Rotational Kinetic Energy Density and an Effective Mass Relation in Incompressible Fluids

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November 17, 2025

Abstract

Kinetic energy contributes to inertia and gravitational mass through the relativistic relation $E = mc^2$. In extended media such as fluids, this contribution can be expressed as an effective mass density associated with internal motion. We consider an incompressible, inviscid Newtonian fluid undergoing rigid-body rotation in a finite cylinder and compute the volume-averaged rotational kinetic energy density. By associating this energy density with an effective mass density via $E = mc^2$ in the nonrelativistic limit, we obtain the closed-form relation

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2,$$

where v_{edge} is the tangential speed at the cylinder boundary and ρ is the rest-mass density. The result provides a transparent classical example of how rotational motion modifies the mass density at order $(v/c)^2$. We discuss the connection with relativistic continuum mechanics and provide numerical estimates for laboratory and astrophysical regimes.

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DOI: 10.5281/zenodo.17629932

1 Introduction

The equivalence between energy and mass, expressed by $E = mc^2$, implies that kinetic, field, and binding energies contribute to the inertia and gravitational mass of extended systems [1, 2, 3]. In the context of relativistic continuum mechanics, this statement is encoded in the stress–energy tensor $T^{\mu\nu}$: the total mass–energy is obtained by integrating T^{00} over a spatial hypersurface, and T^{00} includes both rest-mass and kinetic contributions [3, 4].

While this viewpoint is standard in high-energy physics and general relativity, explicit examples in simple fluid configurations remain pedagogically useful. In particular, it is instructive to make the contribution of *rotational* kinetic energy to an effective mass density quantitatively explicit in a setting where the flow field is analytically tractable.

In this paper we analyze a canonical configuration from classical fluid mechanics [5, 6]: an incompressible, inviscid fluid in rigid-body rotation inside a finite cylinder. Within this model we:

1. compute the local and volume-averaged rotational kinetic energy density;
2. define an effective mass density $\Delta\rho_{\text{eff}}$ via $E = mc^2$ in the regime $v \ll c$;
3. derive a compact expression for $\Delta\rho_{\text{eff}}/\rho$ in terms of the edge speed v_{edge};
4. discuss how this classical result fits within the framework of relativistic continuum mechanics.

The derivation uses only incompressible Euler flow and the special-relativistic mass–energy relation. No modifications of Newtonian or relativistic theory are proposed.

2 Rigid-body rotation in an incompressible, inviscid fluid

2.1 Flow configuration

We consider a Newtonian fluid of constant rest-mass density ρ occupying a right circular cylinder of radius R and height L. The fluid undergoes steady rigid-body rotation with constant angular velocity Ω about the z-axis. In cylindrical coordinates (r, θ, z), with $0 \le r \le R$, the velocity field is

$$\boldsymbol{v}(r) = \Omega r\, \hat{\boldsymbol{\theta}}. \tag{1}$$

This flow is incompressible and inviscid:

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \text{viscosity} = 0, \tag{2}$$

and it satisfies the steady Euler equations with an appropriate pressure distribution [5, 6].

2.2 Local kinetic energy density

The local kinetic energy density of the fluid is

$$e_{\text{kin}}(r) = \frac{1}{2}\,\rho \|\boldsymbol{v}(r)\|^2 = \frac{1}{2}\,\rho\, \Omega^2 r^2. \tag{3}$$

This quantity is position-dependent and increases quadratically with radius.

2.3 Total rotational energy and volume-averaged energy density

The total rotational kinetic energy is obtained by integrating Eq. (3) over the fluid volume:

$$
\begin{aligned}
E_{\mathrm{rot}} &= \int_V e_{\mathrm{kin}}\,\mathrm{d}V \\
&= \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{2}\,\rho\,\Omega^2 r^2\, r\,\mathrm{d}r \\
&= \frac{1}{2}\,\rho\,\Omega^2 (2\pi L) \int_0^R r^3\,\mathrm{d}r \\
&= \frac{\pi}{4}\,\rho\,\Omega^2\, L\, R^4.
\end{aligned}
\tag{4}
$$

The cylinder volume is $V = \pi R^2 L$, so the volume-averaged kinetic energy density is

$$
\langle e_{\mathrm{kin}} \rangle = \frac{E_{\mathrm{rot}}}{V} = \frac{\frac{\pi}{4}\,\rho\,\Omega^2\, L\, R^4}{\pi R^2 L} = \frac{1}{4}\,\rho\,\Omega^2 R^2. \tag{5}
$$

It is convenient to express this in terms of the edge speed

$$
v_{\mathrm{edge}} := \Omega R. \tag{6}
$$

Then Eq. (5) becomes

$$
\langle e_{\mathrm{kin}} \rangle = \frac{1}{4}\,\rho\, v_{\mathrm{edge}}^2. \tag{7}
$$

For comparison, the kinetic energy density at the boundary is

$$
e_{\mathrm{kin}}(R) = \frac{1}{2}\,\rho\, v_{\mathrm{edge}}^2, \tag{8}
$$

so the volume average is exactly one half of the boundary value, reflecting the quadratic radial profile.

3 Effective mass density from $E = mc^2$

3.1 Nonrelativistic limit and effective density

In special relativity, the total relativistic energy of a fluid element with rest-mass density ρ and small velocity $v \ll c$ can be decomposed as [3, 4]

$$
\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \cdots, \tag{9}
$$

where ε is the total energy density in the local rest frame, and the ellipsis denotes higher-order terms in v^2/c^2 and internal energy contributions. To leading order in v^2/c^2, the kinetic part of the energy density can therefore be regarded as an *effective mass density* via

$$
\Delta\rho_{\mathrm{eff}}(\boldsymbol{x}) = \frac{e_{\mathrm{kin}}(\boldsymbol{x})}{c^2}. \tag{10}
$$

This interpretation is consistent with the structure of the stress–energy tensor for a perfect fluid [3].

For the rigidly rotating configuration considered here, we focus on the volume-averaged effective mass density

$$
\Delta\rho_{\mathrm{eff}} := \frac{\langle e_{\mathrm{kin}} \rangle}{c^2}. \tag{11}
$$

Inserting Eq. (7) into Eq. (11), we obtain

$$\Delta\rho_{\text{eff}} = \frac{1}{4c^2}\,\rho\, v_{\text{edge}}^2. \tag{12}$$

Dividing by the rest-mass density ρ yields

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2. \tag{13}$$

Thus, in the nonrelativistic regime, the rotational contribution to the mass density is second order in v_{edge}/c, with a geometric coefficient $1/4$ specific to rigid-body rotation in a cylinder.

3.2 Dimensional check

The dimensions of Eq. (12) are

$$[\Delta\rho_{\text{eff}}] = \frac{[\rho][v]^2}{[c]^2} = \frac{\mathrm{kg\,m^{-3}\,(m/s)^2}}{\mathrm{(m/s)^2}} = \mathrm{kg\,m^{-3}},$$

as required for a mass density. The ratio $\Delta\rho_{\text{eff}}/\rho$ in Eq. (13) is dimensionless, as expected.

4 Numerical estimates

4.1 Laboratory-scale example

Consider water with $\rho \approx 1.0\times10^3\,\mathrm{kg\,m^{-3}}$, a cylinder of radius $R = 0.10\,\mathrm{m}$, and angular velocity $\Omega = 1.0\times10^3\,\mathrm{s^{-1}}$, corresponding to $v_{\text{edge}} = 100\,\mathrm{m\,s^{-1}}$. Then

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{100}{3.0\times10^8}\right)^2 \approx 3\times10^{-14}, \tag{14}$$

and

$$\Delta\rho_{\text{eff}} \sim 3\times10^{-11}\,\mathrm{kg\,m^{-3}}. \tag{15}$$

The effect is many orders of magnitude below typical experimental resolution in laboratory fluids.

4.2 Astrophysical order-of-magnitude

In astrophysical settings, rotational velocities can be relativistic. For example, in the inner regions of accretion flows or rapidly rotating compact stars, characteristic speeds may reach $v \sim 0.1c$ or higher [4]. If one naively substitutes $v_{\text{edge}} = 0.1c$ into Eq. (13), one finds

$$\frac{\Delta\rho_{\text{eff}}}{\rho} \sim \frac{1}{4}(0.1)^2 = 2.5\times10^{-3}, \tag{16}$$

already approaching the percent level. However, in such regimes a fully relativistic treatment of the fluid is required, and higher-order terms in v^2/c^2 as well as strong-gravity effects must be included. Equation (13) should therefore be regarded as illustrating the leading-order trend rather than providing a quantitatively accurate model for relativistic flows.

5 Relation to relativistic continuum mechanics

In relativistic hydrodynamics, a perfect fluid is described by the stress–energy tensor [3, 4]

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{17}$$

where ε is the total energy density in the fluid rest frame, p is the pressure, and u^μ is the four-velocity. In the nonrelativistic limit and for small internal energy, one has

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \ldots, \tag{18}$$

consistent with the decomposition used in Eq. (10).

The contribution of kinetic energy to the gravitational mass of an extended system can be derived by integrating T^{00} over space in an appropriate frame [2, 3]. Our treatment effectively isolates the rotational part of this contribution for the specific case of rigid-body rotation in an incompressible fluid. Equation (13) can therefore be viewed as the nonrelativistic limit of the rotational piece of T^{00}/c^2, evaluated in a simple geometry.

6 Discussion and outlook

We have derived a compact relation between the rotational kinetic energy of an incompressible, inviscid fluid in rigid-body rotation and an associated effective mass density. The key steps are:

1. computation of the local kinetic energy density $e_{\text{kin}}(r) = \frac{1}{2}\rho\Omega^2 r^2$;
2. volume averaging over a finite cylinder, yielding $\langle e_{\text{kin}}\rangle = \frac{1}{4}\rho v_{\text{edge}}^2$;
3. definition of an effective mass density via $E = mc^2$ in the nonrelativistic limit, resulting in $\Delta\rho_{\text{eff}}/\rho = \frac{1}{4}(v_{\text{edge}}/c)^2$.

The analysis is fully contained within classical fluid mechanics and special relativity. It provides a transparent example of how rotational motion contributes to the mass density of an extended medium at order $(v/c)^2$. The coefficient 1/4 is specific to rigid-body rotation in a cylinder and reflects the radial structure of the velocity field. For other velocity profiles or geometries, different geometric factors would appear, though the basic scaling $\Delta\rho_{\text{eff}}/\rho \propto \langle v^2\rangle/c^2$ remains.

Potential applications include:

- pedagogical demonstrations of mass–energy equivalence in continuum systems;
- benchmark problems for numerical schemes that couple incompressible fluid dynamics to relativistic mass–energy accounting in the low-velocity regime;
- conceptual comparisons with fully relativistic treatments of rotating fluids in astrophysical contexts.

As a complementary single-particle example, Appendix A summarises a semi-classical model in which a photon confined on a toroidal path reproduces the electron charge to within about ten per cent, using only classical electromagnetism and the Compton wavelength as input [7].

Any attempt to attribute fundamental rest mass to internal rotational motion would require additional structural assumptions and a fully relativistic framework, and lies beyond the scope of the present work.

Acknowledgments

The author thanks the authors of standard texts on fluid mechanics and relativistic field theory for providing the background material on which this analysis is based.

Appendix A: Semi-classical charge estimate from a confined electromagnetic mode

In the main text we considered how rotational kinetic energy in an incompressible fluid contributes to an effective mass density via the relation $E = mc^2$ in the nonrelativistic regime. For completeness, we record here a simple semi-classical construction, due to Williamson and van der Mark [7], which shows how a confined electromagnetic mode can also reproduce the observed magnitude of the electron charge using only classical electromagnetism and the Compton wavelength.

Consider a photon of wavelength λ whose energy

$$E_\gamma = \frac{hc}{\lambda}$$

is confined to a finite volume V of characteristic size comparable to λ. In the model of Ref. [7], the photon is confined on a toroidal path of length one Compton wavelength $\lambda_C = h/(m_e c)$, with a characteristic radius

$$r = \frac{\lambda_C}{4\pi},$$

so that the confinement volume is of order $V \sim r^3$. Equating the photon energy to the integral of the electromagnetic energy density over this volume leads to an average electric-field amplitude of the form

$$\langle E \rangle = \sqrt{\frac{6hc}{\pi\varepsilon_0\lambda_C^4}}, \tag{19}$$

where ε_0 is the vacuum permittivity.[1]

To relate this confined field to an effective point charge, one compares $\langle E \rangle$ to the magnitude of the Coulomb field at a radius r,

$$E_\mathrm{C}(r) = \frac{q}{4\pi\varepsilon_0 r^2}. \tag{20}$$

Identifying $r = \lambda_C/(4\pi)$ as above and setting $E_\mathrm{C}(r) = \langle E \rangle$ yields

$$q = 4\pi\varepsilon_0 r^2 \langle E \rangle = 4\pi\varepsilon_0 \left(\frac{\lambda_C}{4\pi}\right)^2 \sqrt{\frac{6hc}{\pi\varepsilon_0\lambda_C^4}}. \tag{21}$$

Using $\lambda_C = h/(m_e c)$ and $h = 2\pi\hbar$, one finds that the dependence on λ_C and m_e cancels, giving the closed-form expression

$$q_\mathrm{model} = \frac{1}{2\pi}\sqrt{3\varepsilon_0\hbar c}. \tag{22}$$

Numerically,

$$q_\mathrm{model} = \frac{1}{2\pi}\sqrt{3\varepsilon_0\hbar c} \simeq 1.46 \times 10^{-19}\,\mathrm{C} \simeq 0.91\,e,$$

so that this simple confinement picture reproduces the observed elementary charge $e \simeq 1.60 \times 10^{-19}$ C at the ten-percent level.

[1] The numerical factor depends on the detailed choice of confinement volume and field profile; Eq. (19) corresponds to the specific toroidal geometry considered in [7].

Very short analogy (for intuition): think of squeezing one "loop" of light into a tiny doughnut; the more tightly you squeeze it, the stronger its electric field gets, until that field at the surface looks almost exactly like the field of a point charge the size of an electron.

The purpose of this appendix is not to advocate a specific microscopic model of the electron, but to illustrate that, even within classical electromagnetism, geometric confinement of a single-wavelength mode can naturally generate particle-like mass and charge scales. This complements the continuum calculation in the main text, where rotational kinetic energy in an extended medium leads to an effective mass density proportional to $\langle v^2 \rangle / c^2$.

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