Chapter 2

Stable Discretization

The formulation of good discretization schemes is of course the first step in any numerical solution of continuous equations. For multigrid solutions, some additional considerations enter. First, discrete equations should be written for general meshsizes h, including large ones (to be used on coarse grids). Also, the multigrid processes offer several simplifications of the discretization procedures.

(A) Only *uniform-grid discretization* is often needed. Non-uniform discretization can be effected through multigrid interactions between uniform grids (see §9). In fact, if the basic grids are *non*-uniform, various structural and algorithmic complications are introduced in multigrid, as well as and even more than, in unigrid processes. The uniform discretization can be made either with finite element or finite difference formulations. For finite elements, it is preferable to use either piecewise uniform partitions as in [Ban81] and [Bra77a, §7.3], or uniform partitions modified at special parts (e.g., at boundaries) as in [Bra79a, Fig. 2], and produce local refinements by the multigrid process (§9). When general non-uniform partitions must be used, *algebraic* multigrid solvers (see §13.1) are recommended; their set-up processing (coarse-levels assembly) is much more costly, but it is still small compared with the very expensive processing required for assembling the (finest-level) equations.

(B) Only *low-order (first or second order)* discretization need to be developed. Higher-order schemes can later be superposed for little extra programming effort and computer time (§10). The low order makes it easier to write stable equations, easier to devise and analyze relaxation schemes, and less expensive to operate those schemes. The superposed higher-order schemes need not be stable on their own.

(C) Designing stable discretization with efficient relaxation is only required in terms of the *(sub)principal part of the operator* (see §2.1).

(D) An easier implementation and more flexible control of *global constraints* is achieved by effecting them only at the coarsest levels of the multigrid processing (with suitable inter-grid transfers. See §5.6). This makes it possible to free local differencing from complicated forms aimed at precise conservation of global quantities (such as total kinetic energy and square vorticity, as in [Ara66]), and to add suitable controls to ill-posed problems, etc.

Low order finite elements on uniform grids yield in fact different kinds of difference equations. The description below will therefore be in terms of finite difference formulations only. It should be emphasized, however, that variational formulations, where appropriate, automatically yield good prescriptions for the main multigrid processes [Bra77a, App. A.5], [Nic77], [Bra79a]. This is especially useful in some complicated situations, as in [ABDP81]. We revisit this issue in §§4.5, 4.6 and 11.

For a boundary-value problem to be solvable by a fast multigrid algorithm, its discretization should be suitably stable. More precisely, the type of stability determines the type of multigrid algorithm that can be efficient. *The simplest difference equations of regular elliptic PDEs are stable in every respect, so the reader interested only in such problems can skip the rest of this chapter.* But remember:

Unstable (or very inaccurate) discretization must lead to slow multigrid convergence (unless special techniques are adopted to the case, or algebraic multigrid is used). It is indeed an important *advantage* of multigrid solvers that bad discretization cannot be passed unnoticed; it must show up as slow algebraic convergence.

2.1 Interior stability measures: h-ellipticity

Numerical stability is a *local* property, i.e., a property significant only for non-smooth components of the solution (components that change significantly over a meshsize), whereas the smooth-component stability depends, by consistency, on the *differential* system, not on its discretization. Indeed, in multigrid solvers, stability of the discrete operator is needed only in the *local* process of relaxation (cf. §10.2). Moreover, what really counts is not the stability of the static difference operator itself, but the overall efficiency with which the dynamic process of relaxation smoothes the differential error (cf. §12); numerical stability of the operator is just a necessary condition for achieving that smoothing.

Because of the local character of the required stability (corresponding to the local task of relaxation), it is easy to get a very good quantitative idea about it, for any given difference operator L^h, by **local mode analysis**, analogous to the Von-Neumann stability analysis for time-dependent

problems. It turns out, however, that for steady-state problems, especially non-elliptic or singular-perturbation ones, the distinction between stable and unstable discrete operators is not enough. More important is the *measure* of stability. When that measure, for a given meshsize, is small, the scheme is still formally stable, but its actual behavior can be intolerably bad (see example in [Bra81a, §3]).

Briefly, the basic relevant measure of stability of an interior (not at boundaries) linear difference operator L^h with constant coefficients is its **h-ellipticity measure** $E^h(L^h)$, defined for example by

$$E^h(L^h) := \frac{\min_{\hat{\rho}\pi \le |\underline{\theta}| \le \pi} \left| \tilde{L}^h(\underline{\theta}) \right|}{|L^h|}, \tag{2.1}$$

where the complex function $\tilde{L}^h(\underline{\theta})$ is the "symbol" of L^h, i.e.,

$$L^h e^{i\underline{\theta}\cdot\underline{x}/\underline{h}} = \tilde{L}^h(\underline{\theta}) e^{i\underline{\theta}\cdot\underline{x}/\underline{h}};$$

$\underline{\theta} := (\theta_1, \ldots, \theta_d)$; $\underline{\theta}\cdot\underline{x}/\underline{h} := \theta_1 x_1/h_1 + \cdots + \theta_d x_d/h_d$, $|\underline{\theta}| := \max\{|\theta_1|, \ldots, |\theta_d|\}$; d is the dimension; h_j is the meshsize in direction x_j; and $|L^h|$ is any measure of the size of L^h, e.g., $|L^h| = \max_{\underline{\theta}} |\tilde{L}^h(\underline{\theta})|$. The constant $0 < \hat{\rho} < 1$ is in fact arbitrary, but for convenient multigrid applications a natural choice is the meshsize ratio, hence usually $\hat{\rho} = \frac{1}{2}$. The range $\hat{\rho}\pi \le |\underline{\theta}| \le \pi$ is then the range of "high frequency" components on grid h, i.e., components $e^{i\underline{\theta}\cdot\underline{x}/\underline{h}}$ which on the next coarser grid, with meshsize $h/\hat{\rho}$, coincide (alias) with lower components.

For systems of equations, L^h and $\tilde{L}^h(\underline{\theta})$ are matrices and $|\tilde{L}^h(\underline{\theta})|$ should then be understood as a measure of the non-singularity of $\tilde{L}^h(\underline{\theta})$ (e.g., its smallest eigenvalue, or its determinant). See more details and explanations in [Bra80b, §3], [Bra81a, §3.1]. In fact, good discretization schemes can generally be arrived at by requiring $\det(L^h)$ to be a good discretization for $\det(L)$, the determinant of the given differential operator: see the examples in §17.2, 18.2, 19.2, 20.2. We will use the notation $\tilde{\det} L^h(\underline{\theta})$ to denote the symbol of the operator $\det(L^h)$.

In case the differential operator L, and hence also L^h, have **variable coefficients**, L^h is called h-elliptic if $E^h(L^h) = O(1)$ for each combination of coefficients appearing in the domain. If L is **nonlinear**, L^h is called h-elliptic if its linearizations around all approximate solutions encountered in the calculations are h-elliptic.

A major simplification in selecting the discretization scheme for complicated systems is the fact that, being interested in local properties only, we can confine our considerations to those terms which are locally important. In the discretized and linearized operator L^h, the locally important terms, called the **h-principal terms**, are simply those with large coefficients (relative to other coefficients, if any). In case of a system, the h-principal terms are those contributing to the h-principal term of $\det(L^h)$. Other terms are

not important in relaxation; namely, they need not satisfy any stability conditions, they can actually be transferred to the right hand-side of the relaxed equations, and they need not even be updated each sweep (only each multigrid cycle; cf. §10.3).

The h-principal terms all normally come from discretizing **subprincipal terms** of L. These are defined as the principal terms (the terms contributing to the highest order derivatives in the determinant of the linearized operator) plus the principal terms of the reduced operator (the operator without singular perturbation terms). Thus, *in discretizing any differential operator L, we can confine our attention to its subprincipal part.* See the examples in §19.1 and §20.1. Note however that on very coarse grids, terms corresponding to lower-order derivatives may become h-principal.

Regular discretizations of elliptic systems should, and usually do have good (i.e., O(1)) h-ellipticity measures (but see a counterexample in §17.2). Singular perturbation or non-elliptic systems can also have such good measures, e.g., by using artificial viscosity or by upwind (upstream) differencing (note that a regular elliptic system with lower-order terms may be a singular-perturbation problem on a sufficiently coarse grid).

If, however, characteristic or *sub*characteristic directions (i.e., characteristic directions of the reduced equations, in the case of singular perturbation problems) coincide with grid directions, upwind differencing schemes are only semi h-elliptic. That is, they have a bad h-ellipticity measure E^h, yet they still have a good *semi* h-ellipticity measure in the characteristic direction, defined as follows.

Let $S \subseteq \{1, \ldots, d\}$ be a subset of grid directions. The measure of **semi h-ellipticity in directions** S, or briefly **S-h-ellipticity**, of a difference operator L^h, is

$$E^h_S(L^h) = \frac{\min_{\hat{\rho}\pi \le |\underline{\theta}|_S \le \pi} \left| \tilde{L}^h(\underline{\theta}) \right|}{L^h}, \tag{2.2}$$

where $|\underline{\theta}|_S := \max_{j \in S} |\theta_j|$. Full h-ellipticity is the special case $S = \{1, \ldots, d\}$. If $S_2 \subset S_1$, then clearly $E^h_{S_1} \le E^h_{S_2}$, hence S_1-h-ellipticity entails S_2-h-ellipticity.

In case (sub)characteristics are aligned with grid directions, full h-ellipticity is not needed for stability. The corresponding S-h-ellipticity is enough; it allows large local oscillations perpendicular to the characteristics, but those oscillations are also allowed by the differential equations.

Fully h-elliptic approximations can be constructed even for non-elliptic or semi-elliptic differential equations, by using isotropic artificial viscosity. In various cases, however, semi h-elliptic approximations are preferable, because they entail much less cross-stream smearing. These are mainly cases of **strong alignment**, that is, cases where (sub)characteristic lines are non-locally (i.e., for a length of many meshsizes) aligned with a grid-line, and where this non-local alignment occurs either for many gridlines, or even for one gridline, if that line is adjacent to a boundary layer or a

similar layer of sharp change in the solution (for a method to obtain strong alignments and thus avoid smearing – see §9.3).

A convenient way of **constructing** h-elliptic and semi h-elliptic operators is by term-by-term R-elliptic or semi R-elliptic approximations [Bra79b, §5.2], [BD79, §3.6]. Another, more physical, way is to regard the given boundary-value problem as a limit of an elliptic problem (usually this is physically so anyway), and enlarge the elliptic singular perturbation to serve as artificial-viscosity terms [Bra82c], [Bra81a]. When solving a steady-state problem of originally time-dependent equations, the artificial elliptic terms should conform to the original time-dependent problem, i.e., with those terms that problem should still be well posed. This requirement often determines the sign of the artificial terms. Such physical artificial viscosity terms ensure that computed solutions will exhibit (as $h \to 0$) only those discontinuities allowed physically (hence, with this approach, explicit entropy conditions are not needed). A proper amount of anisotropic artificial viscosity gives the correct upstream differencing whenever desired.

The desirable amount of artificial viscosity (either isotropic or anisotropic) is mainly determined not by stability considerations, but by the smoothing properties of relaxation. Below a certain level of viscosity, more costly, distributive relaxation will have to be used. Increasing the artificial viscosity slightly larger beyond the minimum required for convergence of the simplest scheme makes the relaxation ordering-free (see §3.6 and [Bra81a, §5.7], [Bra80b, §4.2]), which is desirable, except perhaps near discontinuities. Considerably larger artificial viscosity makes the algebraic smoothing faster, but impedes the differential smoothing (cf. §12).

Interior difference equations that are not even semi h-elliptic should be used with care. Their solutions may show large numerical oscillations (giving nice solutions only on the average), and their fast multigrid solvers must have more complicated fine-to-coarse interactions (see for example §4.2.2). Some **quasi-elliptic** equations, i.e., cases where $\tilde{L}^h(\underline{\theta})/|L^h|$ does vanish for some $|\underline{\theta}| = \pi$, but not for other $|\underline{\theta}| \neq 0$, can be solved without much trouble. All that is needed is to average out the bad components (see for example §§18.6, 19.5).

2.2 Boundaries, discontinuities

We have so far discussed the stability conditions related to the *interior* difference equations, away from boundaries. To gain overall stability, some additional conditions should be placed at the boundaries. These can be analyzed by mode analysis in case the boundaries are parallel to grid directions (cf. §7.5). More general boundaries are however difficult to analyze.

Usually, however, h-elliptic approximations consistent with a well-posed problem and employing low-order approximations to boundary conditions, are stable. The order can be then raised in a stable way by one of the methods of §10. At any rate, the boundary stability is not related to

the stages of developing the main (interior) multigrid processes.

More critical than discretization near boundaries is the treatment of *discontinuities*, whether at boundaries (e.g., boundary layers) or in the interior (e.g., shocks). The basic rule, in multigrid as in unigrid processes, is to try not to straddle the discontinuity by any difference operator (during relaxation as well as in residual transfers. The rule also applies to the interpolation operators). More precisely, the rule is not to difference any quantity which is discontinuous in the interval of differencing. This can be fully achieved only in cases where the location of the discontinuity is known or traced (at such discontinuities the above rule overrides upstream differencing if they happen to conflict), or when the discontinuity is more or less parallel to grid directions (so that upstream differencing will automatically satisfy the rule). Captured discontinuities that are not in grid directions must perhaps be smeared; a multigrid way to get high accuracy then is by local refinements (see §9). Generally, multigrid procedures for discontinuities are now under active investigation (see [Bra81a, §4] and a remark in §8.5 below).