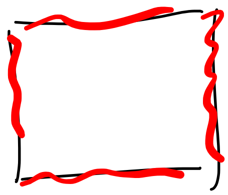
Poisson's equation



$$\begin{cases} -\Delta u = f, & \text{in } (0,1)^2 \\ \underbrace{u = 0}, & \text{on } \partial(0,1)^2 \end{cases}$$

$u(x,1) = 0, \quad u(x,0) = 0$

$u(1,y) = 0, \quad u(0,y) = 0$

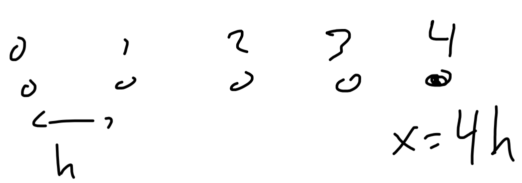
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$$

1-D $\quad f(x), \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$h > 0 \quad f'(x) \simeq \frac{f(x+h) - f(x)}{h}$ (forward differen)

$h > 0 \quad f'(x) \simeq \frac{f(x) - f(x-h)}{h}$ (backward difference)

Grid with spacing $h > 0$, $f_i \simeq f(ih)$



$f_{i+1} = f((i+1)h)$

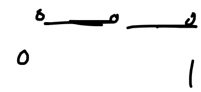
$= f(ih + h)$

$$f'_i \simeq \underbrace{\frac{f_{i+1} - f_i}{h}}_{\text{forward}} \quad \text{or} \quad \underbrace{\frac{f_i - f_{i-1}}{h}}_{\text{backward}}$$

$$f_i'' = \frac{f'_{i+1} - f'_i}{h}$$

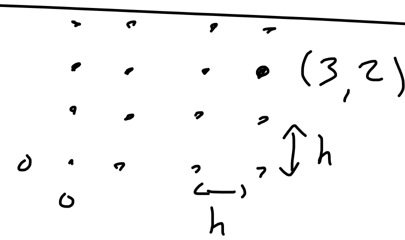
$$= \frac{\left(\frac{f_{i+1} - f_i}{h}\right) - \left(\frac{f_i - f_{i-1}}{h}\right)}{h}$$

$$\boxed{f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}}$$



$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

<u>2-D</u> $u(x,y)$



$$U_{ij} = u(ih, jh)$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{h^2}$$

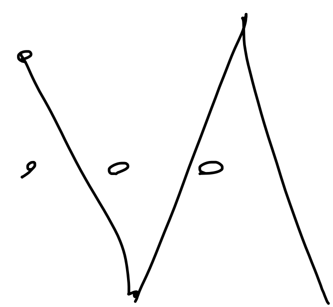
$$\Delta u \simeq \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2}$$

$$\begin{cases} \dfrac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \dfrac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} + f_{ij} = 0, & ij \text{ not on boundary} \\ u_{ij} = 0 & \text{on boundary} \end{cases}$$

$$S[u]_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2} + f_{ij}$$

Gradient descent:

$$\begin{cases} u^0_{ij} = 0 \quad \text{(or anything you like)} \\ u^{n+1}_{ij} = u^n_{ij} + \Delta t\, S[u^n]_{ij} \\ u^{n+1}_{ij} = 0 \quad \text{on boundary} \end{cases}$$



Run until $u^{n+1}_{ij} = u^n_{ij}$, then $S[u^n]_{ij} = 0$

Run until $\left| S[u^n]_{ij} \right| \leq \varepsilon = h^2$