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Rotational Kinetic Energy Density and an Effective Mass Relation in Incompressible Fluids

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Abstract

Kinetic energy contributes to inertia and gravitational mass through the relativistic relation $E = mc^2$. In extended media such as fluids, this contribution can be expressed as an effective mass density associated with internal motion. We consider an incompressible, inviscid Newtonian fluid undergoing rigid-body rotation in a finite cylinder and compute the volume-averaged rotational kinetic energy density. By associating this energy density with an effective mass density via $E = mc^2$ in the nonrelativistic limit, we obtain the closed-form relation

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2,$$

where v_{edge} is the tangential speed at the cylinder boundary and ρ is the rest-mass density. The result provides a transparent classical example of how rotational motion modifies the mass density at order $(v/c)^2$. We discuss the connection with relativistic continuum mechanics and provide numerical estimates for laboratory and astrophysical regimes.

1 Introduction

The equivalence between energy and mass, expressed by $E = mc^2$, implies that kinetic, field, and binding energies contribute to the inertia and gravitational mass of extended systems [1, 2, 3]. In the context of relativistic continuum mechanics, this statement is encoded in the stress–energy tensor $T^{\mu\nu}$: the total mass–energy is obtained by integrating T^{00} over a spatial hypersurface, and T^{00} includes both rest-mass and kinetic contributions [3, 4].

While this viewpoint is standard in high-energy physics and general relativity, explicit examples in simple fluid configurations remain pedagogically useful. In particular, it is instructive to make the contribution of *rotational* kinetic energy to an effective mass density quantitatively explicit in a setting where the flow field is analytically tractable.

In this paper we analyze a canonical configuration from classical fluid mechanics [5, 6]: an incompressible, inviscid fluid in rigid-body rotation inside a finite cylinder. Within this model we:

1. compute the local and volume-averaged rotational kinetic energy density;
2. define an effective mass density $\Delta\rho_{\text{eff}}$ via $E = mc^2$ in the regime $v \ll c$;
3. derive a compact expression for $\Delta\rho_{\text{eff}}/\rho$ in terms of the edge speed v_{edge};
4. discuss how this classical result fits within the framework of relativistic continuum mechanics.

The derivation uses only incompressible Euler flow and the special-relativistic mass–energy relation. No modifications of Newtonian or relativistic theory are proposed.

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2 Rigid-body rotation in an incompressible, inviscid fluid

2.1 Flow configuration

We consider a Newtonian fluid of constant rest-mass density ρ occupying a right circular cylinder of radius R and height L. The fluid undergoes steady rigid-body rotation with constant angular velocity Ω about the z-axis. In cylindrical coordinates (r, θ, z), with $0 \leq r \leq R$, the velocity field is

$$\boldsymbol{v}(r) = \Omega r\, \hat{\boldsymbol{\theta}}. \tag{1}$$

This flow is incompressible and inviscid:

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \text{viscosity} = 0, \tag{2}$$

and it satisfies the steady Euler equations with an appropriate pressure distribution [5, 6].

2.2 Local kinetic energy density

The local kinetic energy density of the fluid is

$$e_{\text{kin}}(r) = \frac{1}{2}\,\rho \|\boldsymbol{v}(r)\|^2 = \frac{1}{2}\,\rho\,\Omega^2 r^2. \tag{3}$$

This quantity is position-dependent and increases quadratically with radius.

2.3 Total rotational energy and volume-averaged energy density

The total rotational kinetic energy is obtained by integrating Eq. (3) over the fluid volume:

$$\begin{aligned} E_{\text{rot}} &= \int_V e_{\text{kin}}\,\mathrm{d}V \\ &= \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{2}\,\rho\,\Omega^2 r^2\, r\,\mathrm{d}r \\ &= \frac{1}{2}\,\rho\,\Omega^2 (2\pi L) \int_0^R r^3\,\mathrm{d}r \\ &= \frac{\pi}{4}\,\rho\,\Omega^2\, L\, R^4. \end{aligned} \tag{4}$$

The cylinder volume is $V = \pi R^2 L$, so the volume-averaged kinetic energy density is

$$\langle e_{\text{kin}} \rangle = \frac{E_{\text{rot}}}{V} = \frac{\frac{\pi}{4}\,\rho\,\Omega^2\, L\, R^4}{\pi R^2 L} = \frac{1}{4}\,\rho\,\Omega^2 R^2. \tag{5}$$

It is convenient to express this in terms of the edge speed

$$v_{\text{edge}} := \Omega R. \tag{6}$$

Then Eq. (5) becomes

$$\langle e_{\text{kin}} \rangle = \frac{1}{4}\,\rho\, v_{\text{edge}}^2. \tag{7}$$

For comparison, the kinetic energy density at the boundary is

$$e_{\text{kin}}(R) = \frac{1}{2}\,\rho\, v_{\text{edge}}^2, \tag{8}$$

so the volume average is exactly one half of the boundary value, reflecting the quadratic radial profile.

3 Effective mass density from $E = mc^2$

3.1 Nonrelativistic limit and effective density

In special relativity, the total relativistic energy of a fluid element with rest-mass density ρ and small velocity $v \ll c$ can be decomposed as [3, 4]

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \cdots , \tag{9}$$

where ε is the total energy density in the local rest frame, and the ellipsis denotes higher-order terms in v^2/c^2 and internal energy contributions. To leading order in v^2/c^2, the kinetic part of the energy density can therefore be regarded as an *effective mass density* via

$$\Delta\rho_{\text{eff}}(\boldsymbol{x}) = \frac{e_{\text{kin}}(\boldsymbol{x})}{c^2}. \tag{10}$$

This interpretation is consistent with the structure of the stress–energy tensor for a perfect fluid [3].

For the rigidly rotating configuration considered here, we focus on the volume-averaged effective mass density

$$\Delta\rho_{\text{eff}} := \frac{\langle e_{\text{kin}} \rangle}{c^2}. \tag{11}$$

Inserting Eq. (7) into Eq. (11), we obtain

$$\Delta\rho_{\text{eff}} = \frac{1}{4c^2}\, \rho\, v_{\text{edge}}^2. \tag{12}$$

Dividing by the rest-mass density ρ yields

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2. \tag{13}$$

Thus, in the nonrelativistic regime, the rotational contribution to the mass density is second order in v_{edge}/c, with a geometric coefficient $1/4$ specific to rigid-body rotation in a cylinder.

3.2 Dimensional check

The dimensions of Eq. (12) are

$$[\Delta\rho_{\text{eff}}] = \frac{[\rho][v]^2}{[c]^2} = \frac{\text{kg}\,\text{m}^{-3}\,(\text{m/s})^2}{(\text{m/s})^2} = \text{kg}\,\text{m}^{-3},$$

as required for a mass density. The ratio $\Delta\rho_{\text{eff}}/\rho$ in Eq. (13) is dimensionless, as expected.

4 Numerical estimates

4.1 Laboratory-scale example

Consider water with $\rho \approx 1.0 \times 10^3\,\text{kg}\,\text{m}^{-3}$, a cylinder of radius $R = 0.10\,\text{m}$, and angular velocity $\Omega = 1.0 \times 10^3\,\text{s}^{-1}$, corresponding to $v_{\text{edge}} = 100\,\text{m}\,\text{s}^{-1}$. Then

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{100}{3.0 \times 10^8}\right)^2 \approx 3 \times 10^{-14}, \tag{14}$$

and

$$\Delta\rho_{\text{eff}} \sim 3 \times 10^{-11}\,\text{kg}\,\text{m}^{-3}. \tag{15}$$

The effect is many orders of magnitude below typical experimental resolution in laboratory fluids.

4.2 Astrophysical order-of-magnitude

In astrophysical settings, rotational velocities can be relativistic. For example, in the inner regions of accretion flows or rapidly rotating compact stars, characteristic speeds may reach $v \sim 0.1c$ or higher [4]. If one naively substitutes $v_{\text{edge}} = 0.1c$ into Eq. (13), one finds

$$\frac{\Delta\rho_{\text{eff}}}{\rho} \sim \frac{1}{4}(0.1)^2 = 2.5 \times 10^{-3}, \tag{16}$$

already approaching the percent level. However, in such regimes a fully relativistic treatment of the fluid is required, and higher-order terms in v^2/c^2 as well as strong-gravity effects must be included. Equation (13) should therefore be regarded as illustrating the leading-order trend rather than providing a quantitatively accurate model for relativistic flows.

5 Relation to relativistic continuum mechanics

In relativistic hydrodynamics, a perfect fluid is described by the stress–energy tensor [3, 4]

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{17}$$

where ε is the total energy density in the fluid rest frame, p is the pressure, and u^μ is the four-velocity. In the nonrelativistic limit and for small internal energy, one has

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \ldots, \tag{18}$$

consistent with the decomposition used in Eq. (10).

The contribution of kinetic energy to the gravitational mass of an extended system can be derived by integrating T^{00} over space in an appropriate frame [2, 3]. Our treatment effectively isolates the rotational part of this contribution for the specific case of rigid-body rotation in an incompressible fluid. Equation (13) can therefore be viewed as the nonrelativistic limit of the rotational piece of T^{00}/c^2, evaluated in a simple geometry.

6 Discussion and outlook

We have derived a compact relation between the rotational kinetic energy of an incompressible, inviscid fluid in rigid-body rotation and an associated effective mass density. The key steps are:

1. computation of the local kinetic energy density $e_{\text{kin}}(r) = \frac{1}{2}\rho\Omega^2 r^2$;
2. volume averaging over a finite cylinder, yielding $\langle e_{\text{kin}}\rangle = \frac{1}{4}\rho v_{\text{edge}}^2$;
3. definition of an effective mass density via $E = mc^2$ in the nonrelativistic limit, resulting in $\Delta\rho_{\text{eff}}/\rho = \frac{1}{4}(v_{\text{edge}}/c)^2$.

The analysis is fully contained within classical fluid mechanics and special relativity. It provides a transparent example of how rotational motion contributes to the mass density of an extended medium at order $(v/c)^2$. The coefficient 1/4 is specific to rigid-body rotation in a cylinder and reflects the radial structure of the velocity field. For other velocity profiles or geometries, different geometric factors would appear, though the basic scaling $\Delta\rho_{\text{eff}}/\rho \propto \langle v^2\rangle/c^2$ remains.

Potential applications include:

- pedagogical demonstrations of mass–energy equivalence in continuum systems;

- benchmark problems for numerical schemes that couple incompressible fluid dynamics to relativistic mass–energy accounting in the low-velocity regime;
- conceptual comparisons with fully relativistic treatments of rotating fluids in astrophysical contexts.

Any attempt to attribute fundamental rest mass to internal rotational motion would require additional structural assumptions and a fully relativistic framework, and lies beyond the scope of the present work.

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