1 Symmetry Classification of Knot-based Swirl String Structures in Swirl–String Theory (SST)

In SST, these symmetries classify the invariance properties of knotted swirl strings, constraining their physical stability, energy quantization, and possible transformation pathways.
Remarks.
Any D_{2k} symmetry ($k \geq 2$) implies $D_2(r)$ symmetry; if k is even, period 2 is also present. D_{2k} symmetry further implies D_{2j} for divisors j of k. Z_{2k} symmetry entails positive amphichirality; $D_2(r)$ guarantees reversibility. I_2 symmetry implies negative amphichirality. These properties map directly to constraints on swirl string energy spectra, fusion/interconversion rules, and topological charge conservation in SST. The "full symmetry group" (FSG) is tabulated for comparison, though it may not capture all SST-relevant invariances.

Note.
Knots such as 8_{10}, 8_{16}, 8_{17}, and 8_{20}, for which period 2 is absent, also uniquely have FSG D_2 among prime knots with 8 or fewer crossings, reflecting special restrictions on allowable swirl string periodicities and energy levels in the foliation. Exceptional knots $12a_{1202}$ and 15331 are included for their rare Z_2 symmetry, potentially corresponding to novel or unanticipated swirl-field states.

2 Glossary of Symmetry Table Symbols

$D_2(r)$ **Order-2 Dihedral (Reflectional) Symmetry.** The knot (or swirl string) admits a dihedral symmetry of order 2, meaning it is invariant under a 180° rotation and a reflection; this often guarantees *reversibility.*

D_{2k} **Higher-Order Dihedral Symmetry.** The knot is invariant under the full dihedral group of order $2k$, i.e., all rotations by $2\pi/k$ and reflections. In SST, this corresponds to invariance under both cyclic flows and chirality-reversing operations.

Z_{2k} **Cyclic Symmetry of Order $2k$.** The knot admits rotational symmetry by $2\pi/(2k)$ (and its multiples), but not necessarily reflection symmetry. In SST, such symmetry is associated with periodic phase cycling and often positive amphichirality.

I **Icosahedral Symmetry or Inversion.** I often indicates additional point group symmetries (such as icosahedral, dodecahedral, or inversion symmetries), depending on the context. In tables, it may specify inversion axes or particular symmetry orders, e.g., I_8, I_4.

reversible Reversible Knot (Swirl String). The knot is topologically equivalent to itself with the orientation reversed; in SST, this reflects invariance under reversal of circulation or swirl clock direction.

amphichiral Amphichiral (Mirror-Image) Symmetry. The knot is equivalent to its mirror image:

- *Positive amphichirality* usually corresponds to cyclic (Z_{2k}) symmetry.
- *Negative amphichirality* is sometimes indicated by special inversion (I_2).

Table 1: **Known Symmetries of Prime Knots as SST Swirl Strings.** This table catalogs the discrete symmetries of low-crossing-number prime knots, interpreted as possible stable knotted swirl string configurations in Swirl–String Theory (SST). Columns show the principal symmetry groups ($D_2(r)$, D_{2k}, Z_{2k}, I), reversibility, amphichirality, allowed periods, and the full symmetry group (FSG).

	$\mathrm{D}_2(r)$	D_{2k}	Z_{2k}	I	reversible	amphichiral	periods	FSG
3_1	✓	D_4, D_6	×	×	✓	×	$2, 3$	Z_2
4_1	✓	D_4	Z_4	I_8	✓	✓	2	D_8
5_1	✓	D_4, D_{10}	×	×	✓	×	$2, 5$	Z_2
$5_2, 6_1, 6_2$	✓	D_4	×	×	✓	×	2	D_4
6_3	✓	D_4	Z_4		✓	✓	2	D_8
7_1	✓	D_4, D_{14}	×	×	✓	×	$2, 7$	Z_2
$7_2, 7_3$	✓	D_4	×	×	✓	×	2	D_4
7_4	✓	D_4	×	×	✓	×	2	D_8
$7_5, 7_6$	✓	D_4	×	×	✓	×	2	D_4
7_7	✓	D_4	×	×	✓	×	2	D_8
$8_1, 8_2$	✓	D_4	×	×	✓	×	2	D_4
8_3	✓	D_4	Z_4	I_8	✓	✓	2	D_8
$8_4, 8_5, 8_6, 8_7, 8_8$	✓	D_4	×	×	✓	×	2	D_4
8_9	✓	D_4		I_4	✓	✓	2	D_8
8_{10}	✓	×	×	×	✓	×	none	D_2
8_{11}	✓	D_4	×	×	✓	×	2	D_4
8_{12}	✓	D_4	Z_4		✓	✓	2	D_8
$8_{13}, 8_{14}, 8_{15}$	✓	D_4	×	×	✓	×	2	D_4
8_{16}	✓	×	×	×	✓	×	none	D_2
8_{17}	×	×	×		×	✓	none	D_2
8_{18}	✓	D_4, D_8	Z_8		✓	✓	$2, 4$	D_{16}
8_{19}	✓	D_4, D_6, D_8	×	×	✓	×	$2, 3, 4$	Z_2
8_{20}	✓	×	×	×	✓	×	none	D_2
8_{21}	✓	D_4	×	×	✓	×	2	D_4
$12a_{1202}$	✓		Z_2, Z_6		✓	✓		D_{12}
15331			Z_2			✓		

periods Periods of Symmetry. Lists the possible orders of cyclic symmetry—i.e., the integer n for which the knot is invariant under a $2\pi/n$ rotation. In SST, this relates to allowed quantized mode numbers.

FSG Full Symmetry Group (FSG). The maximal discrete symmetry group of the knot, encoding all rotational, reflectional, and inversion symmetries. In SST, the FSG constrains the topological conservation laws and fusion/annihilation selection rules for knotted swirl strings.

Torus Knots (Lepton Sector) — Invariants Added

Knot	$D_2(r)$	D_{2k}	Z_{2k}	I	reversible	amphichiral	Dark	periods	FSG
SM mapping (SST default, torus ladder): $e^- \leftrightarrow T(2,3)$ $(=3_1)$, $\mu^- \leftrightarrow T(2,5)$ $(=5_1)$, $\tau^- \leftrightarrow T(2,7)$ $(=7_1)$.									
3_1 $(T(2,3))$, $b{=}2$, $g{=}1$	✓	D_4, D_6	×	×	✓	×	no	$2,3$	Z_2
5_1 $(T(2,5))$, $b{=}2$, $g{=}2$	✓	D_4, D_{10}	×	×	✓	×	no	$2,5$	Z_2
7_1 $(T(2,7))$, $b{=}2$, $g{=}3$	✓	D_4, D_{14}	×	×	✓	×	no	$2,7$	Z_2

Torus invariants (formula). For coprime integers $p, q \geq 2$, the torus knot $T(p,q)$ has braid index $b = \min(p,q)$ and genus $g = \frac{(p-1)(q-1)}{2}$.

Torus Knots (Lepton Sector)

Knot	$D_2(r)$	D_{2k}	Z_{2k}	I	reversible	amphichiral	Dark	periods	FSG
SM mapping (SST default, torus ladder): $e^- \leftrightarrow T(2,3)$ $(=3_1)$, $\mu^- \leftrightarrow T(2,5)$ $(=5_1)$, $\tau^- \leftrightarrow T(2,7)$ $(=7_1)$.									
3_1	✓	D_4, D_6	×	×	✓	×	no	$2,3$	Z_2
5_1	✓	D_4, D_{10}	×	×	✓	×	no	$2,5$	Z_2
7_1	✓	D_4, D_{14}	×	×	✓	×	no	$2,7$	Z_2

Hyperbolic Knots (Quark Sector)

Knot	$D_2(r)$	D_{2k}	Z_{2k}	I	reversible	amphichiral	Dark	periods
SM mapping (SST default, hyperbolic chiral): up/down/strange $\leftrightarrow$ chiral hyperbolics (reps among $6_x, 7_x, 8_x, \ldots$								
4_1	✓	D_4	Z_4	I_8	✓	✓	yes+	2
$5_2, 6_1, 6_2$	✓	D_4	×	×	✓	×	no	2
6_3	✓	D_4	Z_4		✓	✓	yes+	2
$7_2, 7_3$	✓	D_4	×	×	✓	×	no	2
7_4	✓	D_4	×	×	✓	×	no	2
$7_5, 7_6$	✓	D_4	×	×	✓	×	no	2
7_7	✓	D_4	×	×	✓	×	no	2
$8_1, 8_2$	✓	D_4	×	×	✓	×	no	2
8_3	✓	D_4	Z_4	I_8	✓	✓	yes+	2
$8_4, 8_5, 8_6, 8_7, 8_8$	✓	D_4	×	×	✓	×	no	2
8_9	✓	D_4		I_4	✓	✓	yes+	2
8_{10}	✓	×	×	×	✓	×	no	none

8_{11}	✓	D_4	×	×	✓	×	no	2
8_{12}	✓	D_4	Z_4		✓	✓	yes+	2
$8_{13}, 8_{14}, 8_{15}$	✓	D_4	×	×	✓	×	no	2
8_{16}	✓	×	×	×	✓	×	no	none
8_{17}	×	×	×		×	✓	yes–	none
8_{18}	✓	D_4, D_8	Z_8		✓	✓	yes+	$2, 4$
8_{19}	✓	D_4, D_6, D_8	×	×	✓	×	no	$2, 3, 4$
8_{20}	✓	×	×	×	✓	×	no	none
8_{21}	✓	D_4	×	×	✓	×	no	2
$12a_{1202}$	✓		Z_2, Z_6		✓	✓	yes+	
15331			Z_2			✓	yes–	

Remarks. Any D_{2k} symmetry ($k \geq 2$) implies $D_2(r)$ and, if k is even, period 2; it also implies D_{2j} for each divisor j of k. Any Z_{2k} symmetry typically entails *positive* amphichirality (dark sector = yes+); $D_2(r)$ implies reversibility. I_2 symmetry indicates *negative* amphichirality (dark sector = yes–). Among prime knots with ≤ 8 crossings, the ones lacking period 2 ($8_{10}, 8_{16}, 8_{17}, 8_{20}$) have FSG D_2. Multiple 3D realizations can witness different symmetry subgroups; FSG (KnotInfo) does not encode periodicity.

3 Definitive Symmetry and Topological Taxonomy of Knots in Swirl–String Theory (SST)

This taxonomy fuses three strands of data:

1. Discrete symmetries of prime knots (from KnotInfo and Fremlin [?]).
2. Invariants from the SST Canon and Appendix G (crossing number, braid index, genus, hyperbolic volume).
3. Dark-sector assignment via amphichirality (positive/negative amphichiral knots).

3.1 Unified Table

Knot	$D_2(r)$	D_{2k}	Z_{2k}	I	reversible	amphichiral	Dark Sector	periods	FSG	Invaria
3_1	✓	D_4, D_6	×	×	✓	×	no	$2, 3$	Z_2	$b = 2, g$
4_1	✓	D_4	Z_4	I_8	✓	✓	yes+	2	D_8	$b = 2, g = 1, \mathrm{Vol}_{\mathbb{H}}($
6_3	✓	D_4	Z_4		✓	✓	yes+	2	D_8	$b = 3, g = 1, \mathrm{Vol}_{\mathbb{H}}$
8_3	✓	D_4	Z_4	I_8	✓	✓	yes+	2	D_8	$b = 3, g = 2, \mathrm{Vol}_{\mathbb{H}}$
8_9	✓	D_4		I_4	✓	✓	yes+	2	D_8	$b = 3, g = 2, \mathrm{Vol}_{\mathbb{H}}$
8_{12}	✓	D_4	Z_4		✓	✓	yes+	2	D_8	$b = 3, g = 2, \mathrm{Vol}_{\mathbb{H}}($
8_{18}	✓	D_4, D_8	Z_8		✓	✓	yes+	$2, 4$	D_{16}	$b = 3, g = 2, \mathrm{Vol}_{\mathbb{H}}($
8_{17}	×	×	×		×	✓	yes–	none	D_2	$b = 3, g = 2, \mathrm{Vol}_{\mathbb{H}}($
$12a_{1202}$	✓		Z_2, Z_6		✓	✓	yes+		D_{12}	amphichiral e
15331			Z_2			✓	yes–			prime, negative

3.2 Remarks

- All knots with $D_2(r)$ symmetry are strongly invertible.
- Amphichiral knots define the *dark sector*: positive amphichirality (Z_{2k}-type) vs negative amphichirality (I_2-type).
- Fully amphichiral: 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{18}, $12a_{1202}$.
- Negatively amphichiral: 8_{17}, 15331 (prime).
- Of the knots with 8 or fewer crossings, those lacking period 2 (8_{10}, 8_{16}, 8_{17}, 8_{20}) uniquely have FSG D_2.

3.3 Glossary Updates

The glossary from the Canon remains unchanged, except:

- **Dark sector** $\equiv$ the amphichiral subsector of SST swirl strings.
- Positive amphichirality = cyclic symmetry (Z_{2k}).
- Negative amphichirality = inversion symmetry (I_2).

References

References

[1] D. Fremlin. *Symmetry classification of prime knots*, online table. URL: `https://david.fremlin.de/knots/table.htm`. Accessed Sept 2025.

[2] D. Fremlin and J. Mala. Symmetry and measurability. *Acta Mathematica Hungarica*, 155(2):449–459, 2018. Springer. DOI: 10.1007/s10474-017-0778-3.