Analysis II
Summary

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Chapter 1

Ordinary differential equations

1.1 Differential Equation:

An equation for a function f that relates the values of f at x, f(x) to the values of its derivatives at the same point x. We distinguish between the number of variables present in the function:

- **One variable:** Ordinary differential equations (ODE)
- **Several Variables:** Partial differential equations (PDE)

Examples:

- $f'(x) = f(x)$
- $f''(x) = -f(x)$

Notation: We write $y, y', y'', y^{(3)}, \ldots$ instead of $f(x), f'(x), f''(x), f^{(3)}(x)$

Order: The largest derivative present in the equation. Examples:

- $y' = 2xy$ order 1
- $y^{(3)} + 2xy'' + e^x y + 1 = 0$ order 3

The solution to an ODE is not unique in general. When given initial conditions then we can find unique solutions. E.g:

$$y' = x + 1$$
$$y = \frac{x^2}{2} + x + c$$

is a solution for any c. If we are also given y(0) = 1 then c= 1 is a unique solution.

1.2 Linear Differential equations

A linear ODE of order k on an interval $I \subset \mathbb{R}$ is an eqn of the form:

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$$

where a(x) and b(x) are continuous functions from I to $\mathbb{C}$.
For a linear ODE the following hold:

- y and all its derivatives appear in order 1
- there are no products of the function y and its derivatives
- neither the function nor its derivatives are inside another function e.g $\sqrt{y}$, sin(y),...

If b = 0 then we say the equation is **homogeneous** otherwise **inhomogeneous**

Solving a linear ODE means finding all functions $f : I \to \mathbb{C}$ that are k times differentiable such that $\forall x \in I$ the function satisfies the differentiable equation.

Initial Condition A set of equations specifying the values of the derivatives at some initial point.

Theorem 2.2.3 Let $I \subset \mathbb{R}$ and open interval $k \geq 1$ and integer. Consider the linear ODE

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$$

where coefs $a_i(x), b(x)$ are continous functions

1. Let S_0 be the set of solutions for b=0, then S_0 is a vector space of dimension k.
2. For any initial conditions, i.e for any choice of $x_0 \in I$ and $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ there is a unique solution $f \in S$ such that $f_(x_0) = y_0, \ldots f^{(k)}(x_0) = y_k$
3. For an arbitrary b the set of solutions of the linear ODE is $S_b = \{f + f_p | f \in S_0\}$ where f_p is one **particular** solution. S_b is not a vector space.
4. For any initial condition there is a unique solution.

The linearity of the diff equation alsos implies a **superposition** principle. Suppose we have 2 different functions $b_1(x), b_2(x)$ on the RHS with solutions $f_1, f_2 : Df_1 = b_1, Df_2 = b_2$ then $f_1 + f_2$ solves $Df = b_1 + b_2$

Given a diff eqn and a possible solution we can always verify whether it is indeed a solution or not.

1.3 Linear differential equations of order 1

We consider y'+ay = b, where a,b are continous functions. 2 steps:

- Find solutions of the corresponding homogeneous equation y' + ay = 0.
- Find a particular solution $f_p : I \to \mathbb{C}$ such that $f_p + af_p = b$

If f is a solution then so is zf for any constant $z \in \mathbb{C}$

Homogeneous solution: $y' + ay = 0$
$\Rightarrow y' = -ay$
$\Rightarrow \frac{y'}{y} = -a$
$\Rightarrow \int \frac{y'(x)}{y(x)}dx = -\int a(x)dx := A(x)$
$\Rightarrow ln|y(x)| = -A(x) + c$
$\Rightarrow y = z \cdot e^{-A(x)}$ for some constant z

Solution of inhomogeneous equation $y' + ay = b$
There are two methods to solve this:

- Educated guess: the LHS tries to imitate the RHS i.e if b(x) is a polynomial we guess that f_p is also a polynomial or if b is a trig function then we guess f_p is also a trig function
- Variation of constants: Assume

$$f_p = z(x)e^{-A(x)}$$

for some function $z : I \to \mathbb{C}$. We then put this into the equation and see what it forces z(x) to satisfy

The same particular solution can also be obtained by the method of **Integration factor** (IF). Given a ODE of the following form:

$$\frac{dy}{dx} + a(x)y = b(x)$$

one multiplies both sides of the equation by an IF of:

$$e^{\int a(x)dx}$$

$\Rightarrow$

$$\frac{dy}{dx}e^{\int a(x)dx} + a(x)ye^{\int a(x)dx} = b(x)e^{\int a(x)dx}$$

The left hand side simplifies to:

$$\frac{d}{dx}(ye^{\int a(x)dx}) := z(x)$$
$$\Rightarrow y = z(x)e^{-A(x)}$$

$\Rightarrow$

$$z'(x) = b(x)e^{\int a(x)dx} = b(x)e^{A(x)}$$

Example:

$$x\frac{dy}{dx} - 2y = x^2$$

Assume $x \neq 0$. We now put the equation in the above form.

$$\frac{dy}{dx} - \frac{2}{x}y = x$$

- $a(x) = \frac{-2}{x}$
- $b(x) = x$
- $A(x) = -2\int \frac{1}{x}dx = -2ln(x) = ln(x)^{-2}$
- $e^{A(x)} = e^{lnx^{-2}} = \frac{1}{x^2}$

$z'(x) = b(x)e^{A(x)} = x \cdot \frac{1}{x^2} = \frac{1}{x}$
$\Rightarrow z(x) = ln(x)$

- $y_h = ze^{-A(x)} = zx^2$
- $y_p = z(x)e^{-A(x)} = ln(x)x^2$

$\Rightarrow y = y_p + y_h = x^2ln(x) + zx^2$

1.4 Linear differential equations with constant coefficients

For a linear ODE with constant coefficients

$$y^{(k)} + a_{k-1}y^{k-1} + ... + a_0 y = 0$$

The Polynomial

$$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + ... + a_0$$

is called the **companion/charateristic polynomial** of the equation. The zeroes of $P(\lambda)$ are called the **eigenvalues**
Example:
$y'' - y = 0$
$\Rightarrow P(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$
$\Rightarrow$ 2 solutions: e^{-x}, e^x
Any solution of the equation are of the form

$$y(x) = z_1 e^{-x} + z_2 e^x$$

Theorem let $\lambda_1, ...\lambda_r$ be pairwise distince eigenvalues of $P(\lambda)$, characteristic polynomial of

$$(*) \quad y^k + a_{k-1}y^{k-1} + ... + a_0 y = 0$$

with corresponding multiplicities $m_1, ..., m_r$ Then the functions

$$f_{j,l} : \mathbb{R} \to \mathbb{C} \quad x \mapsto x^l e^{\lambda_j x}$$

for $1 \leq j \leq r, 0 \leq l < m_j$
form a system of solutions of the homogeneous D.E (*).

Example $y'' - 2y' + 1 = 0$
$\Rightarrow P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$
$\Rightarrow \lambda = 1$ has multiplicity of 2
$\Rightarrow$ the solutions are e^x, xe^x

If a_i's are real then we consider the real solutions. IF $\alpha = \beta + i\gamma$ is a complex root of $P(\lambda)$ then so is the complex conjugate $\overline{\alpha} = \beta - i\gamma$. Hence

$$f_1 := e^{\alpha x}, f_2 = e^{\overline{\alpha x}}$$

are 2 solutions. We have

$$e^{\alpha x} = e^{\beta x} \cdot e^{i\gamma x} = e^{\beta x}[cos(\gamma x) + isin(\gamma x)]$$
$$e^{\overline{\alpha} x} = e^{\beta x} \cdot e^{-i\gamma x} = e^{\beta x}[cos(\gamma x) - isin(\gamma x)]$$

hence we can replace any solution $af_1 + bf_2$ with a linear combination of

$$\tilde{f_1} = e^{\beta x} cos(x)$$
$$\tilde{f_2} = e^{\beta x} sin(x)$$

Theorem: If $y^k + a_{k-1}^{k-1} + ... + a_0 y = 0$ has real coefficients, then each pair of complex conjugate roots $\beta_j \pm i\gamma_j$ of $P(\lambda)$ with multiplicity m_j leads to solutions:

$$x^l e^{\beta_j x}(cos(\gamma_j x) + isin(\gamma_j x))$$

for $0 \leq l < m_j$
Which can then be replaced by the solutions:

$$x^l e^{\beta_j x} cos(\gamma_j x), x^l e^{\beta_j x} sin(\gamma_j x)$$

example:
$y'' + y = 0$
$\Rightarrow P(\lambda) = \lambda^2 + 1 \Rightarrow \lambda_{1\backslash 2} = i, -i$ multiplicity 1
$\Rightarrow y_h(x) = z_1 e^{ix} + z_2 e^{-ix}$ which we can convert to real by taking different coefficients
$\Rightarrow = \tilde{z}_1 cos(x) + \tilde{z}_2 sin(x)$

Inhomogeneous equations Given

$$y^{(k)} + a_{k-1}y^{k-1} + \cdots + a_0 y = b(x) \quad (*)$$

Goal is to find a particular solution y_p. Any solution of $*$ will be of the form

$$y = y_h + y_p$$

We have 2 methods to solve this:

- **Method of undetermined coefficients ("Ansatz" method):** The solution will be similar to the disturbance function b(x)

$b(x)$	Ansatz
$ae^{\alpha x}$	$be^{\alpha x}$
$a\sin(\beta x)$ $b\cos(\beta x)$	$c\sin(\beta x)+d\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$ $be^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left[c\sin(\beta x)+d\cos(\beta x)\right]$
$P_n(x)e^{\alpha x}$	$R_n(x)e^{\alpha x}$
$P_n(x)e^{\alpha x}\sin(\beta x)$ $Q_n(x)e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left[R_n(x)\sin\beta x + S_n(x)\cos\beta x\right]$

$P_n(x)$, $R_n(x)$, $Q_n(x)$, $S_n(x)$ are polynomials of degree n.

example: $b(x) = a \cdot e^{\alpha x}$

1. $y_p = Be^{\alpha x}$ for some B
2. put y_p into the diff equation $(*)$ resulting in conditions on B
3. Solve for B

If b(x) is a linear combination of the above functions then we try the corresponding linear combination of the "Ansatz" functions.
If $\lambda = \alpha + \beta i$ is a zero of the char poly $P(\lambda)$ of multiplicity m, then the "Ansatz" must be multiplied by x^m

- **Variation of constants:**
The idea is to change the constants z_1, z_2 into the functions $z_1(x), z_2(x)$ (assuming k =2 i.e $y'' + a_1 y' + a_0 y = b(x)$). We have

$$f_h = z_1 f_1 + z_2 f_2$$

for the homogeneous solution where f_1, f_2 are linearly independant. For f_p we now apply the variation:

$$f_p = z_1(x) f_1 + z_2(x) f_2$$

To determine the 2 unknown functions $z_1(x), z_2(x)$ we define the following equations:

$$z_1'(x) f_1 + z_2'(x) f_2(x) = 0$$
$$z_1'(x) f_1' + z_2'(x) f_2'(x) = b$$

Which gives us the following System of linear equations:

$$\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

The Matrix

$$\text{A} = \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}$$

is invertable hence

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

This gives us the equations for $z_1'(x), z_2'(x)$ we can then integrate and get $z_1(x), z_2(x)$. giving us our particular solution. $y_p = z_1(x) f_1 + z_2(x) f_2$

1.5 Separation of variables

A differential equation of first order is called **separable** if it is of the form

$$y' = b(x)g(y)$$

hence we can seperate the variables x, y i.e all y's on one side and all x's on the other:

$$\frac{dy}{dx} = b(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int b(x)dx$$

For any y_0 such that $g(y_0) = 0$ the constant function $y = y_0$ is a solution.
Example:

$$e^{2y}y' = x \Rightarrow y' = x \cdot e^{-2y}$$

Hence x= b(x) and $e^{-2y} = g(y)$. In this case g(y) is never zero since the exp function is never zero.
$\int e^{2y}dy = \int x \ dx$
$\Rightarrow e^{2y} = x^2 + c' \Rightarrow 2y = log(x^2 + c) \Rightarrow y = \frac{log(x^2+c)}{2}$

Chapter 2

Differential Calculus in $\mathbb{R}^n$

Polynomials in n variables: Given $d \geq 0$ a polynomial in n variables of degree $\leq d$ is a finite sum of "monomials" of degree $e \leq d$
Monomial: A monomial of degree e is a function

$$f : \mathbb{R}^n \to \mathbb{R} \quad (x_1, ..., x_n) \mapsto \alpha x_1^{d_1} x_2^{d_2} ... x_n^{d_n}$$

such that $d_1 + ... + d_n = e$
Example:
$P(x, y, z) = x^3 + 2x^2 + yx + xyz + z^4$
with the monomials:

- x^3 degree 3
- $2x^2$ degree 2
- yx degree 2
- xyz degree 3
- z^4 degree 4

degree of P is 4
Degree of a polynomial: The max of the degrees of the monomials in P

We can obtain new functions from old ones by:

- cartesian product of 2 functions
- Functions with separated variables
- Composition of 2 functions

Continuity in $\mathbb{R}^n$

For

$$f : \mathbb{R} \to \mathbb{R}$$

f is continuous at $x_0 \in \mathbb{R}$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$
Hence for $\mathbb{R}^n$ we also have a distance function. From linear algebra:

$$x \in \mathbb{R}^n, x = (x_1, ..., x_n) \Rightarrow ||x|| := \sqrt{x_1^2 + ... + x_n^2}$$

with:

- $\|x\| > 0 \forall x \neq 0$
- $\|tx\| = |t|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Convergence Let $(x_k)_{k\in\mathbb{N}}, x_k \in \mathbb{R}^n$ with $x_k = (x_{k,1}, x_{k,2}, ..., x_{k_n})$
let $y \in \mathbb{R}^n, y = (y_1, .., y_n)$
We say the sequence converges to y as $k \to \infty$. We write $x_k \to y$ or $\lim\limits_{k\to\infty} x_k = y$ if

$$\forall \epsilon > 0, \exists N \geq 1 \text{ s.t } \forall n \geq N \text{ we have } \|x_k - y\| < \epsilon$$

This definition is equivalent to:

- For each $i, 1 \leq i \leq n$ the sequence $(x_k, i)_k$ of real numbers converge to y_i
- The sequence of real numbers $\|x_k - y\| converges to 0$

Limit let
$$f : X \subset \mathbb{R}^n \to \mathbb{R}^m, \quad x_0 \in X, y \in \mathbb{R}^m$$
We say f has a lmit as $x \to x_0$ with $x \neq x_0$ if
$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t } \forall x \in X, x \neq x_0 \text{ such that } \|x - x_0\|_n < \delta, \text{ we have } \|f(x) - y\|_n < \epsilon$$
We write $\lim\limits_{x \to x_0, x \neq x_0} f(x) = y$

Proposition
$f : X \subset \mathbb{R}^n \to \mathbb{R}^m, x_0 \in X \subset \mathbb{R}^n, y \in \mathbb{R}^m$
$$\lim_{x \to x_0} f(x) = y \iff \forall \text{ sequences } (x_k) \text{ in } X \text{ such that } \lim_{x_k} = x_0 \text{ and } x_k \neq x_0 \text{ the sequence } f(x_k) \text{ converge to y}$$

Continuity
$f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ let $x_0 \in X$
We say f is continuous at x_0 if $\forall \epsilon > 0 \exists \delta > 0$ s.t if $x \in X$ satisfies $\|x - x_0\|_n < \delta$ then $\|f(x) - f(x_0)\| < \epsilon$
f is continuous in X if f is continuous in every point $x_0 \in X$ For a continuous function f we have:
$$\lim f(x_k) = f(\lim x_k)$$

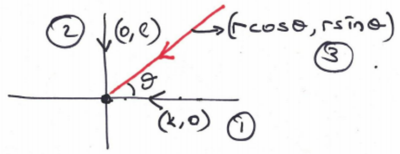
Examples:
- Linear Functions
- Polynomials
- Sums,products of continuous functions are continuous
- Functions of separated variables are continuous if the factors are continuous
- Composition of continuous functions are continuous

The discontinuity of functions of 2 variables can be points (e.g $log(x^2 + y^2)$) or a collection of curves (e.g $log(cos(x^2 + y^2))$

WARNING: If you start with a continuous function and then fix one of the variables at a constant then you obtain a function that has less variables and it will be continuous but the converse that if you start with a function of 2 variables and you fix one of the variables and you get a continous function does not imply that the original function was continuous.

Examples

$$f(x,y) = \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



① $(x_k, y_k) = (k, 0) \to (0,0)$

$f(x_k, y_k) = \frac{k \cdot 0}{k^2 + 0^2} = 0 \to 0.$

② $(x_\ell, y_\ell) = (0, \ell) \to (0,0)$

$f(x_\ell, y_\ell) = \frac{0 \cdot \ell}{0^2 + \ell^2} \to 0$

③ $(x_r, y_r) = (r\cos\theta, r\sin\theta) \to (0,0)$

$f(x_r, y_r) = \frac{(r\cos\theta)(r\sin\theta)}{r^2}$

$= \cos\theta \sin\theta$

for ex if $\theta = 45^\circ$ it converges to $1/2$.

if $\theta = 60 \Rightarrow \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

④. $\frac{\cos k}{k}, \frac{\sin k}{k} \to (0,0)$

$\Rightarrow f$ does not have a limit as $(x,y) \to (0,0)$.

For studying limits of functions $f: \mathbb{R}^n \to \mathbb{R}$ the following lemma is useful.

Lemma (Sandwich lemma).

If $f, g, h: \mathbb{R}^n \to \mathbb{R}$ where $f(x) < g(x) < h(x)$ $\forall x \in \mathbb{R}^n$.

let $a \in \mathbb{R}^n$.

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x)$ also exists and is equal to L.

Ex.: $g(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$h(x,y) = \frac{x^2}{x^2+y^2}$ $\quad g(x,y) = y\, h(x,y)$

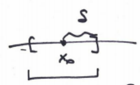
$0 \leq \frac{x^2}{x^2+y^2} < 1$

$0 < |g(x,y)| < |y|$

Then by sandwich lemma.

$g(x,y) \to 0$ as $(x,y) \to (0,0)$

$\|x - x_0\| < \delta.$



$|x - x_0| \leq \delta$

Rk. Sometimes it is helpful to use polar coordinates especially with rational functions $f: \mathbb{R}^2 \to \mathbb{R}$.

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Min -max theorem If $f: [a,b] \to \mathbb{R}$ with $[a,b]$ a compact interval and continuous, then f takes its max and min. i.e

$$\exists v^+ \in [a,b] \text{ s.t } f(x) \leq f(v^+) \forall x \in [a,b], \exists v^- \in [a,b] \text{ s.t } f(x) \geq f(v^-) \forall x \in [a,b]$$

The we define the analog for $\mathbb{R}^n$

bounded: A subset $X \subset \mathbb{R}^n$ is bounded if the set $\{\|x\| | x \in X\}$ is bounded in $\mathbb{R}$

closed: A subset $X \subset \mathbb{R}^n$ is closed if for every sequence $(x_k)_k \subset X$ that converges in $\mathbb{R}^n$, converges to a point $y \in X$
Example:

$$(\frac{1}{k})_k \subset (0,1] = X \subset \mathbb{R}$$

This sequence converges to 0 but 0 is not contained in the X, hence X is not closed.

Compact: $X \subset \mathbb{R}^n$ if it is closed and bounded.

If $X \subset \mathbb{R}^n, Y \in \mathbb{R}^m$ are bounded (resp. closed, resp. compact) then

$$X \times Y = \{(x,y) \in \mathbb{R}^{n+m} | x \in X, y \in Y\}$$

is bounded (resp. closed, compact) in $\mathbb{R}^{n+m}$

If $f = \mathbb{R}^n \to \mathbb{R}^m$ continuous, then for every $Y \subset \mathbb{R}^m$ closed, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n | f(x) \in Y\} \subset \mathbb{R}^n\}$ is closed. The Inverse image of closed sets under continuous maps are closed.

Example:
if $f : \mathbb{R}^n \to \mathbb{R}$ continuous then for any $a \leq b$

$$X := \{x \in \mathbb{R}^n | a \leq f(x) \leq b\}$$

is closed, $X = f^{-1}([a,b])$ the same applies to:

- $\{x \in \mathbb{R}^n | f(x) \geq a\}$
- $\{x \in \mathbb{R}^n | f(x) \leq b\}$

WARNING: If f is continuous then the set

$$\{x \in \mathbb{R}^n | a \leq f(x) \leq b\}$$

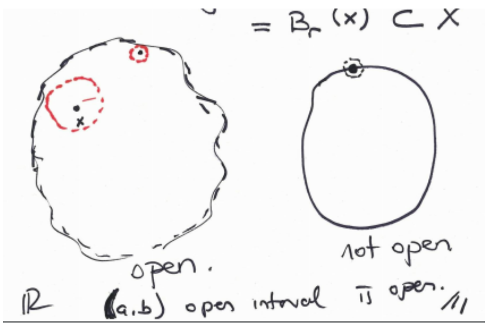
is not always compact $f^{-1}([a,b])$ the inverse image of a closed interval hence its closed, but it is also the inverse image of the compact interval $[a,b]$. But we can not say that it is compact.
Example: $f : \mathbb{R}^3 \to \mathbb{R}, \quad (x,y,z) \mapsto sin(xyz)$
$\{(x,y,z) | -1 \leq f(x,y,z) \leq 1\} = f^{-1}([-1,1]) = \mathbb{R}^3$ closed but not bounded hence not compact

Min-Max theorem for functions of several variables
Let $x \subset \mathbb{R}^n$ compact set $f : X \to \mathbb{R}$ a continuous function. Then f is bounded and attains its max and min i.e $\exists x^+ \in X$ and $x^- \in X$ s.t

- $f(x^+) = supf(x)$
- $f(x^-) = inff(x)$

Open: A set $X \subset \mathbb{R}^n$ is called open if its complement $\mathbb{R}^n \backslash X$ is closed. This is equivalent to

$$\forall x \in X, \exists r > 0 \text{ s.t the set } \{y \in \mathbb{R}^n | \|y - x\| < r\} = B_r(x) \subset X$$



Examples:

- $(a,b) \subset \mathbb{R}$ is open
- $[a,b) \subset \mathbb{R}$ neither open nor closed
- $\mathbb{R}, \emptyset$ both open
- $(a_1,b_1) \times (a_2,b_2) \subset \mathbb{R}^2$ is open
- Inverse image of open sts under continuous maps are open

2.1 Partial derivatives

Goal is to define the analog of the derivative for $f : \mathbb{R}^n \to R^m$ which we can then use to say something about

- how the function changes around a given point
- how can we give an approximation to the value of the function $f(x_0 + h)$ if we know $f(x_0)$

- $f: \mathbb{R} \to \mathbb{R}$.

  $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

- $f: \mathbb{R} \to \mathbb{R}^n$

  $x \mapsto \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$

  Then $f'(x_0) = \begin{pmatrix} f_1'(x_0) \\ f_2'(x_0) \\ \vdots \\ f_n'(x_0) \end{pmatrix}$

  $f_i : \mathbb{R} \to \mathbb{R}$.

- $f: \mathbb{R}^n \to \mathbb{R}^m$

  $x \mapsto \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

  where each $f_i: \mathbb{R}^n \to \mathbb{R}$. $1 \leq i \leq m$.

- How do we study a function $f: \mathbb{R}^n \to \mathbb{R}$ near a point $x_0 = (x_{0,1}, \cdots, x_{0,n})$

- How does f change near x_0?

General case:

Let $f: X \subset \mathbb{R}^n \to \mathbb{R}$

we want to study f around $x_0 = (x_{0,1}, \cdots, x_{0,n})$.

For each j, we consider

$$g_j(t) := f(x_{0,1}, x_{0,2}, \ldots x_{0,j-1}, t, x_{0,j+1}, \ldots x_{0,n})$$

$g_j: \mathbb{R} \to \mathbb{R}$

defined on the set

$I := \{t \in \mathbb{R} \mid (x_{0,1}, \ldots, t, \ldots x_{0,n}) \in X\}$

Then we ask if $\frac{dg_j}{dt}(x_{0,j})$ exists?

$$\frac{dg}{dt}(x_{0,j}) = \lim_{h \to 0} \frac{g(x_{0,j}+h) - g(x_{0,j})}{h}$$

$$= \lim \frac{\left(f(x_{0,1} \cdots, x_{0,j-1}, x_{0,j}+h, x_{0,j+1} \cdots, x_{0,n}) - f(x_{0,1}, \ldots x_{0,j}, \ldots x_{0,n})\right)}{h}.$$

If the limit exists then we say that f has **partial derivative with respect to x_j** at the point x_0, and we write $\frac{\partial f}{\partial x_j}(x_0)$, $(\partial_{x_j} f)(x_0)$ or $(\partial_j f)(x_0)$

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If $f: \mathbb{R}^n \to \mathbb{R}^m$ $x_0 \in \mathbb{R}^n$

then
$$g_j(t) := f(x_{0,1} \ldots, x_{0,j-1}, t, x_{0,j+1}, \ldots x_{0,n})$$

$$= \begin{pmatrix} f_1(x_{0,1} \cdots x_{0,j-1}, t, x_{0,j+1}, \cdots x_{0,n}) \\ f_2(- - - - -) \\ \vdots \\ f_m(- - - - -) \end{pmatrix}$$

$g: \mathbb{R} \to \mathbb{R}^m : \begin{pmatrix} g_1(t) \\ \vdots \\ g_m(t) \end{pmatrix}$

$$g'(x_{0,j}) = \begin{pmatrix} g_1'(x_{0,j}) \\ g_2'(x_{0,j}) \\ \vdots \\ g_m'(x_{0,j}) \end{pmatrix} =: \frac{\partial f}{\partial x_j}(x_0).$$

To evaluate partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to x_j at a point $a = (a_1, ..., a_n)$ we differentiate $f(a_1, ..a_{j-1}, x_j, a_{j+1}, ..., a_n)$ with respect to x_j treating all other variables as a constant with respect to x_j

Example :

$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto (x^2 + xy)sin(y)$

$\Rightarrow \frac{\delta f}{\delta x} = sin(y)(2x + y)$

$\Rightarrow \frac{\delta f}{\delta y} = xsin(y) + (x^2 + xy)cos(y)$

Jacobi Matrix: Let $X \subset \mathbb{R}^n$ open $f : X \to \mathbb{R}^m$ with partial derivatives on X we write:

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

for any $x \in X$ the matrix

$$J_f(x) = \frac{\delta f_i}{\delta x_j} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

$J_f(x)$ is a matrix of m rows and n columns i.e it is a $m \times n$ Matrix

$$f : X \subset \mathbb{R}^n \to \mathbb{R}^m$$

It is called the Jacobi matrix of f at point x

Gradient In the special case $f : X \to \mathbb{R} \quad X \subset \mathbb{R}^n$, the column vector

$$(\nabla f)(x_0) = \begin{pmatrix} \frac{\delta f(x_0)}{\delta x_1} \\ \vdots \\ \frac{\delta f(x_0)}{\delta x_n}) \end{pmatrix}$$

properties:

Properties · 1) If $f, g : X \subset \mathbb{R}^n \to \mathbb{R}^m$
X open have partial derivatives
at x_0 wrt variable x_j in X
with respect to
then so does $f+g$

$$\frac{\partial (f+g)}{\partial x_j}(x_0) = \frac{\partial f}{\partial x_j}(x_0) + \frac{\partial g}{\partial x_j}(x_0)$$

2) If $f, g : \mathbb{R}^n \to \mathbb{R}$
have part. derivatives at x_0
wrt x_j
then so does fg
and if $g(x) \neq 0$ then
also f/g.

$$\frac{\partial (fg)}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$$

$$\frac{\partial (f/g)}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$$

We can differentiate with respect to the same or another variable once we have already differentiated once.

$$\delta_{x_j}(\delta_{x_i} f) = \frac{\delta}{\delta x_j} \cdot \frac{\delta}{\delta x_i} f$$

Example:

$f(x, y, z) = x^3yz^2 + cos(x) + z$

$\frac{\delta f}{\delta x} = 3x^2yz^2 - sin(x)$

$\Rightarrow \frac{\delta}{\delta y}(\frac{\delta f}{\delta x} = 3x^2z^2$

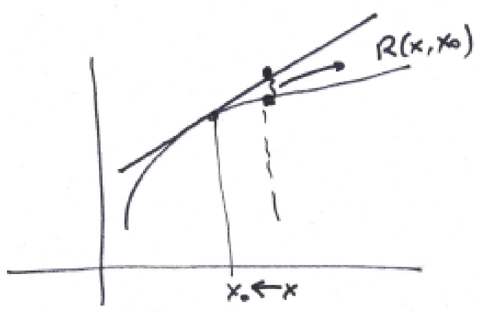
If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0 then f is continuous at x_0. f continuous at x_0 does not imply that f is differentiable at x_0

Partial derivatives are not strong enough to take as an anolog of the derivative from the 1-variable case.

Well Approximated: $f(x) = f(x_0) + f'(x_0)(x - x_0) + R(x, x_0)$ with

$$\lim_{x \to x_0} \frac{R(x, x_0)}{|x - x_0|} = 0$$

i.e $R(x, x_0)$ goes to zero faster than $|x - x_0|$ as $x \to x_0$



$f : \mathbb{R} \to \mathbb{R}$ is differntiable in x_0 if it can be well approximated by the affine linear map

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

when x is near x_0

Differentiable at x_0 Let $X \subset \mathbb{R}^n$ open $x_0 \in X, f : X \to \mathbb{R}^m$ a function we say f is differntiable at x_0 if ther exists a linear map

$$U : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{x \to x_0, x \neq x_0} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

The linear map $U : \mathbb{R}^n \to \mathbb{R}^m$ is called the **total differential** of f at x_0 and is denoted by

$$df(x_0) or d_{x_0} f$$

For $f : \mathbb{R}^n \to \mathbb{R}^m$ the total differential is NOT a number but a linear map!

Directional Derivative: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $v \in \mathbb{R}^n$ a vector. When it exists, the limit

$$D_v f(x) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

is called the directional derivative of f along v at the point $x \in \mathbb{R}^n$

$$D_v f(x) \text{ exists} \iff \text{the function } \phi : [-\delta, \delta] \to \mathbb{R} \text{ given by } \phi(t) = f(x + tv) \text{ is differentiable at } t = 0$$

i.e

$$\phi'(0) = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h} = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} = D_v f(x)$$

We also have

$$\frac{d}{\delta f} f(x_0 + t\overline{v}) \mid_{t=0} = df(x_0)(v) = J_f(x_0) \cdot v$$

$f : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at $x_0 \in X$ then we have

- f is continuous at x_0
- f has all partial derivatives at x_0 and the matrix represents the linear map $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto Ax$ in the canonical basis is given by the Jacobi matrix of f at x_0 i.e

  $$A = J_f(x_0) = (\frac{\delta f_i}{\delta x_j})1 \leq i << m, 1 \leq j \leq n$$

  A is an $m \times n$ matrix
- $f, g : X \to \mathbb{R}^n$ are differentiable in x_0 then so is f+g and

  $$d(f + g)(x_0) = df(x_0) + dg(x_0)$$

  the sum of 2 linear maps
- If m $= 1$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ differentiable in x_0 then so is $f \cdot g : \mathbb{R}^n \to \mathbb{R}$ and if $g \neq 0$ then also $\frac{f}{g}$ is differentiable in x_0

- If $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ has all parital derivatives $\frac{\delta f_i}{\delta x_j} : X \to \mathbb{R}^m$ and if these functions are continuous on X then f is differentiable on X

A linear map $L : \mathbb{R} \to \mathbb{R}$ has the form L(x)=ax sor some $a \in \mathbb{R}$. The Graph of L must go through (0,0). An affine function is a linear function shifted by an amount

We have seen that if the partial derivatives exist in x_0 then

- $\not\Rightarrow$ f is differentiable
- $\not\Rightarrow$ f is continuous

But we have

Partial derivatives exist and they are continuous $\Rightarrow$ f is differentiable

Tangent space at x_0 to the graph of f $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ idfferentiable at x_0 with the differential

$$df(x_0) = u = \mathbb{R}^n \to \mathbb{R}^m$$

The graph of the affine linear approximation

$$g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = f(x_0) + u(x - x_0)$$

The graph of g is

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y = f(x_0) + u(x - x_0)\}$$

In particular f

$f = \mathbb{R}^2 \to \mathbb{R}$. diff at $\overline{x_0} = (x_0, y_0)$

$$df(x_0) = u : \mathbb{R}^2 \to \mathbb{R}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$g(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot (\overline{x} - \overline{x_0})$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

$\overline{x} = (x,y)$ $\quad \overline{x_0} = (x_0, y_0)$

Chain Rule Let $X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open and $f : X \to Y, g : Y \to \mathbb{R}^p$ differentiable functions then $g \circ f : X \to \mathbb{R}^p$ is differentiable in X And for any $x_0 \in X$ ist differential is given by the composition

$$d(g \circ f)(x_0) : X \to \mathbb{R}^p d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

In particular the Jacobi Matrix of $g \circ f$ at x_0 satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0)$$

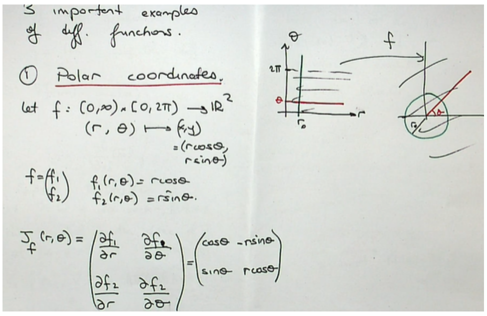
Geometric meaning of gradient: Let $\overline{v}$ be a unit vector i.e ($\|v\| = 1$) The directional derivative of f in teh direction of $\overline{v}$ at the point $\overline{x_0}$ is given by

$$\langle \nabla f(x_0), \overline{v} \rangle = \|\nabla f(x_0)\| \cdot \|v\| cos(\theta)$$

We maximizie the directional derivative at x_0 if we maximize $cos(\theta)$ i.e when $\theta = 0$ i.e in the direction of the gradient. The gradient is the direction of largest change.

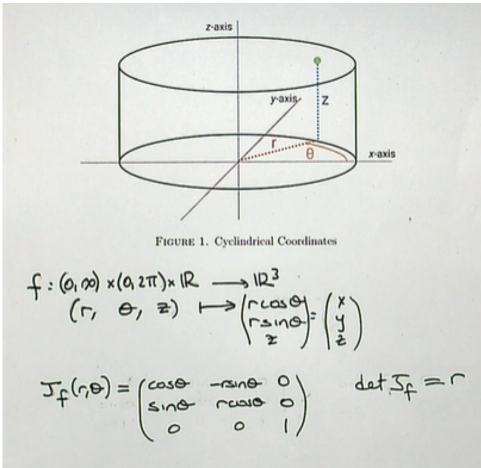
2.2 3 important examples of differential functions

2.2.1 Polar coordinates

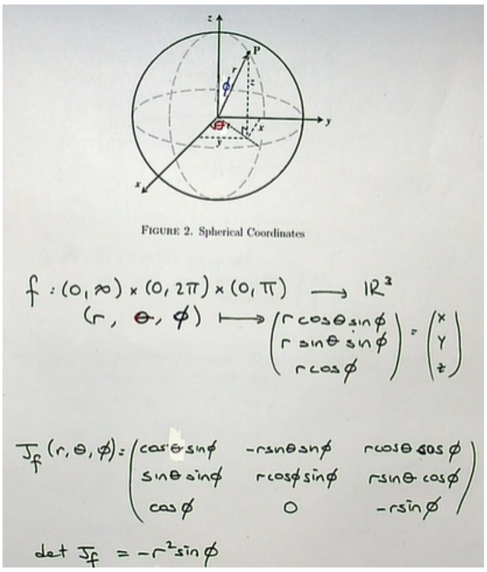


$det(J_f(r, \theta)) = r$

2.2.2 cylindrical coordinates



2.2.3 Spherical coordinates



2.2.4 Change of variables

Let $X \subset \mathbb{R}^n$ an open set and $f : X \to \mathbb{R}^n$ differentiable we say f is a change of variables around x_0 if there is a radius $\rho > 0$ such that the restriction of f to the Ball around x_0 of radius ρ

$$B = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \rho\}$$

so that the image $Y = f(B)$ is open in $\mathbb{R}^n$ and $\exists$ a differential map $g : Y \to B$ such that $f \circ g = id_Y, g \circ f = id_B$ i.e $f\mid_{B_\rho(x_0)}$ is a bijection to the image with a inverse g which is also differentiable

2.2.5 Inverse function theorem

Let $X \subseteq \mathbb{R}^n$ open $f : X \to \mathbb{R}^n$ differentiable if $x_0 \in X$ is such that $det(J_f(x_0)) \neq 0$ i.e $J_f(x_0)$ is invertible then f is a change of variables around x_0 Moreover the Jacobian of g at x_0 is defined by $J_g(f(x_0)) = J_f(x_0)^{-1}$

2.3 Higher Derivatives

$X \subset \mathbb{R}^n, f : X \to \mathbb{R}^m$ we say of class C^1 if f is differentiable on X and all of its partial derivatives are continuous. The set of such functions we denote

$$C^1(X : \mathbb{R}^m)$$

let $k \geq 2$, we say $f \in C^k$ if it is differentiable and each $\delta_{x_i} f : x \to \mathbb{R}^m$ is of class C^{k-1} The set of such functions we denote

$$C^k(X, \mathbb{R}^m)$$

f is **smooth** or C^∞ if $f \in C^k \forall k$
The composition of smooth functions are smooth.
All polynomials,trig functions and exponentials are smooth.

2.3.1 Hessian Matrix

For $f \in C^k, k \geq 2$ then the partial derivatives of order $\leq k$ are independant of order of differentiation i.e Mixed partial derivatives up to order k all commute
Example: If $k = 2, f \in C^2$ then

$$\frac{\delta^2 f}{\delta x_i \delta x_j} = \frac{\delta^2 f}{\delta x_j \delta x_i}$$

If $f \in C^2(X \to \mathbb{R}), X \subset \mathbb{R}^n$ then the $n \times n$ matrix

$$(\frac{\delta^2 f}{\delta_{x_i} \delta_{x_j}}(x_0)) =: Hess_f(x_0)$$

is called the **Hessian** of f at x_0
Since it is C^2 and the order of differentiation does not matter $H = Hess_f(x_0)$ is a symetric matrix $H^T = H$
Example:

eg: $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x, y, z) \longmapsto x^2y + yz$

$\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2 + z \quad \frac{\partial f}{\partial z} = y$

$\frac{\partial^2 f}{\partial x^2} = 2y \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad \frac{\partial^2 f}{\partial z^2} = 0.$

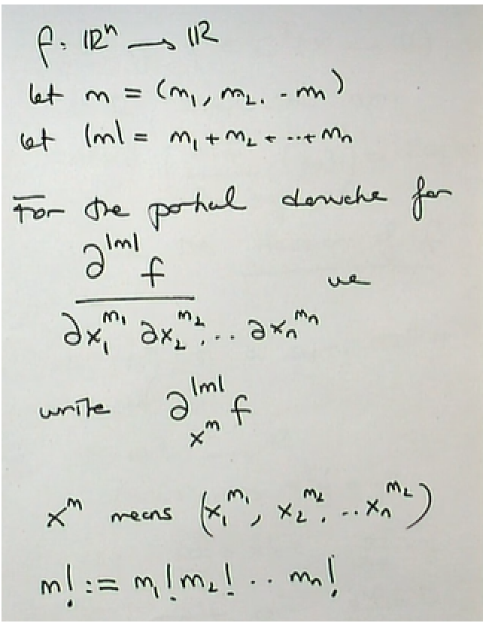
$$\frac{\partial^2 f}{\partial x \partial y} = 2x = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0 = \frac{\partial^2 f}{\partial z \partial x}$$

$$\frac{\partial^2 f}{\partial y \partial z} = 1 = \frac{\partial^2 f}{\partial z \partial y}.$$

$$Hess_f(x, y, z) = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 0 & 1 \\ 0 & 1. & 0 \end{pmatrix}$$

Notation: When we are dealing with partial derivatives of higher order we use multi index notation:



All the eigenvalues of the Hessian matrix are real valued.

2.4 Affine Linear Approximations to f

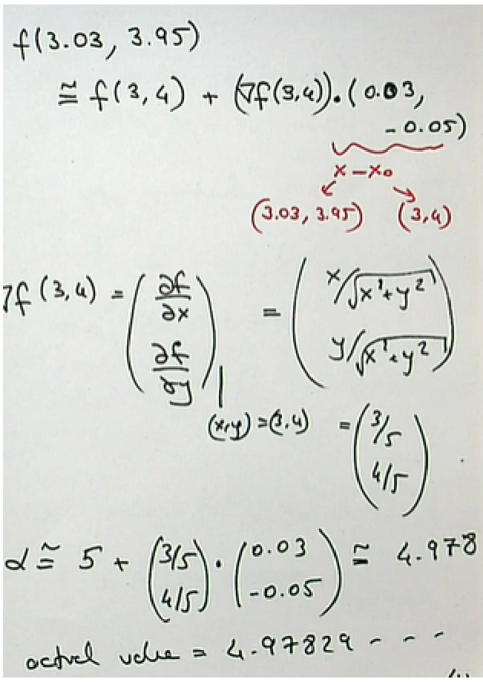
The first order approximation to $f : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x}_0$ is given by:

$$f(x) = f(\overline{x}_0 + \nabla f(x_0) \cdot (\overline{x} - \overline{x}_0) + E_1 f(x, x_0)$$

Example: Find an approximate value for the number

$$\alpha = \sqrt{(3.03)^2 + (3.95)^2}$$

$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \sqrt{x^2 + y^2}$ at $x_0 = (3, 4)$
We have $f(x_0) = 5$



2.4.1 Taylor polynomial of f at x_0 of order 1

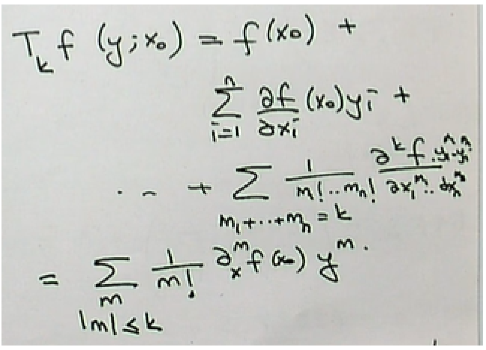
For $f : \mathbb{R}^n \to \mathbb{R}$

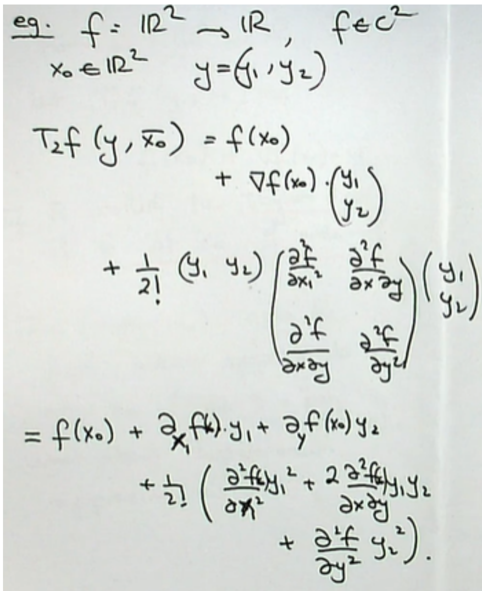
$$T_1 f(y; x_0) := f(x_0) + \nabla f(x_0) \cdot y$$

$T_1 f(x - x_0, x_0)$gives the first order approximation to f at x_0

2.4.2 k-th Taylor polynomial of f at x_0

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}, f \in C^k, x_0 \in X$
The k-the Taylor polynomial of f at x_0 is a polynomial in n-variables of degree $\leq k$ given by:





2.5 Critical points and extrema of functions $f : \mathbb{R}^n \to \mathbb{R}$

2.5.1 Local Maxima/Minima

$f : X \subset \mathbb{R}^n \to \mathbb{R}$ differentiable we say $x_0 \in X$ is a local maximum(min) if we can find a neighbourhood

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$$

at $B_r(x_0) \subset X$ and $\forall x \in B_r(x_0), f(x) \leq f(x_0)($ resp $f(x) \geq f(x_0)$

$f : \mathbb{R} \to \mathbb{R}$ if f has a local min or max at x_0 then $f'(x_0) = 0$
The same is for mulitvariable functions i.e let $X \subset \mathbb{R}^n$ open $f : X \to \mathbb{R}$ is differentiable if $x_0 \in X$ is a local extrema (i.e a min or max) then $\nabla f(x_0) = 0$ i.e

$$\frac{\delta f}{\delta x_1}(x_0) = \frac{\delta f}{\delta x_2}(x_0) = ... = \frac{\delta f}{\delta x_n}(x_0) = 0$$

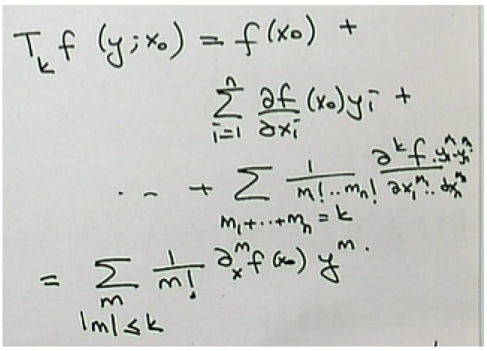
2.5.2 Critical Point

A point $x_0 \in X$ if $\nabla f(x_0) = 0$
Critical points are candidates for local extrema

2.5.3 Saddle Point

A critical point which is not a local min or max is called a saddle point

if $f : [a, b] \to \mathbb{R}$ the global extrema of f is either at an interior point $x_0 \in (a, b)$ for which $f'(x_0) = 0$ or at $x = a, x = b$


$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) y_i + \ldots + \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1!\ldots m_n!} \frac{\partial^k f}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}} y_1^{m_1}\ldots y_n^{m_n}$
$= \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m.$

non-degenerate critical point: A critical point x_0 of $f \in C^2$ for which $det(Hess_f(x_0)) \neq 0$

2.5.4 Role of the second derivative

In several veriables, the role of $f''(x_0)$ is played by the $Hess_f(x_0)$

A symmetric matrix A with $\det(A) \neq 0$ is

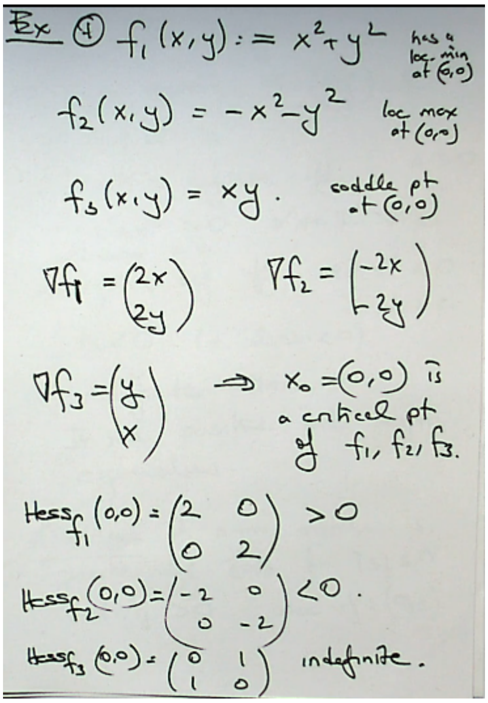
- **Positive definite:** iff $xAx^T > 0, \forall x \in \mathbb{R}^n \backslash \{0\}$ i.e all eigenvalues are greater than 0
- **negative definite:** iff $xAx^T < 0$ i.e all eigenvalues are less than 0
- **Indefinite:** If A has positive and negative eigenvalues

A symmetric matrix is positive definit iff $i \leq j \leq n, det(A_j) > 0$ where $A_j = (a_{kl}) 1 \leq k \leq i, 1 \leq l \leq j$

Theorem: $f : X \subset \mathbb{R}^n \to \mathbb{R}, f \in C^2$ Let x_0 be a non-degenerate critical point of f i.e $(\nabla f(x_0) = 0, det(Hess_f(x_0)) \neq 0)$ then

- If $Hess_f(x_0) > 0$(positive definite) then x_0 is a local minimum
- If $Hess_f(x_0) < 0$(negative definite) then x_0 is a local maximum
- If $Hess_f(x_0)$ is indefinite then x_0 is a saddle

Example:



2.5.5 Degenerate critical point

A critical point with $\det(Hess_f(x_0)) = 0$
We cannot use the above theorem to decide on the nature of x_0. We have to decide case by case.

Chapter 3

useful equations

- $\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$
- $sin(t) = t + o(t)$ as $t \to 0$
- **Formula for a tangent plane:** $z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Chapter 4

Integration in $\mathbb{R}^n$

4.1 Derivatives and Integrals $f : \mathbb{R} \to \mathbb{R}^n$

we have $t \mapsto \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$
The derivative of f is simply:

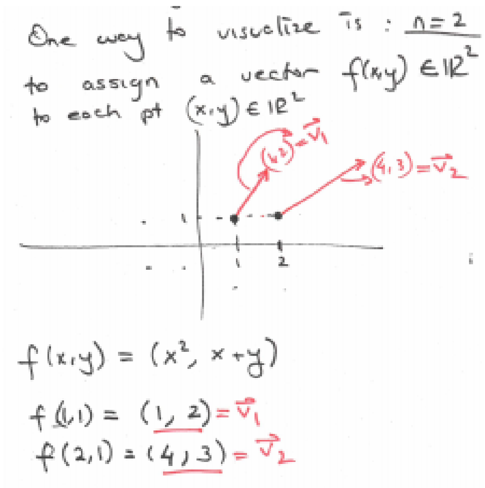
$$f'(t) := \begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix}$$

Similarly the integral of f from a to b is given by:

$$\int_a^b f(t)dt := \begin{pmatrix} \int_a^b f_1(t)dt \\ \vdots \\ \int_a^b f_n(t)dt \end{pmatrix}$$

4.2 Vector Fields

Mappings $f : \mathbb{R}^n \to \mathbb{R}^n$
A vectorfield can be visualized in $\mathbb{R}^2$ as follows:

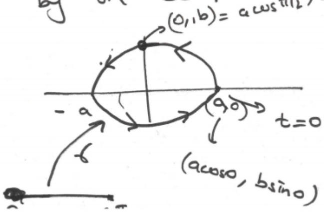


4.2.1 Curves in $\mathbb{R}^n$

This is simply the image of a function $\gamma : [a,b] \to \mathbb{R}^n$, where the function γ, is continous and its piecewise C^1 i.e $\exists k > 1$ and a partition $a = t_0 < t_1 ... < t_k = b$ such that $\gamma\,|_{t_j, t_j+1} \in C^1$. In general γ is called a **parameterization** of the curve.
Examples:

<u>example</u> $\gamma : [0, 2\pi] \longrightarrow (a\cos t, b\sin t)$

is a curve which is given by an ellipse.



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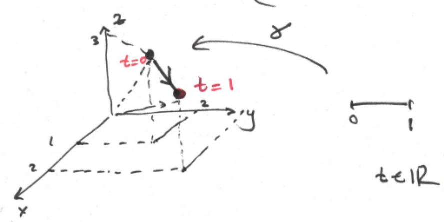
② $\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longrightarrow (1+t,\ 2+t,\ 3-t)$



In general if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$

$\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longmapsto (a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t)$

is the parametrization of the line segment between the vector $\vec{a}$, and $\vec{a} + \vec{b}$.

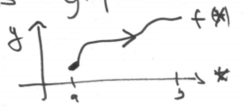
we get a line segment

$r(t) = \vec{a} + \vec{b}t \qquad t \in [0,1]$.

2.

③ $f : [a,b] \longrightarrow \mathbb{R} \ . \in C^1$

Then its graph is a curve in $\mathbb{R}^2$.



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A parametrization

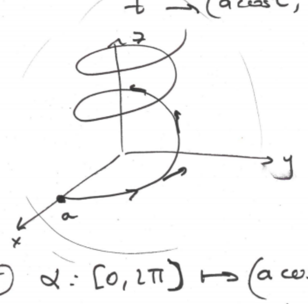
$\gamma : [a,b] \longrightarrow \mathbb{R}^2$

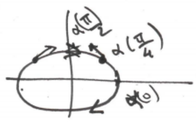
$t \longmapsto (t, f(t))$

3.

④. $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longrightarrow (a\cos t, a\sin t, t)$





⑤ $\alpha : [0, 2\pi] \longmapsto (a\cos(2\pi - t),\ b\sin(2\pi - t))$

traces the same ellipse but in the opposite direction.

$\gamma'(t)$ gives the tangent vector at t, to the curve.

4.

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parametrization of a curve, let $X \subset \mathbb{R}^n$ be a subset which contains the image of γ, let $f : X \to \mathbb{R}^n$ a continuous function. The integral

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \in \mathbb{R}$$

is called the **line integral** of f along γ
It is denoted by

$$\lim_{\gamma} f(s)ds$$

If $f := \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$, $f : \mathbb{R}^n \to \mathbb{R}^n, f_i = \mathbb{R}^n \to \mathbb{R}$, another notation that is used for the line integral is:

$$\int f(s)ds = \int f_1(x)dx_1 + ... + f_n(x)dx_n$$

$$f: \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

eg. $\gamma : I \to \mathbb{R}^2$
$t \to (\gamma_1(t), \gamma_2(t)) = (x, y)$
$x = \gamma_1(t) \quad y = \gamma_2(t)$
$dx = \gamma_1'(t)dt \quad dy = \gamma_2'(t)$

$$\int_\gamma f ds = \int_a^b \begin{pmatrix} f_1(\gamma(t)) \\ f_2(\gamma(t)) \end{pmatrix} \cdot \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$$

$$= \int_a^b \left(f_1(\gamma(t)) \cdot \gamma_1'(t) + f_2(\gamma(t)) \cdot \gamma_2'(t)\right) dt$$

$$\int_a^b f_1(x,y)\, dx + f_2(x,y)\, dy.$$

Ex: ① $f(x,y) = (-y, x)$
$\gamma(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$$\int_\gamma f(s)ds = \int_0^{2\pi} \underbrace{f(\gamma(t))} \cdot \underbrace{\gamma'(t)}\, dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

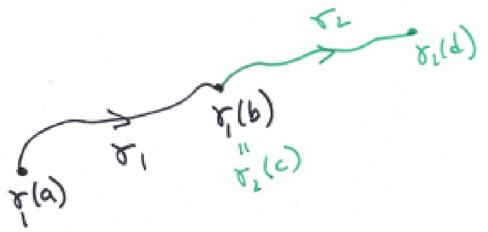
When we reverse the curve, the line integral changes sign.

4.3 Properties of the line integral

- Independent of orientation preserving reparametrization of the curve.
  i.e if $\gamma : [a, b] \to \mathbb{R}^n$ a C^1 curve and let $\tilde{\gamma} : [c, d] \to \mathbb{R}^n$ such that $\tilde{\gamma} = \gamma \circ \phi$ where $\phi : [c, d] \to [a, b]$. ϕ is C^1 such that $\phi(c) = a, \phi(d) = b$ with $\phi' > 0, \forall t \in [c, d]$, then

$$\int_\gamma f ds = \int_{\tilde{\gamma}} f ds$$

- path concatination preserving.
  Let $\gamma_1 : [a, b] \to \mathbb{R}^n, \gamma_2 : [c, d] \to \mathbb{R}^n$ 2 paths with $\gamma_1(b) = \gamma_2(c)$

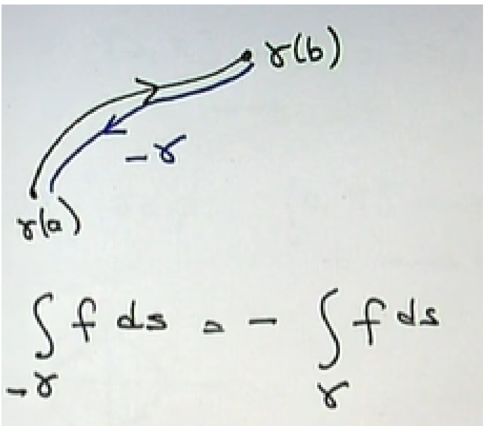


we define $\gamma_1 + \gamma_2$ as the path formed by concatination of the 2 curves.

$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t - b + c) & t \in [b, d + b - c] \end{cases}$$

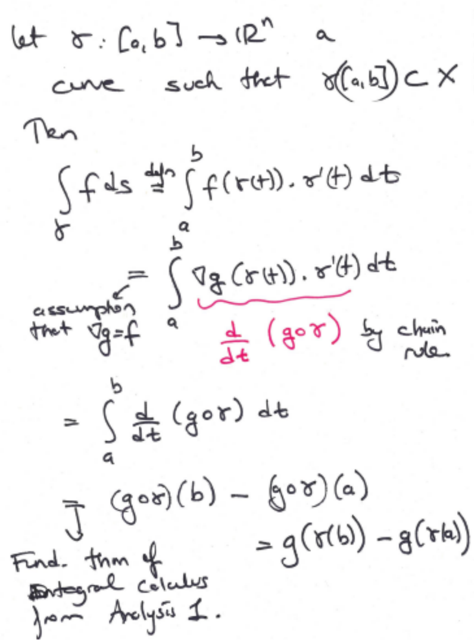
then $\int\limits_{\gamma_1+\gamma_2} fds = \int\limits_{\gamma_1} fds + \int\limits_{\gamma_2} fds$

- Reversal of the path is opposite sign.
  if $\gamma : [a, b] \to \mathbb{R}^n$ a path, let $-\gamma : [a, b] \to \mathbb{R}^n$ same path traced in the opposite direction i.e $(-\gamma)(t) := \gamma(a + b - t)$



4.4 Potential of a function f

A differentiable scalar field $g : X \subset \mathbb{R}^n \to \mathbb{R}$ such that $\nabla g = f, f : X \to \mathbb{R}^n$



Remarks:

- If n=1, a potential is same as the primitive of f. g is a primitive of f if $g' = f$
- If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f always has a primitive namely $g(x) := \int\limits_a \to xf(t)dt$

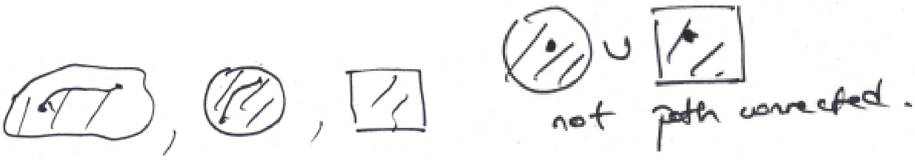
For multivariable functions continuouty is not enough to guarantee the existance of a potential.

4.5 Conservative

A Line integral of f is independant of the path of integration when it only depends on the end points of the path Let $X \subset \mathbb{R}^n, f : X \to \mathbb{R}^n$ be a continous vector field. If for any $x_1, x_2 \in X$ the line integral $\int\limits_\gamma fds$ is independant of the curve in x from x_1 to x_2 then we say the vector field f is conservative.

4.5.1 Path connected

Let $X \subset \mathbb{R}^n$ open X is said to be path connected if for every pair of points $x, y \in X, \exists$ a C^1 path $\gamma : (0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ Examples

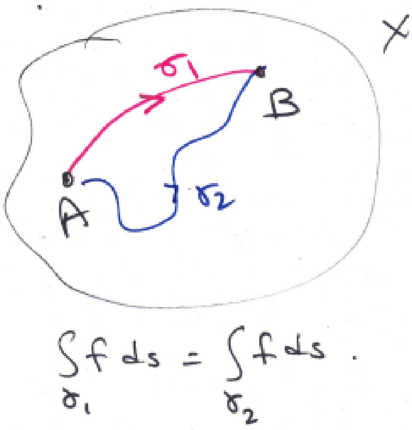


the left picture shows sets which are path connected. The right picture shows that the union of those sets is not.

Theorem

Let f ba a continuous vector field on an open path connected set $X \subset \mathbb{R}^n$, then the following are equivalent

1. f is the gradient of a function $g : X \to \mathbb{R}$ i.e $f = \nabla g$, g is a potential for f.
2. the line integral of f is independant of the path between 2 pts i.e if $\gamma_1[a,b] \to X, \gamma_2 : [c,d] \to X$ are 2 curves and both have the same beginning and end points i.e $\gamma_1(a) = \gamma_2(c) = A, \gamma_1(b) = \gamma_2(d) = B$

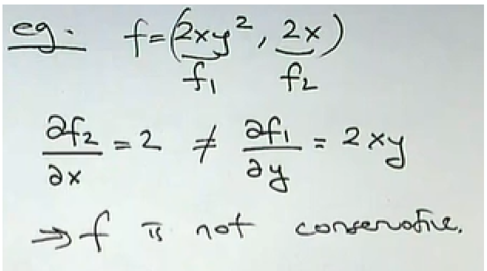


3. The line integral of f around any closed curve is 0

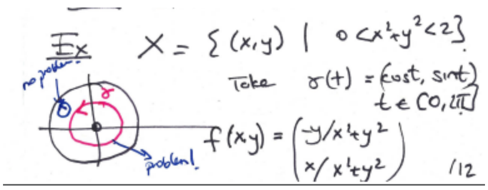
4.5.2 Checking if a vector field is conservatice

Let $X \subset \mathbb{R}^n$ be open $f : X \to \mathbb{R}^n$ a C^1 Vector field $f = (f_1, ..., f_n)$ if f is conservative then

$$\frac{\delta f_i}{\delta x_j} = \frac{\delta f_j}{\delta x_i}$$



WARNING: This criteria is necessary but not sufficient to proof that f is conservative. i.e



Then check.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$

But $\int_\gamma f \, ds \neq 0$. $\quad \| \quad 2\pi$.

ie f is not conservative

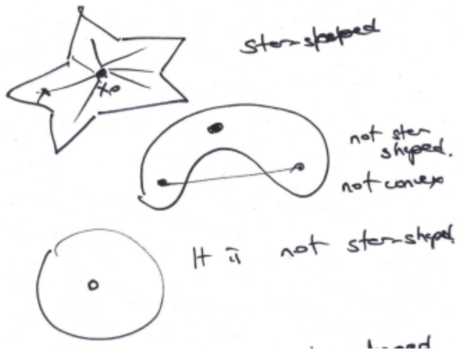
yet $\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$.

4.5.3 Star shaped

A subset $X \subset \mathbb{R}^n$ is called star shaped if $\exists x_0 \in X$ such that $\forall x \in X$, the line segment joining x_0 to x is contained in x

Convex $\Rightarrow$ starshaped

Convex means any 2 points in the vector field can be joined by a line in the set.



Let X be a star-shaped open subset of $\mathbb{R}^n, f \in C^1$ a vector field such that $\frac{\delta f_i}{\delta x_j} = \frac{\delta f_j}{\delta x_i}$ on x $\forall i, j$ then f is conservative

4.5.4 Curl

Let $X \subset \mathbb{R}^3$ open $f : X \to \mathbb{R}^3, C^1$ a vectorfield. Then the curl of f is the vector field on X defined by

$$\boxed{\underline{\mathrm{curl}(f)} := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_3 \end{pmatrix}.}$$

where $f : X \longrightarrow \mathbb{R}^3$

$\bar{x} \longrightarrow \begin{matrix} f_1(\bar{x}) \\ f_2(\bar{x}) \\ f_3(\bar{x}) \end{matrix}$

$\bar{x} = (x, y, z)$

<u>Rk</u> This can be remembered more easily by the formal determinant

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$