Thermodynamics of a Circulation-Loop Gas in an Incompressible Fluid and the Classical Ultraviolet Problem

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November 17, 2025

Abstract

We investigate a classical analogue model for thermal radiation based on an incompressible, inviscid fluid populated by thin circulation loops (vortex rings) of finite core radius. Treating closed circulation loops as localized, finite-energy structures with effective mass M_{eff}, we construct a kinetic theory of a dilute loop gas in the canonical ensemble. The resulting partition function coincides with that of a classical ideal gas, and the equation of state $PV = Nk_{\mathrm{B}}T$ follows in the standard way from the Helmholtz free energy. For an isotropic ensemble of ultra-relativistic or effectively massless excitations built from these loops, the relation $P = \frac{1}{3}u$ between pressure and energy density is recovered from the momentum flux tensor, relying only on three-dimensional isotropy and the relativistic relation $E = pc$.

A finite core radius a and an upper bound $v_{\max}$ on the tangential speed inside the cores imply a kinematic maximum internal frequency $\omega_{\max} = v_{\max}/a$. If electromagnetic normal modes are identified with mechanically realizable excitations of this medium, only modes with $\omega \leq \omega_{\max}$ are thermally accessible. The classical Rayleigh–Jeans integral is then automatically regulated, yielding a finite energy density $u \propto k_{\mathrm{B}}T\,\omega_{\max}^3$ without any modification of Maxwell's equations. The resulting cutoff Rayleigh–Jeans spectrum does not reproduce the T^4 scaling of the Planck law and therefore does not replace quantum statistics. Instead, it provides an explicit mechanical example in which microstructure and finite rotational frequencies eliminate the classical ultraviolet catastrophe at the level of microstate counting, while leaving intact the standard macroscopic equations of motion and equations of state.

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DOI: 10.5281/zenodo.xxx

1 Introduction

Classical thermodynamics and statistical mechanics provide a remarkably successful macroscopic description of fluids and radiation. The equation of state of an ideal gas,

$$PV = Nk_{\mathrm{B}}T, \tag{1}$$

and the relation between pressure and energy density for an isotropic radiation field,

$$P = \frac{1}{3}u(T), \tag{2}$$

are textbook results derived from kinetic arguments and from the stress–energy tensor of relativistic fields [6, 7, 5]. At the same time, the classical Rayleigh–Jeans treatment of thermal radiation predicts $u(T) \propto T\,\omega_{\max}^2$, diverging as the high-frequency cutoff $\omega_{\max} \to \infty$. The resolution of this "ultraviolet catastrophe" by Planck's quantization of oscillator energies is one of the historical starting points of quantum theory [2, 3, 4].

In parallel with these developments, classical vortex dynamics has established that incompressible, inviscid fluids support localized vortical structures—thin vortex rings and more general closed vortex loops—with finite energy, impulse, and self-induced translational velocity [8, 9]. In recent work, the present author has summarized standard results for the energy and impulse of thin vortex loops and emphasized their "particle-like" attributes in purely classical Euler flow [11]. A closed filament of circulation Γ and core size a carries a finite kinetic energy $E(\Gamma, R, a)$, an impulse $\boldsymbol{I}$, and an effective mass $M_{\mathrm{eff}} = \|\boldsymbol{I}\|/U$, where U is the self-induced velocity along the loop axis. The existence of such finite-energy, localized structures invites the question of how far one can push a classical "vortex-particle" picture within standard continuum mechanics.

A second, complementary observation is that kinetic energy in extended media contributes to inertia and gravitational mass through the relativistic relation $E = mc^2$. While this statement is encoded in the stress–energy tensor of relativistic fluids, explicit examples in simple incompressible configurations remain pedagogically useful. In Ref. [10], an incompressible, inviscid fluid undergoing rigid-body rotation in a finite cylinder was analyzed, and the volume-averaged rotational kinetic energy density $\langle e_{\mathrm{kin}} \rangle$ was related to an effective mass density $\Delta\rho_{\mathrm{eff}} = \langle e_{\mathrm{kin}} \rangle / c^2$ in the nonrelativistic limit. For that geometry one finds the compact result

$$\frac{\Delta\rho_{\mathrm{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\mathrm{edge}}}{c}\right)^2, \tag{3}$$

where v_{edge} is the tangential speed at the cylinder boundary and ρ is the rest-mass density. This example shows, within standard special relativity, how rotational motion modifies the effective mass density of an incompressible medium at order $(v/c)^2$.

The purpose of the present paper is to combine these two strands—finite-energy vortex loops and effective mass density from rotational kinetic energy—and to explore the thermodynamics of a dilute gas of vortex loops in an incompressible, inviscid fluid. Our analysis is entirely classical and remains within the framework of Eulerian fluid mechanics, special relativity, and textbook statistical mechanics. We show that:

1. A gas of weakly interacting, randomly oriented vortex loops with fixed circulation and core size admits a kinetic description in which the usual ideal-gas equation of state $PV = Nk_{\mathrm{B}}T$ is recovered from the momentum flux, without any modification of Boltzmann's constant or of the definition of temperature.

2. For an isotropic ensemble of ultra-relativistic or effectively massless excitations built from such loops, the standard relation $P = \frac{1}{3}u$ follows from the three-dimensional structure of the momentum flux tensor, in direct analogy with the radiation case.

3. The mechanical spectrum of admissible loop configurations is effectively bounded from above in frequency by the combination of finite core radius, finite circulation, and a characteristic swirl speed, so that the classical Rayleigh–Jeans integral for the energy density remains finite when evaluated over this mechanically constrained mode set. No modification of Maxwell's equations or of the relativistic field equations is assumed; the regularization arises from the kinematics of the vortex degrees of freedom.

From a thermodynamic standpoint, the main result is thus that a purely classical, incompressible fluid with vortex-loop microstructure can reproduce the familiar pressure–volume–temperature relations and the $P = \frac{1}{3}u$ relation for an isotropic radiation-like gas, while avoiding the ultraviolet divergence of the naive continuum mode counting. From a conceptual standpoint, the construction provides a concrete mechanical model in which the energy, pressure, and effective mass associated with internal rotational motion can be treated on the same footing as in relativistic continuum mechanics, without invoking any nonstandard dynamics or postulates beyond those already present in classical fluid mechanics and special relativity.

Notation and assumptions

Notation. Throughout the paper we use the following symbols:

Symbol	Meaning
ρ_f	Mass density of the (incompressible, inviscid) fluid
a	Core radius of a circulation loop (vortex-ring core thickness)
R	Ring radius (distance from ring centreline to symmetry axis)
Γ	Circulation of a loop, $\Gamma = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell}$
M_{eff}	Effective mass of a circulation loop, defined from energy/impulse
$v_{\max}$	Upper bound on tangential speed inside the loop core
$\omega_{\max}$	Maximal internal angular frequency, $\omega_{\max} = v_{\max}/a$
u	Energy density of the effective gas or radiation field
P	Pressure of the gas / radiation field
T	Temperature, k_{B} Boltzmann constant
c	Speed of light in vacuum

Table 1: Main symbols used in the circulation-loop model and its thermodynamic description.

Assumptions. The analysis in the main text rests on the following simplifying assumptions:

1. **Incompressible, inviscid background.** The fluid hosting the circulation loops is taken to be homogeneous, incompressible, and inviscid; its density is ρ_f and its mean flow vanishes in the rest frame of the container.

2. **Thin, non-reconnecting loops.** Vorticity is confined to thin, closed circulation loops (vortex rings) with core radius a and ring radius $R \gg a$. Loops are assumed not to reconnect, break, or merge on the timescales of interest.

3. **Dilute gas of loops.** The number density of loops is sufficiently low that mutual induction, deformation, and collisions can be neglected in the first approximation. Each loop can therefore be treated as a quasiparticle with effective mass M_{eff} and centre-of-mass momentum $\boldsymbol{p}$.

4. **Bounded tangential speed and maximal internal frequency.** The tangential speed of fluid elements within each core obeys $v_\theta \le v_{\max}$, with $v_{\max} < c$. Together with the finite core radius a, this implies a maximal internal angular frequency $\omega_{\max} = v_{\max}/a$ for rotational motions.

5. **Equipartition / classical limit.** When we assign an average energy $k_B T$ per mode or degree of freedom, we are explicitly in the classical, high-temperature limit of statistical mechanics (equipartition), and we do *not* impose any quantum occupancy constraints on the circulation-loop gas.

6. **Isotropy.** For the radiation-like sector, the ensemble of effective excitations is assumed to have an isotropic momentum distribution in three dimensions, so that angular averages such as $\langle \cos^2 \theta \rangle = 1/3$ apply. In d spatial dimensions this generalizes to $\langle \cos^2 \theta \rangle = 1/d$.

2 Vortex-loop gas and kinetic theory

In this section we set up a minimal kinetic model for a dilute gas of vortex loops in an incompressible, inviscid fluid and derive its thermodynamic equation of state. The construction follows standard kinetic theory, with the only nonstandard ingredient being the identification of the microscopic "particles" with finite-energy vortex loops in Euler flow.

2.1 Vortex-loop kinematics and effective mass

We consider a homogeneous, incompressible, inviscid fluid of density ρ_f, described by the Euler equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_f}\nabla p, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{4}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity field and $p(\boldsymbol{x}, t)$ the pressure. Vorticity is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}. \tag{5}$$

We assume that vorticity is confined to thin tubes (vortex filaments) of core radius a, within which the tangential speed is bounded by a characteristic value v_0, and that the flow outside the cores is irrotational.

Classical vortex dynamics shows that a closed vortex filament of circulation Γ and large-scale radius $R \gg a$ has finite kinetic energy $E(\Gamma, R, a)$, finite impulse $\boldsymbol{I}$, and a self-induced translational velocity U along its symmetry axis [8, 9, 11]. It is natural to define an effective mass

$$M_{\text{eff}} \equiv \frac{\|\boldsymbol{I}\|}{U}, \tag{6}$$

so that the loop behaves kinematically like a particle of mass M_{eff} moving with velocity U along its axis. In addition to this translational motion, the fluid inside and near the core carries internal rotational kinetic energy. In the nonrelativistic regime, the contribution of this internal energy to the effective mass density can be expressed, following Ref. [10], as

$$\Delta \rho_{\text{eff}} = \frac{\langle e_{\text{kin}} \rangle}{c^2}, \tag{7}$$

where $\langle e_{\text{kin}} \rangle$ is the volume-averaged rotational kinetic energy density and c is the speed of light.

In what follows, we treat each vortex loop as a structure with fixed circulation and core size, whose internal degrees of freedom are either frozen or accounted for as a fixed contribution to M_{eff}. The centre-of-mass motion of the loops then admits a standard kinetic description. The explicit thin-ring expressions for impulse, kinetic energy, translational speed, and M_{eff} used in this interpretation are collected in Appendix A.

2.2 Phase space and Hamiltonian

We consider N well-separated vortex loops in a container of volume V. Let $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$ denote the centre-of-mass position and momentum of the i-th loop ($i = 1, \ldots, N$). In the dilute limit we neglect mutual induction and other interactions between loops, so that the total Hamiltonian factorizes:

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2M_{\text{eff}}}. \tag{8}$$

The phase space is the usual $6N$-dimensional space of coordinates and momenta,

$$\Gamma = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)\}, \tag{9}$$

endowed with the Liouville measure

$$\mathrm{d}\mu(\Gamma) = \prod_{i=1}^{N} \mathrm{d}^3 x_i \, \mathrm{d}^3 p_i. \tag{10}$$

Importantly, the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$ is imposed at the level of the underlying Euler flow and affects the detailed form of M_{eff} and the internal energy of each loop. It does not modify the Liouville measure or the Hamiltonian structure for the centre-of-mass coordinates, which remain those of a dilute gas of classical particles.

2.3 Canonical ensemble and partition function

We now place the vortex-loop gas in contact with a heat bath at temperature T and describe it within the canonical ensemble. The canonical partition function is

$$Z_N(T, V) = \frac{1}{N!\, h^{3N}} \int \exp[-\beta H] \, \mathrm{d}^{3N} p \, \mathrm{d}^{3N} x, \qquad \beta = \frac{1}{k_{\text{B}} T}, \tag{11}$$

where h is Planck's constant and k_{B} Boltzmann's constant. Inserting the Hamiltonian (8),

$$Z_N(T, V) = \frac{1}{N!\, h^{3N}} \prod_{i=1}^{N} \left[\int_V \mathrm{d}^3 x_i \int_{\mathbb{R}^3} \exp\left(-\beta \frac{\boldsymbol{p}_i^2}{2M_{\text{eff}}} \right) \mathrm{d}^3 p_i \right]. \tag{12}$$

The position integrals each yield a factor of V, and the momentum integrals are Gaussian:

$$\int_{\mathbb{R}^3} \exp\left(-\beta \frac{\boldsymbol{p}^2}{2M_{\text{eff}}} \right) \mathrm{d}^3 p = (2\pi M_{\text{eff}} k_{\text{B}} T)^{3/2}. \tag{13}$$

Hence

$$Z_N(T, V) = \frac{1}{N!} \left[\frac{V}{\lambda_T^3} \right]^N, \qquad \lambda_T = \frac{h}{\sqrt{2\pi M_{\text{eff}} k_{\text{B}} T}}, \tag{14}$$

where λ_T is the thermal de Broglie wavelength associated with M_{eff}. This is the standard partition function of a classical ideal gas with mass M_{eff}; the vortex nature of the underlying structures is encoded entirely in the value of M_{eff} and in possible internal-state degeneracies, which we neglect here.

2.4 Ideal-gas equation of state

The Helmholtz free energy $F(T, V, N)$ is

$$F(T, V, N) = -k_{\text{B}} T \ln Z_N(T, V) = -k_{\text{B}} T \left[-\ln N! + N \ln V - 3N \ln \lambda_T \right]. \tag{15}$$

The pressure is obtained in the usual way as

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = k_{\mathrm{B}}T\left(\frac{\partial \ln Z_N}{\partial V}\right)_{T,N} = k_{\mathrm{B}}T\,\frac{N}{V}. \tag{16}$$

Thus the dilute vortex-loop gas obeys the ideal-gas law

$$PV = Nk_{\mathrm{B}}T, \tag{17}$$

with no modification of the equation of state due to the incompressibility of the underlying fluid. The incompressibility affects the value of M_{eff} through the kinetic energy of the core and the surrounding flow, but once M_{eff} is fixed, the centre-of-mass dynamics yields the same P–V–T relation as for point particles.

2.5 Isotropic momentum flux and $P = \frac{1}{3}u$

We now turn to the relation between pressure and energy density for an isotropic ensemble of excitations. Consider a gas of particles (here, effective vortex-loop quasiparticles) with energies E and momenta $\boldsymbol{p}$, moving with speed $v = |\boldsymbol{v}|$. In kinetic theory, the pressure on a plane normal to the x-axis is given by the momentum flux in the x-direction,

$$P = \frac{1}{V}\left\langle \sum_i p_{x,i}v_{x,i}\right\rangle, \tag{18}$$

where the average is over all particles in the volume V and $p_{x,i}$, $v_{x,i}$ are the x-components of the momentum and velocity of particle i. For an isotropic distribution, one can write

$$p_x v_x = (\boldsymbol{p}\cdot\boldsymbol{v})\cos^2\theta, \tag{19}$$

where θ is the angle between $\boldsymbol{p}$ and the x-axis. In three dimensions the angular average of $\cos^2\theta$ over the unit sphere is

$$\langle\cos^2\theta\rangle = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\cos^2\theta\,\sin\theta\,\mathrm{d}\theta = \frac{1}{3}. \tag{20}$$

Thus,

$$P = \frac{1}{3V}\left\langle\sum_i \boldsymbol{p}_i\cdot\boldsymbol{v}_i\right\rangle. \tag{21}$$

In the ultra-relativistic or effectively massless limit, one has $E_i = c\|\boldsymbol{p}_i\|$ and $\|\boldsymbol{v}_i\| = c$, so that $\boldsymbol{p}_i\cdot\boldsymbol{v}_i = E_i$. The pressure therefore becomes

$$P = \frac{1}{3V}\left\langle\sum_i E_i\right\rangle = \frac{1}{3}u, \tag{22}$$

where

$$u = \frac{1}{V}\left\langle\sum_i E_i\right\rangle \tag{23}$$

is the energy density. This is the familiar relation for an isotropic gas of massless particles or photons [6, 5]. In the present context, it applies to any regime in which the relevant vortex-loop excitations propagate at speeds close to c and their energy is dominated by translational motion rather than rest energy.

The crucial point is that Eq. (**??**) follows entirely from the three-dimensional isotropy of the momentum distribution and the kinematics of relativistic particles. The generalization to d

spatial dimensions, yielding $P = u/d$, is derived in Appendix B; there the factor 1/3 is seen to be a special case of the purely geometric factor $1/d$.

In summary, a dilute gas of vortex loops in an incompressible, inviscid fluid admits a standard kinetic description in which the ideal-gas law and the radiation-like relation $P = \frac{1}{3}u$ emerge in precisely the same way as for point particles and photons in conventional kinetic theory. The incompressibility of the underlying fluid constrains the internal structure and effective mass of the loops, but does not alter the macroscopic P–V–T relations obtained from their centre-of-mass motion and isotropic momentum flux.

3 Mechanically allowed mode set and the classical ultraviolet problem

The derivation of the Rayleigh–Jeans law in a cavity rests on two ingredients: (i) the normal-mode structure of Maxwell's equations in a finite volume, and (ii) the assumption that each normal mode behaves as an independent harmonic oscillator carrying an average energy $k_B T$ in the classical limit. The number of electromagnetic modes with angular frequency between ω and $\omega + \mathrm{d}\omega$ in a volume V is [5]

$$g_{\mathrm{EM}}(\omega)\,\mathrm{d}\omega = \frac{V}{\pi^2 c^3}\,\omega^2\,\mathrm{d}\omega, \tag{24}$$

so that the classical spectral energy density is

$$u_{\mathrm{RJ}}(\omega, T)\,\mathrm{d}\omega = \frac{\omega^2}{\pi^2 c^3} k_B T\,\mathrm{d}\omega. \tag{25}$$

The total energy density $u(T) = \int_0^\infty u_{\mathrm{RJ}}(\omega, T)\,\mathrm{d}\omega$ diverges as ω^3 at the upper limit, giving the well-known ultraviolet catastrophe in the purely classical field picture [3, 4, 2].

In the present framework, the electromagnetic field is regarded as a macroscopic, irrotational manifestation of the motion of an underlying incompressible fluid with circulation-loop microstructure. Maxwell's equations and their cavity eigenmodes are left intact at the macroscopic level. What changes is the interpretation of the *microstates* underlying a given field configuration: instead of independent field oscillators, the microstates are mechanical configurations of circulation loops. We now show that the kinematics of these loops restricts the set of physically realisable high-frequency modes, effectively regularizing the Rayleigh–Jeans integral without any modification of Maxwell's equations.

3.1 Maximal rotational frequency from core size and tangential speed

As in Sec. 2, we consider thin circulation loops (vortex rings) of core radius a embedded in a homogeneous incompressible fluid of density ρ_f. Inside the core, the tangential speed $v_\theta(r)$ of the fluid elements is bounded by a characteristic value v_0, so that

$$v_\theta(r) \le v_0 \quad \text{for } 0 \le r \le a. \tag{26}$$

The associated angular frequency at radius r is

$$\Omega(r) = \frac{v_\theta(r)}{r}, \tag{27}$$

so that the maximal angular frequency attainable within the core is bounded by

$$\Omega_{\max} = \max_{0<r\le a} \frac{v_\theta(r)}{r} \lesssim \frac{v_0}{a}. \tag{28}$$

This bound is purely kinematic: it expresses the fact that no fluid element can circulate around the core with tangential speed exceeding v_0 at a radius smaller than a. In particular, it implies a minimal time scale

$$\tau_{\min} \sim \frac{1}{\Omega_{\max}} \gtrsim \frac{a}{v_0}, \tag{29}$$

below which the fluid cannot respond to external forcing or internal excitations without violating the speed bound.

In the present context, we will denote by

$$\omega_{\max} \equiv \Omega_{\max} \tag{30}$$

the characteristic maximal angular frequency associated with internal rotational motion in the circulation-loop cores. This frequency provides a mechanical upper bound on how rapidly the substrate can be made to oscillate locally. Any putative field mode with $\omega \gg \omega_{\max}$ would require fluid motions inside the cores with periods shorter than $\tau_{\min}$ and hence tangential speeds above v_0, which are excluded by construction.

3.2 Mechanically realizable field modes

Maxwell's equations in vacuum (or in a homogeneous dielectric) admit harmonic solutions with arbitrarily large wave number k and frequency $\omega = ck$. In the purely field-theoretic picture, nothing prevents one from populating modes with $k \to \infty$. In the present mechanical picture, however, such field configurations must be realizable as coarse-grained manifestations of the motion of the underlying fluid.

We will not attempt to derive Maxwell's equations from the fluid model. Instead, we make the following conservative, kinematic assumption:

> *For a given cavity geometry and boundary conditions, we restrict attention to those electromagnetic normal modes whose spatial and temporal variations can be generated by configurations of circulation loops whose internal rotational frequencies do not exceed $\omega_{\max}$.*

Operationally, this means that among the continuum of formal cavity eigenmodes, only those with angular frequency ω up to some effective cutoff

$$\omega \lesssim \omega_{\max} \tag{31}$$

can actually be excited in thermal equilibrium with the mechanical substrate. Modes with $\omega \gg \omega_{\max}$ exist as mathematical solutions of the field equations, but there is no mechanical microstate in the circulation-loop gas that corresponds to their excitation at finite amplitude; they are therefore excluded from the thermodynamic counting of microstates.

Equivalently, one may say that the *effective* density of thermally accessible modes is modified from Eq. (24) to

$$g_{\text{eff}}(\omega) = g_{\text{EM}}(\omega)\, f\left(\frac{\omega}{\omega_{\max}}\right), \tag{32}$$

where $f(x)$ is a dimensionless cutoff function satisfying

$$f(x) \to 1 \quad (x \ll 1), \qquad f(x) \to 0 \quad (x \gg 1), \tag{33}$$

and decaying sufficiently rapidly as $x \to \infty$ to render the total energy finite. The detailed form of f depends on the microphysics of the circulation-loop gas and on how electromagnetic excitations couple to it; for the present discussion, only the existence of such an f with the stated asymptotics is required.

3.3 Modified Rayleigh–Jeans spectrum and finiteness of energy

With the mechanically constrained mode set (32), the classical Rayleigh–Jeans spectral energy density becomes

$$u_{\text{eff}}(\omega, T)\,\mathrm{d}\omega = \frac{\omega^2}{\pi^2 c^3} k_{\mathrm{B}} T\, f\!\left(\frac{\omega}{\omega_{\max}}\right) \mathrm{d}\omega. \tag{34}$$

The total energy density is

$$u(T) = \int_0^\infty u_{\text{eff}}(\omega, T)\,\mathrm{d}\omega = \frac{k_{\mathrm{B}} T}{\pi^2 c^3} \int_0^\infty \omega^2 f\!\left(\frac{\omega}{\omega_{\max}}\right) \mathrm{d}\omega. \tag{35}$$

By changing variables $x = \omega/\omega_{\max}$, this becomes

$$u(T) = \frac{k_{\mathrm{B}} T}{\pi^2 c^3}\, \omega_{\max}^3 \int_0^\infty x^2 f(x)\,\mathrm{d}x. \tag{36}$$

Under the mild assumption that the integral

$$C_f \equiv \int_0^\infty x^2 f(x)\,\mathrm{d}x \tag{37}$$

converges (which follows from the decay of f as $x \to \infty$), we obtain

$$u(T) = \frac{k_{\mathrm{B}} T}{\pi^2 c^3}\, \omega_{\max}^3 C_f, \tag{38}$$

which is finite for all finite T and scales linearly with temperature in the regime where equipartition holds. The ultraviolet divergence of the naive Rayleigh–Jeans integral is thus removed by the mechanical restriction of the mode set; the formal field-theoretic density of states (24) is replaced, for thermodynamic purposes, by the effective density (32) that accounts for the finite response time and finite core size of the substrate.

As a simple illustration, consider the sharp-cutoff choice

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1, \end{cases} \tag{39}$$

which encodes the assumption that modes with $\omega \le \omega_{\max}$ are fully accessible, while those with $\omega > \omega_{\max}$ are inaccessible. In that case,

$$C_f = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}, \tag{40}$$

and Eq. (38) reduces to

$$u(T) = \frac{k_{\mathrm{B}} T}{3\pi^2 c^3}\, \omega_{\max}^3. \tag{41}$$

This is precisely the Rayleigh–Jeans result with a hard frequency cutoff at $\omega_{\max}$, now justified mechanically by the finite core size and maximal tangential speed of the circulation loops. For more realistic, smooth cutoff functions $f(x)$ the prefactor C_f changes, but the qualitative conclusion remains: the energy density is finite and scales as $\omega_{\max}^3$.

3.4 Maxwell's equations and the role of the substrate

It is important to emphasize what has and has not been modified in this construction. At the macroscopic level:

- The form of Maxwell's equations and their normal-mode solutions in a cavity are left unchanged. The usual mode density (24) remains valid as a statement about the spectrum of solutions to the field equations.
- The relations $PV = Nk_{\mathrm{B}}T$ and $P = \frac{1}{3}u$ for the effective quasiparticles (Sec. 2) are derived from standard kinetic theory and from the isotropy of the momentum distribution, without any modification of the field equations.

What *does* change is the set of microstates that are admitted when one asks how a given electromagnetic field configuration is realized mechanically. In the usual classical treatment, each normal mode is assigned an independent harmonic-oscillator degree of freedom and hence contributes $k_{\mathrm{B}}T$ to the energy in the classical limit. In the present fluid-based model, the independent microstates are not field amplitudes but configurations of circulation loops obeying the kinematic constraints of incompressible Euler flow. These constraints imply a finite maximal rotational frequency $\omega_{\max}$ and thereby restrict the subset of cavity modes that can be populated in thermal equilibrium with the substrate.

In this sense, the incompressible fluid with circulation-loop microstructure acts as a *mechanical regulator* of the ultraviolet behavior of classical radiation. The Rayleigh–Jeans divergence does not arise, not because the field equations are altered, but because the assumption of an infinite set of independent harmonic oscillators is replaced by a finite-density set of mechanically realizable excitations whose internal time scales cannot be shorter than $1/\omega_{\max}$.

From the thermodynamic point of view, the key result of this section is that the ultraviolet behavior of the classical radiation spectrum is rendered finite once the microstate space is defined in terms of circulation loops with finite core radius and bounded tangential speed. The subsequent sections will explore how this mechanical cutoff scale compares with known microscopic scales in high-energy physics and what, if any, observable consequences might arise in astrophysical or cosmological contexts.

4 Microphysical scale and numerical estimates

In Sec. 3 we treated the electromagnetic field, in a purely kinematic sense, as being represented by a gas of mechanically admissible modes built from closed circulation loops (thin vortex rings) in an incompressible, inviscid medium. Each loop was assumed to have a finite core radius a and a bounded tangential speed along the core. In this section we translate these mechanical assumptions into (i) an upper bound on the angular frequencies that can be realized by such loops and (ii) the corresponding ultraviolet cutoff for the Rayleigh–Jeans integral. Maxwell's equations and their normal modes are not modified; the change enters only through the identification of which modes are physically realizable given the microstructure.

4.1 Kinematic upper bound on angular frequency

Consider a single closed circulation loop with core radius a embedded in an incompressible, inviscid fluid of density ρ_f. Let r denote the local radius of curvature of the loop centreline, and let $v_\theta(r)$ be the local tangential speed of the circulating fluid inside the core. Locally, the motion is equivalent to rigid-body rotation in a small disc of radius $O(a)$, so that the angular frequency associated with this local rotation is

$$\omega_{\mathrm{loc}}(r) = \frac{v_\theta(r)}{r}. \tag{42}$$

We impose two mechanical conditions:

1. *Finite core radius:*

$$r \geq a, \tag{43}$$

since the curvature radius of the centreline cannot be smaller than the core radius without leaving the thin-core regime.

2. *Bounded tangential speed:*
$$v_\theta(r) \le v_{\max}, \tag{44}$$
where $v_{\max}$ is a characteristic upper bound, e.g. set by relativistic constraints $v_{\max} < c$ or by the stiffness of the underlying medium.

Combining these two inequalities gives an upper bound on the local rotational frequency,

$$\omega_{\mathrm{loc}}(r) = \frac{v_\theta(r)}{r} \le \frac{v_{\max}}{a}. \tag{45}$$

Since any normal mode of the loop gas is built from superpositions of such local rotational degrees of freedom, it is natural to define the mechanically.allowed maximal angular frequency

$$\omega_{\max} := \frac{v_{\max}}{a}. \tag{46}$$

Equation (46) is a purely kinematic consequence of finite core radius and bounded tangential speed in incompressible Euler flow; no assumption about electromagnetism has been used at this stage.

For later comparison it is helpful to note that a closely related frequency scale already appears as a kinematic reference in rotating-flow benchmarks, where one defines

$$\Omega_0 \equiv \frac{C_e}{r_c}, \qquad E_0 = \hbar\Omega_0, \tag{47}$$

with C_e a characteristic speed and r_c a microscopic length scale. In the present notation, $\omega_{\max}$ plays the same structural role as Ω_0 if one identifies $v_{\max} \leftrightarrow C_e$ and $a \leftrightarrow r_c$.

4.2 Rayleigh–Jeans energy density with a mechanical cutoff

In the standard derivation of the Rayleigh–Jeans law, the electromagnetic field in a cavity is decomposed into normal modes; the number of modes per unit volume with angular frequency between ω and $\omega + \mathrm{d}\omega$ is
$$g(\omega)\,\mathrm{d}\omega = \frac{\omega^2}{\pi^2 c^3}\,\mathrm{d}\omega, \tag{48}$$
where the factor of 2 for polarization is already included in the prefactor. Equipartition assigns an average energy $k_{\mathrm{B}}T$ per mode, so that the spectral energy density in the classical Rayleigh–Jeans limit reads
$$u_{\mathrm{RJ}}(\omega, T) = g(\omega)\,k_{\mathrm{B}}T = \frac{k_{\mathrm{B}}T}{\pi^2 c^3}\,\omega^2. \tag{49}$$
If all frequencies $0 \le \omega < \infty$ are allowed, the total energy density

$$u_{\mathrm{tot}}^{(\mathrm{RJ})}(T) = \int_0^\infty u_{\mathrm{RJ}}(\omega, T)\,\mathrm{d}\omega \tag{50}$$

diverges as ω^3 at the upper limit: the classical ultraviolet catastrophe.

In the circulation-loop picture, the *mechanically realizable* modes are restricted by the bound (46). If one assumes that only modes with $\omega \le \omega_{\max}$ can be populated in thermal equilibrium with the substrate, the Rayleigh–Jeans spectrum is automatically cut off:

$$u_{\mathrm{tot}}^{(\mathrm{RJ,cut})}(T) = \int_0^{\omega_{\max}} \frac{k_{\mathrm{B}}T}{\pi^2 c^3}\,\omega^2\,\mathrm{d}\omega = \frac{k_{\mathrm{B}}T}{3\pi^2 c^3}\,\omega_{\max}^3. \tag{51}$$

Substituting Eq. (46) gives

$$u_{\rm tot}^{(\rm RJ,cut)}(T) = \frac{k_{\rm B}T}{3\pi^2 c^3}\left(\frac{v_{\rm max}}{a}\right)^3. \tag{52}$$

A dimensional check confirms that Eq. (52) has the correct units: $[k_{\rm B}T] = {\rm J}$ and $[v_{\rm max}^3/(a^3c^3)] = {\rm m}^{-3}$, so that $[u_{\rm tot}] = {\rm J\,m^{-3}}$ as required.

More generally, one can write the effect of the mechanical constraint in terms of a dimensionless cutoff function $f(\omega/\omega_{\rm max})$,

$$u_{\rm eff}(\omega, T) = \frac{k_{\rm B}T}{\pi^2 c^3}\,\omega^2\, f\left(\frac{\omega}{\omega_{\rm max}}\right), \tag{53}$$

with $f(x) \to 1$ for $x \ll 1$ and $f(x) \to 0$ for $x \gg 1$. The total energy density is then

$$u(T) = \int_0^\infty u_{\rm eff}(\omega, T)\,{\rm d}\omega = \frac{k_{\rm B}T}{\pi^2 c^3}\,\omega_{\rm max}^3 \int_0^\infty x^2 f(x)\,{\rm d}x, \tag{54}$$

which is finite provided the integral $\int_0^\infty x^2 f(x)\,{\rm d}x$ converges. The sharp cutoff used in Eq. (51) corresponds to $f(x) = 1$ for $0 \le x \le 1$ and $f(x) = 0$ for $x > 1$, giving

$$\int_0^\infty x^2 f(x)\,{\rm d}x = \int_0^1 x^2\,{\rm d}x = \frac{1}{3}, \tag{55}$$

and reproducing Eq. (52).

4.3 Numerical scales and relation to a kinematic reference frequency

The expression (52) becomes quantitatively meaningful once concrete values for $v_{\rm max}$ and a are specified. In the absence of a fully microscopic model, we keep these parameters symbolic but note that a natural identification is

$$v_{\rm max} \sim C_e, \qquad a \sim r_c, \tag{56}$$

where C_e and r_c are a characteristic speed and length scale that already appear in rotating-flow benchmarks, via Eq. (47). With this choice,

$$\omega_{\rm max} \equiv \frac{v_{\rm max}}{a} \longrightarrow \Omega_0 = \frac{C_e}{r_c}, \qquad E_0 = \hbar\Omega_0. \tag{57}$$

For illustrative numerical values one may adopt, following Ref. [Iskandarani],

$$C_e \sim 10^6\ {\rm m\,s^{-1}}, \qquad r_c \sim 10^{-15}\ {\rm m}, \tag{58}$$

so that

$$\Omega_0 \sim \frac{10^6}{10^{-15}}\ {\rm s^{-1}} \sim 10^{21}\ {\rm s^{-1}}, \tag{59}$$

and

$$E_0 = \hbar\Omega_0 \sim 10^{-13}\ {\rm J} \sim 0.5\ {\rm MeV}. \tag{60}$$

Numerically, using CODATA values, this is of the same order as the electron rest energy $m_e c^2 \approx 0.511$ MeV; this numerical proximity is used here only as a reference point for the scale of E_0.

Substituting $\omega_{\rm max} = \Omega_0$ into Eq. (51), one finds

$$u_{\rm tot}^{(\rm RJ,cut)}(T) = \frac{k_{\rm B}T}{3\pi^2 c^3}\,\Omega_0^3. \tag{61}$$

For laboratory blackbodies at $T \sim 10^3$–10^4 K, the characteristic frequencies contributing appreciably to the Planck spectrum lie far below Ω_0, so the presence of such a mechanical cutoff would have no observable effect on standard thermal-radiation measurements.

4.4 Comparison with Planck's law and observational constraints

The exact blackbody spectrum is given by Planck's law,

$$u_{\mathrm{Planck}}(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}, \tag{62}$$

with total energy density

$$u_{\mathrm{tot}}^{(\mathrm{Planck})}(T) = a_{\mathrm{R}} T^4, \qquad a_{\mathrm{R}} = \frac{\pi^2 k_{\mathrm{B}}^4}{15\hbar^3 c^3}. \tag{63}$$

By contrast, the cutoff Rayleigh–Jeans result (52) scales linearly with T for fixed ω_{max}. No choice of ω_{max} can reconcile this T-scaling with the empirically verified T^4 scaling over a broad range of temperatures. The mechanical cutoff therefore *cannot* replace quantum statistics or Planck's law; it merely shows that, in any mechanical substrate with finite core radius and bounded rotational speeds, the classical equipartition argument does not lead to an ultraviolet divergence at the level of the microstates.

From an observational standpoint, the absence of any high-frequency cutoff in laboratory, nuclear, and astrophysical photon spectra implies a lower bound

$$\omega_{\mathrm{max}} = \frac{v_{\mathrm{max}}}{a} \gg \omega_{\gamma,\mathrm{max}}^{(\mathrm{obs})}, \tag{64}$$

where $\omega_{\gamma,\mathrm{max}}^{(\mathrm{obs})}$ is the highest photon frequency reliably measured. This constraint translates into a bound on the ratio v_{max}/a: either the core radius a must be extremely small, or the mechanical speed scale v_{max} extremely large, or both.

In summary, the mechanical mode restriction encoded in Eq. (46) yields a natural ultraviolet cutoff $\omega_{\mathrm{max}} = v_{\mathrm{max}}/a$ and a finite Rayleigh–Jeans energy density, without altering Maxwell's equations or the macroscopic equations of state derived in Sec. 2. The cutoff does not obviate the need for quantization to recover the Planck spectrum and its T^4 scaling, but it provides a concrete example of how a specific class of underlying rotational microphysics regularizes the classical ultraviolet catastrophe while remaining compatible, in principle, with standard electrodynamics at accessible frequencies.

5 Discussion and conclusions

5.1 Summary of results

We have considered a classical, incompressible, inviscid fluid in which vorticity is confined to thin circulation loops (vortex rings) of finite core radius a. Within this framework, the main results of the paper are:

1. *Vortex loops as quasiparticles and the ideal-gas law.* Treating closed circulation loops as localized, finite-energy structures with effective mass M_{eff} and centre-of-mass momentum $\boldsymbol{p}$, we showed that a dilute gas of such loops admits a standard kinetic description. In the canonical ensemble, the partition function reduces to that of a classical ideal gas, and the equation of state
$$PV = Nk_{\mathrm{B}}T \tag{65}$$
follows from the usual derivative of the Helmholtz free energy with respect to volume. The incompressibility of the underlying medium affects the value of M_{eff} through the kinetic energy of the core and the surrounding flow but does not modify the macroscopic P–V–T relation.

2. *Isotropic momentum flux and $P = \frac{1}{3}u$.* For an isotropic ensemble of ultra-relativistic or effectively massless excitations built from circulation loops, we recovered the standard relation between pressure and energy density,

$$P = \frac{1}{3}u, \tag{66}$$

by computing the momentum flux on a plane and using the three-dimensional angular average $\langle \cos^2\theta \rangle = 1/3$. This derivation depends only on the isotropy of the momentum distribution and the relativistic relation $E = pc$; it is independent of the detailed microstructure of the underlying fluid.

3. *Mechanical mode restriction and ultraviolet regularization.* By imposing a finite core radius a and a bound $v_{\max}$ on the tangential speed of fluid elements within the core, we obtained a purely kinematic upper bound on the internal rotational frequency,

$$\omega_{\max} = \frac{v_{\max}}{a}. \tag{67}$$

If electromagnetic normal modes in a cavity are taken to be in one-to-one correspondence with mechanically realizable excitations of this medium, then only modes with $\omega \le \omega_{\max}$ can be populated in thermal equilibrium. The classical Rayleigh–Jeans integral for the energy density, when restricted to this mechanically allowed mode set, becomes

$$u_{\text{tot}}^{(\text{RJ,cut})}(T) = \frac{k_B T}{3\pi^2 c^3}\,\omega_{\max}^3, \tag{68}$$

which is finite for all finite temperatures. More generally, introducing a smooth cutoff function $f(\omega/\omega_{\max})$ yields

$$u(T) = \frac{k_B T}{\pi^2 c^3}\,\omega_{\max}^3 \int_0^\infty x^2 f(x)\,\mathrm{d}x, \tag{69}$$

so that the ultraviolet divergence of the naive Rayleigh–Jeans spectrum is removed once the microstate space is defined in terms of circulation loops with finite core size and bounded rotational speeds.

4. *Consistency with Maxwell's equations.* Throughout, Maxwell's equations and their cavity eigenmodes were left unchanged at the macroscopic level. The density of modes $g(\omega) = \omega^2/(\pi^2 c^3)$ remains valid as a statement about the spectrum of solutions to the field equations. The modification enters only through the identification of which modes are physically realizable given the mechanical properties of the underlying medium; the ultraviolet behaviour of the spectrum is thus regularized at the level of microstate counting rather than by altering the field equations.

Taken together, these results demonstrate that a classical incompressible medium with circulation-loop microstructure can reproduce the familiar ideal-gas and radiation equations of state and can regularize the ultraviolet behaviour of the classical Rayleigh–Jeans spectrum, entirely within the framework of Eulerian fluid mechanics, special relativity, and textbook statistical mechanics.

5.2 Relation to classical and quantum descriptions of radiation

It is important to distinguish clearly between three levels of description:

1. The *macroscopic field description*, in which the electromagnetic field is governed by Maxwell's equations in a given background and admits a continuum of normal modes in a cavity, with density $g(\omega) = \omega^2/(\pi^2 c^3)$.

2. The *classical statistical description*, in which each normal mode is treated as an independent harmonic oscillator with average energy k_BT in the high-temperature limit, leading to the Rayleigh–Jeans spectrum and its ultraviolet divergence.

3. The *quantum statistical description*, in which the energy of each mode is quantized in units $\hbar\omega$ and Bose–Einstein statistics yields Planck's law and the T^4 dependence of the blackbody energy density.

The analysis in this paper modifies only the second of these points. We have shown that if one insists on assigning a classical mechanical substrate to the electromagnetic field—here, an incompressible medium with circulation loops of finite core radius and bounded tangential speeds—then the assumption of an infinite set of independent harmonic oscillators, each carrying k_BT, is no longer tenable. Instead, the set of thermally accessible modes is limited by the kinematics of the medium, and the Rayleigh–Jeans integral is rendered finite without altering Maxwell's equations.

At the same time, the mechanical cutoff introduced by $\omega_{\max} = v_{\max}/a$ does *not* reproduce the correct temperature scaling of the blackbody energy density. For fixed $\omega_{\max}$, the cutoff Rayleigh–Jeans result scales as $u_{\text{tot}}^{(\text{RJ,cut})} \propto T$, whereas Planck's law yields $u_{\text{tot}}^{(\text{Planck})} \propto T^4$.[1] No choice of $\omega_{\max}$ can reconcile these scalings over a broad range of temperatures. The mechanical regularization discussed here therefore does not remove the need for quantization; it simply illustrates how a particular class of underlying microphysics prevents the classical ultraviolet catastrophe from arising at the level of the substrate.

In this sense, the circulation-loop model should be viewed as a *classical analogue* or toy model: it exhibits, in a purely mechanical setting, how a finite microstructure and a maximum internal frequency can regulate the ultraviolet behaviour of a classical field theory, while leaving intact the macroscopic equations of motion and the standard kinetic derivations of $PV = Nk_BT$ and $P = \frac{1}{3}u$.

5.3 Conceptual and pedagogical implications

Even if the actual microscopic origin of electromagnetism lies in quantum fields rather than classical incompressible media, explicit mechanical models can be conceptually useful. The circulation-loop construction offers several such benefits:

- It provides a concrete example in which *internal motion* in an extended medium (here, rotational motion within vortex cores) contributes to effective mass density via $E = mc^2$. This complements textbook discussions of relativistic fluids by exhibiting a specific configuration in which the volume-averaged mass density is shifted by $\Delta\rho_{\text{eff}} = \langle e_{\text{kin}}\rangle/c^2$.

- It illustrates in a tangible way why the naive continuum assumption of infinitely many independent modes can be misleading. In the circulation-loop gas, the irrotational field outside the cores is not a collection of independent oscillators; it is constrained by the finite set of vortex degrees of freedom. The ultraviolet divergence of the Rayleigh–Jeans law is therefore seen as an artefact of overcounting microstates, rather than as an unavoidable pathology of classical physics.

- It connects the statistical mechanics of gases with the dynamics of vortical structures in fluids: the same circulation loops that serve as "particles" in the kinetic theory also carry the rotational energy that underlies the effective mass density and the mechanical cutoff scale.

[1] Here and throughout we assume the usual equilibrium photon gas in flat spacetime and neglect chemical potentials.

For teaching purposes, one might imagine a fluid analogue: a tank of water seeded with smoke-ring-like vortices, illuminated and tracked via particle-image velocimetry. As smaller and smaller scales are probed, one eventually reaches the minimal core size and maximal rotation rate of the rings. No matter how finely one tries to decompose the flow into "modes", there is a physical limit set by the underlying structures. This provides an intuitive picture of how ultraviolet cutoffs can arise from microstructure, independent of any quantum postulate.

5.4 Possible extensions and future work

The present work raises several avenues for further investigation:

- *Detailed microphysical models.* We have treated the circulation loops in a schematic way, assuming a fixed core radius a, a fixed speed bound $v_{\max}$, and a simple relation $\omega_{\max} = v_{\max}/a$. A more detailed analysis could derive a, $v_{\max}$, and the effective cutoff function $f(\omega/\omega_{\max})$ from a specific model of the medium, potentially including compressibility, elasticity, or additional internal degrees of freedom.
- *Fluid analogues and laboratory tests.* In classical fluids, wave spectra in the presence of a bath of vortices often exhibit saturation or steepening at high wavenumber, reflecting the finite size of coherent structures. It would be interesting to design controlled experiments in which the energy spectrum of surface or internal waves is measured in a flow populated by vortices of known core size, to test how closely the observed saturation resembles the simple cutoff picture developed here.
- *Coupling to electromagnetism and relativistic field theory.* We have deliberately avoided deriving Maxwell's equations from the underlying fluid dynamics. One possible extension is to embed the circulation-loop picture in a more general effective field theory, where the fluid degrees of freedom appear as an emergent sector coupled to electromagnetism. This could clarify whether the mechanical cutoff $\omega_{\max}$ can be related to known microscopic scales, such as the electron Compton frequency, without conflicting with precision tests of quantum electrodynamics.
- *Connection to broader microstructural frameworks.* In separate work, circulation loops of the type considered here have been discussed under the name "swirl strings" as part of a broader attempt to represent matter and radiation in terms of structured rotational degrees of freedom in a continuous medium. The present paper can be viewed as the classical, incompressible limit of that wider programme, restricted to thermodynamic and ultraviolet questions. Exploring the full implications of such a framework lies beyond our present scope, but the circulation-loop gas studied here may serve as a useful benchmark sector.

5.5 Final remarks

We have shown that a simple, classical model—an incompressible fluid populated by finite-core circulation loops—is sufficient to reproduce the ideal-gas and radiation equations of state and to regularize the classical ultraviolet behaviour of thermal radiation at the level of microstate counting. The ultraviolet catastrophe is thus seen not as an inevitable failure of classical reasoning, but as a consequence of assigning an infinite set of independent degrees of freedom to the field without regard to the mechanical limitations of any underlying medium. While the correct blackbody spectrum and its T^4 scaling undoubtedly require quantum statistics, the circulation-loop model provides a concrete example of how microstructure and finite rotational frequencies can render the classical ultraviolet problem moot, without modifying Maxwell's equations or the familiar thermodynamic relations that emerge from kinetic theory.

A Thin vortex-ring formulas

For completeness we collect here the standard expressions for the impulse, kinetic energy, and translational speed of a thin circular vortex ring in an ideal incompressible fluid, as used in the main text. These formulas go back to Kelvin, Helmholtz, Lamb, and are summarized in modern form by Batchelor and Saffman.

We consider a single circular vortex ring of radius R and core radius a, with $a \ll R$, embedded in a homogeneous incompressible fluid of density ρ_f. The circulation around the core is denoted by Γ.

A.1 Impulse

The hydrodynamic impulse of an isolated thin-core ring, directed along the symmetry axis, has magnitude

$$I = \rho_f \, \Gamma \, \pi R^2. \tag{70}$$

This result is independent of the detailed vorticity distribution inside the core, provided the ring is thin and axisymmetric.

A.2 Kinetic energy

The kinetic energy associated with the flow field of a thin vortex ring with uniform vorticity in the core is, to leading order in a/R,

$$E = \frac{1}{2} \rho_f \, \Gamma^2 R \left(\ln \frac{8R}{a} - \frac{7}{4} \right). \tag{71}$$

The logarithmic term reflects the long-range nature of the velocity field; the constant offset $7/4$ depends weakly on the assumed core model (e.g. uniform vs. Gaussian vorticity).

A.3 Translational speed

The self-induced translational speed of the ring along its symmetry axis is

$$U = \frac{\Gamma}{4\pi R} \left(\ln \frac{8R}{a} - \frac{1}{4} \right), \tag{72}$$

again valid for $a/R \ll 1$ and for standard core models. Different core profiles change only the constant term inside the parentheses at this order.

A.4 Effective mass

In the main text we treat the ring as a quasiparticle with effective mass M_{eff} moving at speed U along the symmetry axis, carrying impulse I. It is therefore natural to define the effective mass via 6 $M_{\text{eff}} \equiv \frac{I}{U}$. Using Eqs. (70) and (72), one obtains

$$M_{\text{eff}} = \frac{\rho_f \, \Gamma \, \pi R^2}{\dfrac{\Gamma}{4\pi R} \left(\ln \dfrac{8R}{a} - \dfrac{1}{4} \right)} = \frac{4\pi^2 \rho_f R^3}{\ln \dfrac{8R}{a} - \dfrac{1}{4}}. \tag{73}$$

For very slender rings, $\ln(8R/a) \gg 1$, the constant term $1/4$ can be neglected and the effective mass simplifies to

$$M_{\text{eff}} \approx \frac{4\pi^2 \rho_f R^3}{\ln(8R/a)}. \tag{74}$$

In the main text, only the scaling $M_{\text{eff}} \propto \rho_f R^3$ and the presence of a weak logarithmic dependence on a/R are used; the precise numerical coefficients play no qualitative role in the thermodynamic arguments.

B Pressure and energy density in d dimensions

In the main text we use the relation

$$P = \frac{1}{3}u \tag{75}$$

for an isotropic ensemble of ultra-relativistic or effectively massless excitations in three spatial dimensions. Here we show that in d spatial dimensions the corresponding relation is

$$P = \frac{1}{d}\,u, \tag{76}$$

so that the factor $1/3$ is a direct consequence of three-dimensional isotropy rather than any special feature of the circulation-loop model.

Consider a gas of non-interacting, massless particles (or effective excitations) in d spatial dimensions, contained in a volume V_d. Let the energy of a particle with momentum $\boldsymbol{p}$ be

$$E = c\,|\boldsymbol{p}|, \tag{77}$$

so that the group velocity is

$$\boldsymbol{v} = \frac{\partial E}{\partial \boldsymbol{p}} = c\,\frac{\boldsymbol{p}}{|\boldsymbol{p}|}. \tag{78}$$

We assume an isotropic momentum distribution, so that all directions of $\boldsymbol{p}$ are equally probable.

B.1 Energy density

Let $f(\boldsymbol{p})$ be the occupation number (particles per phase-space volume) in momentum space. The energy density is

$$u = \frac{1}{V_d}\int E(\boldsymbol{p})\,f(\boldsymbol{p})\,\mathrm{d}^d p. \tag{79}$$

B.2 Pressure as momentum flux

The pressure is the normal component of the momentum flux across a plane. Without loss of generality, consider the plane orthogonal to the x-axis. The contribution of a single particle with momentum $\boldsymbol{p}$ and velocity $\boldsymbol{v}$ to the x-component of the momentum flux is $p_x v_x/V_d$. Summing over all particles,

$$P = \frac{1}{V_d}\int p_x v_x\,f(\boldsymbol{p})\,\mathrm{d}^d p. \tag{80}$$

For massless particles, $v_x = c\,p_x/|\boldsymbol{p}|$, so

$$p_x v_x = c\,\frac{p_x^2}{|\boldsymbol{p}|}. \tag{81}$$

It is convenient to write $\boldsymbol{p}$ in spherical coordinates in momentum space: $|\boldsymbol{p}| = p$, and let θ be the angle between $\boldsymbol{p}$ and the x-axis. Then $p_x = p\cos\theta$ and

$$p_x v_x = c\,\frac{p^2\cos^2\theta}{p} = c\,p\,\cos^2\theta. \tag{82}$$

Substituting into Eq. (80),

$$P = \frac{c}{V_d}\int p\,\cos^2\theta\,f(\boldsymbol{p})\,\mathrm{d}^d p. \tag{83}$$

B.3 Isotropic angular average

For an isotropic distribution, the angular dependence factorizes:

$$f(\boldsymbol{p}) = f(p), \tag{84}$$

and the average of $\cos^2\theta$ over the unit sphere in d dimensions is

$$\langle \cos^2\theta \rangle = \frac{1}{d}. \tag{85}$$

This is a standard result: by symmetry, $\langle p_x^2 \rangle = \langle p_y^2 \rangle = \cdots = \langle p_d^2 \rangle$, and $\sum_{i=1}^{d} \langle p_i^2 \rangle = \langle p^2 \rangle$, hence $\langle p_x^2 \rangle = \langle p^2 \rangle / d$ and $\langle \cos^2\theta \rangle = \langle p_x^2/p^2 \rangle = 1/d$.

Using this in Eq. (83), we can write

$$P = \frac{c}{V_d} \left\langle \cos^2\theta \right\rangle \int p\, f(p)\, \mathrm{d}^d p = \frac{c}{d V_d} \int p\, f(p)\, \mathrm{d}^d p. \tag{86}$$

Since $E = cp$, the integral $\int p\, f(p)\, \mathrm{d}^d p = (1/c) \int E f(p)\, \mathrm{d}^d p$, and comparing with Eq. (79) we obtain

$$u = \frac{1}{V_d} \int E(\boldsymbol{p}) f(\boldsymbol{p})\, \mathrm{d}^d p = \frac{c}{V_d} \int p\, f(p)\, \mathrm{d}^d p. \tag{87}$$

Therefore Eq. (86) becomes

$$P = \frac{1}{d}\, u. \tag{88}$$

Specializing to $d = 3$ recovers the familiar relation $P = \frac{1}{3}u$ used in the main text. The result is completely general for massless particles with isotropic momentum distributions and does not depend on the details of the circulation-loop model; it is a kinematic statement about momentum flux in d-dimensional space.

Acknowledgments

The author thanks the authors of standard texts on fluid mechanics and relativistic field theory for providing the background material on which this analysis is based.

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