Class Exercise (Kalman Filter)
Introduction to Data Assimilation

Exercise 8b (KF with a Linear Advection Equation)

Consider a one-dimensional linear advection model on a periodic domain of length 360 m ($L_a = -180 \leq x_{axis} < L_b = 180$). The model has a constant advection speed, μ=1 m/s, the grid spacing is Δx = 1 m and the time step Δt = 1 s.

The true initial state $\mathbf{x}_0^t$ is sampled from a normal distribution with mean equal to a square wave (use sqrwv.m in D2L), variance equal to zero and spatial de-correlation length of 45 m. Yes, we assume here that the true state, $\mathbf{x}^\mathrm{t}$ is without system noise.

The initial forecast states $\mathbf{x}_0^f$, however are generated by drawing a sample from a normal distribution with mean $\mathbf{x}_0^t$, variance equal to 0.5 and spatial de-correlation length of 45 m similar to the length scale of $\mathbf{x}_0^t$.

1) Construct $\mathbf{x}_0^t$, $\mathbf{x}_0^f$ and $\mathbf{P}_0^f = \mathbf{Q}$. Use gcorr.m (and gauss.m) – can be found in D2L -- to generate the correlation matrix, $\boldsymbol{\rho}$. Also use mvnrnd.m to draw the sample.

2) Construct a time-independent state transition matrix **M** using Lax-Wendorff scheme to solve the advection equation:

$$\frac{\partial a}{\partial t} = -\mu \frac{\partial a}{\partial x}$$

$$a_i^{k+1} = a_i^k - \frac{\mu \Delta t}{2\Delta x}\left(a_{i+1}^k - a_{i-1}^k\right) + 2\left(\frac{\mu \Delta t}{2\Delta x}\right)^2 \left(a_{i+1}^k + a_{i-1}^k - 2a_i^k\right)$$

   where *k* and *i* denote time and position, respectively.

3) Construct the observation operator assuming that there are a) zero measurements, b) 1 measurement at grid point 10 and c) 4 measurements at grid points 10, 100, 190 and 280. Assume a measurement error variance of 0.01 (and a diagonal error covariance, **R**).

4) Find an optimal estimate $\mathbf{x}_{k=1,300}^a$ using a Kalman Filter and using measurements in (3) and assuming for now that $\mathbf{Q} = \mathbf{0}$. Plot your results (truth, forecast, analysis and observation/s) for all grid points and for iteration 5, iteration 150, and iteration 300. Provide comments on your results.

5) Do (4) but now assume $\mathbf{Q} = \boldsymbol{\sigma}^2 \boldsymbol{\rho}$.

Note: This is a continuation of Exercise 8a (fonts in red are the new tasks).