LA Lecture 7

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1 Topics Covered

1. Computing the null space (Ax = 0)
2. Pivot variables-free variables
3. Special solutions-rref(A) = R

Rank of a matrix = Number of pivots in that matrix.

1.1 Algorithm to obtain null space of a matrix

Note that the given matrix (A) can be a rectangular matrix. Steps of the algorithm are as follows:

1. Apply elimination procedure on the matrix A. The form of the matrix obtained through this procedure is termed as Echelon form.
2. Obtain Row Reduced Echelon form (rref) through this matrix.
3. The null space is all the linear combinations of the columns of matrix

$$\begin{bmatrix} -F \\ I \end{bmatrix}$$

   where I is the identity matrix and F is the coefficients of free variables in the rref.
   Number of columns in this matrix = Number of free variables.

Some key point regarding the algorithm :

1. In step 1 of the algo, we don't have to do row exchanges even if we get 0's in the pivot positions. **Verify this for some other example.**
2. With elimination we are not changing the null space. Since the solution to Ax = 0 won't change with it as the same steps are applied on the RHS. What changes with elimination is the column space.

3. The columns containing the pivot are termed as **pivot columns** and the corresponding variables are termed as **pivot variables** while the remaining columns are termed as **free columns** and the corresponding variables are termed as **free variables**. They are termed as free because we can assign any value to the variables associated with those columns. And then solve for the variables associated with pivot variables.

Let's apply this algorithm on an instance. Let A =

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Applying step E_{21} and E_{31} on A we get :

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

The first pivot is at A_{11} and second is at A_{23}. Thus we do E_{33} in the next step and get :

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}$, which is called the **echelon form**. Clearly, the number of pivots in this matrix = 2, which is called the **rank** of the matrix.
Ux = 0 looks like :

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
$2x_3 + 4x_4 = 0$
Let's try to obtain null space using Ux = 0:
Since x_2 and x_4 are free variables we can assign any values to them. Let's assign 1 and 0 to them respectively. We get :

$x_1 + 2 + 2x_3 = 0$
$2x_3 = 0$
It implies x_3 = 0 and x_1 = -2.
Now, let's assign 0 and 1 to x_2 and x_4 respectively. We get :

$x_1 + 2x_3 + 2 = 0$
$2x_3 + 4 = 0$
It implies x_3 =-2 and x_1 = 2.

What we get is

c $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and d $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ are solutions of Ax = 0 or Ux = 0 where c and d are any constants.

Thus, the linear combinations of columns of $\begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$ is the null space of A.

Let us obtain row reduced echelon form of the above matrix. In this we try to get zeroes above pivot positions as well and make pivot elements = 1.

Performing E_{13} on the above matrix we get:

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Making pivot at $A_{23} = 1$ we get

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= R (in rref) The solutions to Ax = 0, Ux = 0 and Rx = 0 are all the same.
The equations obtained from Rx = 0 are as :

$x_1 + 2x_2 - 2x_4 = 0$

$1x_3 + 2x_4 = 0$

Let's look at the coefficients in a rearranged manner.

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ \multicolumn{2}{c}{Identity\ part} & \multicolumn{2}{c}{Free\ part} \end{bmatrix}$$

If we write in the form, $\begin{bmatrix} -F \\ I \end{bmatrix}$, we get $\begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is same as the columns of the matrix obtained above (in a reordered form) as the solution of null space of A.

What do we get as null space of A^T? What is the rank of A^T? **complete notes and verify this lecture from book**