

# Rotational Kinetic Energy Density and an Effective Mass Relation in Incompressible Fluids

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Kinetic energy contributes to inertia and gravitational mass through the relativistic relation  $E = mc^2$ . In extended media such as fluids, this contribution can be expressed as an effective mass density associated with internal motion. I compute the volume-averaged rotational kinetic energy density for an incompressible, inviscid Newtonian fluid in rigid-body rotation inside a finite cylinder. Associating this energy density with an effective mass density via  $E = mc^2$  (nonrelativistic limit) yields the closed-form relation

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{v_{\text{edge}}}{c} \right)^2,$$

where  $v_{\text{edge}}$  is the tangential speed at the cylinder boundary and  $\rho$  is the rest-mass density. The result provides a transparent classical example of how rotational motion modifies the mass density at order  $(v/c)^2$ . We discuss the connection with relativistic continuum mechanics and provide numerical estimates for laboratory and astrophysical regimes.

## I. INTRODUCTION

The equivalence between energy and mass, expressed by  $E = mc^2$ , implies that kinetic, field, and binding energies contribute to the inertia and gravitational mass of extended systems [1–3]. In the context of relativistic continuum mechanics, this statement is encoded in the stress–energy tensor  $T^{\mu\nu}$ : the total mass–energy is obtained by integrating  $T^{00}$  over a spatial hypersurface, and  $T^{00}$  includes both rest-mass and kinetic contributions [3, 4].

Although standard in high-energy and gravitational physics, an explicit fluid-mechanics example remains pedagogically useful. Here I make the rotational contribution to an effective mass density explicit in an analytically tractable flow.

In this paper we analyze a canonical configuration from classical fluid mechanics [5, 6]: an incompressible, inviscid fluid in rigid-body rotation inside a finite cylinder. Within this model we:

1. compute the local and volume-averaged rotational kinetic energy density;
2. define an effective mass density  $\Delta\rho_{\text{eff}}$  via  $E = mc^2$  in the regime  $v \ll c$ ;
3. derive a compact expression for  $\Delta\rho_{\text{eff}}/\rho$  in terms of the edge speed  $v_{\text{edge}}$ ;
4. discuss how this classical result fits within the framework of relativistic continuum mechanics.

The derivation uses only incompressible Euler flow plus the special-relativistic mass–energy relation. No modifications of Newtonian or relativistic theory are proposed.

## II. RIGID-BODY ROTATION IN AN INCOMPRESSIBLE, INVISCID FLUID

### A. Flow configuration

We consider a Newtonian fluid of constant rest-mass density  $\rho$  occupying a right circular cylinder of radius  $R$  and height  $L$ . The fluid undergoes steady rigid-body rotation with constant angular velocity  $\Omega$  about the  $z$ -axis. In cylindrical coordinates  $(r, \theta, z)$ , with  $0 \leq r \leq R$ , the velocity field is

$$\mathbf{v}(r) = \Omega r \hat{\boldsymbol{\theta}}. \tag{1}$$

This flow is incompressible and inviscid:

$$\nabla \cdot \mathbf{v} = 0, \quad \text{viscosity} = 0, \tag{2}$$

and it satisfies the steady Euler equations with an appropriate pressure distribution [5, 6].

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### B. Local kinetic energy density

The local kinetic energy density of the fluid is

$$e_{\text{kin}}(r) = \frac{1}{2} \rho \|\mathbf{v}(r)\|^2 = \frac{1}{2} \rho \Omega^2 r^2. \quad (3)$$

This quantity is position-dependent and increases quadratically with radius.

### C. Total rotational energy and volume-averaged energy density

The total rotational kinetic energy is obtained by integrating Eq. (3) over the fluid volume:

$$\begin{aligned} E_{\text{rot}} &= \int_V e_{\text{kin}} dV \\ &= \int_0^L dz \int_0^{2\pi} d\theta \int_0^R \frac{1}{2} \rho \Omega^2 r^2 r dr \\ &= \frac{1}{2} \rho \Omega^2 (2\pi L) \int_0^R r^3 dr \\ &= \frac{\pi}{4} \rho \Omega^2 L R^4. \end{aligned} \quad (4)$$

The cylinder volume is  $V = \pi R^2 L$ , so the volume-averaged kinetic energy density is

$$\langle e_{\text{kin}} \rangle = \frac{E_{\text{rot}}}{V} = \frac{\frac{\pi}{4} \rho \Omega^2 L R^4}{\pi R^2 L} = \frac{1}{4} \rho \Omega^2 R^2. \quad (5)$$

Define the edge speed

$$v_{\text{edge}} := \Omega R. \quad (6)$$

Then Eq. (5) becomes

$$\langle e_{\text{kin}} \rangle = \frac{1}{4} \rho v_{\text{edge}}^2. \quad (7)$$

At the boundary

$$e_{\text{kin}}(R) = \frac{1}{2} \rho v_{\text{edge}}^2, \quad (8)$$

the volume average is half of that, as expected from the quadratic radial profile.

## III. EFFECTIVE MASS DENSITY FROM $E = mc^2$

### A. Nonrelativistic limit and effective density

In special relativity, the total relativistic energy of a fluid element with rest-mass density  $\rho$  and small velocity  $v \ll c$  can be decomposed as [3, 4]

$$\varepsilon \simeq \rho c^2 + \frac{1}{2} \rho v^2 + \dots, \quad (9)$$

where  $\varepsilon$  is the total energy density in the local rest frame, and the ellipsis denotes higher-order terms in  $v^2/c^2$  and internal energy contributions. To leading order in  $v^2/c^2$ , the kinetic energy density acts as an *effective mass density*,

$$\Delta \rho_{\text{eff}}(\mathbf{x}) = \frac{e_{\text{kin}}(\mathbf{x})}{c^2}. \quad (10)$$

This interpretation is consistent with the structure of the stress–energy tensor for a perfect fluid [3].



For the rigidly rotating configuration considered here, we focus on the volume-averaged effective mass density

$$\Delta\rho_{\text{eff}} := \frac{\langle e_{\text{kin}} \rangle}{c^2}. \quad (11)$$

Inserting Eq. (7) into Eq. (11), we obtain

$$\Delta\rho_{\text{eff}} = \frac{1}{4c^2} \rho v_{\text{edge}}^2. \quad (12)$$

Dividing by the rest-mass density  $\rho$  yields

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{v_{\text{edge}}}{c} \right)^2. \quad (13)$$

Thus, in the nonrelativistic regime the rotational contribution scales as  $(v_{\text{edge}}/c)^2$  with geometry-specific coefficient 1/4 for a rigidly rotating cylinder.

## B. Dimensional check

The dimensions of Eq. (12) are

$$[\Delta\rho_{\text{eff}}] = \frac{[\rho][v]^2}{[c]^2} = \frac{\text{kg m}^{-3} (\text{m/s})^2}{(\text{m/s})^2} = \text{kg m}^{-3},$$

as required. The ratio  $\Delta\rho_{\text{eff}}/\rho$  in Eq. (13) is dimensionless.

## IV. NUMERICAL ESTIMATES

### A. Laboratory-scale example

Consider water with  $\rho \approx 1.0 \times 10^3 \text{ kg m}^{-3}$ , a cylinder of radius  $R = 0.10 \text{ m}$ , and angular velocity  $\Omega = 1.0 \times 10^3 \text{ s}^{-1}$ , corresponding to  $v_{\text{edge}} = 100 \text{ m s}^{-1}$ . Then

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{100}{3.0 \times 10^8} \right)^2 \approx 3 \times 10^{-14}, \quad (14)$$

and

$$\Delta\rho_{\text{eff}} \sim 3 \times 10^{-11} \text{ kg m}^{-3}. \quad (15)$$

The effect ( $\sim 3 \times 10^{-11} \text{ kg m}^{-3}$ ) is far below laboratory density resolution.

### B. Astrophysical order-of-magnitude

In astrophysical settings, rotational velocities can be relativistic. For example, in the inner regions of accretion flows or rapidly rotating compact stars, characteristic speeds may reach  $v \sim 0.1c$  or higher [4]. If one naively substitutes  $v_{\text{edge}} = 0.1c$  into Eq. (13), one finds

$$\frac{\Delta\rho_{\text{eff}}}{\rho} \sim \frac{1}{4} (0.1)^2 = 2.5 \times 10^{-3}, \quad (16)$$

already approaching the percent level. However, in such regimes a fully relativistic treatment of the fluid is required, and higher-order terms in  $v^2/c^2$  as well as strong-gravity effects must be included. Equation (13) should therefore be regarded as illustrating the leading-order trend rather than providing a quantitatively accurate model for relativistic flows.



## V. RELATION TO RELATIVISTIC CONTINUUM MECHANICS

In relativistic hydrodynamics, a perfect fluid is described by the stress–energy tensor [3, 4]

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (17)$$

where  $\varepsilon$  is the total energy density in the fluid rest frame,  $p$  is the pressure, and  $u^\mu$  is the four-velocity. In the nonrelativistic limit and for small internal energy, one has

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \dots, \quad (18)$$

consistent with the decomposition used in Eq. (10).

The contribution of kinetic energy to the gravitational mass of an extended system can be derived by integrating  $T^{00}$  over space in an appropriate frame [2, 3]. This calculation isolates the rotational contribution for rigid-body rotation of an incompressible fluid. Equation (13) is the nonrelativistic limit of the rotational part of  $T^{00}/c^2$  in this geometry.

## VI. DISCUSSION AND OUTLOOK

We have derived a compact relation between the rotational kinetic energy of an incompressible, inviscid fluid in rigid-body rotation and an associated effective mass density. The key steps are:

1. computation of the local kinetic energy density  $e_{\text{kin}}(r) = \frac{1}{2}\rho\Omega^2 r^2$ ;
2. volume averaging over a finite cylinder, yielding  $\langle e_{\text{kin}} \rangle = \frac{1}{4}\rho v_{\text{edge}}^2$ ;
3. definition of an effective mass density via  $E = mc^2$  in the nonrelativistic limit, resulting in  $\Delta\rho_{\text{eff}}/\rho = \frac{1}{4}(v_{\text{edge}}/c)^2$ .

The analysis stays within classical incompressible flow and special relativity. It provides a transparent example of how rotational motion contributes to the mass density of an extended medium at order  $(v/c)^2$ . The coefficient 1/4 is specific to rigid-body rotation in a cylinder and reflects the radial structure of the velocity field. For other velocity profiles or geometries, different geometric factors would appear, though the basic scaling  $\Delta\rho_{\text{eff}}/\rho \propto \langle v^2 \rangle / c^2$  remains.

Potential applications include:

- pedagogical demonstrations of mass–energy equivalence in continuum systems;
- benchmark problems for numerical schemes that couple incompressible fluid dynamics to relativistic mass–energy accounting in the low-velocity regime;
- conceptual comparisons with fully relativistic treatments of rotating fluids in astrophysical contexts.

As a complementary single-particle example, Appendix A summarizes a semiclassical model in which a photon confined on a toroidal path reproduces the electron charge to within about ten per cent, using only classical electromagnetism and the Compton wavelength as input [7].

Any attempt to attribute fundamental rest mass to internal rotational motion would require additional structural assumptions and a fully relativistic framework, and lies beyond the scope of the present work.

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## APPENDIX A: SEMI-CLASSICAL CHARGE ESTIMATE FROM A CONFINED ELECTROMAGNETIC MODE

In the main text we considered how rotational kinetic energy in an incompressible fluid contributes to an effective mass density via the relation  $E = mc^2$  in the nonrelativistic regime. For completeness, we record here a simple semiclassical construction, due to Williamson and van der Mark [7], which shows



how a confined electromagnetic mode can also reproduce the observed magnitude of the electron charge using only classical electromagnetism and the Compton wavelength.

Consider a photon of wavelength  $\lambda$  whose energy

$$E_\gamma = \frac{hc}{\lambda}$$

is confined to a finite volume  $V$  of characteristic size comparable to  $\lambda$ . In the model of Ref. [7], the photon is confined on a toroidal path of length one Compton wavelength  $\lambda_C = h/(m_e c)$ , with a characteristic radius

$$r = \frac{\lambda_C}{4\pi},$$

so that the confinement volume is of order  $V \sim r^3$ . Equating the photon energy to the integral of the electromagnetic energy density over this volume leads to an average electric-field amplitude of the form

$$\langle E \rangle = \sqrt{\frac{6hc}{\pi\epsilon_0\lambda_C^4}}, \quad (19)$$

where  $\epsilon_0$  is the vacuum permittivity.<sup>1</sup>

To relate this confined field to an effective point charge, one compares  $\langle E \rangle$  to the magnitude of the Coulomb field at a radius  $r$ ,

$$E_C(r) = \frac{q}{4\pi\epsilon_0 r^2}. \quad (20)$$

Identifying  $r = \lambda_C/(4\pi)$  as above and setting  $E_C(r) = \langle E \rangle$  yields

$$q = 4\pi\epsilon_0 r^2 \langle E \rangle = 4\pi\epsilon_0 \left(\frac{\lambda_C}{4\pi}\right)^2 \sqrt{\frac{6hc}{\pi\epsilon_0\lambda_C^4}}. \quad (21)$$

Using  $\lambda_C = h/(m_e c)$  and  $h = 2\pi\hbar$ , one finds that the dependence on  $\lambda_C$  and  $m_e$  cancels, giving the closed-form expression

$$q_{\text{model}} = \frac{1}{2\pi} \sqrt{3\epsilon_0 \hbar c}. \quad (22)$$

Numerically,

$$q_{\text{model}} = \frac{1}{2\pi} \sqrt{3\epsilon_0 \hbar c} \simeq 1.46 \times 10^{-19} \text{ C} \simeq 0.91 e,$$

so that this simple confinement picture reproduces the observed elementary charge  $e \simeq 1.60 \times 10^{-19} \text{ C}$  at the ten-percent level.

*a. Intuition:* compress a single-wavelength loop of light into a small torus; the surface field approaches that of a point charge.

This appendix does not advocate a microscopic electron model; it illustrates that, even within classical electromagnetism, geometric confinement of a single-wavelength mode can generate particle-like mass and charge scales. This complements the continuum calculation in the main text, where rotational kinetic energy in an extended medium leads to an effective mass density proportional to  $\langle v^2 \rangle / c^2$ .

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<sup>1</sup> The numerical factor depends on the detailed choice of confinement volume and field profile; Eq. (19) corresponds to the specific toroidal geometry considered in [7].