

A Unified Electron Scale Relation from Classical Radius, Compton Frequency, and the Hydrogen Ground State Energy

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Abstract

We show that three standard, independently defined electron scales—the classical electron radius r_e , the Compton angular frequency ω_C , and the ground-state energy of hydrogen E_B —combine, within a simple harmonic oscillator construction, to produce an exact, dimensionally consistent identity. Using only textbook definitions and CODATA values of m_e , α , \hbar , and c , we construct a maximal Hooke-law force

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha} \right)^2 r_e$$

and exhibit a Compton-scale radius r_c for which

$$F_{\max} r_c = \frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}.$$

The derivation is strictly algebraic and relies solely on well-established formulas. We discuss the historical origin of the underlying scales (classical electron models, Compton scattering, Bohr's hydrogen theory) and comment on the structural interdependence of atomic, relativistic, and classical electromagnetic quantities revealed by this identity.

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1 Introduction

The electron occupies a central role in both classical and quantum theories of matter. Over the last century, several characteristic length and energy scales associated with the electron have emerged, each from a different theoretical and experimental context:

- The *classical electron radius* r_e , originating in early electron models of Lorentz and Abraham [1, 2, 3].
- The *Compton wavelength* and associated angular frequency ω_C , derived from Compton's scattering experiments and their quantum interpretation [4, 5].
- The *Bohr radius* and hydrogen ground-state energy E_B , obtained in the Bohr model and later justified within nonrelativistic quantum mechanics and quantum electrodynamics [6, 7, 8].

These scales are usually discussed in their respective domains: classical electrodynamics, relativistic quantum mechanics, and atomic physics. It is therefore of conceptual interest to examine how they combine in simple dynamical constructions.

In this work we consider a purely classical harmonic oscillator with electron mass m_e , frequency ω_* , and amplitude x_{\max} . Choosing ω_* to be a rescaling of the Compton frequency and x_{\max} equal to the classical electron radius, we obtain a maximal restoring force

$$F_{\max} = m_e \omega_*^2 x_{\max}$$

that can be expressed solely in terms of m_e , α , \hbar , and c . When this force is multiplied by a Compton-scale radius r_c , one finds that the resulting energy coincides with half the electron rest energy and, equivalently, with the hydrogen ground-state energy divided by α^2 .

The purpose of this article is limited and sharply defined:

- to state and prove this identity using only mainstream, peer-reviewed formulas;
- to check dimensional consistency and evaluate the resulting expressions numerically;
- to place the ingredients in historical context, without proposing any new physical interpretation or modification of existing theories.

2 Standard electron scales: definitions

We collect the standard definitions used throughout, following e.g. Refs. [3, 7, 9].

[Fine-structure constant] The fine-structure constant α is defined by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}. \quad (1)$$

[Classical electron radius] The classical electron radius r_e is defined in SI units by

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2}. \quad (2)$$

Combining Eq. (1) with (2) yields the equivalent form

$$r_e = \frac{\alpha\hbar}{m_e c}. \quad (3)$$

[Compton wavelength and angular frequency] The Compton wavelength and associated angular frequency of the electron are defined by

$$\lambda_C = \frac{h}{m_e c}, \quad \omega_C = \frac{2\pi c}{\lambda_C} = \frac{m_e c^2}{\hbar}. \quad (4)$$

[Hydrogen ground-state energy] In the Bohr model, and equivalently in the solution of the nonrelativistic Schrödinger equation for the hydrogen atom, the ground-state binding energy E_B is

$$E_B = \frac{\alpha^2}{2} m_e c^2. \quad (5)$$

Equations (1)–(5) are standard and experimentally well-validated relations in atomic and high-energy physics [9].

3 Harmonic oscillator construction

We now introduce a classical harmonic oscillator whose parameters are chosen from the electron scales above.

[Oscillator ansatz] Consider a one-dimensional oscillator of mass m_e , angular frequency ω_* , and maximal displacement x_{\max} . Hooke's law gives the maximal restoring force

$$F_{\max} = m_e \omega_*^2 x_{\max}. \quad (6)$$

We define

$$\omega_* := \frac{\omega_C}{\alpha}, \quad x_{\max} := r_e, \quad (7)$$

where ω_C and r_e are given by Eqs. (4) and (3).

[Maximal force expressed in fundamental constants] With the choices (7), the maximal force (6) can be written as

$$F_{\max} = \frac{m_e^2 c^3}{\alpha \hbar}. \quad (8)$$

Proof. Substituting Eq. (7) into (6) gives

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha} \right)^2 r_e. \quad (9)$$

Using $\omega_C = m_e c^2 / \hbar$ from Eq. (4) and $r_e = \alpha \hbar / (m_e c)$ from Eq. (3), we obtain

$$F_{\max} = m_e \left(\frac{m_e c^2 / \hbar}{\alpha} \right)^2 \left(\frac{\alpha \hbar}{m_e c} \right) = m_e \frac{m_e^2 c^4}{\alpha^2 \hbar^2} \frac{\alpha \hbar}{m_e c}. \quad (10)$$

Canceling factors m_e and \hbar ,

$$F_{\max} = \frac{m_e^2 c^3}{\alpha \hbar}, \quad (11)$$

which is Eq. (8). \square

3.1 Dimensional check

The dimensions of Eq. (8) are

$$[F_{\max}] = \frac{[m_e]^2 [c]^3}{[\alpha] [\hbar]} = \frac{\text{kg}^2 (\text{m/s})^3}{1 \cdot \text{J} \cdot \text{s}} = \frac{\text{kg}^2 \text{m}^3 \text{s}^{-3}}{\text{kg m}^2 \text{s}^{-1}} = \text{kg m s}^{-2},$$

which is dimensionally consistent with a force.

4 Associated energy scales

To connect this force to energy scales, we multiply by a characteristic length.

[Energy scale from F_{\max}] Let r_c be a positive length scale. Define

$$E_{\text{osc}}(r_c) := F_{\max} r_c = \frac{m_e^2 c^3}{\alpha \hbar} r_c. \quad (12)$$

We now exhibit a specific choice of r_c that yields a familiar energy scale.

[Half rest energy from Compton-scale radius] Let

$$r_c = \frac{\alpha \hbar}{2m_e c}. \quad (13)$$

Then

$$E_{\text{osc}}(r_c) = \frac{1}{2} m_e c^2. \quad (14)$$

Proof. Using Eq. (8) and the definition (13),

$$E_{\text{osc}}(r_c) = F_{\max} r_c = \frac{m_e^2 c^3}{\alpha \hbar} \cdot \frac{\alpha \hbar}{2m_e c} = \frac{1}{2} m_e c^2. \quad (15)$$

□

[Relation to the hydrogen ground-state energy] Using the standard hydrogen ground-state energy

$$E_B = \frac{\alpha^2}{2} m_e c^2, \quad (16)$$

we have

$$\frac{1}{2} m_e c^2 = \frac{E_B}{\alpha^2}. \quad (17)$$

Thus the energy scale $E_{\text{osc}}(r_c)$ from Eq. (14) can also be written as

$$E_{\text{osc}}(r_c) = \frac{E_B}{\alpha^2}. \quad (18)$$

4.1 Dimensional and numerical consistency

The quantity $E_{\text{osc}}(r_c)$ is an energy, with units

$$[E_{\text{osc}}] = [F_{\max}][r_c] = \text{kg m s}^{-2} \cdot \text{m} = \text{kg m}^2 \text{s}^{-2},$$

as expected.

Numerically, using CODATA values [9]:

$$m_e c^2 \approx 511 \text{ keV}, \quad (19)$$

$$E_B \approx 13.6 \text{ eV}, \quad (20)$$

$$\alpha^{-1} \approx 137.035999. \quad (21)$$

We then have

$$\frac{1}{2} m_e c^2 \approx 255.5 \text{ keV}, \quad \frac{E_B}{\alpha^2} \approx 255.5 \text{ keV}, \quad (22)$$

in agreement to within numerical precision. This confirms the consistency of the analytic result.

5 Historical and conceptual remarks

The ingredients entering the identity

$$E_{\text{osc}}(r_c) = \frac{1}{2}m_e c^2 = \frac{E_B}{\alpha^2}$$

are all well-established:

- The *classical electron radius* r_e was introduced in early electron models by Lorentz and Abraham, who considered the electromagnetic self-energy of a charged sphere [1, 2, 3].
- The *Compton wavelength* λ_C and frequency ω_C emerged from Compton’s explanation of X-ray scattering, providing early evidence for the particle-like behavior of light [4, 5].
- The *Bohr model* of the hydrogen atom, and its later derivation from the Schrödinger equation, yields the binding energy E_B and the fine-structure constant α as key atomic parameters [6, 7, 8].

In standard pedagogy, these scales are often presented in isolation: r_e in classical electrodynamics, λ_C in relativistic quantum mechanics, and E_B in atomic physics and spectroscopy. The identity derived here shows that, once combined in a simple harmonic oscillator ansatz, these three regimes are algebraically intertwined.

We emphasize that the construction is strictly classical on the dynamical side (a Hooke-law oscillator) and uses only standard quantum-electrodynamic definitions of the constants involved. No new interactions, no modifications of Maxwell’s equations or the Dirac equation, and no speculative assumptions are invoked. The result is thus best viewed as a compact consistency relation among established electron scales.

6 Intuitive picture

For intuition, imagine the electron as a mass on a spring whose “natural” distance scale is set by the classical radius r_e , and whose oscillation rate is set by the Compton frequency. If one computes the largest restoring push that such a spring can exert (Hooke’s law) and then lets that push act over a Compton-scale distance, the resulting energy turns out to match familiar electron and hydrogen energy scales that were originally derived from entirely different arguments.

7 Conclusion

We have shown that a simple Hooke-law construction, using the classical electron radius r_e as an amplitude and a Compton-rescaled frequency ω_C/α , produces a maximal force

$$F_{\text{max}} = m_e \left(\frac{\omega_C}{\alpha} \right)^2 r_e$$

that can be expressed purely in terms of m_e , α , \hbar , and c . Multiplying by a Compton-scale length $r_c = \hbar/(2m_e c)$ yields an energy

$$E_{\text{osc}}(r_c) = \frac{1}{2}m_e c^2 = \frac{E_B}{\alpha^2},$$

linking relativistic, atomic, and classical electromagnetic scales in a single, dimensionally consistent relation.

The derivation uses only mainstream, peer-reviewed formulas and constants. Any deeper physical interpretation of this coincidence would require additional assumptions beyond the scope of the present work.

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Appendix: A candidate gravitational coupling from the electron-scale force

The unified electron scale relation of Appendix A shows that

$$F_{\max} = m_e \left(\frac{\omega_C}{\alpha} \right)^2 r_e = \frac{m_e^2 c^3}{\alpha \hbar} = \frac{\alpha \hbar c}{r_e^2} = \frac{4E_B^2}{\alpha^5 \hbar c}, \quad (23)$$

where all symbols have their standard meaning as in the main text and in Appendix A.

In this appendix we sketch how one can interpret this same force scale as a *structural* force in a fluid-based or “swirl-string” model of the electron, and how, given the Planck time, this leads to a candidate expression for an effective gravitational coupling G_{swirl} . This is an interpretive construction that goes beyond conventional quantum electrodynamics and is therefore confined to this appendix.

B.1 Structural force scale from a microscopic core

Suppose we supplement the usual electron scales with two additional structural parameters:

- a characteristic tangential speed v_s associated with microscopic circulation (“swirl speed”),
- a core length scale r_c (“core radius”).

A simple Hooke-law-like structural ansatz for a maximal restoring force is

$$F_{\text{swirl}}^{\max} = \frac{v_s \hbar}{2 r_c^2}. \quad (24)$$

This expression has the correct dimensions:

$$[v_s] = \text{m s}^{-1}, \quad [\hbar] = \text{J s}, \quad [r_c^2] = \text{m}^2 \Rightarrow [F_{\text{swirl}}^{\max}] = \text{J m}^{-1} = \text{N}.$$

Equation (??) can be viewed as defining a structural force scale once v_s and r_c are specified.

The main-text construction provides an *independent* force scale F_{\max} purely from standard electron quantities [Eq. (??)]. Identifying the two,

$$F_{\max} = F_{\text{swirl}}^{\max}, \quad (25)$$

yields a constraint relating the microscopic parameters v_s and r_c to the usual electron scales.

From (??) and (??) we may solve for \hbar in terms of F_{\max} , v_s , and r_c :

$$\hbar = \frac{2F_{\max} r_c^2}{v_s}. \quad (26)$$

Combining (??) with the electron-scale expressions of Appendix A is algebraically equivalent to the unified identity (??). What is new here is the possibility to *reinterpret* F_{\max} as a structural force scale F_{swirl}^{\max} .

B.2 Planck time and a candidate G_{swirl}

The Planck time t_p is defined in terms of the reduced Planck constant \hbar , Newton’s gravitational constant G , and the speed of light c by

$$t_p = \sqrt{\frac{\hbar G}{c^5}}. \quad (27)$$

This is standard and may be found, for example, in CODATA compilations of fundamental constants.[?]

Solving (??) for G gives

$$G = \frac{c^5 t_p^2}{\hbar}. \quad (28)$$

If we now substitute the expression (??) for \hbar , we obtain an effective gravitational coupling expressed entirely in terms of F_{max} , the structural parameters v_s and r_c , and the Planck time:

$$\begin{aligned} G_{\text{swirl}} &\equiv \frac{c^5 t_p^2}{\hbar} = \frac{c^5 t_p^2}{2F_{\text{max}} r_c^2 / v_s} \\ &= \frac{v_s c^5 t_p^2}{2F_{\text{max}} r_c^2}. \end{aligned} \quad (29)$$

By construction this has the correct dimensions of a gravitational constant:

$$[G_{\text{swirl}}] = \frac{\text{m s}^{-1} \cdot \text{m}^5 \text{s}^{-5} \cdot \text{s}^2}{\text{N} \cdot \text{m}^2} = \frac{\text{m}^6 \text{s}^{-4}}{\text{kg m s}^{-2} \text{m}^2} = \text{m}^3 \text{kg}^{-1} \text{s}^{-2}.$$

Equation (??) is the candidate “swirl” gravitational coupling G_{swirl} that can be associated with the electron-scale force F_{max} . Formally, combining (??), (??), and (??) yields a closed algebraic system linking:

$$\{r_e, \omega_C, E_B\} \longleftrightarrow F_{\text{max}} \longleftrightarrow \{v_s, r_c\} \longleftrightarrow G_{\text{swirl}}, t_p.$$

B.3 Numerical consistency check

To illustrate the internal consistency of this construction, take

$$v_s = 1.09384563 \times 10^6 \text{ m s}^{-1}, \quad (30)$$

$$r_c = 1.40897017 \times 10^{-15} \text{ m}, \quad (31)$$

$$t_p = 5.391247 \times 10^{-44} \text{ s}, \quad (32)$$

and

$$F_{\text{max}} = 2.9053507 \times 10^1 \text{ N}, \quad (33)$$

where F_{max} is evaluated from any of the equivalent forms in Eq. (??) using CODATA-2018 values for m_e , α , \hbar , c , and E_B .[?]

Inserting these numbers into (??) gives

$$G_{\text{swirl}} = (6.6743 \dots) \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}, \quad (34)$$

which is numerically indistinguishable, at the quoted precision, from the CODATA value of Newton’s gravitational constant G .[?]

From the standpoint of the present, primarily electrodynamic paper, this agreement should be read as an algebraic and dimensional coincidence that *allows* a unified description of electromagnetic and gravitational scales via the electron force F_{max} . Whether Eqs. (??) and (??) correspond to a physically correct microscopic mechanism is a question for a separate, more speculative fluid-mechanical or “swirl-string” study.

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