FLOW POLYTOPES

NILS OLSSON

Based upon the book *Combinatorial Reciprocity Theorems* by Matthias Beck and Raman Sanyal; 2010 *Mathematics Subject Classification*.

Topic:

(13) *Flow polytopes.* A *directed graph* is a collection of dots (called *vertices*) and arrow between them (called *edges*). A *flow* on a directed graph is a way to label each edge of G by an element of $\mathbb{Z}$ (or $\mathbb{Z}_n$) so that at each vertex v, the sum of the edges entering v equals the sum of the edges leaving v (i.e. the in *flow* at each vertex equals the *out flow*). The flows on a given directed graph can be counted using a certain family of polytopes, whose integer points each coincide with a flow on G.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

Introduction

Out of all the various topics in mathematics, graph theory is a particularly interesting one, and is relevant to a wide variety of other mathematical topics as well as computer science. What this paper intends to do is, in the first section, is introduce the topic of graphs. Many different families of graphs exist, but for this paper we will concern ourselves with directed graphs, and $\mathbb{Z}_n$-flow functions which act upon these graphs.

Then, in the second section, we will briefly explore the material of chapter 7 of Beck and Sanyal's book *Combinatorial Reciprocity Theorems*, and specifically a single main reciprocity expression theorem:

Theorem (Flow counting function)**.** *Let $G = (V, E)$ be a graph with a fixed base orientation. Then $\varphi_G(n)$ counts the number of nowhere-zero $\mathbb{Z}_n$-flows of G, and is defined:*

$$\varphi_G(n) = \sum_{\vec{b} \in \mathcal{C}(G)} \mathrm{ehr}_{\mathsf{P}^\circ_G(\vec{b})}(n) = (-1)^{\xi(G)} \sum_{\vec{b} \in \mathcal{C}(G)} \mathrm{ehr}_{\mathsf{P}_G(\vec{b})}(-n).$$

What this theorem provides us with is a method of traveling from the topics of graphs and combinatorics and into the study of a seemingly unrelated mathematical topic: polytopes. Specifically, using polytopes to count the number of possible flows on directed graphs, subject to a few constraints, and even more specifically we will be looking at a way to count only the nowhere-zero $\mathbb{Z}_n$-flows of a directed graph. This should of course require a brief introduction in polytopes as well, but as time permitted this paper wont include an introducting into polytopes or Ehrhart reciprocity.

What this paper does *not* intend to do (at least, not anymore) is thoroughly explain the proofs of this theorem, in all of its intricacies. Instead the second section is a restructuring of parts of chapter 7 in Beck and Sanyal's book, in an attempt to provided a clearly progression of concepts.

Additionally, this paper will *not* work on a specific example of a graph, as I had originally intended. I had hoped to use a very simple directed graph to demonstrate some of the underlying mechanics of the propositions and theorems of the book, but found it to be beyond my own scope.

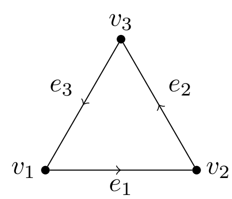

FIGURE 1. The intended example graph

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1. GRAPHS AND FLOWS

1.1. **Graphs.** We set off by defining and briefly explaining the topics of both graphs and flows. A vertex is most intuitively understood as a coordinate in space. For instance the coordinates $x = 1$ and $y = 1$ can define a vertex in Euclidean, $\mathbb{R}^2$, space. An edges then is a connection between two vertices. Edges do not necessarily need to have any particular geometric representation, but the concept that two edges can cross over one-another will be required later.

> **Definition** (Graph)**.** A *graph* $G = (V, E)$ is a collection of V, a set of vertices, and E, a set of edges between the vertices.

Now, a graph does not need to have any vertices. In fact a graph with no vertices is still a graph, but it is *the* empty graph, and will not be useful in the development of the main theorem. Nor does a graph need to have any edges (except that if a graph has edges, then there must be vertices to which those edges connect). Also, since a graph is just a collection of vertices and edges, we can have graphs within graphs, i.e. a *subgraph*, which consist of a subset of the vertices and edges of the whole graph (in particular, every graph necessarily has the empty graph as a subgraph, and every graph has itself as a subgraph).

As mentioned before, vertices, and by extension a graph, can exist within some arbitrary d-dimensional space. However, for this report we will only concern ourselves with graphs in two-dimensional, Euclidean space (or at least something analogous to two dimensional). (I will also point out that the words "vertex" and "node" are equivalent. This paper at times switches between either word, interchangeably.)

So, given a graph with vertices and edges between vertices, we define one classification of a subset with at least a few of its vertices, and with possibly a few of its edges:

Definition (Component, connected component)**.** A *connected component* (or simply a *component*) of a graph G is a subgraph in which any two vertices are either directly connected via an edge, or are connected to one-another via a *path* (a series of vertices and edges). A single vertex with no connecting edges is in itself a connected component.

Any graph then is the disjoint union of connected components. For example Figure 2 can be called a single graph that is the disjoint union of three distinct connected components. Traversing from one vertex of a graph to another by way

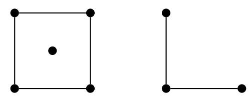


FIGURE 2

of edges only, we see that it may be impossible to travel to all of its vertices, hence a graph consisting of potentially disjoint components.

Lemma. *A connected component of the graph G is a maximal subgraph of G in which any two nodes are connect by a path.*

Lemma. *A graph G is* connected *if it has only one connected component.*

Definition (Bridge)**.** A *bridge* is a single edge which serves as a connection between two connected components which, if removed, would increase the number of connected components of the graph by one.

For example, removing the bridge in Figure 3 turns a single connected component into two disjoint connected components: one complete graph into a graph with two complete subgraphs.

FIGURE 3. An undirected graph with and without a bridge.

1.2. **Directed graphs.** So far we've only looked at *unoriented* graphs, but now we introduce a notion of direction, or orientation, to graphs.

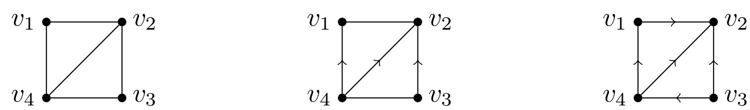
Definition (Directed, oriented graph)**.** A *directed* or *oriented graph* is a graph on which we define an *orientation* via a subset ρ of the edges E of the graph. For each edge $e = v_i v_j \in E$ (connecting the vertices $v_i, v_j \in V$) with $i < j$ we direct:

$$v_i \xleftarrow{e} v_j \text{ if } e \in \rho, \text{ and } v_i \xrightarrow{e} v_j \text{ if } e \notin \rho.$$

The graph G with orientation ρ is denoted ${}_\rho G$.

This definition may be confusing at first glance because ρ may be missing some of the edges of E. It only includes the edges with their direction pointing from the vertex of higher index to the vertex of lower index (which does certainly feel backwards). Then, all of the edges not in ρ are directed in the opposite direction: from lower to higher index.

Figure 4 demonstrate how this encapsulation works by applying an orientation $\rho = \{14, 23, 24\}$ onto a particular graph in three steps.



(A) The undirected graph. (B) ρ edges marked. (C) $E \setminus \rho$ edges marked.

FIGURE 4. A directed graph.

As an aside, there are many algorithms in computer science pertaining to graphs, some for finding optimal ways to traverse the paths of graphs. In some models each edge is considered to be of equal "weight" to any other. That is, if two unweighted edges both originate from a node v_1 and both point to a second node v_2, then there is no reason to choose one edge over the other in traveling from v_1 to v_2. On the other hand, we might assign a weight to each edge, presenting a cost-benefit analysis in choosing a path to take (as in Figure 5, where the edge with weight 5 now becomes most "expensive" to traverse versus the one with weight 2).

FIGURE 5. A directed graph, with and without weights.

Definition (Directed path, directed cycle)**.** A *directed path* in ${}_\rho G$ is a sequence $v_0, v_1, \ldots, v_s$ of distinct nodes such that $v_{j-1} \to v_j$ is a directed edge in ${}_\rho G$ for all $j = 1, \ldots, s$. If $v_s \to v_0$ is also a directed edge, then $v_0, v_1, \ldots, v_s, v_{s+1} = v_0$ is called a *directed cycle*.

Definition (Acyclic orientations)**.** An orientation ρ of G is acyclic if there are no directed cycles in ${}_\rho G$.

Dual to the notion of acyclic orientations are orientations which are *totally cyclic*, where every edge in ${}_\rho G$ is contained in a directed cycle.

Definition (Cyclotomic number)**.** We define the *cyclotomic number* of G as:

$$\xi(G) = |E| - |V| + c.$$

Where $c = c(G)$ is the number of connected components of G.

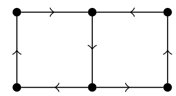


FIGURE 6. A totally cyclic graph with two cycles.

Another topic of graphs is coloring, where every vertex is assigned a color value, and a proper coloring is where no vertices connected by an edge share the same color. Directions and cycles provide a particular way to induce a coloring onto the vertices of a graph, but this is outside the scope of this paper. However, acyclic and cyclic orientations are a critical component of the main focus of this paper, the nowhere-zero $\mathbb{Z}_n$-flow counting function, which we now introduce.

1.3. **Flows.** With a notion of orientations and weights upon the edges of graphs, we will now introduce a particular type of weight value that is specific to living within an Abelian group $\mathbb{Z}_n = {}^{\mathbb{Z}}\!/_{n}$, and a function which gives to every edge of a graph this particular type of weight:

Definition ($\mathbb{Z}_n$-flow)**.** A $\mathbb{Z}_n$*-flow* is a map $f : E \to \mathbb{Z}_n$ which to each edge, e, of a directed graph assigns a flow value, $f(e) \in \mathbb{Z}_n$, such that there is "conservation of flow" at every vertex, v, by which me mean the sum of the flows into a vertex is congruent to the sum of the flows out it:

$$\sum_{\xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e}} f(e).$$

It is these flow functions that are the subject of the main theorem of this paper, in particular counting how many different flows can be constructed for a particular directed graph given the constraint of conservation. It is especially important to note that because the flow values are elements of an Abelian group $\mathbb{Z}_n$ that there is necessarily a finite number of flow functions that can be applied to any particular graph since each flow value has to be one of n possible values, modulus n.

Definition (f support)**.** The *support* of f, denoted $\mathrm{supp}(f)$, is the subset of edges e whose flows are non-zero:

$$\mathrm{supp}(f) = \{e \in E : f(e) \neq 0\} \subseteq E$$

If $\mathrm{supp}(f) = E$ (all e edges have $f(e) \neq 0$), then f is considered "nowhere-zero". For the rest of this report we'll be concerned with these nowhere-zero $\mathbb{Z}_n$-flows, in particular with counting how many of them can be made for any particular graph and Abelian group. Figure 7 illustrates two different but satisfactory $\mathbb{Z}_5$-flows on a graph. As simple integer weights, the second flow would not fullfil the conservation requirement, since from vertex v_1 we have in input of 0 and an output of $2 + 3 = 5$, but as elements of the Abelian group Z_5 they work, since $5 \equiv 0 \bmod 5$.

And now we introduce the main focus of this paper.

Definition (Nowhere-zero $\mathbb{Z}_n$-flow counting function)**.** Define a counting function:

$$\varphi_G(n) = \text{the number of } f \text{ nowhere-zero } \mathbb{Z}_n\text{-flows on } G$$

FIGURE 7. An everywhere-zero $\mathbb{Z}_5$-flow and a nowhere-zero $\mathbb{Z}_5$-flow.

As it happens...

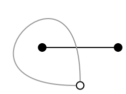
Proposition 1. *The flow-counting function $\varphi_G(n)$ is independent of the orientation ρ of G.*

Proposition 2. *G will not have any nowhere-zero flow if G has a* bridge*—an edge whose removal increases the number of connected components of G.*

1.4. **Dual graphs.** Consider a graph G as subdividing the plane into connected regions. If two points lie within a region then they can be connected without intersecting an edge of G. Regions which are neighbors have an edge of G which separate them (their topological closures sharing a proper 1-dimensional part of their boundaries).

Definition (Dual graph)**.** The *dual graph* of a graph $G = (V, E)$ is the graph $G^* = (V^*, E^*)$ with vertices corresponding to the regions of G. Two vertices $u^*, v^* \in V^*$ which correspond to two regions R_{u^*} and R_{v^*} of G are connected via edge $e^* \in E^*$ if an original edge $e \in E$ is properly contained within the boundaries of R_{u^*} and R_{v^*}.

In other words there is one e^* for every edge e that lies between two regions of G; it connects the G^* vertices corresponding to the two regions; and it crosses over its corresponding e edge (e and e^* intersect). But as a single illustration says a thousand words, Figure 8 illustrates a few examples of graphs (solid dot vertices) and their duals (hollow dot vertices).



(A) Simplest graph with a bridge.

FIGURE 8. Various graphs and their duals.

What naturally follows is that if G is a directed graph, we should want G^* to also be some sort of directed graph. So, given an orientation of G, an orientation on G^* can be induced by "rotating" the direction of the edge clockwise. That is, the dual edge e^* will "point" east assuming that the primal edge e points north.

2. Reciprocity in counting nowhere-zero $\mathbb{Z}_n$-flows

Originally my intention of this second section of the paper was to demonstrate, by example, the underlying mechanisms of the theorems and propositions in the Beck and Sanyal book. Instead what I found was that my grasp on the topic was shaky at best. What proved the most useful, to me at least, was to piece together a different order of the book's chapter 7 definitions, propositions and theorems; an order which feels much more natural in moving from a graph and combinatorial centric topic of flows, into a more polytope centric topic of reciprocity expressions. It was also at this time that I recognized that many topic introduced in the first second of this paper would now have no connection to this second section, specifically dual graphs and cyclic/acyclic orientations. They are in fact critical in the proofs of the propositions behind the $\mathbb{Z}_n$-flow counting function, but by limitations of time wont find any use here-on-out.

We begin by recalling that flows are functions which assign to every edge of a directed graph a value such that there is conservation of flow at every vertex:

$$\sum_{u \to v} f(uv) - \sum_{u \leftarrow v} f(uv) = 0.$$

What may not be obvious from the presentation of this equation is that it is specific to the vertex v and not to u. Rather, u pertains to the other side of an edge, pointing towards v or away from it; there may be many different u vertices, depending on how many edges connect to a particular v.

We will also pay attention to the specific order of the vertices in V and the specific order of the edges in E. What this provides us with is a way to think of flow functions as coordinates, where each flow value is a component of a m-dimensional coordinate, indexed by the edge onto which it is assigned:

$$f = (f(e_1), f(e_2), \ldots, f(e_m)).$$

This is extremely important, as it allows us to eventually build polytopes from collections of possible flows.

Also recall that $\mathbb{Z}_n$-flows (a specific kind of flow function) assign to edges flow values which are elements of an Abelian group $\mathbb{Z}_n$. The conservation equation then is modified to look more like the remainder algorithm:

$$\sum_{u \to v} f(uv) - \sum_{u \leftarrow v} f(uv) = nb_v. \tag{1}$$

Where b_v is an integer, and each b_v corresponds to a specific vertex v of our graph. That is, b_1 corresponds to v_1, b_2 to v_2, and b_3 to v_3. We will collect these b_v values into a vector $\vec{b} = (b_1, b_2, b_3) \in \mathbb{Z}^V$ in order of the vertices, just as we considered the order of the edges in calling a flow f a coordinate.

We should note that the $\mathbb{Z}_n$-flows, $f \in \mathbb{Z}_n^E$, are just one particular solution to this inequality (it's actually surprising to me that the book never uses the notation $\mathbb{Z}_n^E$ for flow values corresponding to the vertices). Many more solutions may exist, in particular for real-valued flows, $f \in \mathbb{R}^E$. Certainly it is confusing having the symbol f change family like this, and the book makes little-to-no distinction. Even so, for $\vec{b} \in \mathbb{Z}^V$ we define the following:

Definition (Subspace of flows)**.** Let $G = (V, E)$ be a graph with a fixed base orientation. We define $\mathcal{F}(\vec{b}) \subseteq \mathbb{R}^E$ to be the affine subspace of all $f \in \mathbb{R}^E$ real-valued flows (also considering f as a real-valued coordinate) satisfying (1) when $n = 1$.

Moving towards our goal, the number of nowhere-zero $\mathbb{Z}_n$-flows of G, $\varphi_G(n)$, then (hinting to polytopes) is the number of lattice points in

$$n(0,1)^E \cap \bigcup_{\vec{b} \in \mathbb{Z}^V} \mathcal{F}_G(n\vec{b}).$$

However this collection contains much more than just the lattice points, since $\mathcal{F}_G(n\vec{b})$ contains all real-valued flows.

Another important property of the flow subspaces to note is when $\vec{b} = \vec{0}$. In fact we give this particular subspace its own definition:

Definition (Flow space)**.** Let $G = (V, E)$ be a graph with a fixed base orientation. We call $\mathcal{F}_G(\vec{0})$ the *flow space* of G, and simply denote it $\mathcal{F}_G$.

Notice that, for the flow space, equation (1) turns into an inequality:

$$\sum_{u \to v} f(uv) = \sum_{u \leftarrow v} f(uv).$$

This condition is called the *conservation of flow* at every v, hearkening back to the original definition of weights and flows, in the first section of this paper. An important takeaway from this fact is that for any $\vec{b} \in \mathbb{Z}^E$, $\mathcal{F}_G(\vec{b}) = t + \mathcal{F}_G(\vec{0})$ for some $t \in \mathbb{R}^E$, meaning that the dimension of the flow subspace $\mathcal{F}_G(\vec{b})$ (not to be confused with the flow space $\mathcal{F}$) is actually independent of $\vec{b}$. As I wont be including any of the proofs of the book in the paper, the following proposition may not be obvious.

Proposition 3. *Let $G = (V, E)$ be a graph with a fixed base orientation. Then the dimension of the flow space* $\dim \mathcal{F}_G = \xi(G)$.

But what this proposition hints at is the relation between cycles in a graph and the flow space (recall that $\xi(G)$ is the cyclotomic number of a graph G).

Going back to the $\vec{b}$ vectors, we make the following set definition by recognizing that although many of the $\vec{b}$ vectors may satisfy equation (1), many still may be somewhere-zero flows:

Definition (Nowhere-zero flow vectors)**.** Let $\mathcal{C}(G) \subset \mathbb{Z}^V$ be the set of all $\vec{b}$ vectors such that:

$$(0,1)^E \cap \mathcal{F}_G(\vec{b}) \neq \emptyset.$$

The book makes no name for this set, but we will refer to it as the nowhere-zero flow vector set.

Finally we are ready move into the world of polytopes proper. In particular we can start to see that the polytope we are constructing will have vertices corresponding to the flows of G. Thus we also define:

Definition (Flow polytope).

$$\mathsf{P}_G(\vec{b}) = [0,1]^E \cap \mathcal{F}_G(\vec{b}).$$

As an intersection of half-spaces and hyperplanes, $\mathsf{P}_G(\vec{b})$ is indeed a polytope, and note that since $\mathsf{P}_G(\vec{b}) \subseteq [0,1]^E$, the lattice points of $\mathsf{P}_G(\vec{b})$ are exactly its vertices.

Proposition 4. *Let $G = (V, E)$ be a graph with a fixed base orientation. Then $\mathsf{P}_G(\vec{b})$ is a lattice polytope for every $\vec{b} \in \mathbb{Z}^V$.*

By dilating this polytope by a factor n corresponding to the Abelian group $\mathbb{Z}_n$, $n\,\mathsf{P}_G(\vec{b})$ represents the number of $\mathbb{Z}_n$-flows on G, and in particular the nowhere-zero $\mathbb{Z}_n$-flows correspond to the lattice points laying within the interior of this polytope. Also note that when $n = 1$, this polytope *has no* nowhere-zero $\mathbb{Z}_n$-flows (specifically $\mathbb{Z}_1$-flows) since no lattice points sit within the unit cube $[0,1]^E$. Only when $n \geq 2$ will nowhere-zero $\mathbb{Z}_n$-flows exist.

Then finally we reach our goal of a reciprocity expression:

Theorem 1 (Flow counting function). *Let $G = (V, E)$ be a graph with a fixed base orientation. Then $\varphi_G(n)$ counts the number of nowhere-zero $\mathbb{Z}_n$-flows of G, and is defined:*

$$\varphi_G(n) = \sum_{\vec{b} \in \mathcal{C}(G)} \operatorname{ehr}_{\mathsf{P}^\circ_G(\vec{b})}(n) = (-1)^{\xi(G)} \sum_{\vec{b} \in \mathcal{C}(G)} \operatorname{ehr}_{\mathsf{P}_G(\vec{b})}(-n).$$

Beck and Sanyal don't state this definition as it's own theorem but I think it's worthy of being called a theorem, especially with respect to the main topic of their book: combinatorial reciprocity theorems.

Conclusion and farewell

Ultimately, I end the paper with the statement of this function, as time has become the most limiting factor. To fully appreciate this last theorem requires an full reintroduction polytopes and Ehrhart reciprocity (especially since without this we don't exactly have a way to even calculate the $\operatorname{ehr}_{\mathsf{P}_G(\vec{b})}(-n)$ expression!). But hopefully what we are able to take away is the ability of polytopes to act as a bridge between seemingly unrelated fields of mathematics.

At many times, this reading project has been an absolute blast, and at others has been an absolute slog of feeling lost and confused. But this last draft does have be recommitted to the understanding that I can in-fact grasp these topics, given enough time and energy, and hopefully by developing my own structure of chapter 7 of Beck and Sanyal's book I will be able to come back to this in the future and have an easier time getting caught up, instead of getting lost in what I thought was an organization mess in the original.

Cheers!