LA Lecture 6

swati.allabadi

1 Topics Covered

1. Vector spaces and Null spaces
2. Column space of A : Solving Ax = B
3. Null space of A

1.1 Vector Space Requirements

For any 2 vectors, V and W lying in the space, V + W should also lie in the space.
For any vector V in the vector space, cV also lies in the vector space where c is any constant.
Above 2 points can be written together as, for any 2 vectors A and B lying in the vector space, c_1A + c_2B should also lie in the vector space, where c_1 and c_2 are any constants.

1.2 Sub Spaces:

Some vectors inside a vector space that forms a vector space of its own.
Consider R^3 and 2 sub spaces of it. One being a plane, P and one being a line, L (not lying in the previous plane). Is $P \cup L$ a sub space?
No. Reason: For any 2 vectors, V and W lying in $P \cup L$, V + W may not lie in $P \cup L$.
For given any 2 vector spaces S_1 and S_2, $S_1 \cap S_2$ always is a vector space. For any vectors V and W lying in $S_1 \cap S_2$, cV and V + W lie in S_1 as well as S_2, where c is any constant. While $S_1 \cup S_2$ may or may not be a vector space.

1.3 Column Spaces

Consider a matrix A. For what values of b, does the system Ax = b has a solution?
We can solve Ax = b exactly when b lies in the column space of A, C(A) because it implies that b is a linear combination of the columns of A.
Are the columns of a matrix independent? (or) Does each column contribute something new? (or) Can we throw away some column(s) of the matrix

and still have the same column space?
Consider the matrix A =

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

The 3 columns of this matrix are dependent. Col 1 + Col 2 = Col 3 of the matrix. Thus we can get any of the columns of matrix A using the rest 2 columns. We will call column 1 and column 2 as pivot columns, while column 3 to be non pivot. It is just a convention to choose pivots whichever comes first in order.
The column space of A is a 2-d vector space. Hence, it's a plane. Since there are 4 components of each vector in A (or say A has 4 rows), C(A) is a subspace of R^4.
For an m*n matrix, column space of a matrix is a subspace of R^m.

1.4 Null Space of a matrix

= All solution vectors, x =

$$\begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \cdots \\ \mathrm{x}_n \end{bmatrix}$$

for Ax = 0.

Null space of an m*n matrix is a subspace of R^n.
Why does null space has term 'space' in its name? It is because **null space is a sub space of R^n** where n is the number of columns in an m*n matrix. It is vector space because: Consider 2 solutions x_1 and x_2 of Ax = 0. It implies $Ax_1 = 0$ and $Ax_2 = 0$. It implies $A(x_1) + A(x_2) = 0$. It implies $A(x_1 + x_2) = 0$. It follows from the distributive law.

We have seen two ways to obtain vector space. One is by taking all possible linear combinations of columns of a matrix. With this we get the column space. Second is by putting conditions which x should satisfy in Ax = 0. With this we get null space.

Do the solutions of Ax = b for a non zero b, form a vector space?
No, because zero vector doesn't satisfy Ax = b for a non zero b. And we can't get a vector space without including the origin.