Finite Difference Methods for Ordinary and Partial Differential Equations

Finite Difference Methods for Ordinary and Partial Differential Equations

Steady-State and Time-Dependent Problems

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To my family,

Loyce, Ben, Bill, and Ann

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Preface

This book evolved from lecture notes developed over the past 20+ years of teaching this material, mostly in Applied Mathematics 585–6 at the University of Washington. The course is taken by first-year graduate students in our department, along with graduate students from mathematics and a variety of science and engineering departments.

Exercises and student projects are an important aspect of any such course and many have been developed in conjunction with this book. Rather than lengthening the text, they are available on the book's Web page:

`www.siam.org/books/OT98`

Along with exercises that provide practice and further exploration of the topics in each chapter, some of the exercises introduce methods, techniques, or more advanced topics not found in the book.

The Web page also contains MATLAB® m-files that illustrate how to implement finite difference methods, and that may serve as a starting point for further study of the methods in exercises and projects. A number of the exercises require programming on the part of the student, or require changes to the MATLAB programs provided. Some of these exercises are fairly simple, designed to enable students to observe first hand the behavior of numerical methods described in the text. Others are more open-ended and could form the basis for a course project.

The exercises are available as PDF files. The LaTeX source is also provided, along with some hints on using LaTeX for the type of mathematics used in this field. Each exercise is in a separate file so that instuctors can easily construct customized homework assignments if desired. Students can also incorporate the source into their solutions if they use LaTeX to typeset their homework. Personally I encourage this when teaching the class, since this is a good opportunity for them to learn a valuable skill (and also makes grading homework considerably more pleasurable).

Organization of the Book

The book is organized into two main parts and a set of appendices. Part I deals with steady-state boundary value problems, starting with two-point boundary value problems in one dimension and then elliptic equations in two and three dimensions. Part I concludes with a chapter on iterative methods for large sparse linear systems, with an emphasis on systems arising from finite difference approximations.

Part II concerns time-dependent problems, starting with the initial value problem for ODEs and moving on to initial-boundary value problems for parabolic and hyperbolic PDEs. This part concludes with a chapter on mixed equations combining features of ordinary differential equations (ODEs) and parabolic and hyperbolic equations.

Part III consists of a set of appendices covering background material that is needed at various points in the main text. This material is collected at the end to avoid interrupting the flow of the main text and because many concepts are repeatedly used in different contexts in Parts I and II.

The organization of this book is somewhat different from the way courses are structured at many universities, where a course on ODEs (including both two-point boundary value problems and the initial value problem) is followed by a course on partial differential equations (PDEs) (including both elliptic boundary value problems and time-dependent hyperbolic and parabolic equations). Existing textbooks are well suited to this latter approach, since many books cover numerical methods for ODEs or for PDEs, but often not both. However, I have found over the years that the reorganization into boundary value problems followed by initial value problems works very well. The mathematical techniques are often similar for ODEs and PDEs and depend more on the steady-state versus time-dependent nature of the problem than on the number of dimensions involved. Concepts developed for each type of ODE are naturally extended to PDEs and the interplay between these theories is more clearly elucidated when they are covered together.

At the University of Washington, Parts I and II of this book are used for the second and third quarters of a year-long graduate course. Lectures are supplemented by material from the appendices as needed. The first quarter of the sequence covers direct methods for linear systems, eigenvalue problems, singular values, and so on. This course is currently taught out of Trefethen and Bau [91], which also serves as a useful reference text for the material in this book on linear algebra and iterative methods.

It should also be possible to use this book for a more traditional set of courses, teaching Chapters 1, 5, 6, 7, and 8 in an ODE course followed by Chapters 2, 3, 9, 10, and 11 in a PDE-oriented course.

Emphasis of the Book

The emphasis is on building an understanding of the essential ideas that underlie the development, analysis, and practical use of finite difference methods. Stability theory necessarily plays a large role, and I have attempted to explain several key concepts, their relation to one another, and their practical implications. I include some proofs of convergence in order to motivate the various definitions of "stability" and to show how they relate to error estimates, but have not attempted to rigorously prove all results in complete generality. I have also tried to give an indication of some of the more practical aspects of the algorithms without getting too far into implementation details. My goal is to form a foundation from which students can approach the vast literature on more advanced topics and further explore the theory and/or use of finite difference methods according to their interests and needs.

I am indebted to several generations of students who have worked through earlier versions of this book, found errors and omissions, and forced me to constantly rethink my understanding of this material and the way I present it. I am also grateful to many

colleagues who have taught out of my notes and given me valuable feedback, both at the University of Washington and at more than a dozen other universities where earlier versions have been used in courses. I take full responsibility for the remaining errors.

I have also been influenced by other books covering these same topics, and many excellent ones exist at all levels. Advanced books go into more detail on countless subjects only briefly discussed here, and I give pointers to some of these in the text. There are also a number of general introductory books that may be useful as complements to the presentation found here, including, for example, [27], [40], [49], [72], [84], and [93].

As already mentioned, this book has evolved over the past 20 years. This is true in part for the mundane reason that I have reworked (and perhaps improved) parts of it each time I teach the course. But it is also true for a more exciting reason—the field itself continues to evolve in significant ways. While some of the theory and methods in this book were very well known when I was a student, many of the topics and methods that should now appear in an introductory course had yet to be invented or were in their infancy. I give at least a flavor of some of these, though many other developments have not been mentioned. I hope that students will be inspired to further pursue the study of numerical methods, and perhaps invent even better methods in the future.

Randall J. LeVeque