CS/EE-712: Image Processing

2D DFT and Properties

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Readings: Chapter 4.6

From Last Lecture

- 2D Impulse and Sifting Property
- 2D Fourier Transform in Continuous Domain
- 2D Sampling Theory and Aliasing Artifacts

2D Discrete Fourier Transform (DFT)

- The 2D DFT is defined as:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M+vy/N)},$$

for $u = 0,1,2,\ldots M-1$ and $v = 0,1,2,\ldots N-1$

- The inverse DFT is defined similarly:

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M+vy/N)},$$

for $x = 0,1,2,\ldots M-1$ and $y = 0,1,2,\ldots N-1$

2D DFT: Some Properties

- Sampling intervals in space and frequency domains

$$\begin{cases} \Delta u = \dfrac{1}{M\Delta T} \\ \Delta v = \dfrac{1}{N\Delta Z} \end{cases}$$

- Translation

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

- Rotation

Let $x = r\cos\theta, y = r\sin\theta, u = \omega\cos\varphi, v = \omega\sin\varphi$

then

$$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$$

2D DFT: Periodicity Properties

- Periodicity

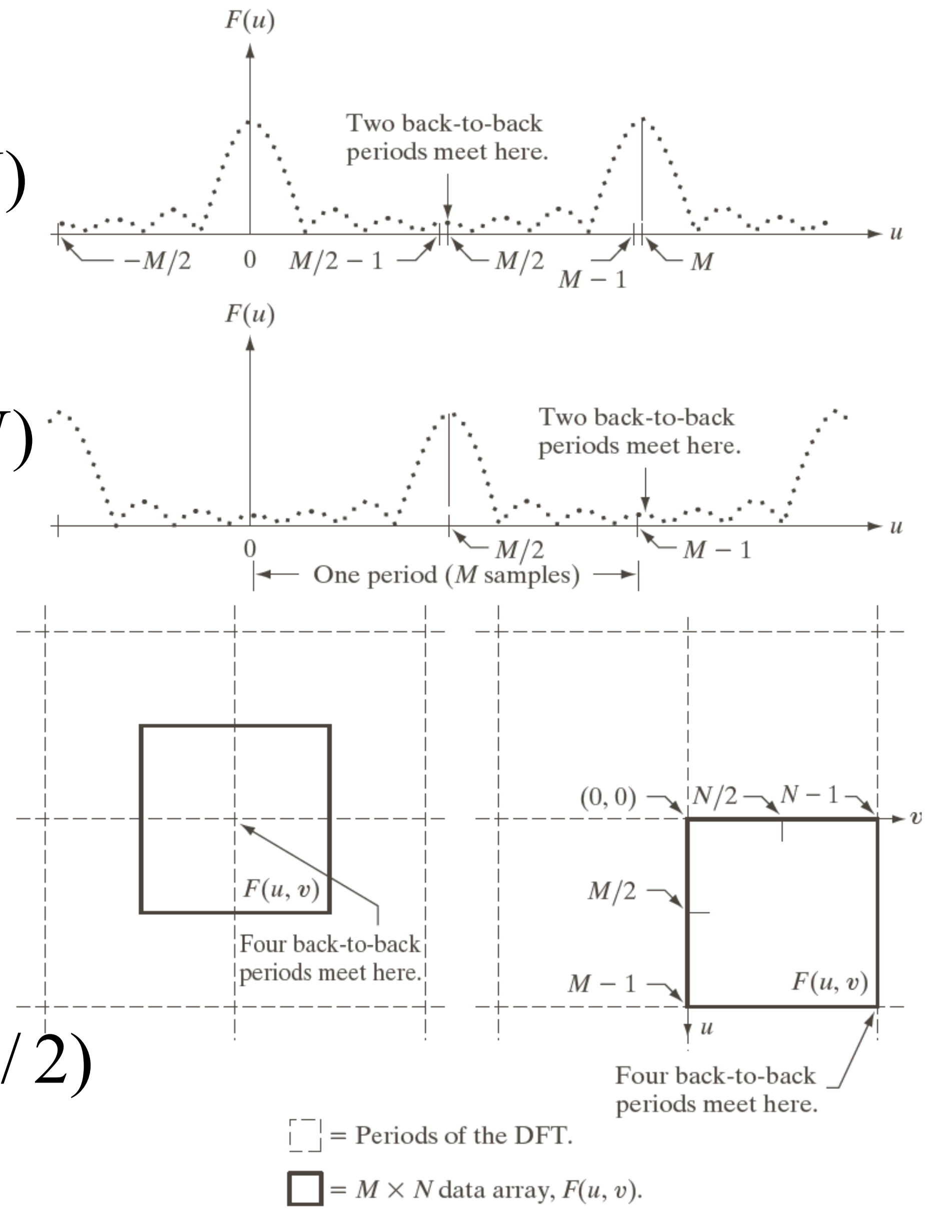
$$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)$$
$$=F(u+k_1M,v+k_2N),$$

$$f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)$$
$$=f(x+k_1M,y+k_2N),$$

- Shifting a signal to its corner

$$f(x)(-1)^x \Leftrightarrow F(u-M/2)$$
$$(u_0=M/2)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2,v-N/2)$$
$$(u_0=M/2,v_0=N/2)$$



Note that $(-1)^x = e^{j\pi x}$.

$$\Im\{f(x,y)(-1)^x\} = \Im\{f(x,y)e^{j\pi x}\} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{j\pi x}e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{j2\pi\frac{xM}{2M}}e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi[\frac{x}{M}(u-\frac{M}{2})+\frac{vy}{N}]}$$

$$= F(u-\frac{M}{2}, v).$$

2D DFT: Symmetry Properties

- Even (or symmetric) functions: $w_e(x,y) = w_e(-x,-y)$
- Odd (or antisymmetric) functions: $w_o(x,y) = -w_o(-x,-y)$
- Any function can be written as the sum of an even function and an odd function:

$$w(x,y) = w_e(x,y) + w_o(x,y)$$

where

$$\begin{cases} w_e(x,y) = \dfrac{w(x,y) + w(-x,-y)}{2} \\ w_o(x,y) = \dfrac{w(x,y) - w(-x,-y)}{2} \end{cases}$$

2D DFT: Symmetry Properties (Cont'd)

- In case of DFT, the signals are periodic. Therefore the definitions are slightly different:

$$w_e(x,y) = w_e(M-x, N-y)$$

$$w_o(x,y) = -w_o(M-x, N-y)$$

- Some facts:
  - Product of two even or two odd functions is even
  - Product of an even and an odd function is odd
  - If a discrete function is odd, then all its samples sum to zero.

  - Therefore, we have:

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} w_e(x,y)\,w_o(x,y) = 0$$

2D DFT: Symmetry Properties (Cont'd)

- Example I: $f = \{f_0, f_1, f_2, f_3\} = \{2,1,1,1\}$
- It can be verified that this is an even function (M=4)
- In general, an even function with 4 items looks like: {a, b, c, b}
- Example II: $f = \{f_0, f_1, f_2, f_3\} = \{0,-1,0,1\}$
- It can be verified that this is an odd function (M=4)
- In general, an odd function with 4 items looks like: {0, -b, 0, b}
- Example III: this 2D array is odd

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

2D DFT: Symmetry Properties (Cont'd)

- Some general properties on DFT
  - If f(x, y) is a real function, then its Fourier transform is conjugate symmetric:

$$F^*(u,v) = F(-u,-v)$$

  - If f(x, y) is an imaginary function, then its Fourier transform is conjugate antisymmetric:

$$F^*(-u,-v) = -F(u,v)$$

  - Proof of the second statement:

$$F^*(-u,-v) = \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(-ux/M-vy/N)}\right]^*$$

$$= \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y)e^{-j2\pi(ux/M+vy/N)}\right] = -\left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}\right] = -F(u,v)$$

2D DFT: Symmetry Properties (Cont'd)

- A summary

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

Proof of Property 8

- Starting from the DFT definition:

$$F(u,v) = \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)} \right]$$

$$= \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_r(x,y) e^{-j2\pi(ux/M)} e^{-j2\pi(vy/N)} \right]$$

$$= \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (even)(even - j \cdot odd)(even - j \cdot odd) \right]$$

$$= \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (even)(even \cdot even - 2j \cdot odd \cdot even - odd \cdot odd) \right]$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} even \cdot even - 2j \sum_{x=0}^{M-1} \sum_{y=0}^{M-1} odd \cdot even - \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} even \cdot even = real$$

$$\& \ even$$

Proof of Property 6

- From the definition of DFT, we can write:

$$\Im\{f(-x,-y)\} = \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(-x,-y)e^{-j2\pi(ux/M+vy/N)}\right]$$

The periodicity implies that $f(-x,-y) = f(M-x, N-y)$

Let's define $m = M-x, n = N-y$. then we have:

$$\Im\{f(-x,-y)\} = \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(u(M-m)/M+v(N-n)/N)}\right]$$

$$= \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(u+v)-j2\pi(-um/M-vn/N)}\right]$$

$$= \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(-um/M-vn/N)}\right] = F(-u,-v)$$