)}{W[Y_1, Y_2]} dt$$

### 3. Examples

**Example:** Consider  $y'' + y = \sec t$ .

Since the characteristic polynomial is  $z^2 + 1$  with roots  $0 \pm 1i$  the fundamental set for the homogeneous version is  $\{\cos t, \sin t\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and then we simply evaluate:

$$\begin{aligned} u_1 &= -\int \frac{(\sin t)(\sec t)}{1} dt = -\int \tan t dt = \ln |\cos t| + C && \text{Choose } u_1 = \ln |\cos t| \\ u_2 &= \int \frac{(\cos t)(\sec t)}{1} dt = \int 1 dt = t + C && \text{Choose } u_2 = t \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = (\ln |\cos t|) \cos t + t \sin t$$

and the general solution to the nonhomogeneous version is

$$Y_p(t) = (\ln |\cos t|) \cos t + t \sin t + c_1 \cos t + c_2 \sin t$$

**Example:** Consider  $y'' - 3y' + 2y = t$ .

Since the characteristic polynomial is  $z^2 - 3z + 2 = (z - 1)(z - 2)$  with roots 1, 2 the fundamental set for the homogeneous version is  $\{e^t, e^{2t}\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

and then we simply evaluate (some IBP here):

$$\begin{aligned} u_1 &= -\int \frac{e^{2t}t}{e^{3t}} dt = -\int te^{-t} dt = te^{-t} - e^{-t} + C && \text{Choose } u_1 = te^{-t} - e^{-t} \\ u_2 &= \int \frac{e^t t}{e^{3t}} dt = \int te^{-2t} dt = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C && \text{Choose } u_2 = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = (te^{-t} - e^{-t}) e^t + \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) e^{2t} = \frac{1}{2}t - \frac{3}{4}$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{2}t - \frac{3}{4} + c_1 e^t + c_2 e^{3t}$$

Side Note: The Method of Undetermined Coefficients is much nicer for this problem.

**Example:** Consider  $(t^2 + 1)y'' - 2ty' + 2y = (t^2 + 1)^2$ .

First we rewrite as  $y'' - \frac{2t}{t^2+1}y' + \frac{2}{t^2+1}y = t^2 + 1$ . It's worth noting that even though this looks uglier the only thing it affects that we need is the right side. We have no technique for finding the fundamental set for the homogeneous version so I'll just give it to you, it's  $\{t, t^2 - 1\}$ .

We find

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2 - 1 \\ 1 & 2t \end{vmatrix} = t^2 + 1$$

and then we simply evaluate:

$$\begin{aligned} u_1 &= -\int \frac{(t^2-1)(t^2+1)}{t^2+1} dt = -\int t^2 - 1 dt = -\frac{1}{3}t^3 + t + C && \text{Choose } u_1 = -\frac{1}{3}t^3 + t \\ u_2 &= \int \frac{(t)(t^2+1)}{t^2+1} dt = \int t dt = \frac{1}{2}t^2 + C && \text{Choose } u_2 = \frac{1}{2}t^2 \end{aligned}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = \left(-\frac{1}{3}t^3 + t\right)t + \left(\frac{1}{2}t^2\right)(t^2 - 1) = \frac{1}{6}t^4 + \frac{1}{2}t^2$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 + c_1 t + c_2(t^2 - 1)$$

We can make this a pretty nice IVP by adding the condition  $Y(1) = 0$  and  $Y'(1) = 1$ . Since  $Y'(t) = \frac{2}{3}t^3 + t + c_1 + 2c_2 t$  we then have

$$\begin{aligned} Y(1) &= \frac{1}{6} + \frac{1}{2} + c_1 = 0 \\ Y'(1) &= \frac{2}{3} + 1 + c_1 + 2c_2 = 1 \end{aligned}$$

so then  $c_1 = -\frac{2}{3}$  and  $c_2 = 0$  so the specific solution to the IVP is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 - \frac{2}{3}t$$

## MATH 246: Chapter 2 Section 8: Mechanical Vibration

### Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Notes
  - Unforced and Undamped Motion
  - Unforced and Damped Motion
  - Forced Motion
- 

#### 1. Introduction

Important: Positive is up and negative is down.

Imagine a spring hanging with no weight on it. We then attach a mass  $m$  which stretches the spring a distance of  $y_R < 0$ . We are now at the rest point. At this point the force of gravity is  $mg$  (negative since  $g < 0$ , think  $g = -9.8$  if it helps) and the force of the spring by Hooke's Law is  $-ky_R$  (the spring force is upwards so we negate against  $y_R < 0$ ). Consequently because we are at rest we have  $mg + (-ky_R) = 0$  and so  $mg = ky_R$ .

Now then, imagine the object and spring system is in motion and at any time  $t$  the displacement from the rest point is given by  $y(t)$ . At any instant now there could be multiple forces acting on the object:

Gravity	$mg$	Acting downwards with $g < 0$ .
Spring	$-k(y + y_R)$	Acting against the displacement.
Damping	$-\gamma y'$	Acting against and proportional to velocity, here $\gamma > 0$ .
External	$f(t)$	Some other external force.

When we put these all together we get:

$$\begin{aligned}F_{Tot} &= F_{Grav} + F_{Spring} + F_{Damping} + F_{External} \\my'' &= mg - k(y + y_R) - \gamma y' + f(t) \\my'' &= ky_r - ky - ky_R - \gamma y' + f(t) \\my'' &= -ky - \gamma y' + f(t)\end{aligned}$$

We finally rewrite this as:

$$my'' + \gamma y' + ky = f(t)$$

If this doesn't look familiar then you've been asleep!

#### 2. A Few Notes

- In the Metric system we may either have length, time, mass and force in meters, seconds, kilograms and newtons ( $\text{newton} = \text{kg} \cdot \text{meter}/\text{s}^2$ ) respectively, or in centimeters, seconds, grams and dynes ( $\text{dyne} = \text{g} \cdot \text{cm}/\text{s}^2$ ) respectively. In the British system we may have feet, seconds, slugs and pounds ( $\text{lb} = \text{slug} \cdot \text{ft}/\text{s}^2$ ). Note also in the British system that weight is also in pounds with  $\text{lb} = \text{slug} \cdot \text{gravity}$ .
- If  $k$  is not given we may need to find it using  $mg = ky_R$ . We would be given the mass  $m$  of the object and the displacement  $y_R$ . We can then find  $k$ . For example if an object of mass 2 kilograms displace a spring 0.5 meters downwards then  $(2)(-9.8) = k(-0.5)$ .
- If  $\gamma$  is not given we may need to find it using  $F_{Damping} = \gamma y'$ . We would be given the damping force for a certain velocity. For example if a mass traveling at  $0.1\text{m/s}$  upwards invokes a damping force of  $0.3\text{N}$  downwards then  $-0.3 = -\gamma(0.1)$ .

### 3. Unforced and Undamped

The simplest situation is when there is no external force and no damping. In this case we have  $my'' + ky = 0$ . The characteristic polynomial has roots  $0 \pm i\sqrt{\frac{k}{m}}$  and so the solution is given by

$$y(t) = c_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m}}\right)$$

This can be rewritten using the Subtraction Formula for Cosine as

$$y(t) = A \cos\left(t\sqrt{\frac{k}{m}} - \delta\right)$$

where  $A = \sqrt{c_1^2 + c_2^2}$  is the amplitude and  $\delta$  satisfies  $\cos \delta = \frac{c_1}{A}$  and  $\sin \delta = \frac{c_2}{A}$ . The graph of this makes good sense for a spring that's bouncing up and down forever.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$ . It is pulled down  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards.

We have  $0.4y'' + 0.1y = 0$  or  $y'' + 0.25y = 0$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

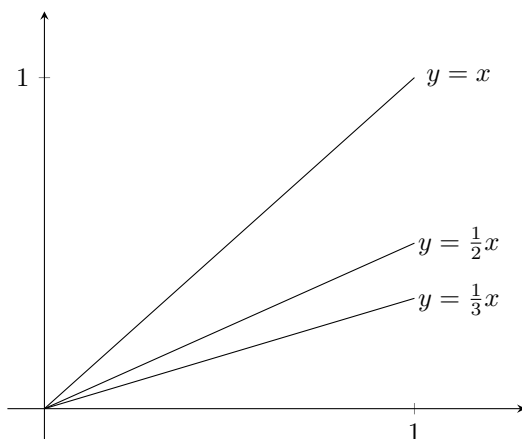
The characteristic polynomial has roots  $0 \pm i\sqrt{\frac{0.1}{0.4}} = 0 \pm 0.5i$ . The general solution is then

$$y(t) = c_1 \cos 0.5t + c_2 \sin 0.5t$$

For the initial value observe  $y'(t) = -0.5c_1 \sin 0.5t + 0.5c_2 \cos 0.5t$  and so  $y(0) = c_1 = -0.2$  and  $y'(0) = 0.5c_2 = -0.3$  so  $c_2 = -0.6$ . This gives us the specific solution

$$y(t) = -0.2 \cos 0.5t - 0.6 \sin 0.5t$$

The amplitude is  $A = \sqrt{(-0.2)^2 + (0.6)^2} = \sqrt{0.4} \approx 0.63$ . We can even draw a nice sketch.



Sketch omitted, but this starts at  $(0, -0.2)$  with a slope of  $-0.3$  and settles into a regular oscillation.



#### 4. Unforced with Damping

Now we have  $my'' + \gamma y' + ky = 0$ . The characteristic polynomial has roots  $-\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$ . the behavior of this depends strongly on  $\gamma^2 - 4mk$ .

- (a) **Underdamped:** When  $\gamma^2 - 4mk < 0$  (meaning the damping coefficient is small) we have complex roots and our solution has both exponential and trigonometric components. The function starts out oscillating but then the amplitude drops, limiting to zero.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.15$ . It is pulled down  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards.

We have  $0.4y'' + 0.15y' + 0.1y = 0$  or  $y'' + 0.375y' + 0.25y = 0$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

The characteristic polynomial has roots  $z = \frac{-0.15 \pm \sqrt{(0.15)^2 - 4(0.4)(0.1)}}{2(0.4)} = -0.1875 \pm \frac{\sqrt{0.1375}}{0.8}i$  The general solution is then

$$y(t) = c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{0.8}t\right) + c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{0.8}t\right)$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, -0.2)$  with a slope of  $-0.3$  and settles into an oscillation which reduces over time and limits to zero.

- (b) **Critically Damped:** When  $\gamma^2 - 4mk = 0$  we have a real root of multiplicity two. This is the special critically damped case. It corresponds to the smallest possible  $\gamma$  for which the oscillation stops.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.15625$  in a fluid with damping coefficient  $\gamma = 0.5$ . It is pulled down  $0.7m$  and released with zero velocity.

We have  $0.4y'' + 0.5y' + 0.15625y = 0$  or  $y'' + 1.25y' + 0.390625y = 0$  with  $y(0) = -0.7$  and  $y'(0) = 0$ .

The characteristic polynomial  $z^2 + 1.25z + 0.390625$  has a single root of multiplicity 2 as it factors as  $(z + 0.625)^2$ . The general solution is then

$$y(t) = c_1e^{-0.625t} + c_2te^{-0.625t}$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, -0.7)$  with a slope of 0 and heads directly but asymptotically to the  $t$ -axis.

- (c) **Overdamped:** When  $\gamma^2 - 4mk > 0$  we have two real roots and the system is overdamped.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.5$ . It is pushed up  $0.6m$  and released with zero velocity.

We have  $0.4y'' + 0.5y' + 0.1y = 0$  or  $y'' + 1.25y' + 0.25y = 0$  with  $y(0) = 0.6$  and  $y'(0) = 0$ .

The characteristic polynomial  $z^2 + 1.25z + 0.25$  factors as  $(z + 1)(z + 0.25)$  with roots  $-1, -0.25$  and hence the general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{-0.25t}$$

For the initial value observe  $y'(t) = -c_1 e^{-t} - 0.25c_2 e^{-0.25t}$  and so  $y(0) = c_1 + c_2 = 0.6$  and  $y'(0) = -c_1 - 0.25c_2$ . This yields  $c_2 = 0.8$  and  $c_1 = -0.2$ . This gives us the specific solution

$$y(t) = -0.2e^{-t} + 0.8e^{-0.25t}$$

We can even draw a nice sketch.

Sketch omitted, but this starts at  $(0, 0.6)$  with a slope of 0 and heads directly but asymptotically to the  $t$ -axis.

- (d) **A Note on Critically Damped vs. Overdamped:** These two functions look very similar. The critical thing to note is that a damped system oscillates, an overdamped system doesn't, and a critically damped system doesn't either but lies right on the edge of the other two.

## 5. Forced

With forced motions  $f(t) \neq 0$  and all bets are off. We know we need to find a particular solution  $Y_p$  and then add the general solution to the homogeneous system. This makes sense because the system is governed by both that forcing function and the usual spring motion stuff. In general the behavior will look springy at the start, although the damping might suppress this a bit, and in the long term (assuming damping) will look as if only the forcing function is acting on it.

**Example:** A mass of  $0.4kg$  hangs from a spring with coefficient  $k = 0.1$  in a fluid with damping coefficient  $\gamma = 0.15$ . It is pulled up  $0.2m$  from resting and released at a rate of  $0.3m/s$  downwards. An additional external force  $f(t) = 0.3$  acts downwards on it.

We have  $0.4y'' + 0.15y' + 0.1y = -0.3$  or  $y'' + 0.375y' + 0.25y = -0.75$  with  $y(0) = -0.2$  and  $y'(0) = -0.3$ .

The Method of Undetermined Coefficients gives us one solution  $Y_p(t) = 3$ . The homogeneous version is an earlier problem with general solution

$$c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2}t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2}t\right)$$

and hence the general solution to our forced problem is

$$Y(t) = -3 + c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2}t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2}t\right)$$

Notice that in the long term the exponentials take complicated part to zero so that  $\lim_{t \rightarrow \infty} Y(t) = 3$ . So in the long term only the forcing function remains acting on it.

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at  $(0, 0.2)$  with slope  $-0.3$  and oscillates with reducing amplitude as it settles down and limits to  $y = -3$ .

## MATH 246: Chapter 2 Section 9: Laplace Transforms

Justin Wyss-Gallifent

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Main Topics:

- Formal Definition
  - Rules
  - Reversing the Rules
  - Derivative Rules
  - Solving IVPs
  - Step Functions with Laplace Transforms
- 

1. **Introduction:** Laplace transforms are a way of changing one function into another function. Basically we start with a function of  $t$  and change it to a function of  $s$ . We can also do the reverse. The Laplace transform has some really useful properties which will help us solve initial value problems.
2. **Formal Definition:** If  $f(t)$  is a function then the Laplace transform of this function is formally defined by:

$$\mathcal{L}[y(t)](s) = \int_0^{\infty} y(t)e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b y(t)e^{-st} dt$$

For an unknown  $y(t)$  we will often write  $\mathcal{L}[y(t)]$  or just  $\mathcal{L}[y]$  for readability. This formal definition is used to build a set of rules and the rules are what we'll use.

**Example:** If  $y(t) = 1$  then we get:

$$\begin{aligned}\mathcal{L}[1] &= \int_0^{\infty} 1e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{s}e^{-st} \right|_0^b \\ &= \lim_{b \rightarrow \infty} -\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s(0)} \\ &= \frac{1}{s}\end{aligned}$$

Thus  $\mathcal{L}[1] = \frac{1}{s}$ .

**Example:** If  $y(t) = t$  then we get:

$$\begin{aligned}
 \mathcal{L}[t] &= \int_0^{\infty} t e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} \int_0^b t e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} t \left( -\frac{1}{s} \right) e^{-st} \Big|_0^b - \int_0^b \left( -\frac{1}{s} \right) e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} t \left( -\frac{1}{s} \right) e^{-st} \Big|_0^b - \frac{1}{s^2} e^{-st} \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{b}{s} e^{-sb} - 0 \right] - \left[ \frac{1}{s^2} e^{-sb} - \frac{1}{s^2} \right] \\
 &= \frac{1}{s^2}
 \end{aligned}$$

Thus  $\mathcal{L}[1] = \frac{1}{s^2}$ .

### 3. Function Rules

Using this same approach we can prove the following rules for common functions:

Function	Example
$\mathcal{L}[0] = 0$	n/a
$\mathcal{L}[c] = \frac{c}{s}$	$\mathcal{L}[42] = \frac{42}{s}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}[t^3] = \frac{3!}{s^4}$
$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}[e^{5t}] = \frac{1}{s-5}$
$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$	$\mathcal{L}[\cos(7t)] = \frac{s}{s^2+49}$
$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$	$\mathcal{L}[\sin(7t)] = \frac{7}{s^2+49}$
$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}[e^{2t}t^4] = \frac{4!}{(s-2)^5}$
$\mathcal{L}[e^{at} \cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t} \cos(3t)] = \frac{s-5}{(s-5)^2+9}$
$\mathcal{L}[e^{at} \sin(bt)] = \frac{b}{(s-a)^2+b^2}$	$\mathcal{L}[e^{5t} \sin(3t)] = \frac{3}{(s-5)^2+9}$
$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$	$\mathcal{L}[2 + 5t] = \mathcal{L}[2] + 5\mathcal{L}[t] = \frac{2}{s} + 5\left(\frac{1}{s^2}\right)$

**Notation:** Sometimes if a function is denoted  $y(t)$  then some sources use  $Y(s)$  instead of  $\mathcal{L}[y(t)]$  for convenience. I will tend to not do this much.

4. **Reversing** If we start with  $\mathcal{L}[y]$  we can also work in reverse. Here are some examples with comments because sometimes we need to manipulate the function first. This can also be written with inverse notation.

**Example:** If  $\mathcal{L}[y] = \frac{2}{s}$  then  $y(t) = 2$ . aka  $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = s$ .

**Example:** If  $\mathcal{L}[y] = \frac{1}{s^5}$  then first we rewrite  $\mathcal{L}[y] = \frac{1}{4!} \left(\frac{4!}{s^5}\right)$  and then we see  $y(t) = \frac{1}{24}t^4$ . aka  $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \frac{1}{24}t^4$ .

**Example:** If  $\mathcal{L}[y] = \frac{1}{s+4}$  we think of it as  $\mathcal{L}[y] = \frac{1}{s-(-4)}$  then  $y(t) = e^{-4t}$ .

**Example:** If  $\mathcal{L}[y] = \frac{-18}{s^2+9}$  then first rewrite to get  $\mathcal{L}[y] = -6 \left(\frac{3}{s^2+9}\right)$  and then we see  $y(t) = -6 \sin(3t)$ .

**Example:** If  $\mathcal{L}[y] = \frac{2}{s^2+s}$  then first we see  $\mathcal{L}[y] = \frac{2}{s(s+1)}$  and then we need to rewrite with partial fractions first and then following this we need a bit more rewriting to fit the formulas so  $\mathcal{L}[y] = \frac{2}{s} - \frac{2}{s+1} = \frac{2}{s} - 2 \left(\frac{1}{s-(-1)}\right)$ , and then we see  $y(t) = 2 - 2e^{-t}$ .

**Example:** If  $\mathcal{L}[y] = \frac{4s+3}{s^2+25}$  then we need to break it up and rewrite a little to fit the formulas:  
 $\mathcal{L}[y] = \frac{4s+3}{s^2+25} = \frac{4s}{s^2+25} + \frac{3}{s^2+25} = 4 \left(\frac{s}{s^2+25}\right) + \frac{3}{5} \left(\frac{5}{s^2+25}\right)$  and then we see that  
 $y(t) = 4 \cos(5t) + \frac{3}{5} \sin(5t)$ .

**Example:** If  $\mathcal{L}[y] = \frac{s+1}{s^2-4s+5}$  then the denominator doesn't factor so instead we complete the square and then do a bit more rewriting to get  $\mathcal{L}[y] = \frac{s+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$  and then we see that  $y(t) = e^{2t} \cos(t) + 3e^{2t} \sin(t)$ .

## 5. Derivative Rules

It turns out that the Laplace transfer is nice with derivatives of functions too.

**Example:** Observe that:

$$\begin{aligned}
 \mathcal{L}[y'(t)] &= \lim_{b \rightarrow \infty} \int_0^b y'(t) e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} \left. y(t) e^{-st} \right|_0^b + s \int_0^b y(t) e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} [y(b) e^{-sb} - y(0)] + s \mathcal{L}[y(t)] \\
 &= -y(0) + s \mathcal{L}[y(t)] \\
 &= s \mathcal{L}[y(t)] - y(0)
 \end{aligned}$$

In general we have the following pattern for an unknown  $y(t)$ :

$$\begin{array}{ll}
 \mathcal{L}[y'] = s \mathcal{L}[y(t)] - y(0) & = s \mathcal{L}[y] - y(0) \\
 \mathcal{L}[y''] = s^2 \mathcal{L}[y(t)] - sy(0) - y'(0) & = s^2 \mathcal{L}[y] - sy(0) - y'(0) \\
 \mathcal{L}[y'''] = s^3 \mathcal{L}[y(t)] - s^2 y(0) - sy'(0) - y''(0) & = s^3 \mathcal{L}[y] - s^2 y(0) - sy'(0) - y''(0) \\
 \vdots & \vdots
 \end{array}$$

For example if  $y(t)$  is unknown but we know  $y(0) = 7$  and  $y'(0) = -3$  then the second rule tells us that

$$\begin{aligned}
 \mathcal{L}[y''] &= s^2 \mathcal{L}[y] - sy(0) - y'(0) \\
 &= s^2 \mathcal{L}[y] - s(7) - (-3) \\
 &= s^2 \mathcal{L}[y] - 7s + 3
 \end{aligned}$$

We'll see very soon why this is significant.



## 6. Solving Initial Value Problems

Laplace Transforms are incredibly useful for dealing with IVPs when  $t_I = 0$ . Other values of  $t_I$  can be dealt with using a function shift but we won't deal with those here.

When dealing with such an initial value problem our approach will be the following:

- (a) Take the Laplace transform of each side.
- (b) Apply the rules for functions and for derivatives to eliminate all the  $t$ , all the derivatives and substitute all the initial values.
- (c) Solve the result for  $\mathcal{L}[y]$ .
- (d) Reverse the Laplace transform to get the solution  $y(t)$ .

**Example:** Suppose we have  $y' = 3$  with  $y(0) = 1$ . We do the following:

$$\begin{aligned}y' &= 3 \\ \mathcal{L}[y'] &= \mathcal{L}[3] \\ s\mathcal{L}[y] - y(0) &= \frac{3}{s} \\ s\mathcal{L}[y] - 1 &= \frac{3}{s} \\ s\mathcal{L}[y] &= \frac{3}{s} + 1 \\ \mathcal{L}[y] &= \frac{3}{s^2} + \frac{1}{s} \\ y(t) &= 3t + 1\end{aligned}$$

And we've solved it! Notice that you really need to understand how the various tables are being used here. The Laplace transform table is used at the beginning and end and the derivative rules are also used early on.

**Example:** Suppose we have  $y'' - 2y' - 3y = 0$  with  $y(0) = 1$  and  $y'(0) = 4$ . We do the following:

$$\begin{aligned}
 y'' - 2y' - 3y &= 0 \\
 \mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[0] \\
 (s^2\mathcal{L}[y] - sy(0) - y'(0)) - 2(s\mathcal{L}[y] - y(0)) - 3\mathcal{L}[y] &= 0 \\
 s^2\mathcal{L}[y] - s - 4 - 2s\mathcal{L}[y] + 2 - 3\mathcal{L}[y] &= 0 \\
 \mathcal{L}[y](s^2 - 2s - 3) - s - 2 &= 0 \\
 \mathcal{L}[y](s^2 - 2s - 3) &= s + 2 \\
 \mathcal{L}[y] &= \frac{s + 2}{s^2 - 2s - 3} \\
 \mathcal{L}[y] &= \frac{s + 2}{(s - 3)(s + 1)}
 \end{aligned}$$

Now we need to do some manipulation with partial fractions:

$$\begin{aligned}
 \frac{s + 2}{(s - 3)(s + 1)} &= \frac{A}{s - 3} + \frac{B}{s + 1} \\
 s + 2 &= A(s + 1) + B(s - 3)
 \end{aligned}$$

At this point  $s = -1$  gives us  $B = -1/4$  and  $s = 3$  gives us  $A = 5/4$ . Back to our problem with the most recent line rewritten:

$$\begin{aligned}
 \mathcal{L}[y] &= \frac{s + 2}{(s - 3)(s + 1)} \\
 \mathcal{L}[y] &= \frac{5/4}{s - 3} + \frac{-1/4}{s + 1} \\
 \mathcal{L}[y] &= \frac{5}{4} \left( \frac{1}{s - 3} \right) - \frac{1}{4} \left( \frac{1}{s - (-1)} \right) \\
 y(t) &= \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t}
 \end{aligned}$$

**Example:** Suppose we have  $y'' + 4y = 2t$  with  $y(0) = 1$  and  $y'(0) = 0$ . We do the following:

$$\begin{aligned}
 y'' + 4y &= 2t \\
 \mathcal{L}[y''] + 4\mathcal{L}[y] &= \mathcal{L}[2t] \\
 s^2\mathcal{L}[y] - sy(0) - y'(0) + 4\mathcal{L}[y] &= \frac{2}{s^2} \\
 s^2\mathcal{L}[y] - s - 0 + 4\mathcal{L}[y] &= \frac{2}{s^2} \\
 \mathcal{L}[y](s^2 + 4) &= \frac{2}{s^2} + s \\
 \mathcal{L}[y] &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4}
 \end{aligned}$$

This doesn't look so nice. The second part is okay (it's from  $\cos$ ) but the first part is not in our table. Instead we need to break it up with partial fractions:

$$\begin{aligned}
 \frac{2}{s^2(s^2 + 4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4} \\
 2 &= As(s^2 + 4) + B(s^2 + 4) + (Cs + D)s^2 \\
 2 &= (A + C)s^3 + (B + D)s^2 + 4As + 4B
 \end{aligned}$$

Comparing coefficients gives us  $A + C = 0$ ,  $B + D = 0$ ,  $4A = 0$  and  $4B = 2$  so that  $B = 1/2$ ,  $A = 0$ ,  $D = -1/2$  and  $C = 0$  and so back our process with the most recent line rewritten:

$$\begin{aligned}
 \mathcal{L}[y] &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4} \\
 \mathcal{L}[y] &= \frac{1/2}{s^2} + \frac{-1/2}{s^2 + 4} + \frac{s}{s^2 + 4} \\
 \mathcal{L}[y] &= \frac{1}{2} \left( \frac{1}{s^2} \right) - \frac{1}{4} \left( \frac{2}{s^2 + 4} \right) + \frac{s}{s^2 + 4} \\
 y(t) &= \frac{1}{2}t - \frac{1}{4}\sin(2t) + \cos(2t)
 \end{aligned}$$

Compare this to before where we'd need to find the general solution to the homogeneous version of the differential equation, also find a specific solution to the nonhomogeneous version, add them, then use the initial values to find the constants. This way is significantly faster.

Sometimes it's good practice just to do the first part of a problem:

**Example:** Find the Laplace Transform of the solution to the initial value problem  $y'' + y' - 3y = t + e^{2t} \cos(3t)$  with  $y(0) = -1$  and  $y'(0) = 2$ .

Here all we need to get to is  $\mathcal{L}[y]$ . We do the following:

$$\begin{aligned}
 y'' + y' - 3y &= t + e^{2t} \cos(3t) \\
 \mathcal{L}[y''] + \mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[t] + \mathcal{L}[e^{2t} \cos(3t)] \\
 s^2 \mathcal{L}[y] - sy(0) - y'(0) + s\mathcal{L}[y] - y(0) - 3\mathcal{L}[y] &= \frac{s-2}{(s-2)^2+9} \\
 s^2 \mathcal{L}[y] + s - 2 + s\mathcal{L}[y] + 1 - 3\mathcal{L}[y] &= \frac{2-s}{(s-2)^2+9} \\
 \mathcal{L}[y](s^2 + s - 3) + s - 1 &= \frac{2-s}{(s-2)^2+9} \\
 \mathcal{L}[y](s^2 + s - 3) &= \frac{2-s}{(s-2)^2+9} + 1 - s \\
 \mathcal{L}[y] &= \frac{2-s}{((s-2)^2+9)(s^2+s-3)} + \frac{1-s}{s^2+s-3}
 \end{aligned}$$

To finish, this would need to undergo a partial fractions decomposition and then the rules would need to be applied.

7. **Step Functions** The most basic step function is the function which returns 0 up until (but not including)  $t = 0$  and then 1 after that. More specifically we have

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

There are other options. If we want to use a value other than 0 we denote it  $u_c(t)$ :

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Step functions are useful because they turn other functions on and off. For example the product function  $u_\pi(t) \sin(t - \pi)$  is 0 for  $t < \pi$  and  $\sin(t - \pi)$  for  $t \geq \pi$ .

It may seem odd that we have  $\sin(t - \pi)$  here rather than just  $\sin(t)$  but there's a reason why this will usually happen. When a function "kicks in" at a certain  $t$ -value this usually means that before that  $t$ -value the function is 0 and then at that  $t$ -value the function begins as though 0 were plugged into it. So for example  $u_\pi(t) \sin(t - \pi)$  equals 0 until  $t = \pi$  at which point the  $\sin(t - \pi)$  part starts behaving as if 0 were plugged in (because of the  $t - \pi$  in there).

**Example:** Suppose a function equals 0 until  $t = \pi/4$  and then starts behaving like the sine function, meaning like the sine function does at  $t = 0$ . This new function would be  $u_{\pi/4}(t) \sin(t - \pi/4)$ .

**Example:** Suppose a function equals 0 until  $t = 3$  and then starts behaving like the exponential function  $e^t$ , meaning like the exponential function does at  $t = 0$ . This new function would be  $u_3(t)e^{t-3}$ .

## 8. Laplace Transforms and Step Functions

Step functions have the following Laplace transform related behavior:

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs}\mathcal{L}[f(t)]$$

This is a bit confusing so for the forward and backwards directions think:

**Forward:** Pull out the  $u_c(t)$  which changes to  $e^{-cs}$  and change all  $t - c$  to  $t$  then continue.

**Example:**  $\mathcal{L}[u_3(t)(t - 3)^5] = e^{-3s}\mathcal{L}[t^5] = e^{-3s}\left(\frac{5!}{s^6}\right)$

**Example:**  $\mathcal{L}[u_\pi(t)\sin(t - \pi)] = e^{-\pi s}\mathcal{L}[\sin(t)] = e^{-\pi s}\left(\frac{1}{s^2 + 1}\right)$

**Example:**  $\mathcal{L}[u_2(t)e^{4(t-2)}(t - 2)^5] = e^{-2s}\mathcal{L}[e^{4t}t^5] = e^{-2s}\left(\frac{5!}{(s-4)^6}\right)$

**Backward:** For  $\mathcal{L}[y] = e^{-cs}J(s)$  first find  $j(t)$  with  $\mathcal{L}[j(t)] = J(s)$ , replace the  $t$  by  $t - c$  and put a  $u_c(t)$  in front.

**Example:** If  $\mathcal{L}[y] = e^{-5s}\left(\frac{s}{s^2 + 49}\right)$  then we note  $\mathcal{L}[\cos(7t)] = \frac{s}{s^2 + 49}$ , replace the  $t$  by  $t - 5$  and put  $u_5(t)$  in front, yielding  $y(t) = u_5(t)\cos(7(t - 5))$ .

**Example:** If  $\mathcal{L}[y] = e^{3s}\left(\frac{5!}{s^6}\right)$  then we note  $\mathcal{L}[t^5] = \frac{5!}{s^6}$ , replace the  $t$  by  $t - (-3)$  and put  $u_{(-3)}(t)$  in front, yielding  $y(t) = u_{(-3)}(t)(t + 3)^5$ .

**Example:** If  $\mathcal{L}[y] = e^{-5s}\left(\frac{6!}{(s-3)^7}\right)$  then we note  $\mathcal{L}[e^{3t}t^6] = \frac{6!}{(s-3)^7}$ , replace the  $t$  by  $t - 5$  and put  $u_5(t)$  in front, yielding  $y(t) = u_5(t)e^{3(t-5)}(t - 5)^6$ .

This can then be tied into initial value problems.

**Example:** Suppose  $y' - 2y = f(t)$  where

$$f(t) = \begin{cases} 0 & t < 3 \\ t - 3 & t \geq 3 \end{cases}$$

and where  $y(0) = 0$ .

We first note that  $f(t) = u_3(t)(t - 3)$  and then proceed:

$$\begin{aligned} y' - 2y &= u_3(t)(t - 3) \\ \mathcal{L}[y'] - 2\mathcal{L}[y] &= \mathcal{L}[u_3(t)(t - 3)] \\ s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] &= e^{-3s}\mathcal{L}[t] \\ s\mathcal{L}[y] - 2\mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2}\right) \\ \mathcal{L}[y](s - 2) &= e^{-3s}\left(\frac{1}{s^2}\right) \\ \mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2(s - 2)}\right) \\ &\dots\text{Partial Fractions Not Shown}\dots \\ \mathcal{L}[y] &= e^{-3s}\underbrace{\left(\frac{-1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s - 2}\right)}_{\text{Yields: } -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}} \\ y(t) &= u_3(t)\left(-\frac{1}{4} - \frac{1}{2}(t - 3) + \frac{1}{4}e^{2(t-3)}\right) \end{aligned}$$

**Example:** Suppose  $y'' - y' - 2y = f(t)$  where

$$f(t) = \begin{cases} 0 & t < 3 \\ 7 & t \geq 3 \end{cases}$$

and where  $y(0) = 0$  and  $y'(0) = -2$ .

We first note that  $f(t) = 7u_3(t)$  and then proceed:

$$\begin{aligned} y'' - y' - 2y &= 7u_3(t) \\ \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] &= 7\mathcal{L}[u_3(t)] \\ s^2\mathcal{L}[y] - sy(0) - y'(0) - (s\mathcal{L}[y] - y(0)) - 2\mathcal{L}[y] &= 7e^{-3t} \\ s^2\mathcal{L}[y] - s + 2 - s\mathcal{L}[y] - 2\mathcal{L}[y] &= 7e^{-3t} \\ \mathcal{L}[y](s^2 - s - 2) - s + 2 &= 7e^{-3t} \\ \mathcal{L}[y](s^2 - s - 2) &= 7e^{-3t} + s - 2 \\ \mathcal{L}[y] &= 7e^{-3t} \left( \frac{1}{s^2 - s - 2} \right) + \frac{s - 2}{s^2 - s - 2} \\ \mathcal{L}[y] &= 7e^{-3t} \left( \frac{1}{(s - 2)(s + 1)} \right) + \frac{1}{s + 1} \\ &\dots \text{Partial Fractions Not Shown} \dots \\ \mathcal{L}[y] &= 7e^{-3t} \underbrace{\left( \frac{1/3}{s - 2} - \frac{1/3}{s + 1} \right)}_{\text{Yields: } \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}} + \frac{1}{s + 1} \\ y(t) &= 7u_3(t) \left( \frac{1}{3}e^{2(t-3)} - \frac{1}{3}e^{-1(t-3)} \right) + e^{-t} \end{aligned}$$

**Example:** Suppose  $y'' - y' = f(t)$  where

$$f(t) = \begin{cases} 0 & t < \pi/4 \\ \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$$

and where  $y(0) = 0$  and  $y'(0) = 0$ . We first note that  $f(t) = u_{\pi/4}(t) \cos(t - \pi/4)$  and then proceed:

$$\begin{aligned} y'' - y' &= u_{\pi/4} \cos(t - \pi/4) \\ \mathcal{L}[y''] - \mathcal{L}[y'] &= \mathcal{L}[u_{\pi/4} \cos(t - \pi/4)] \\ (s^2 \mathcal{L}[y] - sy(0) - y'(0)) - (s \mathcal{L}[y] - y(0)) &= e^{-(\pi/4)s} \mathcal{L}[\cos(t)] \\ s^2 \mathcal{L}[y] - s \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{s^2 + 1} \right) \\ \mathcal{L}[y] (s^2 - s) &= e^{-(\pi/4)s} \left( \frac{s}{s^2 + 1} \right) \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{(s^2 + 1)(s^2 - s)} \right) \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{s}{(s^2 + 1)s(s - 1)} \right) \\ &\dots \text{Partial Fractions Not Shown} \dots \\ \mathcal{L}[y] &= e^{-(\pi/4)s} \left( \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{2}}{s - 1} \right) \\ \mathcal{L}[y] &= -\frac{1}{2} e^{-(\pi/4)s} \left( \frac{s + 1}{s^2 + 1} - \frac{1}{s - 1} \right) \\ \mathcal{L}[y] &= -\frac{1}{2} e^{-(\pi/4)s} \underbrace{\left( \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{1}{s - 1} \right)}_{\text{Yields: } \cos(t) + \sin(t) - e^t} \\ y(t) &= -\frac{1}{2} u_{\pi/4}(t) \left( \cos(t - \pi/4) + \sin(t - \pi/4) - e^{(t - \pi/4)} \right) \end{aligned}$$



## MATH 246: Chapter 3 Section 1: Intro to First Order Systems

Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Rewriting Single Higher Order as Systems
  - Multiple Tank Problems
- 

### 1. Introduction

All that we've studied so far are DEs involving a single function  $y$  depending on a single variable  $t$ . In the real world things can get far more complicated. As a classic example consider a predator-prey situation. The rate of growth of the prey depends on both the number of prey and the number of predators, possibly as well as time, and similarly for the rate of growth of the predators.

What we'll study now are systems of first-order DEs. In the most basic case we'll have two functions  $x_1(t)$  and  $x_2(t)$  in which their derivatives  $x'_1$  and  $x'_2$  depend on both  $x_1$  and  $x_2$  themselves and maybe on other functions of  $t$ .

The goal will be to find two functions  $x_1(t)$  and  $x_2(t)$  which simultaneously satisfy the system.

**Example:**

$$\begin{aligned}x'_1 &= x_1 - x_2 \\x'_2 &= -3x_1 - x_2\end{aligned}$$

In this example the pair  $x_1(t) = 2e^{-2t}$  and  $x_2(t) = 6e^{-2t}$  form a solution.

**Example:**

$$\begin{aligned}x'_1 &= 2t^2x_1 - 3x_2 + \cos t \\x'_2 &= x_1 + 4tx_2 - 2t\end{aligned}$$

In this example a solution pair is not easy at all.

We can also have more functions and equations.

**Example:**

$$\begin{aligned}x'_1 &= 2x_1 - 3x_2 - x_3 + \cos t \\x'_2 &= x_1 + 4x_2 - 2t \\x'_3 &= -x_1 + 20x_2 - 7x_3 + e^t\end{aligned}$$

In addition we could have an initial value, which would mean an initial value for each of the functions.

**Example:** Here is an IVP.

$$\begin{aligned}x'_1 &= tx_1 - 3x_2 \\x'_2 &= x_1 + 4t^2x_2\end{aligned}$$

With  $x_1(0) = 2$  and  $x_2(0) = -3$ .

## 2. Rewriting Single Higher-Order as Systems

Single higher-order DEs can be rewritten as systems of first-order DEs. This may be useful as we go on to develop methods of solving systems. The general idea for an  $n^{\text{th}}$  order DE will be to rewrite it as a system of  $n$  first-order DEs.

(a) When dealing with just the DE part it's simple. For an  $n^{\text{th}}$  order system we assign:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy \\x_3 &= D^2y \\&\vdots \\x_{n-1} &= D^{n-2}y \\x_n &= D^{n-1}y\end{aligned}$$

The first  $n - 1$  then give us:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= x_4 \\&\vdots \\x'_{n-1} &= x_n\end{aligned}$$

We get one more  $x'_n = \dots$  from the DE because  $x'_n = y^{(n)}$ , which we can find, and replacing all the other derivatives by their respective  $x_i$ .

**Example:** Consider  $D^2y + tDy - 3y = t$ . We assign:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy\end{aligned}$$

The first then gives us:

$$x'_1 = y' = Dy = x_2$$

The differential equation gives us:

$$x'_2 = D^2y = t - tDy - 3y = t - tx_2 - 3x_1$$

Thus our final system is:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= -3x_1 - tx_2 + t\end{aligned}$$

**Example:** Consider  $D^3y - 2D^2y + tDy - e^t y = \sin t$ . We assign and get:

$$\begin{aligned}x_1 &= y \\x_2 &= Dy \\x_3 &= D^2y\end{aligned}$$

The first two of these give us:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3\end{aligned}$$

The third  $x'_3 = \dots$  comes from the DE and is:

$$x'_3 = D^3y = 2D^2y - tDy + e^t y + \sin t = 2x_3 - tx_2 + e^t x_1 + \sin t$$

All together we have the system:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= 2x_3 - tx_2 + e^t x_1 + \sin t\end{aligned}$$

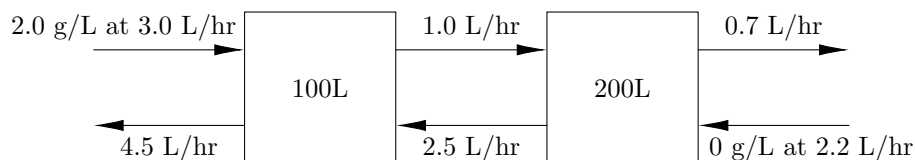
(b) With Initial Values:

Initial values are easy to rewrite. Since we know  $y(t_I), Dy(t_I), \dots, D^{n-1}y(t_I)$  these just become  $x_1(t_I), x_2(t_I), \dots, x_n(t_I)$ .

**Example:** Consider  $D^2y + tDy - 3y = t$  again. Suppose we know that  $y(1) = 2$  and  $y'(1) = -3$ . We set  $x_1 = y$  and  $x_2 = Dy$ . Then we know  $x_1(1) = 2$  and  $x_2(1) = y'(1) = -3$ .

### 3. Tank Problems

A classic example of these are tank problems. Imagine two tanks containing salt water. Water is being pumped into and out of these tanks in a variety of ways. For example in the following scenario there are two tanks. The one on the left contains 100L and the one on the right 200L. These quantities do not change in this example because the liters into each equals the gallons out.



The quantity 2.0 g/L at 3.0 L/hr indicates that salt water with that density is flowing into the left tank at that rate, and the quantity 0 g/L at 2.2 L/hr indicates the same for the right tank.

The other quantities do not have densities because they are assumed to be mixtures from the tank.

Imagine now that  $x_1$  is the amount of salt in the left tank and  $x_2$  is the amount of salt in the right tank, each at time  $t$ . Then the density of salt in the left tank is  $\frac{x_1}{100}$  and in the right tank is  $\frac{x_2}{200}$ . The entire scenario is then modeled by the system:

$$\begin{aligned}x_1' &= [\text{Rate of Salt In}] - [\text{Rate of Salt Out}] \\x_2' &= [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]\end{aligned}$$

which is:

$$\begin{aligned}x_1' &= +(2)(3) + (x_2/200)(2.5) - (x_1/100)(4.5) - (x_1/100)(1) \\x_2' &= +(x_1/100)(1) + (0)(2.2) - (x_2/200)(0.7) - (x_2/200)(2.5)\end{aligned}$$

This simplifies to:

$$\begin{aligned}x_1' &= -0.055x_1 + 0.0125x_2 + 6 \\x_2' &= 0.01x_1 - 0.0475x_2\end{aligned}$$

Suppose in addition we know that at time  $t = 0$  there is 10g of salt in the left tank and 20g of salt in the right tank. Then we can add in the initial value  $x_1(0) = 10$  and  $x_2(0) = 20$ .

**MATH 246: Chapter 3 Section 2: Basic Theory and Notation for Systems**  
**Justin Wyss-Gallifent**

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Main Topics:

- Linear Algebra Notation
  - Theory for Homogeneous
  - The Fundamental Matrix
  - Comment on Nonhomogeneous
- 

**1. Matrix and Vector Notation for Systems of First Order Linear DEs**

Consider the following:

- A first order linear system of differential equations may be written:

$$\bar{x}' = A(t)\bar{x} + \bar{f}(t)$$

- This system is homogeneous when  $\bar{f}(t) = \bar{0}$  and we say the system has constant coefficients when the matrix  $A(t)$  is all constants.
- An initial value can then be written as  $\bar{x}(t_I) = \bar{x}_I$ ,
- The solution can then be given as a single  $\bar{x}$ .

**Example:** Consider the following initial value problem:

$$\begin{aligned}x_1' &= 3x_1 + 2tx_2 + t & x_1(0) &= 1 \\x_2' &= t^2x_1 + 3x_2 & x_2(0) &= -2\end{aligned}$$

This can be rewritten as:

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\bar{x}'} = \underbrace{\begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} t \\ 0 \end{bmatrix}}_{\bar{f}(t)} \text{ with } \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\bar{x}(0)} = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\bar{x}_I}$$

Thus more simply:

$$\bar{x}' = \begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

**Example:** Consider the following solution:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

One solution to this is:

$$\bar{x} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$$

This can be checked with a matrix calculation:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 3e^{5t} + 2e^{5t} \\ 2e^{5t} + 3e^{5t} \end{bmatrix} = \begin{bmatrix} 5e^{5t} \\ 5e^{5t} \end{bmatrix} = \bar{x}'$$

## 2. Theory for Homogeneous - Fundamental Sets and Fundamental Matrices

Note: In what follows I've written  $n = 2$  to mean that I'm giving a specific example that generalizes. You could substitute  $n = 3, 4, \dots$  and the theory would still be good. In cases where it's not clear what happens for  $3, 4, \dots$  I've said more.

Theory: A homogeneous system of  $n = 2$  DEs has a fundamental pair/set consisting of  $n = 2$  solutions  $\bar{x}_1$  and  $\bar{x}_2$  (more if  $n \geq 3$ )

A fundamental set has nonzero Wronskian where

$$W[\bar{x}_1, \bar{x}_2] = |\bar{x}_1 \ \bar{x}_2|$$

That determinant is just found by dumping the vectors  $\bar{x}_1$  and  $\bar{x}_2$  together in a matrix and going from there.

When we have a fundamental pair/set the matrix used to determine the Wronskian is called the *fundamental matrix* and is usually denoted  $\Psi$  or  $\Psi(t)$ .

The general solution to the system then consists of all linear combinations of those  $n = 2$  solutions, this can be written several ways:

$$\bar{x}(t) = c_1 \bar{x}_1 + c_2 \bar{x}_2 = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t) \bar{c}$$

**Example:** Consider the system

$$\bar{x}' = \begin{bmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{bmatrix} \bar{x}$$

This has solutions (calculation omitted)  $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} 1 + t^3 \\ t \end{bmatrix}, \begin{bmatrix} t^2 \\ 1 \end{bmatrix} \right\}$ .

These form a fundamental pair because  $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} 1 + t^3 & t^2 \\ t & 1 \end{vmatrix} = 1 \neq 0$ .

Consequently the general solution to the system is:

$$\bar{x}(t) = c_1 \begin{bmatrix} 1 + t^3 \\ t^2 \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t^2 + c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + t^3 & t^2 \\ t^2 & 1 \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

It's worth noting that if we go back and think of the original problem as:

$$\begin{aligned} x_1' &= t^2 x_1 + (2t - t^4) x_2 \\ x_2' &= x_2 - t^2 x_2 \end{aligned}$$

Then the general solution is:

$$\begin{aligned} x_1 &= c_1(1 + t^3) + c_2 t^2 \\ x_2 &= c_1 t + c_2 \end{aligned}$$

**Example 1:** Consider the system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has solutions (calculation omitted)  $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}, \begin{bmatrix} e^t \\ -e^t \end{bmatrix} \right\}$ .

These form a fundamental pair because  $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{vmatrix} = -2e^{6t} \neq 0$ .

Consequently the general solution to the system is:

$$\bar{x}(t) = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{bmatrix} = \underbrace{\begin{bmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

If we turn this into an initial value problem with the initial condition:

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can find the specific solution using a simple matrix calculation:

$$\begin{aligned} \bar{x}(0) &= \Psi(0)\bar{c} \\ \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{c} \\ \bar{c} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= -\frac{1}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ \bar{c} &= \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Hence the specific solution can be written a few ways:

$$\bar{x}(t) = \Psi(t)\bar{c} = \begin{bmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{bmatrix} \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

It's worth noting that if we go back and think of the original problem as the IVP:

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 & x_1(0) &= 1 \\ x_2' &= 2x_1 + 3x_2 & x_2(0) &= 2 \end{aligned}$$

Then the specific solution is:

$$\begin{aligned} x_1 &= \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ x_2 &= \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{aligned}$$

### 3. Natural Fundamental Sets and Matrices

The natural fundamental matrix associated to a specific  $t_I$  is a specific fundamental matrix which is incredibly useful.

Suppose we solve the two initial value problems:

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{call this solution } x_1$$

and

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \text{call this solution } x_2$$

we get what are known together as the *natural fundamental set associated to  $t_I$*  and if we put these together in a matrix we get the *natural fundamental matrix associated to  $t_I$*  which is denoted  $\Phi(t)$  or just  $\Phi$ . There is only one of these for a given  $t_I$ .

**Example:** Consider the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has natural fundamental matrix associated to  $t_I = 0$  of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

This matrix  $\Phi(t)$  is incredibly useful. To see this, suppose we have a system given by

$$\bar{x}' = A(t)\bar{x}$$

If we have the natural fundamental matrix  $\Phi(t)$  associated to some  $t_I$  and we wish to solve the initial value problem with:

$$\bar{x}(t_I) = \bar{x}_I = \begin{bmatrix} a \\ b \end{bmatrix}$$

Consider the vector:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

Observe that by definition of matrix/vector multiplication we have:

$$\bar{x}(t) = \Phi(t)\bar{x}_I = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1 + b\bar{x}_2$$

Since  $\bar{x}$  is a linear combination of  $\bar{x}_1$  and  $\bar{x}_2$  it is a solution to the DE. Moreover observe that:



$$\bar{x}(t_I) = \Phi(t_I)\bar{x}_I = \Phi(t_I) \begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1(t_I) + b\bar{x}_2(t_I) = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \bar{x}_I$$

It follows that the solution to the IVP is simply given by:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

This is handy when we need to solve repeated initial value problems with the same  $t_I$ .

**Example:**

Revisit the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

We saw this has natural fundamental matrix associated to  $t_I = 0$  of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

So now we can throw out solutions to IVPs easily:

- If we have:  $\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Then the solution is:

$$\bar{x} = \Phi(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

- If we have:  $\bar{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

Then the solution is:

$$\bar{x} = \Phi(t) \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{5t} + 3e^t \\ e^{5t} - 3e^t \end{bmatrix}$$

What's even better is that if we have any fundamental matrix  $\Psi(t)$  then we can find the natural fundamental matrix for any  $t_I$  by calculating:

$$\Phi(t) = \Psi(t)\Psi(t_I)^{-1}$$

**Example:**

The system:

$$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$$

has fundamental matrix:

$$\Psi(t) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$$

Suppose we wish to solve the IVP with  $\bar{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ .

We first find  $\Phi$  associated to  $t_I = 0$  by doing the following:

$$\begin{aligned} \Phi &= \Psi(t)\Psi(0)^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \end{aligned}$$

Then the solution to the IVP is given by:

$$\bar{x} = \Phi \bar{x}_I = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cos x + 7 \sin x \\ -2 \sin x + 7 \cos x \end{bmatrix}$$

## MATH 246: Chapter 3 Section 3: Matrix and Vector Essentials

Justin Wyss-Gallifent

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Main Topics:

- Basic Definitions
  - Combining Matrices and Vectors
  - Inverses
  - Determinants
  - Transpose and Complex Conjugate
  - Eigenstuff
- 

### 1. Introduction:

It's far easier to manage systems of differential equations when we can rephrase them in the language of matrices and vectors. To that end, here are the essentials.

### 2. Basic Definitions:

- (a) A *matrix* is a rectangular array of numbers. It has size  $m \times n$  if there are  $m$  rows and  $n$  columns. Matrices are typically denoted using capital letters:

**Example:** Here is a  $3 \times 4$  matrix:

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 5 & 0 & 17 \\ 2 & 2 & -7 & 3 \end{bmatrix}$$

- (b) Most of the matrices in this class will be *square*, meaning they have the same number of rows and columns. Mostly we'll deal with  $2 \times 2$  and  $3 \times 3$  matrices.
- (c) The *identity matrix*  $I_n$  is the square  $n \times n$  matrix with 1s on the *main diagonal* (upper-left to lower right) and 0s elsewhere. When the size is clear from context we just write  $I$ .

**Example:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (d) The *zero matrix* is matrix of all zeros.
- (e) A *vector* is a matrix which is a single column. Vectors are usually denoted in lower-case with a bar over the letter.

**Example:**  $\bar{a} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

### 3. Combining Matrices and Vectors:

- (a) We can add matrices and vectors by adding matching entries provided they both have the same size.

**Example:** For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 9 & 8 \end{bmatrix}$$

- (b) We can multiply an  $n \times n$  matrix  $A$  by a vector  $\bar{x}$  with  $n$  entries to get a new vector with  $n$  entries. The formal definition of this is that we take the linear combination of the columns of  $A$  using the weights in  $\bar{x}$ . More informally we do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

**Example:** We have:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} &= 5 \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} (1)(5) + (2)(3) + (-3)(2) \\ (0)(5) + (4)(3) + (7)(2) \\ (8)(5) + (-1)(3) + (5)(2) \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 26 \\ 47 \end{bmatrix} \end{aligned}$$

- (c) We can multiply an  $n \times n$  matrix by another  $n \times n$  matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors and putting them together in a new matrix.

**Example:** Here it is with lots of brackets to help you see what's going on:

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 9 \end{bmatrix} &= \left[ \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \right] \\ &= \left[ \begin{bmatrix} (2)(-1) + (1)(5) \\ (4)(-1) + (3)(5) \end{bmatrix} \quad \begin{bmatrix} (2)(3) + (1)(9) \\ (4)(3) + (3)(9) \end{bmatrix} \right] \\ &= \begin{bmatrix} 3 & 15 \\ 9 & 39 \end{bmatrix} \end{aligned}$$

- (d) It's almost always the case that for matrices  $A$  and  $B$  that  $AB \neq BA$ .
- (e) The identity matrix acts like the number 1 in that for any matrix  $A$  we have:

$$AI = IA = A$$

#### 4. Determinants:

- (a) The *determinant* of a matrix, denoted  $\det A$  or by putting the matrix in vertical bars instead of brackets, is a number calculated from the matrix. We've seen this for  $2 \times 2$  and  $3 \times 3$  matrices.
- (b) Properties include:
- A matrix  $A$  has an inverse if and only if  $\det A \neq 0$ .
  - For a  $2 \times 2$  case  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ .

#### 5. Inverses:

- (a) The *inverse* of an  $n \times n$  matrix  $A$  is another matrix denoted  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = I$ . It's like a "reciprocal" for matrices.
- (b) For the  $2 \times 2$  size there is a formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Example:** For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}$$

- (c) Properties include:
- $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$
  - We have  $\det(A^{-1}) = 1/(\det A)$ .

#### 6. Transpose and Complex Conjugate:

- (a) The *transpose* of an  $n \times n$  matrix  $A$ , denoted  $A^T$ , is found by reflecting the matrix in its main diagonal.

**Example:** We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

- (b) A matrix may have complex numbers in it, in which case its *complex conjugate* denoted either  $\bar{A}$  or  $A^c$ , is found by taking the complex conjugate of each entry.

**Example:** We have:

$$\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^c = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}$$

## 7. Eigenstuff:

If we have a matrix, the determinant is the most important number associated to it. After the determinant the next most important items are eigenvalues and eigenvectors.

- (a) If  $A$  is an  $n \times n$  matrix, an *eigenvalue* of  $A$  is a number  $\lambda$  with the property that there is some  $\bar{v} \neq \bar{0}$  such that  $A\bar{v} = \lambda\bar{v}$ . The vector  $\bar{v}$  is then an *eigenvector* associated to  $\lambda$  and we say that  $(\lambda, \bar{v})$  is an *eigenpair* of  $A$ .

**Example:** Observe that:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so we would say that  $\lambda = 3$  is an eigenvalue,  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector and  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  is an eigenpair for the matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ .

- (b) Any nonzero multiple of an eigenvector is also an eigenvector, so in the above example  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,

$\begin{bmatrix} 17 \\ 17 \end{bmatrix}$  and  $\begin{bmatrix} -7 \\ -7 \end{bmatrix}$  are all eigenvectors for the same eigenvalue.

- (c) If we have a complex eigenpair  $(\lambda, \bar{v})$  then  $(\bar{\lambda}, \bar{\bar{v}})$  is also an eigenpair.

- (d) Given an  $n \times n$  matrix  $A$ , the value  $\lambda$  will be an eigenvalue if and only if there is some  $\bar{v} \neq \bar{0}$  such that

$$A\bar{v} = \lambda\bar{v}$$

We can manipulate this:

$$\begin{aligned} \lambda\bar{v} - A\bar{v} &= \bar{0} \\ \lambda I\bar{v} - A\bar{v} &= \bar{0} \\ (\lambda I - A)\bar{v} &= \bar{0} \end{aligned}$$

This will have a nontrivial solution precisely when:

$$\det(\lambda I - A) = 0$$

So what we do is we define the characteristic polynomial of  $A$  as:

$$p(z) = \det(zI - A)$$

Then we know that the eigenvalues of  $A$  are the roots of this characteristic polynomial.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det \left( z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ -2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) - 4 \\ &= z^2 - 6z + 5 \\ &= (z-5)(z-1) \end{aligned}$$

The eigenvalues are then the roots so  $\lambda_1 = 5$  and  $\lambda_2 = 1$ .

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det \left( z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-4 & 1 \\ -1 & z-2 \end{bmatrix} \\ &= (z-4)(z-2) + 1 \\ &= z^2 - 6z + 9 \\ &= (z-3)^2 \end{aligned}$$

The only eigenvalue is the root  $\lambda = 3$ . However this multiplicity counts, so we can think  $\lambda_1 = 3$  and  $\lambda_2 = 3$ .

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ 2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) + 4 \\ &= z^2 - 6z + 13 \end{aligned}$$

This does not factor so we use the quadratic formula:

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2} = 3 \pm 2i$$

The eigenvalues are then  $\lambda_1 = 3 + 2i$  and  $\lambda_2 = 3 - 2i$ .



- (e) Once we find the eigenvalues we take each eigenvalue  $z = \lambda$  and solve the matrix equation  $A\bar{v} = \lambda\bar{v}$ , or  $A\bar{v} - \lambda\bar{v} = \bar{0}$ , or  $(A - \lambda I)\bar{v} = \bar{0}$ . This can be fairly intensive for large cases. For the  $2 \times 2$  case there is a trick, though, which is really useful:

For  $\lambda_1$  choose any nonzero column of  $A - \lambda_2 I$ .

For  $\lambda_2$  choose any nonzero column of  $A - \lambda_1 I$ .

**Example:** We saw that the eigenvalues for  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$  are  $\lambda_1 = 5$  and  $\lambda_2 = 1$ .

Then:

- For  $\lambda_1 = 5$  choose any nonzero column of

$$A - \lambda_2 I = A - 1I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

so  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  will do. Since any multiple of this works, pick the nicer  $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

- For  $\lambda_2 = 1$  choose any nonzero column of

$$A - \lambda_1 I = A - 5I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

so  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  will do. Since any multiple of this works, pick the nicer  $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

We thus have eigenpairs  $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  and  $\left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ .

**Example:** We saw that the eigenvalue for  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$  is  $\lambda_1 = \lambda_2 = 3$ . Then:

- For  $\lambda_1 = 3$  choose any nonzero column of  $A - \lambda_2 I = A - 3I = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  so  $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  will do.
- Notice that  $\lambda_2 = \lambda_1$  so we get nothing new.

We thus have the single eigenpair  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ .

**Example:** We saw that the eigenvalues for  $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$  are  $\lambda_1 = 3 + 2i$  and  $\lambda_2 = 3 - 2i$ . Then:

- For  $\lambda_1 = 3 + 2i$  choose any nonzero column of

$$A - \lambda_2 I = A - (3 - 2i)I = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 + 2i & 0 \\ 0 & 3 + 2i \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ -2 & -2i \end{bmatrix}$$

so  $\begin{bmatrix} 2i \\ -2 \end{bmatrix}$  will do. Since any multiple of this works, we pick the nicer  $\bar{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ .

- We know from earlier that for  $\lambda_2 = 3 - 2i$  we can use the conjugate so  $\bar{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

We thus have eigenpairs  $\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$  and  $\left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ .

## MATH 246: Chapter 3 Section 5: Using Eigenpairs to Construct Solutions

Justin Wyss-Gallifent

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Main Topics:

- Introduction
  - Real Eigenvalues with Multiplicity 1
  - Real Eigenvalues with Multiplicity 2
  - Pairs of Complex Eigenvalues
- 

### 1. Using Eigenpairs to Construct Solutions:

If we go back to  $\bar{x}' = A\bar{x}$  observe that if  $(\lambda, \bar{v})$  is an eigenpair then it turns out that  $\bar{x} = e^{\lambda t}\bar{v}$  is a solution:

$$\bar{x}' = \frac{d}{dt}(e^{\lambda t}\bar{v}) = e^{\lambda t}\lambda\bar{v} = e^{\lambda t}A\bar{v} = Ae^{\lambda t}\bar{v} = A\bar{x}$$

This tells that an eigenpair yields a solution.

However there are some nuances. The process gets significantly harder at the  $3 \times 3$  case and above as we have to start to consider things like eigenspace dimensions and obtaining linearly independent sets of eigenvalues. Consequently we will stay with the  $2 \times 2$  case.

### 2. Two Real Eigenvalues

If we have two real eigenpairs with distinct eigenvectors:

$$(\lambda_1, \bar{v}_1) \text{ and } (\lambda_2, \bar{v}_2) \text{ with } \lambda_1 \neq \lambda_2$$

So our two solutions are:

$$\{\bar{x}_1, \bar{x}_2\} = \{e^{\lambda_1 t}\bar{v}_1, e^{\lambda_2 t}\bar{v}_2\}$$

and they will form a fundamental set.

**Example:** The system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

has a matrix with eigenpairs

$$\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

Therefore we have two solutions

$$\{\bar{x}_1, \bar{x}_2\} = \left\{e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

So the general solution is:

$$\bar{x} = C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

### 3. One Real Eigenvalue

If we have just one eigenpair  $(\lambda, \bar{v})$  with multiplicity 2 then the situation is trickier. This eigenpair will give us one solution

$$\bar{x}_1 = \bar{v}e^{\lambda t}$$

It turns out that a second solution can be obtained by choosing  $\bar{w}$  to be a nonzero multiple of  $\bar{v}$  and then assigning:

$$\bar{x}_2 = e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w})$$

The proof of why this works is not at all obvious.

So our two solutions are:

$$\{\bar{x}_1, \bar{x}_2\} = \{e^{\lambda t} \bar{v}, e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w})\}$$

**Example:**

$$\bar{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpair:

$$\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

This gives us one solution

$$\bar{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find another choose  $\bar{w}$  to be any non-multiple of  $\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , for example  $\bar{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Then a second solution is:

$$\begin{aligned} x_2 &= e^{\lambda t} (\bar{w} + t(A - \lambda I) \bar{w}) \\ &= e^{3t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= e^{3t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ &= e^{3t} \begin{bmatrix} 1+t \\ t \end{bmatrix} \end{aligned}$$

So the general solution is

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1+t \\ t \end{bmatrix}$$

#### 4. Two Complex Conjugate Eigenvalues

If we have two complex conjugate eigenpairs then we do get two solutions but they are not real solutions. We've seen this issue before.

To understand the method we will work through an example the long way and then point out that there's a short method. Then we will work through another example with the short method.

**Example:**

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) \text{ and } \left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$$

**Long Way:** The first gives us the solution:

$$\begin{aligned} \bar{x} &= e^{(3+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{3t} (\cos(2t) + i \sin(2t)) \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ i \cos(2t) - \sin(2t) \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{bmatrix} \quad \textcircled{1} \end{aligned}$$

The second gives us the solution:

$$\begin{aligned} \bar{x} &= e^{(3-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{3t} (\cos(-2t) + i \sin(-2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= (e^{3t} \cos(2t) - i e^{3t} \sin(2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) - i \sin(2t) \\ -i \cos(2t) - \sin(2t) \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} \cos(2t) - i \sin(2t) \\ -\sin(2t) - i \cos(2t) \end{bmatrix} \quad \textcircled{2} \end{aligned}$$

Since linear combinations of solutions are solutions if we take half of the sum of these we get the solution:

$$\bar{x}_1 = \frac{1}{2} [\textcircled{1} + \textcircled{2}] = e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

and if we take  $\frac{1}{2i}$  times the difference we get the solution:

$$\bar{x}_1 = \frac{1}{2i} [\textcircled{1} - \textcircled{2}] = e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

**Short Way:** If we look back at ① and break it into real and imaginary parts then the two solutions can be extracted from the real and imaginary parts:

$$\begin{aligned}\bar{x} &= e^{3t} \begin{bmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{bmatrix} \\ &= e^{3t} \underbrace{\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}_{\bar{x}_1} + i e^{3t} \underbrace{\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}_{\bar{x}_2}\end{aligned}$$

So the general solution is:

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

We can see now that the short method would be to take one eigenpair:

$$(r + si, \bar{v})$$

Use this to calculate ① and extract the real and imaginary parts. But where did ① come from? It came from doing the calculation:

$$e^{rt} (\cos(st) + i \sin(st)) \bar{v}$$

So the key is to do this calculation and extract the real and imaginary parts and those will be  $\bar{x}_1$  and  $\bar{x}_2$ .

**Example:**

$$\bar{x}' = \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(2 + i\sqrt{5}, \begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix}\right) \text{ and } \left(2 - i\sqrt{5}, \begin{bmatrix} -i\sqrt{5} \\ -5 \end{bmatrix}\right)$$

We then calculate:

$$\begin{aligned} & e^{2t} \left( \cos(\sqrt{5}t) + i \sin(\sqrt{5}t) \right) \begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) + i\sqrt{5} \cos(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) - 5i \sin(\sqrt{5}t) \end{bmatrix} \end{aligned}$$

From here we extract the real and imaginary parts:

$$\begin{aligned} \bar{x}_1 &= e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) \end{bmatrix} \\ \bar{x}_2 &= e^{2t} \begin{bmatrix} -\sqrt{5} \cos(\sqrt{5}t) \\ -5 \sin(\sqrt{5}t) \end{bmatrix} \end{aligned}$$

So the general solution is

$$\bar{x} = C_1 e^{2t} \begin{bmatrix} -\sqrt{5} \sin(\sqrt{5}t) \\ -5 \cos(\sqrt{5}t) \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\sqrt{5} \cos(\sqrt{5}t) \\ -5 \sin(\sqrt{5}t) \end{bmatrix}$$

Note: Other valid answers can look quite different from this since any multiple of an eigenvector is an eigenvector and since complex multiples can look quite surprising.

5. **An Initial Value Problem:** Since we haven't done one from start to finish, here is an initial value problem:

**Example:** Solve

$$\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- (a) Find the eigenvalues:

$$\begin{aligned} p(z) &= \det \begin{bmatrix} z-5 & -1 \\ 3 & z-1 \end{bmatrix} = (z-5)(z-1) - (-1)(3) \\ &= z^2 - 6z + 8 = (z-2)(z-4) \end{aligned}$$

So the eigenvalues are  $\lambda_1 = 2$  and  $\lambda_2 = 4$ .

- (b) Find the eigenvectors:

For  $\lambda_1 = 2$  choose a nonzero column of  $A - \lambda_2 I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$  so

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For  $\lambda_2 = 4$  choose a nonzero column of  $A - \lambda_1 I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$  so

$$\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- (c) Write down the general solution:

We have

$$\bar{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- (d) Plug in the initial value and solve:

$$\bar{x}(0) = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \bar{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so that:

$$\bar{c} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

- (e) Write down the answer:

$$\bar{x} = \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

## MATH 246: Chapter 3 Section 6: Graphs of Solutions

Justin Wyss-Gallifent

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Main Topics:

- Basic Idea
  - Gallery of Possibilities
- 

### 1. Introduction:

The goal of this section is to understand what the solutions of the system:

$$\bar{x}' = A\bar{x}$$

look like graphically when we are in the  $n = 2$  case.

To make this a little clearer instead of thinking of  $\bar{x}'$  as  $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  we'll think of  $\bar{x}'$  as  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  because this way a solution can be thought of as a curve which moves around in the  $xy$ -plane as a function of time  $t$ .

In addition we'll think of the matrix  $A$  as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

What we'll do first is go through one example thoroughly and then the rest will be categorized without too much explanation.

### 2. Broad Strokes:

The general idea is that the solutions can always be graphed using just the eigenvalues and sometimes (but not always) the eigenvectors and sometimes (but not always) the matrix. The solutions will not be perfect but they'll give us a lot of insight.

Here the types of solutions have been broken down into five categories to make them easier to remember. Each category has subcategories.

While this seems like a lot there are many similarities and you'll find that patterns repeat over and over and make a lot of sense, so it's really not that terrible!

### 3. First Example:

Consider the system:

$$\bar{x}' = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix} \bar{x}$$

The eigenpairs of  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$  are  $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$  and  $\left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . The general solution is then

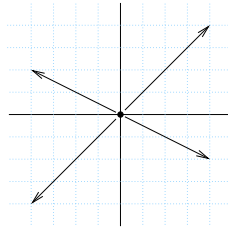
$$\bar{x} = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let's analyze a few solutions:



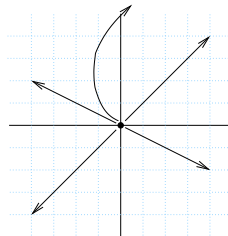
- If  $C_1 = 0$  and  $C_2 = 0$  then we get  $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  which is a constant solution which sits at the origin for all  $t$ .
- If  $C_1 = 0$  and  $C_2 = 1$  then we get  $\bar{x} = e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}$ . Notice that  $x(t) = e^{9t}$  and  $y = e^{9t}$  are always positive and equal. As  $t \rightarrow \infty$  this point moves away from the origin and as  $t \rightarrow -\infty$  this point moves toward but never touches (slows down as it goes) the origin.
- If  $C_1 = 0$  and  $C_2 = -1$  we get a similar thing, the only difference being that both  $x(t)$  and  $y(t)$  are negative.
- If  $C_1 = 1$  and  $C_2 = 0$  then we get  $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{3t} \\ e^{3t} \end{bmatrix}$ . This solution always has  $x = -2y$  but otherwise has the same behavior.
- If  $C_1 = -1$  and  $C_2 = 0$  we get the opposite of the previous.

All together we get the following five solutions:



One more solution:

- If  $C_1 = 1$  and  $C_2 = 1$  then we get  $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . For large negative  $t$  the  $e^{9t}$  is closer to zero than the  $e^{3t}$  and so the function behaves like  $e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ . On the other hand for large positive  $t$  the  $e^{3t}$  still exists and contributes but the  $e^{9t}$  is much more significant and so the function turns out to be approaching parallel to  $e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The result is:



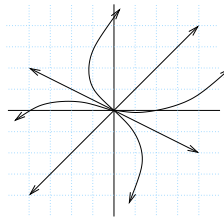
#### 4. Categories of Solutions:

(a) The eigenvalues  $\lambda_1, \lambda_2$  are both real, nonzero and different.

i. Both eigenvalues are positive: **Nodal Source - Unstable**

In this case there are four straight-line solutions moving away from the origin along the vectors  $\pm \bar{v}_1$  and  $\pm \bar{v}_2$ . The other solutions move away from the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

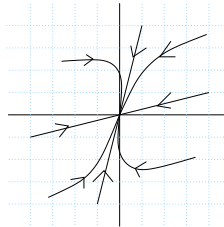
**Example 1:** If  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$  then the epairs are  $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



ii. Both eigenvalues are negative: **Nodal Sink - Stable**

In this case there are four straight-line solutions moving toward the origin along the vectors  $\pm \bar{v}_1$  and  $\pm \bar{v}_2$ . The other solutions move toward the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

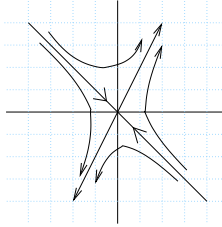
**Example 2:** If  $A = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$  then the epairs are  $\left(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right), \left(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$ . Picture:



iii. One eigenvalue is positive and the other is negative: **Saddle - Unstable**

In this case there are four straight-line solutions. The two corresponding to the positive eigenvalue move away from the origin (along the eigenvector and its opposite) and the two corresponding to the negative eigenvalue move toward the origin (along the eigenvector and its opposite). The other solutions move toward the origin initially parallel to the straight-line solutions moving toward the origin but then curve and move away parallel to the other straight-line solution.

**Example 3:** If  $A = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$  then the epairs are  $\left(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .  
Picture:

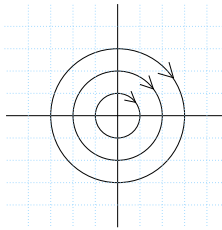


(b) The eigenvalues are complex conjugates.

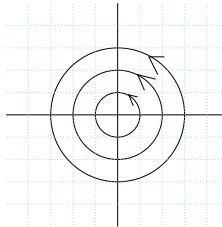
i. They have the form  $0 \pm si$ : **Circle - Stable**

In this case the solutions are circles around the origin. They are clockwise if  $a_{12} > 0$  and counterclockwise if  $a_{12} < 0$ .

**Example 4:** If  $A = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$  then the evals are  $\pm 2i$ . Picture:



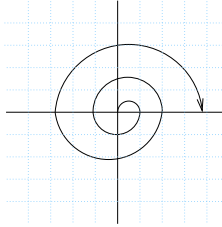
**Example 5:** If  $A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$  then the evals are  $\pm 6i$ . Picture:



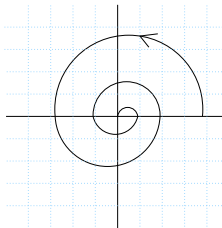
ii. They have the form  $r \pm si$ : **Spiral Source - Unstable (if out) or Sink - Stable (if in)**

In this case the solutions are spirals around the origin. They are clockwise if  $a_{12} > 0$  and counterclockwise if  $a_{12} < 0$  and they spiral in if  $r < 0$  and out if  $r > 0$ .

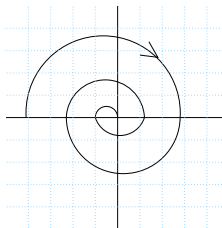
**Example 6:** If  $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$  then the evals are  $3 \pm 2i$ . Picture:



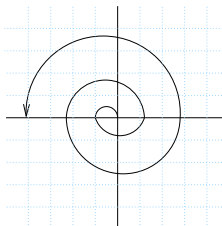
**Example 7:** If  $A = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$  then the evals are  $-3 \pm 2i$ . Picture:



**Example 8:** If  $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$  then the evals are  $-2 \pm 3i$ . Picture:



**Example 9:** If  $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$  then the evals are  $2 \pm 6i$ . Picture:

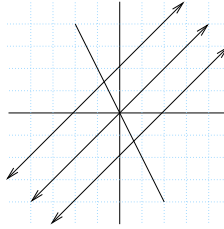


(c) One eigenvalue is 0, the other  $\lambda$  is real and not zero.

i. The other eigenvalue is positive: **Linear Source - Unstable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move away from that line and are parallel to the eigenvector corresponding to  $\lambda$ .

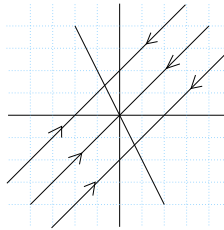
**Example 10:** If  $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$  then the epairs are  $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



ii. The other eigenvalue is negative: **Linear Sink - Stable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move toward that line and are parallel to the eigenvector corresponding to  $\lambda$ .

**Example 11:** If  $A = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$  then the epairs are  $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:

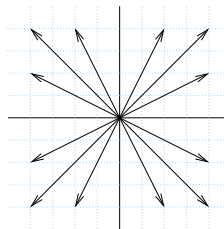


(d) There is a single nonzero eigenvalue  $\lambda$  and  $A$  looks like  $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ :

i. If the eigenvalue is positive: **Radial Source - Unstable**

In this case all the solutions are straight lines moving away from the origin.

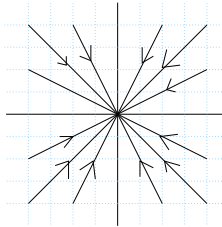
**Example 12:** If  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  then the eval is 2. Picture:



ii. If the eigenvalue is negative: **Radial Sink - Stable**

In this case all the solutions are straight lines moving toward the origin.

**Example 13:** If  $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$  then the eval is  $-3$ . Picture:

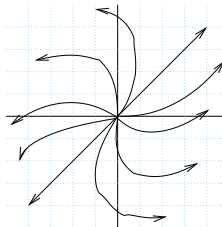


(e) There is a single nonzero eigenvalue  $\lambda$  and  $A$  does not look like that:

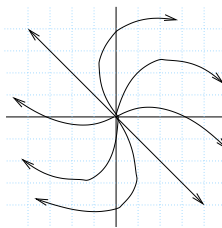
i. If the eigenvalue is positive: **Twist Source - Unstable**

In this case there are two straight-line solutions moving away from the origin along the eigenvector corresponding to  $\lambda$ . The other solutions are all curved solutions which move away from the origin in a clockwise direction if  $a_{12} > 0$  and in a counterclockwise direction if  $a_{12} < 0$ .

**Example 14:** If  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$  then the epair is  $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



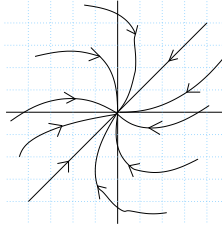
**Example 15:** If  $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$  then the epair is  $\left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ . Picture:



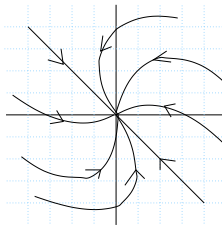
ii. If the eigenvalue is negative: **Twist Sink - Stable**

In this case there are two straight-line solutions moving toward the origin along the eigenvector corresponding to  $\lambda$ . The other solutions are all curved solutions which move toward the origin in a clockwise direction if  $a_{12} > 0$  and in a counterclockwise direction if  $a_{12} < 0$ .

**Example 16:** If  $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$  then the epair is  $\left(-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:



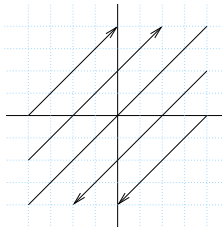
**Example 17:** If  $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$  then the epair is  $\left(-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ . Picture:



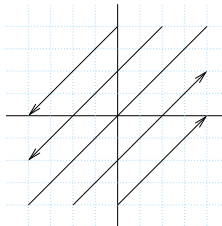
iii. If the eigenvalue is zero: **Parallel Shear - Unstable**

In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions. The other solutions are straight lines parallel to that one, “clockwise” if  $a_{12} > 0$  and “counterclockwise” if  $a_{12} < 0$ .

**Example 18:** If  $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$  then the epair is  $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:

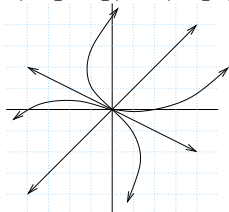


**Example 19:** If  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  then the epair is  $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ . Picture:

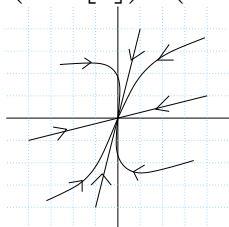


# MATH 246: Chapter 3 Section 6: Graphs of Solutions Cheatsheet

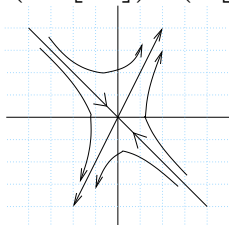
1.  $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$ :  
 $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



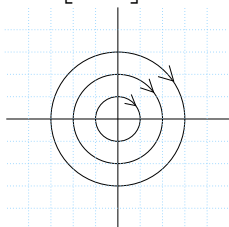
2.  $A = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$ :  
 $\left(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right), \left(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$



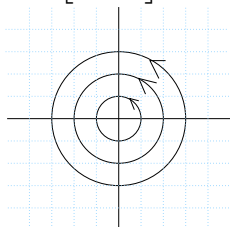
3.  $A = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$ :  
 $\left(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$



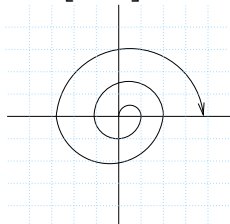
4.  $A = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$ :  $\lambda = 0 \pm 2i$



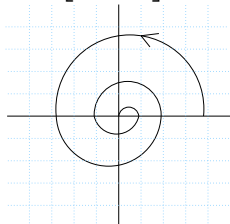
5.  $A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$ :  $\lambda = 0 \pm 6i$



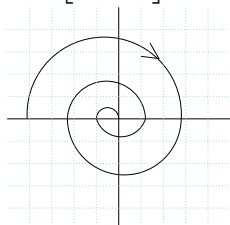
6.  $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$ :  $\lambda = 3 \pm 2i$



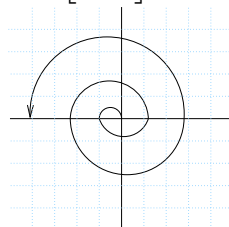
7.  $A = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$ :  $\lambda = -3 \pm 2i$



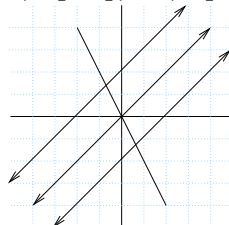
8.  $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$ :  $\lambda = -2 \pm 3i$



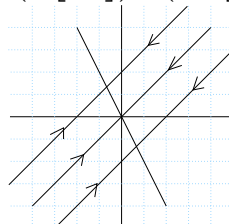
9.  $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ :  $\lambda = 2 \pm 6i$



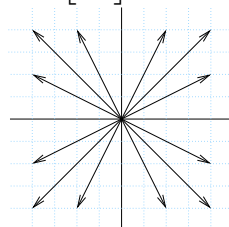
10.  $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ :  
 $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



11.  $A = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$ :  
 $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

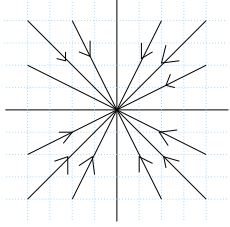


12.  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ :  $\lambda = 2$

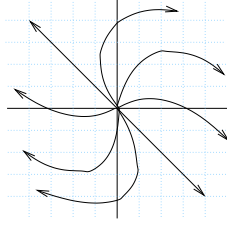




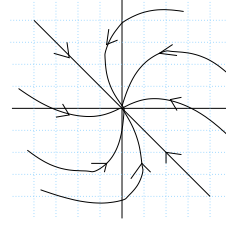
13.  $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} : \lambda = -3$



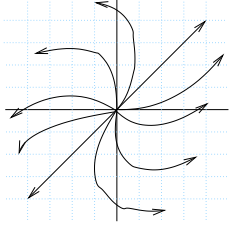
15.  $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} : \left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$



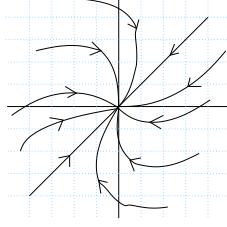
17.  $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix} : \left(-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$



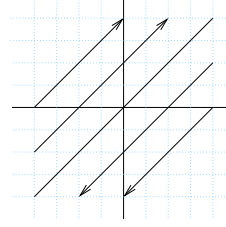
14.  $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} : \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



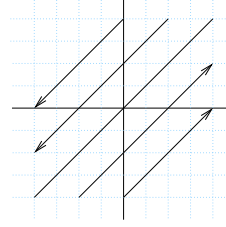
16.  $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} : \left(-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



18.  $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} : \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



19.  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} : \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



## MATH 246: Chapter 3 Section 7: Hamiltonian Systems

Justin Wyss-Gallifent

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Main Topics:

- Preliminaries
  - Hamiltonian Systems
  - Stationary Solutions/Points
  - Analysis with the Hessian
- 

### 1. Preliminaries

The goal of this section is to look at a couple of more specific things related to systems of two differential equations. We're going to first throw out the requirement that the system is linear, meaning we can't assume it looks like  $\bar{x}' = A\bar{x}$ . Instead we'll think of these as:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

**Example:** One example would be something like:

$$\begin{aligned}x' &= x^2 - y^2 \\ y' &= y + 2\end{aligned}$$

2. **Definition of Hamiltonian** One very special type of system of differential equations is a Hamiltonian system. A Hamiltonian system is a system in which there is some function  $H(x, y)$  such that:

$$\begin{aligned}x' &= H_y(x, y) \\ y' &= -H_x(x, y)\end{aligned}$$

The reason that these are nice is that for a Hamiltonian system we have:

$$\begin{aligned}-H_x(x, y) \frac{dx}{dt} &= H_y(x, y) \frac{dy}{dt} \\ H_x(x, y) \frac{dy}{dt} + H_y(x, y) \frac{dx}{dt} &= 0 \\ \frac{d}{dt} H(x, y) &= 0 \\ H(x, y) &= C\end{aligned}$$

For some/any constant  $C$ . This means that solutions to the system of differential equations are level curves for  $H$ .

**Example:** The system:

$$\begin{aligned}x' &= 2y \\ y' &= -2x\end{aligned}$$

is Hamiltonian with  $H(x, y) = x^2 + y^2$  because  $x' = H_y = 2y$  and  $y' = -H_x = -2x$ . The solutions then satisfy  $x^2 + y^2 = C$  and so they're circles.

Notice that we could also have seen this using methods from the previous section. Here  $\bar{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \bar{x}$  so the eigenvalues for  $A$  are  $0 \pm 2i$  and since  $a_{12} > 0$  the solutions are (clockwise) circles.

### 3. Determining if a System is Hamiltonian

It turns out that a system of the form:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

is Hamiltonian if  $f_x + g_y = 0$  and if it is then we can find  $H$  using a process we used before with exact differential equations.

**Example:** Show the following system is Hamiltonian and find  $H(x, y)$ :

$$\begin{aligned}x' &= x^2 + 2y \\ y' &= -2xy\end{aligned}$$

First note that  $f_x + g_y = 2x - 2x = 0$  so the system is Hamiltonian. We wish to find  $H(x, y)$  with  $H_y(x, y) = x^2 + 2y$  and  $-H_x(x, y) = -2xy$ . The latter tells us  $H_x(x, y) = 2xy$  and so  $H(x, y) = x^2y + g(y)$ . From here  $H_y(x, y) = x^2 + g'(y) = x^2 + 2y$  so  $g'(y) = 2y$  and  $g(y) = y^2 + C$ . Then  $H(x, y) = x^2y + y^2 + C$ . Since we can choose any constant we let  $H(x, y) = x^2y + y^2$ . Thus the solutions, when plotted, satisfy the equation  $x^2y + y^2 = C$ , whatever this looks like!

### 4. Analysis of Stationary Solutions/Points

A stationary solution is a solution for which  $x' = 0$  and  $y' = 0$ . These are easy to solve for.

**Example:** Find the stationary solutions to:

$$\begin{aligned}x' &= x^2 - y^2 \\ y' &= y + 2\end{aligned}$$

We set  $x^2 - y^2 = 0$  and  $y + 2 = 0$ . The latter gives us  $y = -2$  and then the former gives us  $x^2 - 4 = 0$  so  $x = \pm 2$ . Therefore there are two stationary solutions,  $(2, -2)$  and  $(-2, -2)$ .

Hamiltonian systems can be analyzed further by looking at the Hessian at each stationary point. The Hessian is the following matrix:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}$$

We'll only look at those for which the determinant of the Hessian is nonzero.

If  $\det \partial^2 H < 0$  then the stationary point is a saddle.

If  $\det \partial^2 H > 0$  then the stationary point is a circle.

Directions can be figured out by testing points.

**Example:** Consider the system

$$\begin{aligned}x' &= 4y - y^3 \\y' &= x\end{aligned}$$

To find the stationary solutions we set  $4y - y^3 = y(4 - y^2) = 0$  and  $x = 0$ . The former gives us  $y = 0, \pm 2$  so there are three stationary solutions at  $(0, 0)$ ,  $(0, 2)$  and  $(0, -2)$ . Noting that  $H_y = 4y - y^3$  and  $-H_x = x$ , so  $H_x = -x$ , we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 - 3y^2 \end{bmatrix}$$

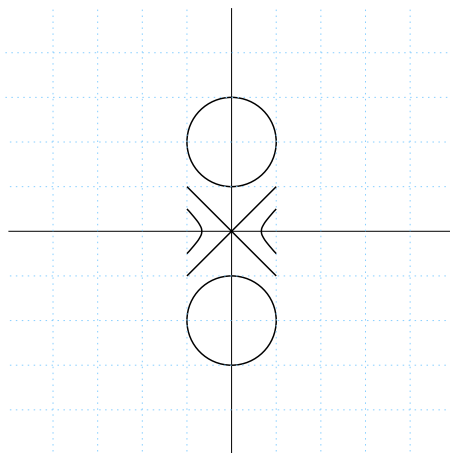
Then at each point:

$$\det \partial^2 H(0, 0) = \det \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = -4 \text{ so } (0, 0) \text{ is a saddle}$$

$$\det \partial^2 H(0, 2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0, 2) \text{ is a circle}$$

$$\det \partial^2 H(0, -2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0, -2) \text{ is a circle.}$$

A preliminary picture is:



Notice how the circles fit nicely with the saddle shape!

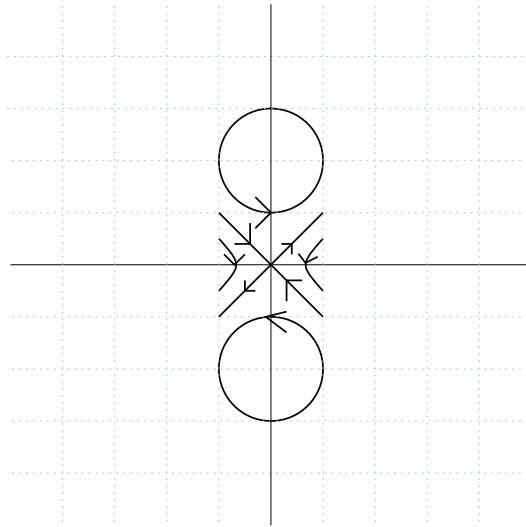
**Example (Continued):**

We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point  $(0, 1)$  we have:

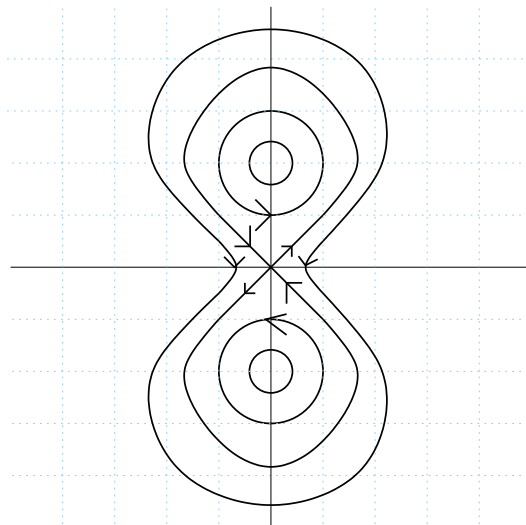
$$x'(0, 1) = 3$$

$$y'(0, 1) = 0$$

Meaning at  $(0, 1)$  the solution is moving to the right. Everything else is filled in according to rules of compatibility.



If we were to fill in more solutions it starts to look pretty:



**Example:** Consider the system

$$x' = -x + y + x^2$$

$$y' = y - 2xy$$

To find the stationary points we set  $-x + y + x^2 = 0$  and  $y - 2xy = y(1 - 2x) = 0$ . The latter gives us  $y = 0$  or  $x = \frac{1}{2}$ . If  $y = 0$  then the former gives us  $x = 0, 1$  and if  $x = \frac{1}{2}$  then the former gives us  $y = \frac{1}{4}$ . So there are three stationary points at  $(0, 0)$ ,  $(1, 0)$  and  $(\frac{1}{2}, \frac{1}{4})$ .

Noting that  $H_y = -x + y + x^2$  and  $-H_x = y - 2xy$ , so  $H_x = 2xy - y$ , we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 1 \\ 2x - 1 & 1 \end{bmatrix}$$

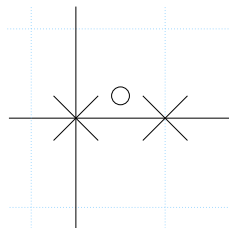
Then at each point:

$$\det \partial^2 H(0, 0) = \det \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = -1 \text{ so } (0, 0) \text{ is a saddle.}$$

$$\det \partial^2 H(1, 0) = \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 \text{ so } (1, 0) \text{ is a saddle.}$$

$$\det \partial^2 H\left(\frac{1}{2}, \frac{1}{4}\right) = \det \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \text{ so } \left(\frac{1}{2}, \frac{1}{4}\right) \text{ is a center.}$$

A preliminary picture is:

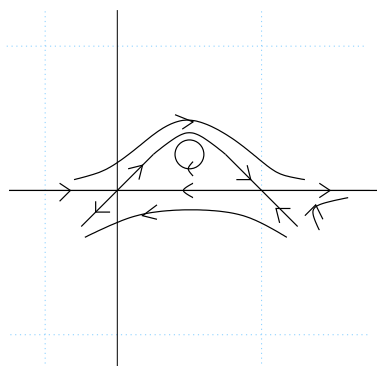


We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point  $(\frac{1}{2}, 0)$  we have:

$$x' \left( \frac{1}{2}, 0 \right) = -$$

$$y' \left( \frac{1}{2}, 0 \right) = 0$$

Meaning at  $(\frac{1}{2}, 0)$  the solution is moving to the left. Everything else is filled in according to rules of compatibility. Some adjustment of the saddles is also needed!



**MATH 246: Chapter 3 Section 8:**  
**Justin Wyss-Gallifent**

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Main Topics:

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**1. Introduction**

The goal of this section is to do a bit of analysis of nonlinear systems which are not necessarily Hamiltonian. The approach is similar though - find stationary solutions, find what they look like, fill them in, figure out what the remaining solutions look like.

**2. Linearization at the Stationary Solutions**

This sounds far more complicated than it sounds. In Calculus suppose you know that  $f(x) = x^2 - 9$  and you're investigating this function. You might notice that the  $x$ -intercepts are  $x = \pm 3$  and you might want to know what happens at those points. You might notice that  $f'(x) = 2x$  so  $f'(-3) = -6$  and  $f'(3) = 6$  and so the function is decreasing at  $x = -3$  and increasing at  $x = 3$ . What you just did was that you *linearized* the function at  $x = \pm 3$ , meaning you sort of made it a line with slope  $\pm 6$  at those points.

What we'll do is precisely the same thing but with a system of nonlinear differential equations.

Given a system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

First a notation point - sometimes we write

$$\bar{f}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Using this notation the linearization matrix for this system is the matrix:

$$\partial \bar{f} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

**Example:** Consider the system:

$$\begin{aligned}x' &= y \\ y' &= 4x - x^3\end{aligned}$$

Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{f} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

### 3. Stationary Point Analysis

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that  $f'(3) = 6$  you know the function is increasing at that point but if you discovered that  $f'(3) = 0$  then the function could be increasing or decreasing or neither at that point.



**Example:** Consider the system:

$$\begin{aligned}x' &= y \\y' &= 4x - x^3\end{aligned}$$

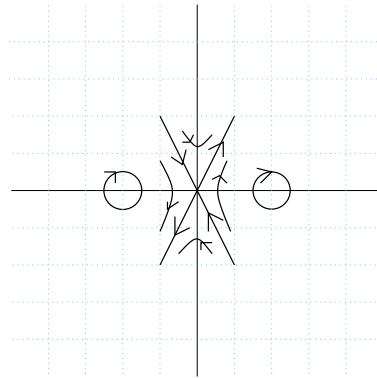
Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{f} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

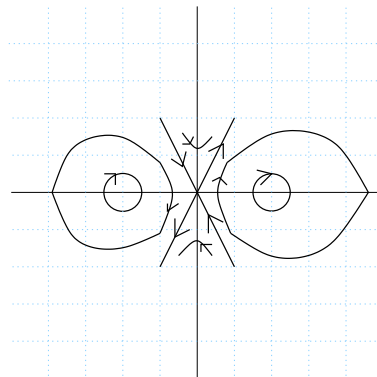
The stationary solutions are  $(0, 0)$ ,  $(2, 0)$  and  $(-2, 0)$ . We check the linearization matrix at those points:

- At  $(0, 0)$ :  $\partial \bar{f}(0, 0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$  and  $\left(2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ . This is a saddle.
- At  $(2, 0)$ :  $\partial \bar{f}(2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .
- At  $(-2, 0)$ :  $\partial \bar{f}(-2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .

Together we get the picture:



From here we can fill in a nice family of solutions:



**Example:** Consider the system:

$$\begin{aligned}x' &= (y - x)(x - 1) \\ y' &= (3 + 2x - x^2)y\end{aligned}$$

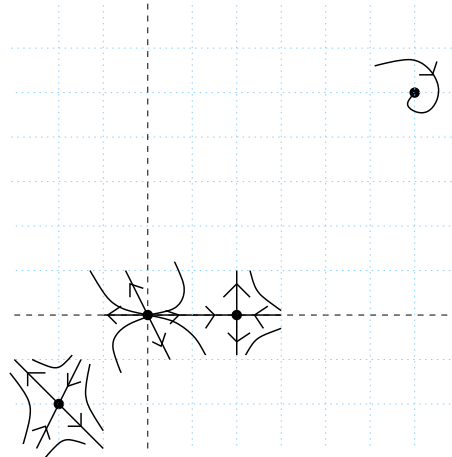
Since  $f(x, y) = xy - x^2 - y + x$  and  $g(x, y) = 3y + 2xy - x^2y$  the linearization matrix is

$$\partial \bar{f} = \begin{bmatrix} y - 2x + 1 & x - 1 \\ 2y - 2xy & 3 + 2x - x^2 \end{bmatrix}$$

The stationary solutions are  $(0, 0)$ ,  $(-1, -1)$ ,  $(1, 0)$  and  $(3, 3)$ . We check the linearization at those point:

- At  $(0, 0)$ :  $\partial \bar{f}(0, 0) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ . The eigenpairs are  $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(3, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ .  
This is a source. Solutions close to  $(0, 0)$  are tangent to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .
- At  $(-1, -1)$ :  $\partial \bar{f}(-1, -1) = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ . This is a saddle.
- At  $(1, 0)$ :  $\partial \bar{f}(1, 0) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ . The eigenpairs are  $\left(-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
This is a saddle.
- At  $(3, 3)$ :  $\partial \bar{f}(3, 3) = \begin{bmatrix} -2 & 2 \\ 12 & 0 \end{bmatrix}$ . The eigenvalues are  $-1 \pm i\sqrt{23}$ . Since  $a_{12} > 0$  this is a clockwise spiral sink.

Together we get the picture:



## MATH 246: Chapter 3 Section 9: Linearization of Nonlinear Systems

Justin Wyss-Gallifent

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Main Topics:

- Linearization
  - Stationary Point Analysis
- 

### 1. Introduction

The goal of this section is to do a bit of analysis of nonlinear systems which are not necessarily Hamiltonian. The approach is similar though - find stationary solutions, find what they look like, fill them in, figure out what the remaining solutions look like.

### 2. Linearization at the Stationary Solutions

This sounds far more complicated than it sounds. In Calculus suppose you know that  $f(x) = x^2 - 9$  and you're investigating this function. You might notice that the  $x$ -intercepts are  $x = \pm 3$  and you might want to know what happens at those points. You might notice that  $f'(x) = 2x$  so  $f'(-3) = -6$  and  $f'(3) = 6$  and so the function is decreasing at  $x = -3$  and increasing at  $x = 3$ . What you just did was that you *linearized* the function at  $x = \pm 3$ , meaning you sort of made it a line with slope  $\pm 6$  at those points.

What we'll do is precisely the same thing but with a system of nonlinear differential equations.

Given a system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

First a notation point - sometimes we write

$$\bar{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Using this notation the linearization matrix for this system is the matrix:

$$\partial \bar{F} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

**Example:** Consider the system:

$$\begin{aligned}x' &= y \\ y' &= 4x - x^3\end{aligned}$$

Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

### 3. Stationary Point Analysis

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that  $f'(3) = 6$  you know the function is increasing at that point but if you discovered that  $f'(3) = 0$  then the function could be increasing or decreasing or neither at that point.

#### 4. Stationary Point Analysis

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that  $f'(3) = 6$  you know the function is increasing at that point but if you discovered that  $f'(3) = 0$  then the function could be increasing or decreasing or neither at that point.

**Example:**

Consider the system:

$$\begin{aligned}x' &= 1 - y \\ y' &= x^2 - y^2\end{aligned}$$

Since  $f(x, y) = 1 - y$  and  $g(x, y) = x^2 - y^2$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$$

The stationary solutions are  $(1, 1)$  and  $(-1, 1)$ . We check the linearization matrix at those points:

- At  $(1, 1)$ :  $\partial \bar{F}(1, 1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$ .

The eigenvalues are  $\lambda = -1 \pm i$  so the point is a counterclockwise spiral sink.

- At  $(-1, 1)$ :  $\partial \bar{F}(-1, 1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ . The eigenpairs are

$$\left( -1 + \sqrt{3}, \begin{bmatrix} 1 + \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} 2.7 \\ -2 \end{bmatrix} \right) \quad \text{and} \quad \left( -1 - \sqrt{3}, \begin{bmatrix} 1 - \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} -1.7 \\ -2 \end{bmatrix} \right).$$

This is a saddle.

**Example:** Consider the system:

$$\begin{aligned}x' &= y \\y' &= 4x - x^3\end{aligned}$$

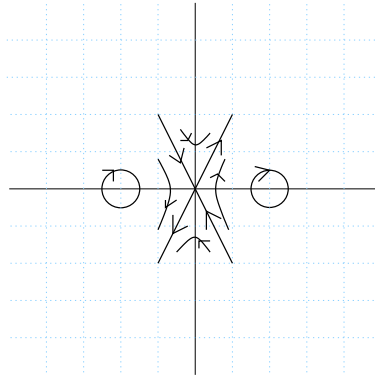
Since  $f(x, y) = y$  and  $g(x, y) = 4x - x^3$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

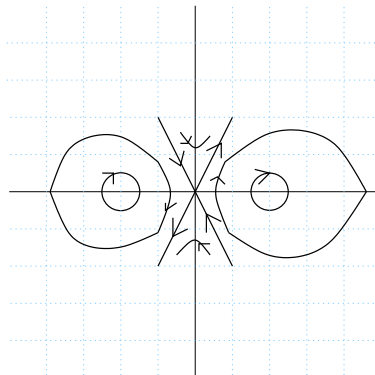
The stationary solutions are  $(0, 0)$ ,  $(2, 0)$  and  $(-2, 0)$ . We check the linearization matrix at those points:

- At  $(0, 0)$ :  $\partial \bar{F}(0, 0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$  and  $\left(2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ . This is a saddle.
- At  $(2, 0)$ :  $\partial \bar{F}(2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .
- At  $(-2, 0)$ :  $\partial \bar{F}(-2, 0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$ . The eigenvalues are  $0 \pm i\sqrt{8}$ . This is a clockwise circle since  $a_{12} > 0$ .

Together we get the picture:



From here we can fill in a nice family of solutions:



**Example:** Consider the system:

$$\begin{aligned}x' &= (y - x)(x - 1) \\y' &= (3 + 2x - x^2)y\end{aligned}$$

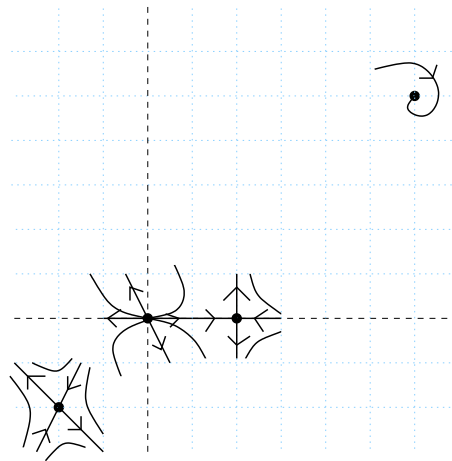
Since  $f(x, y) = xy - x^2 - y + x$  and  $g(x, y) = 3y + 2xy - x^2y$  the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} y - 2x + 1 & x - 1 \\ 2y - 2xy & 3 + 2x - x^2 \end{bmatrix}$$

The stationary solutions are  $(0, 0)$ ,  $(-1, -1)$ ,  $(1, 0)$  and  $(3, 3)$ . We check the linearization at those point:

- At  $(0, 0)$ :  $\partial \bar{F}(0, 0) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ . The eigenpairs are  $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(3, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ . This is a source. Solutions close to  $(0, 0)$  are tangent to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .
- At  $(-1, -1)$ :  $\partial \bar{F}(-1, -1) = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ . The eigenpairs are  $\left(-2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ . This is a saddle.
- At  $(1, 0)$ :  $\partial \bar{F}(1, 0) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ . The eigenpairs are  $\left(-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  and  $\left(4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ . This is a saddle.
- At  $(3, 3)$ :  $\partial \bar{F}(3, 3) = \begin{bmatrix} -2 & 2 \\ 12 & 0 \end{bmatrix}$ . The eigenvalues are  $-1 \pm i\sqrt{23}$ . Since  $a_{12} > 0$  this is a clockwise spiral sink.

Together we get the picture:



**MATH 246: Chapter 3 Section 10: Population Dynamics**  
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Main Topics:

- Predator-Prey Models
  - Competing Species Models
  - Cooperating Species Models
- 

**1. Predator-Prey Models**

Consider a interaction between predators and prey. Suppose the number of prey is  $x(t)$  while the number of predators is  $y(t)$ . A simple but reasonable system of differential equations modeling these could be:

$$\begin{aligned}x' &= (r - ax - by)x \\ y' &= (-s + cx - dy)y\end{aligned}$$

To understand the meaning of these constants, consider that  $r - ax - by$  is the growth rate for prey while  $-s + cx - dy$  is the growth rate for predators. This is the reason they're multiplied by  $x$  and  $y$  respectively to get  $x'$  and  $y'$ . Moreover:

- The constant  $r > 0$  gives the intrinsic growth rate of the prey. This is positive because by default (in absense of predators) the prey will reproduce.
- The growth rate of prey may decline as the number of prey grows due to competetiveness. This is managed by the constant  $a \geq 0$ .
- The growth rate of the prey will decline as the number of predators grows. This is managed by the constant  $d > 0$ .
- The constant  $s > 0$  gives the intrinsic growth rate of the predators. We have  $-s$  because by default (in absense of prey) the predators will die out.
- The growth rate of predators will increase with the number of prey. This is managed by the constant  $c > 0$ .
- The growth rate of predators may decline as the number of predators grows due to competetiveness. This is managed by the constant  $d \geq 0$ .

Our goal will be to analyze such systems and understand what happens to the populations in the long term.



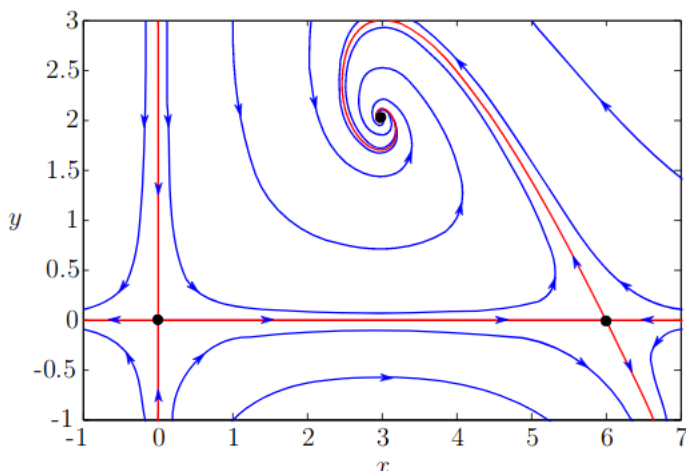
**Example:** Consider the model:

$$\begin{aligned}x' &= (12 - 2x - 3y)x \\ y' &= (-15 + 5x)y\end{aligned}$$

There are three stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$  with eigenpairs  $\left(12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a saddle.
- $(6, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -12 & -18 \\ 0 & 15 \end{bmatrix}$  with eigenpairs  $\left(-12, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(15, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(3, 2)$  has  $\partial \bar{F} = \begin{bmatrix} -6 & -9 \\ 10 & 0 \end{bmatrix}$  with eigenvalues  $-3 \pm 9i$ .  
It follows that this is a counterclockwise spiral sink.

The following picture was pilfered from Levermore's notes:



So now an initial population of  $(0.1, 2)$  will undergo a decrease in predators, resulting in an increase in prey, resulting in an increase in predators, resulting in a decrease in prey, and so on, and will eventually spiral into the stable point  $(3, 2)$ .

**Example:** Consider the model:

$$\begin{aligned}x' &= (6 - 3y)x \\ y' &= (-15 + 5x)y\end{aligned}$$

There are two stationary points which we analyze as follows:

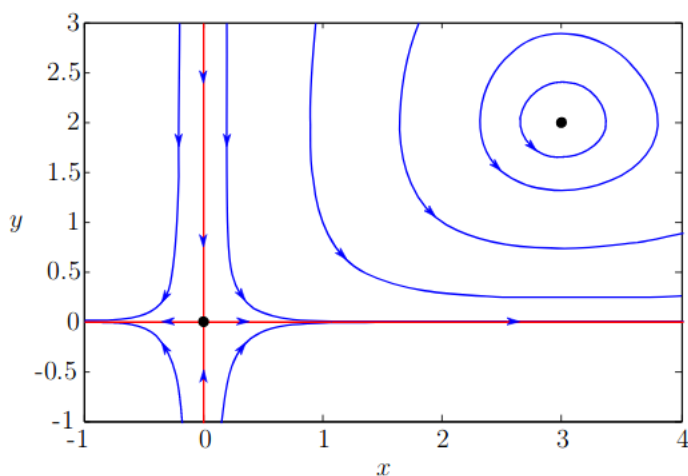
- $(0,0)$  has  $\partial\bar{F} = \begin{bmatrix} 12 & 0 \\ 0 & -15 \end{bmatrix}$  with eigenpairs  $\left(6, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-15, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .

It follows that this is a saddle.

- $(3,2)$  has  $\partial\bar{F} = \begin{bmatrix} 0 & -9 \\ 10 & 0 \end{bmatrix}$  with eigenvalues  $0 \pm 90i$ .

It follows that this is a counterclockwise circle.

The following picture was pilfered from Levermore's notes:



So now an initial population in the first quadrant will tend to circle around  $(3,2)$  but it will not approach it in a spiral sense.

## 2. Competing Species Models

Competing species models look like this:

$$\begin{aligned}x' &= (r - ax - by)x \\ y' &= (s - cx - dy)y\end{aligned}$$

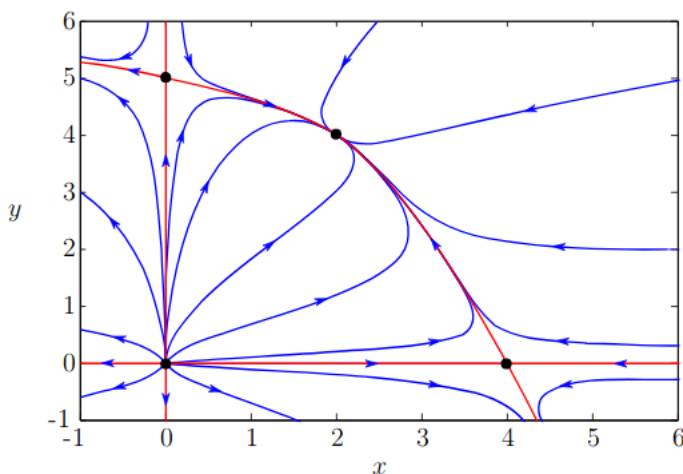
**Example:** Consider the model:

$$\begin{aligned}x' &= (16 - 4x - 2y)x \\ y' &= (10 - x - 2y)y\end{aligned}$$

There are four stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 16 & 0 \\ 0 & 10 \end{bmatrix}$  with eigenpairs  $\left(16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a nodal source..
- $(0, 5)$  has  $\partial \bar{F} = \begin{bmatrix} 6 & 0 \\ -5 & -10 \end{bmatrix}$  with eigenpairs  $\left(6, \begin{bmatrix} 16 \\ -5 \end{bmatrix}\right), \left(-10, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(4, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -16 & -8 \\ 0 & 6 \end{bmatrix}$  with eigenpairs  $\left(-16, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(6, \begin{bmatrix} -4 \\ 11 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(2, 4)$  has  $\partial \bar{F} = \begin{bmatrix} -8 & -4 \\ -4 & -8 \end{bmatrix}$  with eigenpairs  $\left(-12, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(-4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$   
It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both  $x$  and  $y$  positive then it will tend towards  $(2, 4)$  but this can happen in a variety of ways.

For example if we start at  $(10, 0.1)$  this means there are lots of species  $x$  and few of species  $y$ . Because there are a lot of species  $x$  they are constrained by resources and hence their population drops. When it gets close to  $x = 4$  however the rate of drop decreases and at that point resources are not so constraining it tends to start to stabilize. However at that point since  $x$  and  $y$  are competing  $y$  can grow now, since there aren't so many  $x$ . And so it does, and this causes  $x$  to drop more. In the long term (infinity) the pair heads to  $(2, 4)$ .

### 3. Cooperating Species Models

Cooperating species models look like this;

$$\begin{aligned}x' &= (r - ax + by)x \\ y' &= (s + cx - dy)y\end{aligned}$$

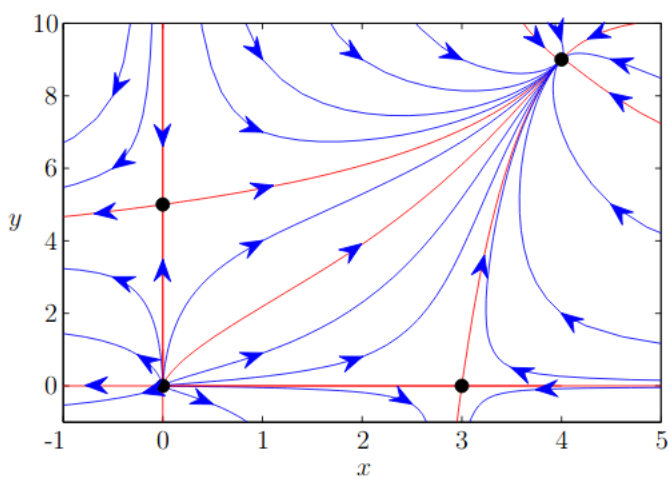
**Example:** Consider the model:

$$\begin{aligned}x' &= (27 - 9x + y)x \\ y' &= (20 + 4x - 4y)y\end{aligned}$$

There are four stationary points which we analyze as follows:

- $(0, 0)$  has  $\partial \bar{F} = \begin{bmatrix} 27 & 0 \\ 0 & 20 \end{bmatrix}$  with eigenpairs  $\left(27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ .  
It follows that this is a nodal source..
- $(0, 5)$  has  $\partial \bar{F} = \begin{bmatrix} 32 & 0 \\ 20 & -20 \end{bmatrix}$  with eigenpairs  $\left(32, \begin{bmatrix} 13 \\ 5 \end{bmatrix}\right), \left(-20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(3, 0)$  has  $\partial \bar{F} = \begin{bmatrix} -27 & 3 \\ 0 & 32 \end{bmatrix}$  with eigenpairs  $\left(-27, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(32, \begin{bmatrix} 3 \\ 59 \end{bmatrix}\right)$   
It follows that this is a saddle.
- $(4, 9)$  has  $\partial \bar{F} = \begin{bmatrix} -36 & 9 \\ 16 & -36 \end{bmatrix}$  with eigenpairs  $\left(-24, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right), \left(-48, \begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$   
It follows that this is a nodal sink.

The following picture was pilfered from Levermore's notes:



We see that if an initial population has both  $x$  and  $y$  positive then it will tend towards  $(4, 9)$  but this can happen in a variety of ways.