LA Lecture 8

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1 Topics Covered

1. Complete solution of Ax = b
   (a) Identify if Ax = b has a solution.
   (b) If it has a solution, how many solutions does it has?
   (c) Find all the possible solution.

1.1 Solvability condition for Ax = b to have a solution (or) Identify if Ax = b has a solution

Let's begin with matrix A as used in previous lecture as example. The equations are as follows:
$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$
What should be the condition on b_1, b_2 *and* b_3 for the above system of equations to have a solution? The augmented matrix [A b] of the above system of equations is as follows:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$$

b has to be tagged along to track the changes in it while applying the elimination process. Applying E_{21} and E_{31} we get:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

Applying E_{33} we get:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

1st and 3rd columns are the pivot columns. **The condition on solvability is** $b_3 - b_1 - b_2 = 0$ because the 3rd row has all zeroes in it. This condition was expected because in the matrix A, 3rd row = 1st row + 2nd row.

Solvability condition on b
Ax = b is solvable if b lies in the column space of A, C(A).
Also, if there's a combination of rows of A which gives zero row, then the same combination of entries of b must give 0.
(Would not the first condition imply the 2nd? Would 2nd condition alone be sufficient?)

1.2 Algorithm to find the complete solution to Ax = b

1. To find $x_{particular}$: Set all the free variables = 0 and solve Ax = b for the pivot variables.
2. Add $x_{null\ space}$ to $x_{particular}$.
3. $x_{complete} = x_{particular} + x_{null\ space}$.

Let's apply the above algorithm to the previous example.

1. In the above eg, x_2 and x_4 are free variables. Putting them = 0, we get: $x_1+2x_3 = 1$ and $2x_3 = 3$. Solving these we get, x_1 = -2, x_3 = 3/2. Hence, $x_p =$
$$\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$
2. The null space obtained for this A from previous lecture: $x_{null\ space} =$ c $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ + d $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ where c and d are any constants.
3. $x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \text{c} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{d} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

We can't multiply x_p by any constant as it won't be the solution any more. x_n works fine because any vector in x_n will give Ax = 0 only. Now, in x_c, x_n is a subspace but x_c is not a subspace because it goes through x_p but not origin. For our example, x_c is a 2-D plane not passing through origin, a shifted plane.

1.3 Determining number of solutions on the basis of relation between rank (r), m and n for an m*n matrix

For an m*n matrix, $r \leq m$, $r \leq n$ because we can't have more than min(m,n) pivots. Basically, we can't have more than n pivots in columns and m pivots in

rows, hence we can't have more than min(m,n) pivots.
Various cases are as follows:

1. **Full column rank i.e. r = n,** $n \leq m$. r = n means there is pivot in very column hence, there is no free variable.

   What will be the null space of such a matrix? It will only be the zero vector. N(A) = zero vector.
   Solution to Ax = b? It's only $x_{particular}$ if it exists. **Hence, it has either 0 or 1 solution.**

2. **Full row rank i.e. r= m,** $m \leq n$. r = m means there is a pivot in every row.

   For every b, there exists a solution for Ax =b (as we don't get any zero row in elimination).

   We have n-r (=n-m) free variables.

3. **r = m =n**. We have a square matrix with a pivot in every row (and column). It's a full rank matrix. **r = m = n means it is an invertible matrix. It's rref form is identity matrix. N(A) = zero vector. It always has a solution, there's no condition on b for Ax = b to have a solution.**

Let us consider an example for each case mentioned above.

1. **r = n**

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

   It's row reduced echelon form (rref) would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. **r=m**.

$$\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

   It's rref =

$$\begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$$

   - means there would be some numbers in that part. These numbers comprise the F part, free variables part.

3. $$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

   It's rref = $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.4 Summary

All the cases are summarised below:

1. **r = m = n**: rref (or R) = I. It has **1 solution**.
2. **r = n < m**: R = $\begin{bmatrix} I \\ 0 \end{bmatrix}$ Here 0 means zero rows. It has **0 or 1 solution**.
3. **r =m < n** : R = [I F]. F means free variables part. Pivots might not appear all in the initial columns, hence I would be scattered. For simplicity, we have written as it's all in the beginning. It has **infinite solutions**.
4. **r < m, r < n**: R = $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ It has **0 or infinite solutions**.