1 Helicity in Vortex Knot Systems under the Vortex Æther Model (VAM)

Objective

Understand and compute the total helicity $\mathcal{H}$ of a knotted or linked vortex system:

$$\boxed{\mathcal{H} = \sum_k \int_{\mathcal{C}_k} \vec{v}_k \cdot \vec{\omega}_k \, dV + \sum_{i<j} 2Lk_{ij} \, \Gamma_i \Gamma_j} \tag{1}$$

This formula splits the helicity into two components:

- Self-helicity: twist + writhe within each vortex
- Mutual helicity: due to linking between different vortices

1. Background Concepts

a. Velocity & Vorticity

- $\vec{v}(\vec{r})$: local fluid velocity
- $\vec{\omega} = \nabla \times \vec{v}$: vorticity vector

b. Circulation (Γ)

$$\Gamma_k = \oint_{\mathcal{C}_k} \vec{v} \cdot d\vec{l} \tag{2}$$

This has units of $[\mathrm{m}^2/\mathrm{s}]$ and represents total swirl.

c. Helicity

$$\mathcal{H} = \int_V \vec{v} \cdot \vec{\omega} \, dV \tag{3}$$

A topological invariant for inviscid, incompressible flows.

2. Derivation of the Full Formula

Assume N disjoint vortex tubes $\mathcal{C}_1, \ldots, \mathcal{C}_N$ with thin cores.

Step 1: Total helicity splits

$$\mathcal{H} = \sum_{i=1}^{N} \mathcal{H}_{\text{self}}^{(i)} + \sum_{i<j} \mathcal{H}_{\text{mutual}}^{(i,j)} \tag{4}$$

Step 2: Self-helicity of vortex $\mathcal{C}_k$

$$\mathcal{H}_{\text{self}}^{(k)} = \int_{\mathcal{C}_k} \vec{v}_k \cdot \vec{\omega}_k \, dV \approx \Gamma_k^2 \cdot SL_k \tag{5}$$

For a trefoil, $SL_k \approx 3$.

Step 3: Mutual helicity

$$\mathcal{H}_{\text{mutual}}^{(i,j)} = 2Lk_{ij}\Gamma_i\Gamma_j \tag{6}$$

Final Form

$$\boxed{\mathcal{H} = \sum_{i=1}^{N} \Gamma_i^2 SL_i + \sum_{i<j}^{N} 2Lk_{ij}\Gamma_i\Gamma_j} \tag{7}$$

Or in integral form:

$$\boxed{\mathcal{H} = \sum_{i=1}^{N} \int_{\mathcal{C}_i} \vec{v}_i \cdot \vec{\omega}_i \, dV + \sum_{i<j} 2Lk_{ij}\Gamma_i\Gamma_j} \tag{8}$$

3. How to Use It

1. Determine vortex configuration: e.g., torus link $T(p,q)$ with $N = \gcd(p,q)$
2. Estimate circulation: $\Gamma \approx 2\pi r_c C_e$
3. Use $SL_k = 3$, $Lk_{ij} = 1$ for trefoil links
4. Evaluate:

$$\mathcal{H} = N \cdot \Gamma^2 \cdot 3 + 2 \cdot \binom{N}{2} \cdot \Gamma^2$$

4. Example: $T(18, 27)$

- $N = 9$, $\Gamma = 2\pi r_c C_e$
- $SL = 3$, $\binom{9}{2} = 36$

$$\mathcal{H} = 9 \cdot \Gamma^2 \cdot 3 + 2 \cdot 36 \cdot \Gamma^2 = 27\Gamma^2 + 72\Gamma^2 = 99\Gamma^2 \tag{9}$$

BibTeX References


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@article{moffatt1969degree,
  author    = {H. K. Moffatt},
  title     = {The degree of knottedness of tangled vortex lines},
  journal   = {Journal of Fluid Mechanics},
  volume    = {35},
  pages     = {117--129},
  year      = {1969},
  doi       = {10.1017/S0022112069000991}
}

@book{arnold1998topological,
  author    = {V. I. Arnold and B. A. Khesin},
  title     = {Topological Methods in Hydrodynamics},
  publisher = {Springer},
  year      = {1998},
  doi       = {10.1007/978-1-4612-0645-3}
}
```


Summary Table

Term	Meaning
$\vec{v} \cdot \vec{\omega}$	Local helicity density
Γ	Circulation around vortex core
SL_k	Self-linking of component k
Lk_{ij}	Gauss linking number between i, j
$\mathcal{H}$	Total helicity (topological + dynamical)