

# Reversible Azimuthal Response to Axisymmetric Vertical Forcing in Rapidly Rotating Fluids: Vorticity Generation, Angle Reversal, and a Fluid Fine-Structure Analogy

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2025

## Abstract

We investigate how a rapidly rotating, incompressible fluid contained in a cylinder responds when a neutrally buoyant spherical control volume on the axis is forced to move vertically. In the limit of small Rossby and Ekman numbers, linear rotating-fluid theory reduces to a compact vorticity-production law,

$$\partial_t \omega_z = 2\Omega \partial_z w,$$

This relation predicts vertical vorticity of opposite sign above and below the driver, leading to instantaneous tracer rotation in opposite directions. For symmetric up-down strokes, the accumulated angle reverses on the return path, so that the net rotation over a full cycle cancels at leading order.

As a speculative extension, we introduce a dimensionless “fluid fine-structure constant” that connects swirl speed to a tracer’s clock-rate rule. Unlike the angle response, this hypothesis predicts a small but non-reversing cycle-averaged proper-time deficit. We present closed-form estimates, dimensional checks, and limiting cases, while stressing that the analogy is offered as a testable conjecture, separate from the rigorous macroscopic results.

## 1 Physical setting and asymptotic regime

A cylinder of radius  $R$  and height  $H$  contains an incompressible fluid of density  $\rho$ . The container rotates at constant angular speed  $\Omega$  about  $\hat{z}$ . In the rotating frame the base state is at rest; the absolute vorticity is uniform,  $\omega_a^{(0)} = 2\Omega$  [?, ?].

A neutrally buoyant spherical control volume of radius  $a$  is centered on the axis  $r = 0$  and is forced vertically with displacement  $Z(t)$ . The forcing is axisymmetric and smooth; denote the induced vertical velocity by  $w(r, z, t)$ . We work in the regime

$$\text{Ro} = \frac{U'}{\Omega L} \ll 1, \quad \text{E} = \frac{\nu}{\Omega L^2} \ll 1,$$

with perturbation speed  $U'$  and length  $L = \mathcal{O}(a)$ , so that linear, inviscid rotating-flow theory applies away from thin boundary layers [?, ?, ?].

## 2 Governing equations and vorticity production

In the rotating frame the inviscid equations are

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla \Pi, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

Taking curl of (1) gives the absolute-vorticity equation [?, ?]

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}_a), \quad \boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\Omega. \quad (3)$$

Linearizing about  $\boldsymbol{\omega}_a^{(0)} = 2\Omega$  (neglect quadratic perturbation terms) yields

$$\partial_t \boldsymbol{\omega} \approx 2(\Omega \cdot \nabla) \mathbf{u}. \quad (4)$$

Axisymmetry implies that the vertical component obeys

$$\boxed{\partial_t \omega_z = 2\Omega \partial_z w.} \quad (5)$$

Introduce a vertical displacement field  $\xi(r, z, t)$  with  $w = \partial_t \xi$ . Integrating (5) from an unperturbed initial state,

$$\boxed{\omega_z(r, z, t) = 2\Omega \partial_z \xi(r, z, t).} \quad (6)$$

Equation (6) is the rotating analogue of vortex-line stretching: regions of column stretching ( $\partial_z \xi > 0$ ) generate cyclonic vorticity, while compression ( $\partial_z \xi < 0$ ) generates anticyclonic vorticity [?, ?].

## 3 From vertical vorticity to azimuthal velocity

Under axisymmetry the kinematic relation between vertical vorticity and azimuthal velocity is

$$\omega_z(r, z, t) = \frac{1}{r} \partial_r (r u_\theta(r, z, t)). \quad (7)$$

Regularity at  $r = 0$  ( $u_\theta \sim r$ ) gives

$$\boxed{u_\theta(r, z, t) = \frac{1}{r} \int_0^r \omega_z(r', z, t) r' dr' = \frac{2\Omega}{r} \int_0^r \partial_z \xi(r', z, t) r' dr'.} \quad (8)$$

### 3.1 A concrete smooth kernel

Let

$$\xi(r, z, t) = Z(t) \psi(r, z), \quad \psi(r, z) = \exp\left(-\frac{r^2 + z^2}{a^2}\right). \quad (9)$$

Then  $\partial_z \xi = -\frac{2Z(t)z}{a^2} \psi$ , and (8) yields

$$\boxed{u_\theta(r, z, t) = -\frac{2\Omega Z(t) z}{a^2} e^{-z^2/a^2} \frac{1 - e^{-r^2/a^2}}{r}.} \quad (10)$$

*Sign structure:* for  $Z(t) > 0$  (up-stroke),  $u_\theta \propto -z$ , so the azimuthal response is cyclonic below ( $z < 0$ ) and anticyclonic above ( $z > 0$ ).

Near the axis,  $1 - e^{-r^2/a^2} \sim r^2/a^2$ , hence  $u_\theta \sim -\frac{2\Omega Z(t)z}{a^4} e^{-z^2/a^2} r$  (regular).

## 4 Angle reversal and reversibility

Define the relative angular rate  $\dot{\theta}_{\text{rel}} = u_\theta/r$ . Over a stroke from  $t_1$  to  $t_2$ ,

$$\Delta\theta_{\text{rel}}(r, z) = \int_{t_1}^{t_2} \frac{u_\theta}{r} dt. \quad (11)$$

Because  $\dot{\theta}_{\text{rel}}$  is *linear* in  $Z(t)$ , a symmetric up-down cycle with zero mean displacement satisfies

$$\boxed{\Delta\theta_{\text{rel}}(r, z; \text{one period}) = 0 \quad (\text{to leading order in Ro, E}).} \quad (12)$$

This is the expected quasi-static reversibility of linear, rapidly rotating flow (the Taylor–Proudman framework) [?, ?, ?]. Deviations arise at higher order from viscosity (Ekman pumping), finite-amplitude advection (steady streaming), or near-resonant inertial waves [?, ?].

## 5 A fluid-inspired kinematic time hypothesis

We now explore a speculative analogy, motivated by the parity property of swirl cancellation. The macroscopic fluid equations remain unchanged; the following is presented only as a *dimensionless coupling hypothesis*.

### 5.1 Definition of a fluid fine-structure constant

Introduce a microscopic length scale equal to half the classical electron radius,

$$L \equiv \frac{1}{2}r_e, \quad r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 2.82 \times 10^{-15} \text{ m}.$$

With vorticity related to swirl velocity by  $\omega = 2u_\theta/C_e$ , we define the *fluid fine-structure constant*

$$\boxed{\alpha_f \equiv \frac{\omega L}{c} = \frac{r_e}{c C_e} u_\theta.} \quad (13)$$

This quantity is dimensionless, and—like the electromagnetic fine-structure constant—it measures the relative strength of a coupling (here, swirl to clock-rate).

## 5.2 Time-rate rule

We propose that the local tracer clock rate is

$$\boxed{\frac{d\tau}{dt} = \sqrt{1 - \alpha_f^2}.} \quad (14)$$

For  $\alpha_f \ll 1$ , expansion yields

$$\frac{d\tau}{dt} \approx 1 - \frac{1}{2}\alpha_f^2 + \mathcal{O}(\alpha_f^4).$$

Thus the leading correction is quadratic in  $\alpha_f$ , so angle reverses on a symmetric stroke but the time deficit does not—a clear falsifiable signature.

## 5.3 Cycle-averaged deficit

For sinusoidal forcing  $Z(t) = Z_0 \sin \sigma t$ ,

$$\Delta\tau = -\frac{1}{2}\alpha_f^2 T + \mathcal{O}(\alpha_f^4), \quad T = \frac{2\pi}{\sigma}.$$

With representative laboratory parameters ( $\Omega \sim 2 \text{ rad/s}$ ,  $a \sim 5 \text{ cm}$ ,  $Z_0 \sim 2 \text{ mm}$ ), we estimate  $\alpha_f \sim 10^{-8}\text{--}10^{-9}$ , giving fractional time-rate shifts of order  $10^{-16}$  per cycle.

## 6 Discussion and experimental considerations

The macroscopic prediction of opposite-sign swirl and cycle reversibility can be demonstrated directly with simple tracer experiments. For instance, dye or ink layers placed at  $\pm z_0$  should rotate in opposite directions during a stroke, with the motions canceling once the driver returns. Such a setup offers a clear and even classroom-level illustration of angle cancellation in rotating flows.

The speculative time-rate analogy points to a different outcome: a cycle-averaged deficit of order  $\alpha_f^2$  that does not reverse. Detecting such a subtle effect would require synchronized tracer clocks capable of fractional timing resolution on the order of  $10^{-16}$  per cycle—far beyond current experimental capabilities, but in principle measurable. Even a null result would be informative, since it would place bounds on  $\alpha_f$  and therefore limit the analogy.

## Conclusion

We have analyzed the azimuthal response of a vertically forced, axisymmetric driver in a rapidly rotating fluid. The theory predicts opposite-sign vertical vorticity above and below the driver, producing tracer motions that cancel over a symmetric cycle. This reversibility is consistent with classical rotating-flow dynamics in the low-Rossby, low-Ekman regime.

As a complementary exploration, we introduced a dimensionless *fluid fine-structure constant*  $\alpha_f$  together with a kinematic time-rate rule. While the resulting prediction—a minuscule but non-reversing time deficit quadratic in  $\alpha_f$ —remains far below present experimental

resolution, it is nonetheless falsifiable in principle. Framed in this way, the analogy connects the present work to the analogue-gravity tradition while keeping a clear distinction between established macroscopic dynamics and speculative microscopic interpretation.

## A Details for the Gaussian kernel

With  $\psi = \exp(-(r^2 + z^2)/a^2)$ ,

$$\partial_z \xi = Z(t) \partial_z \psi = -\frac{2Z(t)z}{a^2} e^{-(r^2+z^2)/a^2}.$$

Inserting into (8):

$$u_\theta(r, z, t) = \frac{2\Omega}{r} \int_0^r \left( -\frac{2Z(t)z}{a^2} e^{-(r'^2+z^2)/a^2} \right) r' dr'.$$

The radial integral evaluates to

$$\int_0^r e^{-r'^2/a^2} r' dr' = \frac{a^2}{2} \left( 1 - e^{-r^2/a^2} \right),$$

giving the stated result (10). The near-axis expansion follows from  $1 - e^{-r^2/a^2} \sim r^2/a^2$ .

## B When reversibility can fail (order estimates)

Let  $\epsilon \sim \text{Ro}$  be a small parameter. Viscous corrections scale as  $\mathcal{O}(E^{1/2})$  in bulk via Ekman pumping. Quadratic advection  $(\mathbf{u} \cdot \nabla) \mathbf{u}$  introduces steady streaming at  $\mathcal{O}(\epsilon^2)$ , yielding a nonzero mean angle per cycle. Near the inertial band  $|\sigma - 2\Omega| \ll 2\Omega$ , wave radiation produces phase lags  $\mathcal{O}(\epsilon)$  that also break exact reversal [?]. Operating with  $\sigma \ll 2\Omega$ ,  $\epsilon \ll 1$ , and small  $E$  ensures the leading-order predictions.