CHAPTER 5

Interest Rates

Chapter Synopsis

5.1 Interest Rate Quotes and Adjustments

Interest rates can compound more than once per year, such as monthly or semiannually. An **annual percentage rate** (APR) equals the periodic interest rate, *r,* times the number of compounding periods per year, *k.* Because it does not include the effect of compounding, an APR understates the amount of interest that will be received if interest compounds more than once per year.

To compute the actual amount of interest earned in one year, an APR can be converted to an **effective annual rate** (EAR), which includes the effect of compounding and provides a measure of the amount of interest that will actually be earned over a year:

Converting an APR to an EAR

$$\text{EAR} = \left(1 + \frac{\text{APR}}{k}\right)^k - 1$$

The more compounding periods, the greater the EAR. For example, suppose a bank offers a certificate of deposit with an interest rate of "6% APR with monthly compounding." In this case, you will earn 6% / 12 = 0.5% every month. To determine the value of $100 invested for one year, you can either compound over 12 months at the monthly rate of 0.5% or you can compound over one year at the EAR = $(1 + .06/12)^{12} - 1$ = 6.17%:

$$FV_1 = \$100(1.005)^{12} = \$100(1.0617) = \$106.17.$$

Many loans, such as home mortgages and car loans, have monthly payments and are quoted in terms of an APR with monthly compounding. These types of loans are typically **amortizing loans** in which each month's payment includes the interest that accrues that month along with some part of the loan's balance. Each monthly payment is the same, and the loan is fully repaid with the final payment. Since the loan balance declines over time, the interest portion of the payment declines over time while the principal repayment portion increases. The number of compounding periods is generally equal to the number of payments per year by convention.

For example, suppose you are offered a $30,000 car loan at "6.75% APR for 60 months." You can find the monthly payment using the PV of an annuity equation:

$$\$30{,}000 = C\frac{1}{.005625}\left[1-\frac{1}{(1.005625)^{60}}\right] \Rightarrow C = \$590.50$$

In the first month, interest equals $30,000(0.005625) = $168.75 and the loan's balance is reduced by $590.50 – 168.75 = $421.75 to $29,578.25. In the second month, interest equals $29,578.25(0.005625) = $166.38 and the loan's balance is reduced by $590.50 – 166.38 = $424.12 to $29,154.13. This process continues until the beginning of the 60th month when the loan balance will be $578.20, so interest equals $587.20(0.005625) = $3.30 and the loan's balance is reduced by $590.50 – 3.30 = $587.20 to $0. A tabular depiction of this process is called an **amortization table.**

5.2 Application: Discount Rates and Loans

To calculate a loan payment, you first equate the outstanding loan balance with the present value of the loan payments using the discount rate from the quoted interest rate of the loan and then solve for the loan payment. Many loans, such as mortgages and car loans, have monthly payments and are quoted in terms of an APR with monthly compounding. These types of loans are amortizing loans, which means that each month you pay interest on the loan plus some part of the loan balance. Each monthly payment is the same, and the loan is fully repaid with the final payment.

Typical terms for a new car loan might be "6.75% APR for 60 months." This quote means that the loan will be repaid with 60 equal monthly payments, computed using a 6.75% APR with monthly compounding. The payment, *C,* is set so that the present value of the cash flows, evaluated using the loan interest rate, equals the original principal amount of $30,000. So, using the annuity formula to compute the present value of the loan payments, the payment *C* must satisfy

$$C \times \frac{1}{0.005625}\left(1-\frac{1}{(1+0.005625)^{60}}\right) = 30{,}000$$

and therefore, $$C = \frac{30{,}000}{\frac{1}{0.005625}\left(1-\frac{1}{(1+0.005625)^{60}}\right)} = \$590.50$$

Alternatively, we can solve for the payment *C* using the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	60	0.5625%	30,000		0	
Solve for PMT				**−590.50**		=PMT(0.005625,60,30000,0)

5.3 The Determinants of Interest Rates

Nominal interest rates, which indicate the actual rate at which interest will accrue, are typically stated in loan agreements and quoted in financial markets. If prices in the economy are also growing due to inflation, the nominal interest rate does not represent the increase in purchasing power that will result from investing at this rate. The rate of growth of purchasing power, after adjusting for inflation, is determined by the **real interest rate,** r_r. If r is the nominal interest rate and i is the rate of inflation, the real rate can be calculated as follows.

The Real Interest Rate

$$r_r = \frac{r - i}{1 + i}$$

Interest rates affect firms' incentives to raise capital and invest as well as individuals' propensities to save. For example, an increase in interest rates will generally decrease an investment's NPV and reduce the number of positive-NPV investments available to firms. The U.S. Federal Reserve as well as central banks in other countries use this idea to try and influence economic activity. Central banks can lower interest rates to stimulate investment if the economy is slowing and raise interest rates to reduce investment if the economy is perceived to be growing too fast ,which may lead to an increase in the inflation rate.

Interest rates generally depend on the horizon, or term, of the investment or loan. The relation between an investment's term and its interest rate is called the **term structure of interest rates,** and it can be plotted on a graph called the **yield curve.** Common equations used for computing present values, such as the annuity and perpetuity formulas, are based on discounting all of the cash flows at the same rate. In situations in which cash flows need to be discounted at different rates depending on when they occur, the following equation can be used:

$$\text{PV} = \frac{C_1}{1+r_1} + \frac{C_2}{1+r_2} \cdots \frac{C_N}{1+r_N} = \sum_{n=0}^{N} \frac{C_n}{\left(1+r_n\right)^n}$$

The Federal Reserve determines short-term interest rates through its influence on the **federal funds rate**, which is the rate at which banks can borrow cash reserves over one night. All other interest rates on the yield curve are set in the market and are adjusted until the supply of lending matches the demand for borrowing at each loan term. Expectations of future interest rate changes have a major effect on investors' willingness to lend or borrow for longer terms and, therefore, on the shape of the yield curve. An increasing yield curve, with long-term rates higher than short-term rates, generally indicates that interest rates are expected to rise in the future. A decreasing (inverted) yield curve, with long-term rates lower than short-term rates, generally signals an expected decline in future interest rates. Because interest rates tend to drop in response to an economic slowdown, an inverted yield curve is often interpreted as a negative economic forecast.

5.4 Risk and Taxes

U.S. Treasury securities are widely regarded as risk-free because there is virtually no chance the U.S. government will fail to pay the interest or default on these bonds; thus, the rate on Treasury securities is often referred to as the **risk-free rate.** All other borrowers are generally assumed to have some risk of default. For these loans, the stated interest rate is the maximum amount that investors will receive. Investors may receive less if the company is unable to fully repay the loan. To compensate for the risk that they will receive less if the firm defaults, investors demand a higher interest rate than the rate on U.S. Treasuries. The difference between the interest rate of the loan and the Treasury rate is called the **credit spread.**

If the cash flows from an investment are taxed, the net cash flow that the investor will receive will be reduced by the amount of the taxes paid. In general, if the interest rate is r and the tax rate is τ , then for each $1 invested you will earn interest equal to r and owe tax of $\tau \times \text{r}$ on the interest. Thus, the equivalent **after-tax interest rate** is $r(1-\tau)$. For example, if an investment pays 8% interest for one year, and you invest \$100 at the start of the year, you will earn 8% × \$100 = \$8 in interest at year-end. If you must pay taxes at 40% on this interest, you will owe 40% × \$8 = \$3.20. Thus you will receive only \$8 − \$3.20 = \$4.80 after paying taxes. This amount is equivalent to earning 4.80% interest and not paying any taxes, so the after-tax interest rate is $r(1-\tau)=8\%(1-.40)=4.80\%$.

5.5 The Opportunity Cost of Capital

The discount rate used to evaluate cash flows is the **cost of capital**, or **opportunity cost of capital,** which is the best available expected return offered in the market on an investment of comparable risk and term to the cash flow being discounted. The cost of capital is the return the investor forgoes when making a new investment. For a risk-free project, it will typically correspond to the interest rate on U.S. Treasury securities with a similar term. For risky projects, it will include a **risk premium.**

Selected Concepts and Key Terms

Amortizing Loan

A loan in which each month you pay interest on the loan plus some part of the loan principal, or amount borrowed. Each monthly payment is the same, and the loan is fully repaid with the final payment. Since the loan balance declines over time, the interest portion of the payment declines over time while the principal repayment portion increases.

Annual Percentage Rate (APR)

The periodic interest rate, *r,* times the number of compounding periods per year, *k.* Because it does not include the effect of compounding, the APR quote is less than the actual amount of interest that will be received if $k > 1$.

Opportunity Cost of Capital

The best available expected return offered in the market on an investment of comparable risk and term to the cash flow being discounted. The cost of capital is the return the investor forgoes when the making a new investment. For a risk-free project, it will typically correspond to the interest rate on U.S. Treasury securities with a similar term. For risky projects, it will include a risk premium.

Credit Spread

The difference between the interest rate of the loan and the risk-free Treasury security rate.

Effective Annual Rate (EAR)

The amount of interest that will be earned over a year. The more compounding periods, the greater the EAR for a given APR.

Nominal Interest Rate

The actual rate at which money will grow. Nominal rates are typically stated in loan agreements and quoted in financial markets. If prices in the economy are also growing due to inflation, the nominal interest rate does not represent the increase in purchasing power that will result from investing at the nominal rate.

Real Interest Rate

The rate of growth of purchasing power after adjusting for inflation.

Term Structure

The relation between an investment's term and its interest rate is called the term structure of interest rates, and it can be plotted on a graph called the yield curve.

Concept Check Questions and Answers

5.1.1. What is the difference between an EAR and an APR quote?

An annual percentage rate is the rate that interest earns in one year before the effect of compounding. An effective annual rate is the rate that the amount of interest actually earns at the end of one year. Because the APR does not include the effect of compounding, it is typically less than the EAR.

5.1.2. Why can't the APR be used as a discount rate?

Because the APR does not reflect the true amount you will earn one year, the APR itself cannot be used as a discount rate.

5.2.1. How can you compute the outstanding balance on a loan?

The outstanding balance can be computed by constructing an amortization table or by finding the present value of the remaining payments.

5.2.2. What is an amortizing loan?

It is a loan in which each month you pay interest on the loan plus some part of the loan principal, or amount borrowed. Each monthly payment is the same, and the loan is fully repaid with the final payment. Since the loan balance declines over time, the interest portion of the payment declines over time while the principal repayment portion increases.

5.3.1. What is the difference between a nominal and real interest rate?

The nominal interest rate is the rate quoted by banks and other financial institutions, whereas the real interest rate is the rate of growth of purchasing power, after adjusting for inflation. The real interest rate is approximately equal to the nominal rate less the rate of inflation.

5.3.2. How do investors' expectations of future short-term interest rates affect the shape of the current yield curve?

The shape of the yield curve tends to vary with investors' expectations of future economic growth and interest rates. It tends to be inverted prior to recessions and to be steep coming out of a recession.

5.4.1. Why do corporations pay higher interest rates on their loans than the U.S. government?

Corporations pay higher interest rates on their loans than the U.S. government does because all corporations have some risk of default, while there is virtually no chance the U.S. government will fail to pay the interest or default on the loans.

5.4.2. How do taxes affect the interest earned on an investment? What about the interest paid on a loan?

The interest the investor earned on an investment is taxable and will be reduced by the amount of the tax payments. In some cases, since the interest on loans is tax deductible, the cost of paying interest on the loan is offset by the benefit of the tax deduction.

5.5.1. What is the opportunity cost of capital?

The opportunity cost of capital is the best available return offered in the market on an investment of comparable risk and term to the cash flow being discounted.

5.5.2. Why do different interest rates exist, even in a competitive market?

The interest rates we observe in the market will vary based on quoting conventions, the term of investment, and risk. The actual return kept by an investor will also depend on how the interest is taxed.

Examples with Step-by-Step Solutions

Solving Problems

Problems using the concepts in this chapter often involve solving problems using the valuation equations in Chapter 4. It is helpful to represent the cash flows involved on a timeline, and it is important to use the correct periodic interest rate. For instance, in example 1 below, the loan involves monthly payments at a 6% APR with monthly compounding, so the correct rate to use is the monthly rate = 6% / 12 = 0.5%. Other problems may involve finding real cash flows. To do this, it is generally necessary to calculate and use real interest rates using the relation between nominal rates, real rates and inflation. Example 2 below provides such an example. Finally, problems may involve understanding the mechanics of an amortizing loan, as in example 3 below.

Examples

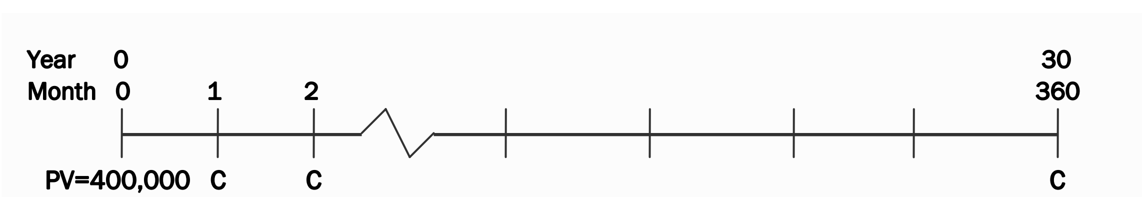
1. You want to buy a vacation house in Hood River, Oregon, by borrowing $400,000.

[A] If you obtain a 30-year loan at 6% APR with monthly compounding, what is your monthly payment? How much goes to interest and how much to principal over the loan's life?

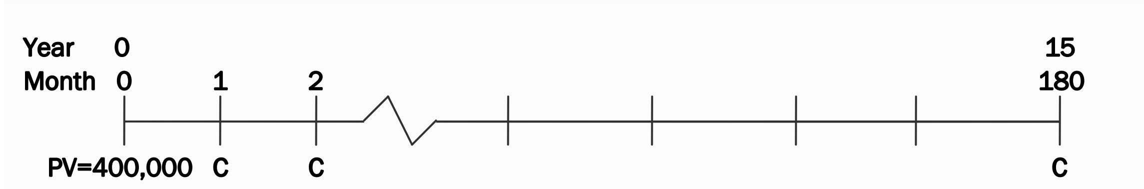
[B] If you obtain a 15-year loan at 6% APR with monthly compounding, what is your monthly payment? How much goes to interest and how much to principal over the loan's life?

Step 1: Put the known and unknown cash flows on a timeline.

The 30-year Loan.



The 15-year Loan.



Step 2: Since this problem involves the present value of an annuity in which *C* is unknown, set the PV of annuity equation equal to $400,000 with r = 0.05 and N = 360 months for part [A] and 180 months for part [B]. Solve for C to get each loan's payment.

$$[\text{A}]\ \$400{,}000 = C\left[\frac{1}{.005} - \frac{1}{.005(1.005)^{360}}\right] \Rightarrow\ C = \$2{,}398.20$$

$$[\text{B}]\ \$400{,}000 = C\left[\frac{1}{.005} - \frac{1}{.005(1.005)^{180}}\right] \Rightarrow C = \$3{,}375.43$$

Step 3. Once the payment is calculated, you can calculate the total interest paid as the total payments made minus the $400,000 principal that was paid.

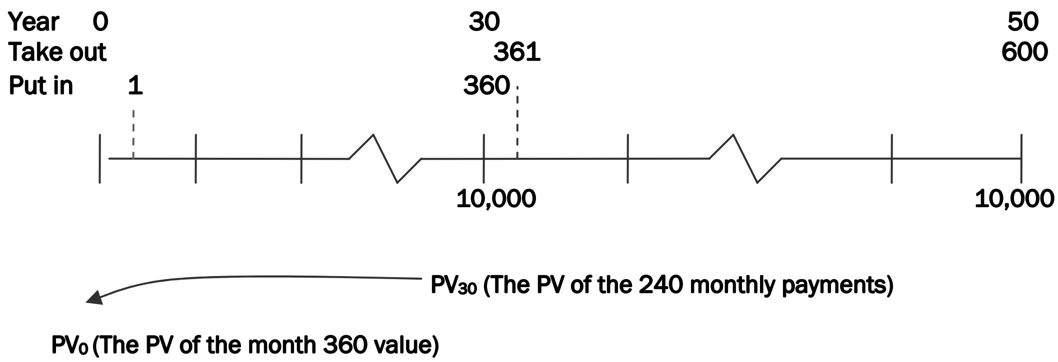
[A] Total = 360(2,398.20) = 863,353
−Principal = 400,000
= Interest = 463,353

[B] Total = 180(3,375.48) = 607,577
−Principal = 400,000
= Interest = 207,577

Thus, the payment and interest paid is $2,398 and $463,353 for the 30-year loan and $3,375 and $207,577 for the 15-year loan.

2. You plan on drawing on retirement income exactly 30 years from today. You have liquidated $1 million worth of investments and are going to move to Spain, taking whatever cash is left after funding your retirement account. You want the retirement account to pay exactly $10,000 per month in today's (real) dollars for 20 years. You are going to place enough into an account yielding a nominal 12% APR, or 1% per month. At retirement, you will place the accumulated money into an account yielding a nominal 6% APR, which compounds monthly at 0.5%, and remove cash from this account according to the payment schedule. You expect that inflation will be 0.25% per month. What payment to the 12% account is required to fund the retirement plan?

Step 1. Put the cash flows that are known and unknown on a timeline.



Step 2. Determine the type or types of valuation problems involved.

This problem can be solved using the following additional steps 3–6:

[3] find the real monthly interest rates in the first 30 years and last 20 years,
[4] find the present value in real terms in year 30 of the $10,000 payments in months 361–600,
[5] find the present value of the year 30 value found in step 4,
[6] subtract the value in step 5 from the $1 million.

Step 3. The real rates are:

$$r_{\text{real}}^{\text{Years 30-50}} = \frac{r-i}{1+i} = \frac{.005-.0025}{1+.025} \Rightarrow r_{\text{real}} \approx 0.0025 = 0.25\% \text{ per month.}$$

$$r_{\text{real}}^{\text{Years 1-30}} = \frac{r-i}{1+i} = \frac{.01-.0025}{1+.025} \Rightarrow r_{\text{real}} \approx 0.75\% \text{ per month.}$$

Step 4. Using the present value of an annuity equation:

$$PV_{30} = 10{,}000\left[\frac{1}{.0025} - \frac{1}{.0025(1.0025)^{240}}\right] = \$1{,}803{,}109 \text{ in real dollars.}$$

Step 5. Using the PV of a single cash flow equation:

$$PV = \frac{1{,}803{,}109}{(1.0075)^{360}} = \$122{,}406.$$

Step 6. Make a conclusion.

You can take $\$1{,}000{,}000 - \$122{,}406 = \$877{,}594$ with you and put the rest in the account.

3. The Boston Beer Company is shopping for a new bottling machine. The machine has a manufacturer's suggested retail price of $350,000.

[A] Dealer A offers to sell them the machine for $290,000 with a 6% APR monthly amortizing 10-year loan. Dealer B will charge the full $350,000 but offers them 0% APR monthly payment loan with financing over 10 years. Which of these two options is a better deal?

[B] If they decide to buy the machine from dealer A, how much of the first two payments goes to paying down the $290,000 principal? How much is interest?

[C] If they decide to buy the machine from dealer A and sell it in three years, how much must they sell it for in order to pay back the remaining balance of the loan? (Ignore tax effects.)

Step 1. To answer part [A], you need to determine which payment option has the lowest present value.

Since both options have the same term and monthly payments, this is the same as finding which option has the lowest payment.

Using the PV of an annuity equation:

$$\$290{,}000 = C\left[\frac{1}{.005} - \frac{1}{.005(1.005)^{120}}\right] \Rightarrow C = \$3{,}219.59$$ is the payment for dealer A.

$$\frac{\$350{,}000}{120} = \$2{,}916.67$$ is the payment for dealer B.

Also, note that the present value of dealer B's payments at 6% APR is $262,714 which is less than the present value of dealer A's payments at 6% APR, $290,000.

Thus, you should select dealer B.

Step 2. To answer part [B] of the problem, you can construct the first two months of an amortization table with a payment of $3,219.59, an original balance of $290,000, and a periodic (monthly) interest rate of 6% / 12 = 0.50%:

Month	Principal	Interest=0.005 x Principal	Payment	Ending Balance
1	290,000.00	1,450.00	3,219.59	288,230.41
2	288,230.41	1,441.15	3,219.59	286,451.97

Now, calculate the principal repaid = 290,000 – 286,451.97 = $3,548.03.

After two months, the total principal repaid is $3,548.03 and the total interest paid is 2(3,219.59) – $3,548.03 = 1,450.00 + 1,441.15 = $2,891.15.

Step 3. To answer part [C] you could construct an amortization table, but without the aid of a spreadsheet this would be too time-consuming. Thus, you can solve for the present value of the remaining payments, which must be the remaining balance of the loan.

Using the present value of an annuity equation with C = $3,219.59, r = 6%/12 = 0.5%, and N = 120 – 36 = 84:

$$\text{PV} = \frac{3{,}219.59}{0.005}\left(1 - \frac{1}{(1.005)^{84}}\right) = \$220{,}390.73.$$

Thus, they must sell it for at least $220,390.73 or else they will have to pay off some of the balance with cash from a different source.

Questions and Problems

1. You won $1 in million the Lottery. The prize is paid out in equal, semi-annual payments over 50 years with the first payment immediately. GenexCapital.com has offered to buy the ticket for $250,000 in cash today. In the contract, they claim to be using an 8% APR with semi-annual compounding. Are they? (Ignore taxes)

2. You have a $50,000 balance on your credit card, and you have set your Wells Fargo checking account bill pay for monthly payments of $1,000. The interest rate is 18% APR with monthly

compounding. How many years until you have paid it off? How long would it take if your balance was $70,000?

3. You are considering paying for a 2006 Mercedes SLK 350 with an MSRP of $50,000 using a 5-year loan. Based on the MSRP, the dealer's finance manager has quoted you a zero down, 4.8% APR (compounded monthly) loan with a payment of $966.64 and your first payment is due one month from today.
   [A] Is the rate you would be paying really 4.8% APR?
   [B] For every $500 that you get the dealer to lower the price of the car at a 4.8% APR, how much does your monthly payment decrease?
   [C] Based on a price of $45,000, how much would your down payment need to be to make your payments equal $700 per month at 4.8% APR?

4. You have decided to refinance your mortgage. You plan to borrow whatever is outstanding on your current mortgage. The current monthly payment is $5,200, and there are exactly 27 years left on the loan. You have just made your 36th monthly payment and the mortgage interest rate is 6% APR. How much do you owe on the mortgage today?

5. You have just sold your house for $2,000,000. Your mortgage was originally a 30-year mortgage with monthly payments, and an initial balance of $400,000. The mortgage is exactly 10 years old, and you have just made a monthly payment. If the fixed interest rate on the mortgage is 3.6% (APR), how much will you have from the sale once you pay off the mortgage?

Solutions to Questions and Problems

1. If they are paying 8% APR (4% per six months), then the PV of the annuity payments at this rate must be $250,000.

$$PV = \$10{,}000\left[\frac{1}{.04} - \frac{1}{.04(1.04)^{99}}\right] + 10{,}000 = \$254{,}852 > \$250{,}00.$$

   Since they are paying less than $250,000, they are using a bit higher rate.

   The actual rate is: $\$10{,}000\left[\frac{1}{r} - \frac{1}{r(1+r)^{99}}\right] + 10{,}000 = \$250{,}000 \Rightarrow \text{APR} \approx 8.175\%$.

2. This is a present value of an annuity problem in which you must solve for N.

$$\$1{,}000\left[\frac{1}{.015} - \frac{1}{.015(1.015)^{N}}\right] = \$50{,}000$$

   $\Rightarrow$ T = 93.11 months, or about 7 years and 10 months.

   For the $70,000 balance, note that if you paid $1,000 in perpetuity:

   $\left(\frac{\$1{,}000}{.015}\right) = \$66{,}667$, so you could never pay it off, since $\$70{,}000 > \$66{,}667$.

3. [A] The actual payment at 4.8% APR would be:

$$50{,}000 = C\left[\frac{1}{.004} - \frac{1}{.004(1.004)^{60}}\right] \Rightarrow C = \$938.99.$$

So the actual rate is higher. The implied rate in the payment can be found as follows:

$$50{,}000 = 966.64\left[\frac{1}{r} - \frac{1}{r(1+r)^{60}}\right] \Rightarrow r = 0.5\%, \text{ or } 6\% \text{ APR}.$$

[B] $$\$500 = C\left(\frac{1}{.004} - \frac{1}{.004(1.004)^{60}}\right) \Rightarrow C = \$9.39.$$

So, for every \$500 reduction, the payment would decrease by \$9.39.

[C] $$\$45{,}000 = 700\left(\frac{1}{.004} - \frac{1}{.004(1.004)^{60}}\right) + \text{Down Payment}$$

$$\Rightarrow \text{Down Payment} = \$45{,}000 - \$37{,}274 = \$7{,}726$$

4. To find out what is owed, compute the PV of the remaining payments:

$$\text{PV} = \frac{5{,}200}{0.005}\left(1 - \frac{1}{(1.005)^{324}}\right) = \$833{,}352.89.$$

5. First compute the original loan payment:

$$C = \frac{400{,}000 \times 0.003}{\left(1 - \frac{1}{(1.003)^{360}}\right)} = \$1{,}818.58.$$

Now compute the PV of continuing to make these payments.

Using the formula for the PV of an annuity:

$$\text{PV} = 1{,}818.58\left(\frac{1}{0.003} - \frac{1}{(1.003)^{240}}\right) = \$310{,}809.15.$$

So you would get to keep \$2,000,000 – \$310,809.15 = \$1,689,190.85.