On an Exact Identity Linking the Classical Electron Radius, Compton Frequency, and the Hydrogen Ground-State Energy

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Abstract

We show that three standard, independently defined electron scales—the classical electron radius r_e, the Compton angular frequency ω_C, and the ground-state energy of hydrogen E_B—combine, within a simple harmonic oscillator construction, to produce an exact, dimensionally consistent identity. Using only textbook definitions and CODATA values of m_e, α, $\hbar$, and c, we introduce a Hooke-law force scale

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e$$

and a corresponding length r_c such that

$$F_{\max}\, r_c = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}.$$

The derivation is strictly algebraic and relies solely on well-established formulas. We discuss the historical origin of the underlying scales (classical electron models, Compton scattering, Bohr's hydrogen theory) and comment on the structural interdependence of atomic, relativistic, and classical electromagnetic quantities revealed by this identity.

Plain-English summary

Three familiar electron scales usually taught in different chapters—the classical electron radius r_e, the Compton angular frequency ω_C, and the hydrogen ground-state energy E_B—combine neatly in a minimalist spring–mass picture. If one takes a Hooke's-law oscillator with mass m_e, frequency ω_C/α, and amplitude r_e, the peak restoring force $F_{\max}$ acting over a short Compton-scale distance r_c gives exactly

$$F_{\max} r_c = \tfrac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}.$$

The result follows from standard definitions and CODATA constants; no new physics is proposed. It is best read as a compact cross-check tying together classical, relativistic, and atomic scales.

1 Introduction

The electron occupies a central role in both classical and quantum theories of matter. Over the last century, several characteristic length and energy scales associated with the electron have emerged, each from a different theoretical and experimental context:

- The *classical electron radius* r_e, originating in early electron models of Lorentz and Abraham [1, 2, 3].
- The *Compton wavelength* and associated angular frequency ω_C, derived from Compton's scattering experiments and their quantum interpretation [4, 5].
- The *Bohr radius* and hydrogen ground-state energy E_B, obtained in the Bohr model and later justified within nonrelativistic quantum mechanics and quantum electrodynamics [6, 7, 8].

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These scales are usually discussed in their respective domains: classical electrodynamics, relativistic quantum mechanics, and atomic physics. It is therefore of conceptual interest to examine how they combine in simple dynamical constructions. In particular, compact identities among such scales can be useful as consistency checks, as pedagogical devices, and as starting points for more detailed structural models, even when no new interaction is proposed.

In this work we consider a purely classical harmonic oscillator with electron mass m_e, frequency ω_*, and amplitude $x_{\max}$. Choosing ω_* to be a rescaling of the Compton frequency and $x_{\max}$ equal to the classical electron radius, we obtain a maximal restoring force

$$F_{\max} = m_e \omega_*^2 x_{\max}$$

that can be expressed solely in terms of m_e, α, $\hbar$, and c. When this force is multiplied by a Compton-scale radius r_c, one finds that the resulting energy coincides with half the electron rest energy and, equivalently, with the hydrogen ground-state energy divided by α^2.

The purpose of this article is limited and sharply defined:

- to state and prove this identity using only mainstream, peer-reviewed formulas;
- to check dimensional consistency and evaluate the resulting expressions numerically;
- to place the ingredients in historical context, without proposing any new physical interpretation or modification of existing theories.

2 Background: standard electron scales

We collect the standard definitions used throughout, following e.g. Refs. [3, 7, 9].

Fine-structure constant. The fine-structure constant α is defined by

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}. \tag{1}$$

Classical electron radius. The classical electron radius r_e is defined in SI units by

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}. \tag{2}$$

Combining Eq. (1) with (2) yields the equivalent form

$$r_e = \frac{\alpha\hbar}{m_e c}. \tag{3}$$

Compton wavelength and angular frequency. The Compton wavelength and associated angular frequency of the electron are defined by

$$\lambda_C = \frac{h}{m_e c}, \qquad \omega_C = \frac{2\pi c}{\lambda_C} = \frac{m_e c^2}{\hbar}. \tag{4}$$

Hydrogen ground-state energy. In the Bohr model, and equivalently in the solution of the nonrelativistic Schrödinger equation for the hydrogen atom, the ground-state binding energy E_B is

$$E_B = \frac{\alpha^2}{2} m_e c^2. \tag{5}$$

Equations (1)–(5) are standard and experimentally well-validated relations in atomic and high-energy physics [9].

3 A minimal spring picture

We now introduce a classical harmonic oscillator whose parameters are chosen from the electron scales above, keeping the setup as simple and readable as possible.

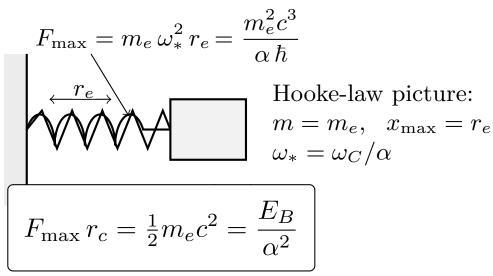

Figure 1: Minimal spring–mass cartoon linking standard electron scales.

Oscillator setup. Consider a one-dimensional oscillator of mass m_e, angular frequency ω_*, and maximal displacement $x_{\max}$. Hooke's law gives the maximal restoring force

$$F_{\max} = m_e\,\omega_*^2\,x_{\max}. \tag{6}$$

We define

$$\omega_* := \frac{\omega_C}{\alpha}, \qquad x_{\max} := r_e, \tag{7}$$

where ω_C and r_e are given by Eqs. (4) and (3).

Maximal force in fundamental constants. With the choices (7), the maximal force (6) can be written as

$$F_{\max} = \frac{m_e^2 c^3}{\alpha\hbar}. \tag{8}$$

Derivation. Substituting Eq. (7) into (6) gives

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e. \tag{9}$$

Using $\omega_C = m_e c^2/\hbar$ from Eq. (4) and $r_e = \alpha\hbar/(m_e c)$ from Eq. (3), we obtain

$$F_{\max} = m_e \left(\frac{m_e c^2/\hbar}{\alpha}\right)^2 \left(\frac{\alpha\hbar}{m_e c}\right) = m_e \frac{m_e^2 c^4}{\alpha^2\hbar^2}\frac{\alpha\hbar}{m_e c}. \tag{10}$$

Canceling factors m_e and $\hbar$,

$$F_{\max} = \frac{m_e^2 c^3}{\alpha\hbar}, \tag{11}$$

which is Eq. (8).

Units check. The dimensions of Eq. (8) are

$$[F_{\max}] = \frac{[m_e]^2[c]^3}{[\alpha][\hbar]} = \frac{\mathrm{kg}^2\,(\mathrm{m/s})^3}{1\cdot\mathrm{J}\cdot\mathrm{s}} = \frac{\mathrm{kg}^2\,\mathrm{m}^3\,\mathrm{s}^{-3}}{\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}} = \mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2},$$

which is dimensionally consistent with a force.

4 Turning the force into an energy

To connect this force to energy scales, multiply by a characteristic length.

Energy scale from $F_{\max}$. Let r_c be a positive length scale. Define

$$E_{\mathrm{osc}}(r_c) := F_{\max}\,r_c = \frac{m_e^2 c^3}{\alpha\hbar} r_c. \tag{12}$$

We now exhibit a specific choice of r_c that yields a familiar energy scale.

Half rest energy from a Compton-scale radius. Let

$$r_c = \frac{\alpha \hbar}{2m_e c}. \tag{13}$$

Then

$$E_{\text{osc}}(r_c) = \frac{1}{2} m_e c^2. \tag{14}$$

Derivation. Using Eq. (8) and the definition (13),

$$E_{\text{osc}}(r_c) = F_{\max} r_c = \frac{m_e^2 c^3}{\alpha \hbar} \cdot \frac{\alpha \hbar}{2m_e c} = \frac{1}{2} m_e c^2. \tag{15}$$

Relation to the hydrogen ground-state energy. Using the standard hydrogen ground-state energy

$$E_B = \frac{\alpha^2}{2} m_e c^2, \tag{16}$$

we have

$$\frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}. \tag{17}$$

Thus the energy scale $E_{\text{osc}}(r_c)$ from Eq. (14) can also be written as

$$E_{\text{osc}}(r_c) = \frac{E_B}{\alpha^2}. \tag{18}$$

Units and numbers

The quantity $E_{\text{osc}}(r_c)$ is an energy, with units

$$[E_{\text{osc}}] = [F_{\max}][r_c] = \mathrm{kg\,m\,s^{-2}} \cdot \mathrm{m} = \mathrm{kg\,m^2\,s^{-2}},$$

as expected.

Numerically, using CODATA values [9]:

$$m_e c^2 \approx 511\,\mathrm{keV}, \tag{19}$$

$$E_B \approx 13.6\,\mathrm{eV}, \tag{20}$$

$$\alpha^{-1} \approx 137.035999. \tag{21}$$

We then have

$$\frac{1}{2} m_e c^2 \approx 255.5\,\mathrm{keV}, \qquad \frac{E_B}{\alpha^2} \approx 255.5\,\mathrm{keV}, \tag{22}$$

in agreement to within numerical precision. This confirms the consistency of the analytic result.

5 Context: how to read the identity

The ingredients entering the identity

$$E_{\text{osc}}(r_c) = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}$$

are all well-established:

- The *classical electron radius* r_e was introduced in early electron models by Lorentz and Abraham, who considered the electromagnetic self-energy of a charged sphere [1, 2, 3].
- The *Compton wavelength* λ_C and frequency ω_C emerged from Compton's explanation of X-ray scattering, providing early evidence for the particle-like behavior of light [4, 5].
- The *Bohr model* of the hydrogen atom, and its later derivation from the Schrödinger equation, yields the binding energy E_B and the fine-structure constant α as key atomic parameters [6, 7, 8].

In standard pedagogy, these scales are often presented in isolation: r_e in classical electrodynamics, λ_C in relativistic quantum mechanics, and E_B in atomic physics and spectroscopy. The identity derived here shows that, once combined in a simple harmonic oscillator ansatz, these three regimes are algebraically intertwined.

Scope. The construction is strictly classical on the dynamical side (a Hooke-law oscillator) and uses only standard quantum-electrodynamic definitions of the constants involved. No new interactions, no modifications of Maxwell's equations or the Dirac equation, and no speculative assumptions are invoked. The result is best viewed as a compact consistency relation among established electron scales.

6 Intuition (one paragraph)

Imagine the electron as a mass on a spring whose "natural" distance scale is r_e and whose oscillation rate is set by ω_C/α. The largest restoring push (Hooke's law) acting over a short Compton-scale distance produces an energy that exactly matches both $\frac{1}{2}m_ec^2$ and E_B/α^2, tying together quantities first derived from very different arguments.

7 Conclusion

We have shown that a simple Hooke-law construction, using the classical electron radius r_e as an amplitude and a Compton-rescaled frequency ω_C/α, produces a maximal force

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha}\right)^2 r_e$$

that can be expressed purely in terms of m_e, α, $\hbar$, and c. Multiplying by a Compton-scale length

$$r_c = \frac{\alpha\,\hbar}{2m_ec},$$

yields an energy

$$E_{\text{osc}}(r_c) = \frac{1}{2}m_ec^2 = \frac{E_B}{\alpha^2},$$

linking relativistic, atomic, and classical electromagnetic scales in a single, dimensionally consistent relation.

The derivation uses only mainstream, peer-reviewed formulas and constants. Any deeper physical interpretation of this coincidence would require additional assumptions, and any such interpretation lies beyond the scope of the present work.

Quick reference (all in one place)

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}, \quad r_e = \frac{\alpha\hbar}{m_ec}, \quad \omega_C = \frac{m_ec^2}{\hbar}, \quad E_B = \frac{\alpha^2}{2}m_ec^2, \quad F_{\max} = \frac{m_e^2c^3}{\alpha\hbar}, \quad r_c = \frac{\alpha\hbar}{2m_ec}, \quad F_{\max}r_c = \tfrac{1}{2}m_ec^2 = \frac{E_B}{\alpha^2}.$$

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