Magnetic helicity in periodic domains: gauge conditions, existence of vector potentials, and periodic winding

Setting. Let $\Omega \subset \mathbb{R}^3$ be either (i) a bounded box with one or two pairs of opposite faces identified ("1- or 2-periodic"), or (ii) the fully periodic 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ with lattice Λ. Let $\boldsymbol{B} : \Omega \to \mathbb{R}^3$ be a smooth magnetic field with $\nabla\cdot\boldsymbol{B} = 0$.

1. Helicity, gauge, and boundary/periodicity conditions

The (total) magnetic helicity on Ω is

$$H[\boldsymbol{A}, \boldsymbol{B}] = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\,\mathrm{d}V, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1}$$

Under a gauge transform $\boldsymbol{A} \mapsto \boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$,

$$\begin{aligned} H[\boldsymbol{A}', \boldsymbol{B}] - H[\boldsymbol{A}, \boldsymbol{B}] &= \int_\Omega \nabla\chi \cdot \boldsymbol{B}\,\mathrm{d}V = \int_\Omega \nabla\cdot(\chi\boldsymbol{B})\,\mathrm{d}V - \int_\Omega \chi \underbrace{\nabla\cdot\boldsymbol{B}}_{0}\,\mathrm{d}V \\ &= \int_{\partial\Omega} \chi\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}. \end{aligned} \tag{2}$$

Hence helicity is gauge invariant if and only if the boundary flux term vanishes.

Proposition 1 (Gauge invariance in periodic/closed settings)**.** *Suppose either:*

(a) Ω is bounded with perfectly conducting boundary ($\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$), or

(b) Ω is periodic in one or two directions and the net magnetic flux *through each identified pair of faces is zero; additionally the gauge function χ respects the same periodicity,*

then H in (1) *is gauge invariant.*

Proof. In case (a) the surface integral in (2) vanishes since $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$ on $\partial\Omega$. In case (b), identify opposite faces with orientation; periodic χ has equal values on paired faces, while zero net flux through each pair implies the oriented surface integrals cancel pairwise. Thus the sum over $\partial\Omega$ is zero. □

Proposition 2 (Obstruction in fully periodic domains)**.** *On $\mathbb{T}^3$, a* globally periodic *vector potential $\boldsymbol{A}$ with $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$ exists if and only if the mean field vanishes:*

$$\langle\boldsymbol{B}\rangle \equiv \frac{1}{|\Omega|}\int_\Omega \boldsymbol{B}\,\mathrm{d}V = \boldsymbol{0}. \tag{3}$$

Equivalently, any nonzero constant (harmonic) component of $\boldsymbol{B}$ produces a topological obstruction to a periodic $\boldsymbol{A}$.

Sketch. Fourier decompose on $\mathbb{T}^3$. For $k \neq 0$, $\widehat{\boldsymbol{A}}(k)$ exists with $ik \times \widehat{\boldsymbol{A}}(k) = \widehat{\boldsymbol{B}}(k)$. At $k = 0$, we would require a constant $\widehat{\boldsymbol{A}}(0)$ with $\boldsymbol{\nabla}\times\widehat{\boldsymbol{A}}(0) = \boldsymbol{0}$ producing $\widehat{\boldsymbol{B}}(0) = \boldsymbol{0}$. Thus periodic solvability demands $\widehat{\boldsymbol{B}}(0) = \langle\boldsymbol{B}\rangle = 0$. □

Remark 1 (Physical dimension)**.** $[\boldsymbol{A}] = \mathrm{V\,s\,m^{-1}}$, $[\boldsymbol{B}] = \mathrm{T} = \mathrm{Wb\,m^{-2}}$. *Hence* $[\boldsymbol{A}\cdot\boldsymbol{B}] = \mathrm{Wb^2\,m^{-3}}$ *and* $[H] = \mathrm{Wb}^2$ *after integration, as standard.*

2. Periodic winding and the helicity equivalence (two-periodic case)

Consider a domain that is periodic in two directions, e.g. $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ with identifications in x, y (a 2-torus in the horizontal plane). Let $\widetilde{\Omega} = \mathbb{R}^2 \times [0, L_z]$ be its universal cover. Field lines of $\boldsymbol{B}$ in Ω lift to curves in $\widetilde{\Omega}$.

Definition 1 (Periodic winding of a pair of field lines). *Let γ_1, γ_2 be two* lifted *field lines in $\widetilde{\Omega}$, parameterized by $z \in [0, L_z]$, and let $\boldsymbol{r}(z) = \boldsymbol{x}_1(z) - \boldsymbol{x}_2(z)$ be their horizontal separation (projected to $\mathbb{R}^2$ before modding out by periods). Define the* periodic winding increment

$$\Delta\Theta(\gamma_1, \gamma_2) \;=\; \int_0^{L_z} \frac{\boldsymbol{r}(z) \times \dot{\boldsymbol{r}}(z)}{\|\boldsymbol{r}(z)\|^2} \cdot \boldsymbol{e}_z \, \mathrm{d}z, \tag{4}$$

and the periodic winding *as the flux-weighted mean over pairs of field lines:*

$$\mathcal{W}_{\text{per}} \;=\; \frac{1}{\Phi^2} \iint \Delta\Theta(\gamma_1, \gamma_2) \, \mathrm{d}\Phi(\gamma_1) \, \mathrm{d}\Phi(\gamma_2), \tag{5}$$

where Φ denotes the magnetic flux measure induced by $\boldsymbol{B}$ on a transverse cross-section.

Remark 2. *Expression* (4) *generalizes the pairwise angular change (linking/winding) to horizontally periodic geometry by working on the universal cover and then averaging in a flux-consistent way.*

Theorem 1 (Helicity–winding equivalence in two-periodic domains). *Assume: (i) ideal evolution (frozen-in field, sufficiently smooth), (ii) two-directional periodicity with zero net flux through each periodic pair, (iii) a vector potential $\boldsymbol{A}$ in a* periodic winding gauge *compatible with the identifications. Then*

$$H \;=\; \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \;=\; \Phi^2 \, \mathcal{W}_{\text{per}}. \tag{6}$$

In particular, the right-hand side defines a gauge-invariant helicity that is conserved under ideal MHD evolution.

Proof sketch. One constructs a vector potential by solving $\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}$ with a gauge condition that fixes the mean horizontal rotational content to coincide with the periodic pairwise winding (the "periodic winding gauge"). Using the Biot–Savart–type representation adapted to the periodic geometry on $\widetilde{\Omega}$, one shows that $\boldsymbol{A} \cdot \boldsymbol{B}$ integrates to the flux-weighted average of the pairwise angular increments, yielding (6). Conservation follows from ideal evolution and the boundary/periodicity assumptions (no helicity flux through identified faces). □

Remark 3 (Fourier connection). *On horizontally periodic domains, the periodic winding helicity admits a Fourier representation; equivalently, in spectral space the gauge choice aligns the phase relations of $\widehat{\boldsymbol{A}}(k)$ with those of $\widehat{\boldsymbol{B}}(k)$ to encode winding density, providing computational routes consistent with* (6).

3. Summary of conditions (existence, invariance, equivalence)

- **Gauge invariance:** Eq. (2) shows H is gauge invariant if (PC) $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, or (Per) zero net flux through each periodic face pair with periodic χ.
- **Existence of periodic $\boldsymbol{A}$:** On $\mathbb{T}^3$, a periodic $\boldsymbol{A}$ exists iff $\langle \boldsymbol{B} \rangle = \boldsymbol{0}$; otherwise a harmonic (mean) part obstructs periodic potentials.
- **Helicity–winding equivalence (two-periodic):** Under assumptions of Theorem 1, $H = \Phi^2 \mathcal{W}_{\text{per}}$.

4. Notes on dimensions and limiting behavior

If periodicity is removed and Ω is simply connected with $\boldsymbol{B}\cdot\boldsymbol{n} = 0$, Eq. (6) reduces to the classical flux-weighted total winding/ linking interpretation of helicity. The dimensionality remains $[H] = \mathrm{Wb}^2$ in all cases.

5. Rosetta Translation: Magnetic Helicity → Swirl–String Theory Canon

Mapping of variables. In Swirl–String Theory (SST), the correspondence is:

$$
\begin{aligned}
\boldsymbol{B} &\longmapsto \boldsymbol{\omega} && \text{(swirl vorticity field, } [\boldsymbol{\omega}] = \mathrm{s}^{-1}\text{)},\\
\boldsymbol{A} &\longmapsto \boldsymbol{\Psi} && \text{(swirl potential / circulation density, } [\boldsymbol{\Psi}] = \mathrm{m}^2\mathrm{s}^{-1}\text{)},\\
H = \int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V &\longmapsto \mathcal{H}_{\mathrm{swirl}} = \int_\Omega \boldsymbol{\Psi}\cdot\boldsymbol{\omega}\,\mathrm{d}V, &&\\
\mathcal{W}_{\mathrm{per}} &\longmapsto \mathcal{L}_{\circlearrowleft}^{\mathrm{per}} && \text{(periodic swirl–linking density)},\\
\Phi = \int \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} &\longmapsto \Gamma && \text{(circulation quantum, flux of vorticity)}.
\end{aligned}
$$

Interpretation.

- The gauge invariance conditions (Prop. 1) translate to: *Swirl helicity $\mathcal{H}_{swirl}$ is invariant under potential shifts $\boldsymbol{\Psi} \mapsto \boldsymbol{\Psi} + \nabla\chi$ whenever the net circulation across identified boundaries vanishes.*
- The obstruction in $\mathbb{T}^3$ (Prop. 2) becomes: *A global swirl potential $\boldsymbol{\Psi}$ exists iff the mean vorticity $\langle\boldsymbol{\omega}\rangle = 0$. Nonzero bias corresponds to an irreducible swirl-clock offset across the periodic foliation.*
- The periodic winding theorem (Thm. 1) translates to:

  $$\mathcal{H}_{\mathrm{swirl}} = \Gamma^2\,\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}, \tag{7}$$

  where $\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}$ measures the pairwise phase-winding of swirl strings in the universal cover of the periodic domain.

Dimensional check.

$$[\Gamma] = \mathrm{m}^2\,\mathrm{s}^{-1}, \quad [\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}] = 1, \quad [\mathcal{H}_{\mathrm{swirl}}] = \mathrm{m}^4\,\mathrm{s}^{-2},$$

consistent with a conserved quadratic invariant of the swirl field.

Canonical status. Equation (7) is classified as:

$$\boxed{\textbf{Theorem (Rosetta)}}$$

In SST, the conserved swirl helicity $\mathcal{H}_{swirl}$ equals the square of the fundamental circulation quantum Γ multiplied by the periodic swirl-linking density $\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}$.

Physical picture (analogy). Swirl helicity is the "knottedness" of swirl strings. Periodicity forces us to lift the foliation to its universal cover: then, every swirl string traces a path whose *relative winding* with others accumulates across layers. The conserved $\mathcal{H}_{\mathrm{swirl}}$ counts exactly this hidden choreography of interwoven clocks.

5.1 Numerical validation (SST units, SI)

We adopt the rigid-swirl estimate for a single core loop:

$$\Gamma \;=\; \oint \mathbf{v}\cdot d\boldsymbol{\ell} \;\approx\; 2\pi\, r_c\, C_e, \quad [\Gamma] = \mathrm{m}^2\,\mathrm{s}^{-1}.$$

With your constants $C_e = 1.09384563 \times 10^6\,\mathrm{m\,s^{-1}}$, $r_c = 1.40897017 \times 10^{-15}\,\mathrm{m}$, we obtain

$$\Gamma \approx 9.683619203 \times 10^{-9}\ \mathrm{m^2\,s^{-1}}.$$

Setting a representative periodic swirl-linking density $\mathcal{L}_{\circlearrowleft}^{\mathrm{per}} = 1$ (dimensionless), the Rosetta identity

$$\mathcal{H}_{\mathrm{swirl}} \;=\; \Gamma^2\, \mathcal{L}_{\circlearrowleft}^{\mathrm{per}}$$

gives

$$\mathcal{H}_{\mathrm{swirl}} \;\approx\; 9.377248088 \times 10^{-17}\ \mathrm{m^4\,s^{-2}}.$$

Dimensional check. $[\,\Gamma^2\,] = \mathrm{m^4\,s^{-2}}$ and $[\,\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}\,] = 1$, so $[\,\mathcal{H}_{\mathrm{swirl}}\,] = \mathrm{m^4\,s^{-2}}$ as required.

Scaling notes. For N coherently linked cores with identical Γ and pairwise periodic winding density contributing additively, one expects $\mathcal{H}_{\mathrm{swirl}} \sim \Gamma^2 \sum_{i<j} \mathcal{L}_{ij}^{\mathrm{per}}$, showing quadratic growth in the circulation scale and linear growth in the effective pair count.

5.2 Sensitivity scalings

Let $r_c \mapsto \lambda r_c$ and $C_e \mapsto \mu C_e$. Then

$$\Gamma(\lambda,\mu) = 2\pi(\lambda r_c)(\mu C_e) = (\lambda\mu)\,\Gamma_0, \qquad \mathcal{H}_{\mathrm{swirl}}(\lambda,\mu) = \Gamma(\lambda,\mu)^2\,\mathcal{L}_{\circlearrowleft}^{\mathrm{per}} = (\lambda\mu)^2\,\Gamma_0^2\,\mathcal{L}_{\circlearrowleft}^{\mathrm{per}}.$$

Hence $\partial \ln \mathcal{H}_{\mathrm{swirl}}/\partial \ln\lambda = 2$ and likewise for μ; the invariant is *quadratic* in each scale.

Numerics (with $\mathcal{L}_{\circlearrowleft}^{\mathrm{per}} = 1$). Using $C_e = 1.09384563 \times 10^6\,\mathrm{m\,s^{-1}}$, $r_c = 1.40897017 \times 10^{-15}\,\mathrm{m}$, we have

$$\Gamma_0 \approx 9.683619203 \times 10^{-9}\ \mathrm{m^2\,s^{-1}}, \quad \Gamma_0^2 \approx 9.377248088 \times 10^{-17}\ \mathrm{m^4\,s^{-2}}.$$

A grid over $\lambda,\mu \in \{0.1, 0.5, 1, 2, 5, 10\}$ confirms $\mathcal{H}_{\mathrm{swirl}} \propto (\lambda\mu)^2$.

5.3 Multi-core aggregate (uniform pairwise winding)

For N identical cores with uniform pairwise periodic winding $L_{ij} = 1$,

$$\mathcal{H}_{\mathrm{swirl}}^{(\mathrm{total})} = \Gamma_0^2 \sum_{i<j} L_{ij} = \Gamma_0^2 \frac{N(N-1)}{2}.$$

This shows linear growth in the number of interacting pairs and quadratic growth in the circulation scale.

Numerics. For $N = 30$ and $L_{ij} = 1$,

$$\mathcal{H}_{\mathrm{swirl}}^{(\mathrm{total})} = \Gamma_0^2 \frac{30 \cdot 29}{2} \approx 4.078 \times 10^{-14}\ \mathrm{m^4\,s^{-2}}.$$

Generalization (non-uniform topology). If L_{ij} depends on geometry (knot type, relative phase, spacing), replace 1 with the measured/estimated L_{ij}:

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2 \sum_{i<j} L_{ij}.$$

In trefoil or Hopf-linked arrays, L_{ij} can deviate from 1 due to phase slippage across periods; the periodic-winding construction (§2) provides the correct L_{ij} on the universal cover.

6. Corollaries (SST extensions beyond original works)

[Numerical grounding of circulation quantum] With $C_e = 1.09384563 \times 10^6\,\mathrm{m\,s^{-1}}$ and $r_c = 1.40897017 \times 10^{-15}\,\mathrm{m}$, the fundamental circulation quantum is

$$\Gamma_0 = 2\pi r_c C_e \;\approx\; 9.68 \times 10^{-9}\ \mathrm{m^2\,s^{-1}},$$

yielding a base swirl helicity scale $\mathcal{H}_{\text{swirl}} = \Gamma_0^2 \approx 9.38 \times 10^{-17}\ \mathrm{m^4\,s^{-2}}$.

[Quadratic scaling in core radius and swirl speed] Under the rescalings $r_c \mapsto \lambda r_c$, $C_e \mapsto \mu C_e$,

$$\mathcal{H}_{\text{swirl}}(\lambda, \mu) = (\lambda\mu)^2\, \Gamma_0^2\, \mathcal{L}_{\circlearrowleft}^{\text{per}}.$$

Thus helicity grows quadratically with either scale, providing explicit control parameters.

[Multi-core helicity growth] For N identical cores with uniform pairwise periodic swirl-linking $L_{ij} = 1$,

$$\mathcal{H}_{\text{swirl}}^{(\text{total})} = \Gamma_0^2\, \frac{N(N-1)}{2}.$$

This shows quadratic growth in Γ and combinatorial growth in N, predicting rapid amplification in linked-knot arrays or coil bundles.

[Canonical SST theorem status] Equation

$$\mathcal{H}_{\text{swirl}} \;=\; \Gamma^2\, \mathcal{L}_{\circlearrowleft}^{\text{per}}$$

is elevated from an MHD identity to a *Theorem in SST Canon*, classifying swirl helicity as a conserved invariant of foliation dynamics in periodic domains. This provides a direct bridge from topological MHD helicity to the SST energy sector ρ_E and establishes the Rosetta dictionary (potential $\boldsymbol{\Psi}$, vorticity $\boldsymbol{\omega}$, circulation Γ).

[Helicity–energy constitutive law for SST experiments] There exists a constitutive coupling between conserved swirl helicity and the energy density (swirl pressure) of the foliation,

$$\rho_E \;=\; \kappa_H\, \frac{\rho_f}{L_H^2}\, \mathcal{H}_{\text{swirl}}, \qquad \mathcal{H}_{\text{swirl}} \;=\; \Gamma^2\, \mathcal{L}_{\circlearrowleft}^{\text{per}}, \tag{C5}$$

where ρ_f is the effective (fluid) density, κ_H is a dimensionless calibration constant, L_H is a helicity coherence length (the transverse scale over which pairwise periodic winding coherently contributes), and $\mathcal{L}_{\circlearrowleft}^{\text{per}}$ is the periodic swirl-linking density of Sec. 5.

Status. *Constitutive (from Canon + calibration).* The identity $\mathcal{H}_{\text{swirl}} = \Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}$ is Theorem (Rosetta). The proportionality to ρ_E is a phenomenological law constrained by units and validated by experiment/simulation via κ_H and L_H.

Dimensional audit. $[\rho_f] = \mathrm{kg\,m^{-3}}$, $[\mathcal{H}_{\text{swirl}}] = \mathrm{m^4\,s^{-2}}$, $[L_H^{-2}] = \mathrm{m^{-2}}$. Thus $[\rho_E] = \mathrm{kg\,m^{-1}\,s^{-2}} = \mathrm{J\,m^{-3}}$ (a pressure), as required.

Scalings and limits. Using $\Gamma = 2\pi r_c C_e$ and (C5),

$$\rho_E = \kappa_H \frac{\rho_f}{L_H^2} (2\pi r_c C_e)^2 \mathcal{L}_{\circlearrowleft}^{\text{per}} \;\propto\; \rho_f \frac{r_c^2 C_e^2}{L_H^2} \mathcal{L}_{\circlearrowleft}^{\text{per}}.$$

Hence $\partial \ln \rho_E / \partial \ln r_c = 2$, $\partial \ln \rho_E / \partial \ln C_e = 2$, $\partial \ln \rho_E / \partial \ln L_H = -2$, and ρ_E grows linearly with $\mathcal{L}_{\circlearrowleft}^{\text{per}}$. For N identical cores with uniform $L_{ij} = 1$, $\mathcal{L}_{\circlearrowleft}^{\text{per}} \sim N(N-1)/2$ (see Sec. 5.3), giving rapid amplification in linked arrays.

Numerical illustration (with your constants). Let $\rho_f = 7.0 \times 10^{-7}\,\mathrm{kg\,m^{-3}}$, $r_c = 1.40897017 \times 10^{-15}$ m, $C_e = 1.09384563 \times 10^6\ \mathrm{m\,s^{-1}}$. Then $\Gamma_0 \approx 9.683619203 \times 10^{-9}\ \mathrm{m^2\,s^{-1}}$ and $\Gamma_0^2 \approx 9.377248088 \times 10^{-17}\ \mathrm{m^4\,s^{-2}}$. For a single-core, unit linking density ($\mathcal{L}_{\circlearrowleft}^{\text{per}} = 1$),

$$\rho_E \;=\; \kappa_H \frac{7.0 \times 10^{-7}}{L_H^2} \left(9.377248088 \times 10^{-17}\right)\ \mathrm{J\,m^{-3}}.$$

Example values:

$$\begin{aligned}
L_H = 10^{-2}\ \mathrm{m}: &\quad \rho_E \approx \kappa_H\, 6.56 \times 10^{-19}\ \mathrm{J\,m^{-3}},\\
L_H = 10^{-4}\ \mathrm{m}: &\quad \rho_E \approx \kappa_H\, 6.56 \times 10^{-15}\ \mathrm{J\,m^{-3}},\\
L_H = 10^{-6}\ \mathrm{m}: &\quad \rho_E \approx \kappa_H\, 6.56 \times 10^{-11}\ \mathrm{J\,m^{-3}}.
\end{aligned}$$

For $N = 30$ with uniform $L_{ij} = 1$ ($\mathcal{L}_{\circlearrowleft}^{\text{per}} = 435$), multiply these by 435.

From energy density to thrust. In the Euler–SST sector, swirl pressure equals energy density: $p_{\text{swirl}} = \rho_E$. A directed pressure gradient produces force $F \approx \Delta p\, A = \Delta \rho_E\, A$ on area A. Thus asymmetric control of L_H and/or $\mathcal{L}_{\circlearrowleft}^{\text{per}}$ (e.g. phase-biased linking across the device) yields a net thrust:

$$F \;\approx\; \kappa_H \frac{\rho_f}{L_H^2}\, \Delta\!\left(\Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}}\right) A.$$

Calibration protocol (minimal experiment).

1. Build a multi-core array with controllable pairwise phase to tune $\mathcal{L}_{\circlearrowleft}^{\text{per}}$.
2. Fix (r_c, C_e) drive; vary N and the phase program to scan $\mathcal{L}_{\circlearrowleft}^{\text{per}}$.
3. Measure net force F on a thrust stand while modulating a boundary layer to set L_H (e.g. dielectric spacing or flow-ring spacing).
4. Fit κ_H and L_H via $F \approx (\rho_f / L_H^2)\, \kappa_H\, \Delta(\Gamma^2 \mathcal{L}_{\circlearrowleft}^{\text{per}})\, A$.

This closes the loop from the conserved topological invariant to an experimentally measurable propulsion observable.

References