IE 501 : Homework Assignment 1

Due: Friday, 23/08/2024

Please follow the following guidelines while working on this homework.

1. You are permitted to consult your class notes, and your classmates. Working in groups is encouraged. However please understand that this is only to further your understanding through collaborative effort, not to encourage cheating. Please indicate the names of all the classmates you consulted with while working on the homework in your submission. Anything apart from this is strictly forbidden.
2. Submissions must either be clearly handwritten or be typeset in LaTeX, illegible submissions will not be graded.
3. Homework solutions should be turned in physically on or before the start of the class at 9:30 AM on Friday, August 23, 2024.

1. (10 points) Consider the following 'districting' problem. A small state is defining constituencies to participate in a national election. There are 25 wards, labeled 1 (top left) to 25 (bottom right) - assume 1,2,3,4,5 on the top row, etc. The symbol in each ward indicates the dominance of a party in that ward, and it would win a ward level election. So party A is dominant in wards 1-5, party B in wards 6-10 etc.

A	A	A	A	A
B	B	B	B	B
A	A	A	A	A
B	B	B	B	B
A	A	A	A	A

Constituencies are 5 wards that are contiguous in having a path with a shared border with other wards (so 1-2-3-6-7 is a valid constituency, but 1-7-8-9-10 is not). In real life, there could be other notions of what is a constituency, but we use this for now. So the state will have 5 constituencies. A party wins a constituency if it wins a majority of wards in that constituency.

   (a) What is the maximum number of constituencies that A can win, depending on how constituencies are defined (best possible scenario for A)? You can suggest such a set of constituencies and also try to show that it is max possible. Try to generalize your result for n wards and a constituency consisting of k wards.

(b) What is the minimum number of constituencies that A will win, no matter how constituencies are defined? You can suggest such a set of constituencies and also try to show that it is minimum. Try to generalize your result for n wards and a constituency consisting of k wards.

(c) Set up a mathematical system of constraints that will allow all feasible definitions of constituencies (this may be used in some later design of constituencies in an appropriate manner - you may have concluded from the above that it is quite important how constituencies are actually defined in the real world).

2. (10 points) The US Open 2018 men's final took place between Novak Djokovic (N) and Juan Martin del Potro (J). Assume that the match format was modified due to unforeseen circumstances to a best-of-three set match, i.e. whoever wins the two sets in the match wins the match and in turn the championship. The betting firm sets odds for each set and also for the match (the following are betting odds not winning odds).

   The returns for the odds α for a set win in favor of (N) are as follows

   1. If you bet \$1 on (N) winning the set, and (N) loses then you lose the \$1. (Net loss \$1)
   2. If you bet \$1 on (N) winning the set, and (N) wins then you get $\$1\frac{1}{\alpha}$ back. (Net gain $\$\frac{1}{\alpha}$).
   3. If you bet \$1 on (J) winning the set, and (J) loses then you lose the \$1. (Net loss \$1)
   4. If you bet \$1 on (J) winning the set, and (J) wins then you get $\$(1+\alpha)$ back. (Net gain $\$\alpha$).

   The same odds apply to all the three sets in the match and for the match as well. The odds offered by the betting firm for (N) winning either of the sets are α and the odds offered for (N) winning the match are β. Not only can you bet on a single player for each set and match, you have the option of hedging your investments by betting on both the players.

   As a IITB undergrad who has just learned formulating LP's you realize this might be a good opportunity to explore arbitrage and make easy money starting with a budget of $\$b$. Of course you can play safe and choose to not involve with the gambling at all in which case your guaranteed winnings are \$0. Can you do better?

3. (10 points) Suppose an investor has \$100 on Monday. At the start of every day of the week (Monday through Friday), the investor has the following investment opportunity available: If he invests x dollars on that day and matches that initial investment with $x/2$ dollars the next day, then he will receive a total return of $2x$ dollars on the third day. Thus, with a total investment of $1.5x$ dollars, the investor receives $2x$ dollars in two days, a gain of $0.5x$ dollars. The investor wishes to determine an investment schedule that maximizes his total cash on Saturday. To facilitate the formulation of a linear program, the investor decides to make the following simplifying assumptions:

   1. If an initial investment is not matched on the subsequent day, the initial investment is lost.
   2. Any return that is due on any given day can be reinvested immediately.
   3. Cash carried forward from one day to the next does not accrue interest.

4. Borrowing money is not allowed.

To understand things better, let us consider the following "naive" strategy: Begin with an investment of (2/3)100 dollars on Monday, while putting aside (1/3)100 dollars in anticipation of the necessary second installment on the next day (Assumption 1). On Tuesday, the investor executes the second installment and, consequently, he won't have any remaining cash to initiate a new investment. On Wednesday, the investor receives a total return of $2 * (2/3)100$ dollars. This completes an investment cycle, during which the total amount invested in two installments ($100) grew by a factor of 4/3. Suppose further that the investor immediately reinvests the yield he receives on Wednesday (Assumption 2) in the same manner as in the just-completed investment cycle. Then, a similar analysis shows that he will receive a total yield of $(4/3)(4/3)100 = 177.8$ dollars on Friday. At that point, since any new investment that starts on Friday won't mature until Sunday (a day too late), the investor simply carries this second yield into Saturday, resulting in a final cash position of 177.8 dollars.

Question: How good is this naive strategy? While we managed to complete two full investment cycles, it seems discomforting to watch cash sit idle from Friday to Saturday. This suggests that we might be able to do better. But how? Note that under the divisibility assumption, the set of possible strategies is a continuum, and hence it cannot even be enumerated. Thus, the task is challenging. To find out what is the best strategy, provide an LP formulation of this problem.

4. (10 points) Suppose that you will play a word construction game. You are assigned $n_a, n_b, \ldots, n_z$ copies of letters $a, b, \ldots, z$ respectively. That is, if $n_b = 8$ then you can use the letter b at most 8 times in your words. These words must be in an English dictionary, say we use D to denote all the words in the dictionary. Then, abacus $\in D$, abode $\in D$, zinc $\in D$, basically D is the set of all English words. Suppose that for every word you construct, you obtain points equivalent to the length of that word. For example, if you are assigned $n_a = 1, n_e = 2, n_i = 5, n_m = 4, n_n = 1, n_x = 1, n_z = 2$ i.e. $(a, e, e, i, i, i, i, i, m, m, m, m, n, x, z, z)$ and no other letter, you should construct words **minimize** and **maximize** scoring 16 points instead of **zen** and **maze** scoring 7 points. For an assignment of letters (i.e., $n_a, n_b, \ldots, n_z$ given and known), provide an linear (integer) program that will choose words so that your score is maximized.

5. (10 points) On graduate school visit days, professors need to meet with visiting students to explain to them how great a school is. Visiting students have certain professors they want to meet with and professors have certain visiting students they want to impress. Over visit day, there are meetings at various fixed time slots, and some professors and students are only available at certain times. Your task is to find an assignment of students, professors, and times that allows as many meetings to take place as possible.

   More formally, we have:

   - A set of visiting students $S = \{s_1, s_2, \ldots, s_m\}$.
   - A set of professors $P = \{p_1, p_2, \ldots, p_m\}$.
   - A set of time slots $T = \{t_1, t_2, \ldots, t_a\}$.
   - A collection of sets $A_{p_i} \subseteq T$, where A_{p_i} gives the time slots that professor p_i is available.
   - A collection of sets $A_{s_i} \subseteq T$, where A_{s_i} gives the time slots that student s_i is available.

- A collection of sets $M_{p_i} \subseteq S$, where M_{p_i} is the set of students with whom professor p_i is interested in meeting.
- A collection of sets $M_{s_i} \subseteq P$, where M_{s_i} is the set of professors that a visiting student s_i wants to meet.

Professor p_i will only meet with students on their list M_{p_i} and student s_i will only meet with professors on their list M_{s_i}. A meeting can only happen at time t_i if both the student and professor are available. During any time slot, each student can meet with at most 1 professor and each professor can meet with at most 1 student.

(a) Formulate the above problem as a linear (integer) program.

(b) Suppose you are now also given, for each professor, a *bigshot score* b_{p_i}, a positive number that says how important it is to satisfy that professor's demands. For every meeting that professor p_i takes with a student in her preference list M_{p_i} , we get b_{p_i} points. Describe how to modify your integer linear program to find a schedule that maximizes the number of *bigshot points* your schedule gets.

(c) Visit days are successful if students meet with a lot of people, even if neither the student or the professor put one another in their preference lists. Describe how to modify your integer linear program so that you get $b_{p_i} > 1$ points if you schedule one of professor p_i's meetings with a student on their list M_{p_i} but only 1 point if you schedule them with someone who is not on their preference list. Assume now that students will meet with anyone they are assigned to, but the time constraints must still be satisfied.