Delay–Induced Mode Discreteness in Nonlinear Ring Systems

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Abstract

Abstract. Nonlinear systems with delayed feedback are known to exhibit spatio–temporal pattern formation despite being governed by low–dimensional equations. In this work, we demonstrate that such delay–induced pattern formation provides a universal classical mechanism for the emergence of discrete circulating modes in nonlinear ring systems. This reinterpretation links delay differential equations, modulational instability, and global bifurcation theory to the observed mode discreteness of optical, electronic, and circulating field resonators. A concrete delay model is analyzed and shown to generate square–wave plateaus and multistability via modulational instability.

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1 Novelty and Conceptual Contribution

While the equivalence between long–delay systems and spatially extended media is well known [1, 2], we propose a new physical interpretation:

Delay–induced pattern formation constitutes a universal mechanism for mode discreteness in circulating fields.

In any nonlinear loop with finite propagation time, feedback forces the field to self–organize into discrete plateau states. This produces quantized circulation modes without invoking microscopic quantization postulates.

2 Governing Delay Model

We consider the minimal nonlinear ring equation:

$$\epsilon \dot{x}(t) = -x(t) + \mu x(t-\tau) - x(t-\tau)^3, \tag{1}$$

where $\epsilon \ll 1$ is the fast relaxation time, τ is the circulation delay, and μ controls the feedback gain.

3 Linear Dispersion and Instability

Linearizing Eq. (1) about the trivial state $x = 0$ yields the characteristic equation:

$$\epsilon\lambda + 1 = \mu e^{-\lambda\tau}. \tag{2}$$

In the long–delay limit ($\tau \gg \epsilon$), the eigenvalues organize into quasi–continuous branches:

$$\lambda_n \approx \frac{1}{\tau}\Big[\ln\mu + i(2\pi n + k)\Big], \tag{3}$$

where k is the continuous pseudo–spatial wavenumber associated with the delay–induced coordinate $\sigma \in [0, \tau]$ [2].

4 Toy Model and Emergent Discreteness

Numerical integration of Eq. (1) reveals that uniform oscillations become modulationally unstable. The system breaks symmetry and settles into square–wave plateau states.

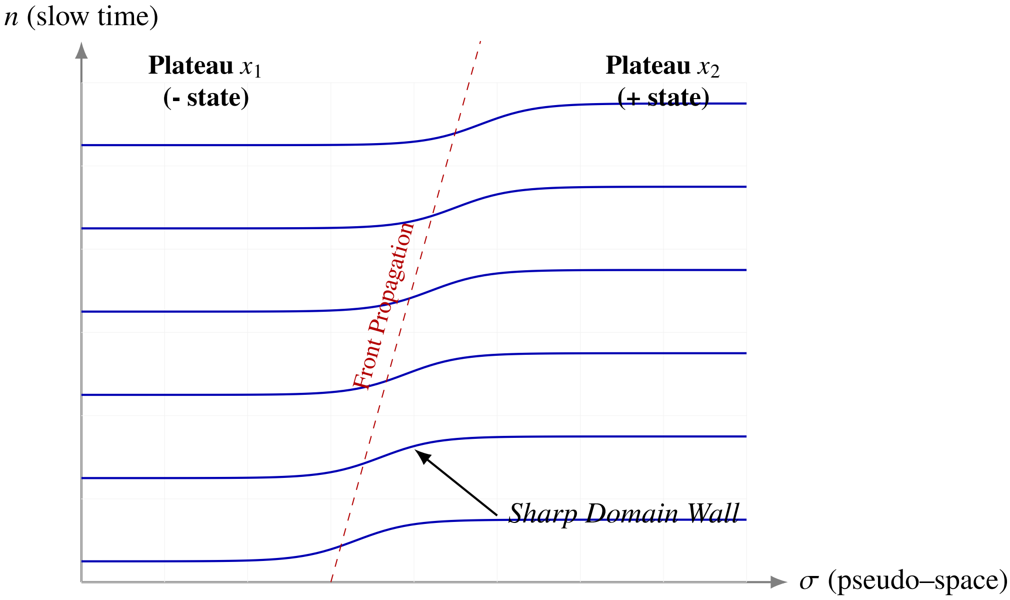

Figure 1: **Space-Time Representation.** Evolution of the pseudo-space coordinate σ versus slow time n. The system spontaneously separates into two stable plateaus (x_1, x_2) separated by sharp transition layers (fronts). Each stable plateau configuration corresponds to a discrete circulating mode.

The system exhibits **multistability**: distinct numbers of plateaus can coexist for the same parameter values, effectively acting as different circulation mode numbers.

5 Global Bifurcations

These square–wave families organize into *collapsed snaking* structures. The global topology of solutions is governed by Bykov T–points, which act as organizing centers for the creation and destruction of coexisting modes [3, 4].

6 Topological Protection of Circulating Modes

The circulating fields of ring systems define Hamiltonian trajectories on a toroidal phase space. As shown by Boozer [5], such Hamiltonian flows generically organize into invariant tori, which act as transport barriers for field lines. Under perturbations, these invariant tori break into *cantori*—fractal remnants that remain extremely effective at suppressing transport—while *turnstiles* provide discrete topological channels for flux transfer between regions.

These structures imply that circulating modes in toroidal systems are not merely dynamically selected by delay–induced modulational instability, but also *topologically protected*. Each plateau configuration identified in Fig. 1 corresponds to a family of field lines trapped between cantori, with transitions between modes mediated by turnstile dynamics. This explains both the robustness of discrete circulating states and their slow decay through rare, topologically constrained transport events.

Thus, delay–induced pattern formation determines *which* modes appear, while toroidal Hamiltonian topology determines *which modes persist*.

7 Circulating Field Interpretation

Any nonlinear loop with finite propagation time—whether optical cavities, electronic delay lines, or electromagnetic resonators—belongs to this universality class.

In circulation–based field models, the delay τ corresponds strictly to the loop transit time. Consequently, **mode discreteness follows directly from classical modulational instability**, removing the need for ad-hoc quantization.

8 Conclusion

Delayed nonlinear feedback transforms time into an effective spatial dimension. The interplay of linear dispersion, modulational instability, and global bifurcations forces circulating systems to self–organize into discrete plateau states. This provides a robust, classical origin for the quantization-like behavior observed in ring systems.

A Sketch of CGLE Reduction

Near the Hopf bifurcation threshold ($\mu \approx 1$), the solution can be decomposed as:

$$x(t) = A(\zeta, \theta)e^{i\omega t} + \text{complex conjugate}.$$

where $\zeta = t/\tau$ is the pseudo-space and $\theta = \epsilon t$ is the slow time. A standard multiple–scale expansion reduces the dynamics to the Complex Ginzburg–Landau Equation (CGLE) [1]:

$$\partial_\theta A = \mu A + (1 + i\alpha)\partial_\zeta^2 A - (1 + i\beta)|A|^2 A. \tag{4}$$

This equation governs the envelope stability and predicts the onset of domain formation.

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