Prof. Dr. Emmanuel Kowalski January 23, 2018

Analysis II
Müsterlösung
D-INFK

Multiple choice questions [6 Points]

(1) If f_1 and f_2 are solutions of the differential equation

$$y'' - xy' + y = \cos(x),$$

then so is $f_1 + 2f_2$.

True ☐ False ☑

(2) If f is C^2 on $\mathbb{R}^2$ and f is maximal at (x_0, y_0), then $\dfrac{\partial^2 f}{\partial x^2}(x, y) = 0$.

True ☐ False ☑

(3) Let $f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$ and $g(u, v, w)$ be differentiable functions $\mathbb{R}^2 \to \mathbb{R}^3$ and $\mathbb{R}^3 \to \mathbb{R}$ respectively. We have

$$\frac{\partial(g \circ f)}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial f_2}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial f_3}{\partial z}.$$

True ☐ False ☑

(4) If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^2, $\nabla f(0, 0, 0) = 0$, and the Hessian matrix of f at $(0, 0, 0)$ is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

then f has at $(0, 0, 0)$

A local minimum ☐ A local maximum ☐ A saddle point ☑

(5) If $f = (f_1, f_2)$ is a conservative C^1 vector field on $\mathbb{R}^2$, then

$$\partial_x f_1 = \partial_y f_2.$$

True ☐ False ☑

(6) For a continuous function f in $\mathbb{R}^2$, we have

$$\int_{[0,2]\times[0,3]} f(x,y)dxdy = 6\int_0^1\int_0^1 f(2x,3y)dxdy.$$

True ☑ False ☐

Exercises

Quick computation 1 [2 Points]

Compute the Hessian of $f(x,y) = \arctan(x^2+y)$ at $(x,y) = (0,0)$. No justification is necessary.

Solution.

As $\arctan(t) = t + O(t^2)$, we get

$$\arctan(x^2+y) = x^2 + y + O((x^2+y^2)^{\frac{3}{2}})$$

Therefore, we have

$$\mathrm{Hess}_{(0,0)} f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Quick computation 2 [2 Points]

For which values of $a \in \mathbb{R}$ is the vector field $f(x,y) = (ay^2e^{2xy^2}, a^2xye^{2xy^2})$ on $\mathbb{R}^2$ conservative? No justification is necessary.

Solution. If $f = (f_1, f_2)$, we have

$$\partial_y f_1(x,y) = (2ay + ay^2 \times 4xy)e^{2xy^2} = 2ay(1+2xy^2)e^{2xy^2}$$
$$\partial_x f_2(x,y) = (a^2y + 2a^2xy^3)e^{2xy^2} = a^2y(1+2xy^2)e^{2xy^2}$$

Therefore, $\partial_y f_1 = \partial_x f_2$ if and only if (taking $x = 0$ and $y = 1$ for example) $2a = a^2$. Therefore the solutions are $a = 0$ and $a = 2$, as $\mathbb{R}^2$ is a starred domain.

Quick computation 3 [2 Points]

Find all *real* solutions of the differential equation

$$u'' - 6u' + 13u = (3x+2)e^x.$$

No justification is necessary.

Solution. As $X^2 - 6X + 13 = (X - (3+2i))(X - (3-2i))$, the real solutions of the homogeneous equation are

$$u(x) = \lambda_1 e^{3x}\cos(2x) + \lambda_2 e^{3x}\sin(2x); \quad \text{with } \lambda_1, \lambda_2 \in \mathbb{R}$$

Then let $u_0(x) = (ax+b)e^x$ be a particular solution to the ODE. Then we compute

$$\begin{aligned} u_0'' - 6u_0' + 13u &= (ax + (2a+b))e^x - 6(ax + (a+b))e^x + 13(ax+b)e^x \\ &= 4(2ax - a + 2b)e^x = (3x+2)e^x \end{aligned}$$

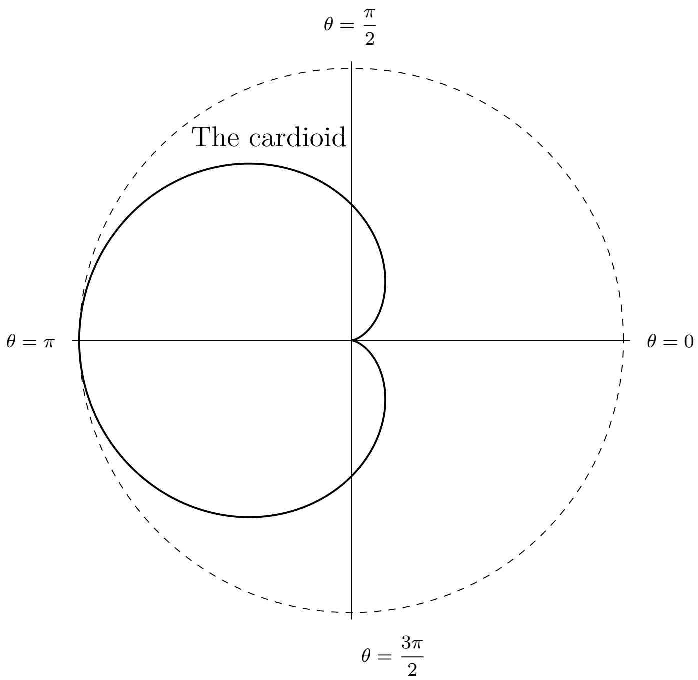
if and only if $a = \dfrac{3}{8}$ and $b = \dfrac{7}{16}$. Therefore, the real solutions of the ODE are

$$u(x) = (\lambda_1\cos(x) + \lambda_2\sin(x))\,e^{3x} + \frac{1}{8}\left(3x + \frac{7}{2}\right)e^x; \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Exercise 4 [2 Points]

Fix $R > 0$. Compute the area of the *compact* region delimited by the cardioid, define by the following parametric equation

$$\gamma(\theta) = \left(R\left(2\cos(\theta) - \cos(2\theta)\right), R\left(2\sin(\theta) - \sin(2\theta)\right)\right) \quad 0 \le \theta \le 2\pi.$$



Solution. The area is by Green's formula and the duplication formulas (and the periodicity)

$$\begin{aligned}
A &= \int_\gamma (-y,0)\cdot d\vec{s} = 2R^2\int_0^{2\pi}(2\sin(\theta)-\sin(2\theta))(\sin(\theta)-\sin(2\theta))d\theta \\
&= 2R^2\int_0^{2\pi}(2\sin^2(\theta)-3\sin(\theta)\sin(2\theta)+\sin^2(2\theta))d\theta \\
&= 2R^2\int_0^{2\pi}\left(1-\cos(2\theta)-6\cos(\theta)\sin^2(\theta)+\frac{1}{2}(1-\cos(4\theta))\right)d\theta \\
&= 2R^2\left(\frac{3}{2}\times 2\pi+\left[-2\sin^3(\theta)\right]_0^{2\pi}\right) = 6\pi R^2.
\end{aligned}$$

Exercise 5 [3 Points]

(a) Show that for all $x\in\mathbb{R}$, we have

$$\sin^3(x) = \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x).$$

(b) Compute the integral

$$\int_D (xy+y^3)dxdy$$

where D is the quarter-disc

$$D = \{(x,y)\in\mathbb{R}^2\ ;\ x\geq 0,\ y\geq 0,\ x^2+y^2\leq 2\}.$$

Solution.

(a) We have

$$\begin{aligned}
\sin^3(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 = \frac{e^{3ix}-e^{-3ix}-3e^{ix}+3e^{-ix}}{-8i} \\
&= \frac{1}{4}\left(-\frac{e^{3ix}-e^{-3ix}}{2i}+3\frac{e^{ix}-e^{-ix}}{2i}\right) \\
&= \frac{1}{4}\left(-\sin(3x)+\sin(x)\right) \\
&= \frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x).
\end{aligned}$$

(b) Using polar coordinates (noticing that $x\geq 0$ and $y\geq 0$)

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta)\end{cases} \qquad 0\leq r\leq\sqrt{2}\ \ 0\leq\theta\leq\frac{\pi}{2}$$

we find by Fubini's theorem, (a)

$$\begin{aligned}
&\int_D \left(xy+y^3\right) dxdy = \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}} (r^2\cos(\theta)\sin(\theta)+r^3\sin^3(\theta))rdrd\theta \\
&= \left(\int_0^{\sqrt{2}} r^3dr\right)\left(\int_0^{\frac{\pi}{2}}\cos(\theta)\sin(\theta)d\theta\right)+\left(\int_0^{\sqrt{2}} r^4dr\right)\left(\int_0^{\frac{\pi}{2}}\frac{1}{4}\left(3\sin(\theta)-\sin(3\theta)\right)d\theta\right) \\
&= \left[\frac{r^4}{4}\right]_0^{\sqrt{2}}\left[\frac{1}{2}\sin^2(\theta)\right]_0^{\frac{\pi}{2}}+\left[\frac{r^5}{5}\right]_0^{\sqrt{2}}\left[-\frac{3}{4}\cos(\theta)+\frac{1}{12}\cos(3\theta)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}+\frac{4\sqrt{2}}{5}\times\left(\frac{3}{4}-\frac{1}{12}\right) \\
&= \frac{1}{2}+\frac{8\sqrt{2}}{15}.
\end{aligned}$$

Exercise 6 [3 Points]

Compute the Taylor polynomial of order 2 of

$$f(x,y,z)=2\exp(x+y^2+z^3)$$

at $(x,y,z)=(0,0,0)$, and the Hessian matrix.

Solution. As $e^t=1+t+\dfrac{t^2}{2}+O(t^3)$, we find

$$\begin{aligned}
f(x,y,z)&=2\left(1+x+y^2+z^3+\frac{1}{2}\left(x+y^2+z^3\right)^2+O(|(x,y,z)|^3)\right) \\
&=2+2x+x^2+2y^2+O(|(x,y,z)|^3)
\end{aligned}$$

Therefore, the Taylor polynomial of f at order 2 at $(0,0,0)$ is given by

$$T_2f(x,y,z,(0,0,0))=2+2x+x^2+2y^2$$

and the Hessian matrix is

$$\mathrm{Hess}_{(0,0,0)}f=\begin{pmatrix}2&0&0\\0&4&0\\0&0&0\end{pmatrix}.$$

Exercise 7 [3 Points]

Let $f(x,y)=\exp(\sin(xy))$. Find the values of $(x,y)\in\mathbb{R}^2$ such that f has a critical point at (x,y).

Solution. We have

$$\nabla f(x,y)=(y,x)\cos(xy)e^{\sin(xy)}.$$

As exp does not vanish, we deduce that $\nabla f(x,y) = 0$ is and only if $(x,y) = (0,0)$ or $\cos(xy) = 0$. As $\cos(0) = 1$, these two alternatives are disjoint. Therefore, $\nabla f(x,y) = 0$ for some $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ if and only if $\cos(xy) = 0$, or

$$xy = \frac{\pi}{2} + k\pi \quad \text{for some } \ k \in \mathbb{Z}.$$

In particular, as $xy \neq 0$, this implies that

$$y = \frac{\frac{\pi}{2} + k\pi}{x}.$$

Finally, the set of critical points of f is

$$(0,0) \cup \left(\mathbb{R}^* \times \mathbb{R}^* \cap \left\{ (x,y) : y = \frac{\frac{\pi}{2} + k\pi}{x} \ \text{ for some } \ k \in \mathbb{Z} \right\} \right).$$

Exercise 8 [2 Points]

Let $f(x,y,z) = x^2 + \cos(y)e^{z^2}$.

(a) Find the critical points of f on $]-1,1[^3$.

(b) Determine whether they are local maxima, minima or saddle points.

Solution.

(a) We have

$$\nabla f(x,y,z) = \left(2x, -\sin(y)e^{z^2}, 2z\cos(y)e^{z^2}\right) = 0$$

if and only if $x = 0$, $\sin(y) = 0$, and $z\cos(y) = 0$. Now, as $\cos^2 + \sin^2 = 1$, $\sin(y) = 0$ implies that $\cos(y) = \pm 1$, so that $z = 0$. Finally, as $\sin(y) = 0$ holds if and only if $y = k\pi$ for some $k \in \mathbb{Z}$ and $1 < \pi$, we deduce that $(x,y,z) = (0,0,0)$ is the only critical point of f on $]-1,1[^3$.

(b) One can either apply the direct criterion with the second derivative, as

$$\mathrm{Hess}_{(0,0,0)} f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

but the following argument is also valid. For all $x \neq 0$, we have

$$f(x,0,0) = 1 + x^2 > 1 = f(0,0,0).$$

For all $y \in -]-1,1[\setminus\{0\}$, we have $\cos(y) < 1$ (using here that $2\pi > 1$). We deduce that for all $y \in -]-1,1[\setminus\{0\}$, we have

$$f(0,y,0) = \cos(y) < 1 = f(0,0,0).$$

Therefore, $(0,0,0)$ is a saddle point.

Exercise 9 [5 Points]

(a) Check that the vector-field on $\mathbb{R}^2$

$$f(x,y) = (6x^5y^2 - 4xy^4 - 7y + 6, 2x^6y - 8x^2y^3 - 7x)$$

is conservative.

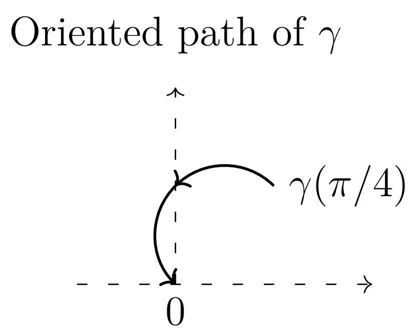
(b) Compute a potential of f.

(c) Compute

$$\int_\gamma f \cdot d\vec{s},$$

where γ is the parametrised curve

$$\begin{cases} \gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \mathbb{R}^2 \\ \theta \mapsto \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(\theta), \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\theta)\right). \end{cases}$$



Solution.

(a) The function f is smooth and writing $f = (f_1, f_2)$, we get

$$\partial_y f_1(x,y) = \partial_x f_2(x,y) = 12x^5y - 16xy^3 - 7$$

Therefore, as $\partial_y f_1 = \partial_x f_2$ and $\mathbb{R}^2$ is a starred domain, the vector-field f is conservative.

(b) If $f = \nabla\varphi$ for some function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, and integrating f_1 in x, we find

$$\varphi_1(x,y) = x^6y^2 - 2x^2y^4 - 7xy + 6x + \eta_1(y)$$

for some function $\eta_1 : \mathbb{R} \to \mathbb{R}$. Likewise, integrating f_2 in y yields

$$\varphi(x,y) = x^6y^2 - 2x^2y^4 - 7xy + \eta_2(x)$$

for some function $\eta_2 : \mathbb{R} \to \mathbb{R}$. Therefore, $\eta_1 = 0$ and $\eta_2(x) = 6x$ is a solution and we find that

$$\begin{aligned} &\varphi : \mathbb{R}^2 \to \mathbb{R} \\ &(x,y) \mapsto x^6y^2 - 2x^2y^4 - 7xy + 6x \end{aligned}$$

is a potential of f.

(c) As f is conservative, the integral does depends only on the end points of γ. Furthermore, as we computed a potential of f, we have

$$\begin{aligned} \int_\gamma f \cdot d\vec{s} &= \varphi\left(\gamma\left(\frac{5\pi}{2}\right)\right) - \varphi\left(\gamma\left(\frac{\pi}{4}\right)\right) \\ &= \varphi(0,0) - \varphi(1,1) = -(1-2-7+6) \\ &= 2. \end{aligned}$$