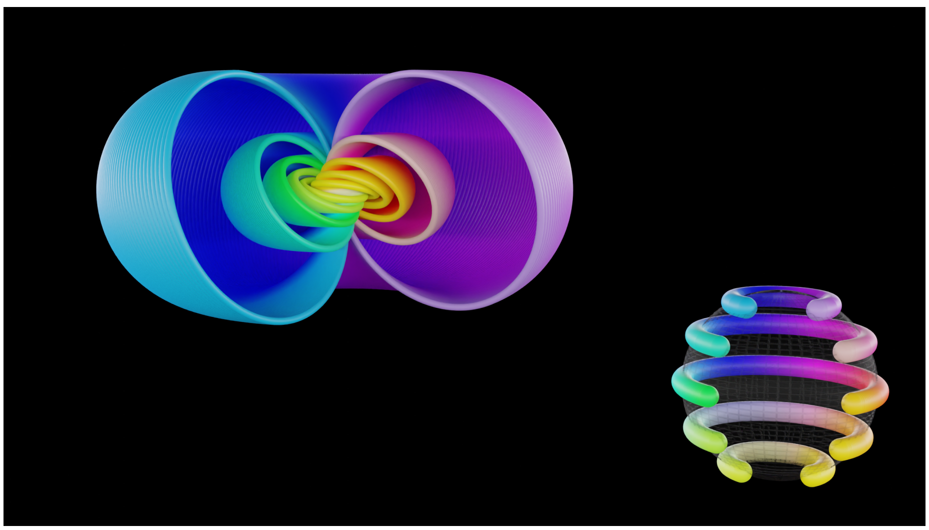
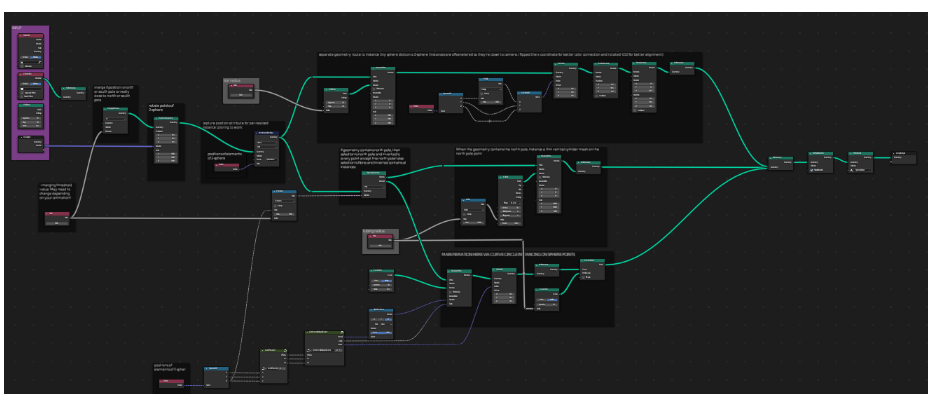
Geometry and Shading Nodes Explanation

Adam Barnett Swearingen (Sweardog)

June 1, 2023

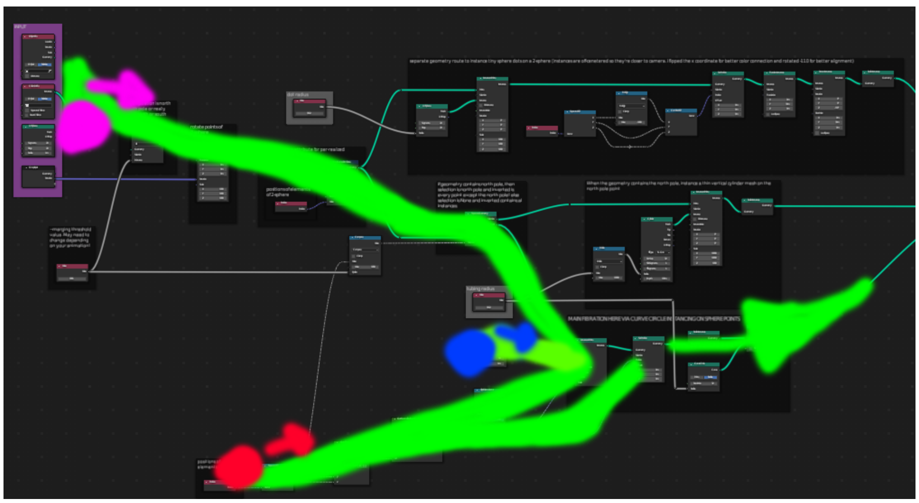
1 Introduction





Shown above is a *Blender Geometry Nodes* modifier that accepts elements of S^2 and instances corresponding Villarceau circles (the stereographically projected Hopf fibers of S^3) on each input element via the "Instance on Points" geometry node. Rather than stereographically projecting each point of a Hopf fiber from S^3 to $\mathbb{R}^3 \cup \{\infty\}$ in a parametric-point-plot approach, it is more effective to determine the center, radius, and normal vector of each Villarceau circle and adjust the circle instances in terms of position, scale, and alignment. This fibration of S^3 will be approached from a complex $\mathbb{C}^2$ interpretation, rather than quaternionic $\mathbb{H}$.

2 Initial Considerations



The primary geometry nodes path for consideration is highlighted in green. Marked by the pink dot is the input geometry - an arbitrary set of elements of S^2. For example, this set could constitute a full uv-sphere, a circle of S^2, or possibly a collection of circles of S^2. The blue dot signifies the instance object; a standard xy equatorial curve circle. The input geometry and instance object feed into the same "Instance on Points" node. Thus, each point of the input geometry is replaced by a circle curve.

At the bottom left, the red dot marks a "Position" node. Each circle instance must be repositioned, scaled, and aligned according to its instancing point. This node provides the x, y, z coordinates of each point of the input geometry, enabling the necessary circle adjustments. The "Instance on Points" node handles only alignment and scaling. Consequently, the path from the red dot must also lead to a "Set Position" node to manage centering.

The main task is to determine the center, radius, and normal vector for each soon-to-be Villarceau circle instance. This process begins with the Hopf Map, mapping great circles of S^3 to points of S^2. This map is not a stereographic projection, which is bijective. The Hopf Map function is surjective, but not injective. Hence, it's not bijective. Both functions are continuous.

3 Understanding the Hopf Map

The Hopf fibers of S^3 constitute a set of great circles that cover S^3 once. Each fiber intersects with no other fiber and each fiber is linked with every other fiber. The Hopf map sends these fibres to points on the 2-sphere S^2 such that every point on S^2 has precisely one pre-image, which is a fiber. Following the interpretation of $\mathbb{C}^2$, consider $(z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in S^3$. The Hopf map, denoted as h, is a composition of two functions: $h(z_1, z_2) = g^{-1}(f(z_1, z_2))$. Here, f represents the generic Hopf map, defined as $f : S^3 \to \mathbb{C} \cup \{\infty\}$.

$$f(z_1, z_2) = \tfrac{z_2}{z_1} = \tfrac{r_2}{r_1} e^{i(\theta_2 - \theta_1)} \in \mathbb{C}$$

When $r_1 = 0$, the corresponding fiber is sent to ∞ in the extended plane (the standard yz unit circle if $\mathbb{R}^4$ is defined by (w, x, y, z); the standard wx unit circle is when $r_2 = 0$). The alternate definition, $f(z_1, z_2) = \frac{z_1}{z_2}$, is also a valid Hopf Map. Radii r_1 and r_2 determine the torus of S^3 and angle $\theta_2 - \theta_1$ determines the torus's fiber of which is mapped to $f(z_1, z_2)$. As θ_1 and θ_2 vary by the same amount, with fixed r_1 and r_2, this defines a travel path around S^3 where $\theta_2 - \theta_1$ remains constant. This implies that points with this choice of r_1, r_2, and $\theta_2 - \theta_1$ all map to the same point, $f(z_1, z_2)$, and hence constitute a Hopf fiber of S^3.

The function g^{-1} represents the, by most common convention, inverse stereographic projection $g^{-1} : \mathbb{C} \cup \{\infty\} \to S^2$. If $Q = (q_1 + q_2 i) \in \mathbb{C}$, then

$$g^{-1}(q_1 + q_2 i) = \left(\tfrac{2(q_1 + i q_2)}{{q_1}^2 + {q_2}^2 + 1}, \tfrac{{q_1}^2 + {q_2}^2 - 1}{{q_1}^2 + {q_2}^2 + 1}\right) \in \mathbb{C} \times \mathbb{R}$$

and $g^{-1}(\infty) = (0, 0, 1)$. The composition $g^{-1} \circ f$ yields a direct map $h : S^3 \to S^2$, where

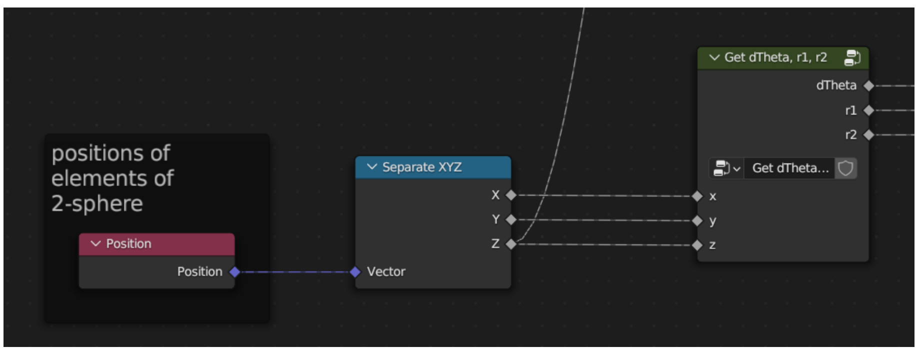
$$h(z_1, z_2) = (2 z_2 z_1^*, r_2^2 - r_1^2) \in \mathbb{C} \times \mathbb{R}$$

4 Obtaining the Pre-Image

Under the Hopf Map h, the pre-image of a point of S^2 can be obtained by recovering $\theta_2 - \theta_1$ and r_1, r_2 for all elements of S^3 that map to it. Let us refer to $\theta_2 - \theta_1$ as $\Delta\theta$. While it is impossible to recover θ_1, θ_2 individually, it is possible to recover their difference, $\Delta\theta$. After fixing r_1, r_2, and $\Delta\theta$, all points of a Hopf fiber can be parameterized through simultaneous component Euler rotations.

To recover $\Delta\theta$, refer to function f. Here, $\Delta\theta$ is simply the angle of $f(z_1, z_2)$ in the extended complex plane. This angle remains the same under stereographic projection as the xy angle of the corresponding point of S^2. Hence, for some $P = (p_1, p_2, p_3) \in S^2$, angle $\Delta\theta$ can be computed as $\Delta\theta = atan2(\frac{p_2}{p_1})$, where this angle is counterclockwise from the $+x$ axis in the xy-plane.

To recover r_1 and r_2, examine function h. If $P = (p_1, p_2, p_3) = h(z_1, z_2)$, then $p_3 = r_2^2 - r_1^2$. A point in the form of $(z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$ is an element of a standard unit S^3, implying $r_1^2 + r_2^2 = 1$. Consequently, r_1 and r_2 can be set as $cos(\alpha)$ and $sin(\alpha)$ respectively, given $cos^2(\alpha) + sin^2(\alpha) = 1$. Thus, $p_3 = sin^2(\alpha) - cos^2(\alpha)$, and one solution for α is $\alpha = sin^{-1}(\pm\sqrt{\frac{p_3+1}{2}})$. This allows for setting $r_1 = cos(sin^{-1}(\sqrt{\frac{p_3+1}{2}}))$ and $r_2 = sin(sin^{-1}(\sqrt{\frac{p_3+1}{2}})) = \sqrt{\frac{p_3+1}{2}}$. If $p_3 = 1$, $r_1 = 0$ and $\frac{r_2}{r_1}$ becomes undefined, which will need to be addressed later in geometry nodes to avoid division by zero.



The positions are fed into node group "Get dTheta, r1, r2" .

Pressing “Tab” on the node group reveals its contents.

5 Obtaining the Villarceau Circle

Having obtained the Hopf fiber great circle pre-image of a $P \in S^2$, the next task is to get the normal vector, radius, and center of the corresponding Villarceau circle down in $\mathbb{R}^3 \cup \{\infty\}$. In a YouTube video, I explained how stereographic projection maps circles to circles, starting with a circle of S^2 and then determining the radius and center of the corresponding circle in $\mathbb{R}^2 \cup \{\infty\}$. If the circle of S^2 contains $(0, 0, 1)$, the corresponding circle in the plane is some $\mathbb{R} \cup \{\infty\}$, which is a line that includes ∞ (but still remains homeomorphic to a 1-sphere). The higher-dimensional analogy of this video only affirms that stereographic projection maps subsets S^2 of S^3 down to subsets of S^2 of $\mathbb{R}^3 \cup \{\infty\}$ (a sphere-to-sphere relation, not circle-to-circle). However, when an S^2 of S^3 includes $(0, 0, 0, 1)$, the S^2 is mapped to some $\mathbb{R}^2 \cup \{\infty\}$, which is an extended plane that contains ∞ (still homeomorphic to a 2-sphere).

In my video, to define a great circle of S^2, I used only the normal vector of the circle of S^2 with a dot product. Great circles of a standard S^2 are centered at the origin, so there is no need to account for distance in their equations. The orthogonal space of a great circle is sufficient to define it. Likewise, Hopf fibers, being great circles, can be defined by their orthogonal space. However, each Hopf fiber of S^3 has a 2-dimensional orthogonal space (akin to having two normal vectors), so we must select two orthogonal vectors of a fiber's orthogonal space to define the fiber and, ultimately, the corresponding Villarceau circle.

To elaborate on this 2-dimensional orthogonal space, consider any plane in $\mathbb{R}^4$, $\mathbb{C}^2$ or $\mathbb{H}$ (4D-space more generally), where an “orthogonal plane” exists to that plane. In these spaces, for a pair of orthogonal basis vectors $\vec{v_1}, \vec{v_2}$ that span a plane, another pair of orthogonal basis vectors $\vec{v_3}, \vec{v_4}$ exists, which span an "orthogonal plane". This indicates that $\vec{v_1} \cdot \vec{v_2} = \vec{v_3} \cdot \vec{v_4} = \vec{v_3} \cdot \vec{v_1} = \vec{v_3} \cdot \vec{v_2} = \vec{v_4} \cdot \vec{v_1} = \vec{v_4} \cdot \vec{v_2} = 0$. Each Hopf Fiber of S^3 is equivalent to the intersection of some $\mathbb{C}$ in $\mathbb{CP}^1$ with S^3, where this $\mathbb{C}$ passes through the origin (a plane intersected with S^3 forms some S^1, just as a plane intersected with S^2 forms some S^1, provided they don't intersect at only one point or yield an empty-set intersection).

Some point element $(z_1, z_2) \in S^3$ is equivalent to $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$. To find the basis vectors $\vec{v_1}, \vec{v_2}$ of the plane this point belongs to, recall that each point of a Hopf fiber must maintain the same r_1, r_2, and $\Delta\theta$. Hence, given r_1, r_2, and $\Delta\theta$ of a point of S^3, and constructing the following two vectors

$$\begin{aligned}
\vec{v_1} &= \left[r_1 e^{i(0)}, r_2 e^{i(\Delta\theta)}\right], \\
\vec{v_2} &= \left[r_1 e^{i\left(\frac{\pi}{2}\right)}, r_2 e^{i\left(\Delta\theta + \frac{\pi}{2}\right)}\right],
\end{aligned}$$

it follows that these vectors maintain the original $\Delta\theta$ difference between their component Euler numbers, and thus, span the plane of the Hopf fiber to which this point of S^3 belongs. In essence, two points of a Hopf fiber have been parameterized here: the first vector is zeroed-out, and the second vector is positioned a standard counterclockwise 90-degree rotation in the complex plane away from the first vector. The vectors $\vec{v_1}$ and $\vec{v_2}$ are elements of $\mathbb{C}^2$, but casting them as elements of $\mathbb{R}^4$ results in

$$\begin{aligned}
\vec{v_1} &= [r_1 cos(0), r_1 sin(0), r_2 cos(\Delta\theta), r_2 sin(\Delta\theta)] \\
&= [r_1, 0, r_2 cos(\Delta\theta), r_2 sin(\Delta\theta)], \\
\vec{v_2} &= \left[r_1 cos\left(\frac{\pi}{2}\right), r_1 sin\left(\frac{\pi}{2}\right), r_2 cos\left(\frac{\pi}{2} + \Delta\theta\right), r_2 sin\left(\frac{\pi}{2} + \Delta\theta\right)\right] \\
&= [0, r_1, -r_2 sin(\Delta\theta), r_2 cos(\Delta\theta)]
\end{aligned}$$

Simplifying these two vectors to $\vec{v}_1 = [a, 0, b, c]$ and $\vec{v}_2 = [0, d, -c, b]$, if these are the first two columns of a matrix, completing the matrix skew-symmetrically forms an orthonormal basis of $\mathbb{R}^4$.

$$\begin{pmatrix} a & 0 & -b & -c \\ 0 & d & c & -b \\ b & -c & a & 0 \\ c & b & 0 & d \end{pmatrix}$$

This implies $\vec{v_3}$ and $\vec{v_4}$ can be set to $\vec{v}_3 = [-b, c, a, 0]$ and $\vec{v}_4 = [-c, -b, 0, d]$. One can verify the orthogonality through the dot product. While other choices for $\vec{v}_3$ and $\vec{v}_4$ are possible, this configuration provides a particularly convenient basis. We can now define a Hopf fiber great circle of S^3 as the set of vectors $\vec{P}$ satisfying

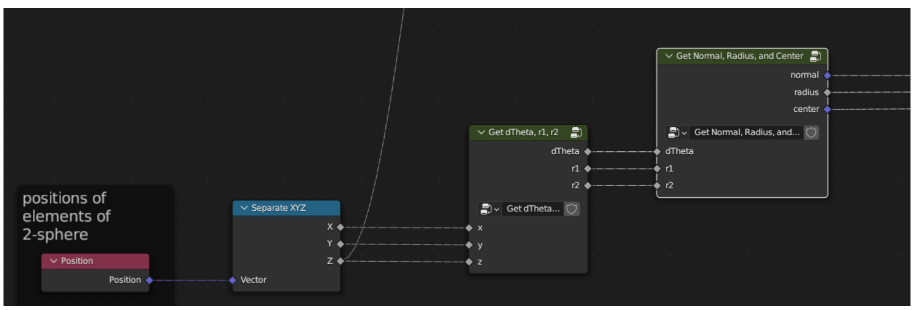
$$\text{Hopf Great Circle} = \left\{ \vec{P} \in S^3 : \begin{pmatrix} \cdots & \vec{v}_3 & \cdots \\ \cdots & \vec{v}_4 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \vec{P} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

To interpret this set, consider that the orthogonal space to a vector in $\mathbb{R}^4$ is some copy of $\mathbb{R}^3$. Hence, each equation of the Hopf fiber great circle describes an intersection of a copy of $\mathbb{R}^3$ with S^3, which forms an S^2 of S^3. The first equation is an S^2 of S^3 that must contain the "north pole" of S^3 (notice that $\vec{P} = [0, 0, 0, 1]$ always satisfies the first equation). The second equation is an S^2 of S^3 that contains $\vec{P} = [0, 0, 0, 1]$. Because the $\vec{P}$ elements must satisfy the equations of two 2-spheres, each Hopf fiber is equal to the intersection of a pair of 2-spheres.

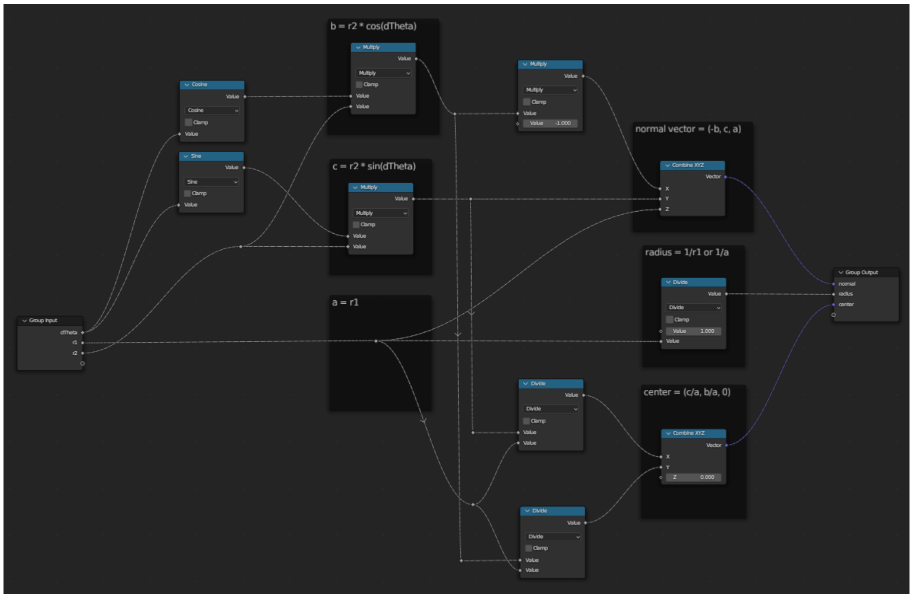
Following the steps of the YouTube video, a Hopf fiber Villarceau circle can be defined as

$$\text{Hopf Villarceau Circle} = \left\{ \begin{matrix} Q \in \mathbb{R}^3 : -bq_1 + cq_2 + aq_3 = 0, \\ (q_1 - \frac{c}{a})^2 + (q_2 - \frac{b}{a})^2 + (q_3)^2 = \frac{1}{a^2} \end{matrix} \right\}.$$

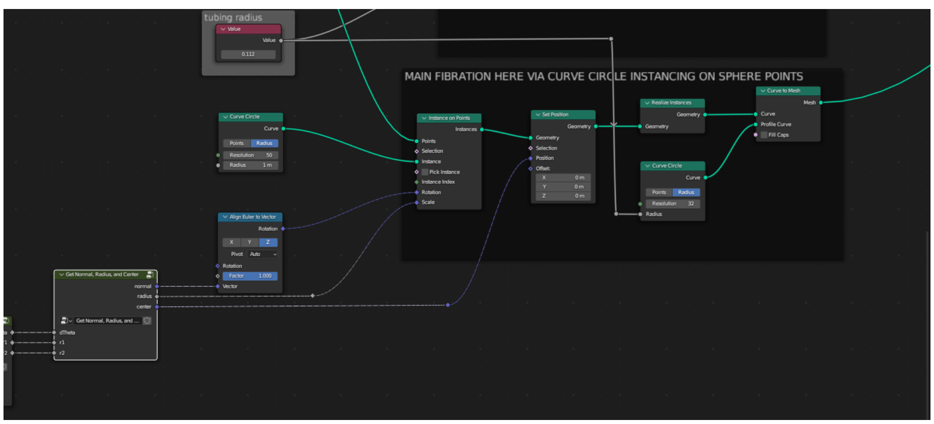
The first equation of the set defines a plane in $\mathbb{R}^3$ through the origin with normal vector $\vec{n} = [-b, c, a]$. The second condition specifies a sphere in $\mathbb{R}^3$ with center $(\frac{c}{a}, \frac{b}{a}, 0)$ and radius $\frac{1}{a}$. Thus, a Hopf fiber Villarceau circle is the intersection of a plane with a sphere. Acknowledging the homeomorphism, this intersection can still be thought of as a sphere-sphere intersection if the plane contains ∞.



The "Get Normal, Radius, and Center" node group comes next.



The contents of the node group.

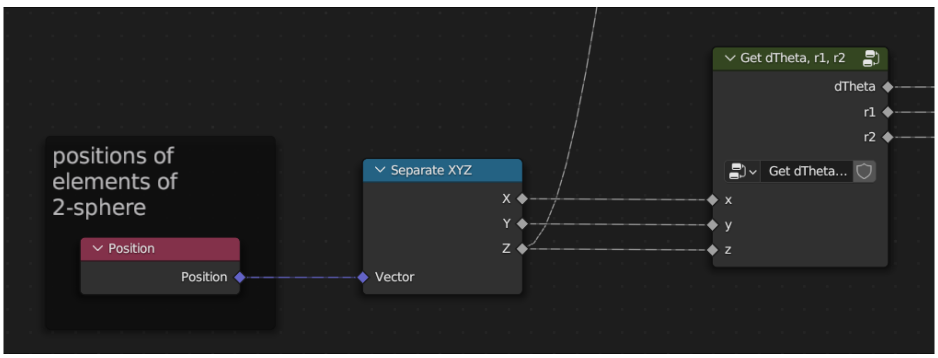


Upon exiting this node group, the normal vector is converted to Euler rotations, which are used to correctly orient the circle via the "Align Euler to Vector" node. The scaling factor is the radius of the Villarceau circle. Note once again that the "Instance on Points" node manages alignment and scaling, while the "Set Position Node" is responsible for centering. For the instance object, a "Curve Circle" is preferred over a "Mesh Circle" because the "Curve to Mesh" node facilitates the addition of a profile curve to the circle (thus creating tubing), using a small curve circle. Furthermore, it is necessary to realize the instances to ensure uniform tubing across all circles. Without realization, the scale of the circle would proportionally affect its tubing.

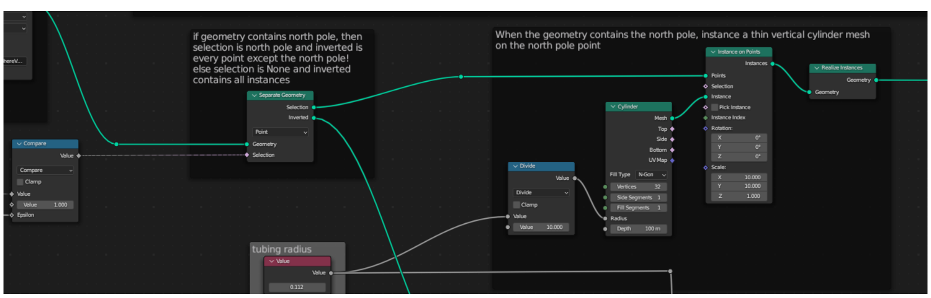
With the core fibration concept elucidated, we now address the issue of division by zero. In the undefined case where $a = 0$, the corresponding Villarceau circle is defined as

$$\mathrm{a} = 0 \text{ Hopf Villarceau Circle} = \left\{ \begin{matrix} Q \in \mathbb{R}^3 : -bq_1 + cq_2 = 0, \\ -cq_1 - bq_2 = 0 \end{matrix} \right\} \cup \{\infty\},$$

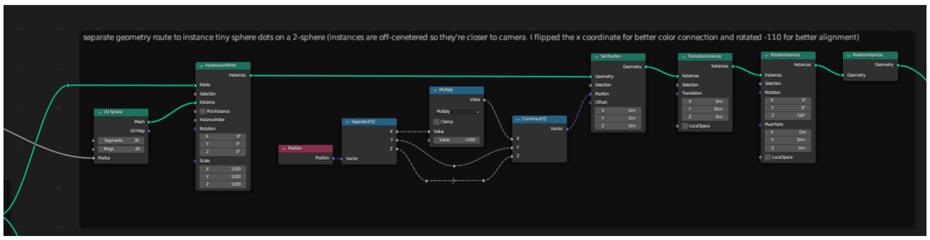
which represents the intersection of two extended planes. This intersection yields a vertically extended line (homeomorphic to a circle) and constitutes the only unbounded set seen amongst all stereographically projected Hopf fibers, justifying its unification with $\{\infty\}$. In the context of Geometry Nodes, this set is not of primary concern, as it is simply instantiated as a lengthy vertical line.



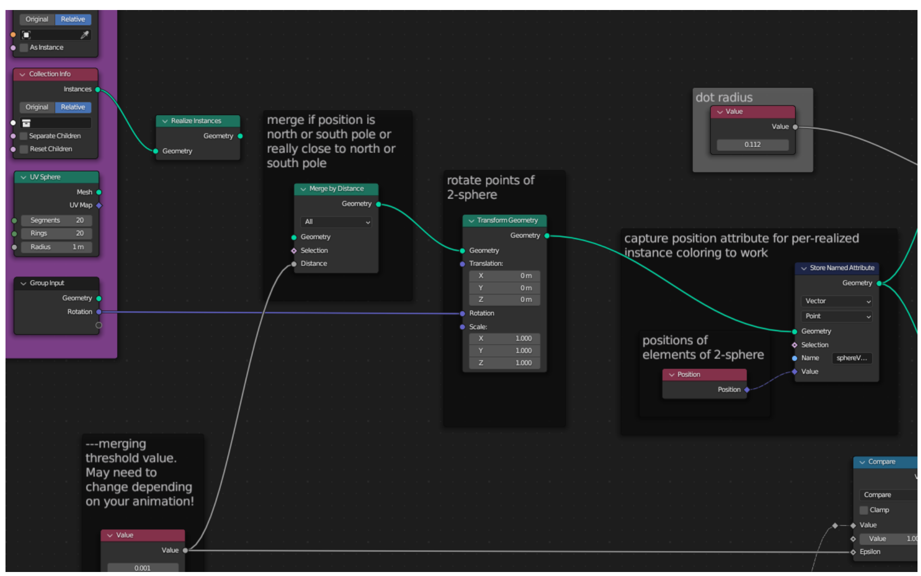
For the $a = 0$ case, caused by $z = 1$, the z-coordinate follows up a separate path.



The separate pathway directs z towards a "Math Compare" node, depicted towards the left in the above figure. This node yields 1 when $z = 1$, and 0 otherwise, within a certain tolerance range. When the output of the compare node is 1 and it is funneled into a "Separate Geometry" node, the latter will select the vertex whose position satisfies the condition of the compare node's output being 1. This will be a vertex at position $(0, 0, 1)$. The inverted geometry comprises all points except the north pole. If the compare node's output is 0, the inverted geometry is identical to the input geometry, because in these cases the north pole is absent from the input geometry. In a situation where a vertex at position $(0, 0, 1)$ is found, a lengthy, slim, vertical cylinder mesh is then situated at the origin. Realization of this vertical line cylinder mesh instance is necessary solely for coloring purposes.



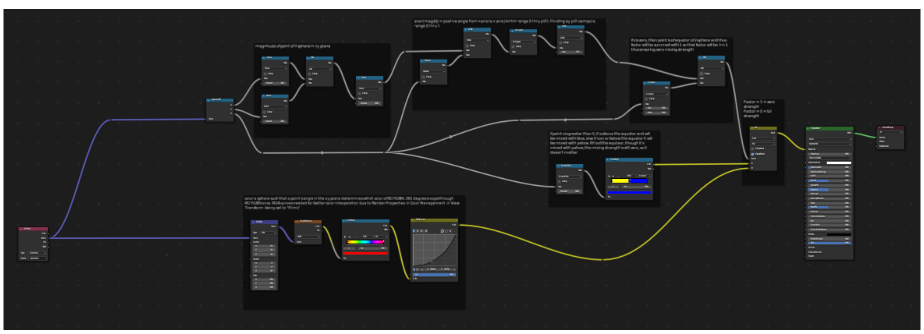
The above image depicts the uppermost route of the three principal geometry-nodes pathways, which simply serves to instantiate numerous small spheres on the input geometry. I discovered that negating the x coordinate improved the color coordination between points and fibers. I also incorporated an offset and several rotations to bring them nearer to the camera and to improve alignment. Ordinarily, realization of these instances would be unnecessary. However, as all operations are conducted in the same geometry nodes modifier, and realization of fiber instances is required for coloring to function properly with the same material, these tiny sphere instances must also be realized for coloring to operate effectively.



As depicted above, a “Merge by Distance” node is employed for cases where input elements occupy the same position. For instance, if multiple input points are situated at the north pole, this could result in overlapping vertical lines, thereby causing distortion. The "Merge by Distance" node collapses identical vertices into a single vertex. A "Capture Attribute" node is also utilized to facilitate per-realized instance coloring. Upon realization of instances, the geometry is constituted of all the points of all the circles rather than merely a collection of circles. Consequently, a unique color must be correctly associated with each point of a unique instance. Also, as you can observe in the above figure, a wire from the "Group Input" node is connected to a "Transform Geometry" node, providing access to certain rotations outside of Geometry Nodes.

Towards the end of the geometry nodes modifier, a number of basic tasks such as joining all the geometry, setting the material, and smoothing the shade are performed. No image will be provided for this phase, but it is worth mentioning that material setting plays a significant role in this process.

6 Shading



About the material, in order to assign a unique color to each Hopf fiber, we need to assign a unique color to every point of S^2. This can be achieved by initiating a standard ROYGBIV rainbow at the equator of S^2. As this rainbow equator progressively moves towards $(0, 0, 1)$, it increasingly blends with blue. Conversely, as it moves towards $(0, 0, -1)$, it increasingly blends with yellow. The above image provides an overview of this "Sphere Rainbow" shading node group. In the "Mix" node, the input "B" corresponds to an associated, non-mixed, rainbow color, which solely depends on the point's xy latitudinal angle. The input "A" is yellow if the point is positioned below or on the equator, and blue otherwise. The "Factor" input indicates the strength of the mix and depends on the longitudinal angle. For further details, I recommend referring to the box comments.