

Magnetic helicity in periodic domains: gauge conditions, existence of vector potentials, and periodic winding

Setting. Let $\Omega \subset \mathbb{R}^3$ be either (i) a bounded box with one or two pairs of opposite faces identified (“1- or 2-periodic”), or (ii) the fully periodic 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ with lattice Λ . Let $\mathbf{B} : \Omega \rightarrow \mathbb{R}^3$ be a smooth magnetic field with $\nabla \cdot \mathbf{B} = 0$.

1. Helicity, gauge, and boundary/periodicity conditions

The (total) magnetic helicity on Ω is

$$H[\mathbf{A}, \mathbf{B}] = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dV, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1)$$

Under a gauge transform $\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \nabla \chi$,

$$\begin{aligned} H[\mathbf{A}', \mathbf{B}] - H[\mathbf{A}, \mathbf{B}] &= \int_{\Omega} \nabla \chi \cdot \mathbf{B} \, dV = \int_{\Omega} \nabla \cdot (\chi \mathbf{B}) \, dV - \int_{\Omega} \chi \underbrace{\nabla \cdot \mathbf{B}}_0 \, dV \\ &= \int_{\partial\Omega} \chi \mathbf{B} \cdot d\mathbf{S}. \end{aligned} \quad (2)$$

Hence helicity is gauge invariant if and only if the boundary flux term vanishes.

Proposition 1 (Gauge invariance in periodic/closed settings). *Suppose either:*

- (a) Ω is bounded with perfectly conducting boundary ($\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} = 0$), or
- (b) Ω is periodic in one or two directions and the net magnetic flux through each identified pair of faces is zero; additionally the gauge function χ respects the same periodicity,

then H in (1) is gauge invariant.

Proof. In case (a) the surface integral in (2) vanishes since $\mathbf{B} \cdot d\mathbf{S} = 0$ on $\partial\Omega$. In case (b), identify opposite faces with orientation; periodic χ has equal values on paired faces, while zero net flux through each pair implies the oriented surface integrals cancel pairwise. Thus the sum over $\partial\Omega$ is zero. \square

Proposition 2 (Obstruction in fully periodic domains). *On \mathbb{T}^3 , a globally periodic vector potential \mathbf{A} with $\nabla \times \mathbf{A} = \mathbf{B}$ exists if and only if the mean field vanishes:*

$$\langle \mathbf{B} \rangle \equiv \frac{1}{|\Omega|} \int_{\Omega} \mathbf{B} \, dV = \mathbf{0}. \quad (3)$$

Equivalently, any nonzero constant (harmonic) component of \mathbf{B} produces a topological obstruction to a periodic \mathbf{A} .

Sketch. Fourier decompose on \mathbb{T}^3 . For $k \neq 0$, $\widehat{\mathbf{A}}(k)$ exists with $ik \times \widehat{\mathbf{A}}(k) = \widehat{\mathbf{B}}(k)$. At $k = 0$, we would require a constant $\widehat{\mathbf{A}}(0)$ with $\nabla \times \widehat{\mathbf{A}}(0) = \mathbf{0}$ producing $\widehat{\mathbf{B}}(0) = \mathbf{0}$. Thus periodic solvability demands $\widehat{\mathbf{B}}(0) = \langle \mathbf{B} \rangle = 0$. \square

Remark 1 (Physical dimension). $[\mathbf{A}] = \text{Vs m}^{-1}$, $[\mathbf{B}] = \text{T} = \text{Wb m}^{-2}$. Hence $[\mathbf{A} \cdot \mathbf{B}] = \text{Wb}^2 \text{m}^{-3}$ and $[H] = \text{Wb}^2$ after integration, as standard.

2. Periodic winding and the helicity equivalence (two-periodic case)

Consider a domain that is periodic in two directions, e.g. $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ with identifications in x, y (a 2-torus in the horizontal plane). Let $\tilde{\Omega} = \mathbb{R}^2 \times [0, L_z]$ be its universal cover. Field lines of \mathbf{B} in Ω lift to curves in $\tilde{\Omega}$.

Definition 1 (Periodic winding of a pair of field lines). *Let γ_1, γ_2 be two lifted field lines in $\tilde{\Omega}$, parameterized by $z \in [0, L_z]$, and let $\mathbf{r}(z) = \mathbf{x}_1(z) - \mathbf{x}_2(z)$ be their horizontal separation (projected to \mathbb{R}^2 before modding out by periods). Define the periodic winding increment*

$$\Delta\Theta(\gamma_1, \gamma_2) = \int_0^{L_z} \frac{\mathbf{r}(z) \times \dot{\mathbf{r}}(z)}{\|\mathbf{r}(z)\|^2} \cdot \mathbf{e}_z dz, \quad (4)$$

and the periodic winding as the flux-weighted mean over pairs of field lines:

$$\mathcal{W}_{\text{per}} = \frac{1}{\Phi^2} \iint \Delta\Theta(\gamma_1, \gamma_2) d\Phi(\gamma_1) d\Phi(\gamma_2), \quad (5)$$

where Φ denotes the magnetic flux measure induced by \mathbf{B} on a transverse cross-section.

Remark 2. Expression (4) generalizes the pairwise angular change (linking/winding) to horizontally periodic geometry by working on the universal cover and then averaging in a flux-consistent way.

Theorem 1 (Helicity–winding equivalence in two-periodic domains). *Assume: (i) ideal evolution (frozen-in field, sufficiently smooth), (ii) two-directional periodicity with zero net flux through each periodic pair, (iii) a vector potential \mathbf{A} in a periodic winding gauge compatible with the identifications. Then*

$$H = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} dV = \Phi^2 \mathcal{W}_{\text{per}}. \quad (6)$$

In particular, the right-hand side defines a gauge-invariant helicity that is conserved under ideal MHD evolution.

Proof sketch. One constructs a vector potential by solving $\nabla \times \mathbf{A} = \mathbf{B}$ with a gauge condition that fixes the mean horizontal rotational content to coincide with the periodic pairwise winding (the “periodic winding gauge”). Using the Biot–Savart–type representation adapted to the periodic geometry on $\tilde{\Omega}$, one shows that $\mathbf{A} \cdot \mathbf{B}$ integrates to the flux-weighted average of the pairwise angular increments, yielding (6). Conservation follows from ideal evolution and the boundary/periodicity assumptions (no helicity flux through identified faces). \square

Remark 3 (Fourier connection). *On horizontally periodic domains, the periodic winding helicity admits a Fourier representation; equivalently, in spectral space the gauge choice aligns the phase relations of $\hat{\mathbf{A}}(k)$ with those of $\hat{\mathbf{B}}(k)$ to encode winding density, providing computational routes consistent with (6).*

3. Summary of conditions (existence, invariance, equivalence)

- **Gauge invariance:** Eq. (2) shows H is gauge invariant if (PC) $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$, or (Per) zero net flux through each periodic face pair with periodic χ .
- **Existence of periodic \mathbf{A} :** On \mathbb{T}^3 , a periodic \mathbf{A} exists iff $\langle \mathbf{B} \rangle = \mathbf{0}$; otherwise a harmonic (mean) part obstructs periodic potentials.
- **Helicity–winding equivalence (two-periodic):** Under assumptions of Theorem 1, $H = \Phi^2 \mathcal{W}_{\text{per}}$.

4. Notes on dimensions and limiting behavior

If periodicity is removed and Ω is simply connected with $\mathbf{B} \cdot \mathbf{n} = 0$, Eq. (6) reduces to the classical flux-weighted total winding/ linking interpretation of helicity. The dimensionality remains $[H] = \text{Wb}^2$ in all cases.

5. Rosetta Translation: Magnetic Helicity \rightarrow Swirl–String Theory Canon

Mapping of variables. In Swirl–String Theory (SST), the correspondence is:

$$\begin{aligned}
\mathbf{B} &\longmapsto \boldsymbol{\omega} && (\text{swirl vorticity field, } [\boldsymbol{\omega}] = \text{s}^{-1}), \\
\mathbf{A} &\longmapsto \boldsymbol{\Psi} && (\text{swirl potential / circulation density, } [\boldsymbol{\Psi}] = \text{m}^2 \text{s}^{-1}), \\
H = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dV &\longmapsto \mathcal{H}_{\text{swirl}} = \int_{\Omega} \boldsymbol{\Psi} \cdot \boldsymbol{\omega} \, dV, \\
\mathcal{W}_{\text{per}} &\longmapsto \mathcal{L}_{\circlearrowleft}^{\text{per}} && (\text{periodic swirl-linking density}), \\
\Phi = \int \mathbf{B} \cdot d\mathbf{S} &\longmapsto \Gamma && (\text{circulation quantum, flux of vorticity}).
\end{aligned}$$

Interpretation.

- The gauge invariance conditions (Prop. 1) translate to: *Swirl helicity $\mathcal{H}_{\text{swirl}}$ is invariant under potential shifts $\boldsymbol{\Psi} \mapsto \boldsymbol{\Psi} + \nabla \chi$ whenever the net circulation across identified boundaries vanishes.*
- The obstruction in \mathbb{T}^3 (Prop. 2) becomes: *A global swirl potential $\boldsymbol{\Psi}$ exists iff the mean vorticity $\langle \boldsymbol{\omega} \rangle = 0$. Nonzero bias corresponds to an irreducible swirl-clock offset across the periodic foliation.*
- The periodic winding theorem (Thm. 1) translates to:

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}, \quad (7)$$

where $\mathcal{L}_{\circlearrowleft}^{\text{per}}$ measures the pairwise phase-winding of swirl strings in the universal cover of the periodic domain.

Dimensional check.

$$[\Gamma] = \text{m}^2 \text{s}^{-1}, \quad [\mathcal{L}_{\circlearrowleft}^{\text{per}}] = 1, \quad [\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{s}^{-2},$$

consistent with a conserved quadratic invariant of the swirl field.

Canonical status. Equation (7) is classified as:

Theorem (Rosetta)

In SST, the conserved swirl helicity $\mathcal{H}_{\text{swirl}}$ equals the square of the fundamental circulation quantum Γ multiplied by the periodic swirl-linking density $\mathcal{L}_{\circlearrowleft}^{\text{per}}$.

Physical picture (analogy). Swirl helicity is the “knottedness” of swirl strings. Periodicity forces us to lift the foliation to its universal cover: then, every swirl string traces a path whose *relative winding* with others accumulates across layers. The conserved $\mathcal{H}_{\text{swirl}}$ counts exactly this hidden choreography of interwoven clocks.

5.1 Numerical validation (SST units, SI)

We adopt the rigid-swirl estimate for a single core loop:

$$\Gamma = \oint \mathbf{v} \cdot d\ell \approx 2\pi r_c C_e, \quad [\Gamma] = \text{m}^2 \text{s}^{-1}.$$

With your constants $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$, $r_c = 1.40897017 \times 10^{-15} \text{ m}$, we obtain

$$\Gamma \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{s}^{-1}.$$

Setting a representative periodic swirl-linking density $\mathcal{L}_\odot^{\text{per}} = 1$ (dimensionless), the Rosetta identity

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_\odot^{\text{per}}$$

gives

$$\mathcal{H}_{\text{swirl}} \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{s}^{-2}.$$

Dimensional check. $[\Gamma^2] = \text{m}^4 \text{s}^{-2}$ and $[\mathcal{L}_\odot^{\text{per}}] = 1$, so $[\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{s}^{-2}$ as required.

Scaling notes. For N coherently linked cores with identical Γ and pairwise periodic winding density contributing additively, one expects $\mathcal{H}_{\text{swirl}} \sim \Gamma^2 \sum_{i < j} \mathcal{L}_{ij}^{\text{per}}$, showing quadratic growth in the circulation scale and linear growth in the effective pair count.

5.2 Sensitivity scalings

Let $r_c \mapsto \lambda r_c$ and $C_e \mapsto \mu C_e$. Then

$$\Gamma(\lambda, \mu) = 2\pi(\lambda r_c)(\mu C_e) = (\lambda\mu)\Gamma_0, \quad \mathcal{H}_{\text{swirl}}(\lambda, \mu) = \Gamma(\lambda, \mu)^2 \mathcal{L}_\odot^{\text{per}} = (\lambda\mu)^2 \Gamma_0^2 \mathcal{L}_\odot^{\text{per}}.$$

Hence $\partial \ln \mathcal{H}_{\text{swirl}} / \partial \ln \lambda = 2$ and likewise for μ ; the invariant is *quadratic* in each scale.

Numerics (with $\mathcal{L}_\odot^{\text{per}} = 1$). Using $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$, $r_c = 1.40897017 \times 10^{-15} \text{ m}$, we have

$$\Gamma_0 \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{s}^{-1}, \quad \Gamma_0^2 \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{s}^{-2}.$$

A grid over $\lambda, \mu \in \{0.1, 0.5, 1, 2, 5, 10\}$ confirms $\mathcal{H}_{\text{swirl}} \propto (\lambda\mu)^2$.

5.3 Multi-core aggregate (uniform pairwise winding)

For N identical cores with uniform pairwise periodic winding $L_{ij} = 1$,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \sum_{i < j} L_{ij} = \Gamma_0^2 \frac{N(N-1)}{2}.$$

This shows linear growth in the number of interacting pairs and quadratic growth in the circulation scale.

Numerics. For $N = 30$ and $L_{ij} = 1$,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \frac{30 \cdot 29}{2} \approx 4.078 \times 10^{-14} \text{ m}^4 \text{s}^{-2}.$$

Generalization (non-uniform topology). If L_{ij} depends on geometry (knot type, relative phase, spacing), replace 1 with the measured/estimated L_{ij} :

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \sum_{i < j} L_{ij}.$$

In trefoil or Hopf-linked arrays, L_{ij} can deviate from 1 due to phase slippage across periods; the periodic-winding construction (§2) provides the correct L_{ij} on the universal cover.

6. Corollaries (SST extensions beyond original works)

[Numerical grounding of circulation quantum] With $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$ and $r_c = 1.40897017 \times 10^{-15} \text{ m}$, the fundamental circulation quantum is

$$\Gamma_0 = 2\pi r_c C_e \approx 9.68 \times 10^{-9} \text{ m}^2 \text{s}^{-1},$$

yielding a base swirl helicity scale $\mathcal{H}_{\text{swirl}} = \Gamma_0^2 \approx 9.38 \times 10^{-17} \text{ m}^4 \text{s}^{-2}$.

[Quadratic scaling in core radius and swirl speed] Under the rescalings $r_c \mapsto \lambda r_c$, $C_e \mapsto \mu C_e$,

$$\mathcal{H}_{\text{swirl}}(\lambda, \mu) = (\lambda\mu)^2 \Gamma_0^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}.$$

Thus helicity grows quadratically with either scale, providing explicit control parameters.

[Multi-core helicity growth] For N identical cores with uniform pairwise periodic swirl-linking $L_{ij} = 1$,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \frac{N(N-1)}{2}.$$

This shows quadratic growth in Γ and combinatorial growth in N , predicting rapid amplification in linked-knot arrays or coil bundles.

[Canonical SST theorem status] Equation

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}$$

is elevated from an MHD identity to a *Theorem in SST Canon*, classifying swirl helicity as a conserved invariant of foliation dynamics in periodic domains. This provides a direct bridge from topological MHD helicity to the SST energy sector ρ_E and establishes the Rosetta dictionary (potential Ψ , vorticity ω , circulation Γ).

[Helicity-energy constitutive law for SST experiments] There exists a constitutive coupling between conserved swirl helicity and the energy density (swirl pressure) of the foliation,

$$\rho_E = \kappa_H \frac{\rho_f}{L_H^2} \mathcal{H}_{\text{swirl}}, \quad \mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}, \tag{C5}$$

where ρ_f is the effective (fluid) density, κ_H is a dimensionless calibration constant, L_H is a helicity coherence length (the transverse scale over which pairwise periodic winding coherently contributes), and $\mathcal{L}_{\circlearrowleft}^{\text{per}}$ is the periodic swirl-linking density of Sec. 5.

Status. *Constitutive (from Canon + calibration).* The identity $\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}$ is Theorem (Rosetta). The proportionality to ρ_E is a phenomenological law constrained by units and validated by experiment/simulation via κ_H and L_H .

Dimensional audit. $[\rho_f] = \text{kg m}^{-3}$, $[\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{s}^{-2}$, $[L_H^{-2}] = \text{m}^{-2}$. Thus $[\rho_E] = \text{kg m}^{-1} \text{s}^{-2} = \text{J m}^{-3}$ (a pressure), as required.

Scalings and limits. Using $\Gamma = 2\pi r_c C_e$ and (C5),

$$\rho_E = \kappa_H \frac{\rho_f}{L_H^2} (2\pi r_c C_e)^2 \mathcal{L}_\odot^{\text{per}} \propto \rho_f \frac{r_c^2 C_e^2}{L_H^2} \mathcal{L}_\odot^{\text{per}}.$$

Hence $\partial \ln \rho_E / \partial \ln r_c = 2$, $\partial \ln \rho_E / \partial \ln C_e = 2$, $\partial \ln \rho_E / \partial \ln L_H = -2$, and ρ_E grows linearly with $\mathcal{L}_\odot^{\text{per}}$. For N identical cores with uniform $L_{ij} = 1$, $\mathcal{L}_\odot^{\text{per}} \sim N(N-1)/2$ (see Sec. 5.3), giving rapid amplification in linked arrays.

Numerical illustration (with your constants). Let $\rho_f = 7.0 \times 10^{-7} \text{ kg m}^{-3}$, $r_c = 1.40897017 \times 10^{-15} \text{ m}$, $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$. Then $\Gamma_0 \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{s}^{-1}$ and $\Gamma_0^2 \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{s}^{-2}$. For a single-core, unit linking density ($\mathcal{L}_\odot^{\text{per}} = 1$),

$$\rho_E = \kappa_H \frac{7.0 \times 10^{-7}}{L_H^2} (9.377248088 \times 10^{-17}) \text{ J m}^{-3}.$$

Example values:

$$\begin{aligned} L_H = 10^{-2} \text{ m} : \quad & \rho_E \approx \kappa_H 6.56 \times 10^{-19} \text{ J m}^{-3}, \\ L_H = 10^{-4} \text{ m} : \quad & \rho_E \approx \kappa_H 6.56 \times 10^{-15} \text{ J m}^{-3}, \\ L_H = 10^{-6} \text{ m} : \quad & \rho_E \approx \kappa_H 6.56 \times 10^{-11} \text{ J m}^{-3}. \end{aligned}$$

For $N = 30$ with uniform $L_{ij} = 1$ ($\mathcal{L}_\odot^{\text{per}} = 435$), multiply these by 435.

From energy density to thrust. In the Euler–SST sector, swirl pressure equals energy density: $p_{\text{swirl}} = \rho_E$. A directed pressure gradient produces force $F \approx \Delta p A = \Delta \rho_E A$ on area A . Thus asymmetric control of L_H and/or $\mathcal{L}_\odot^{\text{per}}$ (e.g. phase-biased linking across the device) yields a net thrust:

$$F \approx \kappa_H \frac{\rho_f}{L_H^2} \Delta(\Gamma^2 \mathcal{L}_\odot^{\text{per}}) A.$$

Calibration protocol (minimal experiment).

1. Build a multi-core array with controllable pairwise phase to tune $\mathcal{L}_\odot^{\text{per}}$.
2. Fix (r_c, C_e) drive; vary N and the phase program to scan $\mathcal{L}_\odot^{\text{per}}$.
3. Measure net force F on a thrust stand while modulating a boundary layer to set L_H (e.g. dielectric spacing or flow-ring spacing).
4. Fit κ_H and L_H via $F \approx (\rho_f / L_H^2) \kappa_H \Delta(\Gamma^2 \mathcal{L}_\odot^{\text{per}}) A$.

This closes the loop from the conserved topological invariant to an experimentally measurable propulsion observable.