Analysis II
Summary

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# Chapter 1

# Ordinary differential equations

## 1.1 Differential Equation:

An equation for a function f that relates the values of f at x, f(x) to the values of its derivatives at the same point x. We distinguish between the number of variables present in the function:

- **One variable:** Ordinary differential equations (ODE)
- **Several Variables:** Partial differential equations (PDE)

Examples:

- $f'(x) = f(x)$
- $f''(x) = -f(x)$

Notation: We write $y, y', y'', y^{(3)}, ...$ instead of $f(x), f'(x), f''(x), f^{(3)}(x)$

**Order:** The largest derivative present in the equation. Examples:

- $y' = 2xy$ order 1
- $y^{(3)} + 2xy'' + e^x y + 1 = 0$ order 3

The solution to an ODE is not unique in general. When given initial conditions then we can find unique solutions. E.g:

$$y' = x + 1$$
$$y = \frac{x^2}{2} + x + c$$

is a solution for any c. If we are also given y(0) = 1 then c= 1 is a unique solution.

## 1.2 Linear Differential equations

A linear ODE of order k on an interval $I \subset \mathbb{R}$ is an eqn of the form:

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where a(x) and b(x) are continuous functions from $I$ to $\mathbb{C}$.
For a linear ODE the following hold:

- y and all its derivatives appear in order 1
- there are no products of the function y and its derivatives
- neither the function nor its derivatives are inside another function e.g $\sqrt{y}$, sin(y),...

If b = 0 then we say the equation is **homogeneous** otherwise **inhomogeneous**

Solving a linear ODE means finding all functions $f : I \to \mathbb{C}$ that are k times differentiable such that $\forall x \in I$ the function satisfies the differentiable equation.

**Initial Condition** A set of equations specifying the values of the derivatives at some initial point.

**Theorem 2.2.3** Let $I \subset \mathbb{R}$ and open interval $k \geq 1$ and integer. Consider the linear ODE

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where coefs $a_i(x), b(x)$ are continous functions

1. Let $S_0$ be the set of solutions for b=0, then $S_0$ is a vector space of dimension k.
2. For any initial conditions, i.e for any choice of $x_0 \in I$ and $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$ there is a unique solution $f \in S$ such that $f_(x_0) = y_0, ...f^{(k)}(x_0) = y_k$
3. For an arbitrary b the set of solutions of the linear ODE is $S_b = \{f + f_p | f \in S_0\}$ where $f_p$ is one **particular** solution
4. For any initial condition there is a unique solution.

The linearity of the diff equation alsos implies a **superposition** principle. Suppose we have 2 different functions $b_1(x), b_2(x)$ on the RHS with solutions $f_1, f_2 : Df_1 = b_1, Df_2 = b_2$ then $f_1 + f_2$ solves $Df = b_1 + b_2$

Given a diff eqn and a possible solution we can always verify whether it is indeed a solution or not.

## 1.3 Linear differential equations of order 1

We consider y'+ay = b, where a,b are continous functions. 2 steps:

- Find solutions of the corresponding homogeneous equation y' + ay = 0.
- Find a particular solution $f_p : I \to \mathbb{C}$ such that $f_p + af_p = b$

If f is a solution then so is zf for any constant $z \in \mathbb{C}$

Homogeneous solution: $y' + ay = 0$
$\Rightarrow y' = -ay$
$\Rightarrow \frac{y'}{y} = a$
$\Rightarrow \int \frac{y'(x)}{y(x)} dx = -\int a(x)dx := A(x)$
$\Rightarrow ln|y(x)| = -A(x) + c$
$\Rightarrow y = z \cdot e^{-A(x)}$ for some constant z

Solution of inhomogeneous equation $y' + ay = b$
There are two methods to solve this:

- Educated guess: the LHS tries to imitate the RHS i.e if b(x) is a polynomial we guess that $f_p$ is also a polynomial or if b is a trig function then we guess $f_p$ is also a trig function
- Variation of constants: Assume

  $$f_p = z(x)e^{-A(x)}$$

  for some function $z : I \to \mathbb{C}$. We then put this into the equation and see what it forces z(x) to satisfy