Chapter 12.9

Problem 13

Find the min and max of $f(x,y) = y^2 - 4x^2$ subject to $x^2 + 2y^2 = 4$.

Let $g(x,y) = x^2 + 2y^2 - 4$, then calculate the gradients of both functions:

$$\nabla f(x,y) = \langle -8x, 2y\rangle;\ \nabla g(x,y) = \langle 2x, 4y\rangle$$

We can express these gradients as an equality using Langrange multipliers and come up with a series of equations:

$$\begin{aligned}
\nabla f(x,y) &= \lambda \nabla g(x,y) \\
\langle -8x, 2y\rangle &= \lambda\langle 2x, 4y\rangle \\
-8x &= \lambda 2x \\
2y &= \lambda 4y
\end{aligned}$$

The system of equations gives us the following possible solutions:

$$\begin{aligned}
x(-4-\lambda) &= 0 & \qquad y(1-2\lambda) &= 0 \\
x &= 0 & y &= 0 \\
\lambda &= -4 & \lambda &= 1/2
\end{aligned}$$

However, because $x = y = 0$ both invalidate the constraint they can both be ignored. But using the solutions for λ give the following:

$$\begin{aligned}
2y - (-4)4y &= 0 & \qquad -8x - (1/2)2x &= 0 \\
\Rightarrow y &= 0 & \Rightarrow x &= 0 \\
x^2 + 2(0)^2 &= 4 & (0)^2 + 2y^2 &= 4 \\
\Rightarrow x &= \pm 2 & \Rightarrow y &= \pm\sqrt{2} \\
\Rightarrow x &= \pm 2 & \Rightarrow y &= \pm\sqrt{2}
\end{aligned}$$

Which leave us with the four points $(\pm 2, 0)$ and $(0, \pm\sqrt{2})$ which when tried in the original function give us:

$$f(2,0) = f(-2,0) = -16;\ f(0,\sqrt{2}) = f(0,-\sqrt{2}) = 2$$

Therefore there is a minimum -16 at $(2,0)$ and $(-2,0)$, and a maximum 2 at $(0,\sqrt{2})$ and $(0,-\sqrt{2})$.