A Difficult Question From My Friend

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Abstract

Def: $f(x)$ is a function increasing monotonically in this interval $(0, +\infty)$, and

$$\lim_{x\to+\infty} \frac{f\left(ax\right)}{f\left(x\right)} = 1$$

proof: $\forall a > 0$, we have

$$\lim_{x\to+\infty} \frac{f(ax)}{f(x)} = 1$$

Solution

We can use the pinch theorem. Obviously $\exists n \in \mathbb{Z},\ s.t.\ a \in [2^n, 2^{n+1}]$. As a result

$$\lim_{x\to+\infty} \frac{f\left(ax\right)}{f\left(x\right)} = \left[\lim_{x\to+\infty} \frac{f\left(2^n x\right)}{f\left(x\right)}, \lim_{x\to+\infty} \frac{f\left(2^{n+1} x\right)}{f\left(x\right)}\right]$$

$$= \left[\lim_{x\to+\infty} \frac{f\left(2^n x\right)}{f\left(2^{n-1} x\right)} \frac{f\left(2^{n-1} x\right)}{f\left(2^{n-2} x\right)} \cdots \frac{f\left(2x\right)}{f\left(x\right)}, \lim_{x\to+\infty} \frac{f\left(2^{n+1} x\right)}{f\left(2^n x\right)} \frac{f\left(2^n x\right)}{f\left(2^{n-1} x\right)} \cdots \frac{f\left(2x\right)}{f\left(x\right)}\right]$$

$$= \left[\lim_{x\to+\infty} \frac{f\left(2^n x\right)}{f\left(2^{n-1} x\right)} \lim_{x\to+\infty} \frac{f\left(2^{n-1} x\right)}{f\left(2^{n-2} x\right)} \cdots \lim_{x\to+\infty} \frac{f\left(2x\right)}{f\left(x\right)},\right.$$

$$\left.\lim_{x\to+\infty} \frac{f\left(2^{n+1} x\right)}{f\left(2^n x\right)} \lim_{x\to+\infty} \frac{f\left(2^n x\right)}{f\left(2^{n-1} x\right)} \cdots \lim_{x\to+\infty} \frac{f\left(2x\right)}{f\left(x\right)}\right]$$

$$\to \{1\}$$