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ETH MADE EASY

Computer Science
Block 3

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Analysis II - Lecture

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Contents

The following was presented in the lecture on 19. September 2019.

2 Differential Equations

Theorem 2.1

An ordinary differential equation has always a solution, at least near the starting time.

2.1 Linear Ordinary Differential Equation

Definition 1. A linear ordinary differential equation of order k has the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b \tag{1}$$

where $b, a_0, \ldots, a_{k-1}$ are continuous functions from I to $\mathbb{R}$ or $\mathbb{C}$.
If $b = 0$ we say that the equation is homogeneous, otherwise inhomogeneous.

Theorem 2.2: 2.2

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b \tag{2}$$

The set S_0 of solutions when $b = 0$ form a vector space of dimension k. For any initial condition, there is a unique solution.
For any b, the set S_b of solutions is

$$S_b = \{f + f_0 \mid f \in S_0\} \quad \text{with } f_0 \in S_b \tag{3}$$

The following was presented in the lecture on 26. September 2019.

Example 1. How to solve equations like this:

$$y' + ay = b \tag{4}$$

Step 1: Solve Homogeneous equation $y' + ay = 0 \Rightarrow \ln(|y|)' = -a \Rightarrow y = z \cdot \exp(-A(x))$ for some $z \in \mathbb{C}$
Step 2: Solve the Inhomogeneous equation $y' + ay = b$ one can either make a guess or do "Variation of constants":

$$f_0(x) = z(x) \cdot \exp(-A(x)) \tag{5}$$

$$z'(x) = b(x) \cdot \exp(A(x)) \tag{6}$$

$$f_0(x) = \left(\int_{x_0}^{x} b(t) \cdot \exp(A(t))dt\right) \cdot \exp(-A(x)) \tag{7}$$

Another trick is to split b into $b_1 + b_2$ make a guess for one of them and solve with variation of constants.

The following was presented in the exercise on 30. September 2019.

2.2 Separation der Variablen

Idea: Separate x and y

$$2yy' + y' = x^2y^2 + x^2y \quad (8)$$

$$y'(2y+1) = x^2y(y+1) \quad (9)$$

$$\frac{dy}{dx}(2y+1) = x^2(y^2+y) \quad (10)$$

$$\int \frac{2y+1}{y^2+y}dy = \int x^2 dxy = \pm\sqrt{c \cdot e^{\frac{1}{3}x^3} + \frac{1}{4}} - \sqrt{1}2 \quad (11)$$

Homogener Fall $b(x) = 0$ kann man wie folgt lösen:

$$e^{\lambda x} \cdot \left(\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0\right) = 0 \quad (12)$$

The following was presented in the lecture on 03. October 2019.

2.3 Variation of constants

$$f = z_1 f_1 + z_2 f_2 \quad (13)$$

Where z_1 and z_2 are functions, such that

$$z_1' f_1 + z_2' f_2 = 0 \quad (14)$$

2.4 Separation of variables

$$2yy' = b \Leftrightarrow (y^2)' = b \quad (15)$$

The following was presented in the exercise on 7.10.19.

3 Differential Calculus

Note 1. Eine abgeschlossene (closed) Menge enthält ihren Rand. Offen (Open) heisst, dass der Rand nicht mehr drin ist. Beschränkt (Bounded) bedeutet, dass es nur eine endliche Ausdehnung im $\mathbb{R}^n$ gibt. Mit einem Ball kann man die Beschränktheit zeigen.

Definition 2. Offener Ball $B_d = \{x \in \mathbb{R}^n | \ \|x - x_0\| < d\}$

The following was presented in the lecture on 10. October 2019.

Definition 3. The set

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \ : y = f(x_0) + u(x - x_0)\}$$
$$(x,y) = (x_0, f(x_0)) + (x - x_0, u(x - x_0))$$

is called the tangent space at x_0

Note 2. Man kann es sich als eine 3D Ableitung vorstellen und statt eine Tangent Linie hat man eine Tangent Ebene.

The following was presented in the lecture on 17. October 2019.

Definition 4. Directional Derivative
We have a vector x_0 and a directional vector v_0.

$$g(t) = f(x_0 + t \cdot v_0) \tag{16}$$

Definition 5. f is differentiable at x_0 with differential

$$\begin{cases} u : \mathbb{R}^n \to \mathbb{R}^m & \text{linear} \\ \text{A matrix with m rows and n columns} & \end{cases} \tag{17}$$

if $f(x) = f(x_0) + A(x - x_0) + E(x)$ where $\lim_{x \to x_0} \frac{\|E(x)\|}{\|x - x_0\|} = 0$

An important property of computing differentials: If $f : x \to \mathbb{R}^m, \quad f = (f_1, ..., f_m)$ has all partial derivatives

$$\frac{\delta f_j}{\delta x_i} : x \to \mathbb{R} \tag{18}$$

and if these are all continuous on X, then f is differentiable on X and

$$df(x) = Jp(x) \tag{19}$$

for every $x \in X$.

The following was presented in the exercise on 21. October 2019.

3.1 Jacobi-Matrix

$$(Df(x))_{i,j} = \frac{\delta f_i}{\delta x_j}(x) \tag{20}$$

3.2 Kettenregel

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad g : \mathbb{R}^m \to \mathbb{R}^k \tag{21}$$
$$h = g \circ f : \quad \mathbb{R}^n \to \mathbb{R}^k \tag{22}$$
$$h(x) = g(f(x)) \tag{23}$$
$$Dh(x) = Dg(y)|_{y=f(x)} D \cdot Df(x) \tag{24}$$

The following was presented in the lecture on 24. October 2019.

$$(g \circ f)'(t) = J_{g \circ f}(t) = J_g(f(t)) \circ J_f(t) = f_1' \frac{\delta g}{\delta y} + \ldots + f_m' \frac{\delta g}{\delta y_m} \tag{25}$$

$$(g \circ f)'(t) = g'(f(t)) \circ f'(t) \Leftrightarrow J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0) \tag{26}$$

The following was presented in the lecture on 31. October 2019.

$$\begin{cases} \delta_x f = \cos(\theta)\delta_r f - \frac{1}{r}\sin(\theta)\delta_\theta f \\ \delta_y f = \sin(\theta)\delta_r f + \frac{1}{r}\cos(\theta)\delta_\theta f \end{cases} \tag{27}$$

Note 3. If one knows f in polar coordinates, then the above formulas say what is $\delta_x f$ in terms of polar coordinates.

3.3 Taylor Polynomials

$$f(x,y) = f(a,b) + \delta_x f(a,b)(x-a) + \delta_y f(a,b)(y-b) + \frac{\delta_{x,x} f(a,b)}{2!}(x-a)^2 + 2 \cdot \frac{\delta_{x,y} f(a,b)}{2!}(x-a)(y-b) + \frac{\delta_{y,y} f(a,b)}{2!}(y-b)^2 \tag{28}$$

Definition 6. $f : U \to \mathbb{R}$ U is open. A critical point of f in U is a point x_0 where

$$J_f(x_0) = 0 \Leftrightarrow \nabla f(x_0) = 0 \tag{29}$$

The following was presented in the lecture on 7. November 2019.

Definition 7. $f \in C^m(U; \mathbb{R}) \quad U \subset \mathbb{R}^n$ open

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^{n} \frac{\delta f}{\delta x_i}(x_0) y_i + \ldots + \sum_{m_i + \ldots + m_n = j \& m_i \geq 0}^{n} \frac{1}{m_1! \cdots m_n!} \frac{\delta^j f}{\delta_{x_1}^{m_1} \cdots \delta_{x_n}^{m_n}}(x_0) x y_1^{m_1} \cdots y_n^{m_n} + \ldots \tag{30}$$

$$T_2 f(x,y) = f(x_0, y_0) + \nabla f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} H_f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \tag{31}$$

Definition 8.

$$Hess(f) = \begin{pmatrix} \frac{\delta^2 f}{\delta x_1^2} & \ldots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \ldots & \ldots & \ldots \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \ldots & \frac{\delta^2 f}{\delta x_n^2} \end{pmatrix} \tag{32}$$

$$H_{i,j} = \frac{\delta^2 f}{\delta x_i \delta x_j} \tag{33}$$

Definition 9. A non-degenerate critical point is has the property

$$\det Hess_f(x_0) \neq 0 \tag{34}$$

Note 4. If H is positive-definite (++++) it is a local minimum, when H is negative-definite (-+-+) it is a local maximum and when H is indefinite it is a saddle point.

Note 5. To compute the negative-definite matrix one has to be really careful as the sign is alternated.

Theorem 3.1: Lagrange Multipliers

If x_0 is a local extemum of the function f, either $\nabla g(x_0) = 0$ or there exists

$$\delta_x f(x_0) = \lambda \delta_x g(x_0) \tag{35}$$

$$\delta_y f(x_0) = \lambda \delta_y g(x_0) \tag{36}$$

$$\ldots = \ldots \tag{37}$$

$$g(x_0) = 0 \tag{38}$$

4 Integration in $\mathbb{R}^n$

4.1 Path integrals

Definition 10. A parameterized curve in $\mathbb{R}^n$ is a continuous function which is piecewise C^1.

Definition 11. Let $\gamma : [a, b]$. The dot is the scalar product. The line integral of f along γ is defined as:

$$\int_\gamma f(s) \cdot d\vec{s} \overset{\text{def}}{=} \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \tag{39}$$

Theorem 4.1

If γ_1, γ_2 are curves with γ_2 a reparametrization of γ_1, then for any f:

$$\int_{\gamma_1} f(s) \cdot d\vec{s} = \gamma_{\gamma_2} f(s) \cdot d\vec{s} \tag{40}$$

The following was presented in the lecture on 21. November 2019.

4.2 Line Integral

$$\int_{\gamma_2} f(s) \cdot d\vec{s} \overset{\text{def}}{=} \int_c^d f(\gamma_1(\sigma(u))) \cdot \gamma_2'(u) du = \int_a^b f(\gamma_1(t)) \gamma_1'(t) dt \tag{41}$$

$$\int_{\gamma} f(s) \cdot d\vec{s} = g(\gamma(b)) - g(\gamma(a)) \tag{42}$$

Definition 12. A vector field f is conservative if the line integrals of f only depend on the extremities of the curves involved. This means $\int_{\gamma} f(s) \cdot d\vec{s}$ is independent of the choice of paramterized curve γ

Definition 13. If $f = \nabla g$ then g is called potential for f.

Definition 14. $x_0 \in X$ is star-shaped if X contains the line-segment joining x_0 to x.

Theorem 4.2

Suppose that $X \subset \mathbb{R}^n$ is open and star-shaped then any C^1 vector field $f : X \to \mathbb{R}^n$ so that:

$$\delta x_i f_j = \delta x_j f_i \tag{43}$$

is conservative.

$$\int_{\gamma} f(s) \cdot d\vec{s} = 0 \Leftrightarrow \text{Conservative} \tag{44}$$

Further the starting point must equal the end point of the shape.

Definition 15.

$$curl(f) = \begin{pmatrix} \delta_y f_3 - \delta_z f_2 \\ \delta_z f_1 - \delta_x f_3 \\ \delta_x f_2 - \delta_y f_1 \end{pmatrix} \tag{45}$$

The following was presented in the lecture on 28. November 2019.

Definition 16. An n-dimensional Volume is defined by:

$$Vol(X) = \int_X dx_1 \cdot \cdot \cdot dx_n \tag{46}$$

4.3 How to compute Riemann Integrals

Integrals have the following properties:

Compatibility, Linearity, Positivity, Triangle inequality, Domain additivity

Fubini formula

$$\int_x f(x_1, x_2) dx = \int_{X_1} g(x_1) dx_1 = \int_{X_1} \left(\int_{Y_{x_1}} f(x_1, x_2) dx_2 \right) dx_1 \tag{47}$$

the function is valid if the inner integral is continuous.

Note 6. If $Y \subset \mathbb{R}^n$ is closed, bounded and negligible then

$$\int_y f(x_1, ..., x_n)dx, ..., dx_n = 0 \tag{48}$$

Corollary 4.1

If $X = Y_1 \cup Y_2$ and $Y_1 \cap Y_2$ is negligible:

$$\int_X f dx = \int_{Y_1} f dx + \int_{Y_2} f dx \tag{49}$$

The following was presented in the lecture on 5. December 2019.

4.4 Improper Integrals

Definition 17.

$$\int_x f(x,y)dxdy = \lim_{n\to\infty} \int_{[a,b],[c,n]} f(x,y)dxdy \tag{50}$$

Note 7.

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \pi \tag{51}$$

4.5 Change of variables

$$\int f(y)dy = \int f(\phi(x))\phi'(x)dx \tag{52}$$

Note 8.

$$y = \phi(x) \tag{53}$$
$$dy = \phi'(x)dx \tag{54}$$
$$dy = |\det J_\phi(x)|dx \tag{55}$$

Theorem 4.3: Change of variables

$$\int_Y f(y)dy = \int_X f(\phi(x))|\det J_\phi(x)|dx \tag{56}$$

The following was presented in the lecture on 12. December 2019.

4.6 Polar Coordinates

$$x = r \cdot \cos(\theta) \qquad y = r\sin(\theta) \tag{57}$$

$$J_p(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{58}$$

$$\int_Y f(x,y)dxdy = \int_X f(r\cos\theta, r\sin\theta) \cdot rdrd\theta \tag{59}$$

4.6.1 Calculate the circle

$$\int_Y f(x,y)dxdy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta) \cdot rd\theta dr \tag{60}$$

4.7 Calculate a Sphere

$$x = \cos(\theta)\sin(\phi)r \tag{61}$$

$$y = \sin(\theta)\sin(\phi)r \tag{62}$$

$$z = \cos(\phi)r \tag{63}$$

$$\int_Y f(x,y,z)dxdydz = \int_X f(\cos(\theta)\sin(\phi)r, \sin(\theta)\sin(\phi)r, \cos(\phi)r) \cdot r^2\sin(\phi)drd\theta d\phi \tag{64}$$

4.7.1 Calculate a sector

$$\int_Y f(x,y)dxdy = \int_0^R \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta) \cdot rd\theta dr \tag{65}$$

4.7.2 Calculate an Annulus

$$\int_Y f(x,y)dxdy = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta) \cdot rd\theta dr \tag{66}$$

Example 2.

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \tag{67}$$

$$\int_Y f(x,y,z)dxdydz = \int_0^R \int_0^{2\pi} \int_0^{\pi} f(r,\theta,\phi)r^2\sin(\phi)d\phi d\theta dr \tag{68}$$

4.8 Surface Area

$$Area(S) = \int_{X\subset\mathbb{R}^2} \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2} dxdy \tag{69}$$

4.9 The Green formula

Theorem 4.4: Green's formula

For any C^1 vector field $f = (f_1, f_2)$ on $\mathbb{R}^2$ we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \int_{\partial X} f \cdot d\overrightarrow{s} = \sum_i \int_{\gamma_i} f \cdot d\overrightarrow{s} \tag{70}$$

The following was presented in the lecture on 19. December 2019.

Definition 18. $f = (f_1, ..., f_n)$ vector field on $\mathbb{R}^n$, then the divergence of f is the function defined by

$$div(f) \stackrel{\text{def}}{=} \frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_n}{\partial x_n} \tag{71}$$

$$\int_X div(f)\ dxdy = \int_{\delta X} \tilde{f} \cdot d\overrightarrow{n} = \sum_i \int_{\gamma_i} \tilde{f} \cdot d\overrightarrow{n} \tag{72}$$

$$div(\nabla g) = \frac{\partial (\nabla g)_1}{\partial x} + \frac{\partial (\nabla g)_2}{\partial y} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \Delta g \tag{73}$$

$$\Delta = \nabla^2 = \sum_{k=1}^{n} \frac{\delta^2}{\delta x_k^2} \tag{74}$$

$$\int_X \Delta g(x, y)\ dxdy = \int_{\delta X} \nabla g \cdot d\overrightarrow{n} \tag{75}$$

$$\int_X \Delta g\ dxdy = 0 \tag{76}$$

5 Analysis I

Definition 19. f ist in x_0 differenzierbar falls der Grenzwert existiert:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{77}$$

Theorem 5.4

$$f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0) \tag{78}$$

Nun definieren wir: $\phi(x) = f'(x_0) + r(x)$

$$f(x) = f(x_0) + \phi(x)(x - x_0) \quad \text{mit} \quad \phi(x_0) = f'(x_0) \tag{79}$$

Theorem 5.9

$$(f+g)'(x_0) = f'(x_0) + g'(x_0) \tag{80}$$
$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \tag{81}$$
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \tag{82}$$

Theorem 5.23: de l'Hôpital

Falls $\lim_{x\to a} f(x) = 0$ und $\lim_{x\to a} g(x) = 0$ folgt

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{83}$$

Proof 5.23

$$\lim_{x\to b} \frac{f(x)}{g(x)} = \lim_{x\to b} \frac{\frac{df(a)}{dx} \cdot dx}{\frac{dg(a)}{dx} \cdot dx} = \lim_{x\to b} \frac{f'(x)}{g'(x)} \tag{84}$$

□

5.1 Potenzreihen und Taylorapproximation

Corollary 5.37

$$f^{(j)}(x) = \sum_{k=j}^{\infty} \frac{f^{(k)}(x_0)}{k!} \cdot \frac{k!}{(k-j)!}(x-x_0)^{k-j} \tag{85}$$

Theorem 5.43: Taylorapproximation

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\epsilon)}{(n+1)!}(x-a)^{n+1} \tag{86}$$

Theorem 5.28: Fundamentalsatz der Differentialrechnung

Es gibt eine Stammfunktion F, die bis auf eine additive Konstante c eindeutig bestimmt ist und es gilt:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{87}$$

Theorem 5.30: Partielle Integration

$$\int_a^b f(x) \cdot g'(x)dx = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f'(x) \cdot g(x)dx \tag{88}$$

Theorem 5.31: Variablenwechsel

$$\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt \tag{89}$$

Theorem 5.32: McLaurin

Sei $f : [1, \infty) \to [0, \infty)$ monoton fallend.

$$\sum_{n=1}^{\infty} f(n) \ \text{ konvergiert} \Leftrightarrow \int_1^{\infty} f(x)\, dx \ \text{ konvergiert} \tag{90}$$

6 Wichtige Formeln

6.1 Bekannte Integrale

$$\int x^s dx = \frac{x^{s+1}}{s+1} + c \tag{91}$$

$$\int \frac{1}{x} = \ln(x) + c \tag{92}$$

$$\int e^x dx = e^x + c \tag{93}$$

$$\int -t \cdot e^{-t \cdot x} dx = e^{-t \cdot x} + c \tag{94}$$

$$\int \sin(x) dx = -\cos(x) + c \tag{95}$$

$$\int \cos(x) dx = \sin(x) + c \tag{96}$$

$$\int \sinh(x) dx = \cosh(x) + c \tag{97}$$

$$\int \cosh(x) dx = \sinh(x) + c \tag{98}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + c \tag{99}$$

$$\int \frac{1}{x^2+1} dx = \arctan(x) + c \tag{100}$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arcsinh}(x) + c \tag{101}$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arccosh}(x) + c \tag{102}$$

$$\int \frac{1}{\cos(x)^2} dx = \tan(x) + c \tag{103}$$

$$\int 1 + \tan(x)^2 dx = \tan(x) + c \tag{104}$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) \tag{105}$$

Substitution: $x = a \cdot u$

$$\int \sqrt{r^2 - x^2} dx = \frac{r^2}{2} \arcsin\left(\frac{x}{r}\right) + \frac{1}{2}\sqrt{r^2-x^2} \tag{106}$$

Substitution: $x = r \cdot \sin(\alpha)$

$$F \circ \phi(u) = \int f(\phi(u)) \phi'(u) du \tag{107}$$

6.2 Partialbruchzerlegung

Theorem 6.65

Seien P, Q Polynome mit:

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{mit grad } P < \text{grad } Q \tag{108}$$

$$R(x) = \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{A_{ij} + B_{ij} \cdot x}{\left((x-\alpha_i)^2 + \beta_i^2\right)^j} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{C_{ij}}{(x-\gamma_i)^j} \tag{109}$$

Algorithmus 6.1: Partialbruchzerlegung

$$Q_1(x) = (x-\alpha_i)^2 + \beta_i^2 \tag{110}$$

$$Q(x) = Q_1(x)^{m_1} \cdot q(x) \tag{111}$$

Es gibt A,B und ein Polynom p mit $grad(p) \leq grad(Q) - 1$ und

$$\frac{P(x)}{Q(x)} = \frac{A + Bx}{Q_1(x)^{m_1}} + \frac{p(x)}{Q_1(x)^{m_1-1} \cdot q(x)} \tag{112}$$

Die Lösung hat die Form:

$$P(x) = (A + Bx) \cdot q(x) + p(x) \cdot Q_1(x) \tag{113}$$

Mit dieser Gleichung kann man dann eindeutig A und B bestimmen.
Der Algorithmus geht dann induktiv fort.
Schlussendlich kommt man auf:

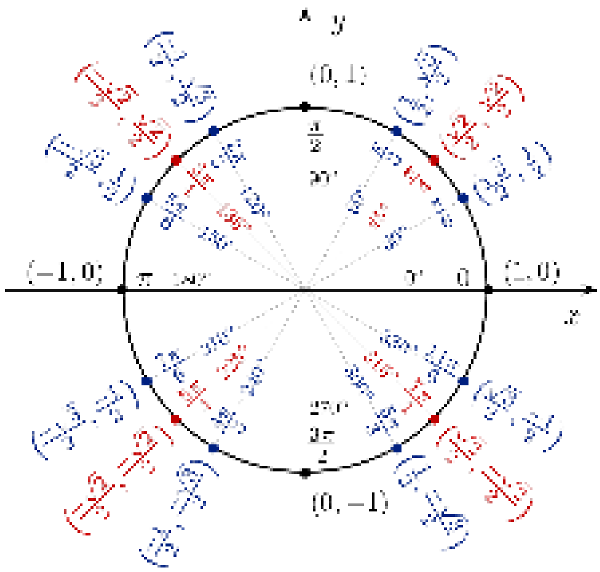
$$\int \frac{(A + B\alpha)dx}{((x-\alpha)^2 + \beta^2)^j} = \frac{A + B \cdot \alpha}{\beta^{\alpha \cdot j - 1}} \cdot \int \frac{dt}{(t^2+1)^j} \tag{114}$$

Note 9.

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{115}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \tag{116}$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \tag{117}$$



6.3 Doppelwinkelsätze

$$\sin(x)^2 = \frac{1-\cos(2x)}{2} \tag{118}$$

$$\sin(2x) = 2\sin(x)\cdot\cos(x) = \frac{2\tan(x)}{1+\tan(x)^2} = \frac{d}{dx}\sin(x)^2 \tag{119}$$

$$\frac{d}{dx}\cos(x)^2 = -2\sin(x)\cdot\cos(x) \tag{120}$$

$$\cos(2x) = \cos(x)^2 - \sin(x)^2 = 1 - 2\sin(x)^2 = 2\cos(x)^2 - 1 = \frac{1-\tan(x)^2}{1+\tan(x)^2} = \frac{d}{dx}\sin(x)\cdot\cos(x) \tag{121}$$

6.4 Additionstheoreme

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \tag{122}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y) \tag{123}$$

6.5 Substitution

$$dx = \phi'(u)du \tag{124}$$

$$u = 5x^2 + 1 \Leftrightarrow du = 10x\ dx \tag{125}$$

$$t = \sqrt{e^x - e^2} \Leftrightarrow x = \ln(t^2 + e^2) \Leftrightarrow dx = \frac{2t}{t^2+e^2}dt \tag{126}$$

$$\Rightarrow \int f(x)dx = \int f(\phi(u))\cdot\phi'(u)du \tag{127}$$

6.6 Sinus Cosinus Vereinfachungen

$$\cos(\arcsin(x)) = \sin(\arccos(x)) = \sqrt{1-x^2} \tag{128}$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}} \tag{129}$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}} \tag{130}$$

Theorem 6.66: Umkehrsatz

An jedem Punkt $f(x)$ differenzierbar
Keine waagrechte Tangente $f'(x) = 0$

6.7 Taylorapproximation

Spezialfall $a = 0$:

$$f(x) = f(0) + f'(0) \cdot x + \frac{1}{2} \cdot f''(0) \cdot x^2 + \frac{1}{6} \cdot f'''(0) \cdot x^3 + \frac{1}{24} \cdot f''''(0) \cdot x^4 + ... \tag{131}$$

6.7.1 Reihenentwicklungen

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \tag{132}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \tag{133}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \tag{134}$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \tag{135}$$

6.8 Geometrische Reihe

$$\sum_{k=0}^{\infty} q^k = \frac{q^{n+1-1}}{q-1} \tag{136}$$

6.9 Bekannte Grenzwerte

$$\lim_{x\to 0} x \cdot \log(x) = \lim_{x\to 0} \frac{\log(x)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -x = 0 \tag{137}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1 \tag{138}$$

$$\lim_{x\to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x\to 0} \frac{1}{2} \cdot \frac{\sin(x)}{x} = \frac{1}{2} \tag{139}$$

7 Aufgaben

Example 3.

$$u'' - 6u' + 13u = (3x + 2)e^x \tag{140}$$

As $X^2 - 6X + 13 = (X - (3 + 2i))(X - (3 - 2i))$, the real solutions of the homogenous equations are:

$$y_h = u(x) = \lambda_1 e^{3x} \cos(2x) + \lambda_2 e^{3x} \sin(2x) \tag{141}$$

Then let $y_p = u_0(x) = (ax + b)e^x$ be a prticular solution to the ODE. Then we compute

$$u_0'' - 6u_0' + 13u = (ax + (2a + b))e^x - 6(ax + (a + b))e^x + 13(ax + b)e^x = 4(2ax - a + 2b)e^x = (3x + 2)e^x \tag{142}$$

Example 4.

$$\sin(x)^3 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{e^{3ix} - e^{-3ix} - 3e^{ix} + 3e^{-ix}}{-8i} \tag{143}$$

$$= \frac{1}{4}\left(-\frac{e^{3ix} - e^{-3ix}}{2i} + 3\frac{e^{ix} - e^{-ix}}{2i}\right) = \frac{1}{4}(-\sin(3x) + 3\sin(x)) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x) \tag{144}$$

Example 5. Tangenital planes

$$f(x,y) = \sin(x) - y^3 + y^2 \tag{145}$$

$$G(f) := \{(x, y, f(x,y))\} \tag{146}$$

Determine the tangential plane at $(0, 3, f(0,3))$

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ \frac{\delta}{\delta x}\sin(x) - y^3 + y^2|_{x=0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{147}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ \frac{\delta}{\delta y}\sin(x) - y^3 + y^2|_{y=3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -21 \end{pmatrix} \tag{148}$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} -1 \\ 21 \\ 1 \end{pmatrix} \tag{149}$$

$$\Rightarrow -x + 21y + z = 45 \tag{150}$$

Example 6. Compute Area with Green Formula
One can compute an area by

$$A = \int_\gamma 1 d\vec{s} \tag{151}$$

$$1 = \underbrace{\delta_x f_2}_{1} - \underbrace{\delta_y f_1}_{0} = f \tag{152}$$

$$\Rightarrow f = \begin{pmatrix} 0 \\ x \end{pmatrix} \tag{153}$$

$$\int_\gamma f d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt = \int_a^b \gamma_1 \cdot \delta_y \gamma_2 dt \tag{154}$$

Example 7. Compute the Integral

$$\int_D (xy + y^3) dx dy \tag{155}$$

where D is the quarter-disc

$$D = \{(x,y) \in \mathbb{R}^2 : \ x \geq 0, \ y \geq 0, \ x^2 + y^2 \leq 2\} \tag{156}$$

Using polar coordinates and noticing $x \geq 0$ and $y \geq 0$

$$x = r\cos(\theta) \qquad y \qquad = r\cos(\theta) \tag{157}$$

$$0 \leq r \leq \sqrt{2} \qquad 0 \leq \theta \leq \frac{\pi}{2} \tag{158}$$

$$
\begin{aligned}
\int_D \left(xy + y^3\right) dxdy &= \int_0^{\sqrt{2}} \int_0^{\frac{\pi}{2}} (r^2 \cos(\theta)\sin(\theta) + r^3 \sin(\phi)^3) r dr d\theta && (159)\\
&= \left(\int_0^{\sqrt{2}} r^3 dr\right)\left(\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta) d\theta\right) && (160)\\
&+ \left(\int_0^{\sqrt{2}} r^4 dr\right)\left(\int_0^{\frac{\pi}{2}} \frac{1}{4}(3\sin(\theta) - \sin(3\theta)) d\theta\right) && (161)\\
&= \left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \left[\frac{1}{2}\sin(\theta)^2\right]_0^{\frac{\pi}{2}} + \left[\frac{r^5}{5}\right]_0^{\sqrt{2}} \left[-\frac{3}{4}\cos(\theta) + \frac{1}{12}\cos(3\theta)\right]_0^{\frac{\pi}{2}} && (162)\\
&= \frac{1}{2} + \frac{8\sqrt{2}}{15} && (163)
\end{aligned}
$$

Example 8. Compute the Taylor polynomial of order 2 of

$$f(x, y, z) = 2exp(x + y^2 + z^3) \tag{164}$$

at $(x, y, z) = (0, 0, 0)$ and the Hessian matrix.
As $e^t = 1 + t + \frac{t^2}{2} + \mathcal{O}(t^3)$ we find:

$$f(x, y, z) = 2 + 2x + x^2 + 2y^2 + \mathcal{O}(|x, y, z|^3) \tag{165}$$

The Hessian matrix is therefore:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{166}$$

Example 9. If F is conservative and one can find a potential, the integral does only depend on the end points. Where ϕ is the potential and γ are the start and end point pluged into the "Abbildung!"

$$
\begin{aligned}
\int_\gamma f d\vec{s} &= \phi\left(\gamma\left(\frac{5\pi}{2}\right)\right) - \phi\left(\gamma\left(\frac{\pi}{4}\right)\right) && (167)\\
&= \phi(0, 0) - \phi(1, 1) = 2 && (168)
\end{aligned}
$$