MATH 240: Introduction to Linear Algebra (1/23)

Instructor: Dr. Wiseley Wong

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Office: MTH 4107

Lecture: Section 03xx: MWF 1-1:50 PM, ESJ 0202

Textbook: *Linear Algebra and its Applications, 6th Edition*, by D. Lay & S. Lay, J. McDonald

Prerequisites : MATH 141

Website: ELMS

Office Hours:

Friday, 3:00-4:30 PM (**in person, MTH 4107**)

Tuesday, 3:00-4:30 PM (**Zoom,** `https://umd.zoom.us/j/2885369947` **(Passcode: J3KCd62QL)**)

Or by appointment.

Discussion Sections:

TA	Section	Time	Location
Shenghao Li	311, 321	TTh 1 PM, 2 PM	MTH 0401, MTH B0427
Qihang Li	312, 322	TTh 1 PM, 2 PM	MTH B0423, MTH 0306
Dohoon Kim	313, 331	TTh 1 PM, 3 PM	MTH 0201, MTH 0103
Yuan-Chih Hsieh	323, 332	TTh 2 PM, 3 PM	PLS 1117, PLS 1119

You must attend your ASSIGNED discussion section.

Course Description/Learning Outcomes

By the end of the course, a student is expected to be familiar with basic concepts of linear algebra: matrix algebra, vector spaces, applications to line and plane geometry, linear equations and matrices, similar matrices, linear transformations, eigenvalues, determinants, inner products, and least-squares problems.

MATLAB/Homework

MATLAB assignments will be given throughout the semester that will be graded. Suggested homework problems will also be given, but are not required for submission. **Matlabs will be submitted online through Gradescope.** Instructions on what to submit will be explained in the assignments.

It is your own responsibility to read over the sections covered in lecture to assure you fully comprehend the material. **(Suggested) Homework problems will be assigned for each section of the textbook that we cover, but it will not be collected or graded.** The homework problems will be posted on ELMS.

Gradescope

Matlabs will be submitted through Gradescope. There is a tab on ELMS that will take you to the website. You can use a scanner or phone app such as Camscanner. It is important to **mark your pages for each problem before submitting.** This allows a smoother process for grading. Moreover, everything must be uploaded by the the time of the due date. **If late assessments are submitted/emailed to me within an hour AFTER the deadline, at least 5 points will be deducted from your score. If it is beyond an hour, the assessment will NOT be accepted. If you do NOT mark your problem numbers by page on Gradescope, the assessment WILL NOT BE GRADED and you will receive a 0.**

Quizzes

There will be about 10 quizzes given in discussion throughout the semester. The **tentative dates** are the following : 2/1, 2/8, 2/15, 2/22, 3/7, 3/14, 3/28, 4/2 (TUESDAY), 4/18, 4/25. Your lowest two quiz scores will be dropped. This includes any absences (excused or not) from illness, interviews, athletics, etc.

Exams

There will be 3 in-class exams and a final exam. The following are **TENTATIVE DATES**:

Exam 1: February 26, 2024 (MONDAY)

Exam 2: April 8, 2024 (MONDAY)

Exam 3: May 1, 2024 (WEDNESDAY)

Final exam: TBA

The final exam will be comprehensive. **The tentative date of the final exam is Saturday May 11, from 1:30-3:30 PM.**

Calculators and electronic devices are prohibited on exams. You must show your work or you will NOT receive full credit.

Any regrade requests must be given BEFORE leaving the classroom the day you are handed back the exam. You must return the exam to your TA with the explanation. The entire exam will be reviewed, which may result in point deductions in other problems.

If there are any conflicts on exam dates, you must notify me at least a week in advance. Formal documentation must be provided for excused absences due to illness or emergencies.

Excused Absences

In the case of a missed assessment due to an excused absence at the time of the assessment, accommodation will be provided for the student, which will be agreed upon up to the discretion of the instructor and student.

Regrading Policy

ANY CLERICAL ERRORS/MISSING SCORES ON ELMS MUST BE REPORTED WITHIN A WEEK WHEN THE SCORES ARE POSTED ON ELMS.

Grade Distribution

Final Exam - 30%

Exam 1-3 - 45% (15% each)

Quizzes - 15%

MATLAB - 10%

Grades will be assigned as follows:

$$A+ : [97 - 100] \qquad C+ : [76 - 80)$$
$$A : [93 - 97] \qquad C : [73 - 76)$$
$$A- : [90 - 93) \qquad C- : [70 - 73)$$
$$B+ : [87 - 90) \qquad D : [62 - 70)$$
$$B : [83 - 87) \qquad F : [0, 62)$$
$$B- : [80 - 83)$$

Academic Integrity and Accommodations:

A description of academic dishonesty can be found here: `https://studentconduct.umd.edu/home/current-students/academic-dishonesty`

The University has a national recognized Honor Code, administered by the Student Honor Council. The Student Honor Council proposed and the University Senate approved an Honor Pledge. It reads:

I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination.

Unless you are specifically advised to the contrary, the Pledge statement should be handwritten and signed on the front cover of all papers, projects, and other academic assignments submitted for evaluation in this course. Students who fail to write and sign the Pledge will be asked to confer with the instructor.

This pledge does not carry the connotation that students cannot be trusted. Rather, the Pledge reflects your public statement of support for academic excellence at UMD, including the highest standards for academic integrity.

Any forms of academic dishonesty, cheating, or illegal activities will be reported and necessary punishments will be administered, which include, but are not limited to suspension or expulsion from the university.

Students with Disabilities

Students with disabilities must register with the Accessibility and Disability Service (ADS) office. Appropriate documentation should be provided to me. The documentation includes an approved ADS form that describes the nature of the requested accommodations, and an individual approval form before every exam. Accommodations recommended by the ADS office for a given student should be discussed with the student.

Sections covered in *Linear Algebra and its Applications, 6th Edition*, by D. Lay, et al.

Chapter 1: 1.1 - 1.5, 1.7 - 1.9

Chapter 2: 2.1 - 2.3, 2.8

Chapter 3: 3.1 - 3.3

Chapter 4: 4.1 - 4.6

Chapter 5: 5.1 - 5.5

Chapter 6: 6.1 - 6.7

Chapter 7: 7.1, 7.4

Sections covered in *Linear Algebra and its Applications, FIFTH Edition*, by D. Lay, et al.

Chapter 1: 1.1 - 1.5, 1.7 - 1.9

Chapter 2: 2.1 - 2.3

Chapter 3: 3.1 - 3.3

Chapter 4: 4.1 - 4.7

Chapter 5: 5.1 - 5.5

Chapter 6: 6.1 - 6.7

Chapter 7: 7.1, 7.4

MATLAB Assignment 1

Due Friday February 16, 2024 11:59 PM EDT (Maryland time) on Gradescope

Instructions:

You can download Matlab here: `https://terpware.umd.edu/Windows/List/232`. On ELMS, see the file MATLAB_basics.pdf to learn how to get MATLAB and do some basic commands first. You may work with up to two other people (**groups of three total**). If you choose to work together, you may simply submit **one copy, and everyone will be receiving the same grade**. **Make sure to include all names when submitting to Gradescope!**

The `Example_Matlab_File.m` in the Files section can be used as your template for your assignment so you don't even have to start from scratch.

Submitting: To get an idea of what you should be submitting, you can first download the file `Example_Matlab_File.m` in the Files section. Open it in Matlab. Then at the top of the program, click on the PUBLISH tab. Click on the Publish button, and it should output an html file with all the code/output. This format is what your Matlab assignment should look like. When you are done with the actual Matlab project, click the Publish button, save this as a PDF, and **upload this to Gradescope.** There is a tab on ELMS that links you to Gradescope. **Remember to separate each problem by a section using the double percent signs.** Even if you have the correct code, **if there is no output, you will NOT receive full credit!**

(separate problems by using double percent signs as shown in the example file!!!!)

1. For this problem, we will keep everything as rational numbers. Copy the following first

```
format rat
```

Consider the following system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 4x_4 &= 3 \\ -3x_1 + x_2 - 2x_4 &= 5 \\ -5x_1 - 3x_2 - 6x_3 + 6x_4 &= 0 \\ -15x_1 - 2x_2 - 9x_3 + 4x_4 &= 13 \end{aligned}$$

(a) Define the augmented matrix for the system in Matlab, and denote it by A.

(b) Apply elementary row operations (see item (10) in MATLAB_basics.pdf) to put the matrix in **ROW REDUCED ECHELON FORM**. Do NOT suppress the output for each operation you do.

(c) Redefine your matrix A as in part (a), and now simply use the `rref` command on the matrix A. **From hereon, we will assume the usage of the command without showing the row operations in Matlab**.

(d) Use the *disp* or *fprintf* command to state what the solution of the system is. If there is no solution, explain why. If there is a unique solution, state what it is. If there are infinite solutions, find the general solution. **You may need to LINE BREAK if your explanation is long. If the grader cannot read what you wrote because the text was chopped off, you won't get full credit!**

2. (Make sure to use double percent signs to make problem 2 a separate section!)
For the following parts, explanations are NOT required, but if you use *disp* or *fprintf* to explain your reasoning, you may get more partial credit if your answer is wrong.

   (a) Let A denote the *augmented matrix* of a linear system with **4 equations with 3 variables that has infinite solutions**. Define a system (i.e. create a matrix A) satisfying this, **and then compute the RREF** of A (this will allow the grader to easily check if your answer makes sense). If such a system is impossible to construct, use *disp* or *fprintf* and say, "`Part (a) is impossible."`

   (b) Let B denote the *augmented matrix* of a linear system with **4 equations with 3 variables that has a unique solution**. Define a system (i.e. create a matrix B) satisfying this, **and then compute the RREF** of B. If such a system is impossible to construct, use *disp* or *fprintf* and say, "`Part (b) is impossible."`

   (c) Let C denote the *augmented matrix* of a linear system with **2 equations with 4 variables that has no solution**. Define a system (i.e. create a matrix C) satisfying this, **and then compute the RREF** of C. If such a system is impossible to construct, use *disp* or *fprintf* and say, "`Part (c) is impossible."`

   You may need to LINE BREAK if your explanation is long. If the grader cannot read what you wrote because the text was chopped off, you won't get full credit! This holds for all parts that you must explain. This is the last warning!

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} -3 \\ 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ -6 \\ -4 \\ -11 \end{bmatrix} \right\}$.

   (a) Look back at the **definition** of a set of vectors being linearly independent. Use *disp* or *fprintf* to explain what it means if set the $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ above is linearly independent. Be exact with the definition.

   (b) Suppose you are given the fact that

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}.$$

   What can you conclude about the set S being linearly independent or linearly dependent? Use *disp* or *fprintf* to briefly explain your answer.

(c) Define a matrix and use appropriate Matlab commands to help you find a non-trivial linear combination of the zero vector using the vectors in S. You may use *disp* or *fprintf* and write something like, '5v1+6v2+7v3+8v4=0" without rewriting all the numbers. Make sure to show all your work.

4. Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 4 \\ 8 \\ -22 \\ 12 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ -33 \\ 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ -23 \\ -2 \end{bmatrix} \right\}.$$

(a) Find TWO different ways to express the vector $\mathbf{z} = \begin{bmatrix} 10 \\ 16 \\ -37 \\ 26 \end{bmatrix}$ as a linear combination of the vectors in S. Use *disp* or *fprintf* and write something like, '5v1+6v2+7v3=z" instead of rewriting all the numbers.

(b) How can you use the previous part to immediately determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{z}\}$ is linearly independent or linearly dependent (and which one is it)? Use *disp* or *fprintf* to briefly explain your answer.

5. Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} -7 \\ 13 \\ -22 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 22 \\ -15 \\ -8 \end{bmatrix} \right\}.$$

(a) Define an appropriate matrix and applying the `rref` command to determine if the vector $\mathbf{v}_1$ can be expressed as a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Make sure to show the output of the RREF. **Then simply state yes or no.**

(b) Define an appropriate matrix and applying the `rref` command to determine if the vector $\mathbf{v}_2$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$. Make sure to show the output of the RREF. **Then simply state yes or no.**

There are parts (c)-(f) continued on the NEXT PAGE.

(c) Define an appropriate matrix and applying the `rref` command to determine if the vector $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$. Make sure to show the output of the RREF. **Then simply state yes or no.**

(d) Define an appropriate matrix and applying the `rref` command to determine if the vector $\mathbf{v}_4$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Make sure to show the output of the RREF. **Then simply state yes or no.**

(e) With knowledge of the previous parts, **use a theorem from class** to deduce whether the set of vectors is linearly independent or linearly dependent. You can use *disp* or *fprintf* to explain your reasoning.

(f) Suppose we only tested if $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$. Can we conclude that S is linear independent or dependent? From your answer in part (e), does this contradict the theorem from class about linear dependence and expressing a vector in terms of the others? You can use *disp* or *fprintf* to explain your reasoning.

There is a short 1 minute video (Gradescope_tutorial.mp4) in the Files section on how to upload to Gradescope. If you are working with other people, MAKE SURE TO ADD THE GROUP MEMBERS WHEN YOU UPLOAD THE FILE. There should be click on a Group Members button for you to list the people in your group that will count as 1 submission. Also make sure to MARK YOUR PAGES AS SHOWN IN THE VIDEO.

See `Example_Matlab_File_Output.pdf` to make sure your submission is of this form. If there is NO OUTPUT, you will receive NO credit!

MATLAB Assignment 2

Due Friday, March 15, 2024 11:59 PM EDT (Maryland time) on Gradescope

Instructions:
On ELMS, see the file MATLAB_basics.pdf to learn how to get MATLAB and do some basic commands first. You may work with up to two other people (**groups of three total**). If you choose to work together, you may simply submit **one copy, and everyone will be receiving the same grade**. **Make sure to include all names when submitting to Gradescope!**

Submitting: To get an idea of what you should be submitting, you can first download the file `Example_Matlab_File.m` in the Files section. Open it in Matlab. Then at the top of the program, click on the PUBLISH tab. Click on the Publish button, and it should output an html file with all the code/output. This format is what your Matlab assignment should look like. When you are done with the actual Matlab project, click the Publish button, save this as a PDF, and **upload this to Gradescope.** There is a tab on ELMS that links you to Gradescope. **Remember to separate each problem by a section using the double percent signs.** Even if you have the correct code, **if there is no output, you will NOT receive full credit!**

(separate problems by using double percent signs as shown in the example file!!!!)

1. Use `format short`.
   (a) Given $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \in \mathbb{R}^2$, define a 2×2 matrix A that scales the x-coordinate of $\mathbf{v}$ by 2 units, and scales the y-coordinate of $\mathbf{v}$ by 3 units.
   (b) Use the matrix A to transform the vector $\mathbf{v}$.
   (c) Define a matrix B that would rotate $\mathbf{v}$ (the vector defined in part (a)) counterclockwise about the origin by an angle $\theta = \pi/3$.
   (d) Use the matrix B to transform the vector $\mathbf{v}$.
   (e) Let T_1 be the transformation that that scales the x-coordinate of a vector by 2 units and scales the y-coordinate by 3 units, and T_2 be the transformation that would rotate a vector counterclockwise about the origin by an angle $\theta = \pi/3$. If T_3 is the composition $T_2 \circ T_1$ i.e. the transformation that first scales the x and y coordinates, and then rotates, find the standard matrix for T_3. Your answer should be a 2×2 matrix. Call this matrix C.
   (f) Find the standard matrix for $T_1 \circ T_2$, and call it D. Does the order matter of scaling or rotating first? Use *disp* or *fprintf* to simply state `YES` or `NO`.
2. Use `format rat`. **The parts are not related.**

(a) Use the inverse matrix command `inv` to help you solve the following *matrix equation for the MATRIX X:*

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}.$$

(b) A square matrix A is said to be **skew-symmetric** if $A^T = -A$ (read carefully!). Let

$$A = \begin{bmatrix} 0 & 2a-3b+c-d & 3a-5b+4c+d \\ -1 & 0 & 5a-8b+5c-d \\ 1 & -8 & a+b-c+d \end{bmatrix}.$$

We are interested in finding what values of a, b, c, and d make A skew-symmetric.

i. Use *disp* or *fprintf* to briefly explain in words all the equations you would need to hold for the matrix to be skew symmetric.

ii. Create an augmented matrix and solve for all the values of a, b, c, and d that would make A skew symmetric. Use *disp* or *fprintf* to state what the values are. If you only have a matrix in RREF with no words, you will NOT receive full credit!

3. Use `format rat`. **The part (c) is not related to (a) and (b).**

(a) Consider the linear system in terms of the constants a and b:

$$\begin{aligned} x_1 + x_2 + ax_3 &= 0 \\ x_1 + x_2 + bx_3 &= 0 \\ ax_1 + bx_2 + x_3 &= 0. \end{aligned}$$

Begin by defining the symbolic variables. Copy the following:

```
syms a b
```

Use determinants to determine when the system has a non-trivial solution. You can use the command `factor` to factor the determinant. Use *disp* or *fprintf* to explain your reasoning. Do NOT just do computations without any explanations!

(b) For the system in part (a), for what values of a and b will there be no solution? Use *disp* or *fprintf* to explain your answer.

(c) Let $A = \begin{bmatrix} 7 & 5 & -10 \\ -4 & -2 & 3 \\ -5 & -4 & 7 \end{bmatrix}$. Suppose B is a 3×3 matrix where $\det(B) = 10$. When necessary, use Matlab to help you compute the following:

i. $\det(AB)$

ii. $\det(2B)$

iii. $\det((A^2B)^{-1})$

Call these 3 values `determinant1`, `determinant2`, `determinant3`. **Do NOT simply write a number as your answer without any work or you will not receive full credit.** You do not have to explain in words what properties you use, but show it in the computation using *disp* or *fprintf* e.g. put `determinant1=det(A)*det(B)` instead of just putting `determinant1=50` (this value is not correct).

4. (a) Create 3×3 matrices A and B **such that neither is the zero matrix** such that (they are NOT equal)

$$(AB)^2 \neq A^2B^2.$$

Compute $(AB)^2$ in one line of code first, then A^2B^2 in another line.

(b) Repeat part (a) now with matrices C and D such that (they ARE equal)

$$(CD)^2 = C^2D^2.$$

(c) If A and B are 2×2 matrices, is it possible for A and B to be both invertible, but $A + B$ to be NOT invertible? If it is impossible, briefly explain why. If it IS possible, provide explicit examples of matrices, and **justify your reasoning using determinants.**

5. (Use `format rat`) Let

$$A = \begin{bmatrix} 2x+1 & x & x+2 \\ x+2 & -2x & -x-3 \\ 2x-1 & x & 3x \end{bmatrix}.$$

(a) Use the `syms` command to symbolically define the variable x.

(b) Look up how to use the `solve` command for Matlab. Use this, along with the determinant to find all values of x for which A does NOT have an inverse.

(c) For any integer $k > 3$, determine all values of x for which A^k will NOT have an inverse. Use *disp* or *fprintf* to **explain** how you deduced your answer. *Hint: Use properties of determinants.*

MATLAB Assignment 3

Due Friday April 12, 2024 at 11:59 PM EDT (Maryland time) on Gradescope

Instructions:

On ELMS, see the file MATLAB_basics.pdf to learn how to get MATLAB and do some basic commands first. You may work with up to two other people (**groups of three total**). If you choose to work together, you may simply submit **one copy, and everyone will be receiving the same grade**. **Make sure to include all names when submitting to Gradescope!**

Submitting: To get an idea of what you should be submitting, you can first download the file `Example_Matlab_File.m` in the Files section. Open it in Matlab. Then at the top of the program, click on the PUBLISH tab. Click on the Publish button, and it should output an html file with all the code/output. This format is what your Matlab assignment should look like. When you are done with the actual Matlab project, click the Publish button, save this as a PDF, and **upload this to Gradescope.** There is a tab on ELMS that links you to Gradescope. **Remember to separate each problem by a section using the double percent signs.** Even if you have the correct code, **if there is no output, you will NOT receive full credit!**

(separate problems by using double percent signs as shown in the example file!!!!)

1. Use `format rat`.

   Let
   $$A = \begin{bmatrix} -3 & -9 & 4 & -7 & -5 & 5 \\ 6 & 18 & -2 & 26 & 3 & 2 \\ -3 & -9 & -1 & -17 & 2 & -5 \\ 1 & 3 & 1 & 7 & 1 & 3 \end{bmatrix}.$$

   (a) Use the `rref` command and then determine a basis for the column space and the kernel for matrix A. You can use *disp* or *fprintf* to show your answer. For simplicity, you may express the vectors using parentheses like it has been done in class.

   (b) Suppose A now is treated as the matrix representation of a linear transformation. Is the transformation one-to-one/injective? Onto/surjective? Use *disp* or *fprintf* to clearly justify your answer.

2. Use `format rat`.

   Let
   $$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the row space of A. Use *disp* or *fprintf*, and **express the vectors as row vectors using brackets** e.g. $\{[3\ \ 4\ \ 6\ \ 1], [1\ \ 3\ \ 5\ \ 7]\}$

(b) Analyze the matrix A closely. Why can we NOT simply looking at the RREF of A and choose the corresponding rows where the pivots are located, to create a basis for row(A)? Use *disp* or *fprintf* to clearly explain your answer. *Hint: What do you observe in the first 3 rows of A, and what did you find in part (a)?*

(c) **Without doing any computations**, does row(A) = col(A)? Use *disp* or *fprintf* is justify your reasoning by either cite a theorem from class, or explaining why they are not equal. You should not be row reducing or altering the matrix in any way.

3. Use `format rat`.

Let

$$W = \text{span}\left(\{(1, 5, -3, -11), (7, 4, -1, 2), (5, 1, 9, 2), (3, -1, 7, 5)\}\right)$$

be a subspace of $\mathbb{R}^4$.

(a) Find a basis for W. Show your work. Use *disp* or *fprintf* to show the basis. For simplicity, you may express the vectors using parentheses like it is done in class.

(b) Let $\mathcal{B}$ denote the basis from the previous part.

(c) Determine $[(19, 18, -13, -10)]_{\mathcal{B}}$. Show your computations/work.

(d) If $[\mathbf{z}]_{\mathcal{B}} = (19, 18, -13)$, find $\mathbf{z}$. Show your computations/work.

4. Use `format rat`. Consider the polynomials (**read carefully**)

$$\begin{aligned}
f_1(x) &= x + x^3 \\
f_2(x) &= -3 + x + 3x^2 + 4x^3 + x^4 \\
f_3(x) &= 3 + x^2 - x^3 + x^5 \\
f_4(x) &= -12 - x - x^2 + 5x^3 + x^4 - 4x^5 \\
f_5(x) &= x^2 + x^5 \\
f_6(x) &= 1 + x
\end{aligned}$$

that lie in $\mathbb{P}_5$. Let

$$W = \text{span}(\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)\}).$$

(a) Denote each of the 6 vectors $\mathbf{v}_i = [f_i(x)]_{\mathcal{B}}$ to be the coordinate vector of $f_i(x)$ relative to the basis $\mathcal{B} = \{1, x, x^2, x^3, x^4, x^5\}$ in $\mathbb{P}_5$. Define these as **column** vectors in Matlab as `v1, v2,...`

(b) Use appropriate commands with the matrix above to help find a basis `S` for W. Use *disp* or *fprintf* to explicitly **show what the basis is as polynomials.**

(c) Does the basis found in the previous part span $\mathbb{P}_5$? Use *disp* or *fprintf* to explain your answer. **PART (d) IS ON THE NEXT PAGE**

(d) Does the entire set $\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)\}$ span $\mathbb{P}_5$? Use *disp* or *fprintf* to explain your answer.

5. Recall the following identities:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\cos(2x) = -1 + 2\cos^2(x)$$

$$\cos(3x) = -3\cos(x) + 4\cos^3(x).$$

Let

$$W = \text{span}\left(\left\{1, \cos^2(x), \sin^2(x), \cos(2x)\right\}\right)$$

be a subspace of the vector space of all real-valued functions.

(a) With the help of the identities, show that $\mathcal{B} = \left(\left\{\cos^2(x), \sin^2(x)\right\}\right)$ is a basis for W. Use *disp* or *fprintf* to explain your reasoning. Look back at the theorem in class above removing vectors to span a set.

(b) Find the $\mathcal{B}$-coordinate vectors for the functions $f(x) = 1$ and $g(x) = \cos(2x)$. Express these as column vectors `u1` and `u2`.

(c) Does $\{f(x), g(x)\}$ form a basis for W? Use *disp* or *fprintf* to explain your reasoning.

(d) Suppose $U = \text{span}(\{1, \cos^2(x), \cos(2x)\})$. Is U a subspace of W? Simply state YES or NO.

(e) With U defined in the previous part, is W a subspace of U? Simply state YES or NO.

MATLAB Assignment 4

Due FRIDAY May 3, 2024 11:59 PM EDT (Maryland time) on Gradescope.

Instructions:

On ELMS, see the file MATLAB_basics.pdf to learn how to get MATLAB and do some basic commands first. You may work with up to two other people (**groups of three total**). If you choose to work together, you may simply submit **one copy, and everyone will be receiving the same grade**. **Make sure to include all names when submitting to Gradescope!**

Submitting: To get an idea of what you should be submitting, you can first download the file `Example_Matlab_File.m` in the Files section. Open it in Matlab. Then at the top of the program, click on the PUBLISH tab. Click on the Publish button, and it should output an html file with all the code/output. This format is what your Matlab assignment should look like. When you are done with the actual Matlab project, click the Publish button, save this as a PDF, and **upload this to Gradescope.** There is a tab on ELMS that links you to Gradescope. **Remember to separate each problem by a section using the double percent signs.** Even if you have the correct code, **if there is no output, you will NOT receive full credit!**

(separate problems by using double percent signs as shown in the example file!!!!)

1. **Use the `short` format for this problem.**

   (a) Input the following matrix in Matlab: $A = \begin{bmatrix} -3 & -2 & -3 & 1 \\ 1 & 2 & 0 & 4 \\ -2 & 1 & 0 & -2 \\ 1 & 3 & -4 & 2 \end{bmatrix}$.

   (b) Input `syms x` to make x a symbolic variable. Treating x as the usual "λ" for an eigenvalue, use the `solve` and `det` commands to solve for the eigenvalues x of A by creating the characteristic equation defined in class. Matlab will not show the roots because the degree 4 polynomial does not factor. You will need to use the `double` command to approximate the roots.

   (c) Compute `eig(A)` to verify your eigenvalues are correct.

   (d) From what you found so far, can you determine if A is diagonalizable? Use *disp* or *fprintf* to explain your answer.

   (e) Compute `[P,D]=eig(A)`, which creates a diagonal matrix D whose diagonal entries are the eigenvalues of A, and P is a matrix whose columns correspond to eigenvectors of the eigenvalues satisfying $AP = PD$. Observe *this does NOT assume A is diagonalizable, as we see in the form it satisfies, it is not written with a* P^{-1}. Matlab tries its "best" to find all possible linearly independent eigenvectors. From observing P, do we have 4 linearly independent eigenvectors? Use *disp* or *fprintf* to explain your answer.

2. **Use the `short` format for this problem.** We look at an example of an inner product that is not the dot product. Consider the inner product over $\mathbb{P}_n$ where

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx.$$

   (a) Define x as a symbolic variable like the previous problem. Look up the `int` (integral) command, and compute the inner product of $f(x) = 7 - x^2$ and $g(x) = 5 + 5x + 4x^2$. (make sure to put an asterisk when multiplying with x).

   (b) Are the two functions orthogonal? Use *disp* or *fprintf* to briefly explain your answer.

   (c) Determine the length/norm of the function $h(x) = 4 - x - x^2 - x^3$. Use the `double` function to express your answer as a decimal.

   (d) Create two **degree one polynomials** $h_1(x)$ and $h_2(x)$ so that $h_1(x)$ and $h_2(x)$ are orthogonal. There is no "trick." Just test some polynomials and play around with the coefficients. Use *disp* or *fprintf* to justify what you did and show why it works. You will not receive full credit otherwise!

3. **Use `short` format**.

   (a) Define the vectors $(2, 3, -3, -6)$, $(6, -1, 4, 1)$, $(0, 5, -3, 6)$, $(-4, 5, -2, 4)$ as **column vectors**. Label them `u1`, `u2`, `u3`, `u4`.

   (b) Define `A=[u1 u2 u3 u4]`

   (c) We now apply the Gram-Schmidt process to the vectors. Input

```
v1=u1
v2=u2-dot(u2,v1)/dot(v1,v1)*v1   %continue defining v3 and v4 similarly
```

   Continue the process and define `v3` and `v4` by the Gram-Schmidt process.

   (d) From the vectors in the previous part, create an **orthonormal basis** by dividing by the magnitude. You can use the `norm` command to help. Denote these vectors as `w1`, `w2`,...

   (e) Define `Q=[w1 w2 w3 w4]`.

   (f) Define $R = Q^T A$, and check that $A = QR$.

   (g) Input

```
[Q1,R1] = qr(A,0)
```

   to find a QR-factorization immediately. Observe your matrices are slightly different from our computation! This is because an orthonormal basis is not unique (think about how the first vector you defined could have been any of the 4).

4. **Use `rat` format.** Suppose

$$\mathcal{B}_1 = \{2 + x + x^2, 1 - x + x^2, 1 + x + x^2, 3 - x^3 + x^4, x + x^2 - x^4\}$$
$$\mathcal{B}_2 = \{1 + x, 2x, 1 + x^2 + x^3, 2x + x^2 + x^4, 1 - x^3 - x^4\}$$

be two bases for $\mathbb{P}_4$.

(a) Find the change of coordinates matrix $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$. It may be efficient to set up an augmented matrix like in class and then extract the appropriate columns (see item 12 in the MATLAB basics PDF in the Files section of ELMS). Denote the matrix by `M1`.

(b) Using whatever method you choose, find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$. Denote this by `M2`.

(c) Determine the coordinate vector $[3 - 7x + x^4]_{\mathcal{B}_1}$. Denote this by `coordB1`. You could generalize the following code to help you set up the system to solve for the coordinate vector (or do whatever method you choose).

```
a=[1 2;3 4] %original matrix
b=[5;6] %new column vector
c=[a b] %creates the (augmented) matrix with the b vector as the last column
```

(d) With the answer in part (c), use either part (a) or part (b) to determine the coordinate vector $[3-7x+x^4]_{\mathcal{B}_2}$. Denote this by `coordB2`. Not required, but how can you check your answer? It may be worth doing anyway to assure you don't lose points from a careless computational mistake!

5. **Use `rat` format.**

(a) Let W be the plane in $\mathbb{R}^3$ given by

$$W = \{(x, y, z) : 5x - 3y + z = 0\}.$$

Find a basis for W, and denote these column vectors as `u1`, etc. Use *disp* or *fprintf* to explain a few of your steps. *Hint: If you are uncertain about finding the basis, what if the question asked you to prove W is a subspace by not using the method where you show the 3 properties of a subspace? Proceed to proving W is a subspace in that manner instead.*

(b) Use the Gram-Schmidt process (like problem 3) to find an **orthogonal basis.**

(c) Determine the point in W that is closest to $\mathbf{y} = (7, 7, 7)$. Define this point as `closest`.

(d) Use the `norm` command to help you determine the distance that $(7, 7, 7)$ is from the plane. Denote this value by `dist`. You may want to draw out a plane and the vector $\mathbf{y}$ to visually see what vector you want so that the length/magnitude of that vector gives the distance from $(7, 7, 7)$ to the plane.

Contents


```
%Bob Smith, Mike Smith, Joe Smith
%Section 101
%Matlab Project 1
```

Problem 1

```
3+4
disp('We just computed a sum')
```

```
ans =

     7

We just computed a sum
```

Problem 2

```
A=[1 2; 3 4; 5 6]

A  %Shows the matrix again
A;  % Suppressed;  Matrix A is not shown again
```

```
A =

     1     2
     3     4
     5     6


A =

     1     2
     3     4
     5     6
```

Problem 3

```
syms x
f(x)=sin(x)
f(pi)

disp('This shows that if we evaluate the sine function at pi, the value is 0')
```

```
f(x) =

sin(x)


ans =

0

This shows that if we evaluate the sine function at pi, the value is 0
```

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Getting started with MATLAB

To get MATLAB, you can go here:
`https://terpware.umd.edu/Windows/List/232`
There should be installation instructions on the website.

After opening MATLAB, one can see the "Command Window" where you can enter basic stuff in the line $fx >>$. You can type in some basic math operations and hit ENTER to get the output. For example, typing in 3/4 and hitting ENTER will yield 0.7500.

Of course, there may be times when you want to do multiple operations and stuff at once, where it will be annoying to have to input one thing at a time and hit ENTER each time. So instead, we look above at the EDITOR window. Create a $.m$ file here and save it on your computer (e.g. assignment1.m). Now this script allows you to enter multiple things until you choose to RUN it. For instance, one can put it $3 + 4$, hit ENTER, $4 + 5$, hit ENTER (hitting enter is just creating a new line of code that will be compiled later). Nothing is evaluated here. Now up at the top, click on the RUN button. The outputs of both the sums should be in the Command Window.

Here are some very basic tips when typing in code in the EDITOR window.

1. If you put a semicolon (;) after a line of code, it will suppress the output. For example, $3 + 4$; would not output 7 (try it!).

2. If you put a percent sign (%), the following line of code will be treated as text (commenting; this can occur when doing multiple step computations where you may have to remind yourself that this equation found volume with disc method, another equation was using shell method, etc)

3. If you put double percent signs (%%), followed by a space, then text (Problem X) and then hit ENTER, it will separate the script into sections. **You MUST do this to separate different problems and parts in the assignment. After the double percent signs, put another percent sign, followed by "Problem X", to distinguish what problem you are doing in each section.**

4. Asterisks ($*$) must be used for products on numbers/functions i.e. if you put (2)(3), MATLAB will not know you want to multiply. Similarly, if you put $2x$, it will not know you are multiplying x by 2. You have to input as $2 * x$.

5. $\sin(3), \cos(5), \pi, e^3$ are inputted as

```
sin(3)
cos(5)
pi
exp(3)
```

6. Matlab does operations often in terms of vectors, **which must be in bracket form**. For example, you can define a vector by typing in the following (**you can also put commas between entries**)

```
a=[4 2 5];
b=[6, 3, 1];
a+b
dot(a,b)
cross(a,b)
norm(a)
```

will output the sum, dot product, and cross product of the two vectors, and the magnitude of the first vector (magnitude/length is given by *norm*). Notice the semicolons for the first two lines are suppressed so they don't get outputted.

7. MATLAB has values approximated automatically (e.g. π), but one can make them "exact" with a symbolic representation. This is also used to define variables. For example, when entering pi, the output is 3.1416. To keep keep it as a symbol, we use *sym* or *syms* (the latter can always be used, and is more handy for multiple things you define as symbolic)

```
syms pi;
3*pi
```

will yield 3*pi, treating pi as a symbol instead of outputting 9.4248. **To remove this symbolic definition (in general for any terms you make symbolic)**, one uses 'clear all'

```
syms pi;
3*pi
clear all
3*pi
```

will output 3*pi and then 9.4248 after interpreting 'clear all'

8. We use the *syms* command to define variables, allowing us to create functions, differentiate, and integrate. We declare functions in this manner.

```
syms f(x);
f(x) = x+3+sin(x);
diff(f(x),x)
int(f(x))
int(f(x),0,pi)
```

will create the function $f(x) = x + 3 + \sin(x)$, differentiate it with respect to x, find the indefinite integral of $f(x)$, and find the definite integral from $x = 0$ to $x = \pi$. Note that

```
diff(f(x),3)
```

would take the third derivative with respect to x. The x from line 3 above isn't required, but is useful in the future when taking partial derivatives.

Note that these bits of code I'm showing are not continuous! When you make a file when hundreds of lines of code, you may need to type in *clear all* when defining more functions and variables!

The following are basic commands for matrices.

9. We can create a matrix using brackets. A semicolon will begin a new row. For example,

```
A=[1 2 3; 4 5 6; 7 8 9]
```

will output $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

To determine a specific entry, say in row 2 column 3, input

```
A(2,3)
```

which will ouput 6.

To determine a specific row, say row 2, we use a colon (:), input

```
A(2,:)
```

which will output

$$4 \;\; 5 \;\; 6$$

To determine a specific column, say column 3, the idea is similar, input

```
A(:,3)
```

which will output

$$\begin{matrix} 3 \\ 6 \\ 9 \end{matrix}$$

To interchange rows, say swap rows 2 and 3, input

```
A([2 3],:) = A([3 2],:)
```

which will output $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$. **This will be defined as the new matrix A now!**

10. We can **apply row operations** to update the matrix. So suppose we first define a matrix A as follows

```
A=[1 2 3; 4 5 6; 7 8 9]
```

If we want to get in RREF, we want to multiply row 1 by -4 and add it to row 2 i.e. -4R1+R2. We input

```
A(2,:)=-4*A(1,:)+A(2,:)
```

Now the matrix A is updated with this elementary row operation i.e. if you type in A in the script following this, it outputs $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$.

11. Applying row operations can still be super tedious. Instead, Matlab can determine the **RREF of a matrix immediately** using the `rref` command. For example, if we input

```
A=[1 2 3; 4 5 6; 7 8 9]; % suppressing the matrix output
rref(A)
```

will output $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

12. To **extract columns** from a matrix, if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$,

```
A=[1 2 3 4; 5 6 7 8; 9 10 11 12];
A(:,[1 3 4])
A(:,2:4)
```

will extract columns 1,3,4, and columns 2 through 4, respectively. So the outputs would be $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 9 & 11 & 12 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 10 & 11 & 12 \end{bmatrix}$

13. Matrix operations in Matlab are done in a similar manner as with numbers. **Make sure to include asterisk (*) when multiplying matrices**. So

```
A+B
A*B
A^4
4*A
```

will add, multiply, take the 4th power of A, and scale the matrix A by 4, respectively.

14. The **rank** of a matrix A can be computed by the command `rank(A)`.

15. The **transpose** of a real matrix A can be computed with either notation:

```
A=[1 2 3; 4 5 6]
A' %this finds the transpose
transpose(A) %this also finds the transpose
```

    The `A'` actually computes the **conjugate transpose** or **adjoint** of A. It will take the complex conjugate of all entries of A^T (but there is obviously no effect if all entries are real).

16. The **eigenvalues** of a square matrix A can be computed by the command `eig(A)`.

17. More efficiently, if we can **diagonalize** the matrix as $A = PDP^{-1}$ (where D is the diagonal matrix whose diagonal are the eigenvalues of A, and the columns of P are the corresponding eigenvectors),then the matrix D and P can be obtained by the command `[P,D]=eig(A)`.

Math 240, NOT A PRACTICE EXAM

Instructor: Dr. Wiseley Wong

Instructions: Five points will be deducted if you are in the wrong classroom.

NO calculators or any external devices are allowed. **Show all your work.** An answer without an explanation may not provide full credit.

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Some practice problems for exam 1

1. Solve the following systems by row reducing the augmented matrix. If there is a unique solution, state what it is. If there is no solution, explain why. If there are infinite solutions, express your solution in parametric vector form.

   (a)

$$\begin{aligned} x + y + z &= 3 \\ y + 2z &= -5 \\ x + 2y + 4z &= -4. \end{aligned}$$

   (b)

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 5x_4 &= 3 \\ 2x_1 - 6x_2 + x_3 - 7x_4 &= 2 \\ x_1 - 3x_2 - 4x_3 + x_4 &= -5. \end{aligned}$$

2. For the following UNRELATED statements, determine if the statement is true or false. If it is TRUE, simply state TRUE. If it is FALSE, **provide an explicit counterexample** i.e. an explicit example that shows the statement is false.

   (a) Suppose A is a 3×3 matrix that is the standard matrix of a linear transformation. Then the transformation is always one-to-one/injective.

   (b) Suppose A is a 3×3 matrix that is the standard matrix of a linear transformation. Then the transformation is sometimes onto.

   (c) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be vectors in $\mathbb{R}^n$. If $n \geq 3$, then the set of vectors must be linearly independent.

   (d) Let A be $m \times n$. If $m > n$, then the set of ROW vectors (each row represents 1 vector) must be linearly dependent.

   (e) If A is the matrix representation of a linear transformation and we know the columns of A form a linearly independent set, then the transformation is always onto.

   (f) If the column vectors of A form a linearly independent set, then so do the column vectors of A^T.

3. Determine if $(0, 10, 8)$ lies in

$$\text{Span}(\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\}).$$

   If it does lie in the span, find an explicit linear combination.

   Is the set $\text{Span}(\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\})$ linearly independent? What about the set $(\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\})$?

4. Suppose $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_m$. If we add another vector $\mathbf{v}_{m+1}$, will $\mathbf{v}$ sometimes, always, or never be in Span$(\{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_m, \mathbf{v}_{m+1}\})$?

5. Matlab assignment 2, problems #1 and #4.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a transformation where $T(x_1, x_2, x_3) = (x_1x_3, x_2)$. Show T is NOT a linear transformation.

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a transformation where $T(x_1, x_2, x_3) = (x_1 + x_3, x_2)$. Find the standard matrix representation of the transformation. Is this transformation onto? Justify.

8. Suppose you have 5 vectors in $\mathbb{R}^7$, and none are the zero vector, and every vector is different. You create a matrix where each column is one of those vectors. Treating this matrix as the standard matrix of a linear transformation, is the transformation sometimes, always, or never one-to-one?

9. Let

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 6 & -3 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 4 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 4 \end{bmatrix}.$$

Compute the following, if it exists. If it does not, just write DNE.

(a) CC^T

(b) $(B + A)^2$

(c) AB

(d) CB^T

(e) A^TC

10. If the matrix of a linear transformation is given by

$$\begin{bmatrix} 1 & -1 & 4 \\ -2 & 0 & 2 \\ -3 & 4 & -8 \end{bmatrix}.$$

Is the transformation one-to-one/injective? Onto/surjective? Justify.

11. Why will you lose points on the exam if you say any of the following?

(a) "...so the matrix is linearly independent."

(b) "There's more columns than rows so it's linearly dependent."

(c) "If the linear transformation is $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, it has a free variable so it's not one-to-one."

(d) "If the linear transformation is $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, there's infinite solutions so it's onto."

(e) "If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, they're not multiples so it's linearly independent."

True or False: F, T, F, T, F, F

Math 240, NOT A PRACTICE EXAM

Instructor: Dr. Wiseley Wong

Name:

Section:

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Some practice problems for exam 2

1. You should memorize the inverse of a general 2×2 matrix and use the formula without showing any steps of the algorithm. This is recommended to save time!

2. Find the unique solution by using the inverse of the coefficient matrix.

   (a)

$$\begin{aligned} 3x + y &= 11 \\ 4x - 2y &= 14 \end{aligned}$$

   (b)

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 1 \\ -x_2 + x_3 &= 0 \\ x_2 - 2x_3 &= -1. \end{aligned}$$

3. For the following UNRELATED statements, determine if the statement is true or false. If it is TRUE, simply state TRUE. If it is FALSE, **provide an explicit counterexample** i.e. an explicit example that shows the statement is false.

   (a) Assume $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3 \in \mathbb{R}^3$ are NOT the zero vector, and no two vectors are scalar multiples of each other. Then the set of 3 vectors form a basis in $\mathbb{R}^3$.

   (b) Let A be a 5×8 matrix. Then the rank of the matrix could be 4.

   (c) Let A be a 10×5 matrix. Then $\dim(\ker(A)) \geq 2$.

   (d) $\{1 + x, 2 + x, 3 + x\}$ form a linearly independent set in $\mathbb{P}_5$.

   (e) Let A and B be *row equivalent* matrices. If A and B are the matrix representations of a linear transformation, then we must have $\ker(A) = \ker(B)$?

4. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation that is the second derivative operator. That is,

$$T(c + bx + ax^2) = 2a.$$

   Find a basis for range(T) and ker(T) as *polynomials*. You do not have to show any work if you can simply see what the bases should be. Verify the rank-nullity theorem by clearly stating the dimensions.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be given by $T(x, y, z) = x + y + z$. Find a basis for the range(T) and ker(T), and verify the rank-nullity theorem.

6. If $\mathcal{B} = \{3x^2, x - 1, 4\}$ is a basis for $\mathbb{P}_2$, find the coordinate vector for $f(x) = 6x^2 + 4$.

7. Let
$$A = \begin{bmatrix} 1 & 4 & 3 & -1 \\ 4 & 8 & 2 & -3 \end{bmatrix}.$$
Find a basis for $\mathrm{col}(A)$, $\mathrm{row}(A)$, and $\ker(A)$.

8. Find a basis for $\mathbb{R}^3$ that includes the vectors $(-1, 0, 2)$ and $(0, 1, 1)$. How would you verify you are correct?

9. Let $S = \{(a, b, c, d) : a^2 = b^2, a, b, c, d \in \mathbb{R}\}$. Is S a subspace of $\mathbb{R}^4$? If it is, justify why it is a subspace. If it is not, state a property it fails.

10. Let set $S = \{(a, b, c) : 5a - 4b + 7c = 0, a, b, c \in \mathbb{R}\}$. Is S a subspace of $\mathbb{R}^3$? If it is, justify why it is a subspace. If it is not, state a property it fails.

11. Let $\mathcal{B}_1 = \{(1, 7), (-3, -3)\}$, and $\mathcal{B}_2 = \{(-2, 4), (5, -1)\}$.

    (a) Find the change of coordinates matrix $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$.

    (b) Find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$.

    (c) By inspection, determine the coordinate vectors $[(-7, 5)]_{\mathcal{B}_2}$.

    (d) Using (c), and the appropriate change of coordinates matrix, determine how to express $(-7, 5)$ as a linear combination in terms of basis $\mathcal{B}_1$.

12. Let $\mathcal{S} = \{(1, 0), (0, 1)\}$, $\mathcal{B}_1 = \{(3, 4), (5, 7)\}$, $\mathcal{B}_2 = \{(2, -1), (3, -1)\}$

    (a) Find the change of coordinates matrix $\underset{\mathcal{S} \leftarrow \mathcal{B}_1}{P}$.

    (b) Find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{S}}{P}$.

    (c) Find the change of coordinates matrix $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$.

    (d) Find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$.

    (e) By inspection, determine the coordinate vectors $[(8, 11)]_{\mathcal{B}_1}$.

    (f) Using (e), and the appropriate change of coordinates matrix, determine how to express $(8, 11)$ as a linear combination in terms of basis $\mathcal{B}_2$.

13. Let $\mathcal{B}_1 = \{1, x, 1 + x + x^2\}$, $\mathcal{B}_2 = \{1, x, 2 + x^2\}$ be two bases for $\mathbb{P}_2$. Find the change of coordinates matrices $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$ and $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$. It can be good practice doing row operations by both solving the system/row reducing AND computing the inverse of the coordinate matrix, and making sure you get the same answer.

14. Matlab 3, problems 1-4.

True and False: F, T, F, F, T

Math 240, NOT A PRACTICE EXAM

Instructor: Dr. Wiseley Wong

Name:

Section:

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Some practice problems for exam 3

1. *Note: None of these parts shouldn't take more than a minute.*

   (a) Determine the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

   (b) Find an eigenvector corresponding to the first eigenvalue you found in part (a).

   (c) Is A diagonalizable? Justify.

   (d) Is A invertible? Justify.

2. (a) Determine the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 5 \\ 3 & 4 & -2 \end{bmatrix}$.

   (b) Find the eigenspace corresponding to the largest eigenvalue.

   (c) Determine the geometric multiplicity of the smallest eigenvalue.

   (d) Is A diagonalizable? Justify. If it is, express it in the form $A = PDP^{-1}$.

3. Suppose A is a square matrix such that $\mathbf{v}$ is an eigenvector with eigenvalue λ. Show that λ^2 is an eigenvalue of A^2.

4. Find the least squares solution to the system

$$\begin{aligned} x_1 + x_2 &= 2 \\ -3x_1 + 3x_2 &= 0 \\ -2x_1 + x_2 &= 1 \\ 2x_1 - 4x_2 &= -3 \end{aligned}$$

5. Find the least squares solution to the system

$$\begin{aligned} x_1 - x_2 &= 3 \\ -2x_1 + x_2 &= -2 \\ x_1 - 3x_2 &= -1. \end{aligned}$$

   Recognize if it is easier to solve using $\text{pinv}(A)$ or not.

6. If $W = \text{Span}(\{(1,1)\})$, determine the point in W that is closest to the point $y = (3,4)$.

7. From the previous problem, express vector $\mathbf{y} = (3,4)$ as a sum of 2 vectors, one in the subspace W, and one in $W^\perp$.

8. If $W = \text{Span}(\{(-2,3,3),(-3,0,-2)\})$, express $\mathbf{y} = (-5,2,2)$ as a sum of two vectors $\mathbf{w}' + \mathbf{z}$, where $\mathbf{w}' \in W, z \in W^\perp$. *Note: There will be fractions, but it should not be too bad to compute.*

9. Consider the subspace W spanned by the vectors $(1,0,1)$ and $(0,1,1)$. Use the Gram-Schmidt process to find an orthogonal basis for W. Then find an *orthonormal* basis for W.

10. For the following UNRELATED statements, determine if the statement is true or false. If it is TRUE, simply state TRUE. If it is FALSE, **provide an explicit counterexample** i.e. an explicit example that shows the statement is false.

    (a) Suppose A has eigenvalue λ with algebraic multiplicity 1. Then there is exactly 1 eigenvector corresponding to λ.

    (b) In $\mathbb{R}^5$, we cannot find an orthogonal set of vectors, none of which is the zero vector, containing 6 vectors.

    (c) Every least squares problem yields a unique solution (no need to find a counterexample if it is false).

    (d) If $(1,1,-1)$ lies in the subspace $W = \text{span}(\{(1,2,3),(-1,0,5)\})$, then the projection of $(1,1,-1)$ onto W is $(1,1,-1)$ itself.

    (e) Every matrix has eigenvalues.

11. Suppose $\mathcal{B}_1 = \{(2,4),(1,7)\}$, and $\mathcal{B}_2 = \{(1,3),(-3,-5)\}$ are bases of $\mathbb{R}^2$.

    (a) If T is a linear transformation such that $T(2,4) = (-2,-2)$ and $T(1,7) = (5,-1)$, determine the matrix for T relative to basis $\mathcal{B}_1$ and $\mathcal{B}_2$.

    (b) Determine the coordinate vector $[(3,11)]_{\mathcal{B}_1}$ by observation.

    (c) Use the previous two parts to help express $T(3,11)$ as a linear combination of the basis vectors in $\mathcal{B}_2$.

12. Suppose $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ is the matrix representation for a linear transformation T. Let $\mathcal{B} = \{(3,-2),(-4,3)\}$ be a basis for $\mathbb{R}^2$.

    (a) Find the $\mathcal{B}$-matrix for this mapping. That is, find $[T]_{\mathcal{B}}$. *Note: It may be useful to know the 2×2 inverse formula.*

    (b) Determine the coordinate vector $[(7,-5)]_{\mathcal{B}}$ by observation.

    (c) Use the previous two parts to help express the mapped vector $T(7,-5)$ as a linear combination of the basis vectors in $\mathcal{B}$.

13. Find the complex eigenvalues $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$. Then find a corresponding eigenvector for each eigenspace.

14. Construct a 3×3 matrix where there is only one distinct eigenvalue, but A is still diagonalizable. If it cannot be done, explain why.

15. Diagonalize the matrix $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$, expressing it similar to a diagonal matrix D.

True and False: F, T, F, T, F

(1) [5] Let **(read carefully)**

$$A^T = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & -5 \\ -1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find $(AB)^T$.

Solution:

$$\begin{bmatrix} 3 & -4 & 6 \\ 0 & 4 & -5 \end{bmatrix}$$

(2) [15] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 - x_3, x_2 + x_3).$$

(a) [7] Find the standard matrix representation A of the transformation.
(b) [4] Is T one-to-one/injective? Justify.
(c) [4] Is T onto/surjective? Justify.

Solution:

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) NO

(c) YES

Quiz 8 Solutions

(1) [20] Let $\mathcal{B}_1 = \{1+2x, x, x^2\}$ and $\mathcal{B}_2 = \{1-2x, x, 2x+x^2\}$ be two bases for $\mathbb{P}^2$.

(a) Find the change of coordinates matrix $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$.

(b) Find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$.

(c) By inspection, determine the coordinate vector $[1+2x+3x^2]_{\mathcal{B}_1}$.

(d) **Using the appropriate parts before**, with the change of coordinates matrix, determine how to express $f(x) = 1+2x+3x^2$ as a linear combination of the vectors in $\mathcal{B}_2$. **Show the linear combination.**

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

(d) $(1-2x) - 2(x) + 3(2x+x^2)$

(2) [20] Let $\mathcal{B}_1 = \{1 + 2x, x, x^2\}$ and $\mathcal{B}_2 = \{1 - 2x, x, 2x + x^2\}$ be two bases for $\mathbb{P}^2$.

(a) Find the change of coordinates matrix $\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$.

(b) Find the change of coordinates matrix $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P}$.

(c) By inspection, determine the coordinate vector $[1 - 6x - 2x^2]_{\mathcal{B}_2}$.

(d) **Using the appropriate parts before**, with the change of coordinates matrix, determine how to express $f(x) = 1 - 6x - 2x^2$ as a linear combination of the vectors in $\mathcal{B}_2$. **Show the linear combination.**

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(d) $(1 + 2x) - 8(x) - 2(x^2)$

Suggested Exercises (no submission) for *Linear Algebra and its Applications, 5th Edition*, by D. Lay & S. Lay, J. McDonald

Section 1.1: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 25, 27, 29, 31, 33, 34

Section 1.2: #1, 3, 5, 7, 9, 11, 13, 15, 16, 17, 19, 21, 22, 23, 31, 33

Section 1.3: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 25

Section 1.4: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 25, 27, 29, 30, 33, 34

Section 1.5: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 35, 37, 39

Section 1.7: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 22, 23, 25, 27, 29, 33-38

Section 1.8: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 22, 23, 24, 25, 26, 32, 33

Section 1.9: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 22, 23, 24, 25-28, 29, 30

Section 2.1: #1, 2, 3, 5, 7, 8, 9, 10, 11, 13, 15, 16, 19, 20, 21, 23, 27

Section 2.2: #1, 3, 7, 9, 10, 11, 13, 15, 17, 19, 29, 31, 33

Section 2.3: #1, 3, 5, 11, 12, 13, 14, 15, 17, 19, 21, 23, 27, 29, 33

Section 3.1: #1, 3, 7, 9, 11, 13, 19, 20, 21, 22, 23, 24, 37, 38, 39, 40

Section 3.2: #1, 2, 3, 4, 5, 9, 15-21, 23, 24, 26-29, 31-35, 39, 40

Section 3.3: #19, 21, 23, 27, 31, 32

Section 4.1: #1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21-31

Section 4.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 26, 27, 29, 31, 33, 35

Section 4.3: #1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 22, 23, 25, 27, 29, 30, 31, 33

Section 4.4: #1, 3, 5, 7, 9, 11, 13, 15, 16, 17, 21, 27, 29, 31, 32

Section 4.5: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 20, 21, 23, 25, 27, 29, 30, 31

Section 4.6: #1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 21, 23, 25, 27, 28, 29

Section 4.7: #1, 3, 5, 7, 9, 11, 12, 13

Section 5.1: #1, 3, 4, 5, 7, 9, 13, 15, 16, 17, 18, 21, 22, 23, 24, 25, 29

Section 5.2: #1, 3, 7, 11, 13, 15, 17, 18, 19, 21, 22, 24

Section 5.3: #1, 3, 4, 5, 7, 9, 11, 13, 17, 19, 21-27

Section 5.4: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 20, 21, 25, 26

Section 5.5: #1, 3, 5, 7, 9, 13, 15, 23, 24

Section 6.1: #1, 5, 7, 9, 11, 13, 14, 15, 17, 19, 20, 22, 23, 24, 25, 29, 31

Section 6.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 27, 29

Section 6.3: #1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 18, 21, 22

Section 6.4: #1, 3, 5, 7, 11, 13, 15, 17, 18, 19

Section 6.5: #1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 21

Section 6.6: #1, 3, 5, 7, 9, 12, 13 (Note: The [**M**] means Matlab is needed for the computation)

Section 7.1: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 25, 26, 27, 33, 34

Section 7.4: #1, 3, 5, 7, 9, 11, 15

Suggested Exercises (no submission) for ***Linear Algebra and its Applications, 6th Edition***, by D. Lay & S. Lay, J. McDonald

Section 1.1: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 24, 25, 27-34, 35, 37, 39, 41

Section 1.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25-34, 35, 37, 41, 43

Section 1.3: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23-32, 33

Section 1.4: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23-34, 35, 39, 40, 42, 44

Section 1.5: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27-36, 39-45, 47, 49, 51

Section 1.7: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21-31, 33, 35, 39-44

Section 1.8: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21-33, 40, 41

Section 1.9: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 22, 23-38

Section 2.1: #1, 2, 3, 5, 7, 8, 9, 10, 11, 13, 15-24, 27-29, 31, 35

Section 2.2: #1, 3, 7, 9, 11-20, 21, 23, 25, 27, 29, 39, 41, 43

Section 2.3: #1, 3, 4, 5, 7, 8, 11-23, 25, 27, 29, 31, 35, 37, 41

Section 3.1: #1, 3, 7, 9, 11, 13, 19-24, 37, 38, 39-42, 45

Section 3.2: #1, 2, 3, 4, 5, 9, 15-21, 23, 24, 26-35, 37, 40, 41, 45, 46, 51, 52

Section 3.3: #19, 21, 23, 27, 31, 32, 36

Section 4.1: #1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21-36

Section 4.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25-39, 47

Section 4.3: #1, 3, 5, 7, 9, 11, 13, 15, 19, 21-33, 35, 39-43

Section 4.4: #1, 3, 5, 7, 9, 11, 13, 15-21, 25, 31, 33, 35, 36

Section 4.5: #1, 3, 5, 7, 9, 11, 13, 15, 17-27, 29, 31, 33, 35, 37, 39, 41, 43-48

Section 4.6: #1, 3, 5, 7, 9, 11-15

Section 5.1: #1, 3, 4, 5, 7, 9, 13, 15, 16, 17, 18, 21-33

Section 5.2: #1, 3, 7, 11, 13, 15, 17, 18, 19, 21-30, 32

Section 5.3: #1, 3, 4, 5, 7, 9, 11, 13, 17, 19, 21-33

Section 5.4: #1, 3, 4, 5, 7, 9, 11, 13, 15, 17-28

Section 5.5: #1, 3, 5, 7, 9, 13, 15, 23-26

Section 6.1: #1, 5, 7, 9, 11, 13, 14, 15-28, 30, 31, 33, 37, 39

Section 6.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23-32, 35, 37

Section 6.3: #1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 18, 21-30

Section 6.4: #1, 3, 5, 7, 11, 13, 15, 17-23

Section 6.5: #1, 3, 5, 7, 9, 11, 13, 15, 17-26

Section 6.6: #1, 3, 5, 7, 9, 13, 15, 18 (Note: The [**T**] means Matlab is needed for the computation)

Section 7.1: #1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 25-32, 33

Section 7.2: #1, 3, 5, 7, 9, 11, 13, 15, 17, 21-31

Section 7.4: #1, 3, 5, 7, 9, 11, 15