CHAPTER 11

Optimal Portfolio Choice and the Capital Asset Pricing Model

Chapter Synopsis

11.1 The Expected Return of a Portfolio

The expected return on an n-asset portfolio is simply the weighted-average of the expected returns of the portfolio's components:

$$R_P = x_1R_1 + x_2R_2 + \cdots + x_nR_n = \sum_{i=1}^{n} x_iR_i \Rightarrow E[R_P] = \sum_{i=1}^{n} x_iE[R_i].$$

where x_i is the value of asset i divided by the portfolio's total value.

11.2 The Volatility of a Two-Stock Portfolio

The standard deviation of an *n*-asset portfolio is generally less than the weighted-average of the standard deviations of the portfolio's components. When two or more stocks are combined in a portfolio, some of their risk will generally be eliminated through diversification. The amount of risk that will remain depends on the degree to which the stocks share systematic risk. Thus the risk of a portfolio depends on more than the risk and return of the component stocks, and the degree to which the stocks' returns move together is important; their **covariance** or **correlation** must be considered:

- Covariance is the expected product of the deviation of each return from its mean, which can be measured from historical data as:

$$\text{Cov}(R_i, R_j) = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - \overline{R}_i)(R_{j,t} - \overline{R}_j).$$

- Correlation is the covariance of the returns divided by the standard deviation of each return:

$$\text{Corr}(R_i, R_j) = \frac{\text{Cov}(R_i, R_j)}{\text{SD}(R_i) \times \text{SD}(R_j)}.$$

Correlation is generally easier to interpret because it always lies between –1 and 1. The closer the correlation is to 1, the more the returns tend to move together. Assets with zero correlation move independently; assets with –1.0 correlation, move in opposite directions.

The variance of a two-asset portfolio can now be calculated as:

$$\begin{aligned}\text{Var}(R_P) &= x_1^2\text{Var}(R_1) + x_2^2\text{Var}(R_2) + 2x_1x_2\text{Cov}(R_1, R_2), \text{ or} \\ &= x_1^2\text{Var}(R_1) + x_2^2\text{Var}(R_2) + 2x_1x_2\text{Cor}(R_1, R_2)\text{SD}(R_1)\text{SD}(R_2).\end{aligned}$$

As can be seen from the equation, only when the correlation between the two stocks is exactly 1.0 is the Var(P) equal to the weighted average of the stocks' variances—otherwise it is less. The portfolio's standard deviation equals $\sqrt{\text{Var}(P)}$ and is often referred to as **volatility**.

11.3 The Volatility of a Large Portfolio

The variance of an equally-weighted n-asset portfolio is:

$$\text{Var}(R_P) = \frac{1}{n}(\textit{Average}\text{ asset variance}) + \left(1 - \frac{1}{n}\right)(\textit{Average}\text{ covariance between assets}).$$

Thus, as the number of stocks, n, grows large, the variance of the portfolio is determined primarily by the average covariance among the stocks (because $1 - \frac{1}{n} \to 1$) while the average variance becomes unimportant (because $\frac{1}{n} \to 0$). Therefore, the risk of a stock in a diversified portfolio depends on its contribution to the portfolio's average covariance.

11.4 Risk Versus Return: Choosing an Efficient Portfolio

The set of efficient portfolios, which offer investors the highest possible expected return for a given level of risk, is called the **efficient frontier**. Investors must choose among the efficient portfolios based on their own preferences for return versus risk.

A positive investment in a security is referred to as a **long position**. It is also possible to invest a negative amount in a stock, called a **short position**, by engaging in a **short sale**, a transaction in which you sell a stock that you do not own and then buy that stock back in the future at hopefully a lower price. When investors can use short sales in their portfolios, the portfolio weights on those stocks are negative, and the set of possible portfolios is extended.

11.5 Risk-Free Saving and Borrowing

Portfolios can be formed by combining borrowing and lending at the risk-free rate and by investing in a portfolio of risky assets. The expected return for this type of portfolio is the weighted average of the expected returns of the risk-free asset and the risky portfolio, or:

$$E[R_{xP}] = (1 - x)r_f + xE[R_P] = r_f + x(E[R_P] - r_f)$$

Since the standard deviation of the risk-free investment is zero, the covariance between the risk-free investment and the portfolio is also zero, and the standard deviation for this type of portfolio equals simply $x\mathrm{SD}(R_P)$ since $\mathrm{Var}(r_f)$ and $\mathrm{Cov}(r_f, R_P)$ are zero:

$$\mathrm{SD}[R_{xP}] = \sqrt{(1-x)^2 \mathrm{Var}(r_f) + x^2 \mathrm{Var}(R_P) + 2(1-x)(x)\mathrm{Cov}(r_f, R_P)} = x\mathrm{SD}(R_P).$$

On a graph, the risk–return combinations of the risk-free investment and a risky portfolio lie on a straight line connecting the two investments. As the fraction x invested in the risk portfolio increases from 0% to 100%, you move along the line from 100% in the risk-free investment, to 100% in the risky portfolio. If you increase x beyond 100%, you reach points beyond the risky portfolio in the graph. In this case, you are borrowing at the risk-free rate and buying stocks on **margin.**

To earn the highest possible expected return for any level of volatility, you must find the portfolio that generates the steepest possible line when combined with the risk-free investment. The slope of the line through a given portfolio P is often referred to as the **Sharpe ratio** of the portfolio:

$$\text{Sharpe Ratio} = \frac{\text{Portfolio Excess Return}}{\text{Portfolio Volatility}} = \frac{E[R_P] - r_f}{\mathrm{SD}(R_P)}.$$

The portfolio with the highest Sharpe ratio is called the **tangent portfolio**. You can draw a straight line from the risk-free rate through the tangent portfolio that is tangent to the efficient frontier of portfolios.

When the risk-free asset is included, all efficient portfolios are combinations of the risk-free investment and the tangent portfolio—no other portfolio that consists of only risky assets is efficient. Therefore, the optimal portfolio of risky investments no longer depends on how conservative or aggressive the investor is. All investors will choose to hold the same portfolio of risky assets, the tangent portfolio, combined with borrowing or lending at the risk-free rate.

11.6 The Efficient Portfolio and Required Returns

The beta of an investment with a portfolio is defined as:

$$\beta_i^P \equiv \frac{\mathrm{Cov}(R_i, R_P)}{\mathrm{Var}(R_P)} = \frac{\mathrm{SD}(R_i) \times \cancel{\mathrm{SD}(R_P)} \times \mathrm{Corr}(R_i, R_P)}{\mathrm{SD}(R_P) \times \cancel{\mathrm{SD}(R_P)}} = \frac{\mathrm{SD}(R_i) \times \mathrm{Corr}(R_i, R_P)}{\mathrm{SD}(R_P)}.$$

Beta measures the sensitivity of the investment's return to fluctuations in the portfolio's return. Buying shares of security i improves the performance of a portfolio if its expected return exceeds the required return:

$$\text{Required Return} \equiv r_i = r_f + \beta_i^P (E[R_P] - r_f).$$

A portfolio is efficient when $E[R_i] = r_i$ for all securities and the following relation holds:

$$E[R_i] = r_i \equiv r_f + \beta_i^{\text{efficient portfolio}} (E[R_{\text{efficient portfolio}}] - r_f).$$

If investors hold portfolio P, this equation yields the return that investors will require to add the asset to their portfolio—i.e. the investment's **cost of capital**. Buying shares of a security improves the performance of a portfolio if its expected return exceeds the required return based on its beta with the portfolio.

11.7 The Capital Asset Pricing Model

The **Capital Asset Pricing Model** (CAPM) makes three assumptions.

- Investors can trade all securities at competitive market prices without incurring taxes or transactions costs, and can borrow and lend at the same risk-free interest rate.
- Investors hold only efficient portfolios of traded securities—portfolios that yield the maximum expected return for a given level of volatility.
- Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities.

It follows that each investor will identify the same efficient portfolio of risky securities, which is the portfolio that has the highest Sharpe ratio in the economy. Because every security is owned by someone, the holdings of all investors' portfolios must equal the portfolio of all risky securities available in the market, which is known as the **market portfolio.**

Thus, when the CAPM assumptions hold, choosing an optimal portfolio is relatively straightforward for individual investors—they just choose to hold the same portfolio of risky assets, the market portfolio, combined with borrowing or lending at the risk-free rate.

The **capital market line** (CML) is the set of portfolios with the highest possible expected return for a given level of standard deviation, or volatility. Under the CAPM assumptions, on a graph in which expected return is on the *y*-axis and standard deviation is on the *x*-axis, the CML is the line through the risk-free security and the market portfolio.

11.8 Determining the Risk Premium

When the CAPM holds, a security's expected return is given by:

$$E[R_i] = r_i = r_f + \beta_i^{Mkt}(E[R_{Mkt}] - r_f).$$

Thus, there is a linear relation between a stock's beta and its expected return. On a graph in which expected return is on the *y*-axis and beta is on the *x*-axis, the line that goes through the risk-free investment (with a beta of 0) and the market (with a beta of 1) is called the **security market line** (SML).

The security market line applies to portfolios as well. For example, the market portfolio is on the SML and, according to the CAPM, other portfolios such as mutual funds must also be on the SML, or else they are mispriced. Therefore, the expected return of a portfolio should correspond to the portfolio's beta, which equals the weighted average of the betas of the securities in the portfolio:

$$\beta_P = \sum_i x_i \frac{\text{Cov}(R_i, R_{Mkt})}{\text{Var}(R_{Mkt})} = \sum_i x_i \beta_i .$$

While the CAPM is based on rather strong assumptions, financial economists find the intuition underlying the model compelling, so it is still the most commonly used to measure systematic risk. It is also the most popular method used by financial managers to estimate their firm's equity cost of capital.

Appendix: The CAPM with Differing Interest Rates

In this chapter, we assume that investors face the same risk-free interest rate whether they are saving or borrowing. However, in practice, investors generally receive a lower rate when they save than they must pay when they borrow. Thus, if borrowing and lending rates differ, investors with different preferences will choose different portfolios of risky securities. So, the first conclusion of the CAPM—that the market portfolio is the unique efficient portfolio of risky investments—is no longer valid.

The more important conclusion of the CAPM for corporate finance is the security market line, which is still valid when interest rates differ. This is because a combination of portfolios on the efficient frontier of risky investments is also on the efficient frontier of risky investments. Because all investors hold portfolios on the efficient frontier, and because all investors collectively hold the market portfolio, the market portfolio must lie on the frontier. As a result, the market portfolio will be tangent for some risk-free interest rate between the efficient portfolios based on different borrowing and lending rates.

Selected Concepts and Key Terms

Correlation

The covariance of two assets' returns divided by the product of the standard deviations of the assets' returns. Correlation is generally easier to interpret than covariance because it always lies between –1 and 1. The closer the correlation is to 1, the more the returns tend to move together. Assets with zero correlation move independently; assets with –1 correlation, move in opposite directions.

Covariance

The expected product of the deviation of each return from its mean, which can be measured from historical data as:

$$\mathrm{Cov}(R_i, R_j) = \frac{1}{T-1} \sum_{t=1}^{T} (R_{i,t} - \overline{R}_i)(R_{j,t} - \overline{R}_j).$$

Efficient Portfolio, Efficient Frontier

An **efficient portfolio** offers investors the highest possible expected return for a given level of risk. The set of efficient portfolios for different levels of risk is called the **efficient frontier**. As investors add stocks to a portfolio, the efficient portfolio generally improves. Since efficient portfolios cannot be easily ranked, investors must choose based on their own preferences for return versus risk.

Capital Market Line (CML).

The set of portfolios with the highest possible expected return for a level of standard deviation, or volatility. Under the CAPM assumptions, on a graph in which expected return is on the *y*-axis and standard deviation is on the *x*-axis, the CML is the line through the risk-free security and the market portfolio.

Security Market Line (SML)

The line that goes through the risk-free investment (with a beta of 0) and the market (with a beta of 1). It is the equation implied by the Capital Asset Pricing Model, and shows the linear relation between the expected return and systematic risk, as measured by beta.

Sharpe Ratio

The ratio of a portfolio's excess return (or expected excess return) to the portfolio's standard deviation, or volatility. The Sharpe ratio was first introduced by William Sharpe as a measure to compare the performance of mutual funds. See William Sharpe, "Mutual Fund Performance," *Journal of Business* (January 1966): 119–138.

Short Position

A transaction in which you sell a stock that you do not own and then buy that stock back in the future at hopefully a lower price. When investors can use short sales in their portfolios, the portfolio weights on those stocks is negative, and the set of possible portfolios is extended.

Tangent Portfolio

The portfolio with the highest Sharpe ratio. Because the tangent portfolio has the highest Sharpe ratio of any portfolio, the tangent portfolio provides the biggest reward per unit of volatility of any portfolio available.

Volatility

A term often used to refer to an asset's or portfolio's standard deviation.

Concept Check Questions and Answers

11.1.1. What is a portfolio weight?

Portfolio weights represent the fraction of the total investment in the portfolio of each individual investment in the portfolio.

11.1.2. How do we calculate the return on a portfolio?

The return on the portfolio is the weighted average of the returns on the investments in the portfolio, where the weights correspond to portfolio weights.

11.2.1. What does the correlation measure?

Correlation measures how returns move in relation to each other. It is between +1 (returns always move together) and –1 (returns always move oppositely).

11.2.2. How does the correlation between the stocks in a portfolio affect the portfolio's volatility?

The lower the correlation between stocks, the lower the portfolio's volatility.

11.3.1. How does the volatility of an equally weighted portfolio change as more stocks are added to it?

The variance of an equally-weighted portfolio equals:

$$\begin{aligned}\text{Var}(R_P) &= \frac{1}{n}(\text{Average Variance of the Individual Stocks}) \\ &\quad + \left(1 - \frac{1}{n}\right)(\text{Average Covariance between the Stocks}).\end{aligned}$$

As the number of stocks grows large, the variance of the portfolio is determined primarily by the average covariance among the stocks.

11.3.2. How does the volatility of a portfolio compare with the weighted average volatility of the stocks within it?

The weighted average volatility of the stocks within a portfolio is greater than the volatility of the portfolio, unless the stocks are perfectly correlated.

11.4.1. How does the correlation between two stocks affect the risk and return of portfolios that combine them?

Correlation between two stocks has no effect on the expected return of the portfolios combining them; however, the volatility of the portfolios will differ depending on the correlation. In particular, the lower the correlation, the lower the volatility of the portfolios.

11.4.2. What is the efficient frontier?

Efficient portfolios offer investors the highest possible expected return for a given level of risk. The set of efficient portfolios is called the efficient frontier.

11.4.3. How does the efficient frontier change when we use more stocks to construct portfolios?

As investors add stocks to a portfolio, the efficient frontier improves.

11.5.1. What do we know about the Sharpe ratio of the efficient portfolio?

The Sharpe ratio of a portfolio is the portfolio excess return divided by the portfolio volatility. The Sharpe ratio measures the ratio of reward-to-volatility provided by a portfolio. The efficient portfolio is the portfolio with the highest Sharpe ratio in the economy.

11.5.2. If investors are holding optimal portfolios, how will the portfolios of a conservative and an aggressive investor differ?

Both will own portfolios that combine investing in the same efficient portfolio and borrowing or lending at the risk-free rate. Aggressive investors will invest a greater portion of their portfolio in the efficient portfolio.

11.6.1. When will a new investment improve the Sharpe ratio of a portfolio?

Increasing the amount invested in an investment will increase the Sharpe ratio of portfolio *P* if its expected return *E*[*Ri*] exceeds its required return given portfolio *P,* defined as:

$$r_i \equiv r_f + \beta_i^P \times (E[R_P] - r_f)\ .$$

11.6.2. An investment's cost of capital is determined by its beta with what portfolio?

The efficient portfolio, which is the portfolio with the highest Sharpe ratio of any portfolio in the economy.

11.7.1. Explain why the market portfolio is efficient according to the CAPM.

Since every investor is holding the tangent portfolio, the combined portfolio of risky securities of all investors must also equal the tangent portfolio. Furthermore, because every security is owned by someone, the sum of all investors' portfolios must equal the portfolio of all risky securities available in the market. Therefore, the efficient tangent portfolio of risky securities (the portfolio that all investors hold) must equal the market portfolio.

11.7.2. What is the capital market line (CML)?

The set of portfolios with the highest possible expected return for a level of standard deviation, or volatility. Under the CAPM assumptions, on a graph in which expected return is on the *y*-axis and standard deviation is on the *x*-axis, the CML is the line through the risk-free security and the market portfolio.

11.8.1. What is the security market line (SML)?

The line that goes through the risk-free investment (with a beta of 0) and the market (with a beta of 1). It is the equation implied by the Capital Asset Pricing Model and shows the linear relation between the expected return and systematic risk, as measured by beta.

11.8.2. According to the CAPM, how can we determine a stock's expected return?

When the CAPM holds, a security's expected return is given by:

$$E[R_i] = r_i = r_f + \beta_i^{Mkt}(E[R_{Mkt}] - r_f)\,.$$

Examples with Step-by-Step Solutions

Solving Problems

Problems using the concepts in this chapter may require calculating the expected return on a portfolio, or the variance, standard deviation, correlation, or covariance of a two-asset portfolio. You should also be able to make conclusions regarding the standard deviation of a large portfolio, given the average variance and correlation (or covariance) of the assets in that portfolio. You also may need to determine the expected return and standard deviation of a portfolio that includes investing in a risk-free asset or a risky portfolio. Finally, you may need to calculate the beta of an asset relative to an efficient portfolio, and determine if adding the asset to your holdings would improve your portfolio.

Examples

1. You are considering investing in Cisco and Yahoo which have never paid dividends and had the following actual end-of-year stock prices (adjusted for splits):

	Cisco	**Yahoo**
12/31/1996	**$7.07**	**$0.71**
12/31/1997	**9.29**	**4.33**
12/31/1998	**23.20**	**29.62**
12/31/1999	**53.56**	**108.17**
12/31/2000	**38.25**	**15.03**
12/31/2001	**18.11**	**8.87**
12/31/2002	**13.10**	**8.18**
12/31/2003	**24.23**	**22.51**
12/31/2004	**19.32**	**37.68**
12/31/2005	**17.12**	**39.18**

Calculate the expected return and variance of an equally weighted portfolio, and graph the efficient frontier, assuming these are the only assets you can invest in.

Step 1. Determine the annual return on each stock.

The annual return $= \dfrac{P_{t+1} - P_t}{P_t}$, for example $\dfrac{P_{1997}^{\text{Cisco}} - P_{1996}^{\text{Cisco}}}{P_{1996}^{\text{Cisco}}} = \dfrac{9.29 - 7.07}{7.07} = 0.314 = 31.4\%$

The returns for 1997–2005 are:

	Cisco	Yahoo
1997	0.314	5.099
1998	1.497	5.841
1999	1.309	2.652
2000	-0.286	-0.861
2001	-0.527	-0.410
2002	-0.277	-0.078
2003	0.850	1.752
2004	-0.203	0.674
2005	0.114	0.040

Step 2. Calculate the expected return on each stock.

$$R_{Cisco}^{\text{Mean}} = \frac{1}{T}\sum_{t=1}^{T} R_t = \frac{\begin{array}{c}0.314+1.497+1.309+-0.286+-0.527+-0.277\\ +0.850+-0.203+-0.114\end{array}}{9}$$
$$= 28.5\%$$

$$R_{Yahoo}^{\text{Mean}} = \frac{1}{T}\sum_{t=1}^{T} R_t = \frac{\begin{array}{c}5.099+5.841+2.652+-0.861+-0.410+-0.078\\ +1.752+0.674+0.040\end{array}}{9}$$
$$= 163\%$$

Step 3. Calculate the standard deviations of each stock.

$$\text{SD}(R_{\text{Cisco}}) = \sqrt{\text{VAR}(R_{\text{Cisco}})} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2} =$$

$$= \sqrt{\frac{\begin{array}{c}(0.314-0.285)^2+(1.497-0.285)^2+(1.309-0.285)^2+(-0.286-0.285)^2\\ +(-0.527-0.285)^2+(-0.277-0.285)^2+(0.850-0.285)^2\\ +(-0.203-0.285)^2+(-0.114-0.285)^2\end{array}}{8}}$$
$$= 0.753 = 75.3\%$$

$$\text{SD}(R_{\text{Yahoo}}) = \sqrt{\text{VAR}(R_{\text{Yahoo}})} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2}$$

$$= \sqrt{\frac{\begin{array}{c}(5.099-1.634)^2+(5.841-1.634)^2+(2.652-1.634)^2+(-0.861-1.634)^2\\ +(-0.410-1.634)^2+(-0.078-1.634)^2+(1.752-1.634)^2\\ +(0.674-1.634)^2+(0.040-1.634)^2\end{array}}{8}}$$
$$= 2.438 = 244\%$$

Step 4. Calculate the expected return on the portfolio.

$$E[R_P] = \sum_{i=1}^{n} x_i E[R_i] = 0.5(0.285) + 0.5(1.63) = 0.958 = 95.8\%$$

Step 5. In order to calculate the portfolio standard deviation, the covariance needs to be calculated.

$$\text{Cov}(R_{\text{Cisco}}, R_{\text{Yahoo}}) = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - \overline{R}_i)(R_{j,t} - \overline{R}_j)$$

$$= \frac{\begin{array}{l}(0.314-0.285)(5.099-1.634)+(1.497-0.285)(5.841-1.634)+(1.309-0.285)(2.652-1.634)\\ +(-0.286-0.285)(-0.861-1.634)+(-0.527-0.285)(-0.410-1.634)+(-0.277-0.285)(-0.078-1.634)\\ +(0.850-0.285)(1.752-1.634)+(-0.203-0.285)(0.674-1.634)+(-0.114-0.285)(0.040-1.634)\end{array}}{8}$$

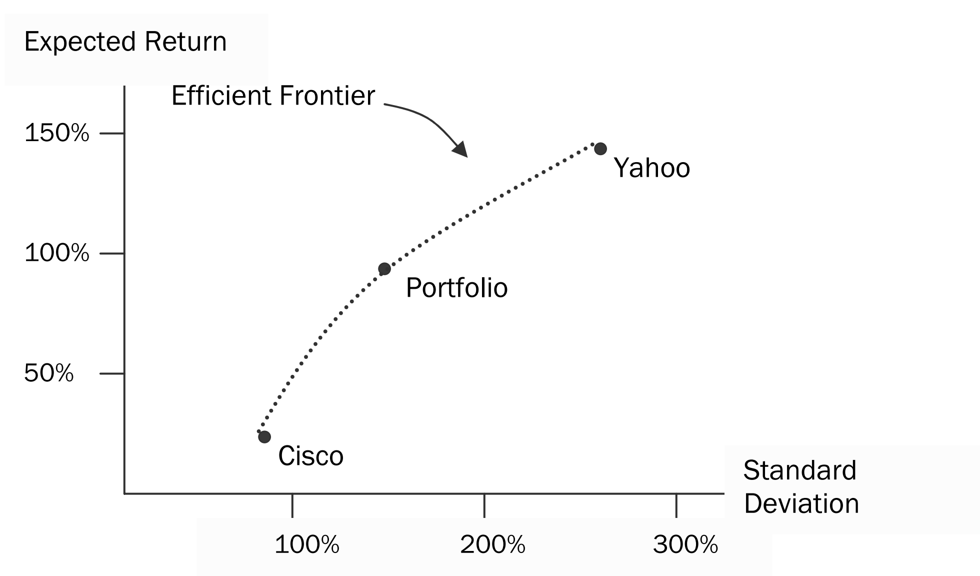
$$= 1.43$$

So, the correlation is: $\text{Corr}(R_{\text{Cisco}}, R_{\text{Yahoo}}) = \dfrac{\text{Cov}(R_{\text{Cisco}}, R_{\text{Yahoo}})}{\text{SD}(R_{\text{Cisco}}) \times SD(R_{\text{Yahoo}})} = \dfrac{1.43}{0.75 \times 2.44} = 0.78$

Step 6. Calculate the portfolio's variance and standard deviation.

$$\begin{aligned}\text{Var}(R_P) &= x_1^2\text{Var}(R_1) + x_2^2\text{Var}(R_2) + 2x_1x_2\text{Cov}(R_1, R_2)\\ &= .5^2 .753^2 + .5^2 2.44^2 + 2(.5)(.5)(1.43)\\ &= .143 + 1.486 + .715 = 2.343\\ \text{SD}(R_P) &= \sqrt{2.343} = 1.153 = 153\%\end{aligned}$$

Step 7. Graph the results.



2. You are considering investing in 10 stocks in two different industries (all 10 stocks you chose will be in the same industry).

- **The first industry is biotechnology. The historical average return on the stocks you are considering is 10%, and the historical average standard deviation is 30%. The stocks have an average correlation of 0.15.**
- **The other industry is S&P 500 exchange-traded funds (ETFs). The historical average return on the ETFs you are considering is 10%, and the historical average standard deviation is 20%. The ETFs have an average correlation of 0.99.**

Which strategy is better?

Step 1. Determine what you should base your decision on.

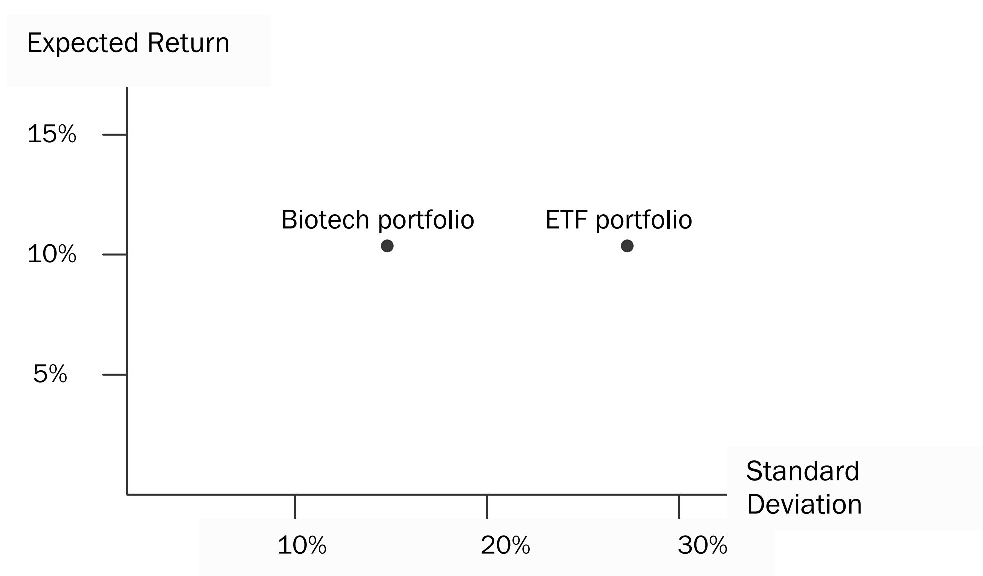
Both strategies have an expected return of 10%. So the portfolio with the lowest standard deviation would be preferred by a risk-averse investor.

Step 2. Calculate the standard deviation of each portfolio using the variance of an *n*-asset portfolio equation.

$$\begin{aligned}\mathrm{Var}(R_{\mathrm{Biotech}}) &= \frac{1}{n}(\text{Average Var}(R_i)) + (1-\frac{1}{n})(\text{Average Cov}(R_i,R_j)) \\ &= \frac{1}{10}(.30^2) + (1-\frac{1}{10})(0.15\times 0.3\times 0.30) \\ &= 0.021,\ \textit{and}\ \mathrm{SD}(R_{\mathit{Biotech}}) = .145 = 14.5\%\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(R_{\mathrm{ETF}}) &= \frac{1}{n}(\text{Average Var}(R_i)) + (1-\frac{1}{n})(\text{Average Cov}(R_i,R_j)) \\ &= \frac{1}{10}(.20^2) + (1-\frac{1}{10})(0.99\times 0.20\times 0.20) \\ &= 0.075,\ \textit{and}\ \mathrm{SD}(R_{\mathrm{ETF}}) = .275 = 27.5\%\end{aligned}$$

Step 3. Make a conclusion.



Since the biotech stock portfolio has a lower standard deviation, and both options offer a 10% return, the biotech stock portfolio is preferred to the ETF portfolio.

3. **You have decided to invest $1 million in the equally weighted portfolio of Cisco and Yahoo in problem 1 above.**

	Expected Return	**Standard Deviation**
Cisco	**28.5%**	**75.3%**
Yahoo	**163%**	**244%**
Equally-weighted portfolio	**95.8%**	**118%**

In addition you are going to borrow $500,000 from your margin account and invest it in the portfolio as well. If the borrowing rate is 6%, what is the expected return and standard deviation of your portfolio? Graph your results.

Step 1. Determine the expected return on the portfolio.

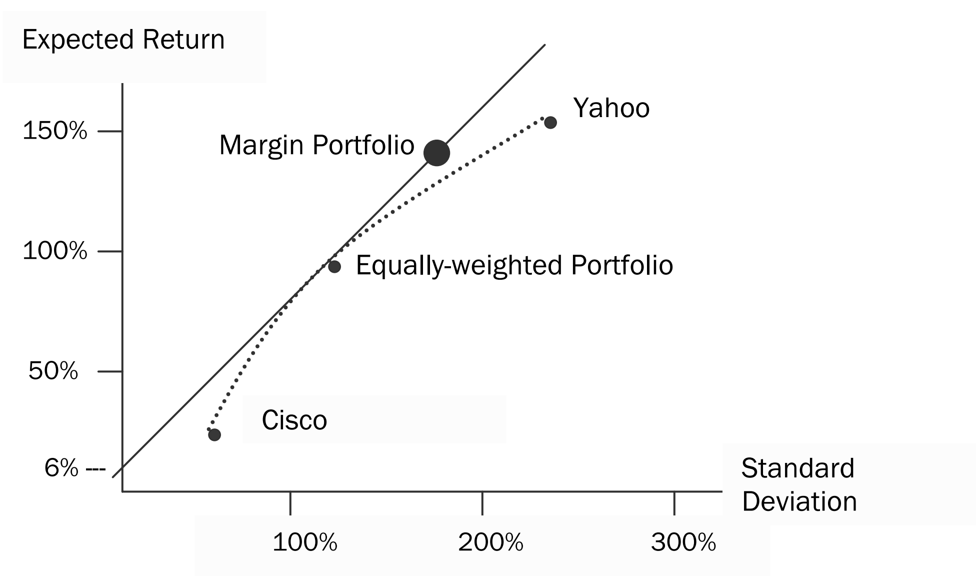
Since $x = \dfrac{\$1{,}000{,}000 \ + \ \$500{,}000}{\$1{,}00{,}000} = 1.5,$

$$E[R_{xP}] = r_f + x(E[R_P] - r_f) = .06 + 1.5(.958 - .06) = 137\%$$

Step 2. Determine the standard deviation of the portfolio.

$$\text{SD}[R_{xP}] = x\text{SD}(R_P) = 1.5(118\%) = 177\%$$

Step 3. Graph the results



Questions and Problems

1. Your TIAA-CREF retirement account contains stock index mutual funds, corporate bond mutual funds, and government bond mutual funds. It has an expected return of 12% and a standard deviation of 18%. You are considering adding a gold exchange traded fund (ETF), which has an 8% expected return, a standard deviation of 25%, and a correlation with your

current retirement holdings of –0.44. If the risk-free rate is 5%, will adding some of the gold ETF improve your portfolio?

2. Your retirement account offers the option of investing in Treasury bills which yield 5% and *one* of the following three diversified mutual funds:

	Expected Return	Standard Deviation
Fidelity Magellan	16%	32%
Vanguard S&P 500 Index	14%	25%
Barclay Total Market	12%	16%

   Which mutual fund should you choose?

3. The following table provides statistics for Microsoft and Anheuser-Busch between 1996 and 2004.

	Microsoft	Anheuser-Busch
Standard deviation	42%	18%
Correlation with Microsoft	1.00	–0.07
Correlation with Anheuser-Busch	–0.07	1.00

   What is the standard deviation of an equally weighted portfolio of these two stocks?

4. The following table provides correlations for Dell and Anheuser-Busch between 1996 and 2004.

	Dell	Anheuser-Busch
Correlation with Dell	1.0	0.10
Correlation with Anheuser-Busch	0.10	1.0

   If all 7,000 stocks in the US stock market had the same correlation, and the average stock had a standard deviation of 50%, what would be the variance of a:

   [A] 2-stock equally weighted portfolio?
   [B] 20-stock equally weighted portfolio?
   [C] 7,000-stock equally weighted portfolio?

5. You have $100,000 in cash and are considering investing in a portfolio with an expected return of 15%, and a standard deviation of 20%.
   [A] If you invest $50,000 in the portfolio and $50,000 in T-bills which yield 5%, what is the expected return and volatility (standard deviation) of your investment?
   [B] If you invest $150,000 in the portfolio by borrowing $50,000 at 5%, what is the expected return and volatility (standard deviation) of your investment?

Solutions to Questions and Problems

1. The beta of the gold ETF with the portfolio is:

$$\beta_i^P = \frac{\text{SD}(R_i) \times \text{Corr}(R_i, R_j)}{\text{SD}(R_P)} = \frac{0.25 \times -0.44}{0.18} = -0.611 .$$

   Beta indicates the sensitivity of the investment's return to fluctuations in the portfolio's return. Buying shares of security *i* improves the performance of a portfolio if its expected return exceeds the required return, which equals:

$$r_i = r_f + \beta_i^P (E[R_P] - r_f) = 0.05 + -0.611(0.12 - 0.05) = 0.007 = 0.7\%.$$

   The gold ETF has an expected return of 8% that exceeds the required return of 0.7%. Therefore, you can improve the performance of your current portfolio by investing in the gold fund.

2. To earn the highest possible expected return for any level of volatility, you must find the portfolio that generates the steepest possible line when combined with the risk-free investment. The slope of the line through a given portfolio *P* is often referred to as the Sharpe ratio of the portfolio.

$$\text{Sharpe Ratio} = \frac{\text{Portfolio Excess Return}}{\text{Portfolio Volatility}} = \frac{E[R_P]\ -\ r_f}{SD(R_P)}$$

$$\text{SR}^{\text{Fidelity}} = \frac{0.16 - 0.05}{0.32} = 0.344$$

$$\text{SR}^{\text{Vanguard}} = \frac{0.14\ -\ 0.05}{0.25} = 0.360$$

$$\text{SR}^{\text{Barclay}} = \frac{0.12\ -\ 0.05}{0.16} = 0.438$$

So the Barclay Total Market fund is the best option.

3. $\text{Var}(R_P) = x_1^2\text{Var}(R_1)\ +\ x_2^2\text{Var}(R_2)\ +\ 2x_1x_2\text{Cov}(R_1,R_2)$

$$= .5^2.42^2 + .5^2 18^2 +\ 2(.5)(.5)(\text{-}.07)(.42)(.18)$$

$$= 0.050$$

$$\text{SD}(R_P) = \sqrt{0.050} = .223 = 22.3\%$$

4. [A] $\text{Var}(R_P^{\text{2 stock}}) = \frac{1}{n}(\text{Average Var}(R_i)) + (1 - \frac{1}{n})(\text{Average Cov}(R_i, R_j))$

$$= \frac{1}{2}(.50^2) + (1 - \frac{1}{2})(0.10 \times 0.50 \times 0.50)$$

$$= 0.138,\ \text{and SD}(R) = .371 = 37.1\%$$

[B] $\text{Var}(R_P^{\text{20 stock}}) = \frac{1}{n}(\text{Average Var}(R_i)) + (1 - \frac{1}{n})(\text{Average Cov}(R_i, R_j))$

$$= \frac{1}{20}(.50^2) + (1 - \frac{1}{20})(0.10 \times 0.50 \times 0.50)$$

$$= 0.036,\ \text{and SD}(R) = .190 = 19.0\%$$

[C] $\text{Var}(R_P^{\text{7,000 stock}}) = \frac{1}{n}(\text{Average Var}(R_i)) + (1 - \frac{1}{n})(\text{Average Cov}(R_i, R_j))$

$$= \frac{1}{7{,}000}(.50^2) + (1 - \frac{1}{7{,}000})(0.10 \times 0.50 \times 0.50)$$

$$= 0.025,\ \text{and SD}(R) = .158 = 15.8\%$$

5. [A] $E[R_{xP}] = r_f + x(E[R_P] - r_f) = .05 + 0.5(.15 - .05) = 10\%$

$SD[R_{xP}] = xSD(R_P) = .5(20\%) = 10\%$

[B] $E[R_{xP}] = r_f + x(E[R_P] - r_f) = .05 + 1.5(.15 - .05) = 20\%$

$\text{SD}[R_{xP}] = \text{xSD}(R_P) = 1.5(20\%) = 30\%\ \text{s}$