Multigrid Techniques

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Preface to the Classics Edition

The *Multigrid Guide* presents the best known practices and techniques for developing multigrid solvers. As best practices evolve with on-going developments, the history of the *Guide* mirrors the history of the field of multigrid research. We delineate between two eras that must be borne in mind when reading this book: 1984 and earlier, and 1984 to the present time.

The earlier period (summarized in the 1984 Guide). Parts I and II of that *Guide* were based on [Bra82b], the first being an expansion of an even earlier mini-guide [Bra80c]. The present Classics edition of the *1984 Guide* includes quite a few minor corrections, additional comments and clarifications of the original manuscript; still, it overall describes the state of the art of multigrid as of 1984 and cites other works of that time. Multigrid solvers for discretized elliptic partial differential equations on well-structured grids, including various CFD systems, are well represented, as they had already matured at that time; but later important multigrid developments are absent. To maintain consistency with the rest of the book, the Introduction (§0) has not been updated, so "recent" developments referenced therein are now nearly thirty years old.

Equipped with the hindsight of contemporary research, yet faithful to the spirit of the *1984 Guide*, only few essential modifications were made. Chapter 14 was thoroughly revised to emphasize general solver performance predictors rather than Local Mode Analysis (LMA). In the early days, the latter was the best approach to practical quantitative performance analysis; hence it is extensively used throughout the entire Part I of this book. While LMA predictions are still perfectly valid today, the new predictors are simpler and preferable in many circumstances. Additionally,

- The original `CycleV` model Fortran program was replaced with a modern object-oriented MATLAB program in §1.5 and Appendix A.

- The local relaxation rule (§5.7) and its FMG application (§9.6) were added.

- The proper usage of a large cycle index, including fractional values, is now explained in §6.2.

Recent Developments (1984 to present). We caution the reader that this edition of the *Guide* falls short of representing later multigrid developments. We regard this book as a baseline for the future *Multigrid Guide 2.0* project, which will be continuously updated to match contemporary research and literature. The *2.0* project is accessible online at `http://www.siam.org/books/CL67`.

In particular, many bibliographical items in the present edition are outdated. Some of the cited technical reports are no longer available. An ever-growing multigrid literature has since emerged, including basic books [BHM00, Hac85, TOS00] and a plethora of works in the proceedings of over thirty Copper Mountain and European conferences on multigrid methods. Reviews of progressively more recent developments have been given in [Bra88, Bra89, Bra02]. These and many other articles are now electronically available at `http://www.wisdom.weizmann.ac.il/~achi/`.

Since 1984, multigrid development has been shifting towards Algebraic Multigrid (AMG), which aims at simplifying complex multigrid design scenarios by automatically constructing a grid hierarchy and inter-grid operators from the given fine-grid matrix. The basic idea is already described in §1.1 of the *1984 Guide* and its present edition, but the dedicated section (§13.1) lacks details as AMG was still in its infancy in 1984. "Classical AMG" was devised over the following decade, the Ruge-Stuben algorithm [RS87] becoming its most popular variant. In the *2.0 Guide* we will focus on a yet more recent approach called Bootstrap Algebraic Multigrid (BAMG) [Bra02, §17.2], which has a wider scope as well as inspires improvements to existing geometric multigrid solvers.

We plan to add new chapters on various generalizations and applications of the multiscale methodology, some of which are outlined in [Bra02]:

- Further work on anisotropic problems and various important PDE systems such as elasticity and magnetohydrodynamics.

- Wave equations, eigenproblems and electronic structures in quantum chemistry.

- Global optimization and stochastic simulations in statistical physics [BR02].

- "Systematic Upscaling," a general multiscaling methodology for deriving macroscopic equations from microscopic physical laws [Bra10].

- Graph problems with applications to image processing [SGS$^+$06], data analysis [RSBss] and transportation networks.

Finally, we wish to invite you, the reader, to take an active role and contribute to the *Multigrid Guide* project. We welcome comments and suggestions. We want the *Guide* to be a reflection of our collective knowledge and understanding of multigrid methods.

Preface

Starting with an elementary exposition of multigrid fast solvers with insights into their analyses and their most general algebraic applicability, detailed practical guidelines are then given how to obtain, stage by stage, the full multigrid efficiency for general elliptic and non-elliptic problems, linear as well as nonlinear, scalar or vectorial, smooth or strongly discontinuous, with various possible singularities, boundary conditions and supplementary global conditions.

Quantitative insights through local mode analyses, combined with gradual algorithm development, are emphasized throughout, and general rules and approaches are explained for the design of relaxation, coarsening and interpolation. Beyond these fast-solver aspects of multigrid, advanced methods are then described, including various applications of the Full Approximation Scheme (FAS), local refinement and local coordinate transformations, error estimation and grid adaptation criteria, small storage algorithms, and the double discretization and other techniques for high-order approximations.

Also briefly outlined are Algebraic Multigrid (AMG); multi-level reduction of complexity for integral equations and for chains of problems; treatment of time-dependent problems; eigenvalue problems; and optimization of PDEs with design parameters.

Dedicated chapters describe in detail the solution of Cauchy-Riemann, Stokes and incompressible and compressible Navier-Stokes equations, with numerical results for staggered and non-staggered grids.