Assignment 1
Probability and Random Variables

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I. Problem

Find the MGF for X ~N(μ, σ^2) .

II. Solution

The Moment generating function for a Normal distribution is given as $M_X(t) = E[e^{-tX}]$.
This is given by the Laplace transform $L_x(t)$ of the density function $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} e^{-tx}\, dx \tag{1}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2 - tx}{2\sigma^2}}\, dx \tag{2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-((x-\mu)^2 + tx)}{2\sigma^2}}\, dx \tag{3}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-((x-\mu)^2 + 2t(x-\mu) + 2t\mu)}{2\sigma^2}}\, dx \tag{4}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma} + t\sigma)^2 + \frac{t^2\sigma^2}{2} - t\mu}\, dx \tag{5}$$

$$= e^{\frac{t^2\sigma^2}{2} - t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma} + t\sigma)^2}\, dx \tag{6}$$

$$= e^{\frac{t^2\sigma^2}{2} - t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{(x+t\sigma^2)-\mu}{\sigma})^2}\, dx \tag{7}$$

Let y = x+ $t\sigma^2$, $dy = dx$ Integral under normal density=1. So,
Now,

$$L_x(t) = e^{\frac{t^2\sigma^2}{2} - t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{y-\mu}{\sigma})^2}\, dy \tag{8}$$

$$= e^{-\mu t + \frac{\sigma^2 t^2}{2}} \tag{9}$$

For the obtained expression, the MGF (0) =1.
The same result is also obtained using the python code.
Download python code from here

https://github.com/Swati−Mohanty/AI5002/blob/main/Assignment%201/codes/mgf.py

```
0th moment :
 [1. 1. 1.]

6th moment :
 [5.20609375e+02 9.13256836e+00 4.26392850e+06]

9th moment :
 55265909588.26437

12th moment :
 [1.53284936e+14 1.63654317e+02 8.83474172e+03 5.17842143e+04]
```

Figure 1: Result obtained from python code

Download latex code from here-

https://github.com/Swati−Mohanty/AI5002/blob/main/Assignment%201/codes/main.tex