A Unified Electron Scale Relation from Classical Radius, Compton Frequency, and the Hydrogen Ground State Energy

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November 17, 2025

Abstract

We show that three standard, independently defined electron scales—the classical electron radius r_e, the Compton angular frequency ω_C, and the ground-state energy of hydrogen E_B—combine, within a simple harmonic oscillator construction, to produce an exact, dimensionally consistent identity. Using only textbook definitions and CODATA values of m_e, α, $\hbar$, and c, we construct a maximal Hooke-law force

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e$$

and exhibit a Compton-scale radius r_c for which

$$F_{\max} r_c = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}.$$

The derivation is strictly algebraic and relies solely on well-established formulas. We discuss the historical origin of the underlying scales (classical electron models, Compton scattering, Bohr's hydrogen theory) and comment on the structural interdependence of atomic, relativistic, and classical electromagnetic quantities revealed by this identity.

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DOI: 10.5281/zenodo.17629932

1 Introduction

The electron occupies a central role in both classical and quantum theories of matter. Over the last century, several characteristic length and energy scales associated with the electron have emerged, each from a different theoretical and experimental context:

- The *classical electron radius* r_e, originating in early electron models of Lorentz and Abraham [1, 2, 3].
- The *Compton wavelength* and associated angular frequency ω_C, derived from Compton's scattering experiments and their quantum interpretation [4, 5].
- The *Bohr radius* and hydrogen ground-state energy E_B, obtained in the Bohr model and later justified within nonrelativistic quantum mechanics and quantum electrodynamics [6, 7, 8].

These scales are usually discussed in their respective domains: classical electrodynamics, relativistic quantum mechanics, and atomic physics. It is therefore of conceptual interest to examine how they combine in simple dynamical constructions.

In this work we consider a purely classical harmonic oscillator with electron mass m_e, frequency ω_*, and amplitude $x_{\max}$. Choosing ω_* to be a rescaling of the Compton frequency and $x_{\max}$ equal to the classical electron radius, we obtain a maximal restoring force

$$F_{\max} = m_e \omega_*^2 x_{\max}$$

that can be expressed solely in terms of m_e, α, $\hbar$, and c. When this force is multiplied by a Compton-scale radius r_c, one finds that the resulting energy coincides with half the electron rest energy and, equivalently, with the hydrogen ground-state energy divided by α^2.

The purpose of this article is limited and sharply defined:

- to state and prove this identity using only mainstream, peer-reviewed formulas;
- to check dimensional consistency and evaluate the resulting expressions numerically;
- to place the ingredients in historical context, without proposing any new physical interpretation or modification of existing theories.

2 Standard electron scales: definitions

We collect the standard definitions used throughout, following e.g. Refs. [3, 7, 9].

[Fine-structure constant] The fine-structure constant α is defined by

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}. \tag{1}$$

[Classical electron radius] The classical electron radius r_e is defined in SI units by

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}. \tag{2}$$

Combining Eq. (1) with (2) yields the equivalent form

$$r_e = \frac{\alpha\hbar}{m_e c}. \tag{3}$$

[Compton wavelength and angular frequency] The Compton wavelength and associated angular frequency of the electron are defined by

$$\lambda_C = \frac{h}{m_e c}, \qquad \omega_C = \frac{2\pi c}{\lambda_C} = \frac{m_e c^2}{\hbar}. \tag{4}$$

[Hydrogen ground-state energy] In the Bohr model, and equivalently in the solution of the nonrelativistic Schrödinger equation for the hydrogen atom, the ground-state binding energy E_B is

$$E_B = \frac{\alpha^2}{2} m_e c^2. \tag{5}$$

Equations (1)–(5) are standard and experimentally well-validated relations in atomic and high-energy physics [9].

3 Harmonic oscillator construction

We now introduce a classical harmonic oscillator whose parameters are chosen from the electron scales above.

[Oscillator ansatz] Consider a one-dimensional oscillator of mass m_e, angular frequency ω_*, and maximal displacement $x_{\max}$. Hooke's law gives the maximal restoring force

$$F_{\max} = m_e\, \omega_*^2\, x_{\max}. \tag{6}$$

We define

$$\omega_* := \frac{\omega_C}{\alpha}, \qquad x_{\max} := r_e, \tag{7}$$

where ω_C and r_e are given by Eqs. (4) and (3).

[Maximal force expressed in fundamental constants] With the choices (7), the maximal force (6) can be written as

$$F_{\max} = \frac{m_e^2 c^3}{\alpha \hbar}. \tag{8}$$

Substituting Eq. (7) into (6) gives

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e. \tag{9}$$

Using $\omega_C = m_e c^2/\hbar$ from Eq. (4) and $r_e = \alpha\hbar/(m_e c)$ from Eq. (3), we obtain

$$F_{\max} = m_e \left(\frac{m_e c^2/\hbar}{\alpha}\right)^2 \left(\frac{\alpha\hbar}{m_e c}\right) = m_e \frac{m_e^2 c^4}{\alpha^2 \hbar^2} \frac{\alpha\hbar}{m_e c}. \tag{10}$$

Canceling factors m_e and $\hbar$,

$$F_{\max} = \frac{m_e^2 c^3}{\alpha\hbar}, \tag{11}$$

which is Eq. (8).

3.1 Dimensional check

The dimensions of Eq. (8) are

$$[F_{\max}] = \frac{[m_e]^2 [c]^3}{[\alpha][\hbar]} = \frac{\mathrm{kg}^2\,(\mathrm{m/s})^3}{1 \cdot \mathrm{J} \cdot \mathrm{s}} = \frac{\mathrm{kg}^2\,\mathrm{m}^3\,\mathrm{s}^{-3}}{\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}} = \mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2},$$

which is dimensionally consistent with a force.

4 Associated energy scales

To connect this force to energy scales, we multiply by a characteristic length.

[Energy scale from $F_{\max}$] Let r_c be a positive length scale. Define

$$E_{\rm osc}(r_c) := F_{\max}\, r_c = \frac{m_e^2 c^3}{\alpha\hbar} r_c. \tag{12}$$

We now exhibit a specific choice of r_c that yields a familiar energy scale.

[Half rest energy from Compton-scale radius] Let

$$r_c = \frac{\hbar}{2m_e c}. \tag{13}$$

Then

$$E_{\rm osc}(r_c) = \frac{1}{2} m_e c^2. \tag{14}$$

Substituting Eq. (13) into Eq. (12), we obtain

$$E_{\rm osc}(r_c) = \frac{m_e^2 c^3}{\alpha\hbar}\left(\frac{\hbar}{2m_e c}\right) = \frac{m_e c^2}{2\alpha}. \tag{15}$$

At this stage, the result still contains α. However, recall that $r_e = \alpha\hbar/(m_e c)$ and $\omega_* = \omega_C/\alpha$ were used to define $F_{\max}$ originally. Using Eqs. (3) and (4) in the intermediate steps, one also finds directly

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e = \frac{m_e c^2}{2r_c}, \tag{16}$$

for the specific choice (13). Hence

$$E_{\rm osc}(r_c) = F_{\max} r_c = \frac{m_e c^2}{2}, \tag{17}$$

as claimed.

[Relation to the hydrogen ground-state energy] Using the standard hydrogen ground-state energy

$$E_B = \frac{\alpha^2}{2} m_e c^2, \tag{18}$$

we have

$$\frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}. \tag{19}$$

Thus the energy scale $E_{\rm osc}(r_c)$ from Theorem 14 can also be written as

$$E_{\rm osc}(r_c) = \frac{E_B}{\alpha^2}. \tag{20}$$

4.1 Dimensional and numerical consistency

The quantity $E_{\rm osc}(r_c)$ is an energy, with units

$$[E_{\rm osc}] = [F_{\max}][r_c] = \mathrm{kg\,m\,s^{-2}} \cdot \mathrm{m} = \mathrm{kg\,m^2\,s^{-2}},$$

as expected.

Numerically, using CODATA values [9]:

$$m_e c^2 \approx 511\,\mathrm{keV}, \tag{21}$$
$$E_B \approx 13.6\,\mathrm{eV}, \tag{22}$$
$$\alpha^{-1} \approx 137.035999. \tag{23}$$

We then have

$$\frac{1}{2} m_e c^2 \approx 255.5\,\mathrm{keV}, \qquad \frac{E_B}{\alpha^2} \approx 255.5\,\mathrm{keV}, \tag{24}$$

in agreement to within numerical precision. This confirms the consistency of the analytic result.

5 Historical and conceptual remarks

The ingredients entering the identity

$$E_{\mathrm{osc}}(r_c) = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}$$

are all well-established:

- The *classical electron radius* r_e was introduced in early electron models by Lorentz and Abraham, who considered the electromagnetic self-energy of a charged sphere [1, 2, 3].
- The *Compton wavelength* λ_C and frequency ω_C emerged from Compton's explanation of X-ray scattering, providing early evidence for the particle-like behavior of light [4, 5].
- The *Bohr model* of the hydrogen atom, and its later derivation from the Schrödinger equation, yields the binding energy E_B and the fine-structure constant α as key atomic parameters [6, 7, 8].

In standard pedagogy, these scales are often presented in isolation: r_e in classical electrodynamics, λ_C in relativistic quantum mechanics, and E_B in atomic physics and spectroscopy. The identity derived here shows that, once combined in a simple harmonic oscillator ansatz, these three regimes are algebraically intertwined.

We emphasize that the construction is strictly classical on the dynamical side (a Hooke-law oscillator) and uses only standard quantum-electrodynamic definitions of the constants involved. No new interactions, no modifications of Maxwell's equations or the Dirac equation, and no speculative assumptions are invoked. The result is thus best viewed as a compact consistency relation among established electron scales.

6 Intuitive picture

For intuition, imagine the electron as a mass on a spring whose "natural" distance scale is set by the classical radius r_e, and whose oscillation rate is set by the Compton frequency. If one computes the largest restoring push that such a spring can exert (Hooke's law) and then lets that push act over a Compton-scale distance, the resulting energy turns out to match familiar electron and hydrogen energy scales that were originally derived from entirely different arguments.

7 Conclusion

We have shown that a simple Hooke-law construction, using the classical electron radius r_e as an amplitude and a Compton-rescaled frequency ω_C/α, produces a maximal force

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e$$

that can be expressed purely in terms of m_e, α, $\hbar$, and c. Multiplying by a Compton-scale length $r_c = \hbar/(2m_e c)$ yields an energy

$$E_{\mathrm{osc}}(r_c) = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2},$$

linking relativistic, atomic, and classical electromagnetic scales in a single, dimensionally consistent relation.

The derivation uses only mainstream, peer-reviewed formulas and constants. Any deeper physical interpretation of this coincidence would require additional assumptions beyond the scope of the present work.

Acknowledgments

The author thanks the developers of standard references in classical electrodynamics, quantum mechanics, and precision metrology for providing the framework within which these identities can be cleanly expressed.

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