NCERT Physics 12.7 Q6

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Question: Obtain the resonance frequency of a series LCR circuit with $L = 2.0\,H$, $C = 32\,\mu F$, and $R = 10\,\Omega$. What is the Q-value of the circuit.

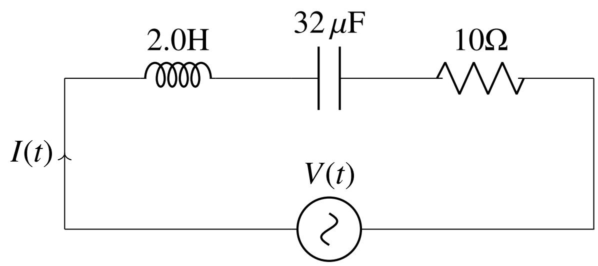

Fig. 1. LCR Circuit

Solution: In Figure Fig. 1 the following information is provided:

Symbol	Value	Description
L	$2.0\ H$	Inductance
C	$32\ \mu F$	Capacitance
R	$10\ \Omega$	Resistance
Q	$\frac{V_L}{V_R}$	Quality Factor
V_L	$sLI(s)$	Voltage across inductance
V_C	$RI(s)$	Voltage across capacitor

TABLE I
PARAMETERS

The equivalent s domain of the circuit is :

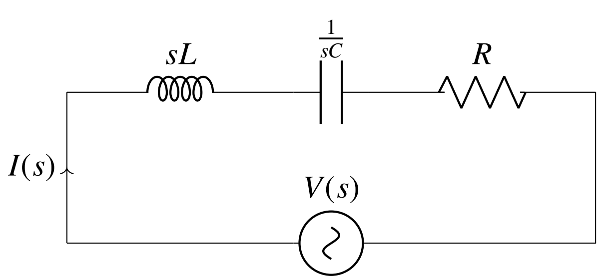

Fig. 2. LCR Circuit in s-domain

Now, the voltage transfer function ($\frac{V(s)}{I(s)} = H(s)$) is given by Ohm's Law in the Laplace domain:

$$\frac{V(s)}{I(s)} = R + sL + \frac{1}{sC} \tag{1}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{2}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{3}$$

s can be expressed in terms of resonant frequency as

$$s = j\omega_0 \tag{4}$$

Comparing equations (3) and (4), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{5}$$

Quality Factor (Q) of an LCR circuit is defined as the ratio of voltage across inductor or capacitor to that across the resistor at resonance.

The voltage across inductor is :

Assuming a sinusoidal current $I_s(t) = I_0 \sin(\omega t + \phi)$, where:

Now, differentiate $I_s(t)$ with respect to time to find $\frac{di}{dt}$:

$$\frac{di}{dt} = I_0\omega\cos(\omega t + \phi). \tag{6}$$

Now, substitute $\frac{di}{dt}$ into the equation for voltage across the inductor:

$$V_s = L\frac{di}{dt} \tag{7}$$

$$= LI_0\omega\cos(\omega t + \phi). \tag{8}$$

Express the cosine term using Euler's formula:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}. \tag{9}$$

Substitute this into the equation:

$$V_s = LI_0\omega\left(\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}\right). \tag{10}$$

Simplify further:

$$V_s = j\omega LI_0 \sin(\omega t + \phi). \tag{11}$$

$$V_s = j\omega LI_s. \tag{12}$$

$$V_s = sLI_s. \tag{13}$$

The voltage across resistor is :

By Ohm's Law, we know that $V = IR$. So, the voltage across a resistor (V_R) is given by $I_s \cdot R$. In mathematical form:

$$V_R = I_s \cdot R \tag{14}$$

$$\tag{15}$$

So, the quality factor from (13) and (15) is :

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{16}$$

$$\implies Q = \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{17}$$

$$\implies Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{18}$$

The voltage transfer function after substituting the values in the (1) :

$$\frac{V(s)}{I(s)} = 10 + 2s + \frac{1}{32 \times 10^{-6}s} \tag{19}$$

$$H(s) = 10 + 2s + \frac{1}{32 \times 10^{-6}s} \tag{20}$$

The resonant frequency of the circuit from (5) :

$$\omega_0 = \frac{1}{\sqrt{2\,\text{H} \times 32\,\mu\text{F}}} = 125\,\text{rad/s} \tag{21}$$

The resonant frequency is 125 rad/s.

The Quality factor of the circuit from (18) :

$$Q = \frac{1}{10}\sqrt{\frac{2}{32 \times 10^{-6}}} = \frac{1}{40 \times 10^{-3}} = 25 \tag{22}$$

Therefore, the quality factor of the LCR circuit is 25.

Plot the amplitude of the transfer function w.r.t omega :

$$H(s) = R + sL + \frac{1}{sC} \tag{23}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{24}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{25}$$

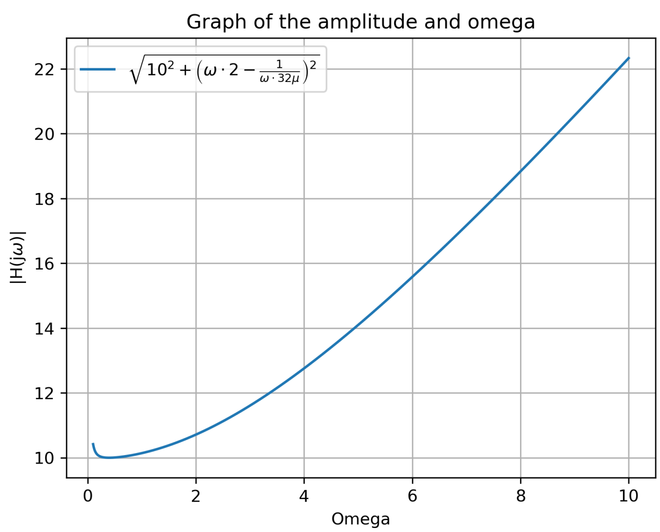

Fig. 3. Impedance vs ω