Discrete Dynamical Systems & Chaos

- toc {:toc} —

Dynamical systems

A *dynamical system* consists of a set of possible states, together with a rule that determines the present state in terms of past states. If the present state can be determined uniquely—without randomness—the rule is *deterministic*. If the present state relies on a element of randomness we do not have a dynamical system and instead a *random*, or *stochastic*, *process*. Two types of dynamical systems will be emphasized: *discrete-time* and *continuous-time*, the rule of the latter involving sets of differential equations. Of course if time in between discrete-time measurements goes to zero, $\Delta t \to 0$, you have a continuous-time system.

Maps

{% include boxed.html text="Definition: Maps" content=" A function with the same domain and range spaces is called a *map*. Fix x a point and f a map. The sequence $\{x, f(x), f^2(x)\}$ is called the *orbit* of x, and x is called the *initial value* of the orbit. A point ρ is a *fixed point* of the map f if $f(\rho) = \rho$. "%}

Kinds of fixed points, $f(p) = p$, based on that pertubation theory. On n^{th} iteration, if $x_n = p + \varepsilon_n$ the distance from the fixed point still small:

$$
\begin{aligned}
|\varepsilon_n| \ll 1 &\Rightarrow \varepsilon_{n+1} \approx f'(p)\varepsilon_n \\
&\Rightarrow \varepsilon_{n+k} \approx f'(p) \int^k \varepsilon_n
\end{aligned}
$$

{% include boxed.html text="Definition: ε-neighborhood" content=" $N_\varepsilon < P1 = \{x \in \mathbb{R} : |x - p| < \varepsilon\}$. "%}

{% include boxed.html text="Definition: Attractor/Repeller" content=" If all points sufficiently close to fixed point p are attracted to p, then p is called a sink or an attracting fixed point.

If all points sufficiently close to fixed point p are repelled from p (except for p itself), then p is called a source or an repelling fixed point. "%}

Taylorizations

Let f be a smooth map, p a fixed point. Then if the derivative at p is less than one, $|f'(p)| < 1$, then p is an sink, or if larger than 1, $|f'(p)| > 1$, then p is a source.

Theorem does not require any particularly sized epsilon. Can be very small, but still be an attractor.

Logistic map, $g_a(x) = ax(1-x)$.

As an example with $a = 2$, $g(x) = 2x(1-x)$:

- Fixed points at $x_1 = 0$ and $x_2 = 1/2$.
- Stability $g'(x) = 2 - 4x = 2(1-2x)$.
- x_1^*:$|g'(x_1^*)| = |g'(0)| = 2 > 1 \Rightarrow$ a repeller.
- x_2^*:$|g'(x_2^*)| = |g'(1/2)| = 0 < 1 \Rightarrow$ an attractor (even "super attracting").