Exam 3 Review (Problems)

1. The series $\sum_{n=1}^{\infty} |a_n|$ is ______ if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ ______.

2. A sequence is *monotonic* if all of its terms are entirely either:

   1. ______ $(a_1$ ______ $\cdots$ ______ $a_n)$, or
   2. ______ $(a_1$ ______ $\cdots$ ______ $a_n)$.

3. Determine the convergence or divergence of the series using the *limit comparison test*

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

4. Write the nth-term formula for the following sequences

   1. $\{3, 7, 11, 15, \dots\}$
   2. $\{2, -1, \frac{1}{2}, -\frac{1}{4}, \dots\}$
   3. $\{1, x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}, \dots\}$

5. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$$

6. Determine convergence or divergence, including absolute/conditional convergence if applicable; state clearly what test is being used; and determine the interval of convergence for x if applicable.

$$\sum_{n=1}^{\infty} n!(n-1)^n$$

7. Find the limit of the sequence and state whether the sequence converges or diverges

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

8. If $\sum_{n=1}^{\infty} b_n$ diverges and a_n ______ b_n, then $\sum_{n=1}^{\infty} a_n$ also diverges.

9. A sequence is bounded if it is ______________________________.

10. Use the integral test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$$

11. Write the first four terms of the sequence

$$a_{n+1} = 2a_n + 1,\ a_1 = 1$$

12. Determine convergence or divergence, including absolute/conditional convergence if applicable; state clearly what test is being used; and determine the interval of convergence for x if applicable.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

13. Does the series converge or diverge, and if it converges then find the sum (use the geometric series test, the telescoping series test, or the nth term test for divergence)

$$\sum_{n=1}^{\infty} \frac{n}{n+3}$$

14. Determine the convergence or divergence of the series using the *integral test*

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

15. Find the limit of the sequence and state whether the sequence converges or diverges

$$a_n = 2 + (-1)^n$$

16. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$$

17. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^n}$$

18. The series $\sum_{n=1}^{\infty} a_n$ is ______ if $\sum_{n=1}^{\infty} |a_n|$ converges.

19. Find the explicit nth term formula for the following sequence

$$\{a_n\} = \{3, 6, 12, 24, 48, \dots\}$$

20. Use the direct comparison test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

21. Determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

22. With $a_n = f(n)$, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both diverge if f is ______, ______, and ______ for $x \geq 1$.

23. Find the limit of the sequence and state whether the sequence converges or diverges

$$\{1, {}^4\!/\!{}_3, {}^9\!/\!{}_7, {}^{16}\!/\!{}_{15}, {}^{25}\!/\!{}_{31}\}$$

24. A sequence is bounded ______ if there is a number M such that $a_n \leq M$ for all n.

25. If you are given an alternating series:
    1. Check for ______ by applying a test on $\sum_{n=1}^{\infty} |a_n|$
    2. If the absolute value of the series ______, then test for ______ using the ______.

26. Given a suitable a_n and b_n to compare against, the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge if $\lim_{n\to\infty} \left(\frac{?}{?}\right) = L$ where L is both ______ and ______.

27. Does the following series converge or diverge? If it converges, find the sum.

$$\sum_{n=0}^{\infty} e^{-n}$$

28. Use the limit comparison test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

29. Does the series converge or diverge, and if it converges then find the sum (use the geometric series test, the telescoping series test, or the nth term test for divergence)

$$\sum_{n=0}^{\infty} (1.2)^n$$

30. If f has n derivatives at center a, then the polynomial

$$P_n(x) = \underline{\qquad} + \cdots + \underline{\qquad}$$

is called the nth degree Taylor polynomial for f at a.

31. Determine convergence or divergence, including absolute/conditional convergence if applicable; state clearly what test is being used; and determine the interval of convergence for x if applicable.

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{n^2}$$

32. A geometric series with ratio r will:
    1. Diverge if ______.
    2. Converge to $S =$ ______ if ______.

33. Given the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^3+1}$$

    1. Use the first 4 terms to approximate the sum; then explain what the maximum error is in this approximation.
    2. Determine how many terms are required to estimate the sum with an error of no more than 0.0001.

34. A geometric series is a series of the form

$$\sum_{n=0}^{\infty} \underline{\quad ? \quad}$$

35. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

36. Find the limit of the sequence and state whether the sequence converges or diverges

$$a_n = \frac{n}{n+1}$$

37. Consider the repeating decimal $0.1313\cdots$; Convert this decimal to a fraction.

38. Does the following series converge or diverge? If it converges, find the sum.

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

39. Determine convergence or divergence, including absolute/conditional convergence if applicable; state clearly what test is being used; and determine the interval of convergence for x if applicable.

$$\sum_{n=1}^{\infty} \frac{(\ln n)(x-3)^n}{n}$$

40. Does the following series converge or diverge? If it converges, find the sum.

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^2$$

41. The series $\sum_{n=1}^{\infty}(-1)^n$ and $\sum_{n=1}^{\infty}(-1)^{n+b}$ converge if the following two conditions are met:

    1. $\lim_{n\to\infty} a_n = \underline{?}$
    2. $a_{n+1} \underline{?}\, a_n$ for all n

42. Does the series converge or diverge, and if it converges then find the sum (use the geometric series test, the telescoping series test, or the nth term test for divergence)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

43. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{n!}{3^n}$$

44. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} 3(\pi)^{-n}$$

45. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3}{2n+1}$$

46. Find the explicit nth term formula for the following sequence

$$\{a_n\} = \{-2, 5, 12, 19, \dots\}$$

47. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$$

48. Does the series converge or diverge, and if it converges then find the sum (use the geometric series test, the telescoping series test, or the nth term test for divergence)

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

49. If $\sum_{n=1}^{\infty} b_n$ converges and a_n ________ b_n, then $\sum_{n=1}^{\infty} a_n$ also converges.

50. A sequence is bounded ______ if there is a number M such that $a_n \geq M$ for all n.

51. Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$$

52. Determine convergence or divergence, including absolute/conditional convergence if applicable. State clearly what test is being used.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Exam 3 Review (Answers)

1. *(Section 9.4)*

   conditionally convergent; diverges.

2. *(Section 9.1–9.3)*

   1. non-decreasing ($a_1 \leqq \cdots \leqq a_n$), or
   2. non-increasing ($a_1 \geqq \cdots \geqq a_n$).

3. *(Quiz 5)*

   By the limit comparison test and selection of the harmonic series/divergent p-series as the comparison series,

   $$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = 1$$

   Because the limit exists and is both positive and finite, the original series behaves similarly to the comparison series, and thus diverges.

4. *(Section 9.1–9.3)*

   1. $a_n = 4n - 1$
   2. $a_n = (-1)^{n+1} 2^{2-n}$
   3. $a_n = \frac{x^{n-1}}{(n-1)!}$

5. *(Exam 3 Studyguide)*

   (Divergence test) $\lim = 1/3$ which $\neq 0$ and therefore diverges.

6. *(Exam 3 Studyguide)*

   (Ratio test) Because $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the series diverges for all x.

7. *(Section 9.1–9.3)*

   $\lim = e$, therefore the sequence converges.

8. *(Section 9.4)*

   and $a_n \geq b_n$

9. *(Section 9.1–9.3)*

   bounded both above and below

10. *(Section 9.4)*

    (Integral test) Because $\int_1^\infty \frac{1}{4x^2+1} dx = \frac{\pi}{4} - \frac{1}{2}\arctan(2) \approx 0.2318$ (converges), the series also converges.

11. *(Quiz 5)*

    $\{1, 3, 7, 15\}$

12. *(Exam 3 Studyguide)*

    (Check for absolute convergence using Ratio Test) Because $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the series converges absolutely for all values of x.

13. *(Quiz 5)*

    By the nth term test, $\lim_{n\to\infty} \frac{n}{n+3} = 1 \neq 0$, therefore the series diverges.

14. *(Quiz 5)*

    By the integral test, $\lim_{b\to x} \int_1^b \frac{x}{x^2+1} dx = \infty$ which is non-finite, therefore the series diverges.

15. *(Section 9.1–9.3)*

    The function oscillates between the values 1 and 3, thus limit does not exist, and therefore the sequence diverges.

16. *(Exam 3 Studyguide)*

    (L.C.T.) Comparing against $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ (convergent p-series),

    $$\lim_{n\to\infty} \frac{1}{n\sqrt{n+1}} \cdot \frac{n\sqrt{n}}{1} = 1$$

    Which is both finite and positive, and so converges similarly.

17. *(Exam 3 Studyguide)*

    (Ratio Test) Because $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = 0 < 1$, the series converges.

18. *(Section 9.4)*

    absolutely convergent

19. *(Quiz 5)*

    $a_n = 3(2)^{n-1}$

20. *(Section 9.4)*

    (D.C.T.) Comparing against $b_n = \sum_{n=1}^\infty \frac{1}{2^n}$ (convergent geometric series), because $a_n < b_n$ the series similarly converges.

21. *(Section 9.4)*

    (D.C.T.) Compared against $\frac{1}{n^2}$ (convergent p-series), because $a_n \leq b_n$ the series similarly converges.

22. *(Section 9.4)*

 positive, continuous, and decreasing for $x \geq 1$

23. *(Section 9.1–9.3)*

 lim = 0, therefore the sequence converges.

24. *(Section 9.1–9.3)*

 above

25. *(Section 9.4)*

 1. absolute convergence
 2. diverges; conditional convergence; alternating series test.

26. *(Section 9.4)*

 $\frac{a_n}{b_n}$; finite and positive.

27. *(Section 9.1–9.3)*

 Converges, and the sum equals $\frac{e}{e-1}$.

28. *(Section 9.4)*

 (L.C.T.) Comparing against $\frac{1}{n^{3/2}}$ (convergent p-series), because $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, which is finite and positive, the series similarly converges.

29. *(Quiz 5)*

 Because $r = |1.2| > 1$ the series diverges (geometric series).

30. *(Section 9.4)*

 $$\underline{f(a)} + \cdots + \underline{\frac{f^{(n)}(a)}{n!}}$$

31. *(Exam 3 Studyguide)*

 The series converges for $-1/5 \leq x \leq 1/5$.

32. *(Section 9.1–9.3)*

 1. Diverge if $|r| \geq 1$.
 2. Converge to $S = \dfrac{a}{1-r}$ if $0 < |r| < 1$.

33. *(Exam 3 Studyguide)*

 1. $S_4 = -\frac{1}{3} + \frac{1}{17} - \frac{1}{55} + \frac{1}{129} \approx -0.2849$, and by alternating series remainder theorem, error $= a_5 = \left|\frac{1}{251}\right|$.
 2. $$a_{n+1} = \left|\frac{(-1)^n}{2n^3+1}\right| \leq 0.0001$$

 $16.099 \leq n$

 $n = 17$

34. *(Section 9.1–9.3)*

 $$\sum_{n=0}^{\infty} ar^n$$

35. *(Quiz 6)*

 (Integral test) Because lim $= \infty$ the series diverges.

36. *(Section 9.1–9.3)*

 lim = 1, therefore the sequence converges.

37. *(Section 9.1–9.3)*

 $$0.1313\cdots = \frac{13}{99}$$

38. *(Section 9.1–9.3)*

 Converges, and the sum equals 2.

39. *(Exam 3 Studyguide)*

 (Check for absolute convergence using Root test; check end points using A.S.T. and L.C.T.) The series converges for $2 \leq x < 4$.

40. *(Section 9.1–9.3)*

 Diverges

41. *(Section 9.4)*

 1. $\lim_{n\to\infty} a_n = \underline{0}$
 2. $a_{n+1} \underline{\leq} a_n$ for all n

42. *(Quiz 5)*

 By means of partial fraction decomposition and evaluation of a telescopic series, the series converges to the value $11/18$.

43. *(Exam 3 Studyguide)*

 (Ratio test) Because $\lim_{n\to\infty} \left|\frac{n+1}{3}\right| = \infty > 1$, the series diverges.

44. *(Exam 3 Studyguide)*

    (Geometric series) The series has the ratio $r = {}^1\!/\!_\pi$, which because $|r| < 0$ converges. The series converges to:

    $$a_1 = {}^3\!/\!_\pi;\ r = {}^1\!/\!_\pi$$
    $$S_n = \frac{{}^3\!/\!_\pi}{1 - {}^1\!/\!_\pi} \approx 1.4008$$

45. *(Exam 3 Studyguide)*

    1. (Check for absolute convergence using L.C.T.) Comparing against $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent p-series):

       $$\lim_{n\to\infty} \left| \frac{3}{2n+1} \cdot \frac{n}{1} \right| = \frac{3}{2}$$

       Which is both finite and positive, and so diverges similarly. We need to check for conditional convergence.
    2. (A.S.T.)
       (a) $\lim_{n\to\infty} \frac{(-1)^n 3}{2n+1} = 0$ ✓
       (b) $a_{n+1} < a_n$ ✓

46. *(Quiz 5)*

    $a_n = 7n - 9$

47. *(Quiz 6)*

    (P-series) Because $p = {}^2\!/\!_3 < 1$ the series diverges.

48. *(Quiz 5)*

    Because $r = |{}^3\!/\!_4| < 1$ the series converges (geometric series), and converges to the value 3.

49. *(Section 9.4)*

    and $a_n \leq b_n$

50. *(Section 9.1–9.3)*

    <u>below</u>

51. *(Quiz 6)*

    (-3,1)

52. *(Quiz 6)*

    Converges conditionally.