Math 252 Final Review (Problems)

1. Using $\mathbf{u} = \langle -2, 4, 4 \rangle$ and $\mathbf{v} = \langle 0, 3, -4 \rangle$,

   a. Find $\mathbf{u} \cdot \mathbf{v}$.

   b. Find $\text{Proj}_{\mathbf{u}}\mathbf{v}$.

   c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   d. Find $\mathbf{u} \times \mathbf{v}$.

2. Find the length of the helix $\mathbf{r}(t) = \langle 6\sin(2t), -5t, -6\cos(2t) \rangle$ for $0 \leq t \leq 4\pi$.

3. Using $\mathbf{r}(t) = \langle 4\cos t, 3t, 4\sin 5 \rangle$ at $t = 0$,

   a. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

   b. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

   c. Find K, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.

4. Given the points $P(2, 1, 2)$, $Q(6, -2, 1)$ and $R(-1, 4, 5)$,

   a. Find an equation of the plane passing through the points.

   b. Find an equation of the line perpendicular to the plane, passing through the point $(8, 2, -1)$.

   c. Find the distance from the point $(-5, -2, 7)$ to the plane.

   d. Find the area of the parallelogram determined by the points.

5. A Projectile is launched at an angle of 30°, with speed 224 feet per second, and from a platform 128 feet above the ground,

   a. Find the position vector of the object at time t.

   b. How far away will it hit the ground?

   c. What is the speed upon impact?

6. Identify each surface by identifing the cross sections in each plane of $\mathbb{R}^3$ space:

   a. $2x^2 - 3y^2 + 6z^2 = 1$

   b. $4x - 2y^2 - 2z^2 = 9$

   c. $4x - 2y^2 = 9$

7. Determine if the planes $ax + by + cz + d_1 = 0$ and $(ak)x + (bk)y + (ck)z + d_2 = 0$ intersect. If they do not, find the distance between them.

8. Prove that all lines and circles (in the xy-plane) have constant curvature.

9. Describe the domain of $f(x, y) = \frac{\ln(xy)}{\sqrt{x+y}}$

10. Evaluate $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy \, dx \, dy$

11. For $f(x, y) = 4x^2 + 2y^2$, find the equation of the level curve that contains the point $P(-2, 4)$ and describe its shape.

12. Use Lagrange multipliers to find the extrema of $f(x,y,z) = x - 2y - 4z$ subject to the constraint $z = 4x^2 + y^2$

13. Evaluate $\int_C 6xt\,dx + x^2y\,dy$ where C is the graph $y = x^2 + 3$ from $(0,3)$ to $(3,12)$.

14. Find the mass of the solid with the density $\delta(x,y,z) = 8xy$ whose base in the xy-plane is bounded by $y = x$, $y = 0$ and $x = 3$ and bounded above by $z = 9 - x^2$.

15. Find the surface area of the paraboloid $z = x^2 + y^2$ between the planes $z = 6$ and $z = 30$.

16. Evaluate $\oint_C (-2xy^2\,dx + 4x^2y\,dy)$ where C is the boundary of the region in the first quadrant bounded by the x-axis, the y-axis and the semicircle $y = \sqrt{16 - x^2}$.

17. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 36$, the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 12$.

18. For $\mathbf{F} = \langle 3x^2y^3, 2y^3z4xz^2 \rangle$ find each of the following:

    a. Div $\mathbf{F}$

    b. Curl $\mathbf{F}$

    c. Div(Curl $\mathbf{F}$)

19. Find the work done by $\mathbf{F} = \langle xy, y, -yz \rangle$, $\mathbf{r}(t) = \langle t, t^2, t \rangle$, $0 \le t \le 4$.

20. Evaluate $\iint_R (x-y)^2(x+y)dA$ where the boundary of R is the rectangle with vertices $(4,0)$, $(8,4)$, $(4,8)$ and $(0,4)$.

21. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \cos(x^2 + y^2)dy\,dx$.

22. Determine if $\mathbf{F}(x,y,z) = \langle 18x^2 \sin y, 6x^3 \cos y - 3e^y \cos z, 3e^y \sin z + 2\cos z \rangle$, is conservative.

23. Using $\mathbf{u} = \langle -4, 6, 5 \rangle$ and $\mathbf{v} = \langle 2, -3, 1 \rangle$,

    a. Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

    b. Find $\mathbf{u} \cdot \mathbf{v}$.

    c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

    d. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

    e. Find $\mathbf{u} \times \mathbf{v}$.

24. Using $P(-4,1,2)$, $Q(1,-3,4)$, $R(-1,0,2)$,

    a. Find an equation of the plane passing through the points.

    b. Find parametric equations for the line through P and parallel to $a = \langle 2, -1, 4 \rangle$.

    c. Find the distance from the point $(5,-3,2)$ to the plane.

    d. Find the area of the parallelogram determined by P, Q, and R.

25. Identify the surface $x = y^2$.

26. Identify the surface $4x^2 + 4y^2 + z^2 = 4$.

27. Identify the surface $2x^2 - 3y^2 + 6z^2 = 6$.

28. Identify the surface $x^2 - 6y + 5z^2 = 0$.

29. A baseball is thrown from the stands 128 feet above the field at an angle of 30 degrees up from the horizontal with an initial speed of 64 feet per second.

    a. Give the position vector for any time t.

    b. When will the ball strike the ground?

    c. How far away will the ball strike the ground?

    d. What is the speed of the ball when it strikes the ground?

30. Using $\mathbf{r}(t) = \langle t\cos t, t\sin t, t^2\rangle$ at $t = 0$,

    a. Find $\mathbf{v}$ and $\mathbf{a}$.

    b. Find $\mathbf{T}$ and $\mathbf{N}$.

    c. Find K.

    d. By first finding $a_{\mathbf{T}}$ and $._{\mathbf{N}}$,
    express $a = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

31. For $f(x, y) = \sqrt{x^2 - y^2}$ find the domain of f and describe the level curves.

32. Find the limit:
$$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}, x \neq y + 1$$

33. Find f_{xy} for $f(x, y) = \ln(xy + y^2)$.

34. If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_x^2 + f_y^2 = \left(\frac{\delta w}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta w}{\delta \theta}\right)^2$.

35. Using $f(x, y) = \frac{x-y}{x+y}$ and $P(2, -1)$,

    a. Find the directional derivative of f in the direction of $\mathbf{v} = \langle 4, -8\rangle$.

    b. Find the direction in which f increases most rapidly.

    c. Find the direction in which f decreases most rapidly.

    d. Find the maximum value of the directional derivative.

36. Using $x^3 - 2xy + z^3 + 7y + 6 = 0$ and $P(1, 4, -3)$,

    a. Find an equation of the tangent plane at P.

    b. Find equations of the normal line at P.

37. A flat metal plate lies on an xy-plane such that the temperature T at (x, y) is given by $T = 10(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Find the instantaneous rate of change of T with respect to distance at $(1, 2)$ in the direction of the x-axis.

38. The total resistance R of three resistances R_1, R_2 and R_3 connected in parallelis given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. If measurements of R_1, R_2 and R_3 are 100, 200 and 400 ohms respectively, with a maximum error of $\pm 1\%$ in each measurement, estimate the maximum error in the calculated value of R.

39. Find the maximum and minimum values of $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ over the triangular region with vertices $(0, 0)$, $(2, 0)$ and $(2, 2)$.

40. Find the volume of the largest rectangular box that has three of its vertices on the positive x, y and z-axes respectively, and a fourth vertex on the plane $3x + 4y + 2z = 24$.

41. Reverse the order of integration of $\int_1^e \int_0^{\ln x} y \, dy \, dx$ and evaluate.

42. Find the volume of the solid bounded by $y = x^3$, $y = x^4$, $z - x - y = 4$, and $z = 0$.

43. Using polar coordinates, evaluate $\iint_R (x^2 + y^2)^{3/2} dA$ where R is the region bounded by the circle of radius a centered at the origin.

44. Find the surface area of S, the part of the paraboloid $z = x^2 + y^2$ under the plane $z = a, a > 0$.

45. Evaluate $\int_{-1}^2 \int_1^x \int_0^{x+y} (3x^2 y) dz \, dy \, dx$.

46. Find the center of mass of the lamina that has the shape of the region bounded by $y = x^2$ and $y = 9$ with density $\delta(x, y) = 12x^2y^2$.

47. For the solid bounded by $z = \sqrt{16x^2 + 16y^2}$, $x^2 + y^2 = 16$, and $z = 0$,

    a. find its volume.

    b. find the center of mass if $\delta = \sqrt{x^2 + y^2}$.

48. Find the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 16$.

49. Evaluate $\iint_R \frac{2y+x}{y-2x} dA$ where R is the region bounded by the trapezoid with vertices $(-1, 0)$, $(-2, 0)$, $(0, 4)$, $(0, 2)$.

50. Find the curl and divergence of $\mathbf{F} = \langle -3 \sin x + \cos y, 6xz^2, 3y + z \rangle$.

51. The force at a point (x, y, z) in three dimensions is given by $\mathbf{F} = \langle y, z, x \rangle$. Find the work done by $\mathbf{F}$ along the twisted cubic $x = t$, $y = t^2$ and $z = t^3$ from $(0, 0, 0)$ to $(2, 4, 8)$.

52. Evaluate $\int_{(1,1,1)}^{(2,3,4)} (4xy) dx + (2x^2 + 2yz^3) dy + (3y^2z^2 + 3) dz$.

53. Use Green's theorem to evaluate $\oint_C (6y) dx + (\frac{5}{3} x^3) dy$, where C is the boundary of the first quadrant region bounded by $y = 36 - x^2$ and the x-axis.

Math 252 Final Review (Answers)

1. a. $\mathbf{u}\cdot\mathbf{v} = -4$

   b. $\text{Proj}_{\mathbf{u}}\mathbf{v} = \left\langle \frac{2}{9}, \frac{-4}{9}, \frac{-4}{9} \right\rangle$

   c. $\theta = \cos^{-1}\left(\frac{-2}{15}\right) \doteq 1.705$ rad

   d. $\mathbf{u}\times\mathbf{v} = \langle -28, -8, -6\rangle$

2. $s = 52\pi$

3. a. $\mathbf{v}(t) = \langle -4\sin t, 3, 4\cos t\rangle$
   $\mathbf{v}(0) = \langle 0, 3, 4\rangle$
   $\mathbf{a}(t) = \langle -4\cos t, 0, -4\sin t\rangle$
   $\mathbf{a}(0) = \langle -4, 0, 0\rangle$

   b. $\mathbf{T}(t) = \left\langle -\frac{4\sin t}{5}, \frac{3}{5}, \frac{4\cos t}{5} \right\rangle$
   $\mathbf{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle$
   $\mathbf{N}(t) = \langle -\cos t, 0, -\sin t\rangle$
   $\mathbf{N}(0) = \langle -1, 0, 0\rangle$

   c. $K = \frac{4}{25}$ $a_{\mathbf{T}} = 0$ $a_{\mathbf{N}} = 4$

4. a. $2x + 3y - z - 5 = 0$

   b. $x = -6t + 8,\ y = -9t + 2,\ z = 3t - 1$

   c. $D = 2\sqrt{14}$

   d. $A = 3\sqrt{14}$

5. a. $\mathbf{r}(t) = \langle 112\sqrt{3}t, -16t^2 + 112t + 128\rangle$

   b. max distance: $893\sqrt{3}$

   c. impact speed: $32\sqrt{57}$

6. a. Elliptical hyperboloid (one sheet)

   b. Circular hyperboloid

   c. Parabolic cylinder

7. (this was extra-credit)

   $\mathbf{n}_1 = \langle a, b, c\rangle,\ \mathbf{n}_2 = \langle ka, kb, kc\rangle = k\langle a, b, c\rangle$

   $\mathbf{n}_1 \times \mathbf{n}_2 = 0,\ \therefore n_1 \parallel n_2$

   point on first plane: $P(0, 0, -\frac{d_1}{c})$

   distance from point to second plane:

   $$\begin{aligned} D &= |\operatorname{Proj}_{\mathbf{n}_1} P|| \\ &= \frac{|ka(0)+kb(0)+kc(-\frac{d_1}{c})+d_2|}{\sqrt{(ka)^2+(kb)^2+(kc)^2}} \\ &= \frac{|-kd_1+d_2|}{|k|\sqrt{a^2+b^2+c^2}} \end{aligned}$$

8. (this was extra-credit)
   (ANSWER)

9. $\{(x, y) : x > 0, y > 0\}$

10. $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy\, dx\, dy = \frac{1}{3}$

11. $4x^2 + 2y^2 = 48$ (an ellipse)

12. Absolute max $f(\frac{1}{32}, -\frac{1}{4}, \frac{17}{256}) = \frac{17}{64}$

13. $\int_0^3 (2t^5 + 12t^3 + 18t)dt$

14. $m = \int_0^3 \int_0^x \int_0^{9-x^2} 8xy\, dz\, dy\, dx = 243$

15. $\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = 201\pi$

16. Using Green's theorem, $\int_0^4 \int_0^{\sqrt{16-x^2}} 12xy\, dy\, dx = 384$.

17. $V = \int_0^{2\pi} \int_0^6 \int_r^{12} r\, dz\, dr\, d\theta = 288\pi$

18. a. $\mathrm{Div}\, \mathbf{F} = 6xy^3 + y^2z + 8xz$
    b. $\mathrm{Curl}\, \mathbf{F} = \langle -2y^3, -4z^2, -9x^2y^2 \rangle$
    c. $\mathrm{Div}(\mathrm{Curl}\, \mathbf{F}) = 0$

19. $W = \int \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = 128$

20. Let $u = x - y$, $v = x + y$, $x = \frac{1}{2}u + \frac{1}{2}v$, and $y = -\frac{1}{2}u + \frac{1}{2}v$, where $-4 \le u \le 4$, $4 \le v \le 8$.

$$J = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{1}{2}$$

$$\iint_R (x-y)^2(x+y)dA = \frac{1}{2} \int_{-4}^{4} \int_{4}^{12} u^2 v\, dv\, du = \frac{4096}{3}$$

21. $\int_0^{\pi/2} \int_0^3 \cos(r^2) r\, dr\, d\theta = \frac{\pi}{4} \sin 9$

22. (Extra credit)
    $M_y = 18x^2 \cos y = N_x$, $M_z = 0 = P_x$ and $N_z = 3e^y \sin z = P_y$, therefore $\mathbf{F}$ is conservative.

$$\begin{aligned} f(x, y, z) &= \int 18x^2 \sin y\, dx = 6x^3 \sin y + g(y, z) \\ f_y &= 6x^3 \cos y + g_y = 6x^3 \cos y - 3e^y \cos z \\ \therefore g(y, z) &= -3e^y \cos z + h(z) \\ f(x, y, z) &= \int 6x^3y^2 - 3e^y \cos z + h(z) \\ f_z &= 3e^y \sin z + h_z = 3e^y \sin z + 2\cos z \\ \therefore h_z &= 2\cos z + C \\ \therefore f(x, y, z) &= 6x^3 \sin y - 3e^y \cos z + 2\sin z + C \end{aligned}$$

23. a. $\|\mathbf{u}\| = \sqrt{77}$
    $\|\mathbf{v}\| = \sqrt{14}$
    b. $\mathbf{u} \cdot \mathbf{v} = -21$
    c. $\theta = \arccos\left(\frac{-21}{7\sqrt{22}}\right)$

24. a. $2x + 6y + 7z - 12 = 0$
    b. $x = 2t - 4,\ y = -t + 1,\ z = 4t + 2$
    c. $D = \dfrac{6}{\sqrt{89}}$
    d. $A = \sqrt{89}$

25. Parabolic cylinder

26. Circular ellipsoid

27. Hyperbaloid (one sheet)

28. Elliptical cone

29. a. $\mathbf{r}(t) = \langle 32\sqrt{3}t, -16t^2 + 32t + 128\rangle$
    b. in 4 seconds
    c. $128\sqrt{3}$ feet away
    d. $64\sqrt{3}$ feet per second

30. a. $\mathbf{v} = \langle -t\sin t + \cos t, t\cos t + \sin t, 2t\rangle$
    $\mathbf{a} = \langle -t\cos t - 2\sin t, -t\sin t + 2\cos t, 2\rangle$
    b. $\mathbf{T}(t) = \left\langle \frac{-t\sin 5+\cos t}{\sqrt{5t^2+1}}, \frac{t\cos t+\sin 5}{\sqrt{5t^2+1}}, \frac{2t}{\sqrt{5t^2+1}} \right\rangle$
    $\mathbf{T}(0) = \langle 1, 0, 0\rangle$
    $\mathbf{N}(0) = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

31. $D = \{(x, y) : |x| \geq |y|\}$
    Hyperbola in xy-plane

32. $L = \frac{1}{4}$

33. $f_{xy} = -\frac{1}{(x+y)^2}$

34. $\frac{\delta w}{\delta r} = f_x(\cos\theta) + f_y(\sin\theta)$
    $\frac{\delta w}{\delta \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta)$
    $(\frac{\delta w}{\delta r})^2 + \frac{1}{r^2}(\frac{\delta w}{\delta \theta})^2 = f_x^2 + f_y^2$

35. a. $\nabla f(x, y) = \langle \frac{2y}{(x_y)^2}, \frac{-2x}{(x_y)^2} \rangle$
    $\nabla f(2, -1) = \langle -2, -4\rangle$
    $\mathbf{u} = \mathbf{v}\frac{1}{|\mathbf{v}|} = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$
    $D_{\mathbf{u}} f(2, -1) = \frac{6\sqrt{5}}{5}$
    b. $\nabla f \frac{1}{|\nabla f|} = \langle \frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5} \rangle$
    c. $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \rangle$

d. $|\nabla f| = 2\sqrt{5}$

36. a. $-5x + 5y + 27z + 66 = 0$

b. $\langle x, y, z\rangle = \langle 1, 4, -3\rangle + t\langle -5, 5, 27\rangle$

$x = -5t + 1;\ y = 5t + 4;\ z = 27t - 3$

37. $T_x = 200$ degrees per centimeters

38. $\left|\frac{dR}{R}\right| = \frac{400}{7}(\frac{1}{100^2} + \frac{2}{200^2} + \frac{4}{400^2}) = 0.01$

39. absolute max $\frac{37}{4}$ at $(1, \frac{3}{2})$

40. $V = \frac{64}{3}$

41. $\int_0^1 \int_{e^y}^{e} y\ dx\ dy = \frac{e}{2} - 1$

42. $V = \frac{157}{630}$

43. (ANSWER)

44. (ANSWER)

45. (ANSWER)

46. (ANSWER)

47. a. (ANSWER)

b. (ANSWER)

48. (ANSWER)

49. (ANSWER)

50. (ANSWER)

51. (ANSWER)

52. (ANSWER)

53. (ANSWER)