Math 252 Cumulative Review (Problems)

1. Using $\mathbf{u} = \langle -4, 6, 5\rangle$ and $\mathbf{v} = \langle 2, -3, 1\rangle$,

   a. Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

   b. Find $\mathbf{u} \cdot \mathbf{v}$.

   c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   d. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

   e. Find $\mathbf{u} \times \mathbf{v}$.

2. Using $P(-4, 1, 2)$, $Q(1, -3, 4)$, $R(-1, 0, 2)$,

   a. Find an equation of the plane passing through the points.

   b. Find parametric equations for the line through P and parallel to $a = \langle 2, -1, 4\rangle$.

   c. Find the distance from the point $(5, -3, 2)$ to the plane.

   d. Find the area of the parallelogram determined by P, Q, and R.

3. Identify the surface $x = y^2$.

4. Identify the surface $4x^2 + 4y^2 + z^2 = 4$.

5. Identify the surface $2x^2 - 3y^2 + 6z^2 = 6$.

6. Identify the surface $x^2 - 6y + 5z^2 = 0$.

7. A baseball is thrown from the stands 128 feet above the field at an angle of 30 degrees up from the horizontal with an initial speed of 64 feet per second.

   a. Give the position vector for any time t.

   b. When will the ball strike the ground?

   c. How far away will the ball strike the ground?

   d. What is the speed of the ball when it strikes the ground?

8. Using $\mathbf{r}(t) = \langle t\cos t, t\sin t, t^2\rangle$ at $t = 0$,

   a. Find $\mathbf{v}$ and $\mathbf{a}$.

   b. Find $\mathbf{T}$ and $\mathbf{N}$.

   c. Find K.

   d. By first finding $a_{\mathbf{T}}$ and $._{\mathbf{N}}$,
   express $a = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

9. Using $\mathbf{u} = \langle -2, 4, 4\rangle$ and $\mathbf{v} = \langle 0, 3, -4\rangle$,

   a. Find $\mathbf{u} \cdot \mathbf{v}$.

   b. Find $\text{Proj}_{\mathbf{u}}\mathbf{v}$.

   c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   d. Find $\mathbf{u} \times \mathbf{v}$.

10. Find the length of the helix $\mathbf{r}(t) = \langle 6\sin(2t), -5t, -6\cos(2t)\rangle$ for $0 \le t \le 4\pi$.

11. Using $\mathbf{r}(t) = \langle 4\cos t, 3t, 4\sin t\rangle$ at $t = 0$,

    a. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

    b. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

    c. Find K, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.

12. Given the points $P(2, 1, 2)$, $Q(6, -2, 1)$ and $R(-1, 4, 5)$,

    a. Find an equation of the plane passing through the points.

    b. Find an equation of the line perpendicular to the plane, passing through the point $(8, 2, -1)$.

    c. Find the distance from the point $(-5, -2, 7)$ to the plane.

    d. Find the area of the parallelogram determined by the points.

13. A Projectile is launched at an angle of 30°, with speed 224 feet per second, and from a platform 128 feet above the ground,

    a. Find the position vector of the object at time t.

    b. How far away will it hit the ground?

    c. What is the speed upon impact?

14. Identify each surface by identifing the cross sections in each plane of $\mathbb{R}^3$ space:

    a. $2x^2 - 3y^2 + 6z^2 = 1$

    b. $4x - 2y^2 - 2z^2 = 9$

    c. $4x - 2y^2 = 9$

15. Determine if the planes $ax + by + cz + d_1 = 0$ and $(ak)x+(bk)y+(ck)z+d_2 = 0$ intersect. If they do not, find the distance between them.

16. Prove that all lines and circles (in the xy-plane) have constant curvature.

17. For $f(x, y) = \sqrt{x^2 - y^2}$ find the domain of f and describe the level curves.

18. Find the limit:

$$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}, x \neq y + 1$$

19. Find f_{xy} for $f(x, y) = \ln(xy + y^2)$.

20. If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_x^2 + f_y^2 = \left(\frac{\delta w}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta w}{\delta \theta}\right)^2$.

21. Using $f(x, y) = \frac{x-y}{x+y}$ and $P(2, -1)$,

    a. Find the directional derivative of f in the direction of $\mathbf{v} = \langle 4, -8\rangle$.

    b. Find the direction in which f increases most rapidly.

    c. Find the direction in which f decreases most rapidly.

    d. Find the maximum value of the directional derivative.

22. Using $x^3 - 2xy + z^3 + 7y + 6 = 0$ and $P(1, 4, -3)$,

    a. Find an equation of the tangent plane at P.

    b. Find equations of the normal line at P.

23. A flat metal plate lies on an xy-plane such that the temperature T at (x, y) is given by $T = 10(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Find the instantaneous rate of change of T with respect to distance at $(1, 2)$ in the direction of the x-axis.

24. The total resistance R of three resistances R_1, R_2 and R_3 connected in parallelis given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. If measurements of R_1, R_2 and R_3 are 100, 200 and 400 ohms respectively, with a maximum error of $\pm 1\%$ in each measurement, estimate the maximum error in the calculated value of R.

25. Find the maximum and minimum values of $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ over the triangular region with vertices $(0, 0)$, $(2, 0)$ and $(2, 2)$.

26. Find the volume of the largest rectangular box that has three of its vertices on the positive x, y and z-axes respectively, and a fourth vertex on the plane $3x + 4y + 2z = 24$.

27. Reverse the order of integration of $\int_1^e \int_0^{\ln x} y\, dy\, dx$ and evaluate.

28. Find the volume of the solid bounded by $y = x^3$, $y = x^4$, $z - x - y = 4$, and $z = 0$.

29. Describe the domain of $f(x, y) = \frac{\ln(xy)}{\sqrt{x+y}}$

30. Evaluate $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy\, dx\, dy$

31. For $f(x, y) = 4x^2 + 2y^2$, find the equation of the level curve that contains the point $P(-2, 4)$ and describe its shape.

32. Use Lagrange multipliers to find the extrema of $f(x, y, z) = x - 2y - 4z$ subject to the constraint $z = 4x^2 + y^2$

33. A closed rectangular box is to have dimensions 40cm $\times$ 30cm $\times$ 60cm with a maximum error of 0.5cm in each measurement. Estimate the maximum error in the volume.

34. Find the volume of the solid bounded by $z = 4x + 2y + 3$, $z = 0$, $x = y^2$ and $x = 2y$.

35. Using $f(x, y, z) = x^3yz^2$ at $P(1, 2, -2)$:

  a. Find the gradient of f at P.

  b. Find the directional derivative of f in the direction of $\mathbf{a} = \langle 9, -12, 20\rangle$.

  c. Find a unit vector in the direction in which f increases most rapidly at P.

  d. Find the equation of the tangent plane of f at P.

  e. Find equations of the normal line to f at P.

  f. Estimate $f(0.99, 2.02, -1.97)$.

36. For $f(x, y) = x^2 + x \sin y - 2x^3y - 6y^4$, find f_{xx}, f_{yy} and f_{xy}.

37. Find $\frac{\delta z}{\delta x}$ if $z = g(x, y)$ is defined implicitly by $x^2z^3 + \cos(xy) = 4 + y^4 - e^z$.

38. Reverse the order of integration of $\int_e^{100} \int_1^{\ln y} g(x, y)\, dx\, dy$.

39. Evaluate $\lim_{(x,y)\to(3,1)} \frac{x^2-9y^2}{x^4-81y^4}$.

40. Find any extrema or saddle points of $f(x, y) = 2x^2 - 6xy + 2y^2 + 10x - 6$.

41. Find the absolute extrema of $f(x, y) = 2x^2 - 6xy + 2y^2 + 10x - 6$ over the region in the xy-plane bounded by $y = 2x$, $y = 0$ and $x = 4$.

42. A function is harmonic if $\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0$ throughout the domain of f. Determine if $f(x, y) = \tan^{-1}(\frac{x}{y})$ is harmonic.

43. Using polar coordinates, evaluate $\iint_R (x^2 + y^2)^{3/2} dA$ where R is the region bounded by the circle of radius a centered at the origin.

44. Find the surface area of S, the part of the paraboloid $z = x^2 + y^2$ under the plane $z = a, a > 0$.

45. Evaluate $\int_{-1}^{2} \int_1^x \int_0^{x+y} (3x^2y) dz\ dy\ dx$.

46. Find the center of mass of the lamina that has the shape of the region bounded by $y = x^2$ and $y = 9$ with density $\delta(x, y) = 12x^2y^2$.

47. For the solid bounded by $z = \sqrt{16x^2 + 16y^2}$, $x^2 + y^2 = 16$, and $z = 0$,

  a. find its volume.

  b. find the center of mass if $\delta = \sqrt{x^2 + y^2}$.

48. Find the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 16$.

49. Evaluate $\iint_R \frac{2y+x}{y-2x} dA$ where R is the region bounded by the trapezoid with vertices $(-1, 0)$, $(-2, 0)$, $(0, 4)$, $(0, 2)$.

50. Find the curl and divergence of $\mathbf{F} = \langle -3 \sin x + \cos y, 6xz^2, 3y + z\rangle$.

51. The force at a point (x, y, z) in three dimensions is given by $\mathbf{F} = \langle y, z, x\rangle$. Find the work done by $\mathbf{F}$ along the twisted cubic $x = t$, $y = t^2$ and $z = t^3$ from $(0, 0, 0)$ to $(2, 4, 8)$.

52. Evaluate $\int_{(1,1,1)}^{(2,3,4)} (4xy)dx + (2x^2 + 2yz^3)dy + (3y^2z^2 + 3)dz$.

53. Use Green's theorem to evaluate $\oint_C (6y)dx + (\frac{5}{3}x^3)dy$, where C is the boundary of the first quadrant region bounded by $y = 36 - x^2$ and the x-axis.

54. Evaluate $\int_C 6xy\, dx + x^2y\, dy$ where C is the graph $y = x^2 + 3$ from $(0, 3)$ to $(3, 12)$.

55. Find the mass of the solid with the density $\delta(x, y, z) = 8xy$ whose base in the xy-plane is bounded by $y = x$, $y = 0$ and $x = 3$ and bounded above by $z = 9 - x^2$.

56. Find the surface area of the paraboloid $z = x^2 + y^2$ between the planes $z = 6$ and $z = 30$.

57. Evaluate $\oint_C (-2xy^2\, dx + 4x^2y\, dy)$ where C is the boundary of the region in the first quadrant bounded by the x-axis, the y-axis and the semicircle $y = \sqrt{16 - x^2}$.

58. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 36$, the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 12$.

59. For $\mathbf{F} = \langle 3x^2y^3, 2y^3z, 4xz^2 \rangle$ find each of the following:

    a. Div $\mathbf{F}$

    b. Curl $\mathbf{F}$

    c. Div(Curl $\mathbf{F}$)

60. Find the work done by $\mathbf{F} = \langle xy, y, -yz \rangle$, $\mathbf{r}(t) = \langle t, t^2, t \rangle$, $0 \le t \le 4$.

61. Evaluate $\iint_R (x-y)^2(x+y)dA$ where the boundary of R is the rectangle with vertices $(4,0)$, $(8,4)$, $(4,8)$ and $(0,4)$.

62. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \cos(x^2+y^2) dy\, dx$.

63. Determine if $\mathbf{F}(x,y,z) = \langle 18x^2 \sin y, 6x^3 \cos y - 3e^y \cos z, 3e^y \sin z + 2\cos z \rangle$, is conservative. If it is, find a potential function f of $\mathbf{F}$.

64. Find the center and radius of the sphere given by $x^2 + y^2 + z^2 - 8x + 6x = 0$

65. Using $\mathbf{u} = \langle 8, 3, -5 \rangle, \mathbf{v} = \langle 4, -4, -2 \rangle$,

    a. Find $3u - 4v$.

    b. Find $\|\mathbf{u}\|, \|\mathbf{v}\|$.

66. Using $\mathbf{u} = \langle 8, -4, 1 \rangle$ and $\mathbf{v} = \langle -4, 4, 2 \rangle$,

    a. Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

    b. Find $\mathbf{u} \cdot \mathbf{v}$.

    c. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

    d. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

    e. Find $\mathbf{u} \times \mathbf{v}$.

67. Using $P(-2,0,3), Q(1,2,4), R(-3,1,0)$,

    a. Find a vector orthogonal to the plane determined by P, Q and R.

    b. Find an equation of the plane passing through P, Q and R.

    c. Find the set of parametric equations for the line through Q and parallel to $\mathbf{a} = \langle 4, -3, -2 \rangle$.

    d. Find the distance from the point $(-4,-1,5)$ to the plane passing through P, Q and R.

68. Indentify via cross-sections the surface defined by $3^2 - y^2 + 3z^2 + 9 = 0$.

69. Indentify via cross-sections the surface defined by $x = 3y^2 + 5z^2$.

70. Indentify via cross-sections the surface defined by $y = x^2$.

71. Indentify via cross-sections the surface defined by $2y^2 = 3z^2 = 12$.

72. Using $r(t) = \langle \cos t, \sin t, t^2 \rangle$, $t = \frac{\pi}{2}$:

    a. Find the velocity vector.

    b. Find the acceleration vector.

73. A projectile is fired at a speed of 448 feet per second at and angle of 30 degrees from a tower 512 feet above the ground.

    a. Give the position vector for any time t.

    b. How far away will the object strike?

74. Using $\mathbf{r}(t) = \langle 4\cos(2t), 4\sin(2t), 6t \rangle$,

    a. Find $\mathbf{T}(t)$

    b. Find $\mathbf{N}(t)$

    c. Find the curvature

75. Find the tangential and normal components of acceleration for the curve $\mathbf{r}(t) = \langle 3t^2, 4t^2, 10t \rangle$ at $t = 2$ and express a in terms of T and N.

76. Describe the domain of $f(x,y) = \frac{\ln(x-y)}{\sqrt{xy}}$

77. Find an equation of the level surface of $f(x,y,z) = xy\sin z + 3xy^2e^z$ at $P(1,2,0)$

78. Determine if the following limit exists; if it does also state the value of the limit: $\lim_{(x,y)\to(2,1)} \frac{x^2-xy-2y^2}{x^2-4y^2}$

79. For $f(x,y) = 3x^4y^2 - x\cos y + 4x^3y^3$, find f_x, f_y, f_{xx} and f_{xy}.

80. For $f(x,y,z) = 4x^z + z^3\sin y$ find $\frac{\delta^3 f}{\delta x \delta y^2}$.

81. Use partial derivatives to find $\frac{dy}{dx}$ if $4x^2y + 2y^3 = 5x^3y^4$.

82. Using $f(x,y) = 3x^2 + 4y^2$, $P(4,-2)$ and $Q(10,6)$:

    a. Find the gradient of f at P.

    b. Find the directional derivative of f at P in the direction from P to Q.

    c. Find the maximum value of the directional derivative of f at P.

83. Using $w = f(x,y,z) = 2xy^2 - 4x^3z$,

    a Find an equation of the tangent plane of w at $(1,3,2)$.

    b Estimate $f(1.02, 3.01, 1.98)$.

84. Without using Lagrange multipliers, find any extrema or saddle points of $f(x,y) = x^3 + 12xy - 3y^2 - 27x + 34$.

85. Use Lagrange multipliers to find any extrema of $f(x,y,z) = 3x^2 - y^2 + 2z^2$ subject to $3x + z + 50 = 4y$.

86. For the integral $\int_0^4 \int_{x^2}^{4x} (6x+12y)dy\ dx$,

    a. evaluate.

    b. rewrite by reversing the order of integration.

87. Use polar coordinates to evaluate the integral $\iint_R x\sqrt{x^2+y^2}\ dA$ where R si the region bounded by the semicircle $x = \sqrt{36-y^2}$.

88. Find the surface area of the part of the paraboloid $z = f(x,y) = 20 - x^2 - y^2$ above $z = 4$.

89. For the solid bounded in the first octant by the plane $4x + 2y + z = 12$ with density $\delta(x,y,z) = 5x^3$,

    a. find it's mass.

    b. set up (but don't solve) the integral to find M_{xz}.

90. Find the volume of the solid that lies outside the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 18$

91. Evaluate $\iint_R (2x+y)e^{(2y-x)}dA$, where R is the rectangle with vertices $(2,1)$, $(6,3)$, $(4,7)$ and $(0,5)$.

92. Find a conservative vector field $\mathbf{F}$ that has the potential $f(x,y,z) = 4x^2y - 2y^2z^3$¿

93. Find the curl and divergence of $\mathbf{F} = \langle xz^2, 2yz, 3xy^2\rangle$.

94. Evaluate the line integral $\int_C (xy^2)dx + (4xy^3)dy$ along C: $x = y^2$ from $(0,0)$ to $(4,2)$.

95. For the following vector fields, determine if it is path independent; and if it is, find a potential function f.

    a. $\mathbf{F} = \langle 6x - 6y^2, \cos y - 12xy\rangle$

    b. $\mathbf{F} = \langle e^y\cos z, xe^y\sin z\rangle$

96. Evaluate $\oint_C (3x^2 - 3x^2y^2)dx + (3x^3y + 2y^4)dy$ where C is the boundary of the region bounded below by the semicircle $y = -\sqrt{9-x^2}$ and above by the x-axis.

97. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F} = \langle 2x, 2y, 4z\rangle$, S is the portion of the paraboloid $z = 16 - x^2 - y^2$ above $z \geq 0$.

Math 252 Cumulative Review (Answers)

1. (Math-252 Practice Exam 1)

   a. $\|\mathbf{u}\| = \sqrt{77}$
      $\|\mathbf{v}\| = \sqrt{14}$

   b. $\mathbf{u} \cdot \mathbf{v} = -21$

   c. $\theta = \arccos\left(\frac{-21}{7\sqrt{22}}\right)$

2. (Math-252 Practice Exam 1)

   a. $2x + 6y + 7z - 12 = 0$

   b. $x = 2t - 4,\ y = -t + 1,\ z = 4t + 2$

   c. $D = \dfrac{6}{\sqrt{89}}$

   d. $A = \sqrt{89}$

3. (Math-252 Practice Exam 1)

   Parabolic cylinder

4. (Math-252 Practice Exam 1)

   Circular ellipsoid

5. (Math-252 Practice Exam 1)

   Hyperbaloid (one sheet)

6. (Math-252 Practice Exam 1)

   Elliptical cone

7. (Math-252 Practice Exam 1)

   a. $\mathbf{r}(t) = \langle 32\sqrt{3}t, -16t^2 + 32t + 128\rangle$

   b. in 4 seconds

   c. $128\sqrt{3}$ feet away

   d. $64\sqrt{3}$ feet per second

8. (Math-252 Practice Exam 1)

   a. $\mathbf{v} = \langle -t\sin t + \cos t, t\cos t + \sin t, 2t\rangle$
      $\mathbf{a} = \langle -t\cos t - 2\sin t, -t\sin t + 2\cos t, 2\rangle$

   b. $\mathbf{T}(t) = \left\langle \frac{-t\sin 5+\cos t}{\sqrt{5t^2+1}}, \frac{t\cos t+\sin 5}{\sqrt{5t^2+1}}, \frac{2t}{\sqrt{5t^2+1}} \right\rangle$
      $\mathbf{T}(0) = \langle 1, 0, 0\rangle$
      $\mathbf{N}(0) = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

9. (Math-252 Exam 1)

   a. $\mathbf{u} \cdot \mathbf{v} = -4$

   b. $\mathrm{Proj}_{\mathbf{u}}\mathbf{v} = \left\langle \frac{2}{9}, \frac{-4}{9}, \frac{-4}{9} \right\rangle$

   c. $\theta = \cos^{-1}\left(\frac{-2}{15}\right) \doteq 1.705$ rad

   d. $\mathbf{u} \times \mathbf{v} = \langle -28, -8, -6\rangle$

10. (Math-252 Exam 1)

    $s = 52\pi$

11. (Math-252 Exam 1)

    a. $\mathbf{v}(t) = \langle -4\sin t, 3, 4\cos t\rangle$
       $\mathbf{v}(0) = \langle 0, 3, 4\rangle$
       $\mathbf{a}(t) = \langle -4\cos t, 0, -4\sin t\rangle$
       $\mathbf{a}(0) = \langle -4, 0, 0\rangle$

    b. $\mathbf{T}(t) = \left\langle -\frac{4\sin t}{5}, \frac{3}{5}, \frac{4\cos t}{5} \right\rangle$
       $\mathbf{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle$
       $\mathbf{N}(t) = \langle -\cos t, 0, -\sin t\rangle$
       $\mathbf{N}(0) = \langle -1, 0, 0\rangle$

    c. $K = \frac{4}{25}$ $a_{\mathbf{T}} = 0$ $a_{\mathbf{N}} = 4$

12. (Math-252 Exam 1)

    a. $2x + 3y - z - 5 = 0$

    b. $x = -6t + 8,\ y = -9t + 2,\ z = 3t - 1$

    c. $D = 2\sqrt{14}$

    d. $A = 3\sqrt{14}$

13. (Math-252 Exam 1)

    a. $\mathbf{r}(t) = \langle 112\sqrt{3}t, -16t^2 + 112t + 128\rangle$

    b. max distance: $896\sqrt{3}$

    c. impact speed: $32\sqrt{57}$

14. (Math-252 Exam 1)

    a. Elliptical hyperboloid (one sheet)

    b. Circular hyperboloid

    c. Parabolic cylinder

15. (Math-252 Exam 1)

    (this was extra-credit)

    $\mathbf{n}_1 = \langle a, b, c\rangle,\ \mathbf{n}_2 = \langle ka, kb, kc\rangle = k\langle a, b, c\rangle$

    $\mathbf{n}_1 \times \mathbf{n}_2 = 0,\ \therefore n_1 \parallel n_2$

    point on first plane: $P(0, 0, -\frac{d_1}{c})$

    distance from point to second plane:

    $D = |\operatorname{Proj}_{\mathbf{n}_1} P||$

    $= \frac{|ka(0)+kb(0)+kc(-\frac{d_1}{c})+d_2|}{\sqrt{(ka)^2+(kb)^2+(kc)^2}}$

    $= \frac{|-kd_1+d_2|}{|k|\sqrt{a^2+b^2+c^2}}$

16. (Math-252 Exam 1)

    (this was extra-credit)
    (ANSWER)

17. (Math-252 Exam 2 Practice)

    $D = \{(x, y) : |x| \geq |y|\}$
    Hyperbola in xy-plane

18. (Math-252 Exam 2 Practice)

    $L = \frac{1}{4}$

19. (Math-252 Exam 2 Practice)

    $f_{xy} = -\frac{1}{(x+y)^2}$

20. (Math-252 Exam 2 Practice)

    $\frac{\delta w}{\delta r} = f_x(\cos\theta) + f_y(\sin\theta)$
    $\frac{\delta w}{\delta \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta)$
    $(\frac{\delta w}{\delta r})^2 + \frac{1}{r^2}(\frac{\delta w}{\delta \theta})^2 = f_x^2 + f_y^2$

21. (Math-252 Exam 2 Practice)

    a. $\nabla f(x, y) = \langle \frac{2y}{(x_y)^2}, \frac{-2x}{(x_y)^2}\rangle$
       $\nabla f(2, -1) = \langle -2, -4\rangle$
       $\mathbf{u} = \mathbf{v}\frac{1}{|\mathbf{v}|} = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\rangle$
       $D_{\mathbf{u}} f(2, -1) = \frac{6\sqrt{5}}{5}$

    b. $\nabla f \frac{1}{|\nabla f|} = \langle \frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\rangle$

    c. $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\rangle$

    d. $|\nabla f| = 2\sqrt{5}$

22. (Math-252 Exam 2 Practice)

    a. $-5x + 5y + 27z + 66 = 0$

    b. $\langle x, y, z\rangle = \langle 1, 4, -3\rangle + t\langle -5, 5, 27\rangle$
       $x = -5t + 1;\ y = 5t + 4;\ z = 27t - 3$

23. (Math-252 Exam 2 Practice)

    $T_x = 200$ degrees per centimeters

24. (Math-252 Exam 2 Practice)

    $\left|\frac{dR}{R}\right| = \frac{400}{7}(\frac{1}{100^2} + \frac{2}{200^2} + \frac{4}{400^2}) = 0.01$

25. (Math-252 Exam 2 Practice)

    absolute max $\frac{37}{4}$ at $(1, \frac{3}{2})$

26. (Math-252 Exam 2 Practice)

    $V = \frac{64}{3}$

27. (Math-252 Exam 2 Practice)

    $\int_0^1 \int_{e^y}^e y\ dx\ dy = \frac{e}{2} - 1$

28. (Math-252 Exam 2 Practice)

    $V = \frac{157}{630}$

29. (Math-252 Exam 2)

    $\{(x, y) : x > 0, y > 0\}$

30. (Math-252 Exam 2)

    $\int_0^1 \int_{y^2}^{\sqrt{y}} 4xy\, dx\, dy = \frac{1}{3}$

31. (Math-252 Exam 2)

    $4x^2 + 2y^2 = 48$ (an ellipse)

32. (Math-252 Exam 2)

    Absolute max $f(\frac{1}{32}, -\frac{1}{4}, \frac{17}{256}) = \frac{17}{64}$

33. (Math-252 Exam 2)

    $x = 40, y = 30, z = 60$
    $dx = dy = dz = 0.5$
    $V(x, y, z) = xyz$
    $\delta V \approx dV = V_x\, dx + V_y\, dy + V_z\, dz = yz\, dx + xz\, dy + xy\, dz$
    $dV = (30)(60)(0.5) + (40)(60)(0.5) + (40)(30)(0.5) = 2700\text{cm}^3$

34. (Math-252 Exam 2)

    $\int_0^2 \int_{y^2}^{2y} (4x + 2y + 3) dx\, dy = \frac{76}{5}$

35. (Math-252 Exam 2)

    a. $\nabla f = \langle f_x, f_y, f_z\rangle = \langle 3x^2yz^2, x^3z^2, 2x^3yz\rangle$,
       $\nabla f(P) = \langle 24, 4, -8\rangle$

b. $\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \langle \frac{9}{25}, -\frac{12}{25}, \frac{20}{25} \rangle$,
$D_{\mathbf{u}}f(P) = \langle 24, 4, -8 \rangle \cdot \langle \frac{9}{25}, -\frac{12}{25}, \frac{20}{25} \rangle = \frac{8}{25}$

c. $\mathbf{u} = \frac{1}{\|\nabla f(P)\|}\nabla f(P) = \langle \frac{6}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-2}{\sqrt{41}} \rangle$

d. $24(x-1) + 4(y-2) - 8(z+2) = 0$,
$6x + y - 2z - 12 = 0$

e. $\langle x, y, z \rangle = \langle 1, 2, -2 \rangle + t\langle 24, 4, -8 \rangle$,
$x = 1 + 24t,\ y = 2 + 4t,\ z = -2 - 8t$

f. $dx = -0.01,\ dy = 0.02,\ dz = 0.03$,
$df = f_x(1)dx + f_y(2)dy + f_z(-2)dz = -0.4$,
$f(0.99, 2.02, -1.97) \approx f(P) + df = 8 + (-0.4) = 7.6$

36. (Math-252 Exam 2)

$f_{xx} = 2 - 12xy$,
$f_{yy} = -x\sin y - 72y^2$,
$f_{xy} = \cos y - 6x^2$

37. (Math-252 Exam 2)

$f(x, y, z) = x^2z^3 + \cos(xy) - 4 - y^4 + e^z$,
$f_x = 2xz^3 - y\sin(xy)$,$f_z = 3x^2z^2 + e^z$,
$\frac{\delta z}{\delta x} = -\frac{f_x}{f_z} = \frac{-2xz^3 + y\sin(xz)}{3x^2z^2 + e^z}$

38. (Math-252 Exam 2)

$\int_1^{\ln 100} \int_{e^x}^{100} g(x, y)\, dy\, dx$

39. (Math-252 Exam 2)

$\lim_{(x,y)\to(3,1)} \frac{x^2 - 9y^2}{x^4 - 81y^4} = \lim_{(x,y)\to(3,1)} \frac{1}{x^2 + 9^2} = \frac{1}{18}$.

40. (Math-252 Exam 2)

$f_x = 4x - 6y + 10 = 0,\ f_y = -6x + 4y = 1$,
$x = 2,\ y = 3,\ z = 4$,
$f_{xx} = 4,\ f_{yy} = 4,\ f_{xy} = -6$,
$f_{xx}f_{yy} - (f_{xy})^2 = -20$,
therefore there is a saddlepoint at $(2, 3, 4)$.

41. (Math-252 Exam 2)

(Extra credit)
$f(x, 2x) = -2x^2 + 10x - 6,\ f_x(x, 2x) = -4x + 10 = 0,\ x = \frac{5}{2},\ y = 5$,
$f(x, 0) = 2x^2 + 10 - 6,\ f_x(x, 0) = 4x + 10 = 0,\ x = -\frac{5}{2}$,
$f(4, y) = 2y^2 - 24y + 66,\ f_y(4, y) = 2y - 12 = 0,\ y = 6$,
$f(\frac{5}{2}, 5) = \frac{13}{2},\ f(4, 6) = -6$,
$f(0, 0) = -6,\ f(4, 0) = 66,\ f(4, 8) = 2$,
absolute max of 66 at $(4, 0)$ and absolute min of -6 at $(0, 0)$ and $(4, 6)$.

42. (Math-252 Exam 2)

(Extra credit)
$f_x = \frac{y}{(x^2+y^2)^2},\ f_{xx} = \frac{-2xy}{(x^2+y^2)^2}$,
$f_y = \frac{-x}{(x^2+y^2)^2},\ f_{yy} \frac{2xy}{(x^2+y^2)^2}$,
$f_{xx} + f_{yy} = 0$

43. (Math-252 Exam 3 Practice)
(ANSWER)

44. (Math-252 Exam 3 Practice)
(ANSWER)

45. (Math-252 Exam 3 Practice)
(ANSWER)

46. (Math-252 Exam 3 Practice)
(ANSWER)

47. (Math-252 Exam 3 Practice)

a. (ANSWER)
b. (ANSWER)

48. (Math-252 Exam 3 Practice)
(ANSWER)

49. (Math-252 Exam 3 Practice)
(ANSWER)

50. (Math-252 Exam 3 Practice)
(ANSWER)

51. (Math-252 Exam 3 Practice)
(ANSWER)

52. (Math-252 Exam 3 Practice)
(ANSWER)

53. (Math-252 Exam 3 Practice)
(ANSWER)

54. (Math-252 Exam 3)
$\int_0^3 (2t^5 + 12t^3 + 18t)dt = 567$

55. (Math-252 Exam 3)

$m = \int_0^3 \int_0^x \int_0^{9-x^2} 8xy\, dz\, dy\, dx = 243$

56. (Math-252 Exam 3)

$\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = 201\pi$

57. (Math-252 Exam 3)

Using Green's theorem, $\int_0^4 \int_0^{\sqrt{16-x^2}} 12xy\, dy\, dx = 384$.

58. (Math-252 Exam 3)

$V = \int_0^{2\pi} \int_0^6 \int_r^{12} r\, dz\, dr\, d\theta = 288\pi$

59. (Math-252 Exam 3)

a. $\operatorname{Div} \mathbf{F} = 6xy^3 + 6y^2 z + 8xz$

b. $\operatorname{Curl} \mathbf{F} = \langle -2y^3, -4z^2, -9x^2y^2 \rangle$

c. $\operatorname{Div}(\operatorname{Curl} \mathbf{F}) = 0$

60. (Math-252 Exam 3)

$W = \int \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt = 128$

61. (Math-252 Exam 3)

Let $u = x - y$, $v = x + y$, $x = \frac{1}{2}u + \frac{1}{2}v$, and $y = -\frac{1}{2}u + \frac{1}{2}v$, where $-4 \le u \le 4$, $4 \le v \le 8$.

$$J = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{1}{2}$$

$$\iint_R (x-y)^2 (x+y) dA = \frac{1}{2} \int_{-4}^{4} \int_{4}^{12} u^2 v\, dv\, du = \frac{4096}{3}$$

62. (Math-252 Exam 3)

$\int_0^{\pi/2} \int_0^3 \cos(r^2) r\, dr\, d\theta = \frac{\pi}{4} \sin 9$

63. (Math-252 Exam 3)

(Extra credit)
$M_y = 18x^2 \cos y = N_x$, $M_z = 0 = P_x$ and $N_z = 3e^y \sin z = P_y$, therefore $\mathbf{F}$ is conservative.

$$\begin{aligned} f(x,y,z) &= \int 18x^2 \sin y\, dx = 6x^3 \sin y + g(y,z) \\ f_y &= 6x^3 \cos y + g_y = 6x^3 \cos y - 3e^y \cos z \\ \therefore g(y,z) &= -3e^y \cos z + h(z) \\ f(x,y,z) &= \int 6x^3 y^2 - 3e^y \cos z + h(z) \\ f_z &= 3e^y \sin z + h_z = 3e^y \sin z + 2\cos z \\ \therefore h_z &= 2\cos z + C \\ \therefore f(x,y,z) &= 6x^3 \sin y - 3e^y \cos z + 2\sin z + C \end{aligned}$$

64. (Math-252 Quiz 1)

$C(4, -3, 0), \rho = 5$

65. (Math-252 Quiz 1)

a. $\langle 8, 25, -7 \rangle$.

b. $\|\mathbf{u}\| = 7\sqrt{2}$, $\|\mathbf{v}\| = 6$.

66. (Math-252 Quiz 2)

a. $\|\mathbf{u}\| = 9$, $\|\mathbf{v}\| = 6$

b. $\mathbf{u} \cdot \mathbf{v} = -46$

c. $\theta = \arccos\left(-\frac{23}{27}\right) = 148.4°$

d. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \langle -\frac{46}{9}, -\frac{46}{9}, -\frac{23}{9} \rangle$

e. $\mathbf{u} \times \mathbf{v} = \langle -12, -20, 16 \rangle$

67. (Math-252 Quiz 3)

a. $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle -7, 8, 5 \rangle$

b. $-7x + 8y + 5z = 29$

c. $x = 1 + 4t, y = 2 - 3t, z = 4 - 2t; t \in \mathbb{R}$

d. $D = \frac{16}{\sqrt{138}}$

68. (Math-252 Quiz 4)

Circular hyperboloid of two sheets

69. (Math-252 Quiz 4)

Elliptical paraboloid

70. (Math-252 Quiz 4)

Parabolic cylinder

71. (Math-252 Quiz 4)

    Elliptical cylinder

72. (Math-252 Quiz 5)

    a. $\mathbf{v}(t) = \langle -\sin t, \cos t, 2t\rangle$
    $\mathbf{v}(\frac{\pi}{2}) = \langle -1, 0, \pi\rangle$

    b. $\mathbf{a}(t) = \langle -\cos t, -\sin t, 2\rangle$
    $\mathbf{a}(\frac{\pi}{2}) = \langle 0, -1, 2\rangle$

73. (Math-252 Quiz 5)

    a. $\mathbf{r}(t) = \langle 224\sqrt{3}t$
    $-16t^2 + 224t + 512\rangle$

    b. $T = 16$
    $x(16) = 224\sqrt{3}(16) \doteq 6207.7$ feet

74. (Math-252 Quiz 6)

    a. $\mathbf{T}(t) = \langle -\frac{4}{5}\sin(2t), \frac{4}{5}\cos(2t), \frac{3}{5}\rangle$

    b. $\mathbf{N}(t) = \langle -\cos(2t), \sin(2t), 0\rangle$

    c. $k = \frac{4}{25}$

75. (Math-252 Quiz 7)

    $\mathbf{a} = 4\sqrt{5}\mathbf{T} + 2\sqrt{5}\mathbf{N}$

76. (Math-252 Quiz 8)

    $\{(x, y) : x > y, xy > 0\}$

77. (Math-252 Quiz 8)

    $xy\sin z + 3xy^2e^z$

78. (Math-252 Quiz 8)

    $L = \frac{3}{4}$

79. (Math-252 Quiz 9)

    $f_x = 12x^3y^2 - \cos y + 12x^2y^3$

    $f_y = 6x^4y + x\sin y + 12x^3y^2$

    $f_{xx} = 36x^2y^2 + 24xy^3$

    $f_{yy} = 6x^4 + x\cos y + 24x^3y$

    $f_{xy} = 24x^3y + \sin y + 36x^2y^2$

80. (Math-252 Quiz 10)

    $\frac{\delta^3 f}{\delta x \delta y^2} = 0$

81. (Math-252 Quiz 10)

    $\frac{dy}{dx} = \frac{15x^2y^4 - 8xy}{20x^3y^3 - 4x^2 + 6y^2}$

82. (Math-252 Quiz 11)

    a. $\nabla f(P) = \langle 24, -16\rangle$

    b. $\mathbf{u} = \frac{1}{\|\overrightarrow{PQ}\|}\overrightarrow{PQ}$; $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u} = \frac{16}{10}$

    c. $\|\nabla f(p)\| = 8\sqrt{13}$

83. (Math-252 Quiz 12)

    a. $-6x + 12y - 4z - 22 = 0$

    b. $f(1.02, 3.01, 1.98) \approx 10.08$

84. (Math-252 Quiz 13)

    Saddle point $f(1, 2) = 20$, local max $f(-9, -18) = 520$

85. (Math-252 Quiz 14)

    Absolute minimum $f(4, 16, 2) = -200$

86. (Math-252 Quiz 15)

    a. $\frac{4736}{5}$

    b. $\int_0^{16}\int_{\frac{1}{4}y}^{\sqrt{y}}(6x + 12y)dx\ dy$

87. (Math-252 Quiz 16)

    $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^6 r\cos\theta\sqrt{r^2}r\ dr\ d\theta = 648$

88. (Math-252 Quiz 17)

    $S = \frac{\pi}{6}(65^{3/2} - 1)$

89. (Math-252 Quiz 18)

    a. $m = 243$

    b. $M_{xz} = \int_0^3\int_0^{6-2x}\int_0^{12-4x-2y}(5x^3)y\,dz\,dy\,dx$

90. (Math-252 Quiz 19)

    $V = 72\pi$

91. (Math-252 Quiz 20)

    $\int_5^{15}\int_0^{10} ue^v\left(\frac{5}{25}\right)dv\,du = 20(e^{10} - 1)$

92. (Math-252 Quiz 21)

    $\mathbf{F} = \nabla f = \langle 8xy, 4x^2 - 4yz^3, -6y^2z^2\rangle$

93. (Math-252 Quiz 21)

    – Curl: $\langle 6xy - 2y, 2xz - 3y^2, 0\rangle$

– Divergence: $z^2 + 2z$

94. (Math-252 Quiz 21)

    $\int_C (xy^2)dx + (4xy^3)dy = \int_0^2 (6t^5)dt = 64$

95. (Math-252 Quiz 22)

    a. $\mathbf{F}$ is conservative and therefore path independent.
       $f = 3x^2 - 6xy + \sin y + C$

    b. $\mathbf{F}$ is not conservative and therefore not path independent.

—

96. (Math-252 Quiz 23)

    $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{0} 15x^2y \; dy \; dx = -486$

97. (Math 252 Quiz 24)

    Note that the surface is not closed, so cannot use divergence theorem.
    $g(x, y, z) = x^2 + y^2 + z - 16$,
    $\nabla g = \langle 2x, 2y, 1\rangle$,
    $\|\nabla g\| = \sqrt{4x^2 + 4y^2 + 1}$,
    $\mathbf{n} = \frac{1}{\|\nabla g\|}\nabla g$,
    $f(x, y) = 16 - x^2 - y^2 - z$, $f_x = -2x$, $f_y = -2y$,
    $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA$,
    $\iint_S \mathbf{F} \cdot \mathbf{v}\, dS = \iint_D \mathbf{F} \cdot \nabla g\, dA$,
    $= \int_0^{2\pi} \int_0^4 64r\, dr\, d\theta = 1024\pi$