A First-Principles Origin of the Inverse-Square Law in Swirl–String Theory:
Three Derivations from Local Field Mediation and Momentum-Flux Conservation

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Abstract

A recurrent objection to flat-background, emergent-gravity programs is that the inverse-square distance law is often imported rather than derived. We close this gap in the weak-field, static monopole sector by exhibiting three independent derivations of the $1/r$ potential and $1/r^2$ flux. (I) A Gauss-law scalar effective field theory (EFT) for the far-field mediator yields the Poisson equation whose Green's function in $\mathbb{R}^3$ is $1/r$. (II) Identifying the specific SST far-field degree of freedom as a foliation/clock scalar, we compute its quadratic EFT stress tensor T_{ij} and show the associated conserved radial flux density scales as $1/r^2$, with charge $Q \propto \int \rho_m \, d^3x$. (III) We replace the Newtonian potential χ by a foliation ("khronon-like") scalar and show that, in the static weak-field monopole limit, the EFT reduces to the same Gauss-law scalar form as in (I), so the $1/r^2$ law follows automatically. These results demonstrate that once SST commits to a local mediator in three spatial dimensions, inverse-square behavior is not an assumption but a consequence of locality, symmetry, and the $\mathbb{R}^3$ Green's function.

1 Setup and scope

We work in a flat operational background with Minkowski causal structure and consider the weak-field, static, spherically symmetric (monopole) sector. Let $\rho_m(\mathbf{x})$ be the rest-mass density of a compact source with total mass

$$M \;=\; \int_{\mathbb{R}^3} \rho_m(\mathbf{x}) \, d^3x. \tag{1}$$

The goal is to derive, from first principles, that the far-field gravitational influence (whatever SST field represents it) must exhibit

$$\Phi(\mathbf{x}) \sim \frac{1}{r}, \qquad \nabla\Phi \sim \frac{1}{r^2}, \qquad r = \|\mathbf{x}\|. \tag{2}$$

We present three derivations that differ in emphasis but agree in content.

2 Derivation I: Gauss-law scalar EFT $\Rightarrow 1/r$ Green's function

2.1 Specify the mediator and write the quadratic EFT

In the static weak-field monopole sector, the minimal local mediator is a scalar field $\phi(\mathbf{x})$ coupled linearly to the source density. The most general rotationally invariant quadratic functional (Euclidean static limit of a Lorentzian EFT) is

$$S_{\text{stat}}[\phi] \;=\; \int_{\mathbb{R}^3} d^3x \left[\frac{\kappa}{2}\,(\nabla\phi)^2 \;-\; \lambda\,\phi\,\rho_m(\mathbf{x})\right], \tag{3}$$

with constants $\kappa > 0$ and coupling λ.

2.2 Euler–Lagrange equation: Poisson form

Varying (3):

$$\delta S_{\text{stat}} = \int d^3x\;[\kappa\,\nabla\phi\cdot\nabla(\delta\phi) \;-\; \lambda\,\rho_m\,\delta\phi] \tag{4}$$

$$= \int d^3x\;\left[-\kappa\,(\nabla^2\phi)\,\delta\phi \;-\; \lambda\,\rho_m\,\delta\phi\right] \quad \text{(integrate by parts, drop boundary term)} \tag{5}$$

so stationarity for arbitrary $\delta\phi$ gives

$$\kappa\,\nabla^2\phi(\mathbf{x}) \;=\; -\,\lambda\,\rho_m(\mathbf{x}). \tag{6}$$

Define the "Gauss-law charge density" $\rho_Q := \rho_m$ and total charge

$$Q \;:=\; \int \rho_Q\,d^3x \;=\; \int \rho_m\,d^3x \;=\; M, \tag{7}$$

so the source is monopolar with charge Q.

2.3 Solve using the Green's function of ∇^2 on $\mathbb{R}^3$

The Green's function $G(\mathbf{x})$ satisfying

$$\nabla^2 G(\mathbf{x}) = -4\pi\,\delta^{(3)}(\mathbf{x}) \tag{8}$$

is

$$G(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}. \tag{9}$$

This is a standard result: $G(r) = 1/r$ is the unique (up to addition of harmonic functions) spherically symmetric fundamental solution on $\mathbb{R}^3$ [1, 2].

Convolving (6) with G gives

$$\phi(\mathbf{x}) = \frac{\lambda}{4\pi\kappa}\int d^3x'\,\frac{\rho_m(\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|}. \tag{10}$$

In the far field $r \gg$ source size, $\|\mathbf{x}-\mathbf{x}'\| \approx r$ and

$$\phi(r) \;\simeq\; \frac{\lambda}{4\pi\kappa}\,\frac{1}{r}\int \rho_m(\mathbf{x}')\,d^3x' \;=\; \frac{\lambda}{4\pi\kappa}\,\frac{Q}{r}. \tag{11}$$

Thus

$$\nabla\phi(r) \;\simeq\; -\frac{\lambda Q}{4\pi\kappa}\,\frac{\hat{\mathbf{r}}}{r^2}, \tag{12}$$

which is the inverse-square flux law.

2.4 Gauss law from divergence theorem

Define the flux density

$$\mathbf{J} := -\,\kappa\,\nabla\phi. \tag{13}$$

Then (6) is precisely

$$\nabla\cdot\mathbf{J} = \lambda\,\rho_m. \tag{14}$$

Integrating over a ball B_R and applying divergence theorem:

$$\oint_{S_R}\mathbf{J}\cdot d\mathbf{A} = \int_{B_R}\nabla\cdot\mathbf{J}\,d^3x = \lambda\int_{B_R}\rho_m\,d^3x \xrightarrow{R\to\infty} \lambda Q. \tag{15}$$

Spherical symmetry implies $\mathbf{J} = J_r(r)\hat{\mathbf{r}}$, hence

$$4\pi r^2\,J_r(r) = \lambda Q \quad\Rightarrow\quad J_r(r) = \frac{\lambda Q}{4\pi}\,\frac{1}{r^2}. \tag{16}$$

Since $\mathbf{J}\propto\nabla\phi$, this is equivalent to (12).

Interpretation (10-year-old analogy). Imagine "influence" leaving a source and spreading equally in all directions. The surface area of a sphere grows like r^2, so the influence per square meter must drop like $1/r^2$.

3 Derivation II: Identify the SST far-field carrier, compute T_{ij}, and extract the $1/r^2$ flux

3.1 Which SST field carries far-field momentum flux?

In SST language, the long-range static field is taken to be carried by a *clock/foliation* mode: a scalar that labels preferred-time hypersurfaces ("swirl-clock"). Denote this field by $T(x)$ and consider small perturbations about an inertial foliation:

$$T(x) = t + \tau(x), \tag{17}$$

where t is the operational background time coordinate and τ is a weak perturbation sourced by matter.

At quadratic order, the most general Lorentz-invariant action for τ (ignoring higher derivatives) is

$$S[\tau] = \int d^4x\,\left[\frac{\kappa}{2}\,\partial_\mu\tau\,\partial^\mu\tau - \lambda\,\tau\,\rho_m(\mathbf{x})\right], \tag{18}$$

with ρ_m treated as static, $\partial_t\rho_m = 0$. The static sector of (18) reduces to (3) with $\phi\equiv\tau$.

3.2 Compute the stress-energy tensor

From (18), the symmetric stress-energy tensor (metric variation, or canonical symmetrized) for the free part is

$$T^{(\tau)}_{\mu\nu} = \kappa\left(\partial_\mu\tau\,\partial_\nu\tau - \frac{1}{2}\eta_{\mu\nu}\,\partial_\alpha\tau\,\partial^\alpha\tau\right). \tag{19}$$

In the static regime, $\partial_0\tau = 0$, so $\partial_\alpha\tau\,\partial^\alpha\tau = -(\nabla\tau)^2$ and

$$T^{(\tau)}_{ij} = \kappa\left(\partial_i\tau\,\partial_j\tau - \frac{1}{2}\delta_{ij}(\nabla\tau)^2\right), \qquad T^{(\tau)}_{00} = \frac{\kappa}{2}(\nabla\tau)^2. \tag{20}$$

3.3 Monopole solution and the *conserved* radial flux density

From Derivation I, for r outside the source,

$$\tau(r) = \frac{\lambda Q}{4\pi\kappa}\,\frac{1}{r}, \qquad \partial_r\tau(r) = -\frac{\lambda Q}{4\pi\kappa}\,\frac{1}{r^2}. \tag{21}$$

Now define the *Gauss-law flux* associated to the foliation scalar (this is the conserved radial "momentum-like" flux density in the static sector):

$$\mathcal{F}_r(r) := -\kappa\,\partial_r\tau(r). \tag{22}$$

Using (21),

$$\mathcal{F}_r(r) = \frac{\lambda Q}{4\pi}\,\frac{1}{r^2}. \tag{23}$$

This is the requested monopole scaling:

$$\boxed{\mathcal{F}_r(r) \propto \frac{1}{r^2}, \qquad Q \propto \int \rho_m\, d^3x.}$$

The proportionality to the enclosed mass is explicit via $Q = M$ from (7). The crucial point is that this $1/r^2$ law is enforced by $\nabla\cdot(\kappa\nabla\tau) = -\lambda\rho_m$ and the divergence theorem, i.e. by local field mediation in $\mathbb{R}^3$.

3.4 How T_{ij} encodes momentum transport

Although the conserved Gauss-law flux (22) is *linear* in $\nabla\tau$ and therefore scales as $1/r^2$, the mechanical stress carried by the field is quadratic and scales as $(\nabla\tau)^2 \sim 1/r^4$. For completeness, the radial traction (pressure/tension) on a sphere is

$$t_r(r) := \hat{r}_i\,T^{(\tau)}_{ij}\,\hat{r}_j = \kappa\left[(\partial_r\tau)^2 - \frac{1}{2}(\partial_r\tau)^2\right] = \frac{\kappa}{2}(\partial_r\tau)^2 \propto \frac{1}{r^4}. \tag{24}$$

However, the *integrated* traction force over the sphere involves the area factor:

$$F_{\text{field}}(r) = \int_{S_r} t_r\, dA \sim 4\pi r^2 \times \frac{1}{r^4} \sim \frac{1}{r^2}. \tag{25}$$

Thus the field's stress tensor is consistent with the inverse-square scaling of net force, while the conserved Gauss-law flux density (23) provides the most direct "flux" statement.

Interpretation (10-year-old analogy). The field is like a stretched rubber sheet. The *slope* (gradient) gets smaller like $1/r^2$ because the "amount of slope" has to spread over bigger spheres. The *stretching energy* depends on slope squared, so it falls off faster.

4 Derivation III: Replace χ by a foliation scalar; static weak-field $\Rightarrow$ Gauss-law scalar form

4.1 Foliation scalar as the gravitational potential variable

Many covariant "preferred foliation" theories introduce a scalar $T(x)$ whose gradient defines a timelike direction. Define the unit timelike vector

$$u_\mu := \frac{\partial_\mu T}{\sqrt{\partial_\alpha T\,\partial^\alpha T}}, \tag{26}$$

and consider a general quadratic action in derivatives of u_μ (the usual Einstein–*aether*/khronon class) [3, 4]. In the weak-field, static, monopole limit, only the scalar (spin-0) sector contributes at leading order, and one can parameterize the effective action for τ by the leading operator $(\nabla\tau)^2$.

Concretely, expand about $T = t$:

$$T = t + \tau, \qquad \partial_\mu T = (1, \nabla\tau), \qquad \sqrt{\partial_\alpha T\, \partial^\alpha T} = \sqrt{1 - (\nabla\tau)^2} \approx 1 - \frac{1}{2}(\nabla\tau)^2. \tag{27}$$

To quadratic order, the effective static Lagrangian necessarily contains

$$\mathcal{L}_{\text{eff,stat}} = \frac{\kappa}{2}\,(\nabla\tau)^2 - \lambda\,\tau\,\rho_m \;+\; (\text{higher derivatives / higher powers}). \tag{28}$$

This is exactly the Gauss-law scalar functional (3). Therefore, independent of the microphysical foliation completion, the static weak-field monopole sector reduces to the same Poisson equation (6).

4.2 Identify χ as a reparameterization of τ

Define the Newtonian potential variable χ as a linear rescaling of τ:

$$\chi := \alpha\,\tau, \tag{29}$$

for some constant α chosen so that test-particle equations match the conventional Newtonian limit. Then (28) becomes

$$\mathcal{L}_{\text{eff,stat}} = \frac{\kappa}{2\alpha^2}\,(\nabla\chi)^2 - \frac{\lambda}{\alpha}\,\chi\,\rho_m. \tag{30}$$

Variation yields

$$\nabla^2\chi = -\left(\frac{\alpha\lambda}{\kappa}\right)\rho_m. \tag{31}$$

Thus, with the identification

$$\frac{\alpha\lambda}{\kappa} = 4\pi G, \tag{32}$$

the foliation scalar reproduces the Gauss-law scalar Poisson equation of Newtonian gravity. Its monopole solution is therefore

$$\chi(r) = -\frac{GM}{r}, \qquad \nabla\chi(r) = +GM\,\frac{\hat{\mathbf{r}}}{r^2}, \tag{33}$$

which is the inverse-square law.

Interpretation (10-year-old analogy). Calling the field χ or calling it a "clock field" does not change the math. If it obeys the same Poisson equation, it must have the same $1/r$ shape.

5 Discussion and outlook (SST identification of G kept here)

The three derivations show that once SST specifies a *local* far-field mediator (identified here with a foliation/clock scalar in the monopole sector), the $1/r^2$ law is forced by:

1. the form of the quadratic EFT and its Euler–Lagrange equation;
2. the unique $\mathbb{R}^3$ Green's function $G(r) = 1/r$ for ∇^2;
3. flux conservation (divergence theorem) implying $4\pi r^2 \mathcal{F}_r = \text{const}$.

In SST, one additionally seeks to *derive the normalization* G (not merely the $1/r^2$ shape) from microphysical constants. In the SST program, this corresponds to matching the EFT coefficient combination in (32) to a derived expression G_{swirl} constructed from SST's canonical constants. This matching fixes λ/κ (and the rescaling α) in terms of SST parameters and thereby provides a first-principles value for G in the low-energy monopole limit. The present paper establishes the distance-law part independently of that normalization step.

Appendix A: How this connects to other SST manuscripts (non-cited in main text)

This appendix is informational and does not enter the main derivations. The "Kelvin-mode suppression" result supports treating the far-field source as effectively rigid in ordinary conditions (so the monopole approximation is stable). Thermodynamic SST results provide a microscopic route to compute the EFT stiffness κ and coupling λ by coarse-graining the condensate's equation of state. Variational particle-structure results motivate that unique stable configurations select unique effective couplings, suggesting that G can be obtained by matching SST microphysics to the EFT coefficients in (32).

References

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