参考最简二足模型的基于一些假设与近似的一步内四足模型的理论建立

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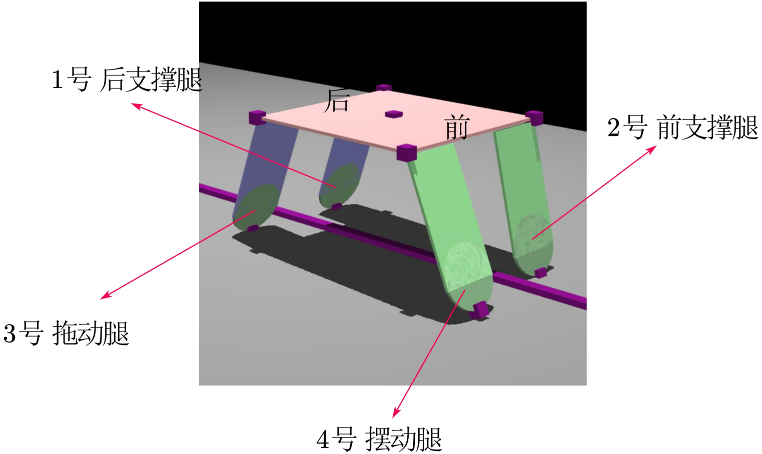
事实、符号与约定

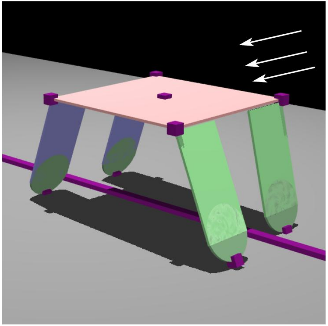
前提与事实的陈述

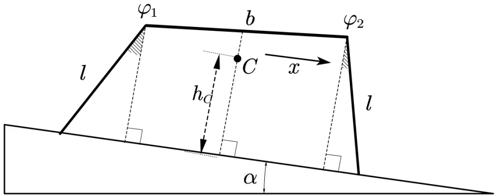
1. 拖地滑行总是发生在后腿，若否则整体打滑，非研究范畴
2. 能量的耗散总是来源于后腿的摩擦，本模型旨在定量分析一次步态内的运动方程，所以触地损耗不被考量

腿的序号的约定

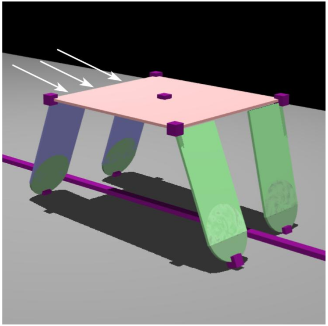


左视图

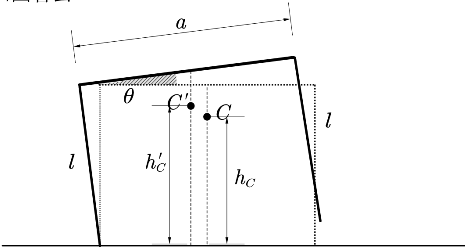




1. C 点是质心. 我们假定没有横向的偏移, 尽管这略微与事实相悖, 在此基础上设 x 是质心相对坐标原点的位移
2. l, b, h_C 分别是腿长, 身长, 质心到斜面的垂直高度
3. φ_1, φ_2, α 分别是后支撑腿 1 号与垂直斜面的夹角, 前支撑腿 2 号与垂直斜面的夹角, 斜面倾角



从后面看去



1. a 是身子的长度
2. θ 是身子沿 x 轴侧转的角度
3. 设关节的角量劲度系数为 k, 关节的平衡位置是 β

各物理量的推导

势能

重力势能为

$$V_G = mg\left[-x\sin\alpha + \left(h_C' - h_C\right)\cos\alpha\right]$$

弹性势能为

$$\begin{aligned} V_k &= \frac{1}{2}k\sum_{i=1}^{4}\left(\varphi_i - \beta\right)^2 \\ &= \frac{1}{2}k\left[\left(\varphi_1 - \beta\right)^2 + \left(\varphi_2 - \beta\right)^2 + \left(\varphi_3 - \beta\right)^2 + \left(\varphi_4 - \beta\right)^2\right] \end{aligned}$$

质心动能

$$T_C = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m \left(h_C^2 + \frac{a^2}{4} \right) \dot{\theta}^2$$

腿的动能

$$T_L = \frac{1}{2} \left(\sum_{i=1}^{4} I\dot{\varphi}_i^2 + \sum_{i=1}^{4} J\dot{\theta}_i^2 \right)$$

用 m_L 表示腿的质量, I 是腿绕质心前后摆动的转动惯量, J 是腿绕质心随身体侧摆的转动惯量

$$I = m_L \left[\frac{1}{12} l^2 + \frac{1}{4} (b + l\sin\varphi)^2 + \frac{1}{4} \left(2h_C' - \frac{l}{2}\cos\varphi \right)^2 \right]$$

$$J = m_L \left[\frac{1}{12} l^2 \cos^2\varphi + \frac{1}{4} (\mathrm{a}\cos\theta - l\cos\varphi\sin\theta)^2 + \frac{1}{4} (2h_C' - l\cos\varphi\sin\theta)^2 \right]$$

拖动腿 3 受到摩擦力为

$$f = \mu m \left(g \cos \alpha \left(\frac{\frac{b}{2} + l \sin \varphi_2}{b + l \left(\sin \varphi_1 + \sin \varphi_2 \right)} \right) \left(\frac{1}{2} - \frac{l}{a} \sin \theta \right) + \left(\frac{h_C^2}{a} + \frac{a}{4} \right) \ddot{\theta} \right)$$

质心位置的几何约束

质心的几何位置满足

$$h'_C = h_C \cos\theta + \frac{a}{2}\sin\theta$$

$$h_C = l\frac{\cos\varphi_1 + \cos\varphi_2}{2}$$

腿的角位移 φ 的几何约束

$$
\begin{aligned}
&l\sin\varphi_1 = l\sin\varphi_{10} + x \\
&l\sin\varphi_2 = l\sin\varphi_{20} + x \\
&l\cos\varphi_3\cos\theta = l\cos\varphi_1\cos\theta + \mathrm{a}\sin\theta \\
(*)&\mathrm{d}\varphi_4 = \lambda\mathrm{d}\theta
\end{aligned}
$$

经验方程: 摆动腿4是腾空的, 无法根据几何关系确定其位置

Euler-Lagrange 方程

设 $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$ 是 Lagrange 量, 其中 q 是广义坐标, 运算 $\dot{x} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ 表示对时间求一次导数, $S[q] = \displaystyle\int_{t_1}^{t_2} \mathcal{L}\mathrm{d}t$ 是作用量, 当作用量取得极值 ($\delta S = 0$) 时, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

Q_i 是 Rayleigh 阻尼项, 是系统的耗散力. 在此系统中

$$\mathcal{L} = T - V$$

整理动能、势能与耗散力

系统的势能为

$$
\begin{aligned}
V &= V_G + V_k = mg\left[-x\sin\alpha + \left(h_C' - h_C\right)\cos\alpha\right] \\
&+ \frac{1}{2}k\left[(\varphi_1-\beta)^2 + (\varphi_2-\beta)^2 + (\varphi_3-\beta)^2 + (\varphi_4-\beta)^2\right]
\end{aligned}
$$

忽略掉腿的动能 T_L, 系统的动能为

$$
T = T_C = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(h_C^2 + \frac{a^2}{4}\right)\dot{\theta}^2
$$

本模型未考虑关节处的阻尼, 所以系统的耗散力就是摩擦力 f

$$
\begin{aligned}
Q = f = \mu m &\left(g\cos\alpha\left(\frac{\frac{b}{2} + l\sin\varphi_2}{b + l\left(\sin\varphi_1 + \sin\varphi_2\right)}\right)\left(\frac{1}{2} - \frac{l}{a}\sin\theta\right)\right. \\
&\left.+\left(\frac{h_C^2}{a} + \frac{a}{4}\right)\ddot{\theta}\right)
\end{aligned}
$$

联立上述所有方程（忽略腿的动能）保留 θ 和 x 为变元

$$
\begin{aligned}
\mathcal{L} = {} & \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \left(\frac{l^2}{4} \left(\sqrt{1 - \left(\sin\varphi_{10} + \frac{x}{l} \right)^2} + \sqrt{1 - \left(\sin\varphi_{20} + \frac{x}{l} \right)^2} \right)^2 + \frac{a^2}{4} \right) \dot{\theta}^2 \\
& - mg \left(- x \sin\alpha + \left(\frac{a}{2} \sin\theta - \frac{l}{2} (1 - \cos\theta) \left(\sqrt{1 - \left(\sin\varphi_{10} + \frac{x}{l} \right)^2} + \sqrt{1 - \left(\sin\varphi_{20} + \frac{x}{l} \right)^2} \right) \right) \cos\alpha \right) \\
& - \frac{1}{2} k \left(\begin{array}{c} \left(\arcsin\left(\sin\varphi_{10} + \frac{x}{l} \right) - \beta \right)^2 + \left(\arcsin\left(\sin\varphi_{20} + \frac{x}{l} \right) - \beta \right)^2 \\ + \left(\arccos\left(\sqrt{1 - \left(\sin\varphi_{10} + \frac{x}{l} \right)^2} + \frac{x}{l} \tan\theta \right) - \beta \right)^2 + (\varphi_{40} + \lambda\theta - \beta)^2 \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= mg\sin\alpha - \frac{1}{2}mg\cos\alpha\,(1-\cos\theta)\left(\frac{\frac{x}{l}+\sin\varphi_{10}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}} + \frac{\frac{x}{l}+\sin\varphi_{20}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right) \\
&+ \frac{k}{l}\left(\begin{array}{c} \dfrac{\beta-\arcsin\left(\frac{x}{l}\sin\varphi_{10}\right)}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}} + \dfrac{\beta-\arcsin\left(\frac{x}{l}\sin\varphi_{20}\right)}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}} \\
+ \dfrac{\left(-\beta+\arccos\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}\right)+\frac{x}{l}\tan\theta\right)\left(-\frac{\frac{x}{l}\sin(\varphi_{10})}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\tan\theta\right)}{\sqrt{1-\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right)^2}}
\end{array}\right) \\
&- \frac{1}{4}ml\left(\frac{\frac{x}{l}\sin\varphi_{10}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}} + \frac{\frac{x}{l}\sin\varphi_{20}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right)\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)
\end{aligned}
$$

$$\frac{\partial\mathcal{L}}{\partial\theta}=\frac{1}{4}\left(\begin{array}{c}-4k\lambda(\beta+\theta\lambda-\phi_{40})-2mga\cos\alpha\cos\theta+2mgl\cos\alpha\sin\theta\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)\\ +\dfrac{4kx\left(-\beta+\arccos\left[\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right]\right)\sec^2\theta}{l\sqrt{1-\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right)^2}}\end{array}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\left(\frac{a^2}{4} + \frac{1}{4}l^2\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2} + \sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)^2\right)\dot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathcal{L}}{\partial \dot{x}} = m\ddot{x}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathcal{L}}{\partial \dot{\theta}} =& m\left(\frac{a^2}{4} + \frac{1}{4}l^2\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2} + \sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)^2\right)\ddot{\theta} \\ & - \frac{1}{2}ml^2\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2} + \sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right) \\ & \cdot \left(\frac{\left(\frac{x}{l}+\sin\varphi_{10}\right)\frac{\dot{x}}{l}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}} + \frac{\left(\frac{x}{l}+\sin\varphi_{20}\right)\frac{\dot{x}}{l}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right)\dot{\theta}\end{aligned}$$

关于广义坐标 x 的 Lagrange 方程

把上面算完的关于 x 各项代入 Euler-Lagrange 方程

$$
\begin{aligned}
& m\ddot{x}+mg\sin\alpha-\frac{1}{2}mg\cos\alpha(1-\cos\theta)\left(\frac{\frac{x}{l}+\sin\varphi_{10}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\frac{\frac{x}{l}+\sin\varphi_{20}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right) \\
& +\frac{k}{l}\left(\begin{array}{c}\dfrac{\beta-\arcsin\left(\frac{x}{l}\sin\varphi_{10}\right)}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\dfrac{\beta-\arcsin\left(\frac{x}{l}\sin\varphi_{20}\right)}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}} \\ +\dfrac{\left(-\beta+\arccos\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}\right)+\frac{x}{l}\tan\theta\right)\left(-\frac{\frac{x}{l}\sin\varphi_{10}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\tan\theta\right)}{\sqrt{1-\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right)^2}}\end{array}\right) \\
& -\frac{1}{4}ml\left(\frac{\frac{x}{l}\sin\varphi_{10}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\frac{\frac{x}{l}\sin\varphi_{20}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right)\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right) \\
& =\mu m\left(g\cos\alpha\left(\frac{\frac{b}{2}+l\sin\varphi_2}{b+l(\sin\varphi_1+\sin\varphi_2)}\right)\left(\frac{1}{2}-\frac{l}{a}\sin\theta\right)+\left(\frac{h_C^2}{a}+\frac{a}{4}\right)\ddot{\theta}\right)
\end{aligned}
$$

关于广义坐标 θ 的 Lagrange 方程

$$
\begin{aligned}
& m\left(\frac{a^2}{4}+\frac{1}{4}l^2\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)^2\right)\ddot{\theta} \\
& -\frac{1}{2}ml^2\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right)\left(\frac{\left(\frac{x}{l}+\sin\varphi_{10}\right)\frac{\dot{x}}{l}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}}+\frac{\left(\frac{x}{l}+\sin\varphi_{20}\right)\frac{\dot{x}}{l}}{\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}}\right)\dot{\theta} \\
& +\frac{1}{4}\left(\begin{array}{c}-4k\lambda(\beta+\theta\lambda-\phi_{40})-2mga\cos\alpha\cos\theta+2mgl\cos\alpha\sin\theta\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{20}\right)^2}\right) \\
+\dfrac{4kx\left(-\beta+\arccos\left[\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right]\right)\sec^2\theta}{l\sqrt{1-\left(\sqrt{1-\left(\frac{x}{l}+\sin\varphi_{10}\right)^2}+\frac{x\tan\theta}{l}\right)^2}}\end{array}\right)=0
\end{aligned}
$$

代码实现

由于原来的方程过于复杂, 就算适当化简也会因为微分方程中含有不可分离的耦合项（例如对于 $\ddot{\theta}$ 的超越方程不能分离其）而导致不能用 solve_ivp 的方法求解, 为此我提出以下简化的方法:

方程常微分化

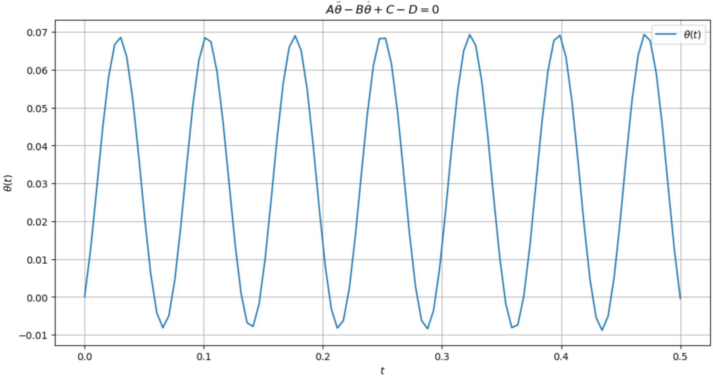
θ 的常微分化方程

由于 x 相对身体的尺度是小量, 不妨对于 θ 及其导数而言, 如果前面的系数中含 x 则直接视作常数, 对于不可分离的变元, 不妨直接视作常数, 即忽略耦合的贡献, 那么关于 θ 的微分方程可写作

$$A\ddot{\theta} - B\dot{\theta} + C\theta - D = 0$$

其中 A, B, C, D 的解析表达式在关于 θ 的 Lagrange 方程中求得, 通过模拟实验我们可以给出一组参考值.
$A = 7.7 \times 10^{-4}, B = 7.53 \times 10^{-5}, C = 5.6, D = 0.17$

模拟实验给出稳定步态时候有 $t = 0$ 时 $\theta = 0, \dot{\theta} = 2$, solve_ivp 给出的数值解为

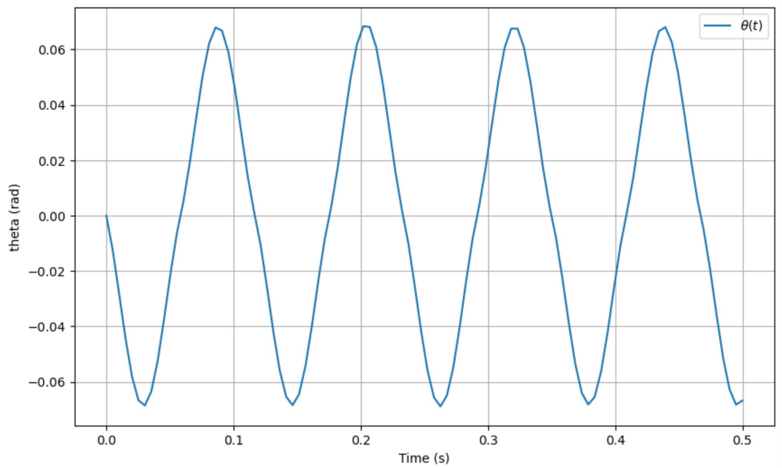


注意这是一次步态的方程对应的解, 在 $\theta < 0$ 的阶段会失去正常的物理意义

步态衔接修正

当完成一次步态（$\theta = 0$, 方向从正到负）后, 有

$$\theta^+ = 0 \qquad \dot{\theta}^+ = 2\text{sign}(\dot{\theta}^-) \qquad (\text{sign表示取符号})$$



附录

参考文献

[1]Garcia, Mariano, Anindya Chatterjee, Andy Ruina, and Michael Coleman. 1998. "The Simplest Walking Model: Stability, Complexity, and Scaling." ASME Journal of Biomechanical Engineering. Accepted April 16, 1997; final version February 10, 1998.

参考数据（一组有代表性的）

$$a = 0.075, \qquad \phi 0 = [0.2, 0, 0, 0]$$

$$b = 0.1, \qquad \beta = 0.3$$

$$l = 0.075, \qquad k = 0.04$$

$$\lambda = 1, \qquad \alpha = 0.4$$

$$m = 0.02, \qquad M = 0.05$$

未加入步态衔接修正

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

A = 7.7e-4
B = 7.53e-5
C = 5.6
D = 0.17

theta_and_theta_dot_0 = [0, 2]

def Dense_Equation(t, theta_and_theta_dot, ):
    theta, theta_dot = theta_and_theta_dot
    theta_ddot = 1/A*(B*theta_dot-C*theta+D)
    return [theta_dot, theta_ddot]

t_span = (0, 0.5)
t_eval = np.linspace(t_span[0], t_span[1], 100)

sol = solve_ivp(Dense_Equation, t_span, theta_and_theta_dot_0, t_eval=t_eval)

plt.figure(figsize=(12, 6))
plt.plot(sol.t, sol.y[0], label=r'$\theta(t)$')
plt.title(r'$A\ddot{\theta}-B\dot{\theta}+C-D=0$')
plt.xlabel('$t$')
plt.ylabel(r'$\theta(t)$')
plt.legend()
plt.grid(True)
plt.show()
```

加入步态衔接修正，利用 solve_ivp 的事件监测功能

```
def Dense_Equation(t, theta_and_theta_dot):
    theta, theta_dot = theta_and_theta_dot
    theta_ddot = 1/A*(B*theta_dot-C*theta+D*np.sign(theta))
    return [theta_dot, theta_ddot]

def monitor(t, theta_and_theta_dot):
    theta, theta_dot = theta_and_theta_dot
    return theta
#monitor.terminal = True
monitor.direction = 0

t_span = [0, 0.5]
times = np.array([])
thetas = np.array([])
theta_dots = np.array([])

while True:
    sol = solve_ivp(Dense_Equation, t_span, theta_and_theta_dot_0, events=monitor, dense_output=True, max_step=0.05)
    print(sol)
    t = np.linspace(sol.t[0], sol.t[-1], num=100)
    theta_and_theta_dot = sol.sol(t)
    times = np.concatenate((times, t))
    thetas = np.concatenate((thetas, theta_and_theta_dot[0]))
    theta_dots = np.concatenate((theta_dots, theta_and_theta_dot[1]))

    if sol.status == 1:
        new_theta = 1e-5 * np.sign(theta_and_theta_dot_0[0])
        new_theta_dot = 2 * np.sign(theta_and_theta_dot_0[1])
        if times[-1] > 0.5:
            break
        theta_and_theta_dot_0 = [new_theta, new_theta_dot]
        t_span[0] = sol.t_events[0][0] - 1e-5
    else:
        break
```

D 的符号也要更新

正负方向经过零点都被监测

END