Rotational Kinetic Energy Density and an Effective Mass Relation in Incompressible Fluids

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Kinetic energy contributes to inertia and gravitational mass through the relativistic relation $E = mc^2$. In extended media such as fluids, this contribution can be expressed as an effective mass density associated with internal motion. I compute the volume-averaged rotational kinetic energy density for an incompressible, inviscid Newtonian fluid in rigid-body rotation inside a finite cylinder. Associating this energy density with an effective mass density via $E = mc^2$ (nonrelativistic limit) yields the closed-form relation

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2,$$

where v_{edge} is the tangential speed at the cylinder boundary and ρ is the rest-mass density. The result provides a transparent classical example of how rotational motion modifies the mass density at order $(v/c)^2$. We discuss the connection with relativistic continuum mechanics and provide numerical estimates for laboratory and astrophysical regimes.

I. INTRODUCTION

The equivalence between energy and mass, expressed by $E = mc^2$, implies that kinetic, field, and binding energies contribute to the inertia and gravitational mass of extended systems [1–3]. In the context of relativistic continuum mechanics, this statement is encoded in the stress–energy tensor $T^{\mu\nu}$: the total mass–energy is obtained by integrating T^{00} over a spatial hypersurface, and T^{00} includes both rest-mass and kinetic contributions [3, 4].

Although standard in high-energy and gravitational physics, an explicit fluid-mechanics example remains pedagogically useful. Here I make the rotational contribution to an effective mass density explicit in an analytically tractable flow.

In this paper we analyze a canonical configuration from classical fluid mechanics [5, 6]: an incompressible, inviscid fluid in rigid-body rotation inside a finite cylinder. Within this model we:

1. compute the local and volume-averaged rotational kinetic energy density;
2. define an effective mass density $\Delta\rho_{\text{eff}}$ via $E = mc^2$ in the regime $v \ll c$;
3. derive a compact expression for $\Delta\rho_{\text{eff}}/\rho$ in terms of the edge speed v_{edge};
4. discuss how this classical result fits within the framework of relativistic continuum mechanics.

The derivation uses only incompressible Euler flow plus the special-relativistic mass–energy relation. No modifications of Newtonian or relativistic theory are proposed.

II. RIGID-BODY ROTATION IN AN INCOMPRESSIBLE, INVISCID FLUID

A. Flow configuration

We consider a Newtonian fluid of constant rest-mass density ρ occupying a right circular cylinder of radius R and height L. The fluid undergoes steady rigid-body rotation with constant angular velocity Ω about the z-axis. In cylindrical coordinates (r, θ, z), with $0 \le r \le R$, the velocity field is

$$\boldsymbol{v}(r) = \Omega r\,\hat{\boldsymbol{\theta}}. \tag{1}$$

This flow is incompressible and inviscid:

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \text{viscosity} = 0, \tag{2}$$

and it satisfies the steady Euler equations with an appropriate pressure distribution [5, 6].

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B. Local kinetic energy density

The local kinetic energy density of the fluid is

$$e_{\text{kin}}(r) = \frac{1}{2}\,\rho\|\boldsymbol{v}(r)\|^2 = \frac{1}{2}\,\rho\,\Omega^2 r^2. \tag{3}$$

This quantity is position-dependent and increases quadratically with radius.

C. Total rotational energy and volume-averaged energy density

The total rotational kinetic energy is obtained by integrating Eq. (3) over the fluid volume:

$$\begin{aligned}
E_{\text{rot}} &= \int_V e_{\text{kin}}\,\mathrm{d}V \\
&= \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{2}\,\rho\,\Omega^2 r^2\, r\,\mathrm{d}r \\
&= \frac{1}{2}\,\rho\,\Omega^2 (2\pi L) \int_0^R r^3\,\mathrm{d}r \\
&= \frac{\pi}{4}\,\rho\,\Omega^2\, L\, R^4.
\end{aligned} \tag{4}$$

The cylinder volume is $V = \pi R^2 L$, so the volume-averaged kinetic energy density is

$$\langle e_{\text{kin}} \rangle = \frac{E_{\text{rot}}}{V} = \frac{\frac{\pi}{4}\,\rho\,\Omega^2\,L\,R^4}{\pi R^2 L} = \frac{1}{4}\,\rho\,\Omega^2 R^2. \tag{5}$$

Define the edge speed

$$v_{\text{edge}} := \Omega R. \tag{6}$$

Then Eq. (5) becomes

$$\langle e_{\text{kin}} \rangle = \frac{1}{4}\,\rho\, v_{\text{edge}}^2. \tag{7}$$

At the boundary

$$e_{\text{kin}}(R) = \frac{1}{2}\,\rho\, v_{\text{edge}}^2, \tag{8}$$

the volume average is half of that, as expected from the quadratic radial profile.

III. EFFECTIVE MASS DENSITY FROM $E = mc^2$

A. Nonrelativistic limit and effective density

In special relativity, the total relativistic energy of a fluid element with rest-mass density ρ and small velocity $v \ll c$ can be decomposed as [3, 4]

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \cdots, \tag{9}$$

where ε is the total energy density in the local rest frame, and the ellipsis denotes higher-order terms in v^2/c^2 and internal energy contributions. To leading order in v^2/c^2, the kinetic energy density acts as an *effective mass density*,

$$\Delta\rho_{\text{eff}}(\boldsymbol{x}) = \frac{e_{\text{kin}}(\boldsymbol{x})}{c^2}. \tag{10}$$

This interpretation is consistent with the structure of the stress–energy tensor for a perfect fluid [3].

For the rigidly rotating configuration considered here, we focus on the volume-averaged effective mass density

$$\Delta\rho_{\text{eff}} := \frac{\langle e_{\text{kin}} \rangle}{c^2}. \tag{11}$$

Inserting Eq. (7) into Eq. (11), we obtain

$$\Delta\rho_{\text{eff}} = \frac{1}{4c^2}\,\rho\, v_{\text{edge}}^2. \tag{12}$$

Dividing by the rest-mass density ρ yields

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{v_{\text{edge}}}{c}\right)^2. \tag{13}$$

Thus, in the nonrelativistic regime the rotational contribution scales as $(v_{\text{edge}}/c)^2$ with geometry-specific coefficient $1/4$ for a rigidly rotating cylinder.

B. Dimensional check

The dimensions of Eq. (12) are

$$[\Delta\rho_{\text{eff}}] = \frac{[\rho][v]^2}{[c]^2} = \frac{\text{kg}\,\text{m}^{-3}\,(\text{m/s})^2}{(\text{m/s})^2} = \text{kg}\,\text{m}^{-3},$$

as required. The ratio $\Delta\rho_{\text{eff}}/\rho$ in Eq. (13) is dimensionless.

IV. NUMERICAL ESTIMATES

A. Laboratory-scale example

Consider water with $\rho \approx 1.0 \times 10^3\,\text{kg}\,\text{m}^{-3}$, a cylinder of radius $R = 0.10\,\text{m}$, and angular velocity $\Omega = 1.0 \times 10^3\,\text{s}^{-1}$, corresponding to $v_{\text{edge}} = 100\,\text{m}\,\text{s}^{-1}$. Then

$$\frac{\Delta\rho_{\text{eff}}}{\rho} = \frac{1}{4}\left(\frac{100}{3.0 \times 10^8}\right)^2 \approx 3 \times 10^{-14}, \tag{14}$$

and

$$\Delta\rho_{\text{eff}} \sim 3 \times 10^{-11}\,\text{kg}\,\text{m}^{-3}. \tag{15}$$

The effect ($\sim 3 \times 10^{-11}\,\text{kg}\,\text{m}^{-3}$) is far below laboratory density resolution.

B. Astrophysical order-of-magnitude

In astrophysical settings, rotational velocities can be relativistic. For example, in the inner regions of accretion flows or rapidly rotating compact stars, characteristic speeds may reach $v \sim 0.1c$ or higher [4]. If one naively substitutes $v_{\text{edge}} = 0.1c$ into Eq. (13), one finds

$$\frac{\Delta\rho_{\text{eff}}}{\rho} \sim \frac{1}{4}(0.1)^2 = 2.5 \times 10^{-3}, \tag{16}$$

already approaching the percent level. However, in such regimes a fully relativistic treatment of the fluid is required, and higher-order terms in v^2/c^2 as well as strong-gravity effects must be included. Equation (13) should therefore be regarded as illustrating the leading-order trend rather than providing a quantitatively accurate model for relativistic flows.

V. RELATION TO RELATIVISTIC CONTINUUM MECHANICS

In relativistic hydrodynamics, a perfect fluid is described by the stress–energy tensor [3, 4]

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{17}$$

where ε is the total energy density in the fluid rest frame, p is the pressure, and u^μ is the four-velocity. In the nonrelativistic limit and for small internal energy, one has

$$\varepsilon \simeq \rho c^2 + \frac{1}{2}\rho v^2 + \dots, \tag{18}$$

consistent with the decomposition used in Eq. (10).

The contribution of kinetic energy to the gravitational mass of an extended system can be derived by integrating T^{00} over space in an appropriate frame [2, 3]. This calculation isolates the rotational contribution for rigid-body rotation of an incompressible fluid. Equation (13) is the nonrelativistic limit of the rotational part of T^{00}/c^2 in this geometry.

VI. DISCUSSION AND OUTLOOK

We have derived a compact relation between the rotational kinetic energy of an incompressible, inviscid fluid in rigid-body rotation and an associated effective mass density. The key steps are:

1. computation of the local kinetic energy density $e_{\text{kin}}(r) = \frac{1}{2}\rho\Omega^2 r^2$;
2. volume averaging over a finite cylinder, yielding $\langle e_{\text{kin}}\rangle = \frac{1}{4}\rho v_{\text{edge}}^2$;
3. definition of an effective mass density via $E = mc^2$ in the nonrelativistic limit, resulting in $\Delta\rho_{\text{eff}}/\rho = \frac{1}{4}(v_{\text{edge}}/c)^2$.

The analysis stays within classical incompressible flow and special relativity. It provides a transparent example of how rotational motion contributes to the mass density of an extended medium at order $(v/c)^2$. The coefficient 1/4 is specific to rigid-body rotation in a cylinder and reflects the radial structure of the velocity field. For other velocity profiles or geometries, different geometric factors would appear, though the basic scaling $\Delta\rho_{\text{eff}}/\rho \propto \langle v^2\rangle/c^2$ remains.

Potential applications include:

- pedagogical demonstrations of mass–energy equivalence in continuum systems;
- benchmark problems for numerical schemes that couple incompressible fluid dynamics to relativistic mass–energy accounting in the low-velocity regime;
- conceptual comparisons with fully relativistic treatments of rotating fluids in astrophysical contexts.

As a complementary single-particle example, Appendix A summarizes a semiclassical model in which a photon confined on a toroidal path reproduces the electron charge to within about ten per cent, using only classical electromagnetism and the Compton wavelength as input [7].

Any attempt to attribute fundamental rest mass to internal rotational motion would require additional structural assumptions and a fully relativistic framework, and lies beyond the scope of the present work.

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APPENDIX A: SEMI-CLASSICAL CHARGE ESTIMATE FROM A CONFINED ELECTROMAGNETIC MODE

In the main text we considered how rotational kinetic energy in an incompressible fluid contributes to an effective mass density via the relation $E = mc^2$ in the nonrelativistic regime. For completeness, we record here a simple semiclassical construction, due to Williamson and van der Mark [7], which shows how a confined electromagnetic mode can also reproduce the observed magnitude of the electron charge using only classical electromagnetism and the Compton wavelength.

Consider a photon of wavelength λ whose energy

$$E_\gamma = \frac{hc}{\lambda}$$

is confined to a finite volume V of characteristic size comparable to λ. In the model of Ref. [7], the photon is confined on a toroidal path of length one Compton wavelength $\lambda_C = h/(m_e c)$, with a characteristic radius

$$r = \frac{\lambda_C}{4\pi},$$

so that the confinement volume is of order $V \sim r^3$. Equating the photon energy to the integral of the electromagnetic energy density over this volume leads to an average electric-field amplitude of the form

$$\langle E \rangle = \sqrt{\frac{6hc}{\pi \varepsilon_0 \lambda_C^4}}, \tag{19}$$

where ε_0 is the vacuum permittivity.[1]

To relate this confined field to an effective point charge, one compares $\langle E \rangle$ to the magnitude of the Coulomb field at a radius r,

$$E_{\mathrm{C}}(r) = \frac{q}{4\pi\varepsilon_0 r^2}. \tag{20}$$

Identifying $r = \lambda_C/(4\pi)$ as above and setting $E_{\mathrm{C}}(r) = \langle E \rangle$ yields

$$q = 4\pi\varepsilon_0 r^2 \langle E \rangle = 4\pi\varepsilon_0 \left(\frac{\lambda_C}{4\pi}\right)^2 \sqrt{\frac{6hc}{\pi\varepsilon_0\lambda_C^4}}. \tag{21}$$

Using $\lambda_C = h/(m_e c)$ and $h = 2\pi\hbar$, one finds that the dependence on λ_C and m_e cancels, giving the closed-form expression

$$q_{\mathrm{model}} = \frac{1}{2\pi}\sqrt{3\varepsilon_0 \hbar c}. \tag{22}$$

Numerically,

$$q_{\mathrm{model}} = \frac{1}{2\pi}\sqrt{3\varepsilon_0\hbar c} \simeq 1.46 \times 10^{-19}\,\mathrm{C} \simeq 0.91\,e,$$

so that this simple confinement picture reproduces the observed elementary charge $e \simeq 1.60 \times 10^{-19}$ C at the ten-percent level.

a. Intuition: compress a single-wavelength loop of light into a small torus; the surface field approaches that of a point charge.

This appendix does not advocate a microscopic electron model; it illustrates that, even within classical electromagnetism, geometric confinement of a single-wavelength mode can generate particle-like mass and charge scales. This complements the continuum calculation in the main text, where rotational kinetic energy in an extended medium leads to an effective mass density proportional to $\langle v^2 \rangle / c^2$.

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[1] The numerical factor depends on the detailed choice of confinement volume and field profile; Eq. (19) corresponds to the specific toroidal geometry considered in [7].