Parametric and Non-parametric Tests

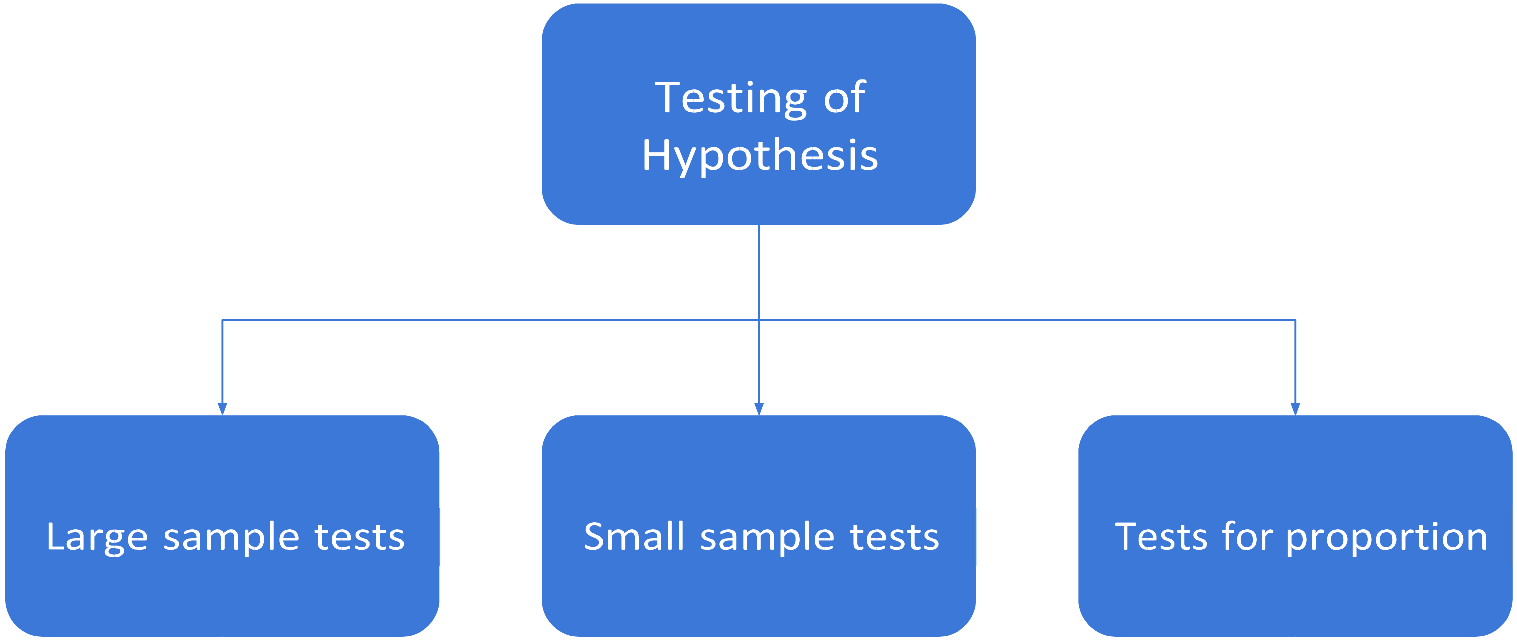
POST READ

Agenda

- Test for Population Proportion
  - One Sample
  - Two Sample

- Non-Parametric Tests
    - One Sample
        - Wilcoxon Signed Rank Test
    - Two Sample
        - Wilcoxon Signed Rank Test
        - Wilcoxon Rank Sum Test
        - Mann-Whitney U Test

In this session…


Testing of Hypothesis
Large sample tests
Small sample tests
Tests for proportion

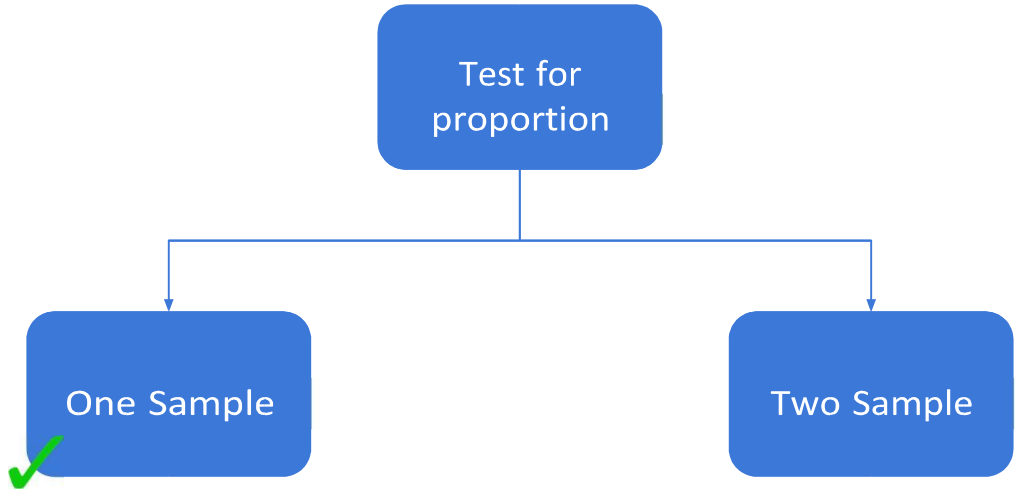
Test for Population Proportion

Test for proportion

- For qualitative data the proportion of a desired characteristic is obtained
- Test for proportion:
  - One sample: Testing population proportion (P) is equal to a specified value (P_0)
  - Two sample: Testing equality of Two population proportions ($P_1 = P_2$)

- Similar to the tests of population mean

Test for proportion



One sample test - hypothesis

- The hypothesis to test the population proportion is equal to a specified value

$$H_0 : P = P_0 \quad \text{against} \quad H_1 : P \neq P_0$$

- It implies

H_0: The population proportion is equal to P_0

against H_1: The population proportion is not equal to P_0

- Failing to reject H_0 implies that the population proportion is equal to P_0

Test for proportions

- The test statistic is given by

Sample proportion

Specified proportion

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}}$$

Sample size

- Under H_0, the test statistic follows standard normal distribution

One sample test for proportion - decision rule

	H_1	Based on critical region	Based on p-value	Based on confidence interval
For two tailed test	$P \neq P_0$	Reject H_0 if $\lvert Z \rvert \geq Z_{\alpha/2}$	Reject H_0 if p-value is less than or	

For left tailed test	$P < P_0$	Reject H_0 if $Z \leq -Z_\alpha$	equal to the level of significance	Reject H_0 if P_0 does not lie in the confidence interval
For right tailed test	$P > P_0$	Reject H_0 if $Z \geq Z_\alpha$		

One sample test for proportion

Question:

From a sample 361 business owners had gone into bankruptcy due to recession. On taking a survey, it was found that 105 of them had not consulted any professional for managing their finance before opening the business. Test the null hypothesis that at most 25% of all businesses had not consulted before opening the business.

Test the claim using p-value technique. [Use $\alpha = 0.05$].

One sample test for proportion

Solution:

From a sample 361 business owners had gone into bankruptcy due to recession. i.e. n = 361

On taking a survey, it was found that 105 of them had not consulted any professional for managing their finance before opening the business.

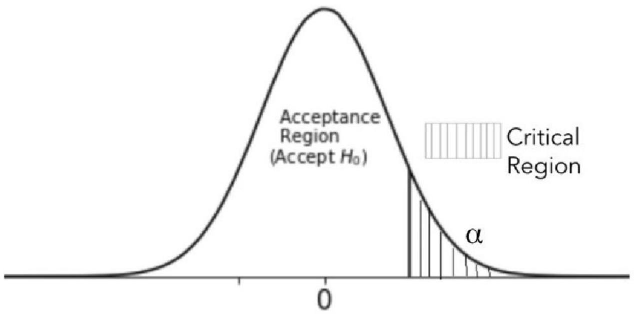
Let X: business which did not consult before x = 105

The sample proportion (p) = x/n = 105/361 = 0.2909

One sample test for proportion

Solution:

To test: The null hypothesis that at most 25% of all businesses had not consulted before opening the business



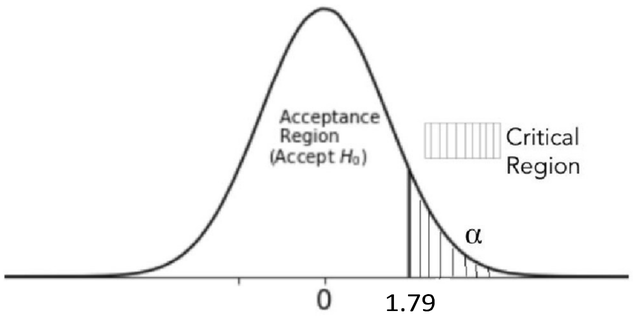
Here $P_0 = 0.25$

To test, H_0: $P \leq 0.25$ against H_1: $P > 0.25$

One sample test for proportion

Solution:

The test statistic $$Z = \frac{p - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{0.2909 - 0.25}{\sqrt{\frac{0.25(0.75)}{316}}} = 1.79$$



The p-value = P(Z > 1.79) = 0.0367

As p-value < 0.05, reject H_0.

We may conclude that at least 25% of all businesses had not consulted before starting the business.

One sample test for proportion

Python solution: Calculate test statistic

```
# sample size
n = 361

# number of business owners that did not consult before
x = 105

# sample proportion
p_samp = x / n

# hypothesized proportion
hypo_p = 0.25

# calculate test statistic value for 1 sample proportion test
z_prop = (p_samp - hypo_p) / np.sqrt((hypo_p * (1 - hypo_p)) / n)

print('Test statistic:', z_prop)
```

```
Test statistic: 1.7928245201151534
```

One sample test for proportion

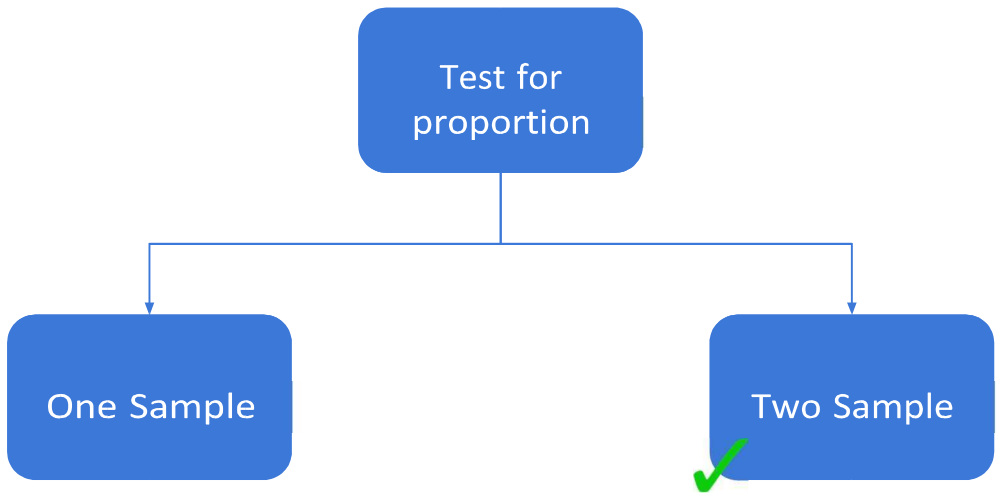
Python solution: Calculate p-value

```python
# calculate the corresponding p-value for the test statistic
# use 'sf()' to calculate P(Z > z_prop)
p_value = stats.norm.sf(z_prop)

print('p-value:', p_value)
```

```
p-value: 0.03650049373124949
```

As the p-value < 0.05, we reject H_0.

Test for proportion



Two sample tests for population proportion

- Let there be two samples sizes n_1 and n_2 from different populations of such that x_1 and x_2 are the number of specific items in each of them respectively

- Suppose these samples have proportions of specific items p_1 and p_2 respectively

- To test the equality of population proportion from which these samples are chosen

Two sample test - hypothesis

- The hypothesis to test the population proportion

$$H_0 : P_1 = P_2 \quad \text{against} \quad H_1 : P_1 \neq P_2$$

- It implies

H_0: The two population proportions are equal ($P_1 = P_2$)

against H_1: The two population proportions are not equal ($P_1 \neq P_2$)

- Failing to reject H_0 implies that the two population proportions are equal

Test for proportions

- The test statistic is given by

$$Z = \frac{p_1 - p_2}{\sqrt{\bar{P}(1-\bar{P})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$$

where $\bar{P}$ is the proportion of pooled sample such that

$$\bar{P} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

- Under H_0, the test statistic follows standard normal distribution

The python code to conduct a Z test for two population proportions is

statsmodels.api.stats.proportions_ztest(Sample_1, Sample_2)

Two sample test for proportion - decision rule

	H_1	Based on critical region	Based on p-value	Based on confidence interval

For two tailed test	$P_1 \neq P_2$	Reject H_0 if $\lvert Z \rvert \geq Z_{\alpha/2}$	Reject H_0 if p-value is less than or equal to the level of significance	Reject H_0 if $P_1 - P_2$ does not lie in the confidence interval
For left tailed test	$P_1 < P_2$	Reject H_0 if $Z \leq -Z_\alpha$		

For right tailed test	$P_1 > P_2$	Reject H_0 if $Z \geq Z_\alpha$		

Two sample test for proportion Question:

Steve owns a kiosk where sells two magazines - A and B in a month. He buys 100 copies of magazine A out of which 78 were sold and 70 copies of magazine B out of which 65 were sold. Is there enough evidence to say that magazine is B is more popular?

Test the claim using p-value technique. [Use $\alpha = 0.05$].

Two sample test for proportion

Solution:

Steve owns a kiosk where sells two magazines - A and B in a month.

Let X: the number of magazines sold

Out of 100 copies of magazine A 78 are sold Here, $x_1 = 78$ and $n_1 = 100$

Let p_1 be the proportion of sell of magazine A $p_1 = x_1/n_1 = 78/100 = 0.78$ Out of 70 copies of magazine B 65 are sold

Here, $x_2 = 65$ and $n_1 = 70$

Let p_2 be the proportion of sell of magazine B

$p_2 = x_2/n_2 = 65/70 = 0.928$

Two sample test for proportion

Solution:

To test, whether magazine B is more popular, i.e H_0:

$P_1 \geq P_2$ against H_1: $P_1 < P_2$

Where P_1: denotes population proportion of magazine A sold P_2: denotes population proportion of magazine B sold

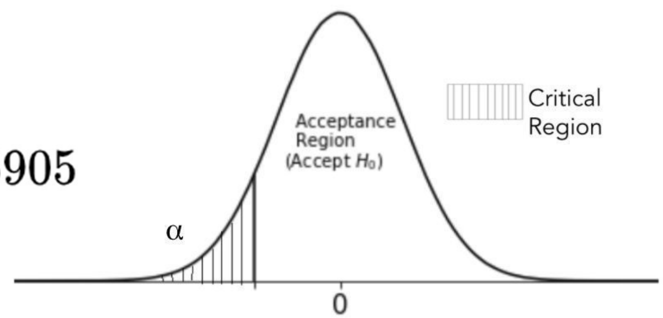
Two sample test for proportion

Solution:

The pooled proportion is $\bar{P} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{78+65}{100+70} = 0.84$

The test statistic is

$$Z = \frac{p_1 - p_2}{\sqrt{\bar{P}(1-\bar{P})(\frac{1}{n_1}+\frac{1}{n_2})}} = \frac{0.78-0.928}{\sqrt{0.84(1-0.84)(\frac{1}{100}+\frac{1}{70})}} = -2.5905$$

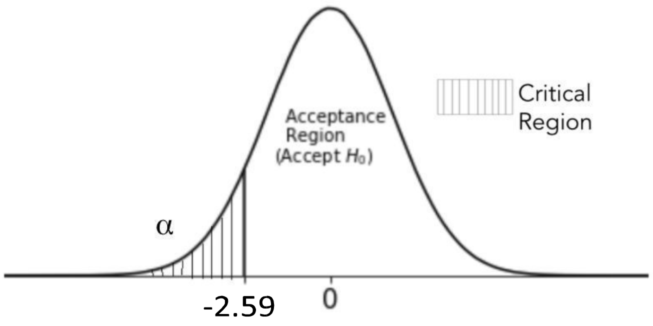


Two sample test for proportion

Solution:

The test statistic Z = -2.5905

The p-value = $P(Z < Z_{calc}$, under $H_0) = P(Z < -2.5905, \mu = 13)$
=



0.0048 Since p-value < 0.05, we reject H_0.

Thus there is enough evidence to conclude that magazine is B is more popular.

Two sample test for proportion

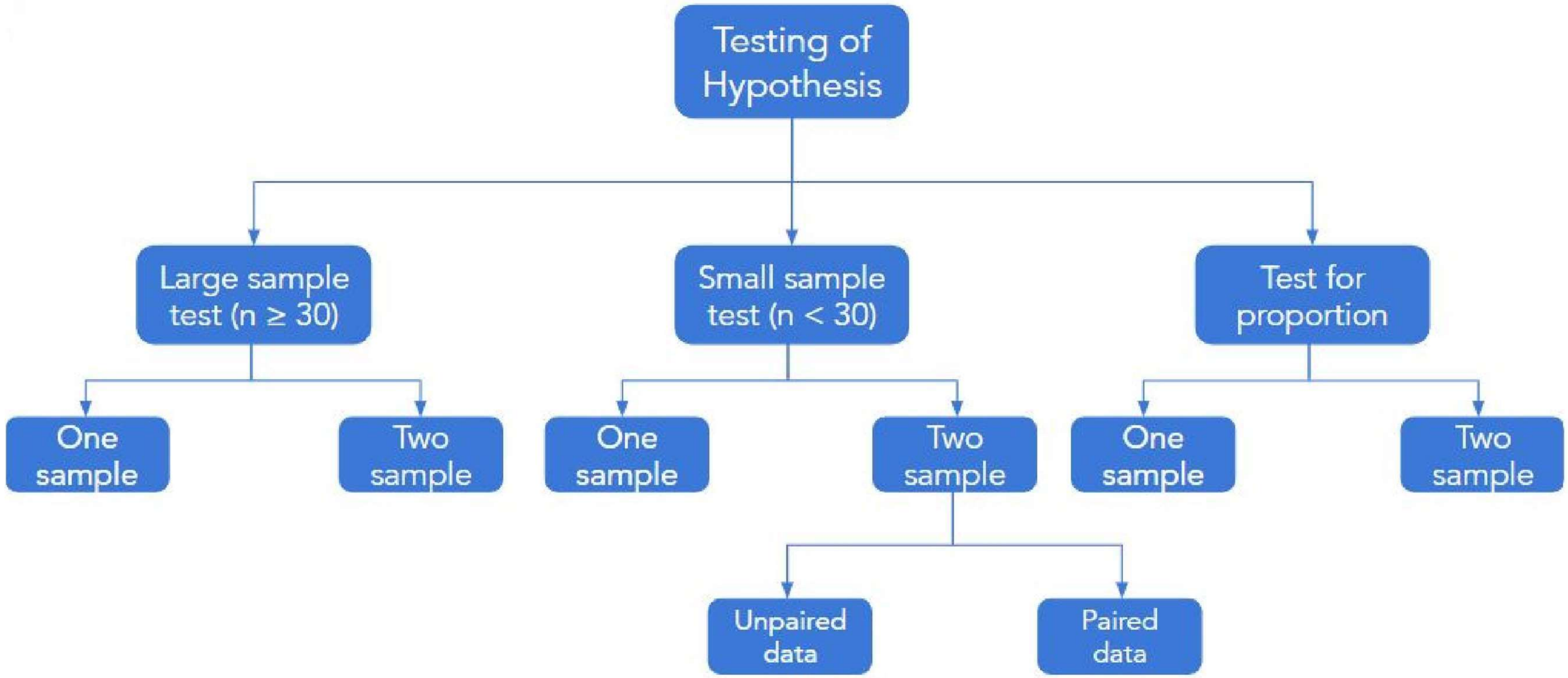
Python solution: Calculate test statistic and p-value

```
# calculate test statistic value for two sample proportion test
# pass the copies sold for both the magazines to the parameter, 'count'
# pass the size of both the samples to the parameter, 'nobs'
# pass the one-tailed condition to the parameter, 'alternative'
z_prop, p_val = sm.stats.proportions_ztest(count = np.array([78, 65]),
                                           nobs = np.array([100, 70]),
                                           alternative = 'smaller')

# print the value of test statistic and the corresponding p-value
print('Test statistic:', z_prop)
print('p-value:', p_val)
```

```
Test statistic: -2.60830803458311
p-value: 0.004549551600547303
```

As the p-value < 0.05, we reject H_0.

Summary


Testing of Hypothesis
Large sample test (n ≥ 30)
Small sample test (n < 30)
Test for proportion
One sample
Two sample
One sample
Two sample
One sample
Two sample
Unpaired data
Paired data

Parametric tests

- The tests considered so far have two features:
  - The probability distribution of the samples was assumed to be known
  - The hypothesis test was about the parameter of the probability distribution
- These tests are known as the parametric tests
- The times when these assumptions are not satisfied, use the non-parametric tests

Non-parametric Tests

Non-parametric tests

- A test that does not depend on the particular form of the basic probability distribution from which the samples are drawn

- The assumptions of a non-parametric test are
  - Sample observations are independent

○ Sample observations are random

- These assumptions are weaker than those associated with the parametric tests
- Applications in Psychometry, Sociology and Educational Statistics

Characteristics of non-parametric tests

- Does not require complicated sampling theory
- Can be applied to nominal or ordinal data, for instance blood group such as A^{+}, AB^{-} and so on
- No assumptions are made about the form of probability distribution of the parent population, hence are also known as Distribution Free methods

- Tests are based on median, range, quartile and so on. Henceforth an ordered sample is desired

Ordered sample

- Let $x_1, x_2, \ldots, x_n$ be a sample

- The ordered sample for $x_1, x_2, \ldots, x_n$ is $x_1 \leq x_2 \leq \ldots \leq x_n$

Example:

Let the sample be 23, 44, 10, 5, 39

The ordered sample is 5, 10, 23, 39, 44 since $5 \leq 10 \leq 23 \leq 39 \leq 44$

Non-parametric Tests-One Sample

One sample test

- To test for the location parameter of the sample equal to a specific value
- The location parameter is considered to be the median
- Use the Wilcoxon sign rank test

Wilcoxon signed rank test

- Let x_1, x_2, .. x_n be a random sample of size n of the random variable X
- Let $F_X(.)$ be the distribution function and M be the median of the random variable
- Assumption of Wilcoxon sign rank test
  - $F_X(.)$ is continuous
  - $F_X(.)$ is symmetric

Wilcoxon signed rank test

To test,

$$H_0 : M = M_0 \quad \text{against} \quad H_1 : M \neq M_0$$

$$H_0 : M \leq M_0 \quad \text{against} \quad H_1 : M > M_0$$

$$H_0 : M \geq M_0 \quad \text{against} \quad H_1 : M < M_0$$

Where, M is the median.

Wilcoxon signed rank test

Procedure:

- Compute the $D_i = X_i - M_0$
- Rank $|D_i|$, where 1 is assigned to the smallest difference
- The test statistic is given by

ided test	T_+
Two sided test	$T = \text{Min}\,(T^+, T^-)$

where

T_+	Sum of positive signed ranks
T_-	Sum of negative signed ranks

- Failing to reject H_0, implies the median is equal to specified median value M_0

The python code to conduct Wilcoxon Signed Rank test for one population is scipy.stats.wilcoxon(Sample, alternative)

H_1	Based on critical region

For left tailed test	$< M_0$	Reject H_0 if $T^{+} \leq T_{\alpha, n}$
For right tailed test	$> M_0$	Reject H_0 if $T^{-} \leq T_{\alpha, n}$
For two tailed test	$\neq M_0$	Reject H_0 if $T^{+} \leq T_{\alpha}$ or $T \geq T^{+}_{\alpha}$

Wilcoxon signed rank test

n	Two-Tailed Test		One-Tailed Test	
	α = .05	α = .01	α = .05	α = .01
5	--	--	0	--
6	0	--	2	--
7	2	--	3	0
8	3	0	5	1
9	5	1	8	3
10	8	3	10	5
11	10	5	13	7
12	13	7	17	9
13	17	9	21	12
14	21	12	25	15
15	25	15	30	19
16	29	19	35	23
17	34	23	41	27
18	40	27	47	32
19	46	32	53	37
20	52	37	60	43
21	58	42	67	49
22	65	48	75	55
23	73	54	83	62
24	81	61	91	69
25	89	68	100	76
26	98	75	110	84
27	107	83	119	92
28	116	91	130	101
29	126	100	140	110
30	137	109	151	120

Decision rule:

How to rank data?

Smallest rank is given to data with smallest magnitude.

Example:

Rank the data 21, 8, 5, 4, 10, 15, 30, 24

Unordered Data	21	8	5	4	10	15	30	24

Ordered Data	4	5	8	10	15	21	24	30
Rank	1	2	3	4	5	6	7	8

What to do in case of ties?

Take the average of ranks

Example: Rank the data 10, 21, 8, 5, 10, 30, 24, 10, 5

Unordered Data	10	21	8	5	10	30	24	10	5

Ordered Data	5	5	8	10	10	10	21	24	30
Rank without ties	1	2	3	4	5	6	7	8	9
Working	$\frac{1+2}{2} = 1.5$			$\frac{4+5+6}{2} = 5$					
Rank with ties	1.5	1.5	3	5	5	5	7	8	9

Wilcoxon signed rank test

- The normal approximation is good for $n \geq 15$
- The limiting distribution of T^+ is normal distribution such that

$$E(T^+) = \frac{n(n+1)}{4} \qquad\qquad Var(T^+) = \frac{n(n+1)(2n+1)}{24}$$

- The Z statistic is $Z = \frac{T^+ - E(T^+)}{\sqrt{Var(T^+)}} \sim N(0,1)$

- Approximation can also be used when critical values are not available in the table

One sample test

Question:

A company manufactures auto ancillaries. One of them are steel rods with median diameter 10 cm. A sample of 10 rods randomly selected from the production process gives the following results

9.1	10.1	9.9	9.9	10	9.8	9.7	9.8	9.9	8.6

Test the hypothesis that the median of population has reduced. [Use $\alpha = 0.05$].

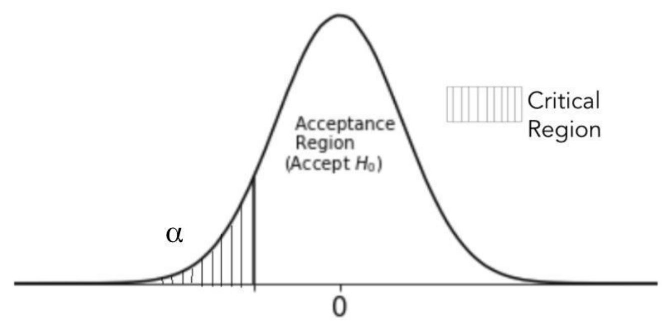
One sample test

Solution:

Let X: diameter of steel rod (in cm)
Let M be the median of X

Here $M_0 = 10$

To test, H_0: $M \geq 10$ against H_1: $M < 10$



One sample test

Solution:

Unordered X	9.1	10.1	9.9	9.9	10	9.8	9.7	9.8	9.9	8.6
Ordered X	8.6	9.1	9.7	9.8	9.8	9.9	9.9	9.9	10	10.1
D_i	-1.4	-0.9	-0.3	-0.2	-0.2	-0.1	-0.1	-0.1	0	0.1
$\lvert D_i \rvert$	1.4	0.9	0.3	0.2	0.2	0.1	0.1	0.1	0	0.1
Rank	9	8	7	5.5	5.5	2.5	2.5	2.5	-	2.5

It is obvious that $\min(T^+, T^-) = T^+$

The test statistic T^+ = Sum of positive signed ranks = 2.5

One sample test

n	Two-Tailed Test		One-Tailed Test	
	α = .05	α = .01	α = .05	α = .01
5	--	--	0	--
6	0	--	2	--
7	2	--	3	0
8	3	0	5	1
9	5	1	8	3
10	8	3	$\boxed{10}$	5
11	10	5	13	7
12	13	7	17	9
13	17	9	21	12
14	21	12	25	15
15	25	15	30	19
16	29	19	35	23
17	34	23	41	27
18	40	27	47	32
19	46	32	53	37
20	52	37	60	43
21	58	42	67	49
22	65	48	75	55
23	73	54	83	62
24	81	61	91	69
25	89	68	100	76
26	98	75	110	84
27	107	83	119	92
28	116	91	130	101
29	126	100	140	110
30	137	109	151	120

Solution:

The test statistic $T^+ = 2.5$ From the table $T^+ = 10$

Since $10 > 2.5$, reject H_0.

There is no sufficient evidence to claim that the median is 10.

One sample test

Python solution: Calculate test statistic and p-value

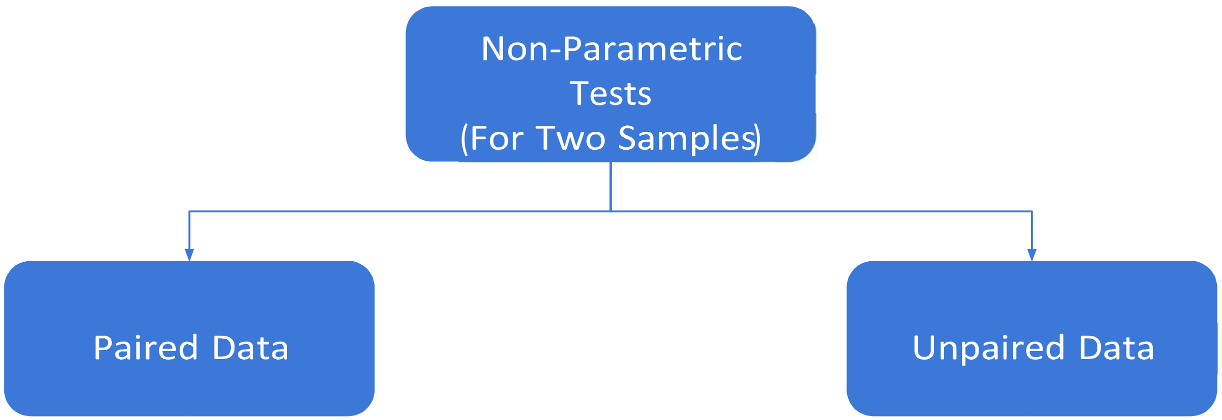
```python
# given diameters
diameter = np.array([9.1, 10.1, 9.9, 9.9, 10, 9.8, 9.7, 9.8, 9.9, 8.6])

# hypothesized median
M_0 = 10

# calculate the difference between diameter and M_0
diff = diameter - M_0

# perform wilcoxon signed rank test
# pass the differnces to the parameter, 'x'
# pass the one-tailed condition to the parameter, 'alternative'
test_stat, p_value = stats.wilcoxon(x = diff, alternative = 'less')

# print the test statistic and corresponding p-value
print('Test statistic:', test_stat)
print('p-value:', p_value)
```

```
Test statistic: 2.5
p-value: 0.008364861494636245
```

As the p-value < 0.05, we reject H_0.

Non-parametric Tests - Two Sample

Non-parametric tests - two samples



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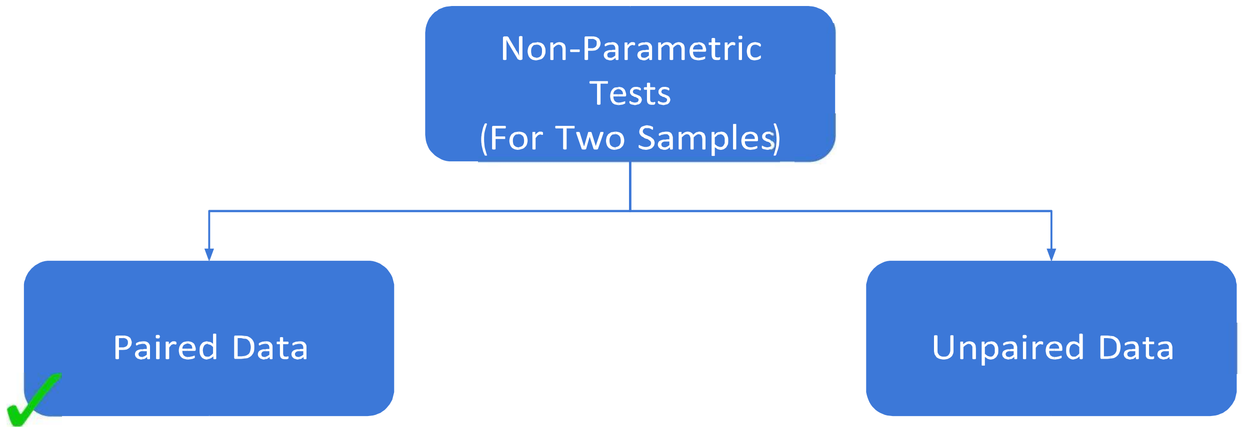
Test

- Wilcoxon Rank Sum Test
- Mann-Whitney U test

Non-parametric tests - two samples



- Wilcoxon Signed Rank Test

- Wilcoxon Rank Sum Test
- Mann-Whitney U test

Wilcoxon signed rank test

- The wilcoxon test is also used for as a non-parametric alternative for a paired t test

- It is done by simply taking the difference in each pair

- Let $x_1, x_2, .. x_n$ be a random sample of size n with distribution $F_X(.)$ of random variable X and $y_1, y_2, .. y_n$ be a random sample of size n with distribution $F_Y(.)$ of random variable Y

- It is assumed that the difference between the paired sample should be symmetric about the median

Wilcoxon signed rank test

- To test,

$H_0 : M_d = M_0$ against $H_1 : M_d \neq M_0$

$H_0 : M_d \leq M_0$ against $H_1 : M_d > M_0$

against

$H_0 : M_d \geq M_0$ against $H_1 : M_d < M_0$

- Failing to reject H_0, implies the null hypothesis is true

Wilcoxon signed rank test

- Compute the $D_i = X_i - Y_i$
- Rank $|D_i|$, where 1 is assigned to the smallest difference
- The test statistic is given by

One sided test	T^+
Two sided test	Min (T^+, T^-)

where

T^+	Sum of positive signed ranks
T^-	Sum of negative signed ranks

The python code to conduct Wilcoxon Signed Rank test for paired data is

```
scipy.stats.wilcoxon(Sample_1, Sample_2, alternative)
```

Wilcoxon signed rank test

- Failing to reject H_0, implies the median is equal to specified median value M_0

- The decision rule remain as that of one sample test

Non-parametric - paired data

Question:

The weights (in kg) of five hens is before and after a special diet of millets was given

Before	2.7	1.1	1.4	0.9	0.9
After	1.3	1.4	1.1	1.3	1.9

Test the hypothesis that the new millet diet has increased the weight of the hens at 5% level of significance.

Non-parametric - paired data

Solution:

Let X: weight (in kg) of hen before a special diet of millets

Let Y: weight (in kg) of hen after a special diet of millets

To test whether the the diet increases the weights i.e to test,
H_0: $M_d \geq 0$ against H_1: $M_d < 0$

Where M_d is the median of difference the weights before and after the diet.

Non-parametric - paired data

Solution:

X_i	2.7	1.1	1.4	0.9	0.9
Y_i	1.3	1.4	1.1	1.3	1.9
D_i	1.4	-0.3	0.3	-0.4	-0.6
$\|D_i\|$	1.4	0.3	0.3	0.4	0.6
Rank	5	1.5	1.5	3	4

It is obvious that $\min(T^+, T^-) = T^+$

The test statistic T^+

= Sum of positive signed ranks

= 1.5 + 5

= 6.5

Non-parametric - paired data

n	Two-Tailed Test		One-Tailed Test	
	α = .05	α = .01	α = .05	α = .01
5	--	--	$\boxed{0}$	--
6	0	--	2	--
7	2	--	3	0
8	3	0	5	1
9	5	1	8	3
10	8	3	10	5
11	10	5	13	7
12	13	7	17	9
13	17	9	21	12
14	21	12	25	15
15	25	15	30	19
16	29	19	35	23
17	34	23	41	27
18	40	27	47	32
19	46	32	53	37
20	52	37	60	43
21	58	42	67	49
22	65	48	75	55
23	73	54	83	62
24	81	61	91	69
25	89	68	100	76
26	98	75	110	84
27	107	83	119	92
28	116	91	130	101
29	126	100	140	110
30	137	109	151	120

Solution:

The test statistic $T^+ = 6.5$

From the table $T^+ = 0$

Since $6.5 > 0$, fail to reject H_0.

There is not enough evidence to conclude that the new millets diet increase the weight of hens.

Non-parametric - paired data

Python solution: Calculate test statistic and p-value

```
# given weights
before_wt = [2.7, 1.1, 1.4, 0.9, 0.9]
after_wt = [1.3, 1.4, 1.1, 1.3, 1.9]

# perform wilcoxon signed rank test for paired data
# pass the before and after weights to the parameter, 'x' and 'y' respectively
# pass the one-tailed condition to the parameter, 'alternative'
test_stat, p_value = stats.wilcoxon(x = before_wt, y = after_wt, alternative = 'less')

# print the test statistic and corresponding p-value
print('Test statistic:', test_stat)
print('p-value:', p_value)
```

```
Test statistic: 6.5
p-value: 0.40625
```

As the p-value > 0.05, we fail to reject H_0.

Non-parametric tests - two samples

Non-Parametric Tests (For Two Samples)

Paired Data

- Wilcoxon Signed Rank Test

Unpaired Data

- Wilcoxon Rank Sum Test
- Mann-Whitney U test

Wilcoxon rank sum test

- Used for unpaired data
- Test for the location parameter - median
- Equivalent to Mann-Whitney U test provided there are no ties in the ranked data

- Let $x_1, x_2, .. x_n$ be a random sample of size n_1 with distribution $F_X(.)$ of random variable X and $y_1, y_2, .. y_n$ be a random sample of size n_2 with distribution $F_Y(.)$ of random variable Y

Wilcoxon rank sum test

- Assumptions:
  - $F_X(.)$ and $F_Y(.)$ are continuous
  - The samples differ only in their locations
  - The samples are independent

- Let n_1 denote the observations of variable X and n_2 denote the observations of variable Y

- Total observations $N = n_1 + n_2$

Wilcoxon rank sum test

- Let m_1 denote the median of variable X and m_2 be the median of variable Y
- Let $M = m_1 - m_2$
- To test,

$H_0 : M = 0$ against $H_1 : M \neq 0$

$H_0 : M \leq 0$ against $H_1 : M > 0$ $H : M$

≥ 0 against $H_1 : M < 0$

Wilcoxon rank sum test

- Sort the combined data in ascending order and then rank it
- The test statistic T is given by sum of ranks assigned to X
- Failing to reject H_0, implies that two samples are drawn from identical distributions •

Decision rule:

○ If H_1: $M < 0$, reject H_0 if $T \leq T_L$

○ If H_1: $M > 0$, reject H_0 if $T \geq T_U$

- If H_1: $M \neq 0$, reject H_0 if $T \leq T_L$ or $T \geq T_U$

Wilcoxon rank sum test

Wilcoxon rank sum table for one sided test (α=0.025) and two sided test (α=0.05)

n_2 \ n_1	3		4		5		6		7		8		9		10	
	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U
3	5	16	6	18	6	21	7	23	7	26	8	28	8	31	9	33
4	6	18	11	25	12	28	12	32	13	35	14	38	15	41	16	44
5	6	21	12	28	18	37	19	41	20	45	21	49	22	53	24	56
6	7	23	12	32	19	41	26	52	28	56	29	61	31	65	32	70
7	7	26	13	35	20	45	28	56	37	68	39	73	41	78	43	83
8	8	28	14	38	21	49	29	61	39	73	49	87	51	93	54	98
9	8	31	15	41	22	53	31	65	41	78	51	93	63	108	66	114
10	9	33	16	44	24	56	32	70	43	83	54	98	66	114	79	131

Wilcoxon rank sum test

Wilcoxon rank sum table for one sided test (α=0.05) and two sided test (α=0.1)

n_2 \ n_1	3		4		5		6		7		8		9		10	
	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U	T_L	T_U
3	6	15	7	17	7	20	8	22	9	24	9	27	10	29	11	31
4	7	17	12	24	13	27	14	30	15	33	16	36	17	39	18	42
5	7	20	13	27	19	36	20	40	22	43	24	46	25	50	26	54
6	8	22	14	30	20	40	28	50	30	54	32	58	33	63	35	67
7	9	24	15	33	22	43	30	54	39	66	41	71	43	76	46	80
8	9	27	16	36	24	46	32	58	41	71	52	84	54	90	57	95
9	10	29	17	39	25	50	33	63	43	76	54	90	66	105	69	111
10	11	31	18	42	26	54	35	67	46	80	57	95	69	111	83	127

Wilcoxon rank sum test

- The normality approximation:

$$T^* = \frac{T - n_1(N+1)/2}{\sqrt{n_1 n_2 (N+1)/12}} \sim N(0,1)$$

as $\min(n_1, n_2) \to \infty$

- In general normal approximation is used when n_1, $n_2 > 15$

The python code to conduct Wilcoxon Rank Sum for unpaired data is

```
scipy.stats.stats.ranksums(Sample_1, Sample_2)
```

Non-parametric - unpaired data

Question:

The lengths of time in minutes spent in the operating room by 9 patients undergoing the same operating method. 4 patients are from hospital A and 5 are from hospital B

A	32	31	33	46	
B	49	20	58	55	52

On the basis of these data can we conclude that for the sample operative method, patients in hospital B tend to be longer in hospital A. Consider 5% level of significance.

Non-parametric - unpaired data

Solution:

Let X: the time spent by patient in hospital A in operating room

Y: the time spent by patient in hospital B in operating room

To test: For the sample operative method, patients in hospital B tend to be longer in hospital A

H_0: Time spent by patients in both the hospital operating rooms is same i.e. $M \geq 0$ Against

H_1: Time spent by patients in hospital B is more than that of patients of hospital A i.e. $M < 0$

Non-parametric - unpaired data

Solution:

Here, $n_1 = 4$ and $n_2 = 5$

Pooled sample	32	31	33	46	

	Data from X
	Data from Y

Ordered Sample	20	31	32	33	
Rank	1	2	3	4	

The test statistic T = Sum of ranks assigned to X = 2 + 3 + 4 + 5 = 14

Non-parametric - unpaired data

Solution:

The test statistic T = 14

Here, $n_1 = 4$ and $n_2 = 5$, thus $T_L = 12$

Decision rule: reject H_0 if $T < T_L$

Since $T > T_L$, we accept H_0.

We can not conclude that the patients of hospital B tend to spend more time in the operating room than patients of hospital A

n_2 \ n_1	3		4		5	
	T_L	T_U	T_L	T_U	T_L	T_U
3	5	16	6	18	6	21
4	6	18	11	25	12	28
5	6	21	$\boxed{12}$	28	18	37
6	7	23	12	32	19	41
7	7	26	13	35	20	45
8	8	28	14	38	21	49
9	8	31	15	41	22	53
10	9	33	16	44	24	56

Non-parametric - unpaired data

Python solution: Calculate test statistic and p-value

```python
# given data
A = [32, 31, 33, 46]
B = [49, 20, 58, 55, 52]

# compute test statistic and corresponding p-value for two-tailed test
test_stat, p_value = stats.ranksums(A, B)

# divide the p-value by 2
req_p_val = p_value/2

# print the test statistic value and corresponding p-value
print('Test Statistic:', test_stat)
print('p-value:', req_p_val)
```

```
Test Statistic: -1.4696938456699067
p-value: 0.0708223451475684
```

Python function uses normal approximation to calculate the test statistic. As the p-value > 0.05, we fail to reject H_0.

Mann-whitney U test

- Used for unpaired data
- Test for the location parameter - median
- Let $x_1, x_2, .. x_{n_1}$ be a random sample of size n_1 with distribution $F_X(.)$ of random variable X and $y_1, y_2, .. y_{n_2}$ be a random sample of size n_2 with distribution $F_Y(.)$ of random variable Y

Mann-whitney U test

- Assumptions:
  - $F_X(.)$ and $F_Y(.)$ are continuous
  - The samples are independent
- Let m_1 denote the median of variable X and m_2 be the median of variable Y

- Total observations $N = n_1 + n_2$
- Let $M = m_2 - m_1$

Mann-whitney U test

- Let m_1 denote the median of variable X and m_2 be the median of variable Y
- Let $M = m_2 - m_1$

- To test,

$H_0 : M = 0$ against $H_1 : M \neq 0$

$H_0 : M \leq 0$ against $H_1 : M > 0$

$H_0 : M \geq 0$ against $H_1 : M < 0$

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Mann-whitney U test

- However, the appropriate null hypothesis is that the two samples are drawn from identical distributions

- The modified hypothesis are

$H_0 : F_X = F_Y$ against $H_1 : F_X \neq F_Y$

$H_0 : F_X \leq F_Y$ against $H_1 : F_X > F_Y$

$H_0 : F_X \geq F_Y$ against $H_1 : F_X < F_Y$

Mann-whitney U test

- The assumptions and test hypothesis are similar Wilcoxon sum rank test
- The test statistics are different but are equivalent
- The test statistic is $U = \sum_i \sum_j \phi(x_i, y_j)$

where $\phi(x_i, y_j) = \begin{cases} 1 & \text{if } x_i < y_j \\ 0 & \text{otherwise} \end{cases}$

- The statistic U is the number of times x_i preceding y_j among all (x_i, y_j) pairs and U' be the number of times x_i succeeding y_j among all (x_i, y_j) pairs

For no ties in the data, $U + U' = n_1.n_2$

Mann-whitney U test

- Wilcoxon test uses the ranks while Mann-Whitney test statistic measures the count

- Decision rule: Reject H_0 if the value of test statistic is less than the table value

Non-parametric - unpaired data

Question:

The lengths of time in minutes spent in the operating room by 9 patients undergoing the same operating method. 4 patients are from hospital A and 5 are from hospital B

A	32	31	33	46	

B	49	20	58	55	52

On the basis of these data can we conclude that for the sample operative method, patients in hospital B tend to be longer than in hospital A. Consider 5% level of significance.

Non-parametric - unpaired data

Solution:

Let F_X: the distribution of time spent by patient in hospital A in operating room

Non-parametric - unpaired data

Solution:

F_Y: the distribution of time spent by patient in hospital B in operating room

To test: For the sample operative method, patients in hospital B tend to be longer in hospital A

$H_0: F_X \geq F_Y$

Against $H_1: F_X < F_Y$

Here, $n_1 = 4$ and $n_2 = 5$

We have our data as

A	32	31	33	46	
B	49	20	58	55	52

Count the number of times x_i precedes y_j

Non-parametric - unpaired data

Solution:

Consider $x_i = 32$

The possible pairs are (32, 49), (32, 20), (32, 58), (32, 55), (32, 52)

T he number of pairs in which $32 < y_j$

A	32	31	33	46	
B	49	20	58	55	52

Thus $\Phi(x_1, y_j) = 4$

Consider all the pair and obtain $\Phi(x_2, y_j)$, $\Phi(x_3, y_j)$, and $\Phi(x_4, y_j)$

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Non-parametric - unpaired data

Solution:

A	32	31	33	46	
B	49	20	58	55	52

We obtain $U = \sum \sum \Phi(x_i, y_j) = 4 + 4 + 4 + 4 = 16$

And we have $U' = 1 + 1 + 1 + 1 = 4$

We see $U + U' = 20$ and $n_1.n_2 = 4 \times 5 = 20$

Mann-whitney U test

- It is seen that for large n_1 and n_2 previous (direct) method is not feasible. Thus we can use the indirect method

- The test statistic is defined as $U = \min\{U_1, U_2\}$

Where

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

R_1 is the sum of ranks of 1st sample and R_2 is the sum of ranks of 2nd sample

- $U_1 + U_2 = n_1.n_2$

The python code to conduct Mann Whitney U test for unpaired data is

scipy.stats.stats.mannwhitneyu(Sample_1, Sample_2, alternative)

Mann-whitney U test - one tailed table

n_2	α	n_1																	
		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	.05	0	0	1	2	2	3	4	4	5	5	6	7	7	8	9	9	10	11
	.01	--	0	0	0	0	0	1	1	1	2	2	2	3	3	4	4	4	5
4	.05	0	1	2	3	4	5	6	7	8	9	10	11	12	14	15	16	17	18
	.01	--	--	0	1	1	2	3	3	4	5	5	6	7	7	8	9	9	10
5	.05	1	2	4	5	6	8	9	11	12	13	15	16	18	19	20	22	23	25
	.01	--	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	.05	2	3	5	7	8	10	12	14	16	17	19	21	23	25	26	28	30	32
	.01	--	1	2	3	4	6	7	8	9	11	12	13	15	16	18	19	20	22
7	.05	2	4	6	8	11	13	15	17	19	21	24	26	28	30	33	35	37	39
	.01	0	1	3	4	6	7	9	11	12	14	16	17	19	21	23	24	26	28
8	.05	3	5	8	10	13	15	18	20	23	26	28	31	33	36	39	41	44	47
	.01	0	2	4	6	7	9	11	13	15	17	20	22	24	26	28	30	32	34
9	.05	4	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54
	.01	1	3	5	7	9	11	14	16	18	21	23	26	28	31	33	36	38	40
10	.05	4	7	11	14	17	20	24	27	31	34	37	41	44	48	51	55	58	62
	.01	1	3	6	8	11	13	16	19	22	24	27	30	33	36	38	41	44	47
11	.05	5	8	12	16	19	23	27	31	34	38	42	46	50	54	57	61	65	69
	.01	1	4	7	9	12	15	18	22	25	28	31	34	37	41	44	47	50	53
12	.05	5	9	13	17	21	26	30	34	38	42	47	51	55	60	64	68	72	77
	.01	2	5	8	11	14	17	21	24	28	31	35	38	42	46	49	53	56	60
13	.05	6	10	15	19	24	28	33	37	42	47	51	56	61	65	70	75	80	84
	.01	2	5	9	12	16	20	23	27	31	35	39	43	47	51	55	59	63	67
14	.05	7	11	16	21	26	31	36	41	46	51	56	61	66	71	77	82	87	92
	.01	2	6	10	13	17	22	26	30	34	38	43	47	51	56	60	65	69	73
15	.05	7	12	18	23	28	33	39	44	50	55	61	66	72	77	83	88	94	100
	.01	3	7	11	15	19	24	28	33	37	42	47	51	56	61	66	70	75	80
16	.05	8	14	19	25	30	36	42	48	54	60	65	71	77	83	89	95	101	107
	.01	3	7	12	16	21	26	31	36	41	46	51	56	61	66	71	76	82	87
17	.05	9	15	20	26	33	39	45	51	57	64	70	77	83	89	96	102	109	115
	.01	4	8	13	18	23	28	33	38	44	49	55	60	66	71	77	82	88	93
18	.05	9	16	22	28	35	41	48	55	61	68	75	82	88	95	102	109	116	123
	.01	4	9	14	19	24	30	36	41	47	53	59	65	70	76	82	88	94	100
19	.05	10	17	23	30	37	44	51	58	65	72	80	87	94	101	109	116	123	130
	.01	4	9	15	20	26	32	38	44	50	56	63	69	75	82	88	94	101	107
20	.05	11	18	25	32	39	47	54	62	69	77	84	92	100	107	115	123	130	138
	.01	5	10	16	22	28	34	40	47	53	60	67	73	80	87	93	100	107	114

Mann-whitney U test - two tailed table

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n_2	α	n_1																	
		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	.05	--	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
	.01	--	0	0	0	0	0	0	0	0	1	1	1	2	2	2	2	3	3
4	.05	--	0	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	14
	.01	--	--	0	0	0	1	1	2	2	3	3	4	5	5	6	6	7	8
5	.05	0	1	2	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20
	.01	--	--	0	1	1	2	3	4	5	6	7	7	8	9	10	11	12	13
6	.05	1	2	3	5	6	8	10	11	13	14	16	17	19	21	22	24	25	27
	.01	--	0	1	2	3	4	5	6	7	9	10	11	12	13	15	16	17	18
7	.05	1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
	.01	--	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	22	24
8	.05	2	4	6	8	10	13	15	17	19	22	24	26	29	31	34	36	38	41
	.01	--	1	2	4	6	7	9	11	13	15	17	18	20	22	24	26	28	30
9	.05	2	4	7	10	12	15	17	20	23	26	28	31	34	37	39	42	45	48
	.01	0	1	3	5	7	9	11	13	16	18	20	22	24	27	29	31	33	36
10	.05	3	5	8	11	14	17	20	23	26	29	33	36	39	42	45	48	52	55
	.01	0	2	4	6	9	11	13	16	18	21	24	26	29	31	34	37	39	42
11	.05	3	6	9	13	16	19	23	26	30	33	37	40	44	47	51	55	58	62
	.01	0	2	5	7	10	13	16	18	21	24	27	30	33	36	39	42	45	48
12	.05	4	7	11	14	18	22	26	29	33	37	41	45	49	53	57	61	65	69
	.01	1	3	6	9	12	15	18	21	24	27	31	34	37	41	44	47	51	54
13	.05	4	8	12	16	20	24	28	33	37	41	45	50	54	59	63	67	72	76
	.01	1	3	7	10	13	17	20	24	27	31	34	38	42	45	49	53	56	60
14	.05	5	9	13	17	22	26	31	36	40	45	50	55	59	64	67	74	78	83
	.01	1	4	7	11	15	18	22	26	30	34	38	42	46	50	54	58	63	67
15	.05	5	10	14	19	24	29	34	39	44	49	54	59	64	70	75	80	85	90
	.01	2	5	8	12	16	20	24	29	33	37	42	46	51	55	60	64	69	73
16	.05	6	11	15	21	26	31	37	42	47	53	59	64	70	75	81	86	92	98
	.01	2	5	9	13	18	22	27	31	36	41	45	50	55	60	65	70	74	79
17	.05	6	11	17	22	28	34	39	45	51	57	63	67	75	81	87	93	99	105
	.01	2	6	10	15	19	24	29	34	39	44	49	54	60	65	70	75	81	86
18	.05	7	12	18	24	30	36	42	48	55	61	67	74	80	86	93	99	106	112
	.01	2	6	11	16	21	26	31	37	42	47	53	58	64	70	75	81	87	92
19	.05	7	13	19	25	32	38	45	52	58	65	72	78	85	92	99	106	113	119
	.01	3	7	12	17	22	28	33	39	45	51	56	63	69	74	81	87	93	99
20	.05	8	14	20	27	34	41	48	55	62	69	76	83	90	98	105	112	119	127
	.01	3	8	13	18	24	30	36	42	48	54	60	67	73	79	86	92	99	105

Non-parametric - unpaired data

Solution:

We have

	Data from A
	Data from B

Pooled sample	32	31	33	46	

49

Ordered Sample	20	31	32	33	

46

Rank	1	2	3	4	5	6	7	8	9

$R_1 = 2+3+4+5 = 14$ and $R_2 = 1+6+7+8+9 = 31$

Non-parametric - unpaired data

Here, $n_1 = 4$ and $n_2 = 5$

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$
$$= 20 + 10 - 14$$
$$= 16$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$
$$= 20 + 15 - 31$$
$$= 4$$

Non-parametric - unpaired data

Solution:

$$U = min(U_1, U_2) = 4$$

The test statistic

$$U = min(U_1, U_2) = 4$$

The table value is 2.

n_2	α	n_1 = 3	4	5	6
3	.05	0	0	1	2
	.01	--	0	0	0
4	.05	0	1	2	3
	.01	--	--	0	1
5	.05	1	$\boxed{2}$	4	5
	.01	--	0	1	2
6	.05	2	3	5	7
	.01	--	1	2	3
7	.05	2	4	6	8
	.01	0	1	3	4

Thus we fail to reject H_0. We can not conclude that the of hospital B tend to spend more time in the operating than patients of hospital A

patients

room

Non-parametric - unpaired data

Python solution: Calculate test statistic and p-value

```
# compute test statistic and corresponding p-value for one-tailed test
test_stat, p_value = stats.mannwhitneyu(A, B, alternative = 'less')

# print the test statistic and corresponding p-value
print('Test statistic:', test_stat)
print('p-value:', p_value)
```

```
Test statistic: 4.0
p-value: 0.08895479749349122
```

As the p-value > 0.05, we fail to reject H_0.

Thank You