

# Rotational Kinetic Energy Density and an Effective Mass Relation in Incompressible Fluids

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November 14, 2025

## Abstract

Kinetic energy contributes to inertia and gravitational mass through the relativistic relation  $E = mc^2$ . In extended media such as fluids, this contribution can be expressed as an effective mass density associated with internal motion. We consider an incompressible, inviscid Newtonian fluid undergoing rigid-body rotation in a finite cylinder and compute the volume-averaged rotational kinetic energy density. By associating this energy density with an effective mass density via  $E = mc^2$  in the nonrelativistic limit, we obtain the closed-form relation

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{v_{\text{edge}}}{c} \right)^2,$$

where  $v_{\text{edge}}$  is the tangential speed at the cylinder boundary and  $\rho$  is the rest-mass density. The result provides a transparent classical example of how rotational motion modifies the mass density at order  $(v/c)^2$ . We discuss the connection with relativistic continuum mechanics and provide numerical estimates for laboratory and astrophysical regimes.

## 1 Introduction

The equivalence between energy and mass, expressed by  $E = mc^2$ , implies that kinetic, field, and binding energies contribute to the inertia and gravitational mass of extended systems [1, 2, 3]. In the context of relativistic continuum mechanics, this statement is encoded in the stress–energy tensor  $T^{\mu\nu}$ : the total mass–energy is obtained by integrating  $T^{00}$  over a spatial hypersurface, and  $T^{00}$  includes both rest-mass and kinetic contributions [3, 4].

While this viewpoint is standard in high-energy physics and general relativity, explicit examples in simple fluid configurations remain pedagogically useful. In particular, it is instructive to make the contribution of *rotational* kinetic energy to an effective mass density quantitatively explicit in a setting where the flow field is analytically tractable.

In this paper we analyze a canonical configuration from classical fluid mechanics [5, 6]: an incompressible, inviscid fluid in rigid-body rotation inside a finite cylinder. Within this model we:

1. compute the local and volume-averaged rotational kinetic energy density;
2. define an effective mass density  $\Delta\rho_{\text{eff}}$  via  $E = mc^2$  in the regime  $v \ll c$ ;
3. derive a compact expression for  $\Delta\rho_{\text{eff}}/\rho$  in terms of the edge speed  $v_{\text{edge}}$ ;
4. discuss how this classical result fits within the framework of relativistic continuum mechanics.

The derivation uses only incompressible Euler flow and the special-relativistic mass–energy relation. No modifications of Newtonian or relativistic theory are proposed.

## 2 Rigid-body rotation in an incompressible, inviscid fluid

### 2.1 Flow configuration

We consider a Newtonian fluid of constant rest-mass density  $\rho$  occupying a right circular cylinder of radius  $R$  and height  $L$ . The fluid undergoes steady rigid-body rotation with constant angular velocity  $\Omega$  about the  $z$ -axis. In cylindrical coordinates  $(r, \theta, z)$ , with  $0 \leq r \leq R$ , the velocity field is

$$\mathbf{v}(r) = \Omega r \hat{\boldsymbol{\theta}}. \quad (1)$$

This flow is incompressible and inviscid:

$$\nabla \cdot \mathbf{v} = 0, \quad \text{viscosity} = 0, \quad (2)$$

and it satisfies the steady Euler equations with an appropriate pressure distribution [5, 6].

### 2.2 Local kinetic energy density

The local kinetic energy density of the fluid is

$$e_{\text{kin}}(r) = \frac{1}{2} \rho \|\mathbf{v}(r)\|^2 = \frac{1}{2} \rho \Omega^2 r^2. \quad (3)$$

This quantity is position-dependent and increases quadratically with radius.

### 2.3 Total rotational energy and volume-averaged energy density

The total rotational kinetic energy is obtained by integrating Eq. (3) over the fluid volume:

$$\begin{aligned} E_{\text{rot}} &= \int_V e_{\text{kin}} dV \\ &= \int_0^L dz \int_0^{2\pi} d\theta \int_0^R \frac{1}{2} \rho \Omega^2 r^2 r dr \\ &= \frac{1}{2} \rho \Omega^2 (2\pi L) \int_0^R r^3 dr \\ &= \frac{\pi}{4} \rho \Omega^2 L R^4. \end{aligned} \quad (4)$$

The cylinder volume is  $V = \pi R^2 L$ , so the volume-averaged kinetic energy density is

$$\langle e_{\text{kin}} \rangle = \frac{E_{\text{rot}}}{V} = \frac{\frac{\pi}{4} \rho \Omega^2 L R^4}{\pi R^2 L} = \frac{1}{4} \rho \Omega^2 R^2. \quad (5)$$

It is convenient to express this in terms of the edge speed

$$v_{\text{edge}} := \Omega R. \quad (6)$$

Then Eq. (5) becomes

$$\langle e_{\text{kin}} \rangle = \frac{1}{4} \rho v_{\text{edge}}^2. \quad (7)$$

For comparison, the kinetic energy density at the boundary is

$$e_{\text{kin}}(R) = \frac{1}{2} \rho v_{\text{edge}}^2, \quad (8)$$

so the volume average is exactly one half of the boundary value, reflecting the quadratic radial profile.

### 3 Effective mass density from $E = mc^2$

#### 3.1 Nonrelativistic limit and effective density

In special relativity, the total relativistic energy of a fluid element with rest-mass density  $\rho$  and small velocity  $v \ll c$  can be decomposed as [3, 4]

$$\varepsilon \simeq \rho c^2 + \frac{1}{2} \rho v^2 + \dots, \quad (9)$$

where  $\varepsilon$  is the total energy density in the local rest frame, and the ellipsis denotes higher-order terms in  $v^2/c^2$  and internal energy contributions. To leading order in  $v^2/c^2$ , the kinetic part of the energy density can therefore be regarded as an *effective mass density* via

$$\Delta\rho_{\text{eff}}(\mathbf{x}) = \frac{e_{\text{kin}}(\mathbf{x})}{c^2}. \quad (10)$$

This interpretation is consistent with the structure of the stress–energy tensor for a perfect fluid [3].

For the rigidly rotating configuration considered here, we focus on the volume-averaged effective mass density

$$\Delta\rho_{\text{eff}} := \frac{\langle e_{\text{kin}} \rangle}{c^2}. \quad (11)$$

Inserting Eq. (7) into Eq. (11), we obtain

$$\Delta\rho_{\text{eff}} = \frac{1}{4c^2} \rho v_{\text{edge}}^2. \quad (12)$$

Dividing by the rest-mass density  $\rho$  yields

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{v_{\text{edge}}}{c} \right)^2. \quad (13)$$

Thus, in the nonrelativistic regime, the rotational contribution to the mass density is second order in  $v_{\text{edge}}/c$ , with a geometric coefficient 1/4 specific to rigid-body rotation in a cylinder.

#### 3.2 Dimensional check

The dimensions of Eq. (12) are

$$[\Delta\rho_{\text{eff}}] = \frac{[\rho][v]^2}{[c]^2} = \frac{\text{kg m}^{-3} (\text{m/s})^2}{(\text{m/s})^2} = \text{kg m}^{-3},$$

as required for a mass density. The ratio  $\Delta\rho_{\text{eff}}/\rho$  in Eq. (13) is dimensionless, as expected.

### 4 Numerical estimates

#### 4.1 Laboratory-scale example

Consider water with  $\rho \approx 1.0 \times 10^3 \text{ kg m}^{-3}$ , a cylinder of radius  $R = 0.10 \text{ m}$ , and angular velocity  $\Omega = 1.0 \times 10^3 \text{ s}^{-1}$ , corresponding to  $v_{\text{edge}} = 100 \text{ m s}^{-1}$ . Then

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4} \left( \frac{100}{3.0 \times 10^8} \right)^2 \approx 3 \times 10^{-14}, \quad (14)$$

and

$$\Delta\rho_{\text{eff}} \sim 3 \times 10^{-11} \text{ kg m}^{-3}. \quad (15)$$

The effect is many orders of magnitude below typical experimental resolution in laboratory fluids.

## 4.2 Astrophysical order-of-magnitude

In astrophysical settings, rotational velocities can be relativistic. For example, in the inner regions of accretion flows or rapidly rotating compact stars, characteristic speeds may reach  $v \sim 0.1c$  or higher [4]. If one naively substitutes  $v_{\text{edge}} = 0.1c$  into Eq. (13), one finds

$$\frac{\Delta\rho_{\text{eff}}}{\rho} \sim \frac{1}{4}(0.1)^2 = 2.5 \times 10^{-3}, \quad (16)$$

already approaching the percent level. However, in such regimes a fully relativistic treatment of the fluid is required, and higher-order terms in  $v^2/c^2$  as well as strong-gravity effects must be included. Equation (13) should therefore be regarded as illustrating the leading-order trend rather than providing a quantitatively accurate model for relativistic flows.

## 5 Relation to relativistic continuum mechanics

In relativistic hydrodynamics, a perfect fluid is described by the stress–energy tensor [3, 4]

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (17)$$

where  $\varepsilon$  is the total energy density in the fluid rest frame,  $p$  is the pressure, and  $u^\mu$  is the four-velocity. In the nonrelativistic limit and for small internal energy, one has

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \dots, \quad (18)$$

consistent with the decomposition used in Eq. (10).

The contribution of kinetic energy to the gravitational mass of an extended system can be derived by integrating  $T^{00}$  over space in an appropriate frame [2, 3]. Our treatment effectively isolates the rotational part of this contribution for the specific case of rigid-body rotation in an incompressible fluid. Equation (13) can therefore be viewed as the nonrelativistic limit of the rotational piece of  $T^{00}/c^2$ , evaluated in a simple geometry.

## 6 Discussion and outlook

We have derived a compact relation between the rotational kinetic energy of an incompressible, inviscid fluid in rigid-body rotation and an associated effective mass density. The key steps are:

1. computation of the local kinetic energy density  $e_{\text{kin}}(r) = \frac{1}{2}\rho\Omega^2 r^2$ ;
2. volume averaging over a finite cylinder, yielding  $\langle e_{\text{kin}} \rangle = \frac{1}{4}\rho v_{\text{edge}}^2$ ;
3. definition of an effective mass density via  $E = mc^2$  in the nonrelativistic limit, resulting in  $\Delta\rho_{\text{eff}}/\rho = \frac{1}{4}(v_{\text{edge}}/c)^2$ .

The analysis is fully contained within classical fluid mechanics and special relativity. It provides a transparent example of how rotational motion contributes to the mass density of an extended medium at order  $(v/c)^2$ . The coefficient  $1/4$  is specific to rigid-body rotation in a cylinder and reflects the radial structure of the velocity field. For other velocity profiles or geometries, different geometric factors would appear, though the basic scaling  $\Delta\rho_{\text{eff}}/\rho \propto \langle v^2 \rangle/c^2$  remains.

Potential applications include:

- pedagogical demonstrations of mass–energy equivalence in continuum systems;

- benchmark problems for numerical schemes that couple incompressible fluid dynamics to relativistic mass–energy accounting in the low-velocity regime;
- conceptual comparisons with fully relativistic treatments of rotating fluids in astrophysical contexts.

Any attempt to attribute fundamental rest mass to internal rotational motion would require additional structural assumptions and a fully relativistic framework, and lies beyond the scope of the present work.

## Acknowledgments

The author thanks the authors of standard texts on fluid mechanics and relativistic field theory for providing the background material on which this analysis is based.

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