LA Lecture 10

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1 Topics Covered

1. 4 fundamental sub spaces of a matrix

1.1 Sub spaces of a matrix A

1. Column space C(A)
2. Null space N(A)
3. Row space = all combinations of the rows of A = all combinations of the columns of $A^T = (C(A^T))$.
4. Null space of $A^T = (N(A^T))$ = left null space of A.

For an m*n matrix A

1. C(A) is in R^m.
2. N(A) is in R^n.
3. $C(A^T)$ is in R^n.
4. $N(A^T)$ is in R^m.

1.2 Basis and Dimension of the spaces

1. Column space (C(A))
   Its dimension is r (rank of A) and the pivot columns of A forms its basis.
2. Row space (C(A^T))
   Its dimension is r (rank of A). Its basis is explained in a while.
   Row space and column space have the same dimension.
3. Null space N(A)
   Its basis is the special solution and its dimension is n-r (number of free variables).
   Note: Dimension of null space + dimension of row space = n-r + r = n. Also, both N(A) and $C(A^T)$ are in R^n. Now since dimension of column space = r and both column space and null space of A^T lie in R^m, should the dimension of null space of A^T be m - r? Ans: Yes.

4. Left null space or Null space of A^T, $N(A^T)$
   Its dimension is m-r, because A^T has m variables and its rank is r. Hence it has (m-r) free variables.

   Basis of row space: 2 ways to obtain it.
   1) Perform row operations on A^T and get it in rref form. We will know which all are the pivot columns of A^T through the rref form and those columns of A^T will form the row space of A.
   2) If we have already performed row operations on A to get it in rref form, we don't need to repeat the same for A^T. Let rref form of A be R. The first r (r = rank of A) rows of R form the basis of row space of A. They obviously form the basis for row space of R as well.
   The first r rows of A may or may not form the row space.

   Why? We might have swapped the rows while getting R?

   $C(R^T) = C(A^T)$. But, $C(A) \neq R(A)$ i.e. **row space of A and R are equal while column space of A and R may not be equal.**
   Why is null space of A^T also called as left null space?
   Let, $A^T y = 0$. Here y represents the null space of A^T. Taking transpose on both sides we get, $y^T A = 0^T$. Since, y is coming on the left when we try to get A in the equation, therefore, it is called the left null space.

   Basis of $N(A^T)$(null space of A^T): To find its basis through rref form of A, we will tack on I alongwith A (as we did in Gauss-Jordan to calculate inverse of A).
   rref[A_{m*n} I_{m*m}] $\rightarrow$ [R$_{m*n}$ E_{m*m}] because EA = R.

   (In case when A is square invertible matrix, E comes out to be A^{-1}.)

   $E_{m*m}[A_{m*n}\ I_{m*m}] = [R_{m*n}\ E_{m*m}]$

   The basis of left null space of A is given by the last (m-r) rows of E. Reason: The last (m-r) rows in R will be all zeroes and last (m-r) rows of E basically tells you what combination of rows of A will result in a all zeros row. Just like to find out the N(A), we find the combination of columns of A which results in all zeroes; similarly for N(A^T), we need to find the combination of rows of A which results in all zeroes and hence the solution.