Helicity-Constrained Stability Beyond Energy Minimization

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Abstract

Stability in classical continuum systems is commonly associated with energetic optimality, whereby persistent configurations correspond to minima of an appropriate energy functional. However, ideal incompressible flows admit long-lived localized structures that do not satisfy this criterion. In particular, knotted vortex filaments have been observed to persist for many characteristic times despite not corresponding to energy-minimizing states.

In this work, we show that the longevity of knotted vortex filaments in ideal incompressible flows arises from topological constraints imposed by helicity conservation, rather than energetic optimality. By analyzing helicity as an active dynamical constraint, we demonstrate that conserved topological invariants restrict admissible decay pathways and obstruct continuous transitions between distinct topological classes. As a result, knotted configurations remain stable in the sense of long-term persistence, even while undergoing continuous dynamical evolution.

We distinguish energetic stability from topological stability and show that the latter provides an independent mechanism for robustness in classical continuum systems. Knotted vortex filaments exemplify this mechanism, persisting far from energetic equilibrium due to the structure of the admissible configuration space defined by the Euler equations. These results clarify the role of topology in classical fluid dynamics and suggest that conserved topological quantities should be regarded as fundamental contributors to stability, robustness, and structure formation in ideal and near-ideal flows.

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1 Introduction

Stability in classical continuum systems is commonly associated with energetic optimality: configurations persist because they minimize an appropriate energy functional subject to external constraints. This paradigm underlies much of classical mechanics, elasticity, and fluid dynamics [12, 13]. However, a growing body of evidence from ideal fluid flows, plasmas, and related systems indicates the existence of long-lived localized structures that do not correspond to global energy minima, yet remain dynamically stable over extended timescales.

In inviscid, incompressible flows, vortex filaments can form closed and knotted configurations that propagate, deform, and interact without rapid decay. Experimental and numerical studies have demonstrated that such knotted structures may persist for many turnover times, even in the absence of confinement or external forcing. [9, 10]. Their longevity cannot be explained solely by energetic considerations, as these configurations are generally not stationary solutions of minimal energy.

Ideal fluid dynamics possesses conserved quantities that go beyond energy and momentum. Among these, helicity plays a distinguished role as a global invariant that encodes the topological structure of the flow. [2]. While helicity has traditionally been employed as a diagnostic measure of knottedness or linkage, its role as an active constraint on the dynamics has received comparatively less attention.

In this work, we show that helicity conservation imposes topological constraints that stabilize localized knotted structures independently of energetic optimality. We argue that the persistence of such structures is best understood as a consequence of topological obstruction to decay, rather than proximity to an energy minimum. This perspective provides a unified explanation for the observed longevity of knotted vortex filaments and clarifies the distinction between energetic and topological stability in ideal continuum systems.

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2 Ideal Flow Framework and Helicity Conservation

We consider a three-dimensional, incompressible, inviscid fluid described by the Euler equations,

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p, \qquad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field and p is the pressure enforcing incompressibility. We assume the flow to be barotropic and free of external forcing and dissipation.

Under these conditions, Kelvin's circulation theorem guarantees the conservation of circulation along material loops advected by the flow. [1]. A direct consequence is the conservation of helicity,

$$H = \int_V \mathbf{v} \cdot (\nabla \times \mathbf{v})\, \mathrm{d}V, \tag{2}$$

provided the velocity field remains smooth and reconnection events are absent.

Helicity is a pseudoscalar quantity that measures the degree of linkage between vortex lines. [2, 7]. Unlike energy, which is sensitive to local deformation, helicity is invariant under continuous deformations of the flow that preserve topology. As such, it encodes global structural information that cannot be altered by smooth dynamics alone.

In many treatments, helicity is regarded as a passive invariant, useful for characterizing flow complexity or diagnosing turbulent cascades. Here, we instead emphasize its role as a dynamical constraint: helicity conservation restricts the class of admissible flow evolutions and forbids continuous decay pathways that would alter the topological structure of vortex filaments.

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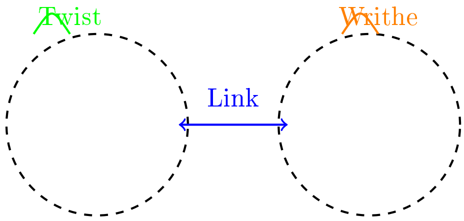

Figure 1: Schematic illustration of helicity decomposition in a vortex filament. Helicity receives contributions from pairwise linking between filaments (*link*), as well as internal geometric structure of individual filaments through *twist* and *writhe*. Continuous deformations may redistribute twist and writhe while preserving total helicity, whereas changes in linking require reconnection or dissipation.

3 Helicity as a Topological Constraint

For flows dominated by thin vortex filaments with circulations Γ_i, helicity admits a decomposition into topological contributions,

$$H = \sum_{i \neq j} \Gamma_i \Gamma_j \, \mathrm{Lk}_{ij} + \sum_i \Gamma_i^2 (\mathrm{Tw}_i + \mathrm{Wr}_i), \tag{3}$$

where Lk_{ij} denotes the Gauss linking number between filaments i and j, while Tw_i and Wr_i represent the twist and writhe of filament i, respectively.

This decomposition makes explicit that helicity is fundamentally topological in nature. The linking numbers are integer-valued invariants, unchanged under smooth deformations of the flow, while twist and writhe may interchange continuously so long as their sum remains fixed. As a result, the total helicity is conserved even as individual geometric features evolve.

The key implication is that knotted or linked vortex configurations cannot be continuously transformed into unknotted states without violating helicity conservation. Any decay process that would eliminate knotting necessarily requires either reconnection or dissipation, both of which lie outside the ideal-flow framework.

We therefore interpret helicity not merely as a measure of flow complexity, but as a topological constraint that actively restricts admissible dynamical evolutions. In particular, helicity conservation obstructs decay pathways that would otherwise lead to topologically simpler configurations.

4 Knotted Structures as Long-Lived Localized Excitations

Closed vortex filaments in ideal incompressible flows admit a wide range of topological configurations, including linked and knotted states. Experimental visualizations and numerical simulations have shown that such structures can propagate, deform, and interact over extended timescales while preserving their topological identity. In contrast to unknotted vortex rings, which may readily expand or relax through smooth deformations, knotted filaments exhibit a marked resistance to topological simplification.

A defining characteristic of knotted vortex configurations is the localization of kinetic energy along the filament core. Although the induced velocity field extends throughout the surrounding fluid, the dominant energetic contribution remains concentrated within a narrow tubular region. This localization persists in the absence of boundaries, external confinement, or imposed potentials, and is maintained solely by the internal structure of the flow.

The energetic properties of knotted filaments are constrained by their topology. For a filament of fixed circulation and core thickness, geometric knot theory implies the existence of lower

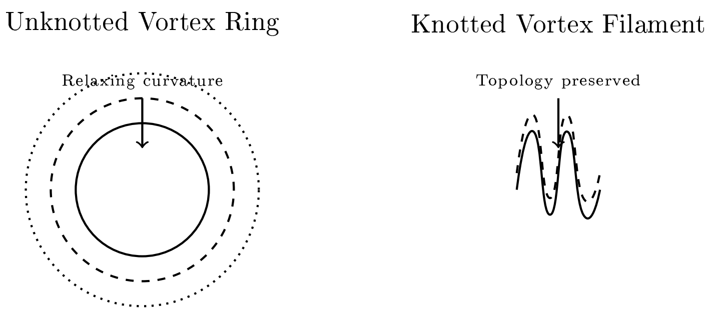

Figure 2: Comparison of the evolution of unknotted and knotted vortex filaments in an ideal incompressible flow. Unknotted vortex rings may deform continuously toward lower-curvature configurations, whereas knotted filaments preserve their topological structure over many characteristic times, exhibiting sustained energy localization.

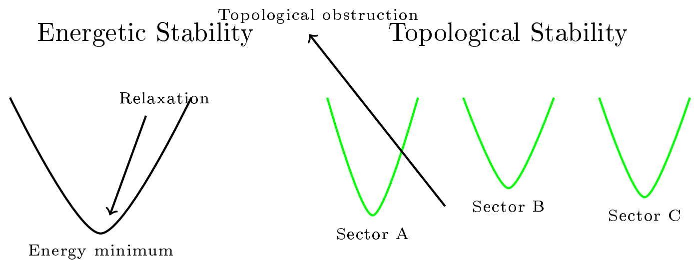

Figure 3: Schematic illustration of stability mechanisms in ideal flows. Energetic stability corresponds to relaxation toward minima of an energy functional, whereas topological stability arises from obstruction of admissible decay pathways by conserved invariants. Knotted vortex filaments exemplify the latter mechanism.

bounds on filament length and curvature. These bounds translate into corresponding constraints on the kinetic energy, such that

$$E(K) \;\geq\; C\,\mathrm{Ropelength}(K), \tag{4}$$

where $E(K)$ denotes the kinetic energy associated with a filament of knot type K, $\mathrm{Ropelength}(K)$ is the minimal length of a tube of fixed radius realizing that knot, and C is a constant determined by the circulation and core structure. This inequality does not imply energetic optimality, but establishes that increased topological complexity carries an unavoidable energetic cost.

Despite this, knotted vortex filaments are not observed to relax toward stationary energy minima. Instead, they persist in dynamically evolving states that continually redistribute curvature, twist, and writhe along the filament while preserving global topological invariants. Their longevity therefore cannot be attributed to energetic trapping in a potential well.

For unknotted structures, continuous deformations may reduce curvature and energy without altering circulation. By contrast, any decay process that would eliminate knotting requires a change in topological invariants and is therefore forbidden within the ideal-flow framework. Reconnection or dissipation is necessary to bypass this obstruction.

We therefore identify knotted vortex filaments as long-lived localized excitations whose persistence arises from topological obstruction rather than energetic stability. This distinction clarifies why such structures remain robust despite not corresponding to energy-minimizing configurations and motivates a reassessment of stability criteria in classical continuum systems.

5 Stability Without Energetic Optimality

In classical continuum systems, stability is most commonly associated with energetic optimality: a configuration is considered stable if it corresponds to a local or global minimum of an appropriate energy functional. Small perturbations increase the energy, and the system relaxes back toward the minimum under admissible dynamics. This framework underlies much of classical mechanics, elasticity, and fluid dynamics.

The behavior of knotted vortex filaments in ideal incompressible flows does not fit naturally within this paradigm. As discussed in the preceding section, knotted configurations do not correspond to stationary points of the kinetic energy, nor do they minimize energy subject to fixed circulation. Instead, they exhibit persistent dynamical evolution while maintaining their topological identity and localized energy content.

The key distinction lies in the nature of admissible perturbations. Energetic stability assumes that decay pathways exist within the allowed configuration space, but are suppressed by energetic barriers. In contrast, topological stability arises when entire classes of decay pathways are forbidden by conservation laws. In ideal flows, helicity conservation restricts the system to a topologically fixed subset of configurations, within which continuous deformations cannot eliminate knotting or linkage.

As a result, knotted vortex filaments may evolve dynamically without approaching an energy minimum, yet remain stable in the sense of long-term persistence. Their stability is not enforced by energetic trapping, but by topological obstruction: the absence of continuous paths in configuration space that connect knotted and unknotted states while preserving the ideal-flow constraints.

This distinction can be formalized by separating two notions of stability. Energetic stability concerns the response of a system to perturbations that explore nearby configurations in energy space. Topological stability, by contrast, concerns the global structure of the admissible configuration space itself. A configuration may be energetically unstable in principle, yet topologically stable if all decay routes require violation of conserved topological invariants.

From this perspective, the longevity of knotted vortex filaments is not exceptional, but expected. Their persistence reflects the structure of the underlying phase space defined by the Euler equations and associated invariants. Only when ideal conditions are relaxed—through viscosity, reconnection, or other non-ideal effects—do topological constraints weaken, allowing decay to proceed.

We therefore conclude that stability in ideal continuum systems cannot be understood solely in terms of energy minimization. Conserved topological quantities such as helicity provide an independent mechanism for stability, enabling long-lived localized structures that persist far from energetic equilibrium. This mechanism complements, rather than replaces, energetic notions of stability and plays a central role in the dynamics of knotted flows.

6 Topological Stabilization as a General Mechanism

The preceding sections establish that the longevity of knotted vortex filaments in ideal incompressible flows arises from topological constraints imposed by helicity conservation, rather than from energetic optimality. This mechanism can be understood as a consequence of the restricted structure of the admissible configuration space: while energetic gradients may drive continuous deformation, topological invariants prohibit transitions between distinct topological classes under ideal dynamics.

This perspective suggests that topological stabilization is not unique to vortex filaments, but reflects a more general principle applicable to a wide class of continuum systems. Whenever the dynamics preserve global topological quantities, the configuration space may decompose into disconnected sectors, within which evolution is confined. Localized structures residing in such sectors may persist indefinitely, even in the absence of energetic minima.

Analogous mechanisms are known to operate in other physical contexts. In magnetohydrodynamics, magnetic helicity constrains the relaxation of magnetic field configurations and can stabilize linked or twisted flux tubes. In superfluid systems, quantized circulation enforces topological protection of vortices. Elastic rods and filaments exhibit geometric constraints that prevent the continuous untangling of knotted configurations without cutting or self-intersection. In each case, conserved topological quantities restrict dynamical evolution and give rise to long-lived, structured states.

Within this broader context, knotted vortex filaments serve as a particularly transparent example of topological stabilization in a purely classical setting. The Euler equations provide an idealized framework in which energetic and topological effects can be cleanly separated, allowing the role of topology to be isolated without ambiguity. The resulting distinction between energetic and topological stability clarifies why certain structures persist far from equilibrium, while others decay rapidly despite comparable energetic content.

Topological stabilization should therefore be regarded as a complementary mechanism to energetic stabilization, rather than an exception to it. In systems where both mechanisms are present, their interplay may determine observed lifetimes, interaction rules, and decay pathways. Recognizing the independent role of topology expands the set of tools available for understanding stability, robustness, and structure formation in classical continuum dynamics.

7 Discussion and Outlook

The results presented in this work clarify the role of topological constraints in the stability of localized structures in ideal incompressible flows. By distinguishing topological stabilization from energetic stabilization, we provide a framework for understanding why certain knotted configurations persist far from energetic equilibrium, while others decay rapidly despite comparable energetic content.

In realistic physical systems, ideal conditions are only approximately realized. Viscosity, finite core thickness, and numerical or physical reconnection events inevitably relax strict topological constraints over sufficiently long timescales. In such cases, helicity is no longer exactly conserved, and decay pathways between topological sectors become accessible. The present analysis therefore applies most directly to regimes in which non-ideal effects act on timescales long compared to the intrinsic dynamical evolution of the flow.

From an experimental perspective, advances in flow visualization and vortex generation have made it possible to create and track knotted vortex filaments in controlled settings. The framework developed here suggests clear diagnostics for distinguishing energetic relaxation from topological decay, such as the onset of reconnection events or abrupt changes in helicity. Quantitative measurements of lifetimes, deformation modes, and interaction rules as a function of knot type would provide direct tests of the role of topology in stabilizing localized structures.

Beyond classical fluid dynamics, the distinction between energetic and topological stability may prove useful in a range of continuum systems, including magnetohydrodynamics, superfluids, and elastic media. In these contexts, topological invariants similarly restrict admissible dynamics and may give rise to robust, long-lived structures whose persistence cannot be understood purely through energy minimization. Exploring the interplay between energetic and topological effects in such systems remains an open direction for future work.

8 Conclusions

We have shown that the persistence of knotted vortex filaments in ideal incompressible flows is a consequence of topological constraints imposed by helicity conservation, rather than energetic optimality. By analyzing helicity as an active dynamical constraint, we demonstrated that knotted configurations are stabilized by obstruction of admissible decay pathways within the ideal-flow framework.

This perspective separates two distinct notions of stability in classical continuum systems: energetic stability, associated with minima of an energy functional, and topological stability, arising from conserved global invariants. Knotted vortex filaments exemplify the latter mechanism, persisting as localized structures despite continuous dynamical evolution and the absence of energetic minimization.

Recognizing topological stabilization as an independent and complementary mechanism provides a unified explanation for the observed longevity of knotted flows and clarifies the role of topology in classical dynamics. These results suggest that topology should be regarded as a fundamental ingredient in the analysis of stability, robustness, and structure formation in ideal and near-ideal continuum systems.

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