Macroeconometrics
Group Assignment 1

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1 Exercise

1.a

In the Keynesian model given in the task, y_t (the national income), c_t (the consumption), i_t (the investment) and g_t (the government consumption) are endogenous variables. Wherein the lags of these variables y_{t-1}, c_{t-1} and i_{t-1} are the predetermined or lagged endogenous variables.

$$
\begin{aligned}
c_t &= \gamma y_{t-1} && (1)\\
i_t &= \alpha(c_t - c_{t-1}) && (2)\\
g_t &= \bar{g} - \delta(y_t - y_{t-1}) && (3)\\
y_t &= c_t + i_t + g_t && (4)
\end{aligned}
$$

Equation (2), (3) and (4) are **structural equations** since the endogenous variables i_t, g_t and y_t depend on the realization of other endogenous variables c; y and c; i, c and g at time t.
To show that in the given model national income follows a second-order linear difference equation, the reduced-form for investment must first be derived. For this purpose, the consumption function (1) is substituted into the equation for investment (2).

$$i_t = \alpha\left[(\gamma y_{t-1}) - c_{t-1}\right]$$

Further, equation (1) is lagged by one period $c_{t-1} = \gamma y_{t-2}$ and also substituted into the equation for investment. From this, the following equation is obtained:

$$i_t = \alpha\left[(\gamma y_{t-1}) - (\gamma y_{t-2})\right] \tag{5}$$

By substituting equations (1), (3) and (5) into the equation for national income (4), we obtain the reduced-from:

$$
\begin{aligned}
y_t &= \gamma y_{t-1} + \alpha(\gamma y_{t-1} - \gamma y_{t-2}) + \bar{g} - \delta(y_t - y_{t-1})\\
y_t(1+\delta) &= y_{t-1}(\gamma + \alpha\gamma + \delta) - \alpha\gamma y_{t-2} + \bar{g}\\
y_t &= \frac{y_{t-1}(\gamma + \alpha\gamma + \delta) - \alpha\gamma y_{t-2} + \bar{g}}{(1+\delta)}\\
y_t &= \frac{(\gamma + \alpha\gamma + \delta)}{(1+\delta)}\, y_{t-1} - \frac{\alpha\gamma}{(1+\delta)}\, y_{t-2} + \frac{1}{(1+\delta)}\, \bar{g}
\end{aligned}
$$

Then y_t can be expressed in the following form

$$y_t = a y_{t-1} + b y_{t-2} + k \tag{6}$$

where $a = \frac{(\gamma+\alpha\gamma+\delta)}{(1+\delta)}$, $b = \frac{\alpha\gamma}{(1+\delta)}$ and the constant $k = \frac{1}{(1+\delta)}\,\bar{g}$.
Equation (6) is a **univariate reduced-form equation** in which the endogenous variable y_t is expressed solely as a function of its own lags and a constant.
Since the dependent variable y at time t in this model depends on 2 previous periods, we were able to show that the national income follows a second-order linear difference equation.

1.b

In this align, it was proposed that one would find the equilibrium solution of the second order difference equation obtain previously, the solution will be denoted by Y^P, and it should be noted that in the state of equilibrium $y^P = y_t = y_{t-1} = y_{t-2}$, it is thus possible to do the following calculations.

$$\begin{aligned} y^P(1+\delta) - y^P(\gamma + \alpha\gamma + \delta) + \alpha^P &= \bar{g} \\ y^P(1-\gamma) &= \bar{g} \\ y^P &= \frac{\bar{g}}{(1-\gamma)} \end{aligned}$$

1.c

In order to find solutions for $t \geq 2$, and given that there are no shocks at any point in time, one needs only to find the equilibrium solution for the given values of $\bar{g} = 10$ and $\gamma = 0,9$.

$$\begin{aligned} y^P &= \frac{\bar{g}}{(1-\gamma)} \\ y^P &= \frac{10}{0,1} \\ y^P &= 100 \end{aligned}$$

This value was further confirmed by the calculations made in Excel.

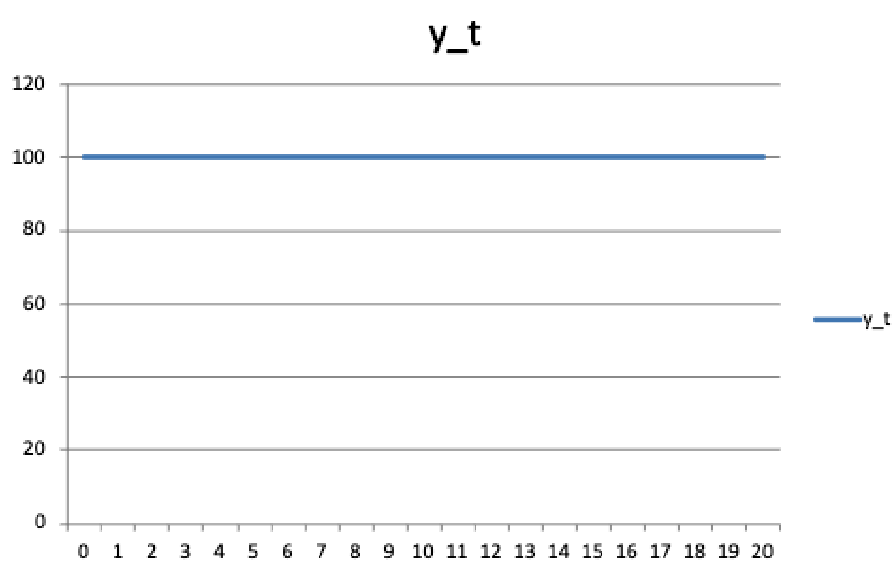

Figure 1: Path of Yt for t>2

1.d

For the following section, we assume that the economy is in the long run equilibrium ($y_0 = y_1 = y^P$), and $\alpha = 0.2,\ \gamma = 0.9,\ \delta = 0.1,\ and\ \bar{g} = 10$.

1.d.1

To compute the path of national income after a given temporary shock, one simply adds the temporary shock to the Excel calculations, the impact of this shock may be analyzed in the figure below and it will be further dissected in the next exercise.

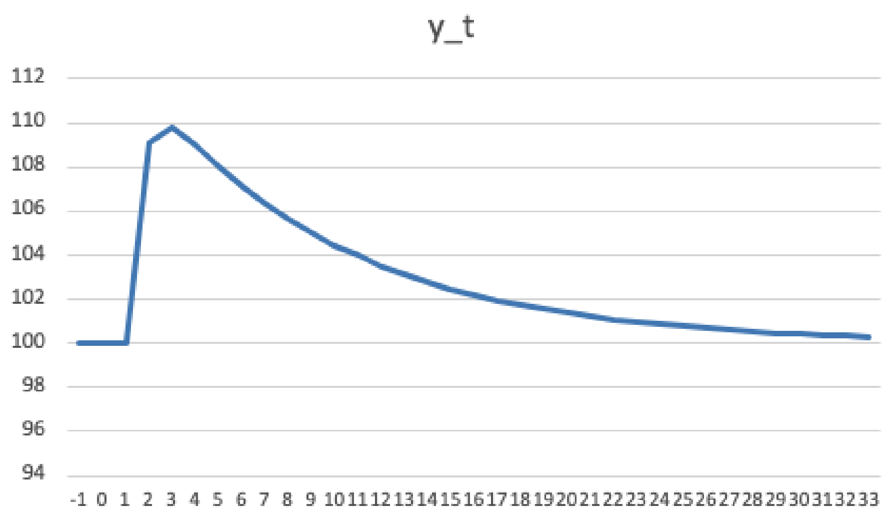

Figure 2: Path of Yt for t>0

1.d.2

The impact of the shock can be calculated in the following manner

$$\frac{\partial y_t}{\partial \bar{g}} = \frac{1}{1+\delta} = a_1$$
$$a_1 = 0,909$$

Given that $0 < a_1 < 1$ it makes sense that the impact dies off exponentially. One can also check this by using the lag operator,using the given values our initial equation may be written in the following way:

$$y_t(1 - \frac{1,18}{1,1}L + \frac{0,18}{1,1}L^2) = \frac{10}{1,1} + \frac{e_t}{1,1}$$

By factorizing the characteristic equation we can infer about the stability conditions

$$\lambda^2 - \frac{1,18}{1,1}\lambda + \frac{0,18}{1,1} = 0$$

$$\lambda = \frac{\frac{1,18}{1,1} \pm \sqrt{(\frac{1,18}{1,1})^2 - 4 * \frac{0,18}{1,1}}}{2}$$

$$\lambda_1 = 0,88857$$

$$\lambda_2 = 0,18415$$

Given that λ's are real numbers with absolute value smaller than one it is possible to predict that the impulse response function will be stable after an initial shock.

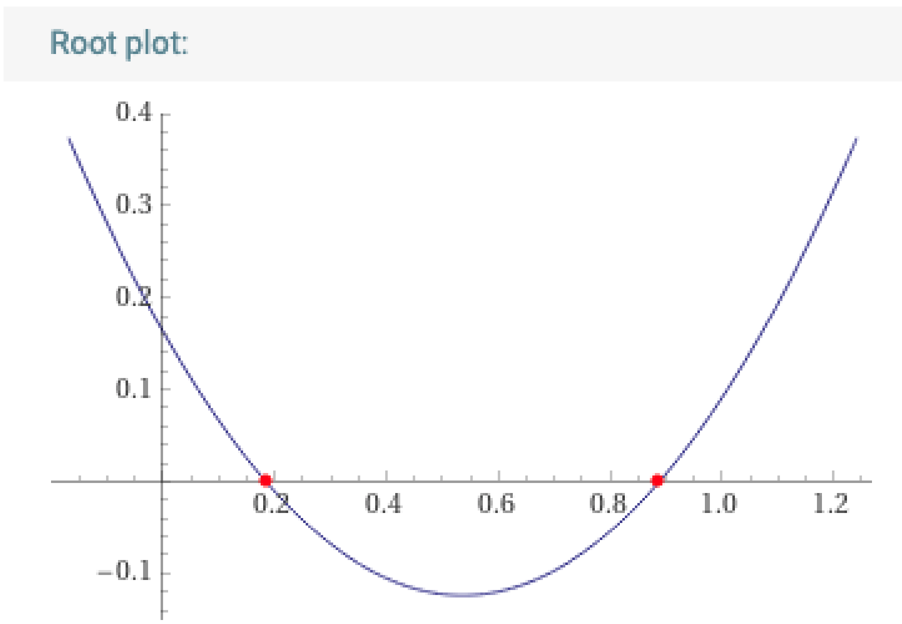

Figure 3: Root Plot

This is in accordance with the figure presented in the previous exercise were indeed the impact of the shock died off exponentially. The impulse response function and it's values at every point in time are presented in the Excel file.

1.d.3

The permanent shock formula for a difference equation of order 2 can be calculated using lag operators:

$$y_t(1 - \frac{1,18}{1+\delta}L + \frac{0,18}{1+\delta}L^2) = \frac{10}{1+\delta} + \frac{e_t}{1+\delta}$$

Using the roots of the characteristic polynomial one can express the above equation in a more general manner :

$$y_t = y^p + \frac{(e_t + \lambda_2 e_{t-1} + (\lambda_2)^2 e_{t-2}...)}{1+\delta} + \frac{\lambda_1(e_{t-1} + \lambda_2 e_{t-2} + (\lambda_2)^2 e_{t-3}...)}{1+\delta} + \frac{\lambda_1^2(e_{t-2} + \lambda_2 e_{t-3} + (\lambda_2)^2 e_{t-4}...)}{1+\delta}...$$

Given that for permanent shock $e_t = e_{t-1} = e_{t-2} = ...$, it's possible to factor out e_t.

$$y_t = y^p + \frac{e_t}{1+\delta} * [(1 + \lambda_2 + \lambda_2^2...) + \lambda_1(1 + \lambda_2 + \lambda_2^2...)\lambda_1^2(1 + \lambda_2 + \lambda_2^2...)]$$
$$y_t = y^p + \frac{e_t}{1+\delta} * (1 + \lambda_2 + \lambda_2^2...) * (1 + \lambda_1 + \lambda_1^2)$$
$$y_t = y^p + \frac{e_t}{1+\delta} * \sum_{n=1}^{\infty} \lambda_2^n * \sum_{n=1}^{\infty} \lambda_1^n$$
$$y_t = y^p + \frac{e_t}{1+\delta} * \frac{1}{1-\lambda_2} * \frac{1}{1-\lambda_1}$$

Where the last step is possible due to the fact that $|\lambda_2| < 1, |\lambda_1| < 1$. Substituting the values we have one gets:

$$y_t = 100 + \frac{10}{1+0,1} * \frac{1}{1-0,18415} * \frac{1}{1-0,88857}$$
$$y_t = 100 + 99,9988$$

This results is coherent with the excel simulation where $y_t \to 200$.

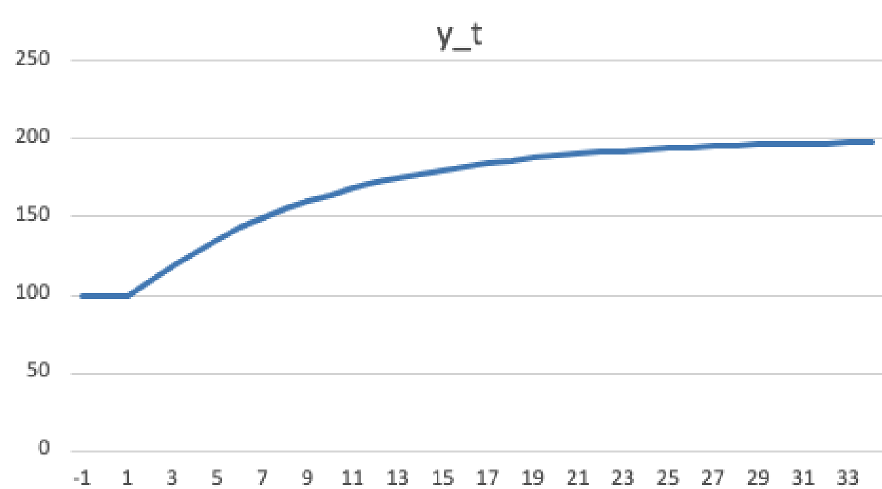

Figure 4: Path of Yt for t>0

1.e

Given that the impact is of the kind $a_1 = \frac{1}{1+\delta}$ we get that as $\delta \to \infty$ the impact of the shock is felt only momentarily. On the other hand if $\delta \to 0$ we get that a $\to 1$ and the shock has a permanent impact. So it's possible to say that δ controls the speed of convergence back to equilibrium after a momentary shock. Therefore, δ is providing information on the persistence of a shock : The larger the δ is, the more persistent will a temporary shock be over the following period of time. There are no positive δ that implicate imaginary values for λ so there will be no cycles coming from imaginary values of λ. For λ to be imaginary:

$$\begin{aligned}\sqrt{(\frac{1,08+\delta}{1+\delta})^2 - \frac{0,72}{1+\delta}} &< 0 \\ (\frac{1,08+\delta}{1+\delta})^2 &< \frac{0,72}{1+\delta} \\ 0,4464 + 1,44\delta + \delta^2 &< 0 \\ \delta < -0,72 \pm 0,268 &< 0\end{aligned}$$

On the other hand there could be cycles if we have values of a_1 and a_2 that are similar in absolute value, but with different signs. So to test what values of δ would bring us such values of a_1 and a_2 one can start with the condition $a_1 + a_2 = 0$.

$$\begin{aligned}a_1 + a_2 &= 0 \\ (\gamma + \alpha\gamma + \delta)/(1+\delta) - \alpha\gamma/(1+\delta) &= 0 \\ (\gamma + \alpha\gamma + \delta)/(1+\delta) &= \alpha\gamma/(1+\delta) \\ (\gamma + \alpha\gamma + \delta) &= \alpha\gamma \\ \delta &= -\gamma = -0,9\end{aligned}$$

Testing this in Excel, we see that, in fact, with this value of δ cycles are introduced, however this value is negative and again we conclude that there are no positive values of δ that introduce cycles while keeping the other variables constant.