Chapter 21

Remarks On Solvers For Transonic Potential Equations

21.1 Multigrid improvements

The multigrid solution of transonic potential flows was first studied in collaboration with Jerry South [SB77], [Bra77a, §6.5]. At the time of that study multigrid research was less advanced, and many of the improved approaches described in this Guide were not implemented. Collected below is a list of important improvements that the present Guide would recommend.

(A) The Neumann boundary condition and the constant-potential jump condition in the wake of an airfoil should not be *enforced* in relaxation, only *smoothed* (see §5.3). Thus, in relaxation, the potential jump at each wake point should be just set to be the average of the jumps at the neighboring wake points. The conditions should only be enforced at the coarsest level. Likewise, Kutta condition should only be applied at the coarsest level (cf. §5.6): the far-field conditions on each level should accordingly be adjusted at the stage of interpolating-and-adding the coarse-grid correction.

(B) Improved rates can be obtained if, before each full relaxation sweep, special local sweeps are made around singular points, such as trailing edges (see §5.7).

(C) Near a strong shock it may be better to use interpolation procedures which take jumps in p into account (see §4.6).

(D) Instead of stretching coordinates (to cover large exterior domains) and other transformations, a better multigrid procedure is to use increasingly coarser grids on increasingly larger domains, possibly with local refinements, anisotropic refinements and local coordinates (see §9). On such grids, simple relaxation schemes can be use: block relaxation is only needed in directions of strong alignment (see §3.3).

(E) Most importantly, because of the non-elliptic nature of the problem, perfect smoothers and good *asymptotic* convergence rates should not be attempted: much simpler and vectorizable schemes can be used if all one attempts is to get below truncation errors (see end of §4.1 and 3.3, 7 and 20.3.1). Correspondingly, the performance of the algorithm should be ascertained through direct measurements of $\|\tilde{\phi}^h - \tilde{\phi}^{2h}\|$ (see §1.6) and of $|F^h(\tilde{\phi}^h) - F^{2h}(\tilde{\phi}^{2h})|$, where $F(\Phi)$ is any solution functional one wants to get as the end result of the computations. If the norm measures discontinuous quantities, such as velocities, it should be an L_1 norm [Bra81a, §4.5].

21.2 Artificial viscosity in quasi-linear formulations

The transonic potential equation can be written in the quasilinear form

$$\left[(\underline{u} \cdot \underline{\nabla})^2 - a^2\Delta\right] \Phi = 0, \qquad (21.1)$$

where $\underline{u} = \underline{\nabla}\Phi$. This operator has appeared above in the discussion of Euler and Navier-Stokes equations (see for instance (20.13), or (20.22)). This physical origin of the operator suggests a physical form for the artificial viscosity which should be added to it, different from the Murman-Cole-type forms. Namely, the artificial viscosity should be added to $\underline{u} \cdot \underline{\nabla}$, before it is squared, using generally the form of Q_0^h (see (20.17)–(20.18) for $\alpha = 0$). In particular, if upwinding is desired, it is the operator $\underline{u} \cdot \underline{\nabla}$ that should be upwinded, before it is squared.

This form of artificial viscosity (or upwinding) is not only smoother and more physical, it is also more straightforward than the usual scheme where the operator should be rotated before it is upwinded. Also, this scheme requires no distinction between subsonic and supersonic points. The main difference between the two schemes is near sonic points ($M \approx 1$), or near shock transition from supersonic to subsonic. The Murman-Cole artificial viscosity vanishes there, and may therefore give rise to non-physical solutions.

In deeply subsonic regions, where (21.1) is uniformly elliptic, the form of the artificial viscosity does not matter of course, and one can switch to fully central approximations. In multigrid processing it is not important to do that, because $O(h^2)$ approximations can be obtained, even in the supersonic regions, by omitting the artificial viscosity (or the upwinding) from the operator used in the fine-to-coarse residuals transfer (see §10.2).

This latter operator should notwithstanding respect shocks as far as possible. Namely, it should be written in conservation form, and the stronger the shock, the weaker should it be straddled by flux calculations.