# CHAPTER 4

The Time Value of Money

Chapter Synopsis

Many financial problems require the valuation of cash flows occurring at different times. However, money received in the future is worth less than money received today because the money received today can be invested to grow to have a larger value in the future. Thus, money has **time value,** and it is only possible to compare cash flows occurring at different times by valuing them at the same point in time.

4.1 The Timeline

The first step in most problems is to put the cash flows involved on a timeline in which each point is a specific date, such as this representation of a $10,000 loan made by a bank to a borrower who promises to pay $6,000 to the bank in each of the next two years:

	← Year 1 →	← Year 2 →
Date	0	1 2
Cash Flow	–$10,000	$6,000 $6,000
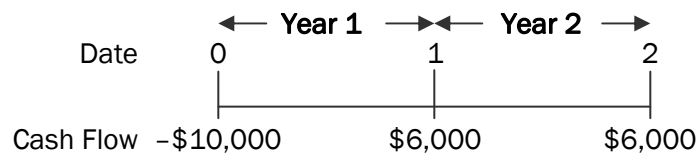
The space between date 0 and date 1 represents the time period between these dates. Date 0 is the beginning of the first year, and date 1 is the end of the first year. Similarly, date 1 is the beginning of the second year, and date 2 is the end of the second year. The signs of the cash flows are important: in the diagram, –$10,000 represents a cash outflow, and $6,000 represents a cash inflow.

4.2 The Three Rules of Time Value

To correctly account for the time value of money, three general rules must be followed:

(1) It is only possible to compare or combine values at the same point in time;

(2) To move a cash flow forward in time, you must compound it; and

(3) To move a cash flow backward in time, you must discount it.

For example, valuation problems often require the determination of the **future value** (FV) of a series of cash flows at a given interest rate, such as in a retirement savings planning application. Invested cash that is earning a positive rate of interest grows at an increasing rate over time in a process called **compounding** in which interest earned in the later periods accrues on both the original value of the cash and the interest earned in the prior periods. The general expressions for the FV of a lump sum, *C,* invested at rate *r* for *n* periods is as follows.

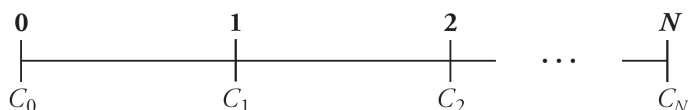### Future Value of a Cash Flow

$$FV_n = C(1+r)^n$$

Other problems seek to determine the **present value** (PV) of a series of future cash flows, such as in the valuation today of a bond that promises to make a series of payments in the future. The process of determining the present value of future cash flows is referred to as **discounting,** and the result is a discounted cash flow value. The general expression for the PV of a lump sum is as follows.

Present Value of a Cash Flow

$$PV = \frac{C_n}{(1+r)^n}$$

4.3 Valuing a Stream of Cash Flows

Applications often involve accurately considering a stream of cash flows occurring at different points in time over N periods:

The PV of such a stream can be found by using:

Present Value of a Cash Flow Stream

$$PV = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N} = \sum_{n=0}^{N} \frac{C_n}{(1+r)^n}$$

While this equation can generally be used to calculate the present value of future cash flows, there are often certain types of cash flow streams, such as annuities and perpetuities discussed below, that make the calculation less tedious.

## 4.4 Calculating the Net Present Value

An investment decision can be represented on a timeline as a stream of cash flows. The Net Present Value (NPV) of the project is thus the present value of the stream of cash flows of the opportunity:

NPV = PV(benefits) – PV(costs).

4.5 Perpetuities, Annuities, and Other Special Cases

A **perpetuity** is a stream of equal cash flows that occurs at regular intervals and lasts forever. For example, suppose you could invest $100 in a bank account paying 5% interest per year forever, and you want to create a perpetuity by taking $5 out each year. At the end of one

year, you will have $105 in the bank, and you can withdraw the $5 interest and reinvest the $100 for a second year. Again you will have $105 after one year, and you can withdraw $5 and reinvest $100 for another year as depicted in the diagram:
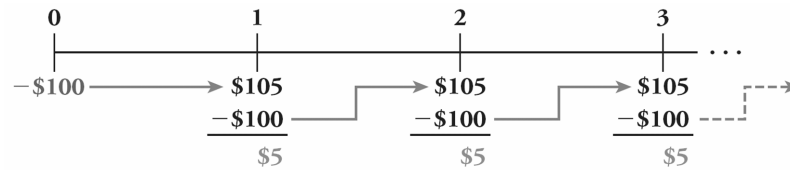

Thus, the PV of the $5 perpetuity with *r* = 5% must be $100 = $5 / .05 = *C/r*—the cost of replicating the cash flow stream. Thus, the present value of receiving *C* in perpetuity is:

Present Value of a Perpetuity

$$PV(C \text{ in perpetuity}) = \frac{C}{r}$$

An **annuity** is a stream of equal cash flows paid each period for *N* periods. Examples of annuities are home mortgage loans and corporate bonds. To determine the PV of an annuity suppose once again that you invest $100 in a bank account paying 5% interest. At the end of one year, you will have $105 in the bank. Using the same strategy as for a perpetuity, suppose you withdraw the $5 interest and reinvest the $100 for a second year. Once again you will have $105 after two years, and you can repeat the process, withdrawing $5 and reinvesting $100 every year for 20 years to close the account and withdraw the principal:
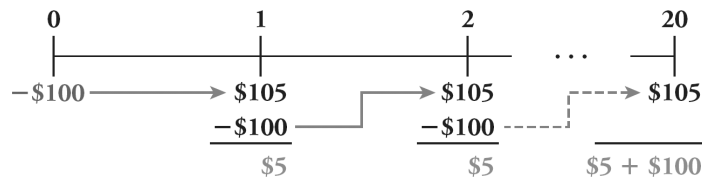
You have created a 20-year, $5 annuity. The value must be the NPV of the cash flows associated with creating it: the initial amount required to fund the annuity minus the PV of the return of the initial amount in *N* years, or $100 − $100 / (1.05)^{20} = $62.31. In general, the PV of an N-year annuity paying *C* per period with the first payment one period from date 0 is as follows.
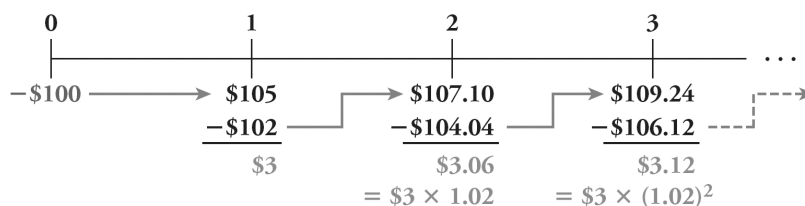
Present Value of an Annuity

$$PV(\text{annuity of } C \text{ for } N \text{ periods}) = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

Based on these equations, the future value of an annuity can now be calculated as:

Future Value of an Annuity

$$FV(\text{annuity of } C \text{ for } N \text{ periods}) = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)(1+r)^N = C \times \frac{1}{r}\left((1+r)^N - 1\right)$$

A **growing perpetuity** is a cash flow stream that occurs at regular intervals and grows at a constant rate forever. For example, suppose you invest $100 in a bank account that pays 5% interest. At the end of the first year, you will have $105 in the bank. If you withdraw $3, you will have $102 to reinvest. This amount will then grow to in the following year to $102(1.05) = $107.10 and you can withdraw $3.06 leaving the balance at $104.04.

You have created a 2% growing perpetuity with a first payment of $3 so the value must equal the cost to create it, $100 = $3/(.05 − .02)= C/(r − g). In general, the PV of a perpetuity growing at g percent that pays C one period from date 0 is as follows.

Present Value of a Growing Perpetuity

$$\text{PV(perpetuity growing at } g) = \frac{C_1}{r - g}$$

Finally, a **growing annuity** is a stream of N growing cash flows paid at regular intervals, where $N < \infty$. Assuming that the first cash flow, *C,* is paid at the end of the first period, the present value of an N-period growing annuity is:

Present Value of a Growing Annuity

$$\text{PV(annuity growing at } g) = C \times \left(\frac{1}{r-g} \right) \left(1 - \left(\frac{1+g}{1+r} \right)^N \right)$$

4.6 Solving Problems with a Spreadsheet Program

Microsoft Excel has time value of money functions based on the variables defined above. In the program: N = NPER, r = RATE, PV = PV, C = PMT, and FV = FV. You must input four of these variables and then Excel finds the fifth (one can be zero). For example, if you invest $20,000 at 8% for 15 years, how much will you have in 15 years?

The Excel function is = FV(RATE,NPER,PMT,PV)=FV(0.08,15,0,-20000) and the resulting FV equals $63,443.

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	15	8.00%	−20,000	0		
Solve for FV					63,443	= FV(0.08,15,0,−20000)

You can also use a financial calculator to do the same calculations. The calculators work in much the same way as the annuity spreadsheet. You enter any four of the five variables, and the calculator calculates the fifth variable.

4.7 Solving for Variables Other Than Present Value or Future Value

Any of the equations above (the future value of a lump sum, the present value of a lump sum, the present value of a perpetuity, the present value of an annuity, the future value of an annuity, the present value of a growing annuity, and the present value of a growing perpetuity) can be solved for any of the variables as long as the remaining variables in the equation are known. For example, the rate of return that makes the net present value of a stream of cash flows equal to zero, the **internal rate of return** (IRR), can be found using the present value of an annuity equation if the present value of the annuity payments (PV), the number of payments (N), and the periodic level cash flow (C) are all known.

Selected Concepts and Key Terms

Time Value of Money

The idea that it is only possible to compare cash flows occurring at different times by bringing them to the same point in time. When the expected rate of return on invested cash is positive, cash received in the future is worth less than cash received today because less cash can be invested today to equal the future amount. Thus, the present value of a future cash flow is less than the amount received in the future, and the future value of a cash flow invested in a previous period is worth more than the amount invested in the past.

Compounding

The process of moving cash forward in time over more than one time period. When cash is invested over multiple periods in the future, interest earned in the later periods grows at an increasing rate because it accrues on both the original value of the cash and the interest earned in the prior periods.

Discounting

The process of moving cash backwards in time. When interest rates are positive, the present value received is less than the value of the future cash flow. The process of calculating such present values is commonly referred to as discounting.

Perpetuity

A stream of cash flows that is received over equal, periodic intervals that lasts forever. A perpetuity can have level cash flows or it can have cash flows that grow at a constant rate, which is referred to as a **growing perpetuity.**

Annuity

A stream of cash flows that is received over equal, periodic intervals that ends at some future time period. An annuity generally has level cash flows, but it can also be a **growing annuity** and have cash flows that grow at a constant rate*.*

Internal Rate of Return

The rate of return that makes the net present value of a stream of cash flows equal to zero. The internal rate of return (IRR) is a popular measure used to evaluate the desirability of an investment based on its projected cash flows.

Concept Check Questions and Answers

4.1.1. What are the key elements of a timeline?

A timeline is a linear representation of the timing of the cash flows. Date 0 represents the present. Date 1 is one period (a month or a year) later; that is, it represents the end of the first period. The cash flow shown below date 1 is the payment you will receive at the end of the first period. You continue until all the cash flows and their timing are shown on the timeline.

4.1.2. How can you distinguish cash inflows from outflows on a timeline?

To differentiate between the two types of cash flows, we assign a positive sign to cash inflows and a negative sign to cash outflows.

4.2.1. Can you compare or combine cash flows at different times?

No, you cannot compare or combine cash flows at different times. A dollar today and dollar in one year are not equivalent.

4.2.2. What is compound interest?

Compound interest is the effect of earning "interest on interest." For compound interest, you can earn interest on the original investment and the interest reinvested from prior periods.

4.2.3. How do you move a cash flow backward and forward in time?

To move a cash flow forward in time, you must compound it. To move a cash flow back in time, you must discount it.

4.3.1. How do you calculate the present value of a cash flow stream?

The present value of cash flow stream is the sum of the present values of each cash flow.

4.3.2. How do you calculate the future value of a cash flow stream?

The future value of a cash flow stream is the sum of the future values of each cash flow.

4.4.1. How do you calculate the net present value of a cash flow stream?

The net present value of a cash flow stream is the present value of all the benefits minus the present value of all the costs. The benefits are the cash inflows and the costs are the cash outflows.

4.4.2. What benefit does a firm receive when it accepts a project with a positive NPV?

When a firm accepts a project with a positive NPV, the value of the firm will increase by the NPV today.

4.5.1. How do you calculate the present value of a

a. Perpetuity?

The present value of a perpetuity is the annual cash flow divided by the appropriate discount rate.

b. Annuity?

The present value of an annuity of C for n periods with interest rate r is:

$$\text{PV(annuity of } C \text{ for } N \text{ periods)} = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right).$$

c. Growing perpetuity?

The present value of a growing perpetuity is:

$$\text{PV(perpetuity growing at } g) = \frac{C_1}{r - g}.$$

d.   Growing annuity?

$$PV(\text{annuity growing at } g) = C \times \left(\frac{1}{r-g} \right) \left(1 - \left(\frac{1+g}{1+r} \right)^{N} \right)$$

4.5.2. How are the formulas for the present value of a perpetuity, annuity, growing perpetuity, and growing annuity related?

The formula for the present value of growing annuity is a general solution. From this formula, we can deduce the formulas for the present value of a perpetuity, annuity, and growing perpetuity.

4.6.1. What tools can you use to simplify the calculation of present values?

Two shortcuts you can use to simplify the calculation of present values are the use of spreadsheets and financial calculators.

4.6.2. What is the process for using the annuity spreadsheet?

Spreadsheet programs such as Excel have a set of functions that performs the calculations that finance professionals do most often. In Excel, the functions are NPER, RATE, PV, PMT, and FV.

4.7.1. How do you solve for the periodic payment of an annuity?

The cash flow of an annuity (or the loan payment) in terms of the amount borrowed P, interest rate r, and number of payments n, can be computed as:

$$C = \frac{P}{\frac{1}{r}\left[1 - \frac{1}{(1+r)^{n}} \right]}.$$

4.7.2. What is the internal rate of return, and how do you calculate it?

The internal rate of return (IRR) is the interest rate that sets the net present value of the cash flows equal to zero. You can guess the IRR and manually calculate its value. An easier solution is to use a spreadsheet or calculator to automate the guessing process.

4.7.3. How do you solve for the number of periods to pay off an annuity?

To solve for the number of periods to payoff an annuity, you use the trial and error method, guessing, and manually calculate the value of N. Alternatively, you can use an annuity spreadsheet to solve for N.

## Examples with Step-by-Step Solutions

Solving Problems

Problems using the concepts in this chapter can be solved by first determining the timeline of the known and unknown cash flows and then determining which valuation equation is necessary. Some problems will involve using two or more of the equations and two or more steps. You just need to make sure that each equation you are solving has only one unknown variable: PV, FV, *C, r, N,* or *g*. The examples below demonstrate the general procedure for solving these types of problems.

Examples

1. You would like to endow a scholarship. In-state tuition is currently $10,000, and the rate will grow by 3% per year. How much would it cost today to endow a scholarship that pays full tuition once every year forever starting 10 years from now? Assume a 5% annual percentage rate (APR) rate of return on your investment to fund the scholarship.

   Step 1. Put the cash flows that are known and unknown on a timeline.
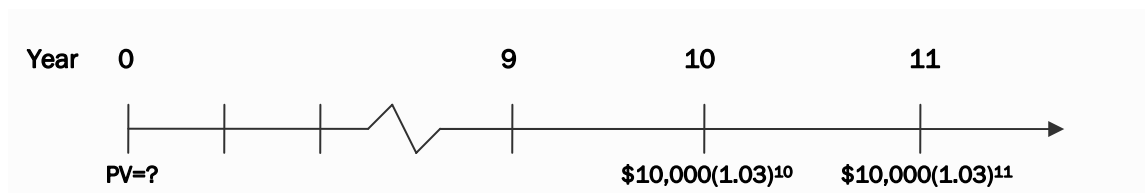
 | Year | 0 | | | | 9 | 10 | 11 |
   |------|---|---|---|---|---|----|----|
   | | PV=? | | | | | $10,000(1.03)^{10}$ | $10,000(1.03)^{11}$ |

   Step 2. Determine the type or types of valuation problems involved.

   This problem involves the following additional steps 3 and 4:

   [3] finding the present value in year 9 of the known tuition payments in years 10 to infinity, and

   [4] finding the present value at time 0 of the time 9 value found in step 3.

   Step 3. Find the present value in year 9 of the known tuition payments in years 10 to infinity.

Year	0				9	10	11
					PV=?	$10,000(1.03)^{10}$	$10,000(1.03)^{11}$
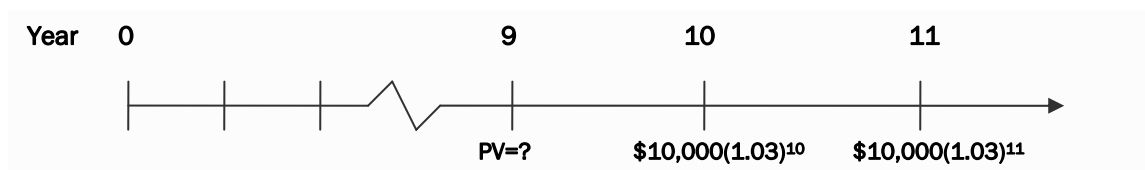
   Using the present value of a growing perpetuity with $C = \$10{,}000(1.03)^{10}$, $r = .05$, and $g = .03$:

   $$PV_9 = \frac{C}{r-g}\left[\frac{10{,}000(1.03)^{10}}{.05-.03}\right] = \left[\frac{13{,}439}{.05-.03}\right] = \$671{,}958$$

   Step 4. Find the present value of at time 0 of the time 9 value found in step 3.

Year	0				9	10	11
	PV=?				$671,958		
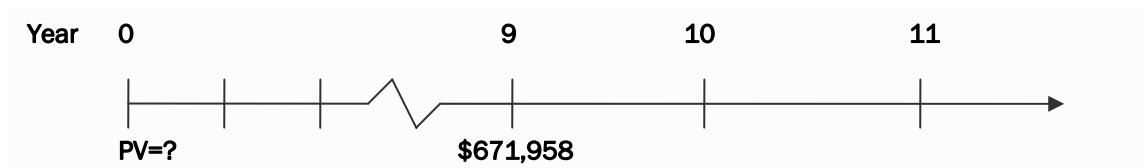
   Using the present value of a lump sum equation with FV = $671,958, $r = .05$, and $N = 9$:

   $$PV_0 = \frac{\$671{,}958}{(1.05)^9} = \$433{,}150$$

   You would need to make invest $433,150 into the account today.

2. You have determined that you will need $50,000 per year for four years to send your daughter to college. You have already saved $10,000 and placed the money in an account that you expect will yield a monthly compounded 12% APR (1% per month). Money for the first of the four payments will be removed from the account exactly 15 years from now and the last withdrawal will be made 18 years from now. You have decided to save more by

making monthly payments into the same account yielding an expected 12% APR (1% per month) over the next 14 years beginning next month. You will take the money out of the 12% account and place it in a 6% APR account in 14 years and take the cash out as needed. How large must these monthly payments be?

Step 1. Put the cash flows that are known and unknown on a timeline.
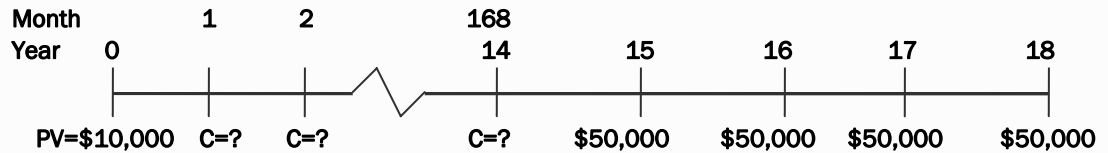
Step 2. Determine the type or types of valuation problems involved. This problem can be solved using the following additional steps 3–5:

[3] Find the present value in year 14 of the known $50,000 payments in years 15–18,
[4] Find the future value of the $10,000 you have today at time 14 years, and
[5] Find the unknown monthly annuity payment that has the future value at time 14 equal to the value found in step 3 minus the value found in step 4.

Step 3. Find the present value of the known $50,000 payments. Since it is only possible to compare values at the same point in time, the first step in a problem like this is to find the value of the known cash flows at one point in time. The most straightforward time to value the cash flows is time 14 because then you can use the present value of an annuity equation, which assumes that the first cash flow occurs one period after it is being valued, time 15.
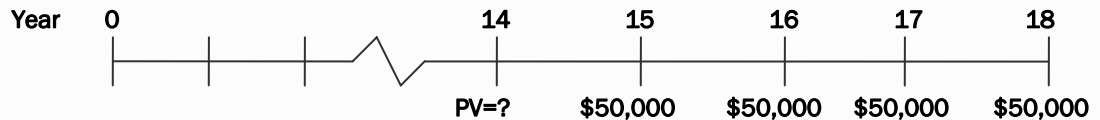
Using the present value of an annuity equation with N = 4, r = 0.06, and C = $50,000:

$$PV = \$50,000 \left(\frac{1}{.06}\right)\left[1 - \frac{1}{.06(1.06)^4}\right] = \$173,255$$

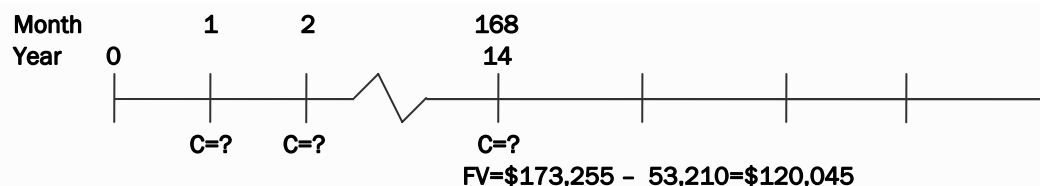**Step 4.** Find the future value of the $10,000 you have today at time 14 years.
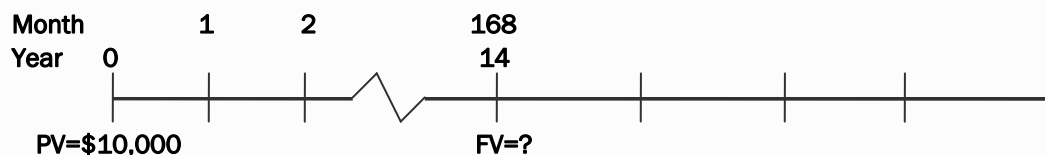
Using the future value of a lump sum equation with N = 12 x 14 = 168, r = 0.01, and PV = $10,000:

$$FV = 10,000(1.01)^{168} = \$53,210$$

Step 5. Find the unknown monthly annuity payment that has the future value at time 14 equal to the amount found in part a minus the amount found in part b.

Month   1   2   168
Year   0
C=?   C=?   C=?
FV=$173,255 – 53,210=$120,045

Using the future value of an annuity equation with $N = 12 \times 14 = 168$, $r = 0.01$, and FV = $120,045:

$$FV = 120,045 = C\left[\frac{1}{.01} - \frac{1}{.01(1.01)^{168}}\right](1.01)^{168} \Rightarrow C = \$278.$$

You would need to make monthly payments of $278 into the account.

3. Some Republicans would like to give those contributing to Social Security the option of investing in their own personal accounts and in assets riskier than Treasury Bonds. Assume that the average worker will contribute $5,000 into his or her retirement account next year, and can choose option 1 and invest in T-bonds, which have a 4% expected return, or option 2 and invest in a stock index fund that has a 12% expected return. In both options at the date of retirement, the money will be placed in an account with an expected return of 3% APR (0.25% per month). Assume that the amount workers contribute will grow by 3% per year, and the average worker is 35 years from retirement age. If the money is withdrawn from the account beginning the month after retirement, and the average worker is expected to live for 20 years after retirment, what size monthly payment would the average worker be able to withdraw in both of the options?

Step 1. Put the cash flows that are known and unknown on a timeline.
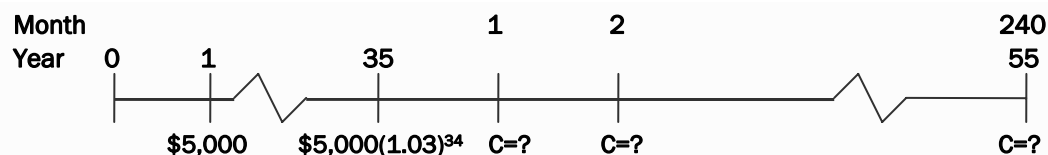
Month   1   2   240
Year   0   1   35   55
 $5,000 $5,000$(1.03)^{34}$ C=?   C=?   C=?

Step 2. Determine the type or types of valuation problems involved.

This problem involves the following additional steps 3–4:

[3] finding the future value of the known growing annuity of payments that will have accumulated in 35 years, and
[4] finding the unknown monthly annuity payment that has that present value.

Step 3. Find the future value of the growing annuity of payments.

Year   0   1   35
        $5,000 $5,000$(1.04)^{34}$
                  FV=?
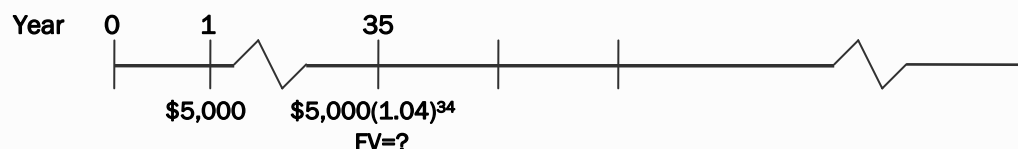
The future value of a growing annuity can be found using the present value of a growing annuity equation and the future value of a lump sum equation as follows:

$$\text{The present value of the payments at time } 0 = PV_0 = \left(\frac{C}{r-g}\right)\left[1-\left(\frac{1+g}{1+r}\right)^N\right],$$

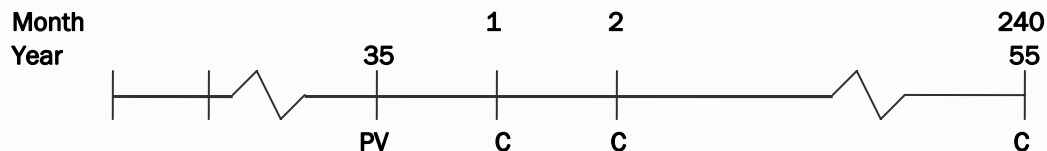so the future value at time $35 = FV_{35} = PV_0(1+r)^{35}$, or

$$FV_{35} = \left(\frac{C}{r-g}\right)\left[1-\left(\frac{1+g}{1+r}\right)^N\right](1+r)^{35}.$$

For both of the options, C = $5,000, g = 3%, and N = 35. For the T-bond option, r = 4%, and for the stock index fund option, r = 12%.

$$FV_{35}^{\text{T-bond}} = \left[\left(\frac{5,000}{.04-.03}\right)\left(1-\left(\frac{1.03}{1.04}\right)^{35}\right)\right](1.04)^{35} = \$566,113$$

$$FV_{35}^{\text{Stock}} = \left[\left(\frac{5,000}{.12-.03}\right)\left(1-\left(\frac{1.03}{1.12}\right)^{35}\right)\right](1.12)^{35} = \$2,776,986$$

Step 4. Find the 240 month annuity payment that has that present value.

Month			1	2		240
Year		35				55

PV   C   C   C

Using the present value of an annuity equation with PV = FV$_{35}$ from step 3, N = 12 x 20 = 240,

r = .03 / 12 = 0.0025, and solving for C, you have:

$$FV_{35}^{\text{T-bond}} = PV = \$566,113 = C\left[\frac{1}{.0025} - \frac{1}{.0025(1.0025)^{240}}\right] \Rightarrow C = \$3,140$$

$$FV_{35}^{\text{Stock}} = PV = \$2,776,986 = C\left[\frac{1}{.0025} - \frac{1}{.0025(1.0025)^{240}}\right] \Rightarrow C = \$15,401$$

Under the 4% T-bond plan, workers could withdraw $3,140 per month, and under the 12% Stock Index plan, workers could withdraw $15,401 per month.

Questions and Problems

1. The historical average return on U.S. T-bills is 3.8% per year, while the average return for small company stocks is 16.9% per year. Assuming these rates occur annually in the future, how much more cash would you have in 20 years by investing $50,000 in small company stocks rather than T-bills?

2. Your daughter is currently 8 years old. You anticipate that she will be going to college in 10 years. You would like to have $100,000 in a savings account at that time. If the account

promises to pay a fixed interest rate of 3% per year, how much money do you need to put into the account today to ensure that you will have $100,000 in ten years?

3. You have determined that you will need $3,000,000 when you retire in 40 years. You plan to set aside a series of payments each year in an account yielding 12% per year to reach this goal. You will put the first payment in the account one year from today, and the payments will grow with your income by 3% per year. Calculate your first annual payment into this account. Calculate the last payment.

4. Like in problem 3, you have determined that you will need $3,000,000 when you retire in 40 years, and you plan to set aside a series of payments each year in an account yielding 12% per year to reach this goal. You will put in the first payment in the account one year from today and the payments will grow with your income by 3% per year. Assuming that the money is placed in a 6% APR account throughout your retirement period, and you plan to withdraw $25,000 per month, approximately how many years will the money last you?

5. You are offering the employees in your small firm a so-called defined benefit pension plan. Beginning exactly 21 years from today you will pay out the first annual payment of a guaranteed 30-year stream of annual payments. The first payment will be $100,000 for 10 employees, or $1 million. The payment stream will grow by 3% per year each year to match expected inflation. You have already started investing in the pension account, which has a balance of $113,971.84 today. You expect that the account will always yield 13% APR, and you will always leave the money in the account, only withdrawing the money as needed by the plan. To supplement the plan, you will make 20 even, annual payments over the next 20 years, beginning one year from today. How big must the annual payment that you will contribute be?

Solutions to Questions and Problems

1. This problem requires using the FV of a lump sum equation:

$$FV_{20} = \$50,000(1.038)^{20} = \$105,418.56$$
$$FV_{20} = \$50,000(1.169)^{20} = \$1,135,691.11$$

2. This problem requires using the PV of a single cash flow equation:

$$PV_0 = \frac{FV_{10}}{(1+r)^{10}} = \frac{100,000}{(1.03)^{10}} = \$74,409.39$$

3. Set $3,000,000 equal to the present value a growing annuity equation and solve for the first payment, C_1.

$$FV_{40} = \left[\frac{C}{.12 - .03} \right] \left[1 - \left(\frac{1.03}{1.12} \right)^{40} \right] (1.12)^{40} = 3,000,000$$

$$= C \left[11.1111 \right] \left[.96494 \right] 93.05$$

$$= C(997.643) \Rightarrow C = \$3,007$$

Thus, the first payment is $3,007 and the last payment is ($3,007)(1.03)^{39} = $9,523.

4. You need to solve the following present value of an annuity equation for N.

$$\$3,000,000 = \$25,000 \left(\frac{1}{.005}\right)\left(1 - \frac{1}{.005(1.005)^N}\right) \Rightarrow N = 184 \text{ months, or 15 years and 4 months.}$$

5. You must contribute $99,648:

$$PV_{20} = \left[\frac{1,000,000}{.13-.03}\right]\left[1 - \left(\frac{1.03}{1.13}\right)^{30}\right] = \$9,379,469$$

$$PV_0 = \frac{\$9,379,469}{(1.13)^{20}} - \$113,971.84 = \$813,972 - 113,971.84 = \$700,000$$

$$\$700,000 = C\left[\frac{1}{.13} - \frac{1}{.13(1.13)^{20}}\right] \Rightarrow C = \$99,648$$