LA Lecture Notes

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1 Lecture 4

Topics Covered

1. Inverse of AB, A^T
2. Product of elimination matrices
3. A = LU (considering U is obtained without any row changes)

1.1 Inverse of AB

Consider A and B to be square invertible matrices, then what is the inverse of AB i.e. $(AB)^{-1} =?$

Now, $B^{-1}A^{-1}AB = I$. Also, $ABB^{-1}A^{-1} = I$.
$\Rightarrow B^{-1}A^{-1}$ is the inverse of AB.
Considering A and B to be square matrices, implies their left and right inverses are same.

1.2 Inverse of A^T

We know that $AA^{-1} = I$. Taking transpose on both sides we get,
$(A^{-1})^T A^T = I$. $\Rightarrow (A^T)^{-1} = (A^{-1})^T$.
Hence, for a single matrix transpose and inverse can be done in any order.

1.3 Product of elimination matrices (obtained without row changes)

From previous lectures, $EA = U$ where A is the matrix obtained from system of equations to be solved, E is elimination matrix and U is upper triangular matrix. We also have, $A = LU$ where L is lower triangular matrix. Hence, $L = E^{-1}$.
A =

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$$

For this A, we get E =

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

and U =

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

For this A and U, L =

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

There is no significant difference between E and L i.e. there is no evident reason why we should prefer L over E (A = LU over EA = U). But it is there in a 3*3 matrix. For a 3*3 matrix, $E_{32}E_{31}E_{21}A = U$ is the equation. Let E_{31} be an identity matrix , E_{32} =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

and E_{21} =

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{32}E_{31}E_{21} = E_{32}E_{21} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

Now $L = E^{-1} = E_{21}^{-1}E_{32}^{-1}$
We have E_{21}^{-1} =

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and E_{32}^{-1} =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$L = E^{-1} = E_{21}^{-1}E_{32}^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Hence, if there are no row exchanges the multiplier goes directly into L. Note that L has 1's on its diagonal. We can also decompose U into 2 matrices to get 1's on its diagonal.

1.4 How many operations (multiplication and subtraction) on an n*n matrix A to get L and U?

We need to perform $O(n)$ operations to make an element 0. Hence we perform $O(n^2)$ operations to get a pivot in a column. To be precise, we perform around $n^2 + (n-1)^2 + ... + 1^2 \approx 1/3n^2 = O(n^2)$ operations to get all the pivots.

Check if t should be $(n-1)^2 + ... + 1^2$ or $n^2 + (n-1)^2 + ... + 1^2$.

1.5 If row changes required (Permutation matrix P)

We can fix this with the help of permutation matrices. It's a matrix with the rows of identity matrix (in some order). Total no of permutation matrices for an n*n matrix is n!. We can call these n! matrices a group as the product of any 2 matrices of those is a matrix within those. Note that, $P^{-1} = P^T$ Hence, if row changes are required PA = LU. Thus we can first change the order of the rows and proceed to get L and U.

1.6 $R^T R$ and RR^T are symmetric matrices where R is any rectangular matrix

$(R^T R)^T = R^T R^{TT} = R^T R$
$(RR^T)^T = R^{TT} R^T = RR^T$