

Math 252 Final Review (Problems)

- Using $\mathbf{u} = \langle -2, 4, 4 \rangle$ and $\mathbf{v} = \langle 0, 3, -4 \rangle$,
 - Find $\mathbf{u} \cdot \mathbf{v}$.
 - Find $\text{Proj}_{\mathbf{u}} \mathbf{v}$.
 - Find the angle θ between \mathbf{u} and \mathbf{v} .
 - Find $\mathbf{u} \times \mathbf{v}$.
- Find the length of the helix $\mathbf{r}(t) = \langle 6 \sin(2t), -5t, -6 \cos(2t) \rangle$ for $0 \leq t \leq 4\pi$.
- Using $\mathbf{r}(t) = \langle 4 \cos t, 3t, 4 \sin 5 \rangle$ at $t = 0$,
 - Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.
 - Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
 - Find K , $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.
- Given the points $P(2, 1, 2)$, $Q(6, -2, 1)$ and $R(-1, 4, 5)$,
 - Find an equation of the plane passing through the points.
 - Find an equation of the line perpendicular to the plane, passing through the point $(8, 2, -1)$.
 - Find the distance from the point $(-5, -2, 7)$ to the plane.
 - Find the area of the parallelogram determined by the points.
- A Projectile is launched at an angle of 30° , with speed 224 feet per second, and from a platform 128 feet above the ground,
 - Find the position vector of the object at time t .
 - How far away will it hit the ground?
 - What is the speed upon impact?
- Identify each surface by identifying the cross sections in each plane of \mathbb{R}^3 space:
 - $2x^2 - 3y^2 + 6z^2 = 1$
 - $4x - 2y^2 - 2z^2 = 9$
 - $4x - 2y^2 = 9$
- Determine if the planes $ax + by + cz + d_1 = 0$ and $(ak)x + (bk)y + (ck)z + d_2 = 0$ intersect. If they do not, find the distance between them.
- Prove that all lines and circles (in the xy-plane) have constant curvature.
- Describe the domain of $f(x, y) = \frac{\ln(xy)}{\sqrt{x+y}}$
- Evaluate $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy \, dx \, dy$
- For $f(x, y) = 4x^2 + 2y^2$, find the equation of the level curve that contains the point $P(-2, 4)$ and describe its shape.
- Use Lagrange multipliers to find the extrema of $f(x, y, z) = x - 2y - 4z$ subject to the constraint $z = 4x^2 + y^2$
- Evaluate $\int_C 6xt \, dx + x^2y \, dy$ where C is the graph $y = x^2 + 3$ from $(0, 3)$ to $(3, 12)$.
- Find the mass of the solid with the density $\delta(x, y, z) = 8xy$ whose base in the xy-plane is bounded by $y = x$, $y = 0$ and $x = 3$ and bounded above by $z = 9 - x^2$.
- Find the surface area of the paraboloid $z = x^2 + y^2$ between the planes $z = 6$ and $z = 30$.
- Evaluate $\oint_C (-2xy^2 \, dx + 4x^2y \, dy)$ where C is the boundary of the region in the first quadrant bounded by the x-axis, the y-axis and the semicircle $y = \sqrt{16 - x^2}$.
- Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 36$, the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 12$.
- For $\mathbf{F} = \langle 3x^2y^3, 2y^3z, 4xz^2 \rangle$ find each of the following:
 - $\text{Div } \mathbf{F}$
 - $\text{Curl } \mathbf{F}$
 - $\text{Div}(\text{Curl } \mathbf{F})$
- Find the work done by $\mathbf{F} = \langle xy, y, -yz \rangle$, $\mathbf{r}(t) = \langle t, t^2, t \rangle$, $0 \leq t \leq 4$.

20. Evaluate $\iint_R (x-y)^2(x+y)dA$ where the boundary of R is the rectangle with vertices $(4, 0)$, $(8, 4)$, $(4, 8)$ and $(0, 4)$.
21. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \cos(x^2 + y^2) dy dx$.
22. Determine if $\mathbf{F}(x, y, z) = \langle 18x^2 \sin y, 6x^3 \cos y - 3e^y \cos z, 3e^y \sin z + 2 \cos z \rangle$, is conservative.
23. Using $\mathbf{u} = \langle -4, 6, 5 \rangle$ and $\mathbf{v} = \langle 2, -3, 1 \rangle$,
- Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.
 - Find $\mathbf{u} \cdot \mathbf{v}$.
 - Find the angle θ between \mathbf{u} and \mathbf{v} .
 - Find $\text{proj}_{\mathbf{v}} \mathbf{u}$.
 - Find $\mathbf{u} \times \mathbf{v}$.
24. Using $P(-4, 1, 2)$, $Q(1, -3, 4)$, $R(-1, 0, 2)$,
- Find an equation of the plane passing through the points.
 - Find parametric equations for the line through P and parallel to $a = \langle 2, -1, 4 \rangle$.
 - Find the distance from the point $(5, -3, 2)$ to the plane.
 - Find the area of the parallelogram determined by P , Q , and R .
25. Identify the surface $x = y^2$.
26. Identify the surface $4x^2 + 4y^2 + z^2 = 4$.
27. Identify the surface $2x^2 - 3y^2 + 6z^2 = 6$.
28. Identify the surface $x^2 - 6y + 5z^2 = 0$.
29. A baseball is thrown from the stands 128 feet above the field at an angle of 30 degrees up from the horizontal with an initial speed of 64 feet per second.
- Give the position vector for any time t .
 - When will the ball strike the ground?
 - How far away will the ball strike the ground?
 - What is the speed of the ball when it strikes the ground?
30. Using $\mathbf{r}(t) = \langle t \cos t, t \sin t, t^2 \rangle$ at $t = 0$,
- Find \mathbf{v} and \mathbf{a} .
 - Find \mathbf{T} and \mathbf{N} .
 - Find K .
 - By first finding $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$, express $a = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.
31. For $f(x, y) = \sqrt{x^2 - y^2}$ find the domain of f and describe the level curves.
32. Find the limit:
- $$\lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}, x \neq y + 1$$
33. Find f_{xy} for $f(x, y) = \ln(xy + y^2)$.
34. If $w = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, show that $f_x^2 + f_y^2 = \left(\frac{\delta w}{\delta r}\right)^2 + \frac{1}{r^2} \left(\frac{\delta w}{\delta \theta}\right)^2$.
35. Using $f(x, y) = \frac{x-y}{x+y}$ and $P(2, -1)$,
- Find the directional derivative of f in the direction of $\mathbf{v} = \langle 4, -8 \rangle$.
 - Find the direction in which f increases most rapidly.
 - Find the direction in which f decreases most rapidly.
 - Find the maximum value of the directional derivative.
36. Using $x^3 - 2xy + z^3 + 7y + 6 = 0$ and $P(1, 4, -3)$,
- Find an equation of the tangent plane at P .
 - Find equations of the normal line at P .
37. A flat metal plate lies on an xy -plane such that the temperature T at (x, y) is given by $T = 10(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Find the instantaneous rate of change of T with respect to distance at $(1, 2)$ in the direction of the x -axis.

38. The total resistance R of three resistances R_1 , R_2 and R_3 connected in parallel is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. If measurements of R_1 , R_2 and R_3 are 100, 200 and 400 ohms respectively, with a maximum error of $\pm 1\%$ in each measurement, estimate the maximum error in the calculated value of R .
39. Find the maximum and minimum values of $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ over the triangular region with vertices $(0, 0)$, $(2, 0)$ and $(2, 2)$.
40. Find the volume of the largest rectangular box that has three of its vertices on the positive x , y and z -axes respectively, and a fourth vertex on the plane $3x + 4y + 2z = 24$.
41. Reverse the order of integration of $\int_1^e \int_0^{\ln x} y \, dy \, dx$ and evaluate.
42. Find the volume of the solid bounded by $y = x^3$, $y = x^4$, $z - x - y = 4$, and $z = 0$.
43. Using polar coordinates, evaluate $\iint_R (x^2 + y^2)^{3/2} dA$ where R is the region bounded by the circle of radius a centered at the origin.
44. Find the surface area of S , the part of the paraboloid $z = x^2 + y^2$ under the plane $z = a$, $a > 0$.
45. Evaluate $\int_{-1}^2 \int_1^x \int_0^{x+y} (3x^2y) dz \, dy \, dx$.
46. Find the center of mass of the lamina that has the shape of the region bounded by $y = x^2$ and $y = 9$ with density $\delta(x, y) = 12x^2y^2$.
47. For the solid bounded by $z = \sqrt{16x^2 + 16y^2}$, $x^2 + y^2 = 16$, and $z = 0$,
- find its volume.
 - find the center of mass if $\delta = \sqrt{x^2 + y^2}$.
48. Find the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 16$.
49. Evaluate $\iint_R \frac{2y+x}{y-2x} dA$ where R is the region bounded by the trapezoid with vertices $(-1, 0)$, $(-2, 0)$, $(0, 4)$, $(0, 2)$.
50. Find the curl and divergence of $\mathbf{F} = \langle -3 \sin x + \cos y, 6xz^2, 3y + z \rangle$.
51. The force at a point (x, y, z) in three dimensions is given by $\mathbf{F} = \langle y, z, x \rangle$. Find the work done by \mathbf{F} along the twisted cubic $x = t$, $y = t^2$ and $z = t^3$ from $(0, 0, 0)$ to $(2, 4, 8)$.
52. Evaluate $\int_{(1,1,1)}^{(2,3,4)} (4xy)dx + (2x^2 + 2yz^3)dy + (3y^2z^2 + 3)dz$.
53. Use Green's theorem to evaluate $\oint_C (6y)dx + (\frac{5}{3}x^3)dy$, where C is the boundary of the first quadrant region bounded by $y = 36 - x^2$ and the x -axis.

Math 252 Final Review (Answers)

1. (Math-252 Exam 1)

a. $\mathbf{u} \cdot \mathbf{v} = -4$

b. $\text{Proj}_{\mathbf{u}} \mathbf{v} = \langle \frac{2}{9}, \frac{-4}{9}, \frac{-4}{9} \rangle$

c. $\theta = \cos^{-1} \left(\frac{-2}{15} \right) \doteq 1.705 \text{ rad}$

d. $\mathbf{u} \times \mathbf{v} = \langle -28, -8, -6 \rangle$

2. (Math-252 Exam 1)

$s = 52\pi$

3. (Math-252 Exam 1)

a. $\mathbf{v}(t) = \langle -4 \sin t, 3, 4 \cos t \rangle$

$\mathbf{v}(0) = \langle 0, 3, 4 \rangle$

$\mathbf{a}(t) = \langle -4 \cos t, 0, -4 \sin t \rangle$

$\mathbf{a}(0) = \langle -4, 0, 0 \rangle$

b. $\mathbf{T}(t) = \langle -\frac{4 \sin t}{5}, \frac{3}{5}, \frac{4 \cos t}{5} \rangle$

$\mathbf{T}(0) = \langle 0, \frac{3}{5}, \frac{4}{5} \rangle$

$\mathbf{N}(t) = \langle -\cos t, 0, -\sin t \rangle$

$\mathbf{N}(0) = \langle -1, 0, 0 \rangle$

c. $K = \frac{4}{25} \quad a_{\mathbf{T}} = 0 \quad a_{\mathbf{N}} = 4$

4. (Math-252 Exam 1)

a. $2x + 3y - z - 5 = 0$

b. $x = -6t + 8, \quad y = -9t + 2, \quad z = 3t - 1$

c. $D = 2\sqrt{14}$

d. $A = 3\sqrt{14}$

5. (Math-252 Exam 1)

a. $\mathbf{r}(t) = \langle 112\sqrt{3}t, -16t^2 + 112t + 128 \rangle$

b. max distance: $893\sqrt{3}$

c. impact speed: $32\sqrt{57}$

6. (Math-252 Exam 1)

a. Elliptical hyperboloid (one sheet)

b. Circular hyperboloid

c. Parabolic cylinder

7. (Math-252 Exam 1)

(this was extra-credit)

$\mathbf{n}_1 = \langle a, b, c \rangle, \quad \mathbf{n}_2 = \langle ka, kb, kc \rangle = k\langle a, b, c \rangle$

$\mathbf{n}_1 \times \mathbf{n}_2 = 0, \quad \therefore n_1 \parallel n_2$

point on first plane: $P(0, 0, -\frac{d_1}{c})$

distance from point to second plane:

$D = |\text{Proj}_{\mathbf{n}_1} P|$

$= \frac{|ka(0) + kb(0) + kc(-\frac{d_1}{c}) + d_2|}{\sqrt{(ka)^2 + (kb)^2 + (kc)^2}}$

$= \frac{|-kd_1 + d_2|}{|k|\sqrt{a^2 + b^2 + c^2}}$

8. (Math-252 Exam 1)

(this was extra-credit)

(ANSWER)

9. (Math-252 Exam 2)

$\{(x, y) : x > 0, y > 0\}$

10. (Math-252 Exam 2)

$\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy \, dx \, dy = \frac{1}{3}$

11. (Math-252 Exam 2)

$4x^2 + 2y^2 = 48$ (an ellipse)

12. (Math-252 Exam 2)

Absolute max $f(\frac{1}{32}, -\frac{1}{4}, \frac{17}{256}) = \frac{17}{64}$

13. (Math-252 Exam 3)

$\int_0^3 (2t^5 + 12t^3 + 18t) dt$

14. (Math-252 Exam 3)

$m = \int_0^3 \int_0^x \int_0^{9-x^2} 8xy \, dz \, dy \, dx = 243$

15. (Math-252 Exam 3)

$\iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA = 201\pi$

16. (Math-252 Exam 3)

Using Green's theorem, $\int_0^4 \int_0^{\sqrt{16-x^2}} 12xy \, dy \, dx = 384$.

17. (Math-252 Exam 3)

$V = \int_0^{2\pi} \int_0^6 \int_r^{12} r \, dz \, dr \, d\theta = 288\pi$

18. (Math-252 Exam 3)

- a. $\text{Div } \mathbf{F} = 6xy^3 + y^2z + 8xz$
- b. $\text{Curl } \mathbf{F} = \langle -2y^3, -4z^2, -9x^2y^2 \rangle$
- c. $\text{Div}(\text{Curl } \mathbf{F}) = 0$

19. (Math-252 Exam 3)

$$W = \int \mathbf{F}(t) \cdot \mathbf{r}'(t) dt = 128$$

20. (Math-252 Exam 3)

Let $u = x - y$, $v = x + y$, $x = \frac{1}{2}u + \frac{1}{2}v$, and $y = -\frac{1}{2}u + \frac{1}{2}v$, where $-4 \leq u \leq 4$, $4 \leq v \leq 8$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2}$$

$$\iint_R (x-y)^2(x+y) dA = \frac{1}{2} \int_{-4}^4 \int_4^{12} u^2 v dv du = \frac{4096}{3}$$

21. (Math-252 Exam 3)

$$\int_0^{\pi/2} \int_0^3 \cos(r^2) r dr d\theta = \frac{\pi}{4} \sin 9$$

22. (Math-252 Exam 3)

(Extra credit)

$M_y = 18x^2 \cos y = N_x$, $M_z = 0 = P_x$ and $N_z = 3e^y \sin z = P_y$, therefore \mathbf{F} is conservative.

$$f(x, y, z) = \int 18x^2 \sin y dx = 6x^3 \sin y + g(y, z)$$

$$f_y = 6x^3 \cos y + g_y = 6x^3 \cos y - 3e^y \cos z$$

$$\therefore g(y, z) = -3e^y \cos z + h(z)$$

$$f(x, y, z) = \int 6x^3 y^2 - 3e^y \cos z + h(z)$$

$$f_z = 3e^y \sin z + h_z = 3e^y \sin z + 2 \cos z$$

$$\therefore h_z = 2 \cos z + C$$

$$\therefore f(x, y, z) = 6x^3 \sin y - 3e^y \cos z + 2 \sin z + C$$

23. (Math-252 Practice Exam 1)

- a. $\|\mathbf{u}\| = \sqrt{77}$
 $\|\mathbf{v}\| = \sqrt{14}$
- b. $\mathbf{u} \cdot \mathbf{v} = -21$
- c. $\theta = \arccos\left(\frac{-21}{7\sqrt{22}}\right)$

24. (Math-252 Practice Exam 1)

- a. $2x + 6y + 7z - 12 = 0$
- b. $x = 2t - 4$, $y = -t + 1$, $z = 4t + 2$
- c. $D = \frac{6}{\sqrt{89}}$
- d. $A = \sqrt{89}$

25. (Math-252 Practice Exam 1)

Parabolic cylinder

26. (Math-252 Practice Exam 1)

Circular ellipsoid

27. (Math-252 Practice Exam 1)

Hyperboloid (one sheet)

28. (Math-252 Practice Exam 1)

Elliptical cone

29. (Math-252 Practice Exam 1)

- a. $\mathbf{r}(t) = \langle 32\sqrt{3}t, -16t^2 + 32t + 128 \rangle$
- b. in 4 seconds
- c. $128\sqrt{3}$ feet away
- d. $64\sqrt{3}$ feet per second

30. (Math-252 Practice Exam 1)

- a. $\mathbf{v} = \langle -t \sin t + \cos t, t \cos t + \sin t, 2t \rangle$
 $\mathbf{a} = \langle -t \cos t - 2 \sin t, -t \sin t + 2 \cos t, 2 \rangle$
- b. $\mathbf{T}(t) = \left\langle \frac{-t \sin 5 + \cos t}{\sqrt{5t^2 + 1}}, \frac{t \cos t + \sin 5}{\sqrt{5t^2 + 1}}, \frac{2t}{\sqrt{5t^2 + 1}} \right\rangle$
 $\mathbf{T}(0) = \langle 1, 0, 0 \rangle$
 $\mathbf{N}(0) = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

31. (Math-252 Exam 2 Practice)

$D = \{(x, y) : |x| \geq |y|\}$
Hyperbola in xy-plane

32. (Math-252 Exam 2 Practice)

$$L = \frac{1}{4}$$

33. (Math-252 Exam 2 Practice)

$$f_{xy} = -\frac{1}{(x+y)^2}$$

34. (Math-252 Exam 2 Practice)

$$\begin{aligned}\frac{\delta w}{\delta r} &= f_x(\cos \theta) + f_y(\sin \theta) \\ \frac{\delta w}{\delta \theta} &= f_x(-r \sin \theta) + f_y(r \cos \theta) \\ \left(\frac{\delta w}{\delta r}\right)^2 + \frac{1}{r^2} \left(\frac{\delta w}{\delta \theta}\right)^2 &= f_x^2 + f_y^2\end{aligned}$$

35. (Math-252 Exam 2 Practice)

$$\begin{aligned}\text{a. } \nabla f(x, y) &= \left\langle \frac{2y}{(x_y)^2}, \frac{-2x}{(x_y)^2} \right\rangle \\ \nabla f(2, -1) &= \langle -2, -4 \rangle \\ \mathbf{u} = \mathbf{v} \frac{1}{|\mathbf{v}|} &= \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle \\ D_{\mathbf{u}} f(2, -1) &= \frac{6\sqrt{5}}{5} \\ \text{b. } \nabla f \frac{1}{|\nabla f|} &= \left\langle \frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5} \right\rangle \\ \text{c. } \left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle \\ \text{d. } |\nabla f| &= 2\sqrt{5}\end{aligned}$$

36. (Math-252 Exam 2 Practice)

$$\begin{aligned}\text{a. } -5x + 5y + 27z + 66 &= 0 \\ \text{b. } \langle x, y, z \rangle &= \langle 1, 4, -3 \rangle + t \langle -5, 5, 27 \rangle \\ x &= -5t + 1; y = 5t + 4; z = 27t - 3\end{aligned}$$

37. (Math-252 Exam 2 Practice)

$$T_x = 200 \text{ degrees per centimeters}$$

38. (Math-252 Exam 2 Practice)

$$\left| \frac{dR}{R} \right| = \frac{400}{7} \left(\frac{1}{100^2} + \frac{2}{200^2} + \frac{4}{400^2} \right) = 0.01$$

39. (Math-252 Exam 2 Practice)

$$\text{absolute max } \frac{37}{4} \text{ at } \left(1, \frac{3}{2}\right)$$

40. (Math-252 Exam 2 Practice)

$$V = \frac{64}{3}$$

41. (Math-252 Exam 2 Practice)

$$\int_0^1 \int_{e^y}^e y \, dx \, dy = \frac{e}{2} - 1$$

42. (Math-252 Exam 2 Practice)

$$V = \frac{157}{630}$$

43. (Math-252 Exam 3 Practice)

(ANSWER)

44. (Math-252 Exam 3 Practice)

(ANSWER)

45. (Math-252 Exam 3 Practice)

(ANSWER)

46. (Math-252 Exam 3 Practice)

(ANSWER)

47. (Math-252 Exam 3 Practice)

a. (ANSWER)

b. (ANSWER)

48. (Math-252 Exam 3 Practice)

(ANSWER)

49. (Math-252 Exam 3 Practice)

(ANSWER)

50. (Math-252 Exam 3 Practice)

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51. (Math-252 Exam 3 Practice)

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52. (Math-252 Exam 3 Practice)

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53. (Math-252 Exam 3 Practice)

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