THERMODYNAMICS ASSIGNMENT FOR THE THIRD TIME

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ABSTRACT. Here is the thermodynamics assignment for the third time which is for the corse given by professor Yuanbo Zhang. In order to practise the expertise in scientific film of physics, students need to practise using LaTeXto composing their own work, even if this is only a ordinary homework.

MAIN TEXT

Q1. Do an intergral:

$$\left\langle \frac{1}{v} \right\rangle = \int_0^{+\infty} \frac{1}{v} f_{\mathrm{M}}(v)\,\mathrm{d}v = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^{+\infty} v \exp\left(-\frac{mv^2}{2kT}\right) \mathrm{d}v = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \zeta_2\left(\frac{m}{2kT}\right)$$

The ζ means Guass integral:

$$\zeta_n(a) = \int_0^{+\infty} x^{n-1} \exp\left(-ax^2\right) \mathrm{d}x$$

It has some properties like

- $\zeta_n(a) = \dfrac{\Gamma\left(\frac{n}{2}\right)}{2a^{\frac{n}{2}}}$ is we use Γ to present Gamma function like $\Gamma(s) = \displaystyle\int_0^{+\infty} x^{s-1} \exp(-x)\,\mathrm{d}x$
- $\zeta_{n+2}(a) = -\dfrac{\partial}{\partial a}\zeta_n(a)$ and with $\zeta_1 = \dfrac{\sqrt{\pi}}{2\sqrt{a}}$, $\zeta_2 = \dfrac{1}{2a}$

As a result:

$$\left\langle \frac{1}{v} \right\rangle = \sqrt{\frac{2}{\pi}}\sqrt{\frac{m}{kT}}$$

It is very diffrent from $\dfrac{1}{\langle v \rangle} = \sqrt{\dfrac{\pi}{8}}\sqrt{\dfrac{m}{kT}}$. What more we have $\left\langle \dfrac{1}{v} \right\rangle > \dfrac{1}{\langle v \rangle}$.

Q2. Use some conclutions we get from class:

$$\begin{aligned} M = (\mathit{\Gamma}_1 - \mathit{\Gamma}_2) A m_0 &= \frac{1}{4} A \frac{M^{\mathrm{mol}}}{N_{\mathrm{A}}} \bar{v}(n_1 - n_2) = \frac{1}{4} A \frac{M^{\mathrm{mol}}}{N_{\mathrm{A}}} \bar{v} \frac{p_1 - p_2}{kT} \\ &= \frac{1}{4} A \frac{M^{\mathrm{mol}}}{R/k} \frac{1}{kT} \sqrt{\frac{8}{\pi}} \sqrt{\frac{RT}{M^{\mathrm{mol}}}} (p_1 - p_2) = \sqrt{\frac{M^{\mathrm{mol}}}{2\pi RT}} A (p_1 - p_2) \end{aligned}$$

Q3. We know that $f_{\mathrm{M}}(v_i) = \sqrt{\dfrac{m}{2\pi kT}} \exp\left(-\dfrac{mv_i^2}{2kT}\right)$ is correct for $i = x, y, z, w$... When it comes to two dimension the Maxwell distribution law of velocity:

$$f_{\mathrm{M}}(v) = 2\pi v f_{\mathrm{M}}(v_x) f_{\mathrm{M}}(v_y) = \frac{mv}{kT} \exp\left(-\frac{mv^2}{2kT}\right)$$

So

$$\left.\frac{\mathrm{d}\ln}{\mathrm{d}v}f_{\mathrm{M}}\left(v\right)\right|_{v=v_p}=\frac{1}{v_p}-\frac{mv_p}{kT}=0$$

$$v_p=\sqrt{\frac{kT}{m}}$$

$$v_{\mathrm{rms}}^2=\int_0^{+\infty}v^2f_{\mathrm{M}}\left(v\right)\mathrm{d}v=\frac{m}{kT}\int_0^{+\infty}v^3\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v=\frac{m}{kT}\zeta_4\left(-\frac{m}{2kT}\right)$$

$$v_{\mathrm{rms}}=\sqrt{2}\sqrt{\frac{kT}{m}}$$

$$v_{\mathrm{mean}}=\int_0^{+\infty}vf_{\mathrm{M}}\left(v\right)\mathrm{d}v=\frac{m}{kT}\int_0^{+\infty}v^2\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v=\frac{m}{kT}\zeta_3\left(-\frac{m}{2kT}\right)$$

$$v_{\mathrm{mean}}=\sqrt{\frac{\pi}{2}}\sqrt{\frac{kT}{m}}$$

Q4. Solve it by Boltzman distribution:

$$p\left(h\right)=p_0\exp\left(-\frac{1}{kT}\left(E\left(h\right)-E\left(0\right)\right)\right)$$

$$\frac{E\left(h\right)}{E\left(0\right)}=\frac{R_E}{h+R_E}$$

$$E\left(0\right)=mgR_E$$

As a result we have

$$p\left(h\right)=p_0\exp\left(-\frac{1}{kT}\frac{h}{h+R_E}mgR_E\right)$$

Q5. Because of $v=\sqrt{\frac{2\varepsilon}{m}}$ so we have $\mathrm{d}v=\frac{1}{2}\left(\frac{2\varepsilon}{m}\right)^{-\frac{1}{2}}\frac{2}{m}\mathrm{d}\varepsilon=\frac{1}{m}\sqrt{\frac{m}{2\varepsilon}}\mathrm{d}\varepsilon$.

$$f_{\mathrm{M}}\left(v\right)\mathrm{d}v=4\pi v^2\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v=4\pi\frac{2\varepsilon}{m}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{\varepsilon}{kT}\right)\frac{1}{m}\sqrt{\frac{m}{2\varepsilon}}\mathrm{d}\varepsilon$$

Just clean up the formula:

$$f_{\mathrm{M}}\left(v\right)\mathrm{d}v=\frac{2}{\sqrt{\pi}}\left(kT\right)^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}\exp\left(-\frac{\varepsilon}{kT}\right)\mathrm{d}\varepsilon$$

Let's rewrite the way we write the quantum

$$f\left(\varepsilon\right)\mathrm{d}\varepsilon=\frac{2}{\sqrt{\pi}}\left(kT\right)^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}\exp\left(-\frac{\varepsilon}{kT}\right)\mathrm{d}\varepsilon$$

Q6.

(1). From the graphic we can know:

$$Nf\left(v\right)=\begin{cases}\dfrac{a}{v_0}v & 0\leqslant v\leqslant v_0\\ a & v_0\leqslant v\leqslant 2v_0\\ 0 & 2v_0<v\leqslant+\infty\end{cases}$$

And normalizing condition:

$$\int_{\mathbb{R}^+}f\left(v\right)\mathrm{d}v=\left(\int_0^{v_0}+\int_{v_0}^{2v_0}\right)f\left(v\right)\mathrm{d}v=1$$

According to the equation we have

$$a=\frac{2N}{3v_0}$$

(2).

$$N(1.5v_0\sim 2v_0)=\int_{1.5v_0}^{2v_0}f(v)\mathrm{d}v=\frac{1}{3}N$$

(3).

$$v_{\mathrm{mean}}=\int_{\mathbb{R}^+}vf\left(v\right)\mathrm{d}v=\int_0^{v_0}\frac{a}{Nv_0}v^2\mathrm{d}v+\int_{v_0}^{2v_0}\frac{a}{N}v\mathrm{d}v=\frac{11}{9}v_0$$

Q7.

(1). Without thinking about it we just do the integral:

$$\Delta N=N\cdot\int_0^{v_{\max}}f_{\mathrm{M}}\left(v_x\right)\mathrm{d}v_x=N\cdot\sqrt{\frac{m}{2\pi kT}}\int_0^{v_{\max}}\exp\left(-\frac{mv_x^2}{2kT}\right)\mathrm{d}v_x=\frac{N}{\sqrt{\pi}}\cdot v_{\max}\int_0^{v_{\max}}\exp\left(-\frac{v_x^2}{v_{\max}^2}\right)\mathrm{d}v_x$$

$$=\frac{N}{\sqrt{\pi}}\cdot\int_0^1\exp\left(-\frac{v_x^2}{v_{\max}^2}\right)\mathrm{d}\frac{v_x}{v_{\max}}=\frac{N}{\sqrt{\pi}}\int_0^1\exp\left(-x^2\right)\mathrm{d}x=\frac{N}{2}\mathrm{erf}\left(1\right)$$

(2). Do as same as above:

$$\begin{aligned}\int_{v_0}^{+\infty}f_{\mathrm{M}}\left(v_x\right)\mathrm{d}v_x&=\frac{1}{2}-\int_0^{v_0}f_{\mathrm{M}}\left(v_x\right)\mathrm{d}v_x=\frac{1}{2}-\frac{1}{\sqrt{\pi}}\frac{1}{v_{\max}}\int_0^{v_0}\exp\left(-\frac{v_x^2}{v_{\max}^2}\right)\mathrm{d}v_x\\&=\frac{1}{2}-\frac{1}{\sqrt{\pi}}\int_0^{\frac{v_0}{v_{\max}}}\exp\left(-\frac{v_x^2}{v_{\max}^2}\right)\mathrm{d}\left(\frac{v_x}{v_{\max}}\right)=\frac{1}{2}-\frac{1}{\sqrt{\pi}}\int_0^{x_0}\exp\left(-x\right)\mathrm{d}x\\&=\frac{1}{2}-\frac{1}{2}\mathrm{erf}\left(x_0\right)\end{aligned}$$

As a result $\Delta N=\dfrac{N}{2}(1-\mathrm{erf}(x_0))$.

(3). According to $\mathrm{d}\left(x\mathrm{e}^{-x^2}\right) = \mathrm{e}^{-x^2}\mathrm{d}x - 2x^2\mathrm{e}^{-x^2}\mathrm{d}x$ we can conclude

$$\int_0^{x_0} x^2\mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\int_0^{x_0}\mathrm{e}^{-x^2}\mathrm{d}x - \frac{1}{2}\int_0^{x_0}\mathrm{d}\left(x\mathrm{e}^{-x^2}\right) = \frac{\sqrt{\pi}}{4}\mathrm{erf}\left(x_0\right) - \frac{1}{2}x_0\mathrm{e}^{-x_0^2}$$

Base on this

$$\begin{aligned}\int_0^{v_0} f_{\mathrm{M}}\left(v\right)\mathrm{d}v &= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^{v_0} v^2\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v = \frac{4}{\sqrt{\pi}}\frac{1}{v_{\max}^3}\int_0^{v_0} v^2\exp\left(-\frac{v^2}{v_{\max}^2}\right)\mathrm{d}v\\ &= \frac{4}{\sqrt{\pi}}\int_0^{\frac{v_0}{v_{\max}}}\frac{v^2}{v_{\max}^2}\exp\left(-\frac{v^2}{v_{\max}^2}\right)\mathrm{d}\left(\frac{v}{v_{\max}}\right) = \frac{4}{\sqrt{\pi}}\int_0^{x_0} x^2\exp\left(-x^2\right)\mathrm{d}x\\ &= \mathrm{erf}\left(x_0\right) - \frac{2}{\sqrt{\pi}}x_0\mathrm{e}^{-x_0^2}\end{aligned}$$

As a result $\Delta N = N\left(\mathrm{erf}\left(x_0\right) - \frac{2}{\sqrt{\pi}}x_0\mathrm{e}^{-x_0^2}\right)$.

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