Quantum Analogue of Bead on a Rotating Hoop

Ashlin V Thomas

1 Introduction

The classical problem of a bead on a rotating hoop is a well-known problem in classical mechanics, nonlinear dynamics, and bifurcation theory. In this problem, a bead is constrained to move on a circular hoop that is rotating about a vertical axis. The model attracted significant attention due to the appearance of non-trivial equilibrium points appearning as the angular velocity of the hoop is increased. This phenomenon is a classical example of a pitchfork bifurcation, and has been studied extensively in the context of classical mechanics. In this article, we will explore the quantum analogue of this problem, where we will consider a quantum particle constrained to move on a rotating hoop.

2 Classical Problem

In the classical problem, we consider a bead of mass m constrained to move on a circular hoop of radius R that is rotating with a constant angular velocity ω about a vertical axis. The position of the bead can be described by the angle θ it makes with the vertical axis. The Hamiltonian of the system can be written as:

$$H = \frac{p_\theta^2}{2mR^2} - mgR\cos\theta - \frac{1}{2}mR^2\omega^2\sin^2\theta \tag{1}$$

where p_θ is the conjugate momentum to θ. The equilibrium points of the system can be found by setting the derivative of the potential energy to zero, which leads to the condition:

$$\theta^* \in \begin{cases} 0, & \text{if } \omega < \sqrt{\frac{g}{R}} \\ \{0, \pm\arccos\left(\frac{g}{R\omega^2}\right)\}, & \text{if } \omega > \sqrt{\frac{g}{R}} \end{cases} \tag{2}$$

This indicates that for $\omega < \sqrt{\frac{g}{R}}$, there is only one equilibrium point at $\theta = 0$, which is stable. However, as ω increases beyond $\sqrt{\frac{g}{R}}$, two additional equilibrium points appear at $\theta = \pm\arccos\left(\frac{g}{R\omega^2}\right)$, which are stable, while the original equilibrium point at $\theta = 0$ becomes unstable. This is a classic example of a pitchfork bifurcation.

3 Quantum Analogue

In the quantum analogue of this problem, we consider a quantum particle of mass m constrained to move on a circular hoop of radius R that is rotating with a constant angular velocity ω about a vertical axis. We quantize the system by promoting θ and p_θ to operators that satisfy the canonical commutation relation:

$$\theta \to \hat{\theta}, \quad p_\theta \to \hat{p}_\theta = -i\hbar\frac{\partial}{\partial\theta}, \quad [\hat{\theta}, \hat{p}_\theta] = i\hbar \tag{3}$$

The quantum Hamiltonian can be written as:

$$\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\theta^2} - mgR\cos\hat{\theta} - \frac{1}{2}mR^2\omega^2\sin^2\hat{\theta} \tag{4}$$

Let us define $U_0 = -\frac{1}{2mR^2}$, $U_1 = -mgR$, and $U_2 = -\frac{1}{2}mR^2\omega^2$. The Hamiltonian can then be expressed as:

$$\hat{H} = U_0 p_\theta^2 + U_1\cos\hat{\theta} + U_2\sin^2\hat{\theta} \tag{5}$$

Let us rewrite the Hamiltonian in the angular momentum basis. The basis states are $|n\rangle$, where n is an integer representing the angular momentum quantum number. The action of the operators on the basis states is given by:

$$\hat{\theta}\,|n\rangle = i\frac{\partial}{\partial n}\,|n\rangle\,, \quad \hat{p}_\theta\,|n\rangle = n\hbar\,|n\rangle \tag{6}$$

The matrix elements of the Hamiltonian in this basis can be computed as follows:

$$\langle m|\,\hat{H}\,|n\rangle = U_0 n^2\hbar^2\delta_{mn} + \frac{U_1}{2}(\delta_{m,n+1} + \delta_{m,n-1}) - \frac{U_2}{4}(\delta_{m,n+2} - 2\delta_{mn} + \delta_{m,n-2}) \tag{7}$$

Let us define $U = U_0\hbar^2$, $t_1 = -\frac{U_1}{2}$ and $t_2 = \frac{U_2}{4}$, and rewrite the Hamiltonian in the angular momentum basis as:

$$\hat{H} = \sum_{n=-\infty}^{\infty} (Un^2 - 2t_2)\,|n\rangle\,\langle n| + (t_1\,|n\rangle\,\langle n+1| + t_2\,|n\rangle\,\langle n+2| + h.c.) \tag{8}$$

This Hamiltonian can be interpreted as a tight-binding model on a one-dimensional lattice with nearest-neighbor and next-nearest-neighbor hopping terms, as well as an on-site potential which is a discrete version of the quantum harmonic oscillator potential.