Math 252 Final Review (Problems)

1. Using $\mathbf{u} = \langle -2, 4, 4\rangle$ and $\mathbf{v} = \langle 0, 3, -4\rangle$,
   a. Find $\mathbf{u} \cdot \mathbf{v}$.
   b. Find $\text{Proj}_{\mathbf{u}}\mathbf{v}$.
   c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.
   d. Find $\mathbf{u} \times \mathbf{v}$.

2. Find the length of the helix $\mathbf{r}(t) = \langle 6\sin(2t), -5t, -6\cos(2t)\rangle$ for $0 \leq t \leq 4\pi$.

3. Using $\mathbf{r}(t) = \langle 4\cos t, 3t, 4\sin t\rangle$ at $t = 0$,
   a. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.
   b. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
   c. Find K, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.

4. Given the points $P(2, 1, 2)$, $Q(6, -2, 1)$ and $R(-1, 4, 5)$,
   a. Find an equation of the plane passing through the points.
   b. Find an equation of the line perpendicular to the plane, passing through the point $(8, 2, -1)$.
   c. Find the distance from the point $(-5, -2, 7)$ to the plane.
   d. Find the area of the parallelogram determined by the points.

5. A Projectile is launched at an angle of 30°, with speed 224 feet per second, and from a platform 128 feet above the ground,
   a. Find the position vector of the object at time t.
   b. How far away will it hit the ground?
   c. What is the speed upon impact?

6. Identify each surface by identifing the cross sections in each plane of $\mathbb{R}^3$ space:
   a. $2x^2 - 3y^2 + 6z^2 = 1$
   b. $4x - 2y^2 - 2z^2 = 9$
   c. $4x - 2y^2 = 9$

7. Determine if the planes $ax + by + cz + d_1 = 0$ and $(ak)x+(bk)y+(ck)z+d_2 = 0$ intersect. If they do not, find the distance between them.

8. Prove that all lines and circles (in the xy-plane) have constant curvature.

9. Describe the domain of $f(x, y) = \frac{\ln(xy)}{\sqrt{x+y}}$

10. Evaluate $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy\, dx\, dy$

11. For $f(x, y) = 4x^2 + 2y^2$, find the equation of the level curve that contains the point $P(-2, 4)$ and describe its shape.

12. Use Lagrange multipliers to find the extrema of $f(x, y, z) = x - 2y - 4z$ subject to the constraint $z = 4x^2 + y^2$

13. A closed rectangular box is to have dimensions 40cm × 30cm × 60cm with a maximum error of 0.5cm in each measurement. Estimate the maximum error in the volume.

14. Find the volume of the solid bounded by $z = 4x + 2y + 3$, $z = 0$, $x = y^2$ and $x = 2y$.

15. Using $f(x, y, z) = x^3yz^2$ at $P(1, 2, -2)$:
    a. Find the gradient of f at P.
    b. Find the directional derivative of f in the direction of $\mathbf{a} = \langle 9, -12, 20\rangle$.
    c. Find a unit vector in the direction in which f increases most rapidly at P.
    d. Find the equation of the tangent plane of f at P.
    e. Find equations of the normal line to f at P.
    f. Estimate $f(0.99, 2.02, -1.97)$.

16. For $f(x, y) = x^2 + x\sin y - 2x^3y - 6y^4$, find f_{xx}, f_{yy} and f_{xy}.

17. Find $\frac{\delta z}{\delta x}$ if $z = g(x, y)$ is defined implicitly by $x^2z^3 + \cos(xy) = 4 + y^4 - e^z$.

18. Reverse the order of integration of $\int_e^{100} \int_1^{\ln y} g(x, y)\, dx\, dy$.

19. Evaluate $\lim_{(x,y)\to(3,1)} \frac{x^2-9y^2}{x^4-81y^4}$.

20. Find any extrema or saddle points of $f(x,y) = 2x^2 - 6xy + 2y^2 + 10x - 6$.

21. Find the absolute extrema of $f(x,y) = 2x^2 - 6xy + 2y^2 + 10x - 6$ over the region in the xy-plane bounded by $y = 2x$, $y = 0$ and $x = 4$.

22. A function is harmonic if $\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0$ throughout the domain of f. Determine if $f(x,y) = \tan^{-1}(\frac{x}{y})$ is harmonic.

23. Evaluate $\int_C 6xy\,dx + x^2y\,dy$ where C is the graph $y = x^2 + 3$ from $(0,3)$ to $(3,12)$.

24. Find the mass of the solid with the density $\delta(x,y,z) = 8xy$ whose base in the xy-plane is bounded by $y = x$, $y = 0$ and $x = 3$ and bounded above by $z = 9 - x^2$.

25. Find the surface area of the paraboloid $z = x^2 + y^2$ between the planes $z = 6$ and $z = 30$.

26. Evaluate $\oint_C (-2xy^2\,dx + 4x^2y\,dy)$ where C is the boundary of the region in the first quadrant bounded by the x-axis, the y-axis and the semicircle $y = \sqrt{16 - x^2}$.

27. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 36$, the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 12$.

28. For $\mathbf{F} = \langle 3x^2y^3, 2y^3z, 4xz^2\rangle$ find each of the following:

    a. Div $\mathbf{F}$

    b. Curl $\mathbf{F}$

    c. Div(Curl $\mathbf{F}$)

29. Find the work done by $\mathbf{F} = \langle xy, y, -yz\rangle$, $\mathbf{r}(t) = \langle t, t^2, t\rangle$, $0 \le t \le 4$.

30. Evaluate $\iint_R (x-y)^2(x+y)dA$ where the boundary of R is the rectangle with vertices $(4,0)$, $(8,4)$, $(4,8)$ and $(0,4)$.

31. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \cos(x^2+y^2)dy\,dx$.

32. Determine if $\mathbf{F}(x,y,z) = \langle 18x^2 \sin y, 6x^3 \cos y - 3e^y \cos z, 3e^y \sin z + 2\cos z\rangle$, is conservative. If it is, find a potential function f of $\mathbf{F}$.

Math 252 Final Review (Answers)

1. (Math-252 Exam 1)

   a. $\mathbf{u} \cdot \mathbf{v} = -4$

   b. $\text{Proj}_{\mathbf{u}}\mathbf{v} = \left\langle \frac{2}{9}, \frac{-4}{9}, \frac{-4}{9} \right\rangle$

   c. $\theta = \cos^{-1}\left(\frac{-2}{15}\right) \doteq 1.705$ rad

   d. $\mathbf{u} \times \mathbf{v} = \langle -28, -8, -6 \rangle$

2. (Math-252 Exam 1)

   $s = 52\pi$

3. (Math-252 Exam 1)

   a. $\mathbf{v}(t) = \langle -4\sin t, 3, 4\cos t \rangle$
   $\mathbf{v}(0) = \langle 0, 3, 4 \rangle$
   $\mathbf{a}(t) = \langle -4\cos t, 0, -4\sin t \rangle$
   $\mathbf{a}(0) = \langle -4, 0, 0 \rangle$

   b. $\mathbf{T}(t) = \left\langle -\frac{4\sin t}{5}, \frac{3}{5}, \frac{4\cos t}{5} \right\rangle$
   $\mathbf{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle$
   $\mathbf{N}(t) = \langle -\cos t, 0, -\sin t \rangle$
   $\mathbf{N}(0) = \langle -1, 0, 0 \rangle$

   c. $K = \frac{4}{25}$ $a_{\mathbf{T}} = 0$ $a_{\mathbf{N}} = 4$

4. (Math-252 Exam 1)

   a. $2x + 3y - z - 5 = 0$

   b. $x = -6t + 8,\ y = -9t + 2,\ z = 3t - 1$

   c. $D = 2\sqrt{14}$

   d. $A = 3\sqrt{14}$

5. (Math-252 Exam 1)

   a. $\mathbf{r}(t) = \langle 112\sqrt{3}t, -16t^2 + 112t + 128 \rangle$

   b. max distance: $896\sqrt{3}$

   c. impact speed: $32\sqrt{57}$

6. (Math-252 Exam 1)

   a. Elliptical hyperboloid (one sheet)

   b. Circular hyperboloid

   c. Parabolic cylinder

7. (Math-252 Exam 1)

   (this was extra-credit)

   $\mathbf{n}_1 = \langle a, b, c \rangle,\ \mathbf{n}_2 = \langle ka, kb, kc \rangle = k\langle a, b, c \rangle$

   $\mathbf{n}_1 \times \mathbf{n}_2 = 0,\ \therefore n_1 \parallel n_2$

   point on first plane: $P(0, 0, -\frac{d_1}{c})$

   distance from point to second plane:

   $$D = |\,\text{Proj}_{\mathbf{n}_1} P||$$
   $$= \frac{|ka(0)+kb(0)+kc(-\frac{d_1}{c})+d_2|}{\sqrt{(ka)^2+(kb)^2+(kc)^2}}$$
   $$= \frac{|-kd_1+d_2|}{|k|\sqrt{a^2+b^2+c^2}}$$

8. (Math-252 Exam 1)

   (this was extra-credit)
   (ANSWER)

9. (Math-252 Exam 2)

   $\{(x, y) : x > 0, y > 0\}$

10. (Math-252 Exam 2)

    $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy\, dx\, dy = \frac{1}{3}$

11. (Math-252 Exam 2)

    $4x^2 + 2y^2 = 48$ (an ellipse)

12. (Math-252 Exam 2)

    Absolute max $f(\frac{1}{32}, -\frac{1}{4}, \frac{17}{256}) = \frac{17}{64}$

13. (Math-252 Exam 2)

    $x = 40, y = 30, z = 60$
    $dx = dy = dz = 0.5$
    $V(x, y, z) = xyz$
    $\delta V \approx dV = V_x\, dx + V_y\, dy + V_z\, dz = yz\, dx + xz\, dy + xy\, dz$
    $dV = (30)(60)(0.5) + (40)(60)(0.5) + (40)(30)(0.5) = 2700\text{cm}^3$

14. (Math-252 Exam 2)

    $\int_0^2 \int_{y^2}^{2y} (4x + 2y + 3) dx\, dy = \frac{76}{5}$

15. (Math-252 Exam 2)

    a. $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 3x^2yz^2, x^3z^2, 2x^3yz \rangle$,
    $\nabla f(P) = \langle 24, 4, -8 \rangle$

b. $\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \langle \frac{9}{25}, -\frac{12}{25}, \frac{20}{25}\rangle$,
$D_{\mathbf{u}}f(P) = \langle 24, 4, -8\rangle \cdot \langle \frac{9}{25}, -\frac{12}{25}, \frac{20}{25}\rangle = \frac{8}{25}$

c. $\mathbf{u} = \frac{1}{\|\nabla f(P)\|}\nabla f(P) = \langle \frac{6}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-2}{\sqrt{41}}\rangle$

d. $24(x-1) + 4(y-2) - 8(z+2) = 0$,
$6x + y - 2z - 12 = 0$

e. $\langle x, y, z\rangle = \langle 1, 2, -2\rangle + t\langle 24, 4, -8\rangle$,
$x = 1 + 24t,\ y = 2 + 4t,\ z = -2 - 8t$

f. $dx = -0.01,\ dy = 0.02,\ dz = 0.03$,
$df = f_x(1)dx + f_y(2)dy + f_z(-2)dz = -0.4$,
$f(0.99, 2.02, -1.97) \approx f(P) + df = 8 + (-0.4) = 7.6$

16. (Math-252 Exam 2)

$f_{xx} = 2 - 12xy$,
$f_{yy} = -x\sin y - 72y^2$,
$f_{xy} = \cos y - 6x^2$

17. (Math-252 Exam 2)

$f(x, y, z) = x^2z^3 + \cos(xy) - 4 - y^4 + e^z$,
$f_x = 2xz^3 - y\sin(xy), f_z = 3x^2z^2 + e^z$,
$\frac{\delta z}{\delta x} = -\frac{f_x}{f_z} = \frac{-2xz^3 + y\sin(xz)}{3x^2z^2 + e^z}$

18. (Math-252 Exam 2)

$\int_1^{\ln 100}\int_{e^x}^{100} g(x, y)\, dy\, dx$

19. (Math-252 Exam 2)

$\lim_{(x,y)\to(3,1)} \frac{x^2 - 9y^2}{x^4 - 81y^4} = \lim_{(x,y)\to(3,1)} \frac{1}{x^2 + 9^2} = \frac{1}{18}$.

20. (Math-252 Exam 2)

$f_x = 4x - 6y + 10 = 0,\ f_y = -6x + 4y = 1$,
$x = 2,\ y = 3,\ z = 4$,
$f_{xx} = 4,\ f_{yy} = 4,\ f_{xy} = -6$,
$f_{xx}f_{yy} - (f_{xy})^2 = -20$,
therefore there is a saddlepoint at $(2, 3, 4)$.

21. (Math-252 Exam 2)

(Extra credit)
$f(x, 2x) = -2x^2 + 10x - 6,\ f_x(x, 2x) = -4x + 10 = 0,\ x = \frac{5}{2},\ y = 5$,
$f(x, 0) = 2x^2 + 10 - 6,\ f_x(x, 0) = 4x + 10 = 0,\ x = -\frac{5}{2}$,
$f(4, y) = 2y^2 - 24y + 66,\ f_y(4, y) = 2y - 12 = 0,\ y = 6$,
$f(\frac{5}{2}, 5) = \frac{13}{2},\ f(4, 6) = -6$,
$f(0, 0) = -6,\ f(4, 0) = 66,\ f(4, 8) = 2$,
absolute max of 66 at $(4, 0)$ and absolute min of -6 at $(0, 0)$ and $(4, 6)$.

22. (Math-252 Exam 2)

(Extra credit)
$f_x = \frac{y}{(x^2+y^2)^2},\ f_{xx} = \frac{-2xy}{(x^2+y^2)^2}$,
$f_y = \frac{-x}{(x^2+y^2)^2},\ f_{yy}\frac{2xy}{(x^2+y^2)^2}$,
$f_{xx} + f_{yy} = 0$

23. (Math-252 Exam 3)

$\int_0^3 (2t^5 + 12t^3 + 18t)dt = 567$

24. (Math-252 Exam 3)

$m = \int_0^3\int_0^x\int_0^{9-x^2} 8xy\, dz\, dy\, dx = 243$

25. (Math-252 Exam 3)

$\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = 201\pi$

26. (Math-252 Exam 3)

Using Green's theorem, $\int_0^4\int_0^{\sqrt{16-x^2}} 12xy\, dy\, dx = 384$.

27. (Math-252 Exam 3)

$V = \int_0^{2\pi}\int_0^6\int_r^{12} r\, dz\, dr\, d\theta = 288\pi$

28. (Math-252 Exam 3)

a. $\text{Div}\,\mathbf{F} = 6xy^3 + 6y^2z + 8xz$
b. $\text{Curl}\,\mathbf{F} = \langle -2y^3, -4z^2, -9x^2y^2\rangle$
c. $\text{Div}(\text{Curl}\,\mathbf{F}) = 0$

29. (Math-252 Exam 3)

$W = \int \mathbf{F}(t)\cdot\mathbf{r}'(t)\, dt = 128$

30. (Math-252 Exam 3)

Let $u = x - y,\ v = x + y,\ x = \frac{1}{2}u + \frac{1}{2}v$, and $y = -\frac{1}{2}u + \frac{1}{2}v$, where $-4 \le u \le 4,\ 4 \le v \le 8$.

$$J = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{1}{2}$$

$$\iint_R (x-y)^2(x+y)dA = \frac{1}{2}\int_{-4}^{4}\int_4^{12} u^2v\, dv\, du = \frac{4096}{3}$$

31. (Math-252 Exam 3)

$\int_0^{\pi/2}\int_0^3 \cos(r^2)r\, dr\, d\theta = \frac{\pi}{4}\sin 9$

32. (Math-252 Exam 3)

(Extra credit)
$M_y = 18x^2 \cos y = N_x$, $M_z = 0 = P_x$ and $N_z = 3e^y \sin z = P_y$, therefore $\mathbf{F}$ is conservative.

$$\begin{aligned}
f(x,y,z) &= \int 18x^2 \sin y \, dx = 6x^3 \sin y + g(y,z) \\
f_y &= 6x^3 \cos y + g_y = 6x^3 \cos y - 3e^y \cos z \\
\therefore g(y,z) &= -3e^y \cos z + h(z) \\
f(x,y,z) &= \int 6x^3 y^2 - 3e^y \cos z + h(z) \\
f_z &= 3e^y \sin z + h_z = 3e^y \sin z + 2\cos z \\
\therefore h_z &= 2\cos z + C \\
\therefore f(x,y,z) &= 6x^3 \sin y - 3e^y \cos z + 2 \sin z + C
\end{aligned}$$