Prompt:

(13) *Flow polytopes.* A *directed graph* is a collection of dots (called *vertices*) and arrow between them (called *edges*). A *flow* on a directed graph is a way to label each edge of G by an element of $\mathbb{Z}$ (or $\mathbb{Z}_n$) so that at each vertex v, the sum of the edges entering v equals the sum of the edges leaving v (i.e. the in *flow* at each vertex equals the *out flow*). The flows on a given directed graph can be counted using a certain family of polytopes, whose integer points each coincide with a flow on G.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

Graphs are a major area of concern in mathematics, having an entire field to study dedicated to them. Many different families of graphs exist, but for this paper we'll concern ourselves with directed graphs and $\mathbb{Z}_n$-flows on these graphs, all concepts which the first section introduces. The second section is dedicated to a reciprocity expression for counting the number of possible flows on directed graphs, and to developing some insight into it and its proof.

1. Introduction to graphs and flows

1.1. **Graphs.** A graph $G = (V, E)$ is a collection of V, a set of vertices, and E, a set of edges between the vertices. Now, a graph doesn't have to have any vertices, nor does it have to to have any edges (except that if it has edges there must be vertices to which those edges connect). Also note that the nodes of a graph can be elements of some arbitrary d-dimensional space, but for this report we'll only concern ourselves with graphs in some two-dimensional space. (Note that the word "node" is analogous to "vertex" and that this paper will do plenty of haphazard switching between the two of them.)

Definition (Component, connected component)**.** A *connected component* (or simply a *component*) of a graph G is a subgraph (a subset of the vertices and edges of G) in which any two vertices are either directly connected via an edge or are connected to one-another via a *path* (a series of vertices and edges). A single vertex with no connecting edges is in itself a connected component.

Any graph then is the disjoint union of connected components. For example Figure 1 can be called a single graph that is the disjoint union of three distinct connected components.

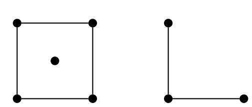


Figure 1

Lemma. *A connected component of the graph G is a maximal subgraph of G in which any two nodes are connect by a path.*

Lemma. *A graph G is* connected *if it has only one connected component.*

> **Definition** (Bridge)**.** A *bridge* is a single edge which serves as a connection between two connected components which, if removed, would increase the number of connected components of the graph by one.

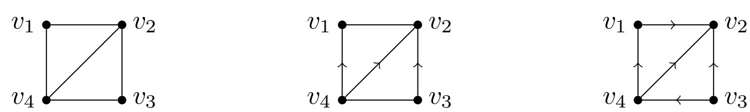
For example, removing the bridge in Figure 2 turns a single connected component into two triangular connected components: one complete graph into a graph with two complete subgraphs.

FIGURE 2. A graph with and without a bridge.

1.2. **Directed graphs.** So far we've only looked at *unoriented* graphs, but now we introduce a notion of direction, or orientation, to graphs.

> **Definition** (Directed, oriented graph)**.** A *directed* or *oriented graph* is a graph on which we define an *orientation* via a subset ρ of the edges E of the graph. For each edge $e = v_i v_j \in E$ (connecting the vertices $v_i, v_j \in V$) with $i < j$ we direct:
> $$v_i \overset{e}{\leftarrow} v_j \text{ if } e \in \rho, \text{ and } v_i \overset{e}{\rightarrow} v_j \text{ if } e \notin \rho.$$

This definition may be confusing at first glance because ρ may be missing some of the edges of E. What this definition attempts to succinctly encapsulate is only the edges with their direction pointing from the vertex of higher index to the vertex of lower index (which does certainly feel backwards), with all the remaining edges not in ρ directed in the opposite direction, from lower to higher index. Figure 3 tries to illustrate this encapsulation for a graph with five vertices and an orientation $\rho = \{14, 23, 24\}$.



(A) The undirected graph. (B) ρ edges marked. (C) $E \setminus \rho$ edges marked.

FIGURE 3. A directed graph.

As an aside, directed graphs are the subject of many algorithms in computer science, often pertaining to finding optimal ways to traverse the paths of said graphs. In some models each edge is considered to be of equal "weight" as any other. That is if two edges originating from the same node both point to the same node, there should be no real preference to traveling down one edge or the other to get to the destination. On the other hand we might assign a weight to each edge, presenting a cost-benefit analysis in choosing a path to take (as in Figure 4 where the edge with weight 5 now becomes most expensive to traverse versus the one with weight 2). Intuitively these weights are often simple $\mathbb{Z}$ integers.

FIGURE 4. A directed graph with and without weights.

1.3. **Flows.** Moving on we introduce *flow* values to edges; similar values to weights, but living not within $\mathbb{Z}$ but instead within an Abelian group $\mathbb{Z}_n = {}^{\mathbb{Z}}/_{n}$.

Definition ($\mathbb{Z}_n$-flow)**.** A $\mathbb{Z}_n$*-flow* is a map $f : E \to \mathbb{Z}_n$ which to each e edge of E assigns a flow $f(e) \in \mathbb{Z}_n$ such that there is "conservation of flow" at every $v \in V$ vertex, by which me mean the sum of the flows into a vertex v is equivalent to the sum of the flows out it:

$$\sum_{\xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e}} f(e).$$

(And it's important to note equivalence here since $\sum f(e) \in \mathbb{Z}_n$.) These functions and more specifically counting them will be the ultimate purpose of this paper.

Definition (f support)**.** The *support* of f, denoted $\text{supp}(f)$, is the subset of edges e whose flows are non-zero:

$$\text{supp}(f) = \{e \in E : f(e) \neq 0\} \subseteq E$$

If $\text{supp}(f) = E$ (all e edges have $f(e) \neq 0$), then f is considered "nowhere-zero". For the rest of this report we'll be concerned with these nowhere-zero $\mathbb{Z}_n$-flows, in particular with counting how many of them can be made for any particular graph and Abelian group. Figure 5 illustrates two different but satisfactory $\mathbb{Z}_5$-flows on a graph. Without the context of the Abelian group Z_5 these flows wouldn't seem to work, but, since $3 + 2 \equiv 0 \bmod 5$, the top and bottom left vertices both have conservation of flow equal to zero.

FIGURE 5. An everywhere-zero $\mathbb{Z}_5$-flow and a nowhere-zero $\mathbb{Z}_5$-flow.

Going forward, G will always refer to a 2-dimensional graph with V vertices and E edges, ρ to an arbitrary (but fixed) orientation of G and $\mathbb{Z}_n$ to an Abelian group to which f a $\mathbb{Z}_n$-flow maps.

Definition. Define a counting function:

$$\varphi_G(n) = \text{the number of } f \text{ nowhere-zero } \mathbb{Z}_n\text{-flows on } G$$

As it happens...

Proposition. *The flow-counting function $\varphi_G(n)$ is independent of the orientation ρ of G.*

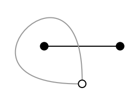
(The proof of this is a "left to the reader" exercise in the original paper.)

Proposition. *G will not have any nowhere-zero flow if G has a* bridge*—an edge whose removal increases the number of connected components of G.*

1.4. **Dual graphs.** Consider a graph G as subdividing the plane into connected regions. If two points lie within a region then they can be connected without intersecting an edge of G. Regions which are neighbors have an edge of G which separate them (their topological closures sharing a proper 1-dimensional part of their boundaries).

> **Definition** (Dual graph)**.** The *dual graph* of a graph $G = (V, E)$ is the graph $G^* = (V^*, E^*)$ with vertices corresponding to the regions of G. Two vertices $u^*, v^* \in V^*$ which correspond to two regions R_{u^*} and R_{v^*} of G are connected via edge $e^* \in E^*$ if an original edge $e \in E$ is properly contained within the boundaries of R_{u^*} and R_{v^*}.

In other words there is one e^* for every edge e that lies between two regions of G; it connects the G^* vertices corresponding to the two regions; and it crosses over its corresponding e edge (e and e^* intersect). But as a single illustration says a thousand words, Figure **??** illustrates a few examples of graphs (solid dot vertices) and their duals (hollow dot vertices).



(A) Simplest graph with a bridge.

FIGURE 6. Various graphs and their duals.

What naturally follows is that if G is a directed graph, we should want G^* to also be some sort of directed graph. So, given an orientation of G, an orientation on G^* can be induced by "rotating" the direction of the edge clockwise. That is, the dual edge e^* will "point" east assuming that the primal edge e points north.

2. Reciprocity in counting graph flows

Ultimately the purpose of this report is in providing some intuition into the following theorem and its proof:

Theorem 1. *Let G be a bridgeless graph. For every positive integer n, the reciprocity statement:*

$$(-1)^{\xi(G)}\varphi_G(-n)$$

counts the number of pairs (f, p), where f is a $\mathbb{Z}_n$-flow and ρ is a totally-cyclic reorientation of $G/\operatorname{supp}(f)$. In particular, when $n = -1$, the statement equals the number of totally-cyclic orientations of G.

Going forward we'll attempt to build some insight into theorem **??** and its proof, and into the propositions upon which the proof relies.

2.1. Flows and flow spaces.

As a matter of notation, for a vertex v, uv is a directed edge pointing either from a vertex u towards v, or from v to u. Now to slightly mangle a previous definition, we now define flows, as opposed to Z_n-flows:

> **Definition** (Flow)**.** For $G = (V, E)$ a graph and $\mathbb{Z}_n$ an Abelian group, we define a *nowhere-zero flow*, $f : E \to \mathbb{Z}$, to be a map such that $0 < f(e) < n$ for all $e \in E$, and which for all $v \in V$ with an integer b_v (corresponding to v) satisfies the equation:
>
> $$\sum_{u\to v} f(uv) - \sum_{v\to u} f(uv) = nb_v.$$

This should appear very similar to the conservation equation in the definition of $\mathbb{Z}_n$-flow functions, but the distinction is that $\mathbb{Z}_n$-flows map to $\mathbb{Z}_n$ and flows map to $\mathbb{Z}$. Nevertheless, since the difference of the sums in the equation of the definition above is equal to a multiple of n, the sums are indeed congruent modulo n, similar to the conservation in the definition of $\mathbb{Z}_n$-flows. (I'll present the justification for this distinction between $\mathbb{Z}_n$-flows and flows soon.)

Next we introduce a specific space of these function (as well as further mangle the definition of the f flow "functions"!):

> **Definition.** For $\vec{b} \in \mathbb{Z}^V$ a vector, we define $\mathcal{F}_G(\vec{b}) \subseteq \mathbb{R}^E$ to be the affine subspace of all $f \in \mathbb{R}^E$ satisfying the equation of the previous definition but specifically for $n = 1$.

What I meant by further mangle is that now we don't consider f only as a function, but now also as a coordinate in $\mathbb{R}^E$ space. What is $\mathbb{R}^E$ space you ask? It is $|E|$-dimensional, but we don't call it $\mathbb{Z}^{|E|}$ space because: each component of a coordinate is indexed by the edges themselves as opposed to a more familiar scheme such as $x_1, x_2, x_3, \ldots$.

I certainly had a hard time processing this dual representation, so I'll attempt to give a brief example to explain this relationship from function to coordinate:

> *Example.* Fix G a directed graph with four edges, $E = \{e_a, e_b, e_c, e_d\}$, and $f \in \mathbb{Z}^E$ a nowhere-zero flow on G for some Abelian group $\mathbb{Z}_n$. Suppose f, being a coordinate in $\mathbb{Z}^E$ space, is the coordinate $(4, 2, 7, 5)$ which maps the edges of

the graph in the order as they appear above. Here then is a sort of condensing of f's various representation:

$$f = (4, 2, 7, 5) \in \mathbb{Z}^4,\ f(e) \in \{4, 2, 7, 5\}$$

And then the action of mapping:

$$f(e_a) = 4,\ f(e_b) = 2,\ f(e_c) = 7,\ f(e_d) = 5$$

But what prompts/justifies this mangling of definitions? Well, what is ultimately being sought after is a way to relate $\mathbb{Z}_n$-flow functions to lattice points in/on some polytope, and vice versa: flows f which are points in $n(0,1)^E \cap \mathbb{Z}^E$. (Note that $n(0,1)^E$ is the n'th dilation of all real-valued points contained within the $|E|$-dimensional open unit cube—no points on the boundaries. Then the intersection with $\mathbb{Z}^E$ yields only those points that are lattice/integer point.)

Definition (Flow space)**.** $\mathcal{F}_G(\vec{0})$ is the *flow space* of G such that each f satisfies the conservation equation:

$$\sum_{u \to v} f(uv) = \sum_{v \to u} f(uv).$$

This should now look very familiar to Z_n-flows, which from the get-go were defined with conservation of flow.

(And eventually I get to actually talking about theorem **??**!)