(13) *Flow polytopes.* A *directed graph* is a collection of dots (called *vertices*) and arrow between them (called *edges*). A *flow* on a directed graph is a way to label each edge of G by an element of $\mathbb{Z}$ (or $\mathbb{Z}_n$) so that at each vertex v, the sum of the edges entering v equals the sum of the edges leaving v (i.e. the in *flow* at each vertex equals the *out flow*). The flows on a given directed graph can be counted using a certain family of polytopes, whose integer points each coincide with a flow on G.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

A graph $G = (V, E)$ is a collection of V, a set of vertices, and E, a set of edges between the vertices. Now, a graph doesn't have to have any vertices, nor does it have to to have any edges (except that if it has edges there must be vertices to which those edges connect). Also note that the nodes of a graph can be elements of some arbitrary d-dimensional space, but for this report we'll only concern ourselves with graphs in some two-dimensional space. (Note that the word "node" is analogous to "vertex" and that this paper will do plenty of haphazard switching between the two of them.)

Definition (Component, connected component)**.** A *connected component* (or simply a *component*) of a graph G is a subgraph (a subset of the vertices and edges of G) in which any two vertices are either directly connected via an edge or are connected to one-another via a *path* (a series of vertices and edges). A single vertex with no connecting edges is in itself a connected component.

Any graph then is the disjoint union of connected components. For example figure 1 can be called a single graph that is the disjoint union of three distinct connected components.

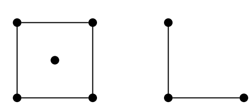


FIGURE 1

Lemma. *A connected component of the graph G is a maximal subgraph of G in which any two nodes are connect by a path.*

Lemma. *A graph G is* connected *if it has only one connected component.*

Definition (Bridge)**.** A *bridge* is a single edge which serves as a connection between two connected components which, if removed, would increase the number of connected components of the graph by one.

For example, removing the bridge in figure 2 turns a single connected component into two triangular connected components: one complete graph into a graph with two complete subgraphs. So far we've only looked at *unoriented* graphs, but now we introduce directions within graphs.

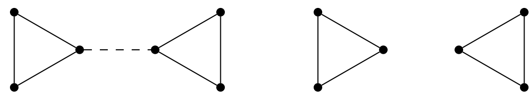
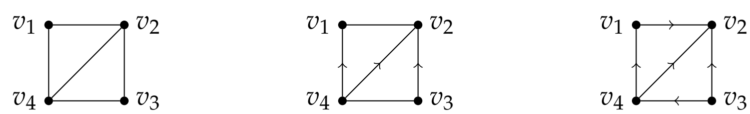


FIGURE 2. A graph with and without a bridge.

Definition (Directed, oriented graph)**.** A *directed* or *oriented graph* is a graph on which we define an *orientation* via a subset ρ of the edges E of the graph. For each edge $e = v_i v_j \in E$ (connecting the vertices $v_i, v_j \in V$) with $i < j$ we direct:

$$v_i \overset{e}{\leftarrow} v_j \text{ if } e \in \rho, \text{ and } v_i \overset{e}{\rightarrow} v_j \text{ if } e \notin \rho.$$

This definition may be confusing at first glance because ρ may be missing some of the edges of E. What this definition attempts to succinctly encapsulate is only the edges with their direction pointing from the vertex of higher index to the vertex of lower index (which does certainly feel backwards), with all the remaining edges not in ρ directed in the opposite direction, from lower to higher index. Figure 3 tries to illustrate this encapsulation for a graph with five vertices and an orientation $\rho = \{14, 23, 24\}$.



(A) The undirected graph. (B) ρ edges marked. (C) $E \setminus \rho$ edges marked.

FIGURE 3. A directed graph.

As an aside, directed graphs are the subject of many algorithms in computer science, often pertaining to finding optimal ways to traverse the paths of said graphs. In some models each edge is considered to be of equal "weight" as any other. That is if two edges originating from the same node both point to the same node, there should be no real preference to traveling down one edge or the other to get to the destination. On the other hand we might assign a weight to each edge, presenting a cost-benefit analysis in choosing a path to take (as in figure 4 where the edge with weight 5 now becomes most expensive to traverse versus the one with weight 2).

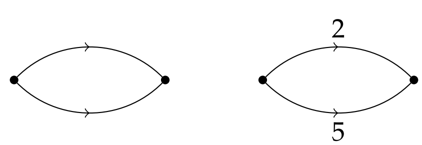


FIGURE 4. A directed graph with and without weights.

Intuitively these weights are often simple $\mathbb{Z}$ integers. But moving on, we now we introduce *flow* values to edges; similar values to weights, but living not within $\mathbb{Z}$ but instead within an Abelian group $\mathbb{Z}_n = \mathbb{Z}/n$. (There may also be optimal path algorithms associated with flow values, but for this paper our focus is elsewhere.)

Definition ($\mathbb{Z}_n$-flow)**.** A $\mathbb{Z}_n$*-flow* is a map $f : E \to \mathbb{Z}_n$ which to each e edge of E assigns a flow $f(e) \in \mathbb{Z}_n$ such that there is "conservation of flow" at every $v \in V$ vertex, by which me mean the sum of the flows into a vertex v is equivalent to the sum of the flows out it:

$$\sum_{\xrightarrow{e} v} f(e) = \sum_{v \xrightarrow{e}} f(e).$$

And it's important to note equivalence here since $\sum f(e) \in \mathbb{Z}_n$.

Definition (f support)**.** The *support* of f, denoted $\mathrm{supp}(f)$, is the subset of edges e whose flows are non-zero:

$$\mathrm{supp}(f) = \{e \in E : f(e) \neq 0\} \subseteq E$$

If $\mathrm{supp}(f) = E$ (all e edges have $f(e) \neq 0$), then f is considered "nowhere-zero". For the rest of this report we'll be concerned with these nowhere-zero $\mathbb{Z}_n$-flows, in particular with counting how many of them can be made for any particular graph and Abelian group. (So going forward, G will always refer to a graph in $\mathbb{Z}^2$ with V vertices and E edges, ρ to an arbitrary (but fixed) orientation of G and $\mathbb{Z}_n$ an Abelian group to which f a $\mathbb{Z}_n$-flow maps.)

Definition. Define a counting function:

$$\varphi_G(n) = \text{the number of } f \text{ nowhere-zero } \mathbb{Z}_n\text{-flows on } G$$

As it happens. . .

Proposition. *The flow-counting function $\varphi_G(n)$ is independent of the orientation ρ of G.*

(Todo:? Describe the proof of this proposition.)

Proposition. *G will not have any nowhere-zero flow if G has a* bridge*—an edge whose removal increases the number of connected components of G.*

Now we introduce the concept of the *dual graph* of a graph. Consider a graph G as subdividing the plane into discrete regions; if two points lie within a region then they can be connected without intersecting an edge of G. Regions which are neighbors have an edge of G which separate them (their topological closures sharing a proper 1-dimensional part of their boundaries).

Definition (Dual graph)**.** The *dual graph* of a graph G is the graph $G^* = (V^*, E^*)$ with vertices corresponding to the regions of G. Two V^* vertices are connected via edge $e^* \in E^*$ if an original edge e of G is properly contained within the boundaries of their corresponding regions.

Figure 5 shows a graph with four vertices (solid) along with the vertices (hollow) of its dual to demonstrate the regions which G subdivides the plane into. (However the edges are omitted. See figure 6 for complete illustrations of graphs and their duals.)

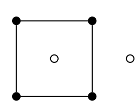
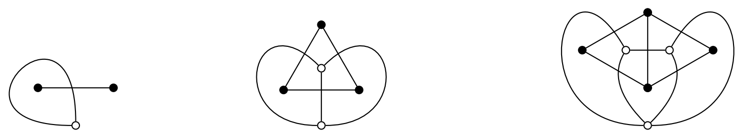


FIGURE 5. A graph and its (incompletely portrayed) dual graph.

Another way to think of the e^* edges is that there is one for every edge e that bounds two regions of G (connecting the G^* vertices corresponding to the two regions), and that e^* crosses-over its corresponding e edge. Figure 6 illustrates some examples of graphs (with solid dot vertices) and their duals (with hollow dot vertices).



(A) Simplest graph with a bridge.

(B) A complete graph.

(C) Graph with two connected components.

FIGURE 6. Various graphs and their duals.

What naturally follows is that if G is a directed graph, we should want G^* to also be some sort of directed graph. So, given an orientation of G, an orientation on G^* can be induced by "rotating" the direction of the edge clockwise. That is, the dual edge e^* will "point" east assuming that the primal edge e points north: