

# Magnetic helicity in periodic domains: gauge conditions, existence of vector potentials, and periodic winding

**Setting.** Let  $\Omega \subset \mathbb{R}^3$  be either (i) a bounded box with one or two pairs of opposite faces identified (“1- or 2-periodic”), or (ii) the fully periodic 3-torus  $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$  with lattice  $\Lambda$ . Let  $\mathbf{B} : \Omega \rightarrow \mathbb{R}^3$  be a smooth magnetic field with  $\nabla \cdot \mathbf{B} = 0$ .

## 1. Helicity, gauge, and boundary/periodicity conditions

The (total) magnetic helicity on  $\Omega$  is

$$H[\mathbf{A}, \mathbf{B}] = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dV, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1)$$

Under a gauge transform  $\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \nabla \chi$ ,

$$\begin{aligned} H[\mathbf{A}', \mathbf{B}] - H[\mathbf{A}, \mathbf{B}] &= \int_{\Omega} \nabla \chi \cdot \mathbf{B} \, dV = \int_{\Omega} \nabla \cdot (\chi \mathbf{B}) \, dV - \int_{\Omega} \chi \underbrace{\nabla \cdot \mathbf{B}}_0 \, dV \\ &= \int_{\partial\Omega} \chi \mathbf{B} \cdot d\mathbf{S}. \end{aligned} \quad (2)$$

Hence helicity is gauge invariant if and only if the boundary flux term vanishes.

**Proposition 1** (Gauge invariance in periodic/closed settings). *Suppose either:*

- (a)  $\Omega$  is bounded with perfectly conducting boundary ( $\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} = 0$ ), or
- (b)  $\Omega$  is periodic in one or two directions and the net magnetic flux through each identified pair of faces is zero; additionally the gauge function  $\chi$  respects the same periodicity,

*then  $H$  in (1) is gauge invariant.*

*Proof.* In case (a) the surface integral in (2) vanishes since  $\mathbf{B} \cdot d\mathbf{S} = 0$  on  $\partial\Omega$ . In case (b), identify opposite faces with orientation; periodic  $\chi$  has equal values on paired faces, while zero net flux through each pair implies the oriented surface integrals cancel pairwise. Thus the sum over  $\partial\Omega$  is zero.  $\square$

**Proposition 2** (Obstruction in fully periodic domains). *On  $\mathbb{T}^3$ , a globally periodic vector potential  $\mathbf{A}$  with  $\nabla \times \mathbf{A} = \mathbf{B}$  exists if and only if the mean field vanishes:*

$$\langle \mathbf{B} \rangle \equiv \frac{1}{|\Omega|} \int_{\Omega} \mathbf{B} \, dV = \mathbf{0}. \quad (3)$$

*Equivalently, any nonzero constant (harmonic) component of  $\mathbf{B}$  produces a topological obstruction to a periodic  $\mathbf{A}$ .*

*Sketch.* Fourier decompose on  $\mathbb{T}^3$ . For  $k \neq 0$ ,  $\widehat{\mathbf{A}}(k)$  exists with  $ik \times \widehat{\mathbf{A}}(k) = \widehat{\mathbf{B}}(k)$ . At  $k = 0$ , we would require a constant  $\widehat{\mathbf{A}}(0)$  with  $\nabla \times \widehat{\mathbf{A}}(0) = \mathbf{0}$  producing  $\widehat{\mathbf{B}}(0) = \mathbf{0}$ . Thus periodic solvability demands  $\widehat{\mathbf{B}}(0) = \langle \mathbf{B} \rangle = \mathbf{0}$ .  $\square$

**Remark 1** (Physical dimension).  $[\mathbf{A}] = \text{V s m}^{-1}$ ,  $[\mathbf{B}] = \text{T} = \text{Wb m}^{-2}$ . Hence  $[\mathbf{A} \cdot \mathbf{B}] = \text{Wb}^2 \text{m}^{-3}$  and  $[H] = \text{Wb}^2$  after integration, as standard.

## 2. Periodic winding and the helicity equivalence (two-periodic case)

Consider a domain that is periodic in two directions, e.g.  $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$  with identifications in  $x, y$  (a 2-torus in the horizontal plane). Let  $\tilde{\Omega} = \mathbb{R}^2 \times [0, L_z]$  be its universal cover. Field lines of  $\mathbf{B}$  in  $\Omega$  lift to curves in  $\tilde{\Omega}$ .

**Definition 1** (Periodic winding of a pair of field lines). *Let  $\gamma_1, \gamma_2$  be two lifted field lines in  $\tilde{\Omega}$ , parameterized by  $z \in [0, L_z]$ , and let  $\mathbf{r}(z) = \mathbf{x}_1(z) - \mathbf{x}_2(z)$  be their horizontal separation (projected to  $\mathbb{R}^2$  before modding out by periods). Define the periodic winding increment*

$$\Delta\Theta(\gamma_1, \gamma_2) = \int_0^{L_z} \frac{\mathbf{r}(z) \times \dot{\mathbf{r}}(z)}{\|\mathbf{r}(z)\|^2} \cdot \mathbf{e}_z dz, \quad (4)$$

and the periodic winding as the flux-weighted mean over pairs of field lines:

$$\mathcal{W}_{\text{per}} = \frac{1}{\Phi^2} \iint \Delta\Theta(\gamma_1, \gamma_2) d\Phi(\gamma_1) d\Phi(\gamma_2), \quad (5)$$

where  $\Phi$  denotes the magnetic flux measure induced by  $\mathbf{B}$  on a transverse cross-section.

**Remark 2.** Expression (4) generalizes the pairwise angular change (linking/winding) to horizontally periodic geometry by working on the universal cover and then averaging in a flux-consistent way.

**Theorem 1** (Helicity–winding equivalence in two-periodic domains). *Assume: (i) ideal evolution (frozen-in field, sufficiently smooth), (ii) two-directional periodicity with zero net flux through each periodic pair, (iii) a vector potential  $\mathbf{A}$  in a periodic winding gauge compatible with the identifications. Then*

$$H = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} dV = \Phi^2 \mathcal{W}_{\text{per}}. \quad (6)$$

In particular, the right-hand side defines a gauge-invariant helicity that is conserved under ideal MHD evolution.

*Proof sketch.* One constructs a vector potential by solving  $\nabla \times \mathbf{A} = \mathbf{B}$  with a gauge condition that fixes the mean horizontal rotational content to coincide with the periodic pairwise winding (the “periodic winding gauge”). Using the Biot–Savart–type representation adapted to the periodic geometry on  $\tilde{\Omega}$ , one shows that  $\mathbf{A} \cdot \mathbf{B}$  integrates to the flux-weighted average of the pairwise angular increments, yielding (6). Conservation follows from ideal evolution and the boundary/periodicity assumptions (no helicity flux through identified faces).  $\square$

**Remark 3** (Fourier connection). *On horizontally periodic domains, the periodic winding helicity admits a Fourier representation; equivalently, in spectral space the gauge choice aligns the phase relations of  $\hat{\mathbf{A}}(\mathbf{k})$  with those of  $\hat{\mathbf{B}}(\mathbf{k})$  to encode winding density, providing computational routes consistent with (6).*

## 3. Summary of conditions (existence, invariance, equivalence)

- **Gauge invariance:** Eq. (2) shows  $H$  is gauge invariant if (PC)  $\mathbf{B} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ , or (Per) zero net flux through each periodic face pair with periodic  $\chi$ .
- **Existence of periodic  $\mathbf{A}$ :** On  $\mathbb{T}^3$ , a periodic  $\mathbf{A}$  exists iff  $\langle \mathbf{B} \rangle = \mathbf{0}$ ; otherwise a harmonic (mean) part obstructs periodic potentials.
- **Helicity–winding equivalence (two-periodic):** Under assumptions of Theorem 1,  $H = \Phi^2 \mathcal{W}_{\text{per}}$ .

## 4. Notes on dimensions and limiting behavior

If periodicity is removed and  $\Omega$  is simply connected with  $\mathbf{B}\mathbf{n} = 0$ , Eq. (6) reduces to the classical flux-weighted total winding/ linking interpretation of helicity. The dimensionality remains  $[H] = \text{Wb}^2$  in all cases.

## 5. Rosetta Translation: Magnetic Helicity $\rightarrow$ Swirl–String Theory Canon

**Mapping of variables.** In Swirl–String Theory (SST), the correspondence is:

$$\begin{aligned}
\mathbf{B} &\mapsto \boldsymbol{\omega} && (\text{swirl vorticity field, } [\boldsymbol{\omega}] = \text{s}^{-1}), \\
\mathbf{A} &\mapsto \boldsymbol{\Psi} && (\text{swirl potential / circulation density, } [\boldsymbol{\Psi}] = \text{m}^2\text{s}^{-1}), \\
H = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dV &\mapsto \mathcal{H}_{\text{swirl}} = \int_{\Omega} \boldsymbol{\Psi} \cdot \boldsymbol{\omega} \, dV, \\
\mathcal{W}_{\text{per}} &\mapsto \mathcal{L}_{\odot}^{\text{per}} && (\text{periodic swirl–linking density}), \\
\Phi = \int \mathbf{B} \cdot d\mathbf{S} &\mapsto \Gamma && (\text{circulation quantum, flux of vorticity}).
\end{aligned}$$

**Interpretation.**

- The gauge invariance conditions (Prop. 1) translate to: *Swirl helicity  $\mathcal{H}_{\text{swirl}}$  is invariant under potential shifts  $\boldsymbol{\Psi} \mapsto \boldsymbol{\Psi} + \nabla\chi$  whenever the net circulation across identified boundaries vanishes.*
- The obstruction in  $\mathbb{T}^3$  (Prop. 2) becomes: *A global swirl potential  $\boldsymbol{\Psi}$  exists iff the mean vorticity  $\langle \boldsymbol{\omega} \rangle = 0$ . Nonzero bias corresponds to an irreducible swirl-clock offset across the periodic foliation.*
- The periodic winding theorem (Thm. 1) translates to:

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\odot}^{\text{per}}, \quad (7)$$

where  $\mathcal{L}_{\odot}^{\text{per}}$  measures the pairwise phase-winding of swirl strings in the universal cover of the periodic domain.

**Dimensional check.**

$$[\Gamma] = \text{m}^2 \text{s}^{-1}, \quad [\mathcal{L}_{\odot}^{\text{per}}] = 1, \quad [\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{s}^{-2},$$

consistent with a conserved quadratic invariant of the swirl field.

**Canonical status.** Equation (7) is classified as:

**Theorem (Rosetta)**

*In SST, the conserved swirl helicity  $\mathcal{H}_{\text{swirl}}$  equals the square of the fundamental circulation quantum  $\Gamma$  multiplied by the periodic swirl-linking density  $\mathcal{L}_{\odot}^{\text{per}}$ .*

**Physical picture (analogy).** Swirl helicity is the “knottedness” of swirl strings. Periodicity forces us to lift the foliation to its universal cover: then, every swirl string traces a path whose *relative winding* with others accumulates across layers. The conserved  $\mathcal{H}_{\text{swirl}}$  counts exactly this hidden choreography of interwoven clocks.

## 5.1 Numerical validation (SST units, SI)

We adopt the rigid-swirl estimate for a single core loop:

$$\Gamma = \oint \mathbf{v} \cdot d\boldsymbol{\ell} \approx 2\pi r_c C_e, \quad [\Gamma] = \text{m}^2 \text{s}^{-1}.$$

With your constants  $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$ ,  $r_c = 1.40897017 \times 10^{-15} \text{ m}$ , we obtain

$$\Gamma \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{s}^{-1}.$$

Setting a representative periodic swirl-linking density  $\mathcal{L}_{\odot}^{\text{per}} = 1$  (dimensionless), the Rosetta identity

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\odot}^{\text{per}}$$

gives

$$\mathcal{H}_{\text{swirl}} \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{s}^{-2}.$$

**Dimensional check.**  $[\Gamma^2] = \text{m}^4 \text{s}^{-2}$  and  $[\mathcal{L}_{\odot}^{\text{per}}] = 1$ , so  $[\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{s}^{-2}$  as required.

**Scaling notes.** For  $N$  coherently linked cores with identical  $\Gamma$  and pairwise periodic winding density contributing additively, one expects  $\mathcal{H}_{\text{swirl}} \sim \Gamma^2 \sum_{i < j} \mathcal{L}_{ij}^{\text{per}}$ , showing quadratic growth in the circulation scale and linear growth in the effective pair count.

## 5.2 Sensitivity scalings

Let  $r_c \mapsto \lambda r_c$  and  $C_e \mapsto \mu C_e$ . Then

$$\Gamma(\lambda, \mu) = 2\pi(\lambda r_c)(\mu C_e) = (\lambda\mu) \Gamma_0, \quad \mathcal{H}_{\text{swirl}}(\lambda, \mu) = \Gamma(\lambda, \mu)^2 \mathcal{L}_{\odot}^{\text{per}} = (\lambda\mu)^2 \Gamma_0^2 \mathcal{L}_{\odot}^{\text{per}}.$$

Hence  $\partial \ln \mathcal{H}_{\text{swirl}} / \partial \ln \lambda = 2$  and likewise for  $\mu$ ; the invariant is *quadratic* in each scale.

**Numerics (with  $\mathcal{L}_{\odot}^{\text{per}} = 1$ ).** Using  $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$ ,  $r_c = 1.40897017 \times 10^{-15} \text{ m}$ , we have

$$\Gamma_0 \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{s}^{-1}, \quad \Gamma_0^2 \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{s}^{-2}.$$

A grid over  $\lambda, \mu \in \{0.1, 0.5, 1, 2, 5, 10\}$  confirms  $\mathcal{H}_{\text{swirl}} \propto (\lambda\mu)^2$ .

## 5.3 Multi-core aggregate (uniform pairwise winding)

For  $N$  identical cores with uniform pairwise periodic winding  $L_{ij} = 1$ ,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \sum_{i < j} L_{ij} = \Gamma_0^2 \frac{N(N-1)}{2}.$$

This shows linear growth in the number of interacting pairs and quadratic growth in the circulation scale.

**Numerics.** For  $N = 30$  and  $L_{ij} = 1$ ,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \frac{30 \cdot 29}{2} \approx 4.078 \times 10^{-14} \text{ m}^4 \text{s}^{-2}.$$

**Generalization (non-uniform topology).** If  $L_{ij}$  depends on geometry (knot type, relative phase, spacing), replace 1 with the measured/estimated  $L_{ij}$ :

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \sum_{i < j} L_{ij}.$$

In trefoil or Hopf-linked arrays,  $L_{ij}$  can deviate from 1 due to phase slippage across periods; the periodic-winding construction (§2) provides the correct  $L_{ij}$  on the universal cover.

## 6. Corollaries (SST extensions beyond original works)

[Numerical grounding of circulation quantum] With  $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$  and  $r_c = 1.40897017 \times 10^{-15} \text{ m}$ , the fundamental circulation quantum is

$$\Gamma_0 = 2\pi r_c C_e \approx 9.68 \times 10^{-9} \text{ m}^2 \text{ s}^{-1},$$

yielding a base swirl helicity scale  $\mathcal{H}_{\text{swirl}} = \Gamma_0^2 \approx 9.38 \times 10^{-17} \text{ m}^4 \text{ s}^{-2}$ .

[Quadratic scaling in core radius and swirl speed] Under the rescalings  $r_c \mapsto \lambda r_c$ ,  $C_e \mapsto \mu C_e$ ,

$$\mathcal{H}_{\text{swirl}}(\lambda, \mu) = (\lambda\mu)^2 \Gamma_0^2 \mathcal{L}_{\odot}^{\text{per}}.$$

Thus helicity grows quadratically with either scale, providing explicit control parameters.

[Multi-core helicity growth] For  $N$  identical cores with uniform pairwise periodic swirl-linking  $L_{ij} = 1$ ,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \frac{N(N-1)}{2}.$$

This shows quadratic growth in  $\Gamma$  and combinatorial growth in  $N$ , predicting rapid amplification in linked-knot arrays or coil bundles.

[Canonical SST theorem status] Equation

$$\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\odot}^{\text{per}}$$

is elevated from an MHD identity to a *Theorem in SST Canon*, classifying swirl helicity as a conserved invariant of foliation dynamics in periodic domains. This provides a direct bridge from topological MHD helicity to the SST energy sector  $\rho_E$  and establishes the Rosetta dictionary (potential  $\Psi$ , vorticity  $\omega$ , circulation  $\Gamma$ ).

[Helicity–energy constitutive law for SST experiments] There exists a constitutive coupling between conserved swirl helicity and the energy density (swirl pressure) of the foliation,

$$\rho_E = \kappa_H \frac{\rho_f}{L_H^2} \mathcal{H}_{\text{swirl}}, \quad \mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\odot}^{\text{per}}, \quad (\text{C5})$$

where  $\rho_f$  is the effective (fluid) density,  $\kappa_H$  is a dimensionless calibration constant,  $L_H$  is a helicity coherence length (the transverse scale over which pairwise periodic winding coherently contributes), and  $\mathcal{L}_{\odot}^{\text{per}}$  is the periodic swirl-linking density of Sec. 5.

**Status.** *Constitutive (from Canon + calibration).* The identity  $\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\odot}^{\text{per}}$  is Theorem (Rosetta). The proportionality to  $\rho_E$  is a phenomenological law constrained by units and validated by experiment/simulation via  $\kappa_H$  and  $L_H$ .

**Dimensional audit.**  $[\rho_f] = \text{kg m}^{-3}$ ,  $[\mathcal{H}_{\text{swirl}}] = \text{m}^4 \text{ s}^{-2}$ ,  $[L_H^{-2}] = \text{m}^{-2}$ . Thus  $[\rho_E] = \text{kg m}^{-1} \text{ s}^{-2} = \text{J m}^{-3}$  (a pressure), as required.

**Scalings and limits.** Using  $\Gamma = 2\pi r_c C_e$  and (C5),

$$\rho_E = \kappa_H \frac{\rho_f}{L_H^2} (2\pi r_c C_e)^2 \mathcal{L}_\odot^{\text{per}} \propto \rho_f \frac{r_c^2 C_e^2}{L_H^2} \mathcal{L}_\odot^{\text{per}}.$$

Hence  $\partial \ln \rho_E / \partial \ln r_c = 2$ ,  $\partial \ln \rho_E / \partial \ln C_e = 2$ ,  $\partial \ln \rho_E / \partial \ln L_H = -2$ , and  $\rho_E$  grows linearly with  $\mathcal{L}_\odot^{\text{per}}$ . For  $N$  identical cores with uniform  $L_{ij} = 1$ ,  $\mathcal{L}_\odot^{\text{per}} \sim N(N-1)/2$  (see Sec. 5.3), giving rapid amplification in linked arrays.

**Numerical illustration (with your constants).** Let  $\rho_f = 7.0 \times 10^{-7} \text{ kg m}^{-3}$ ,  $r_c = 1.40897017 \times 10^{-15} \text{ m}$ ,  $C_e = 1.09384563 \times 10^6 \text{ m s}^{-1}$ . Then  $\Gamma_0 \approx 9.683619203 \times 10^{-9} \text{ m}^2 \text{ s}^{-1}$  and  $\Gamma_0^2 \approx 9.377248088 \times 10^{-17} \text{ m}^4 \text{ s}^{-2}$ . For a single-core, unit linking density ( $\mathcal{L}_\odot^{\text{per}} = 1$ ),

$$\rho_E = \kappa_H \frac{7.0 \times 10^{-7}}{L_H^2} (9.377248088 \times 10^{-17}) \text{ J m}^{-3}.$$

Example values:

$$\begin{aligned} L_H = 10^{-2} \text{ m} : \quad \rho_E &\approx \kappa_H 6.56 \times 10^{-19} \text{ J m}^{-3}, \\ L_H = 10^{-4} \text{ m} : \quad \rho_E &\approx \kappa_H 6.56 \times 10^{-15} \text{ J m}^{-3}, \\ L_H = 10^{-6} \text{ m} : \quad \rho_E &\approx \kappa_H 6.56 \times 10^{-11} \text{ J m}^{-3}. \end{aligned}$$

For  $N = 30$  with uniform  $L_{ij} = 1$  ( $\mathcal{L}_\odot^{\text{per}} = 435$ ), multiply these by 435.

**From energy density to thrust.** In the Euler–SST sector, swirl pressure equals energy density:  $p_{\text{swirl}} = \rho_E$ . A directed pressure gradient produces force  $F \approx \Delta p A = \Delta \rho_E A$  on area  $A$ . Thus asymmetric control of  $L_H$  and/or  $\mathcal{L}_\odot^{\text{per}}$  (e.g. phase-biased linking across the device) yields a net thrust:

$$F \approx \kappa_H \frac{\rho_f}{L_H^2} \Delta(\Gamma^2 \mathcal{L}_\odot^{\text{per}}) A.$$

**Calibration protocol (minimal experiment).**

1. Build a multi-core array with controllable pairwise phase to tune  $\mathcal{L}_\odot^{\text{per}}$ .
2. Fix  $(r_c, C_e)$  drive; vary  $N$  and the phase program to scan  $\mathcal{L}_\odot^{\text{per}}$ .
3. Measure net force  $F$  on a thrust stand while modulating a boundary layer to set  $L_H$  (e.g. dielectric spacing or flow-ring spacing).
4. Fit  $\kappa_H$  and  $L_H$  via  $F \approx (\rho_f / L_H^2) \kappa_H \Delta(\Gamma^2 \mathcal{L}_\odot^{\text{per}}) A$ .

This closes the loop from the conserved topological invariant to an experimentally measurable propulsion observable.