Drawing Triangles

Visual Computing Part II: Computer Graphics

Why should we care about triangles?

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https://www.youtube.com/watch?v=4kOvYQW0AJM

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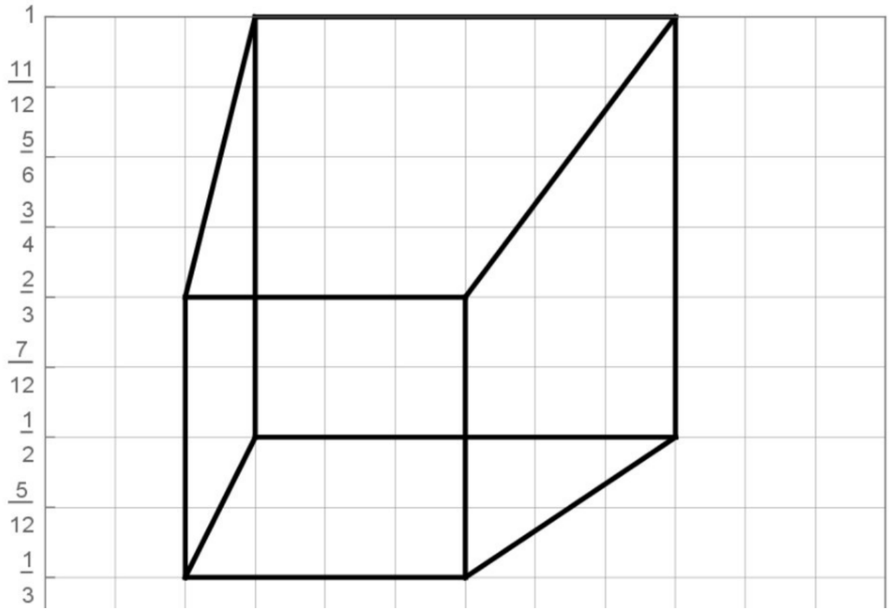
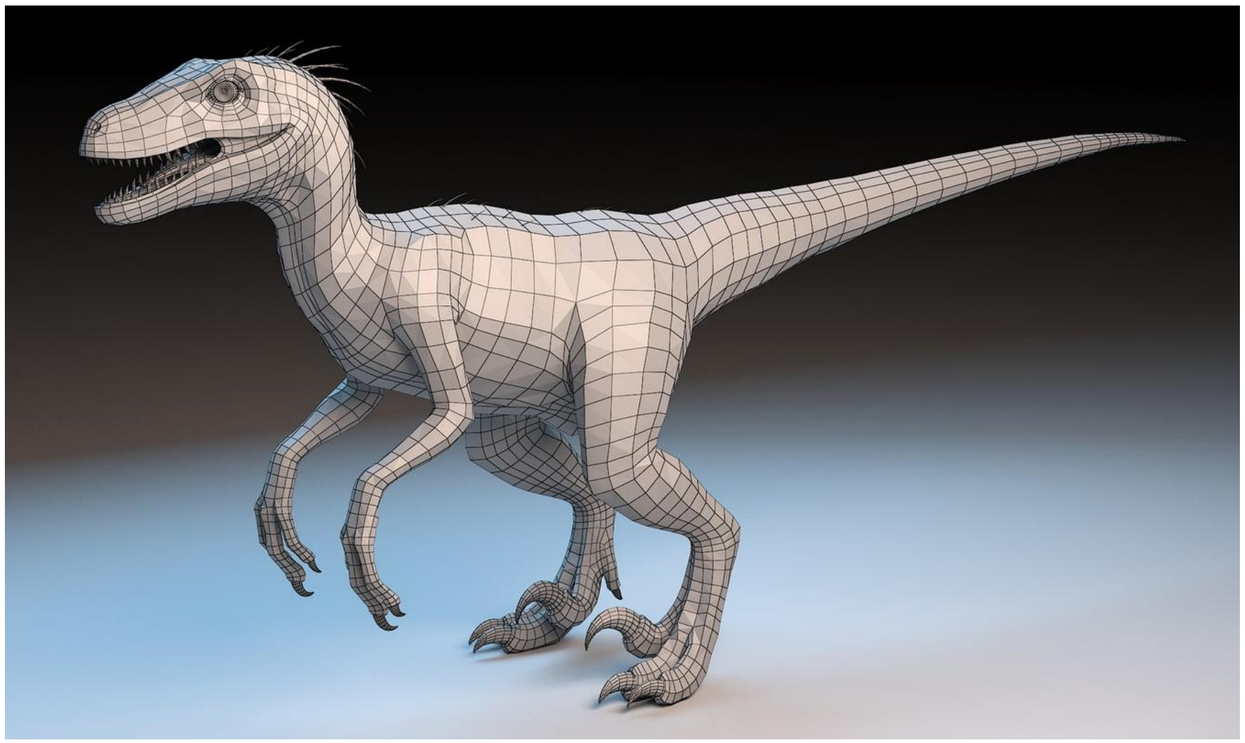


2D coordinates:

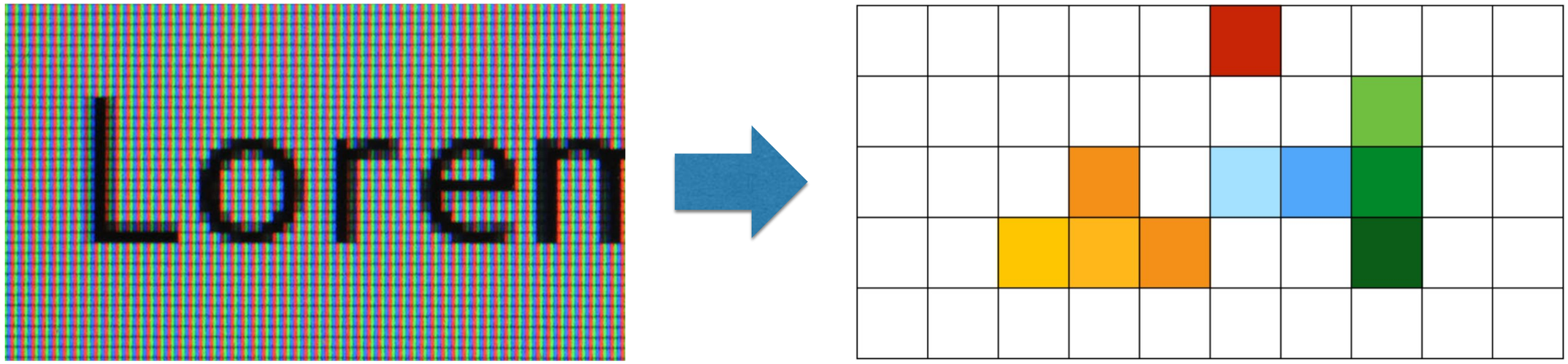
A: 1/4, 1/2
B: 3/4, 1/2
C: 1/4, 1
D: 3/4, 1
E: 1/6, 1/3
F: 1/2, 1/3
G: 1/6, 2/3
H: 1/2, 2/3



If you know how to draw a triangle, you'll go a long way!

Why triangles and not other shapes, like squares, circles, or stars?

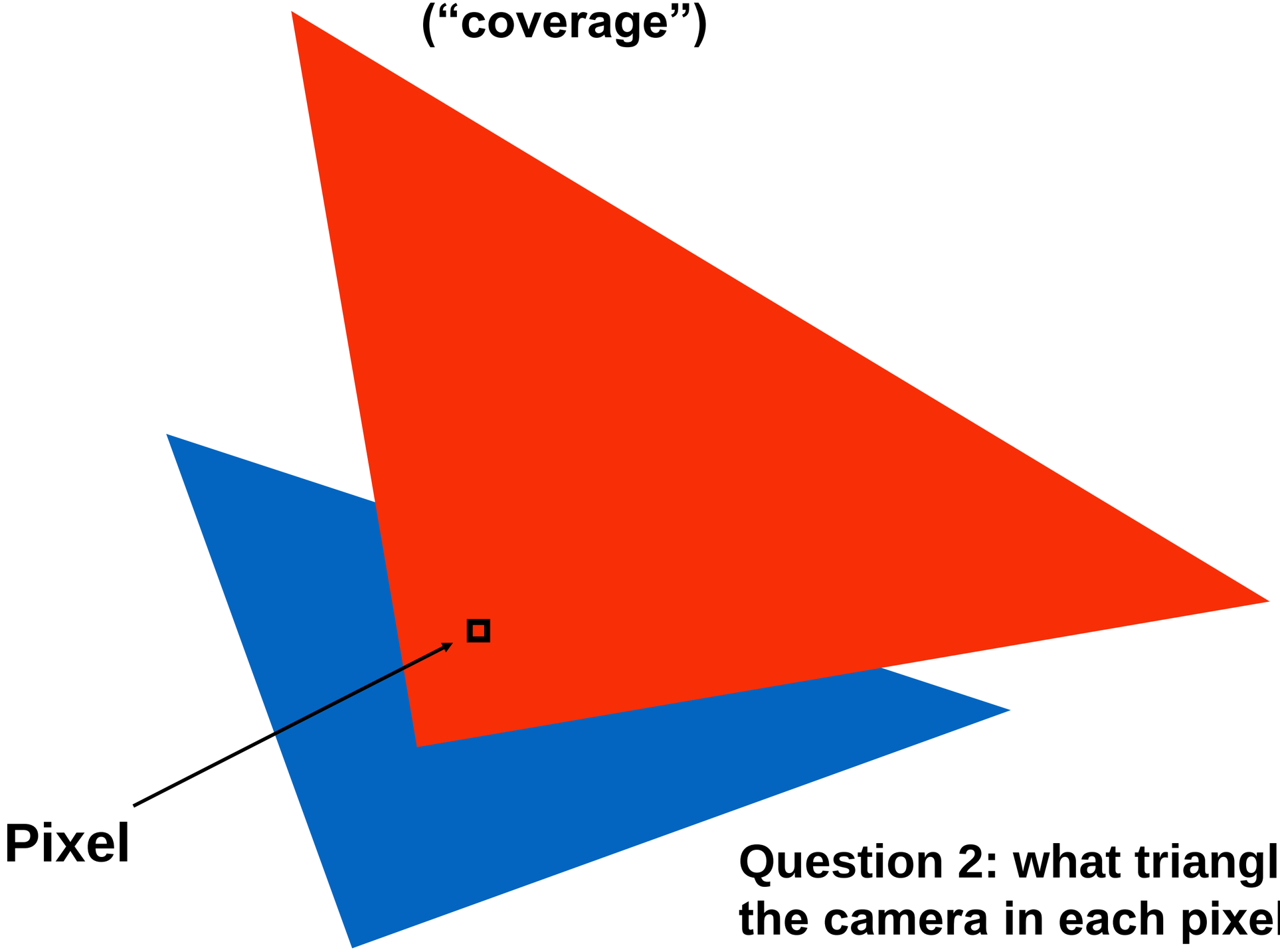
Let’s draw some triangles on the screen!



Must assign a color to each pixel on the screen

Let’s draw some triangles on the screen!

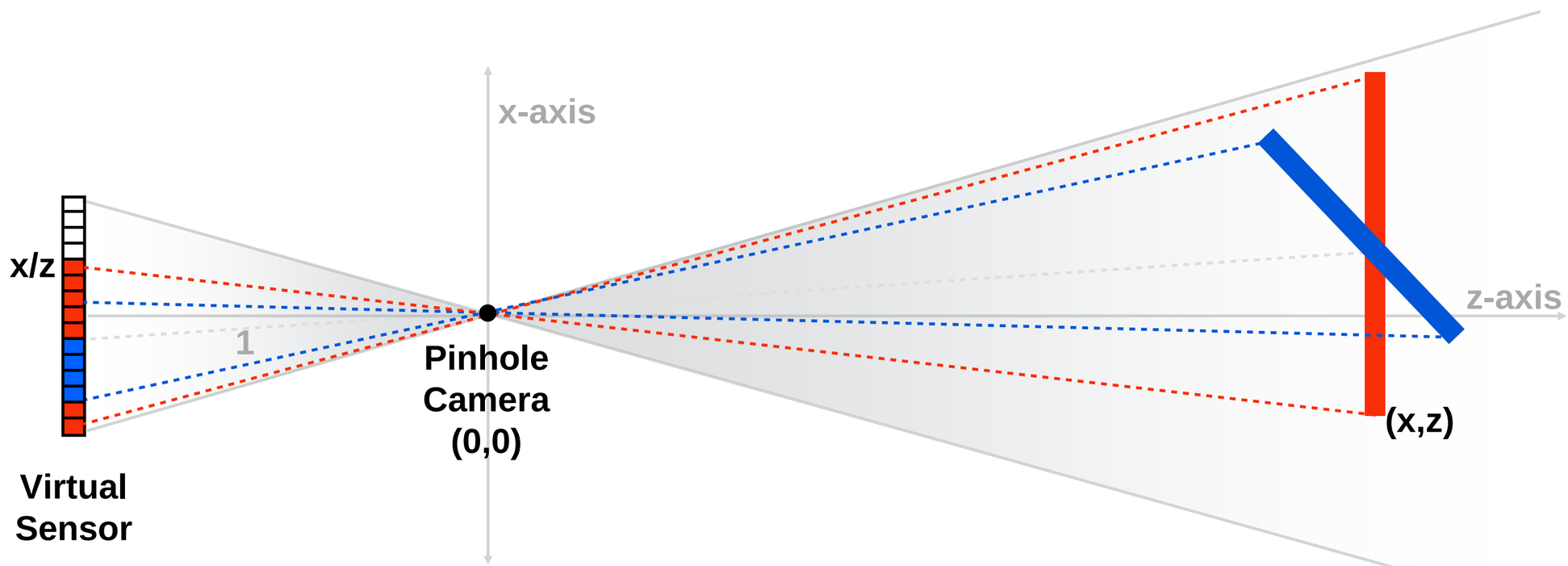
Question 1: what pixels does the triangle overlap? (“coverage”)



Question 2: what triangle is closest to the camera in each pixel? (“occlusion”)

The visibility problem

- **An informal definition: what scene geometry is visible within each screen pixel?**
  - **What scene geometry projects into a screen pixel? (coverage)**
  - **Which geometry is visible from the camera at that pixel? (occlusion)**



Recall perspective projection from first class

The visibility problem

- **An informal definition: what scene geometry is visible within each screen pixel?**
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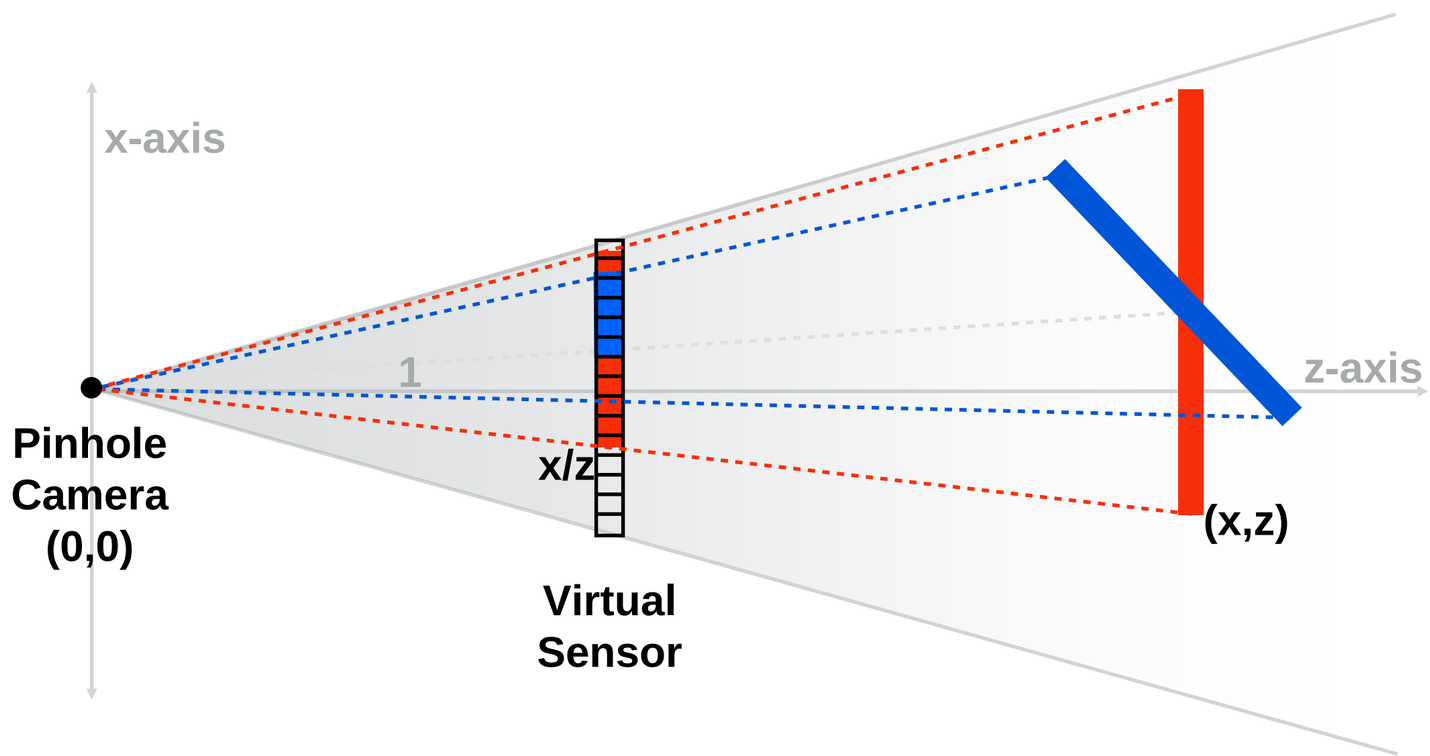

The visibility problem

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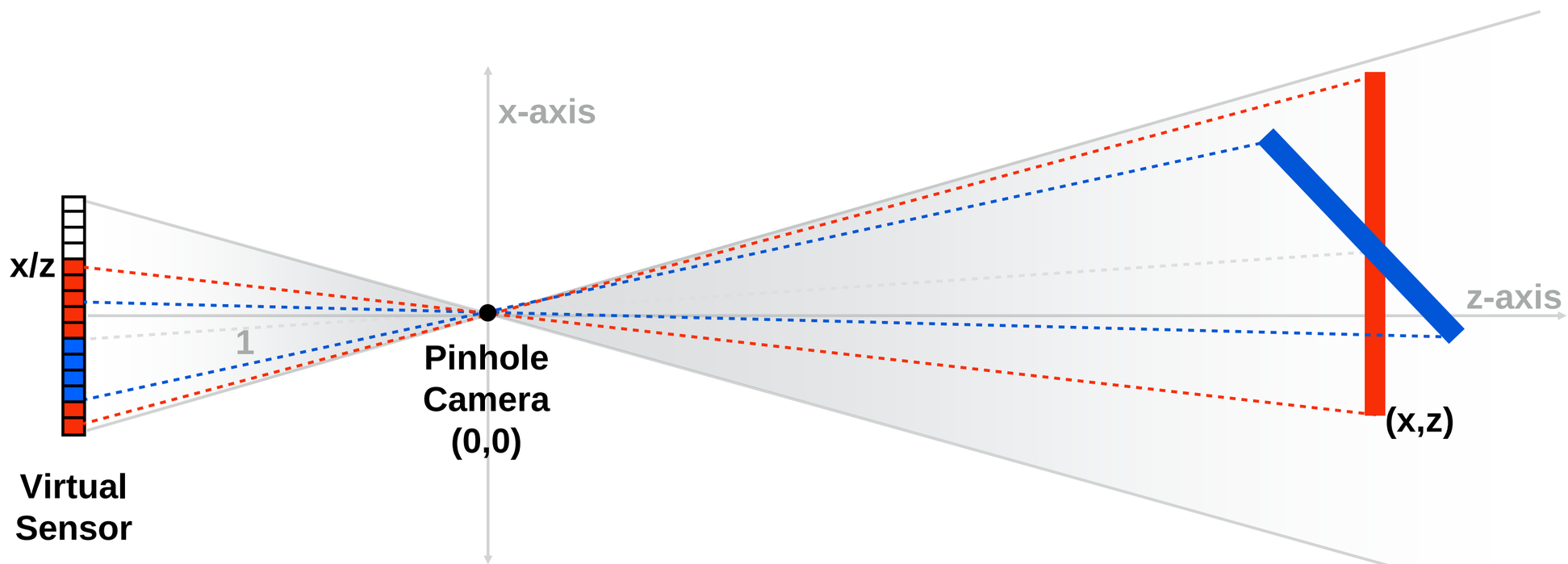
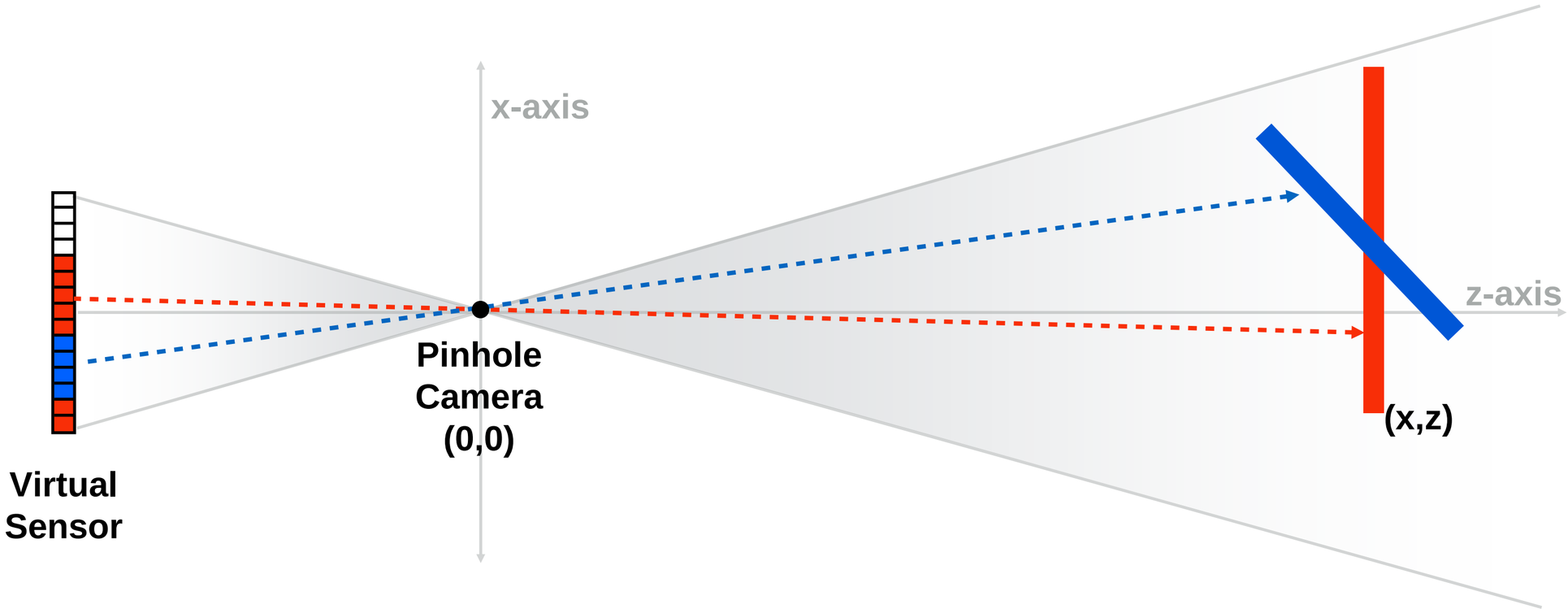


Think about light bouncing around from source, to objects in scene through pinhole to the screen

The visibility problem (said differently)

- **In terms of rays:**
  - **What scene geometry is hit by a ray from a pixel through the pinhole? (coverage)**
  - **What object is the first hit along that ray? (occlusion)**

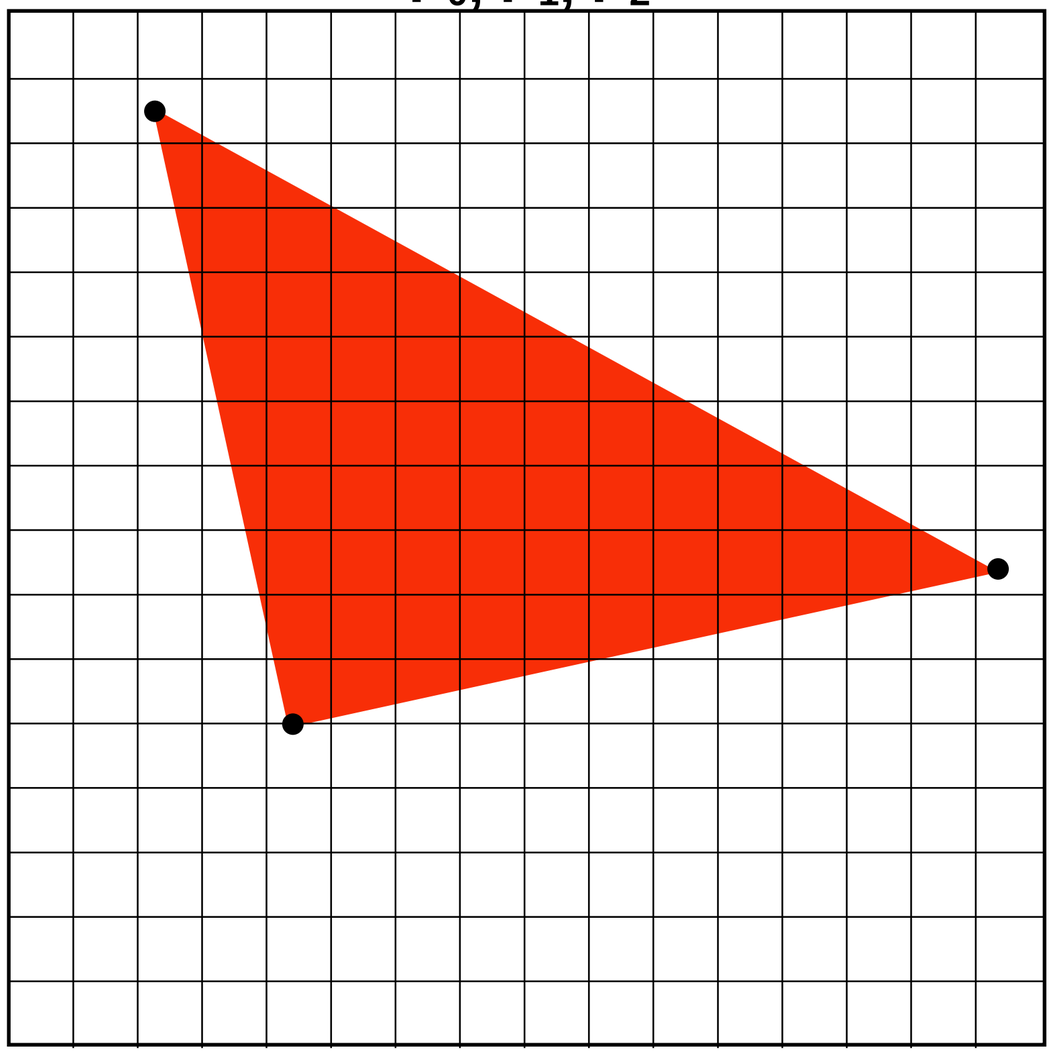


Hold onto this thought for later on in the semester.

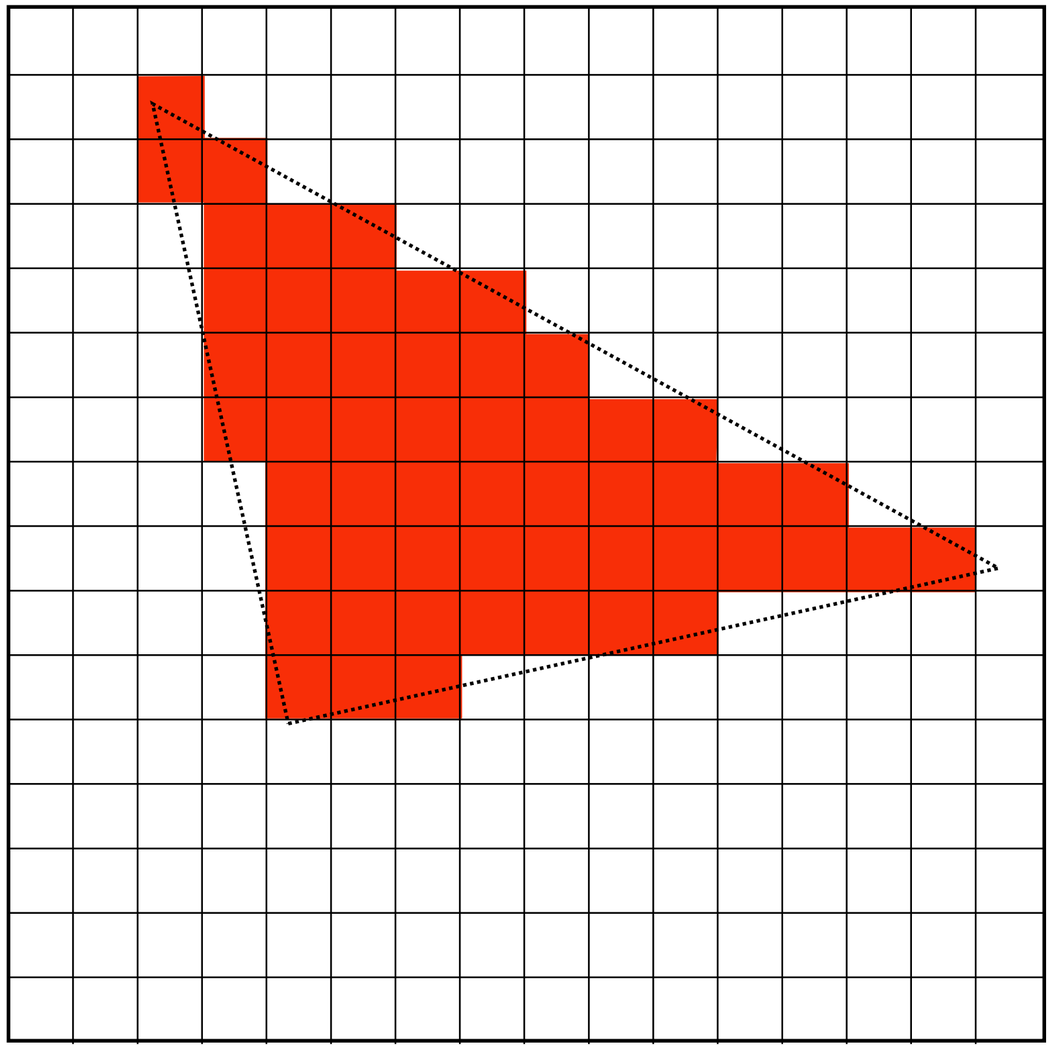
Computing triangle coverage

Which pixels does the triangle overlap?

Input:
projected position of triangle vertices:
P_0, P_1, P_2



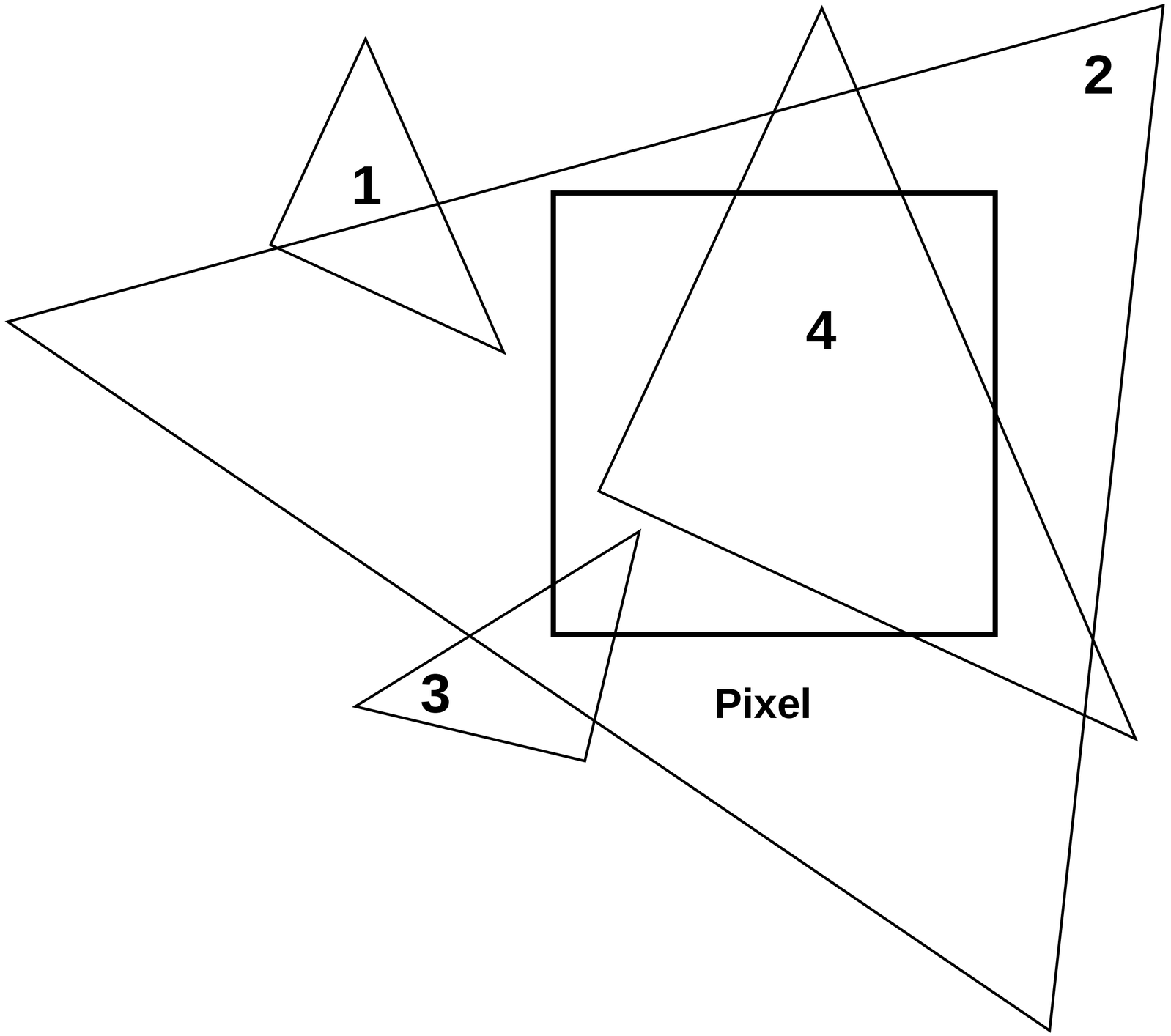
Output:
set of pixels “covered” by the triangle



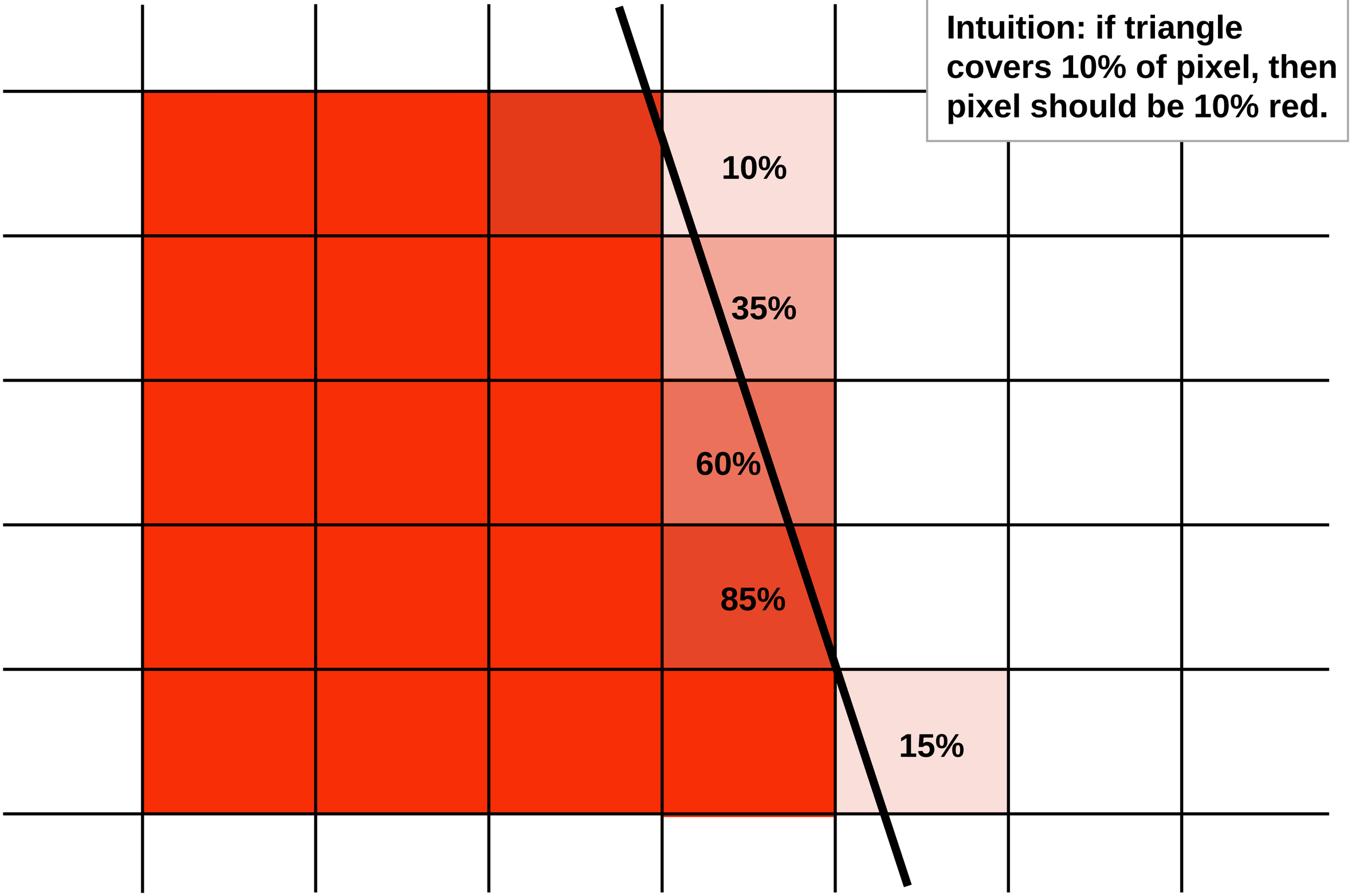
Must represent a continuous signal using a discrete approximation!

What does it mean for a pixel to be covered by a triangle?

Question: which triangles “cover” this pixel?

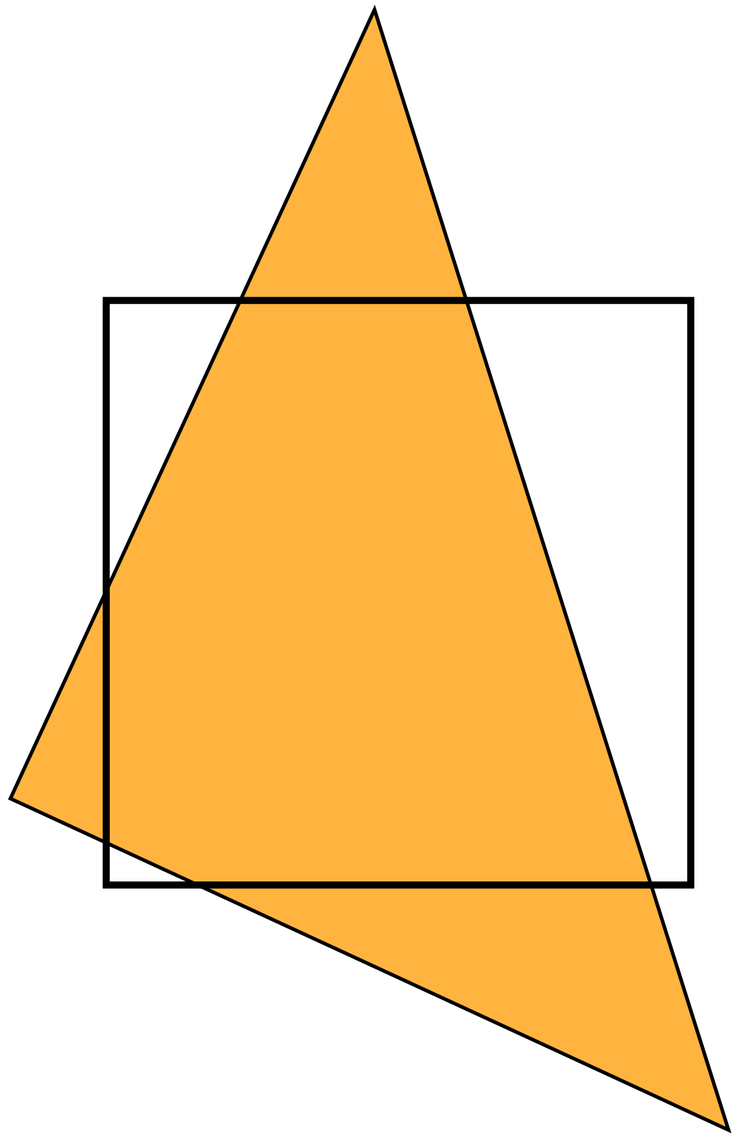


One option: compute fraction of pixel area covered by triangle, then color pixel according to this fraction.

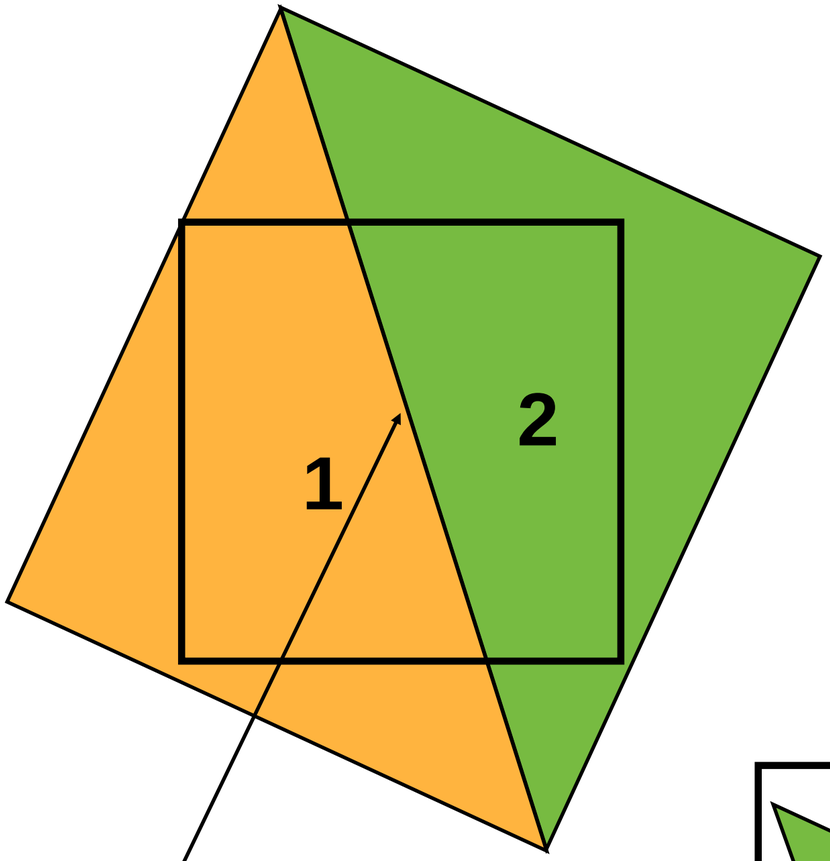


Intuition: if triangle covers 10% of pixel, then pixel should be 10% red.

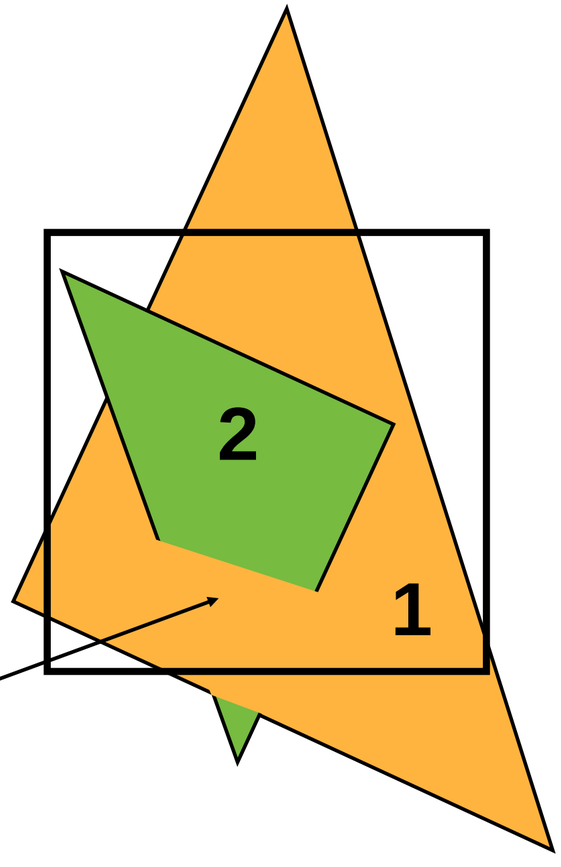
Computing amount of overlap?



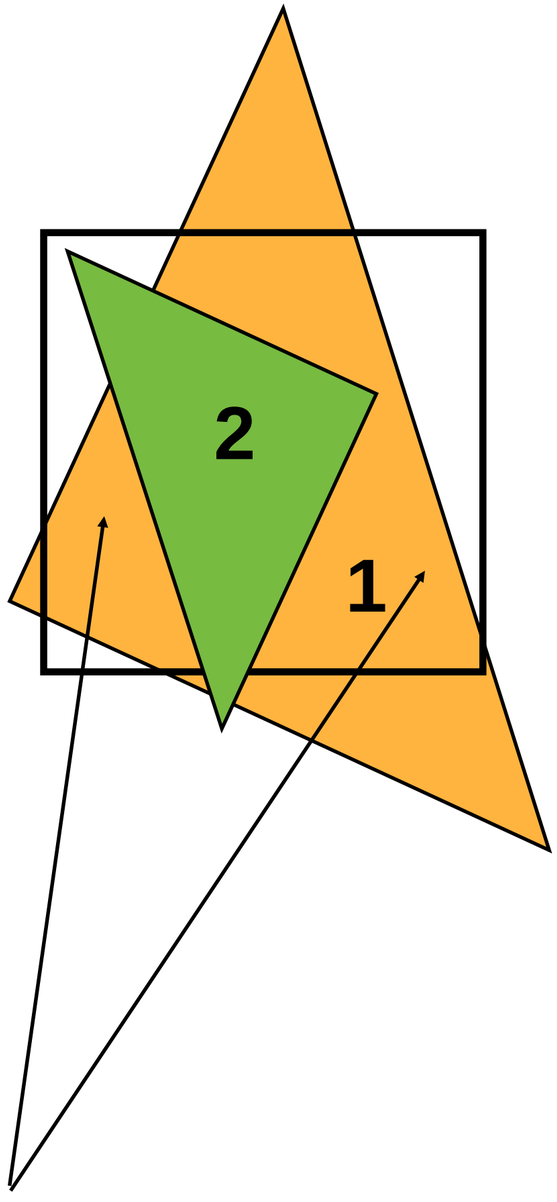
Analytical schemes can get quite tricky, especially when considering interactions between multiple triangles



Pixel covered by triangle 1, other half covered by triangle 2

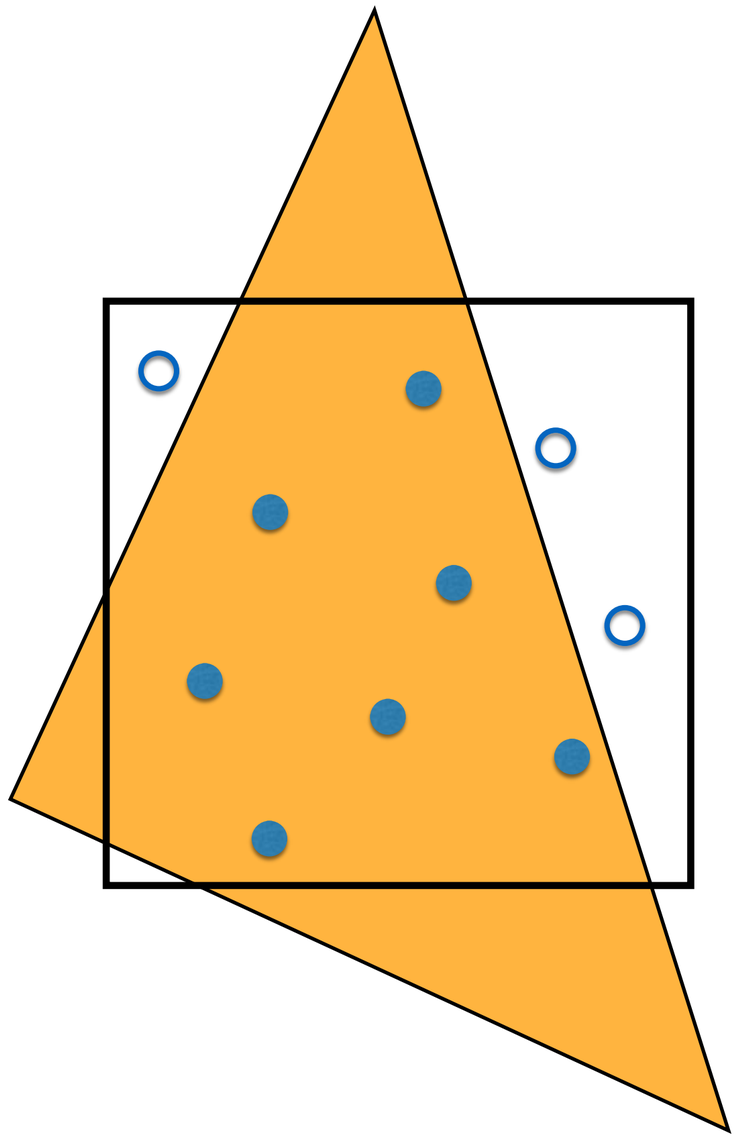


Interpenetration of triangles: even trickier



Two regions of triangle 1 contribute to pixel. One of these regions is not even convex.

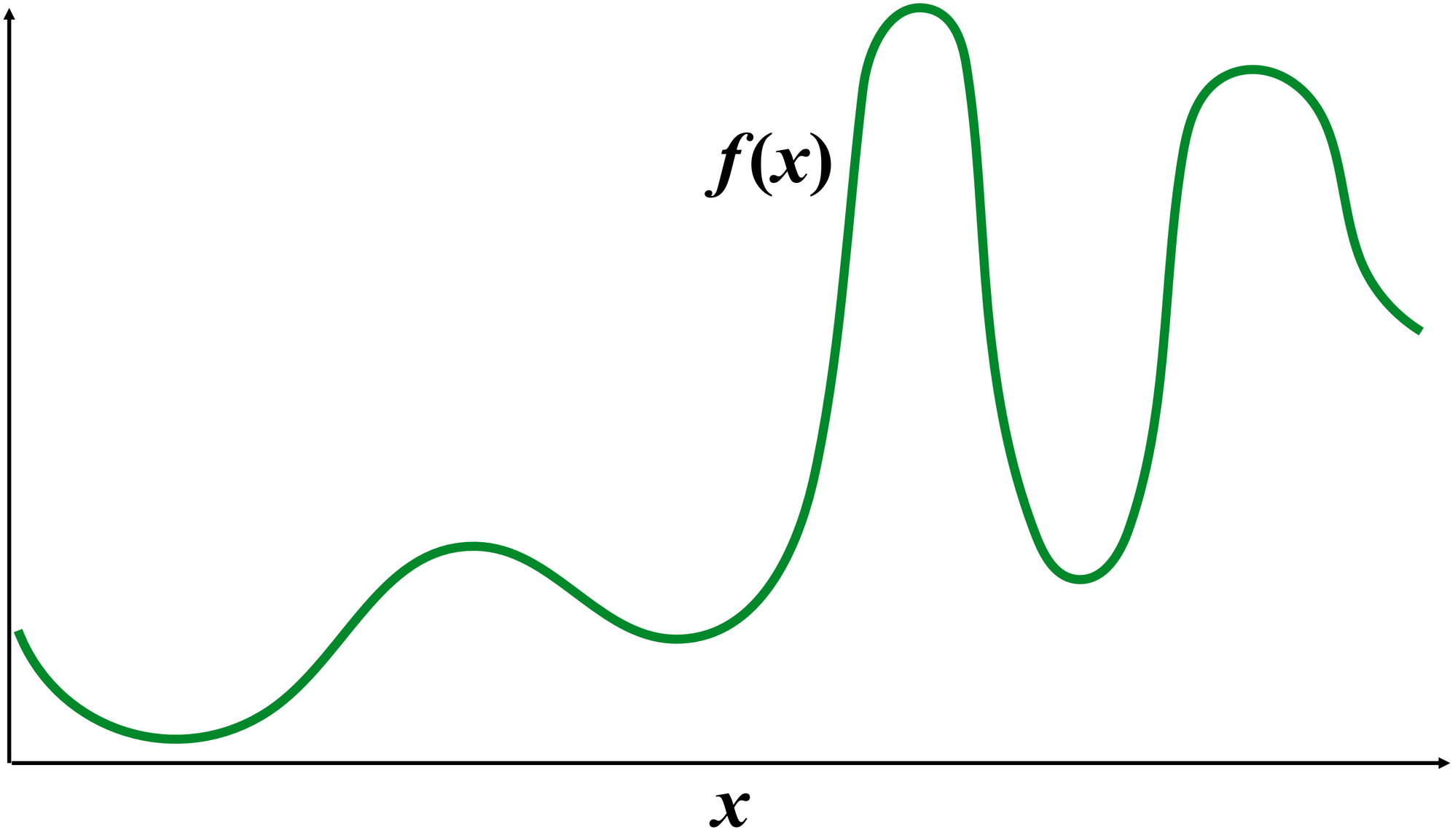
Estimating amount of overlap through sampling



What is a principled approach to think about this process?

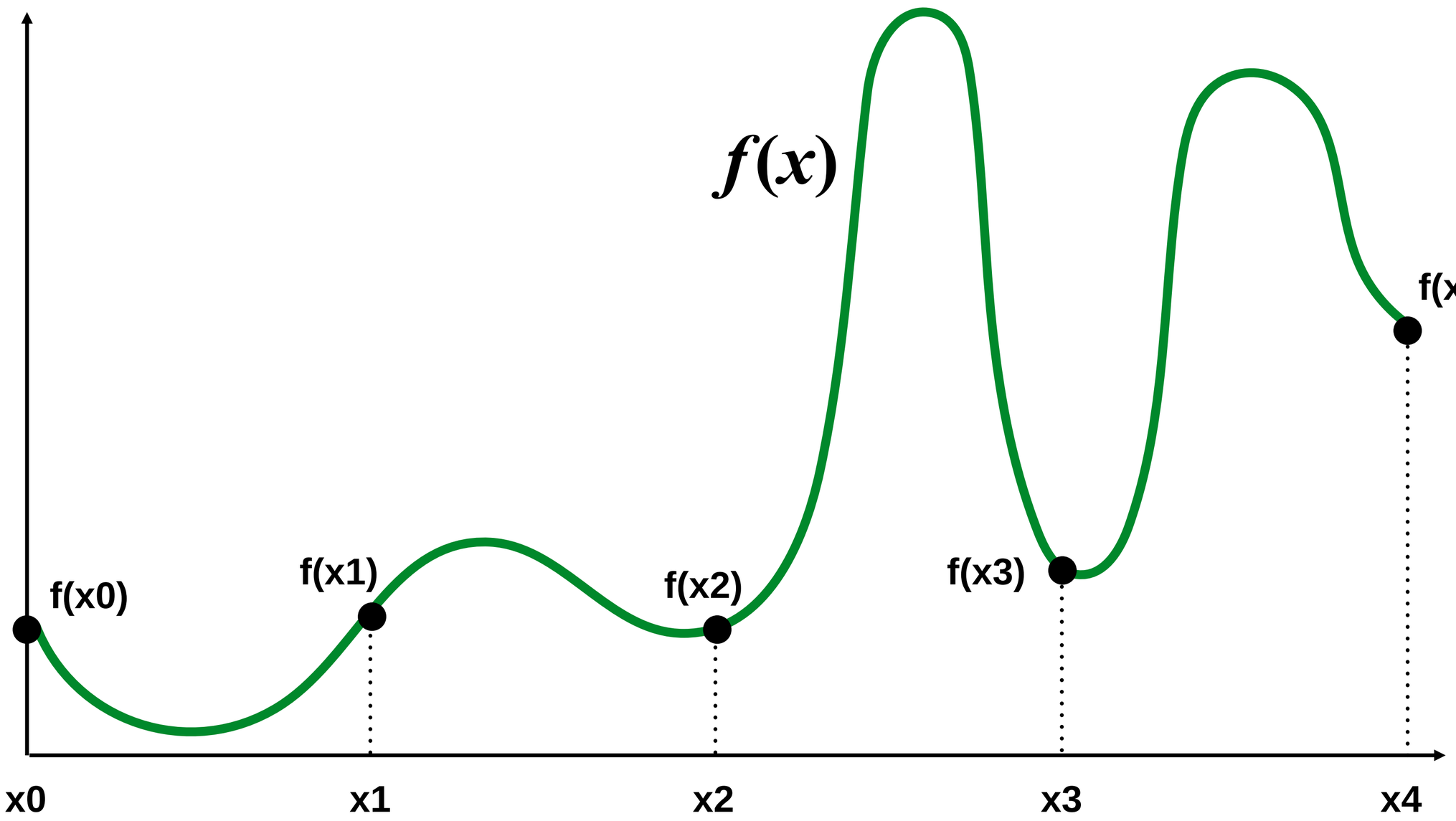
Sampling 101

1D signal



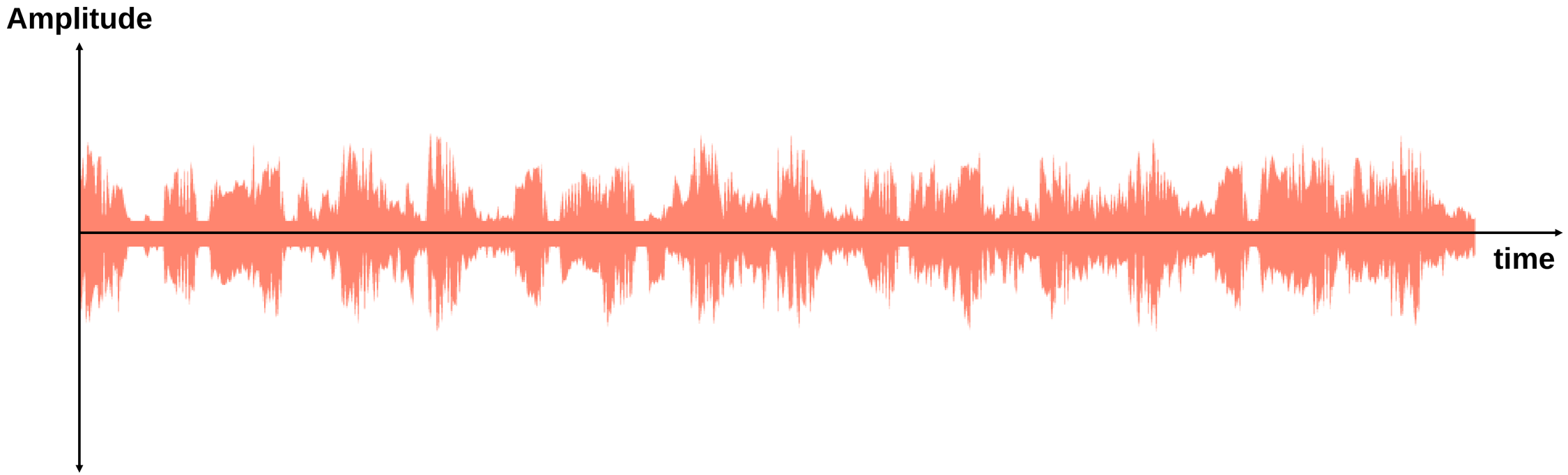
Sampling: from continuous to discrete

Below: 5 measurements (“samples”) of $f(x)$



Audio file: stores samples of a 1D signal

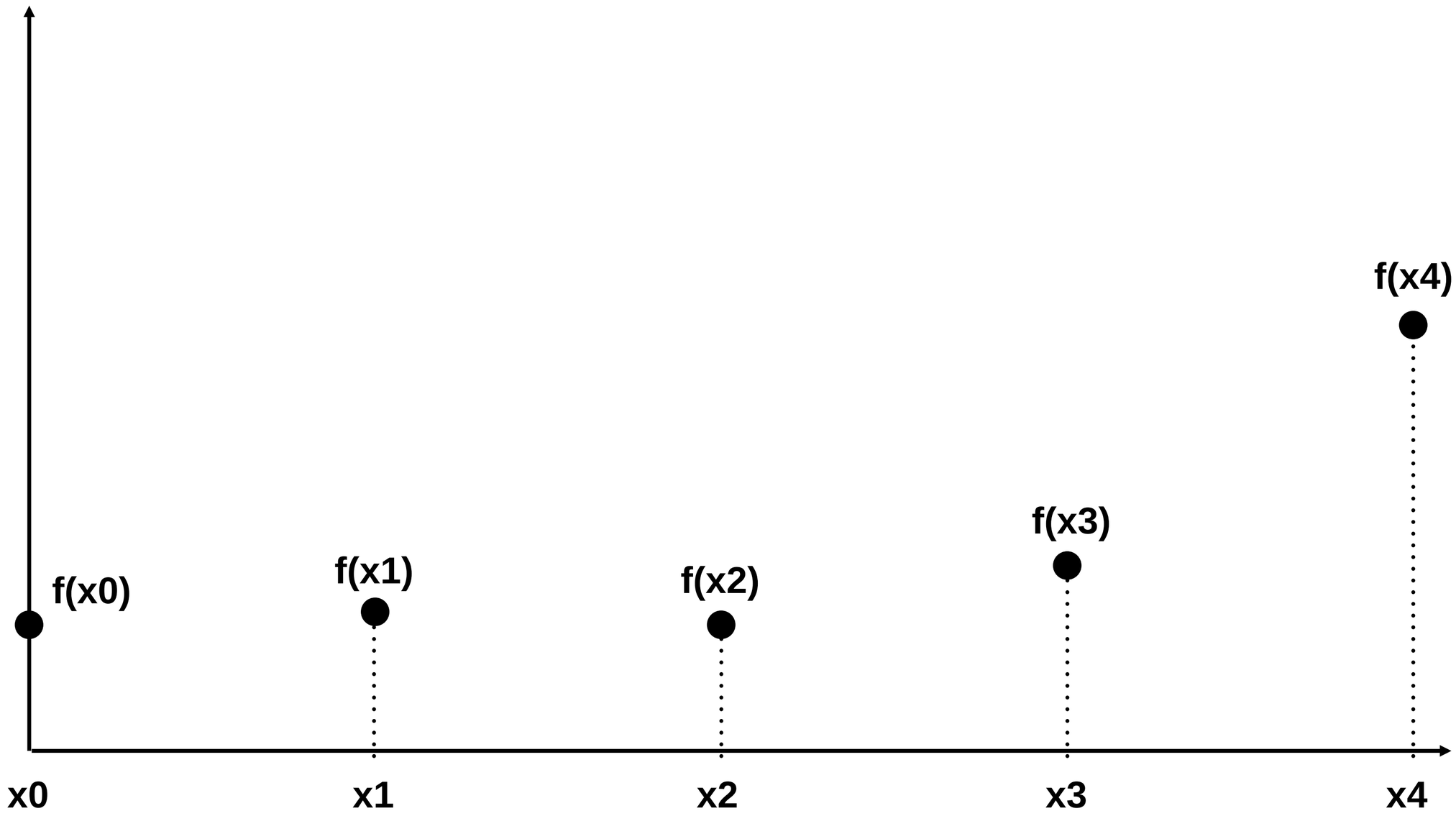
Most consumer audio is sampled at 44.1 KHz



Q: Why 44.1Khz?

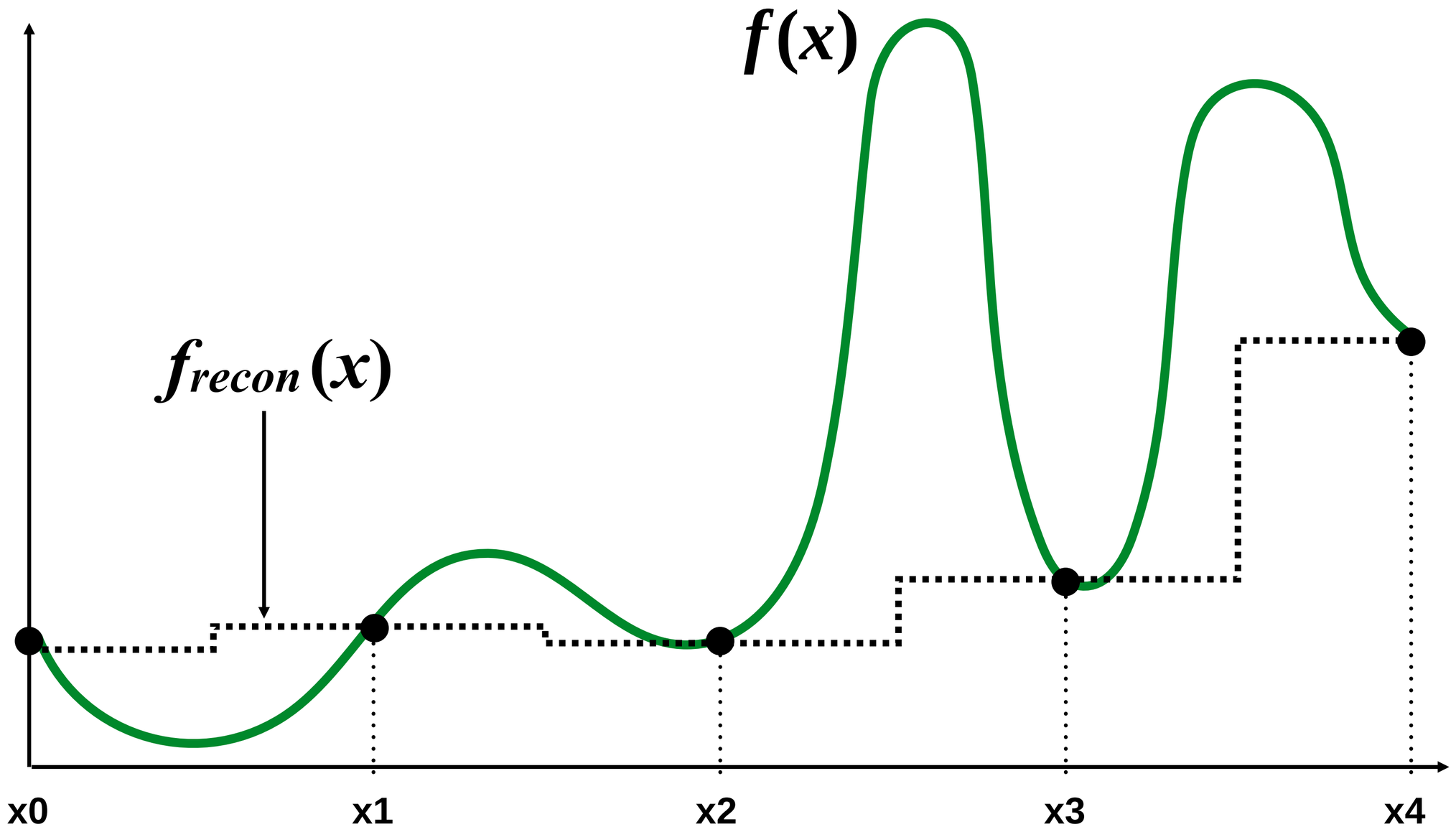
Reconstruction: from discrete to continuous (an interpolation problem)

$f_{recon}(x)$ **is the reconstructed version of the original function** $f(x)$



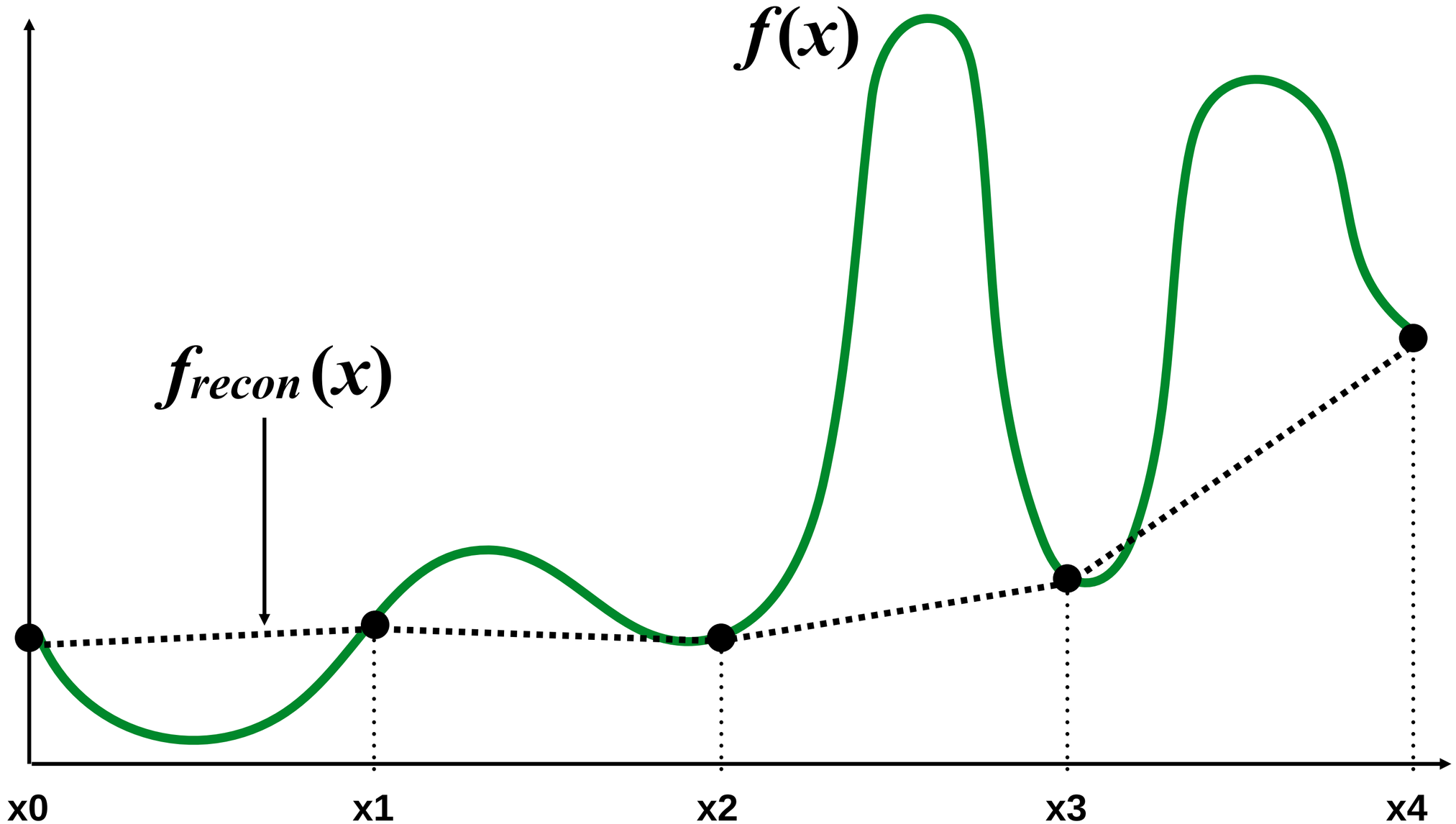
Piecewise constant approximation

$f_{recon}(x)$ = **value of sample closest to** x **(Nearest Neighbor)**

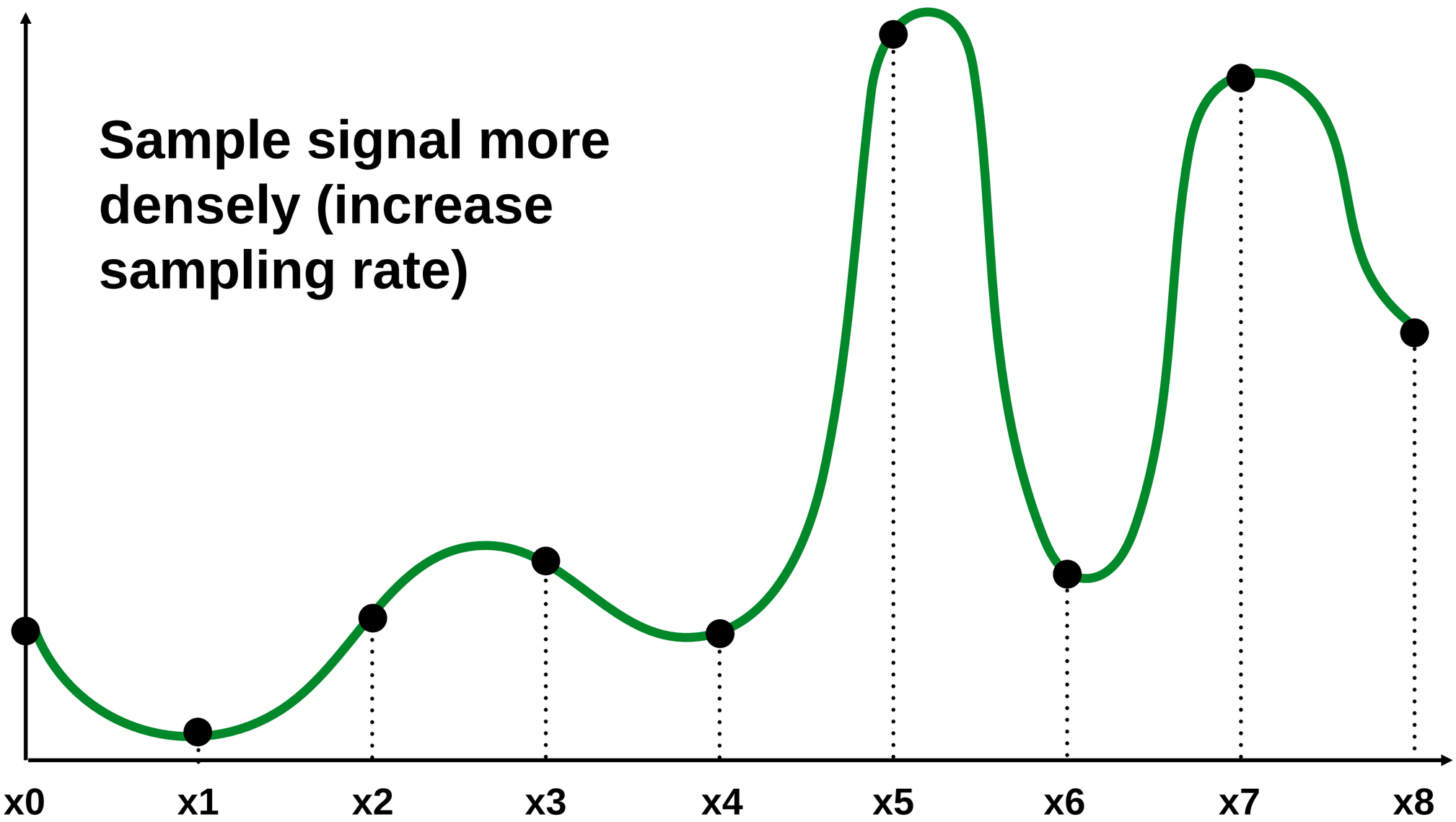


Piecewise linear approximation

$f_{recon}(x)$ = linear interpolation between two samples closest to x

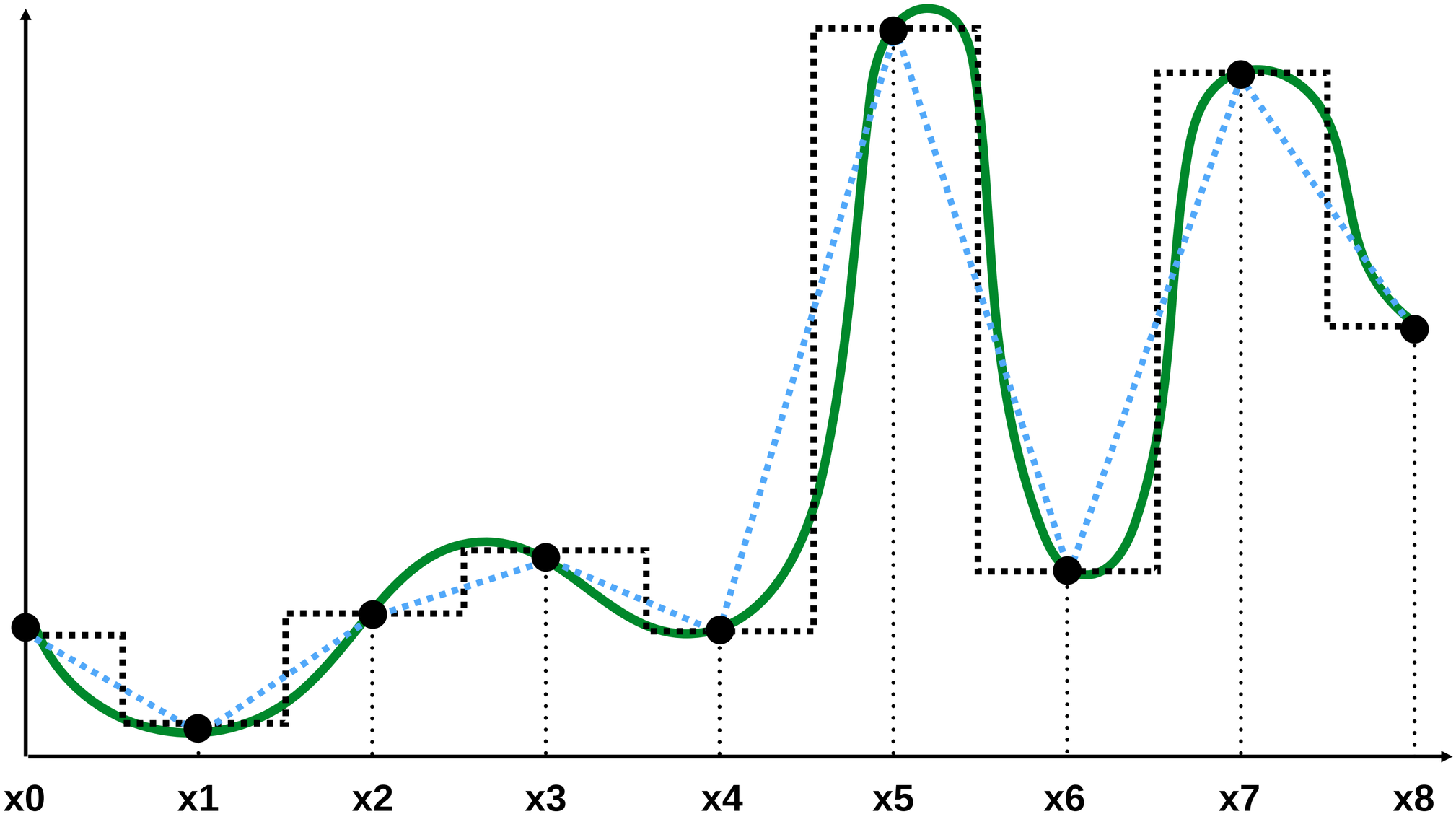


How can we reconstruct the signal more accurately?



Q: What does “increase sampling rate” mean for our problem?

Reconstruction from denser sampling



= reconstruction via nearest neighbor

= reconstruction via linear interpolation

Sampling and Reconstruction

- As an aside
  - Sampling rate is obviously very important
    - Why limit it?
  - In general, what else might you worry about when sampling a signal?
    - Noise
    - Quantization
    - Impulse response

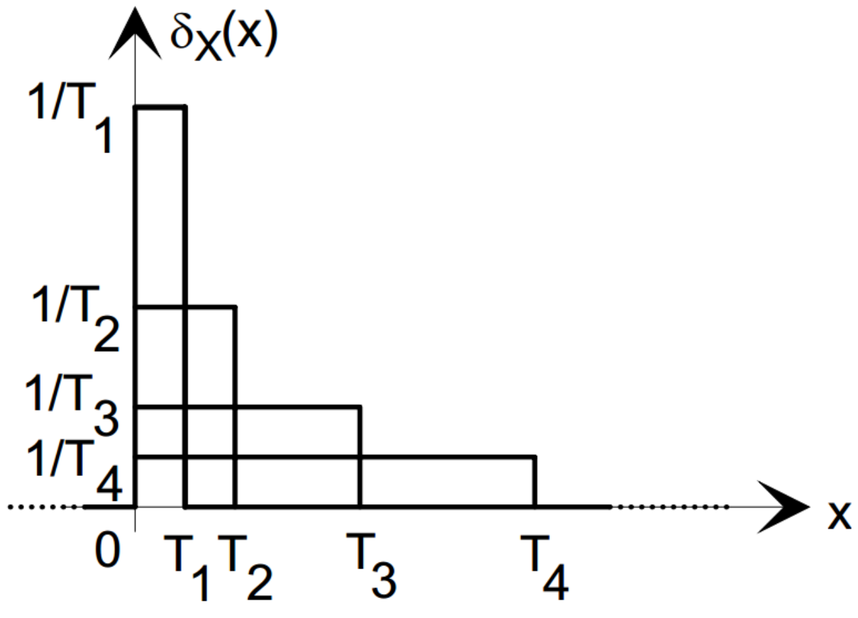
Mathematical representation of sampling

Consider the Dirac delta:

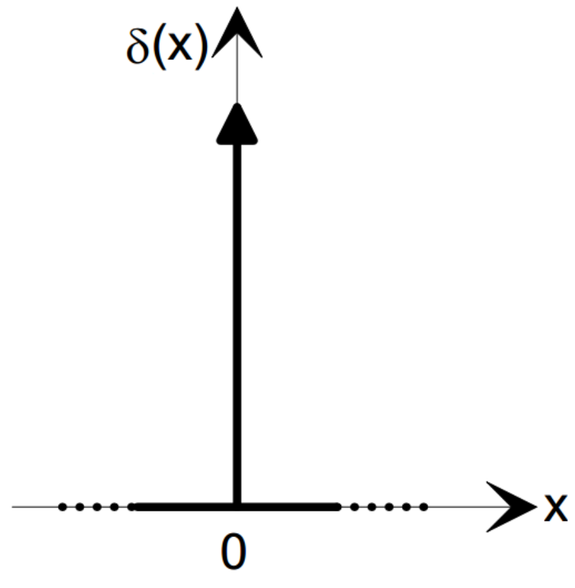
$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \text{undefined} & \text{at } x = 0 \end{cases}$$

s.t.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$



Unit pulse functions



An Impulse

Mathematical representation of sampling

The ‘sifting’ property of the Impulse:

$$\int_{-\infty}^{\infty} f(x)\underbrace{\delta(x-a)}dx = f(a)$$

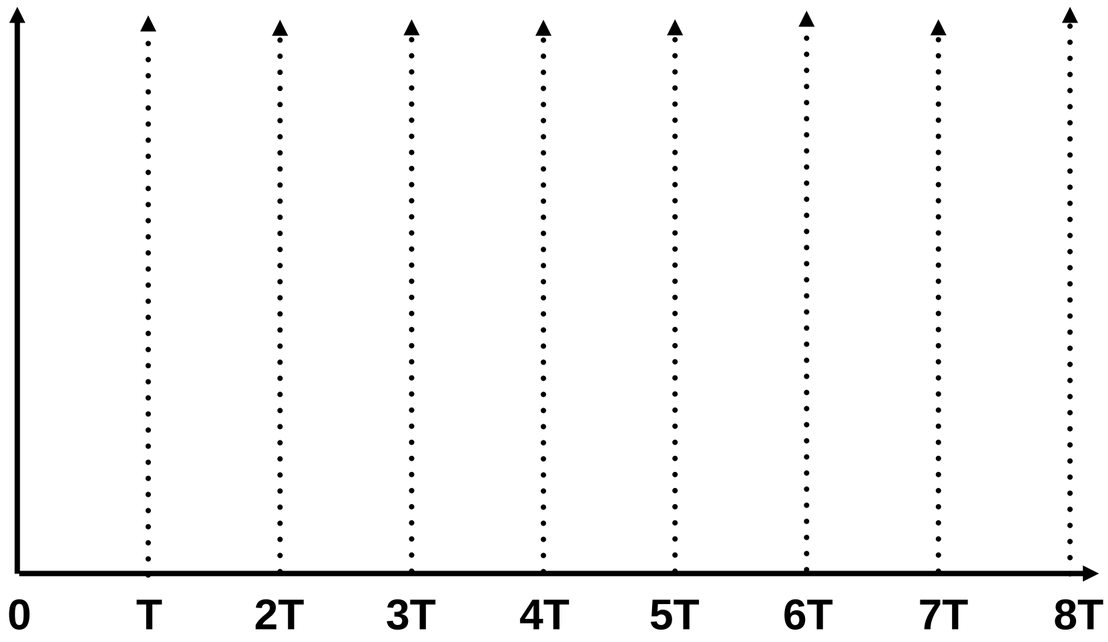
Impulse occurring at $x = a$

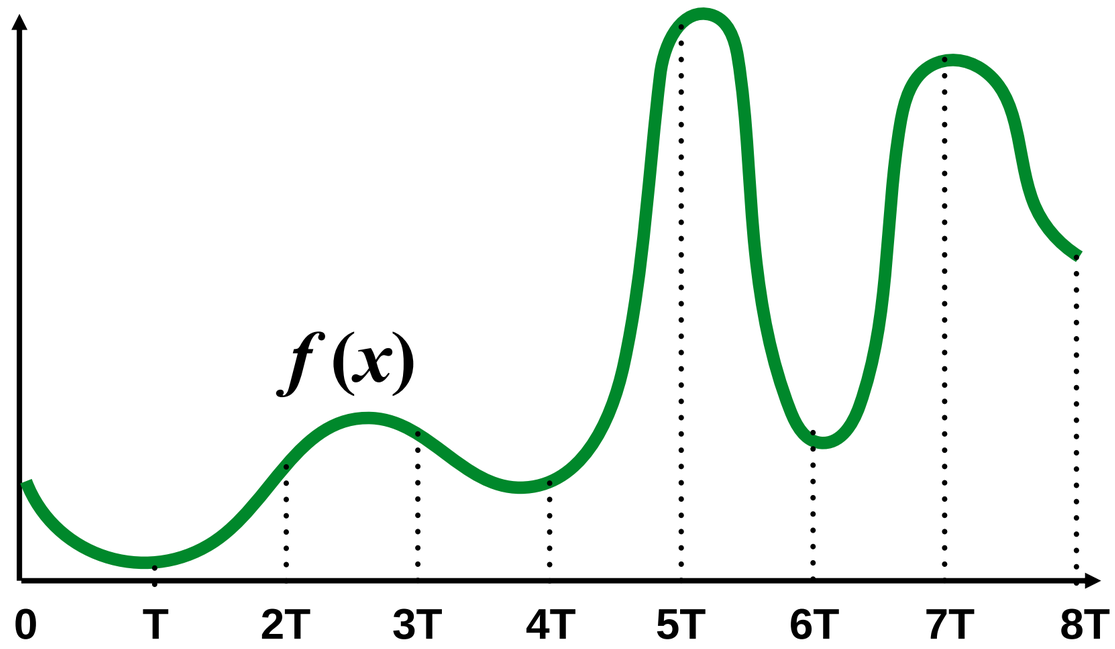
Sampling the function is equivalent to multiplying it (inner product) by the Dirac delta

Sampling function

Consider a sequence of impulses with period T (Dirac comb or impulse train):

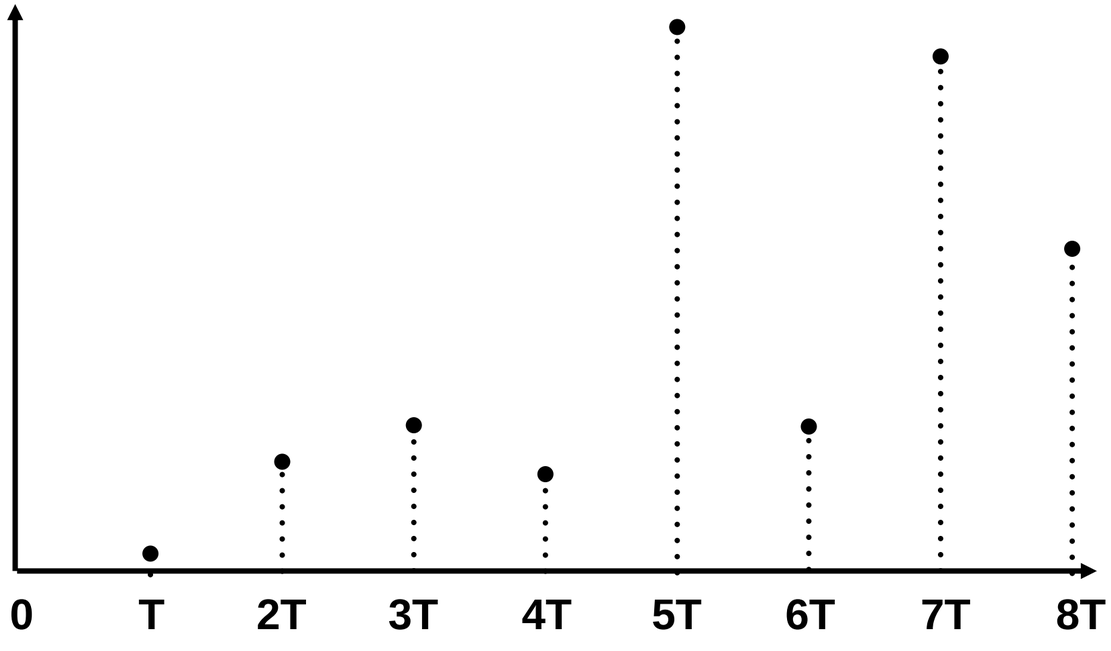
$$\text{Ш}_T(x) = T \sum_{i=-\infty}^{\infty} \delta(x - iT)$$





$$\text{Ш}_T(x) f(x) = T \sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$$

Discrete sampling of a continuous function *f(x)* can be expressed as inner product between *f* and the Dirac comb



Reconstruction as convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

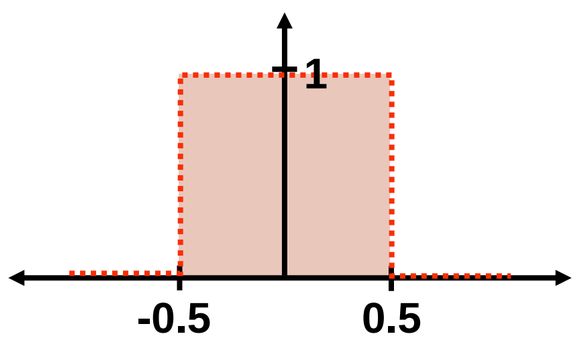
reconstructed signal (“smooth” version of g)

filter

input signal (sampled signal)

It may be helpful to consider the effect of convolution with the simple unit-area “box” function:

$$f(x) = \begin{cases} 1 & |x| \leq 0.5 \\ 0 & otherwise \end{cases}$$



$$(f * g)(x) = \int_{-0.5}^{0.5} g(x-y)dy$$

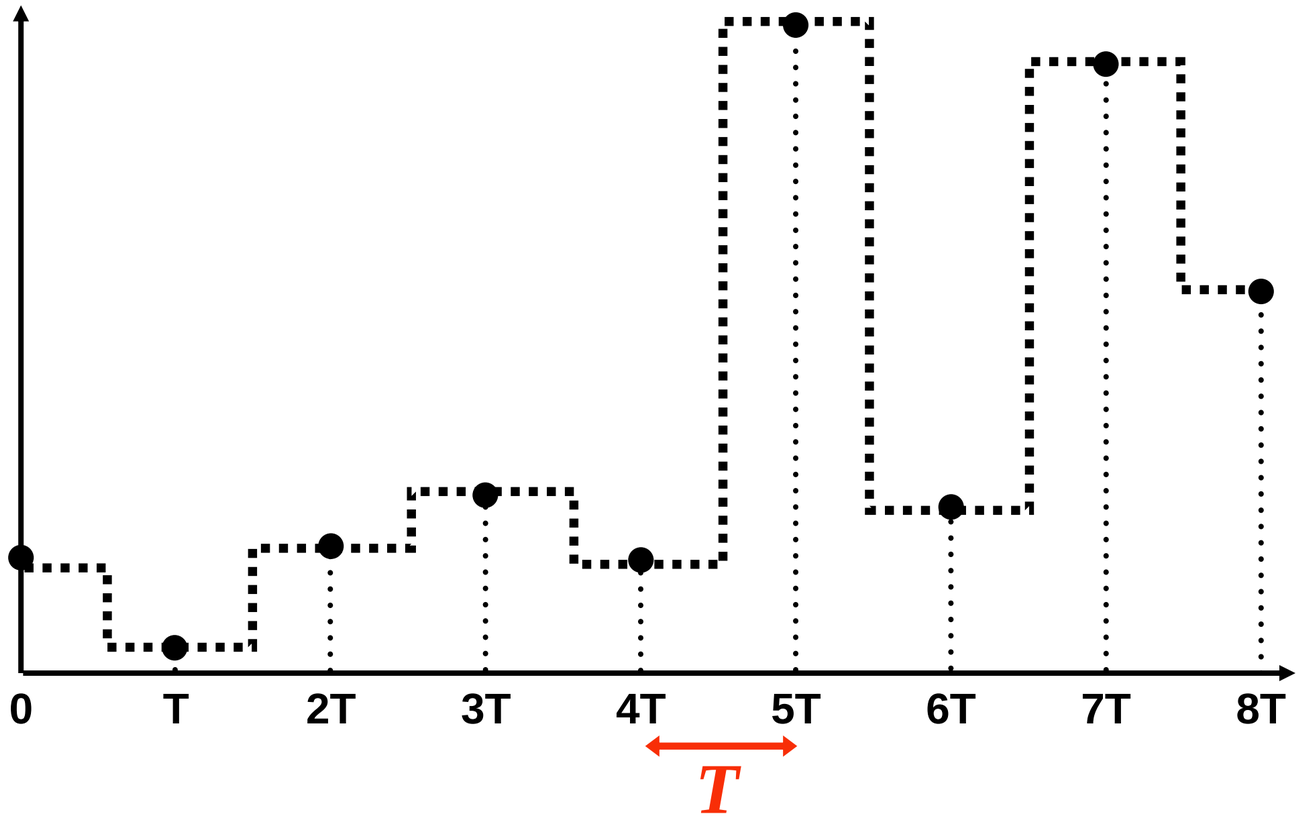
Reconstruction as convolution (box filter)

Sampled signal:
(with period *T*)

$$g(x) = \text{Ш}_T(x) f(x) = T \sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$$

Reconstruction filter:
(unit area box of width T)

$$h(x) = \begin{cases} 1/T & |x| \le T/2 \\ 0 & otherwise \end{cases}$$



Reconstructed signal:
(nearest neighbor)

$$f_{recon}(x) = (h * g)(x) = T \int_{-\infty}^{\infty} h(y) \sum_{i=-\infty}^{\infty} f(iT)\delta(x - y - iT)dy$$

non-zero only for *iT* closest to *x*

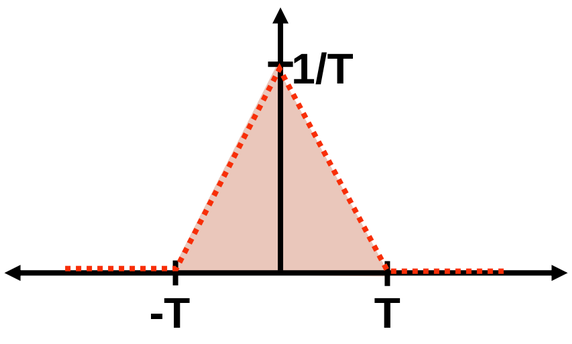
Reconstruction as convolution (triangle filter)

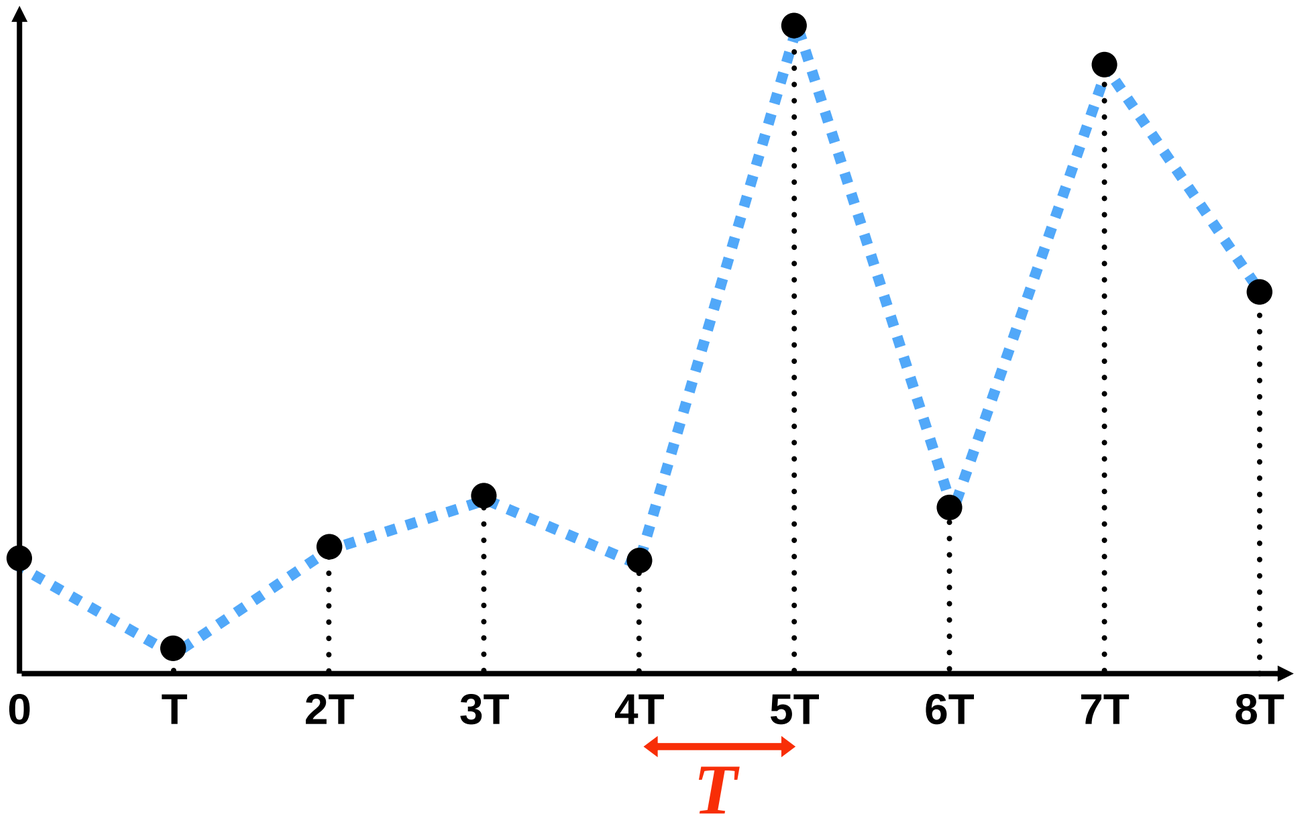
Sampled signal:
(with period T)

$$g(x) = \text{Ш}_T(x) f(x) = T \sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$$

Reconstruction filter:
(unit area triangle of width T)

$$h(x) = \begin{cases} (1 - \frac{|x|}{T})/T & |x| \le T \\ 0 & otherwise \end{cases}$$



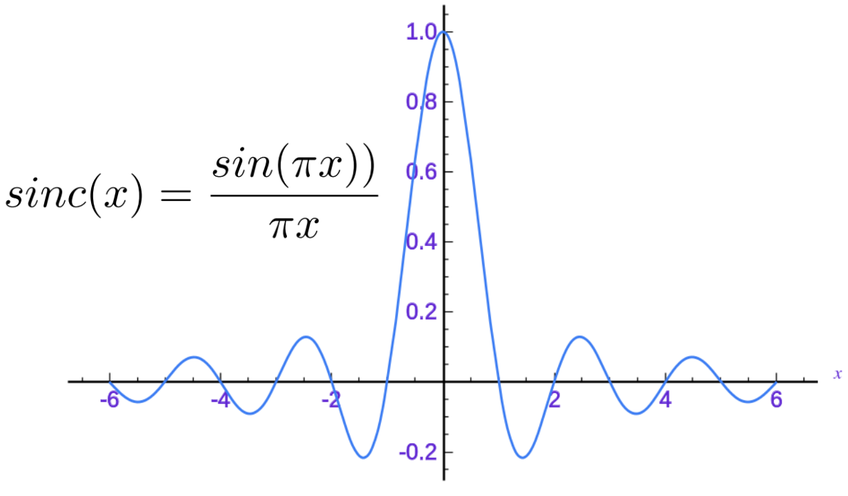


Reconstructed signal:

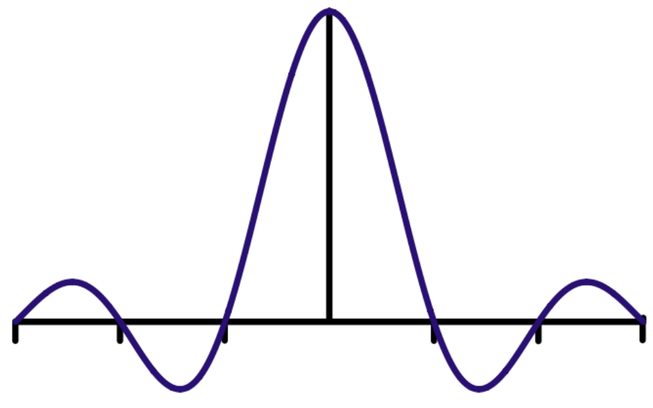
$$f_{recon}(x) = (h * g)(x) = \int_{-\infty}^{\infty} h(y)g(x - y)dy = \ldots$$

Summary

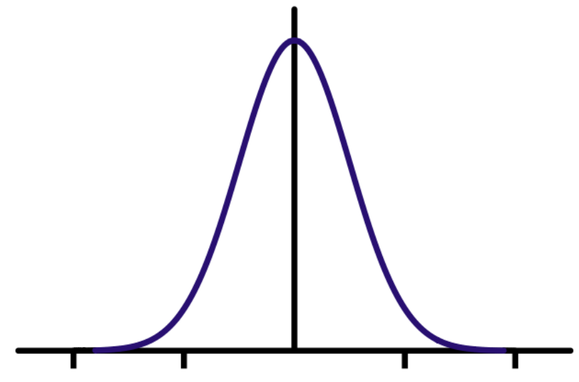
- Sampling = measurement of a signal
  - Represent signal as discrete set of samples
  - Mathematically described as multiplication by impulse train
- Reconstruction = generating signal from a discrete set of samples
  - Convolution of sampled signal with a reconstruction filter
  - Intuition: value of reconstructed function at any point in domain is a combination of sampled values
  - We saw simple box & triangle filters, but there are other, much higher quality filters



Normalized sinc filter



Truncated sinc filter



Truncated gaussian filter

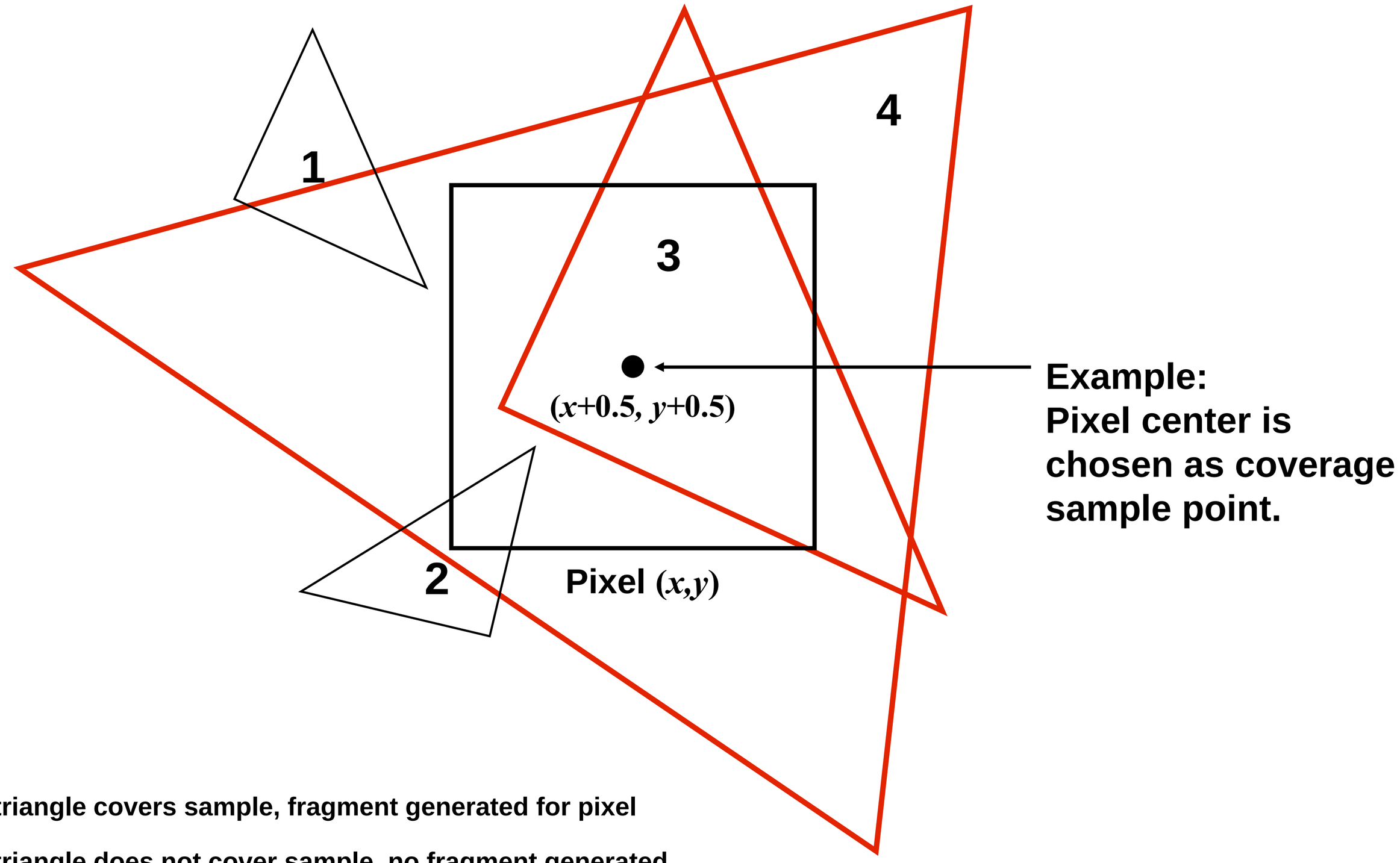
[Image credit: Wikipedia]

Now back to computing coverage

Think of coverage as a 2D signal

$$\text{coverage}(x,y) = \begin{cases} 1 & \text{if the triangle contains point } (x,y) \\ 0 & \text{otherwise} \end{cases}$$

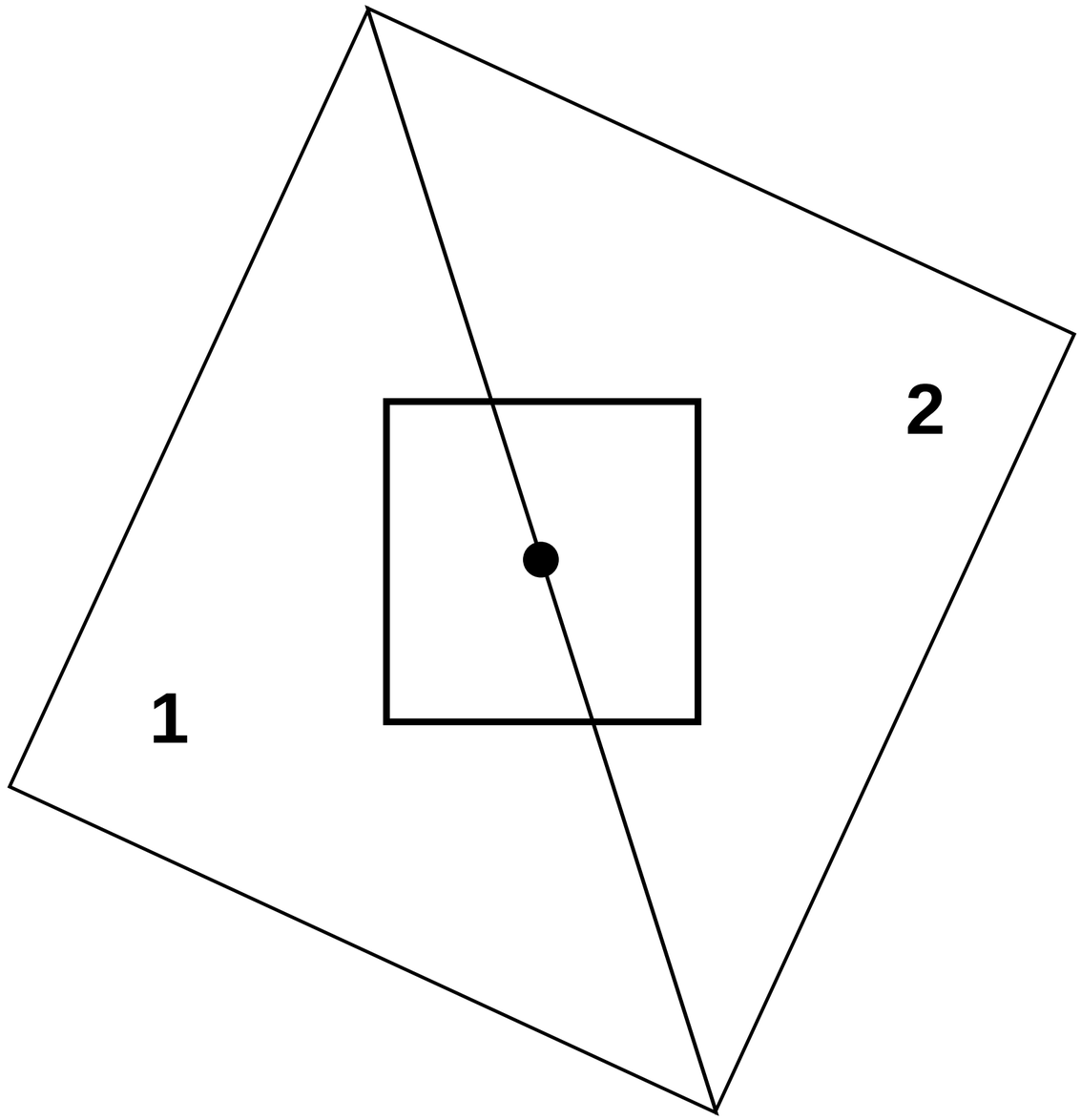
Estimate triangle-screen coverage by sampling the binary function: coverage(x,y)



Edge cases (literally)

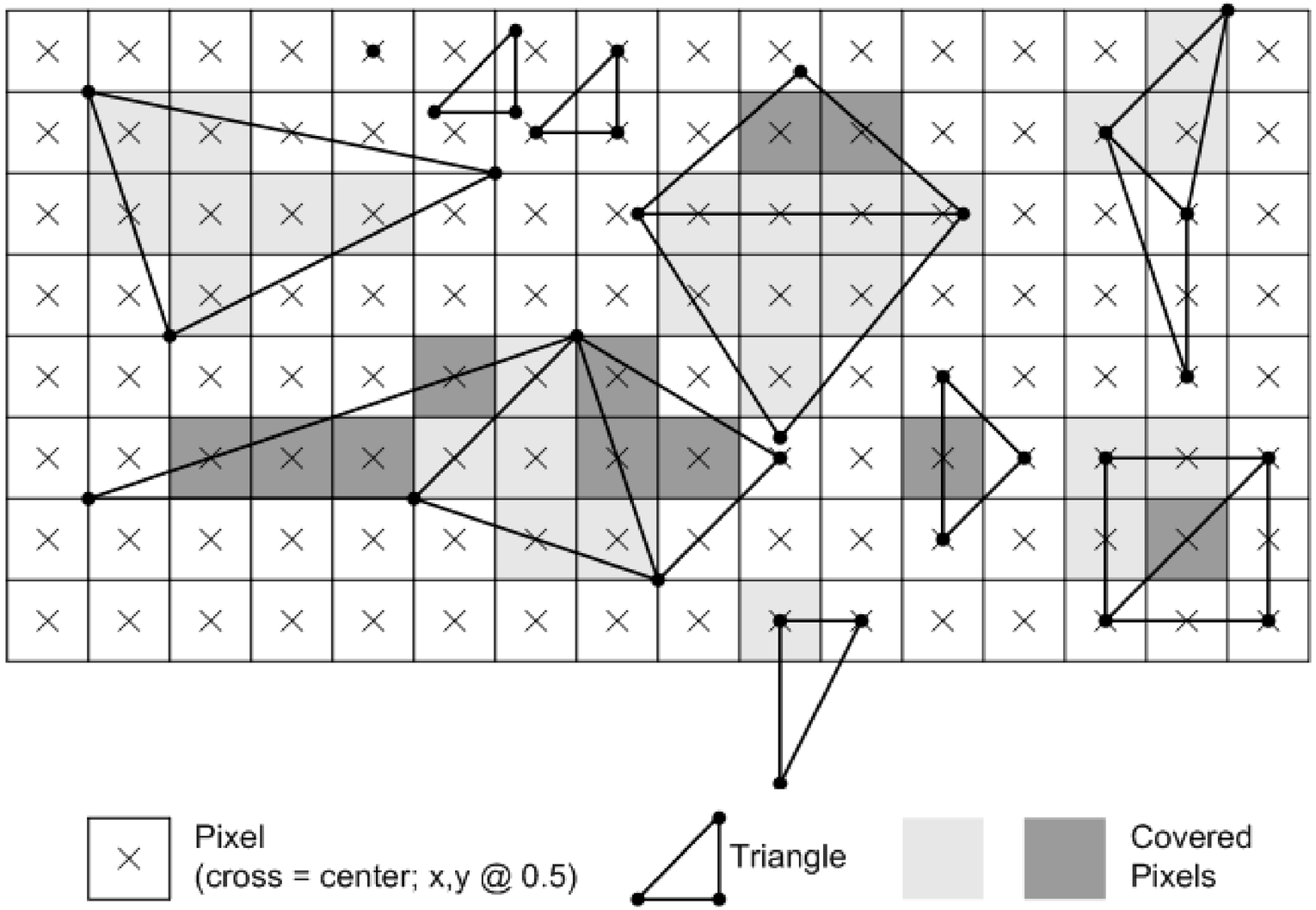
Is this sample point covered by triangle 1? or triangle 2? or both? or none?

Why is it important to decide?



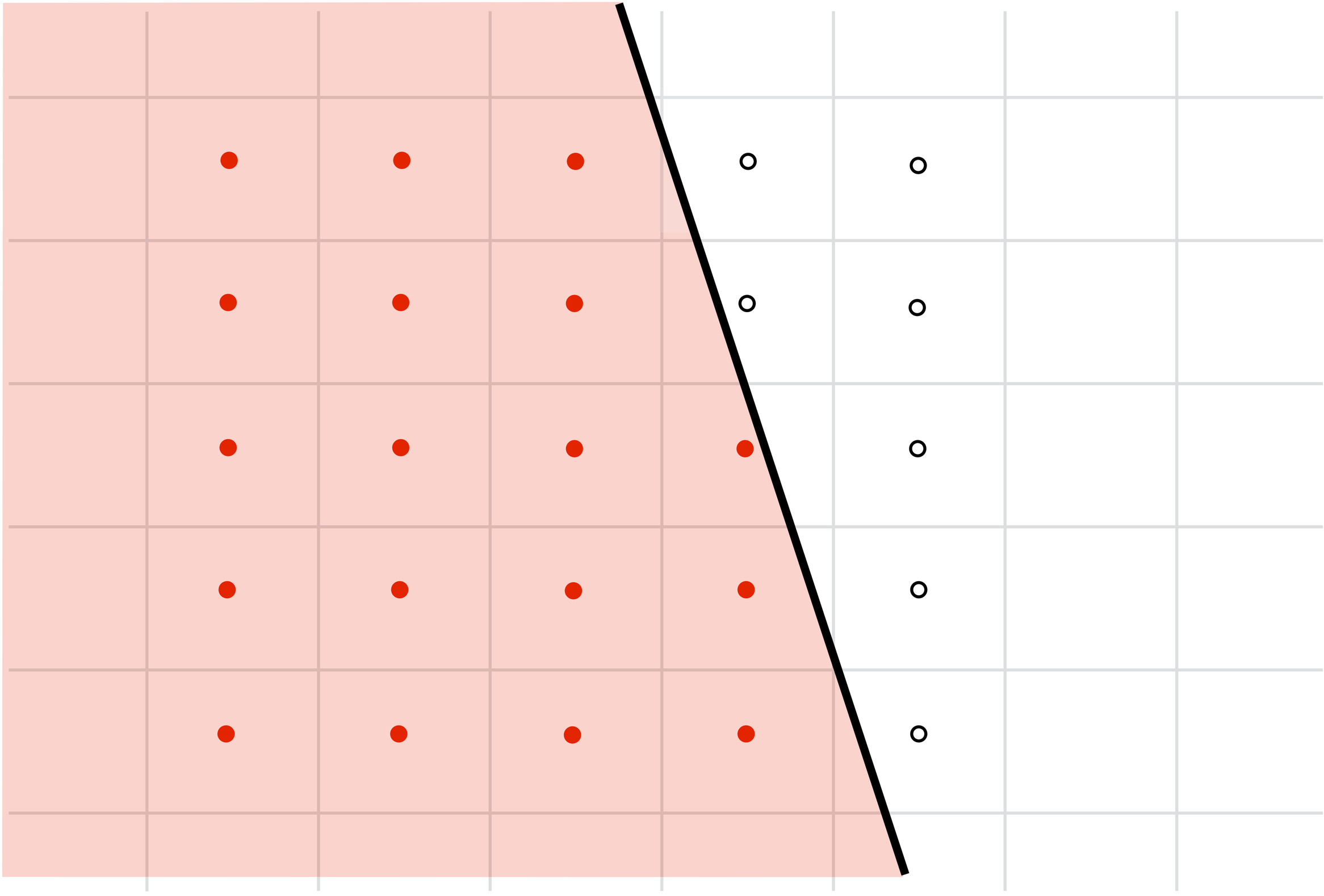
OpenGL/Direct3D edge rules

- When edge falls directly on a screen sample point, the sample is classified as within triangle if the edge is a “top edge” or “left edge”
  - Top edge: horizontal edge that is above all other edges
  - Left edge: an edge that is not exactly horizontal and is on the left side of the triangle. (triangle can have one or two left edges)

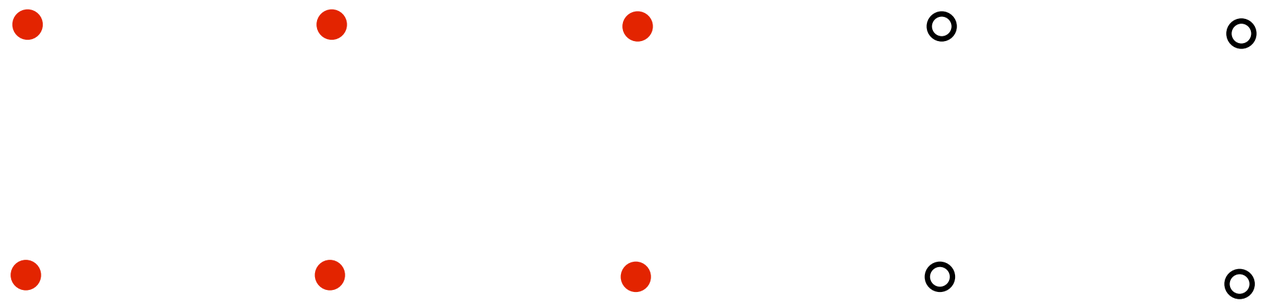


Source: Direct3D Programming Guide, Microsoft

Results of sampling triangle coverage

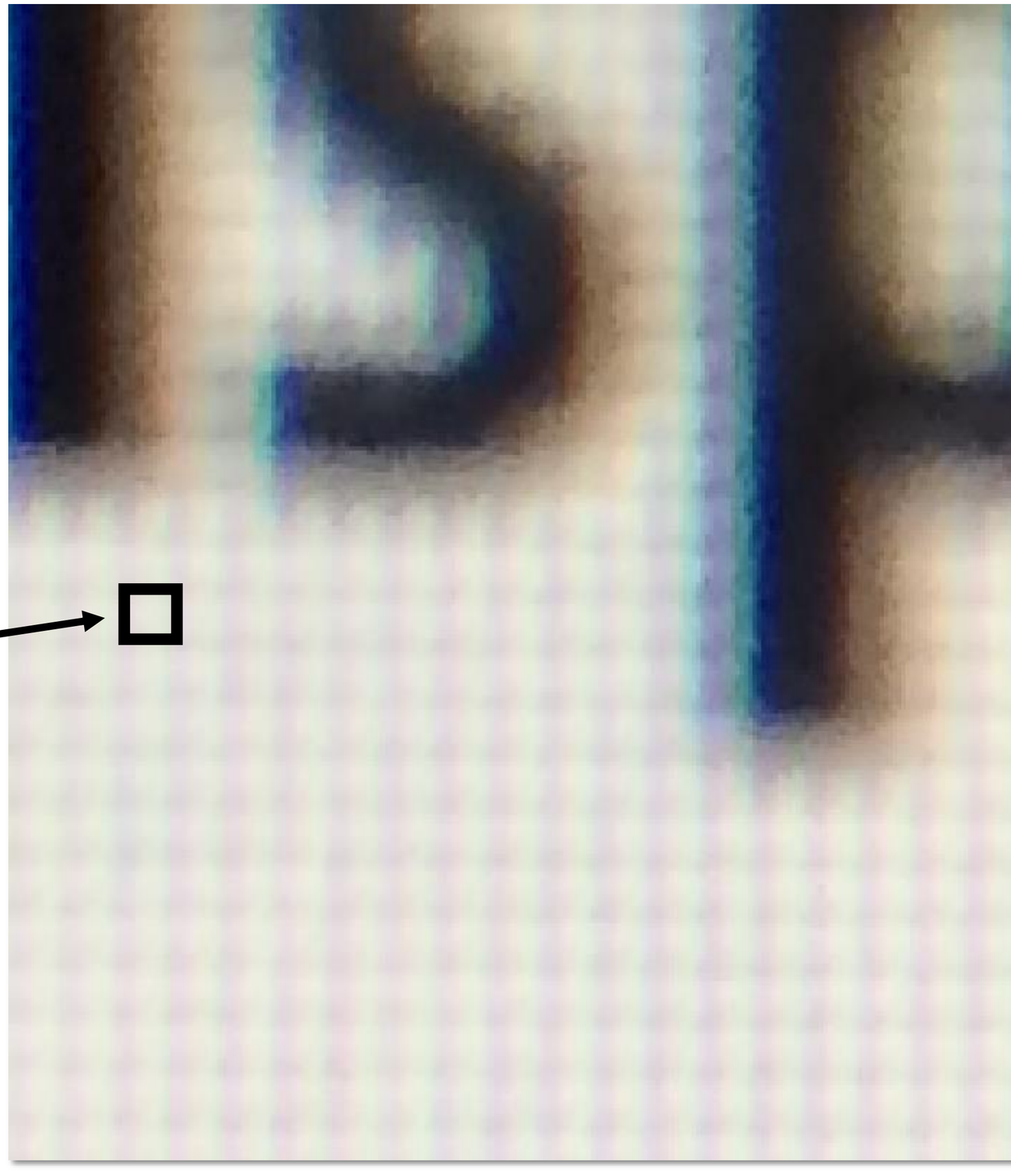


We have a sampled signal, now we want to display it on a screen



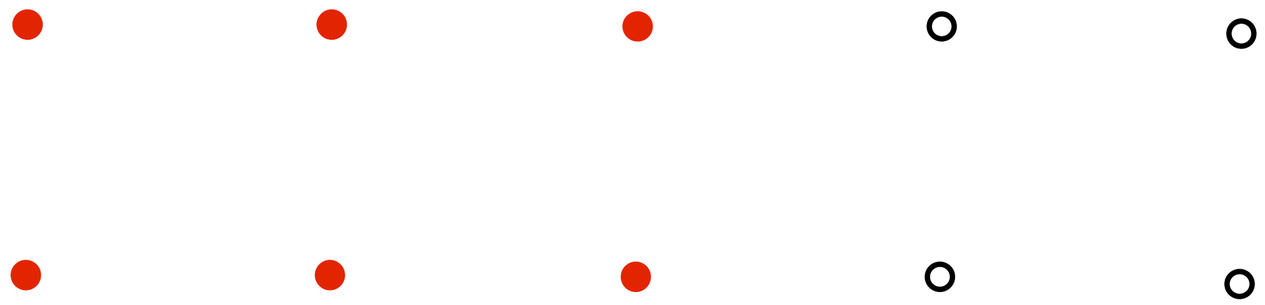
Pixels on a screen

Each image sample sent to the display is converted into a little square of light of the appropriate color:
(a pixel = picture element)

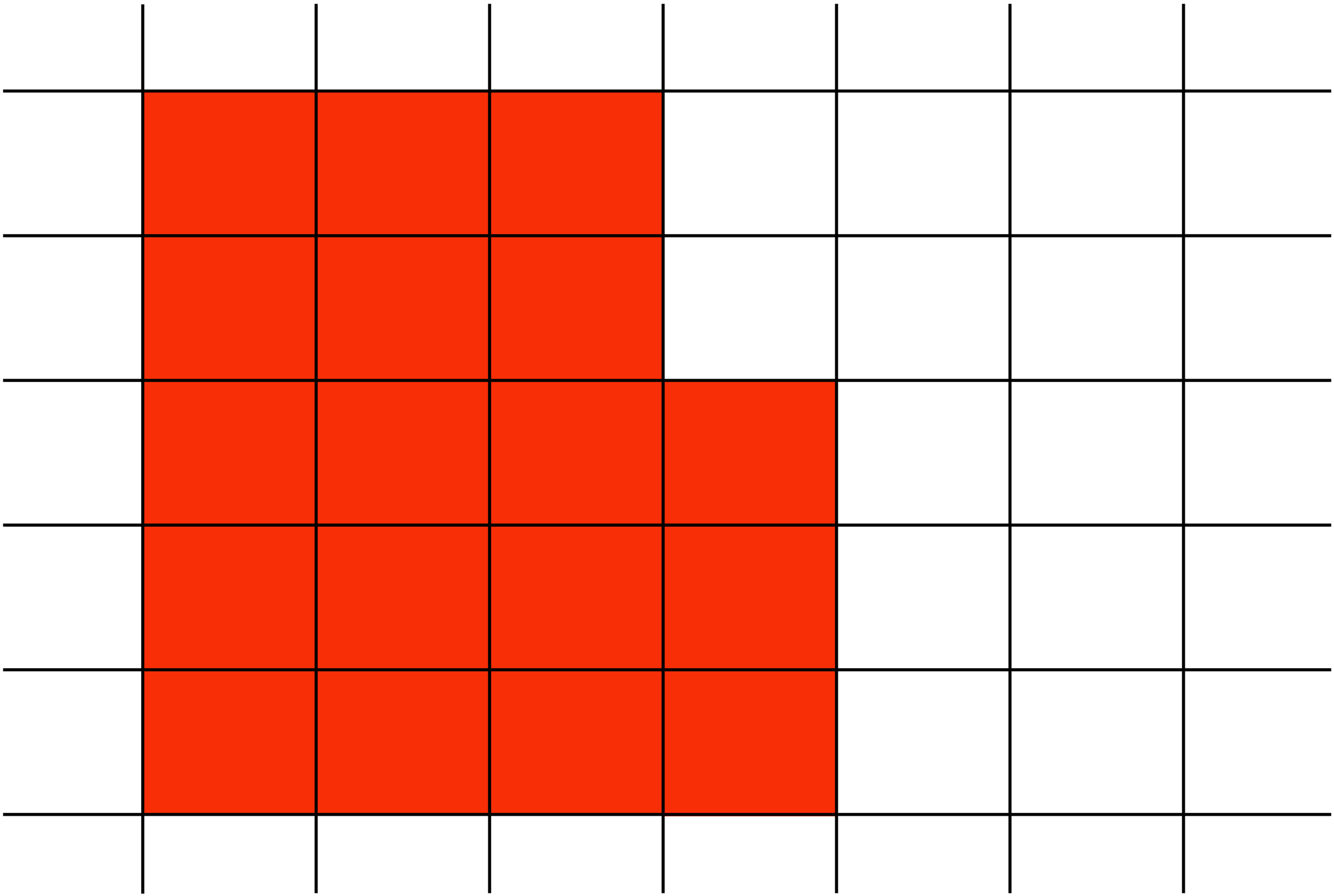


*** Thinking of each pixel as emitting a square of uniform intensity light of a single color is a bit of an approximation to how real displays work, but it will do.**

So if we send the display this:



We see this on the screen

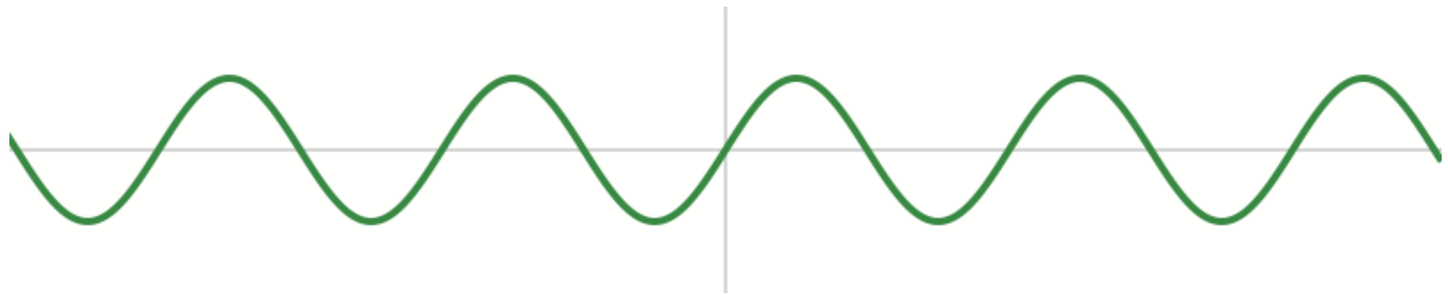


Recall: the real coverage signal

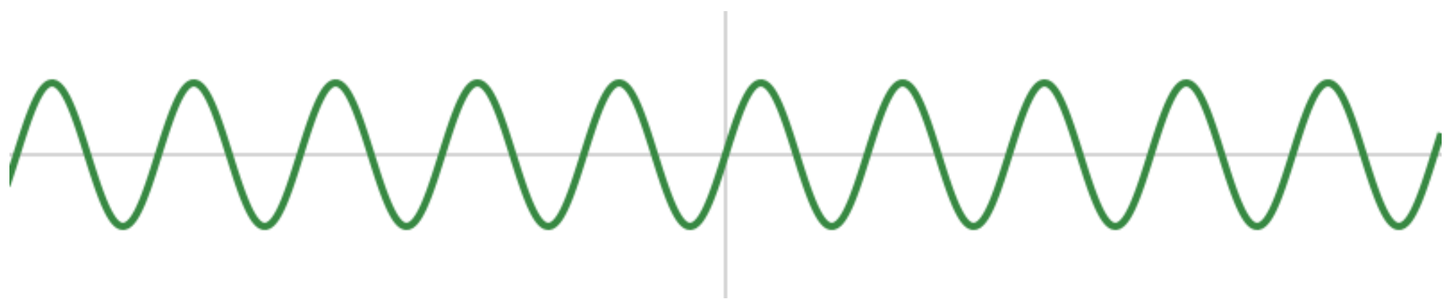
Aliasing

Representing signals as a superposition of frequencies

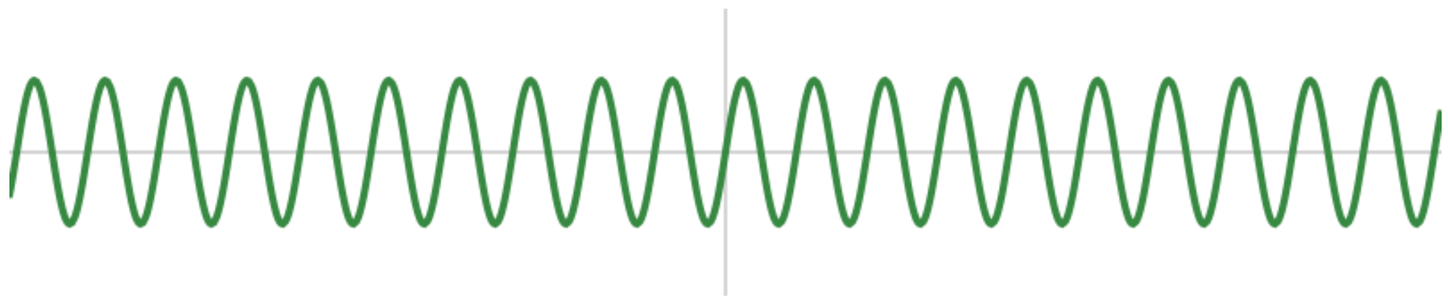
$f_1(x) = sin(\pi x)$



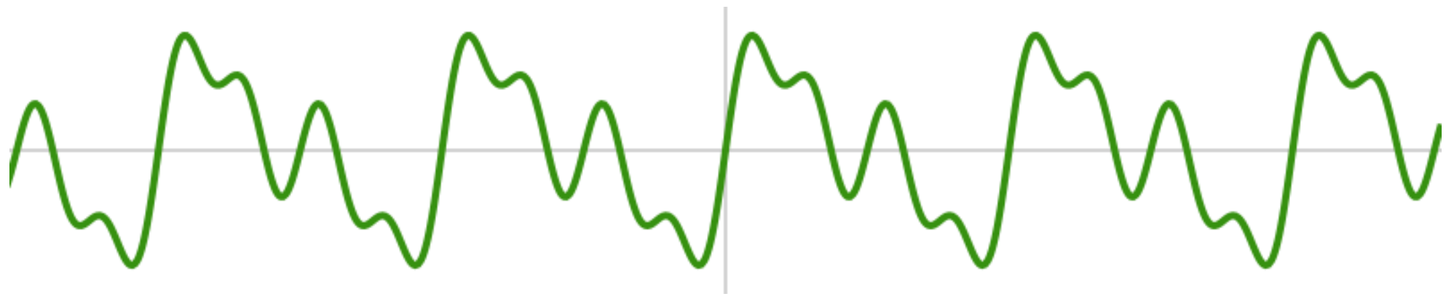
$f_2(x) = sin(2\pi x)$



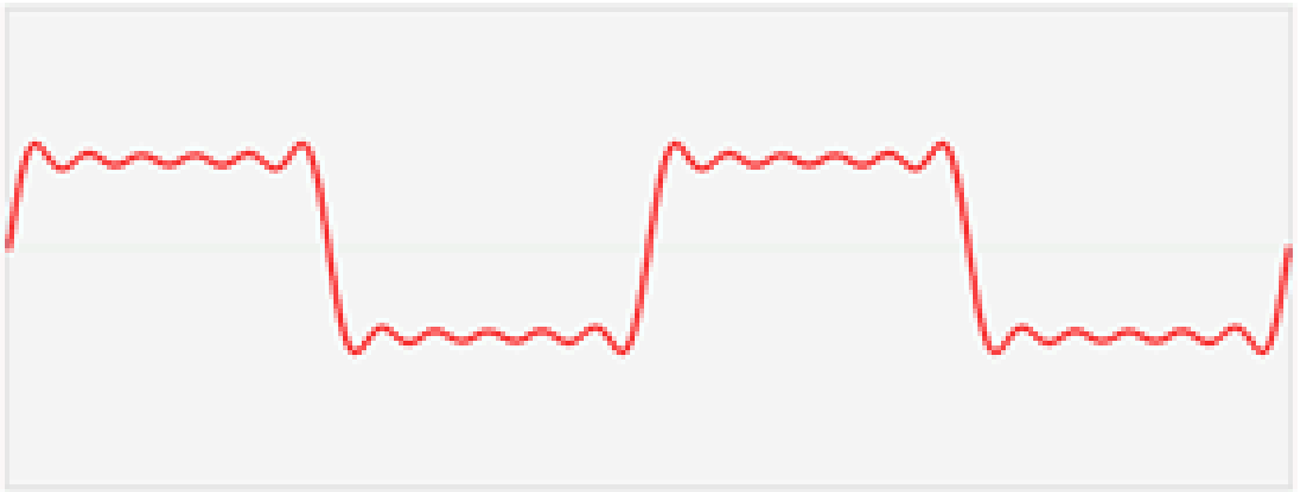
$f_4(x) = sin(4\pi x)$



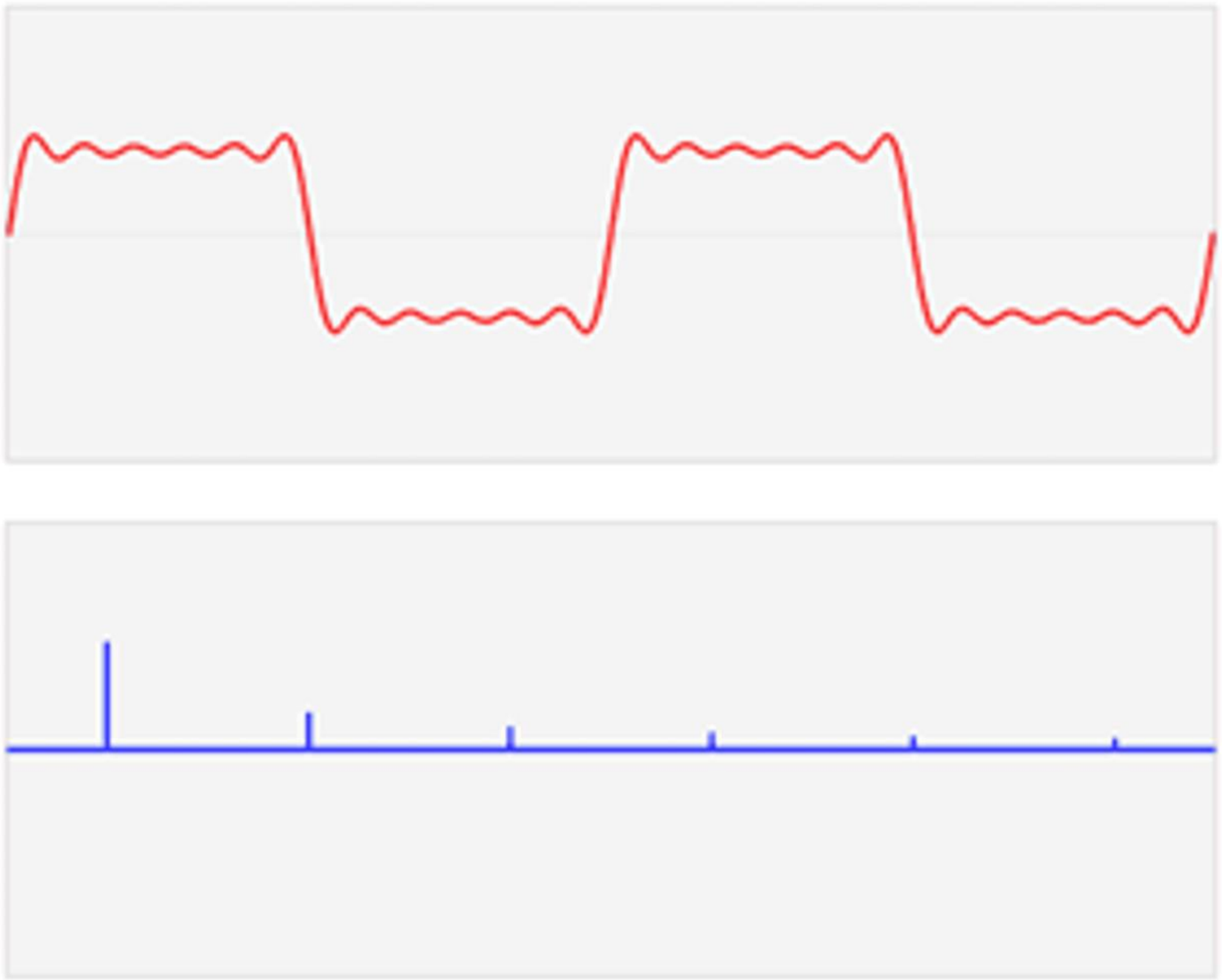
$f(x) = f_1(x) + 0.75\, f_2(x) + 0.5\, f_4(x)$



Representing signals as a superposition of frequencies



Representing signals as a superposition of frequencies

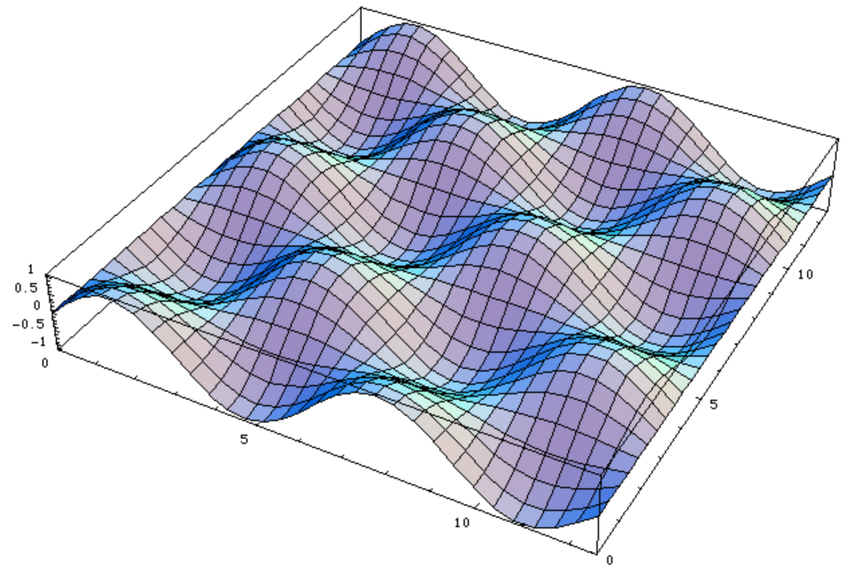


Spatial domain

Frequency domain

It’s exactly the same function!

Representing images (2D signals) as superposition of frequencies

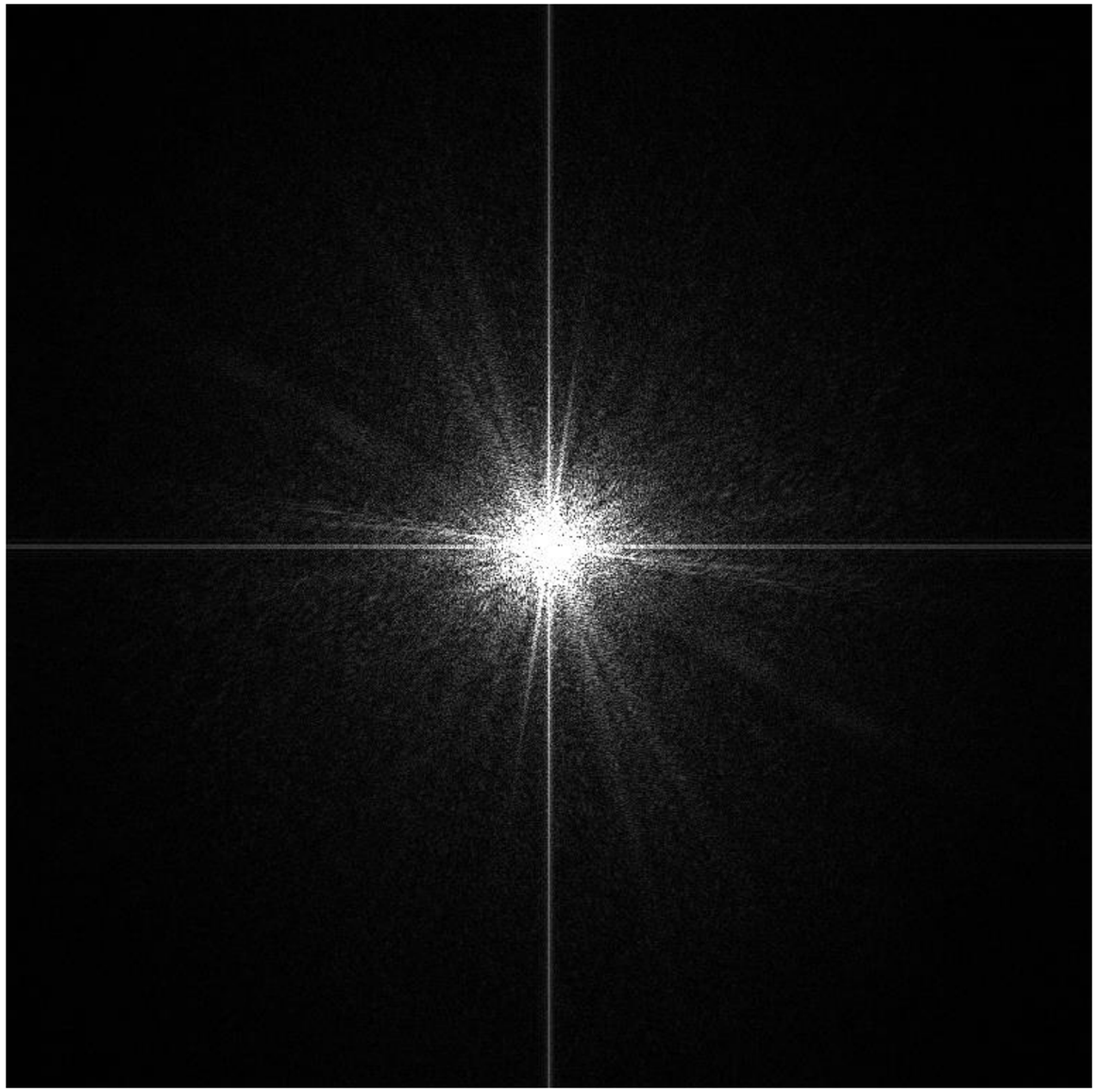


individual frequencies are 2D sinusoids

(e.g. $f(x, y) = \sin(a\pi x) * \sin(b\pi x)$)

Visualizing the frequency content of images

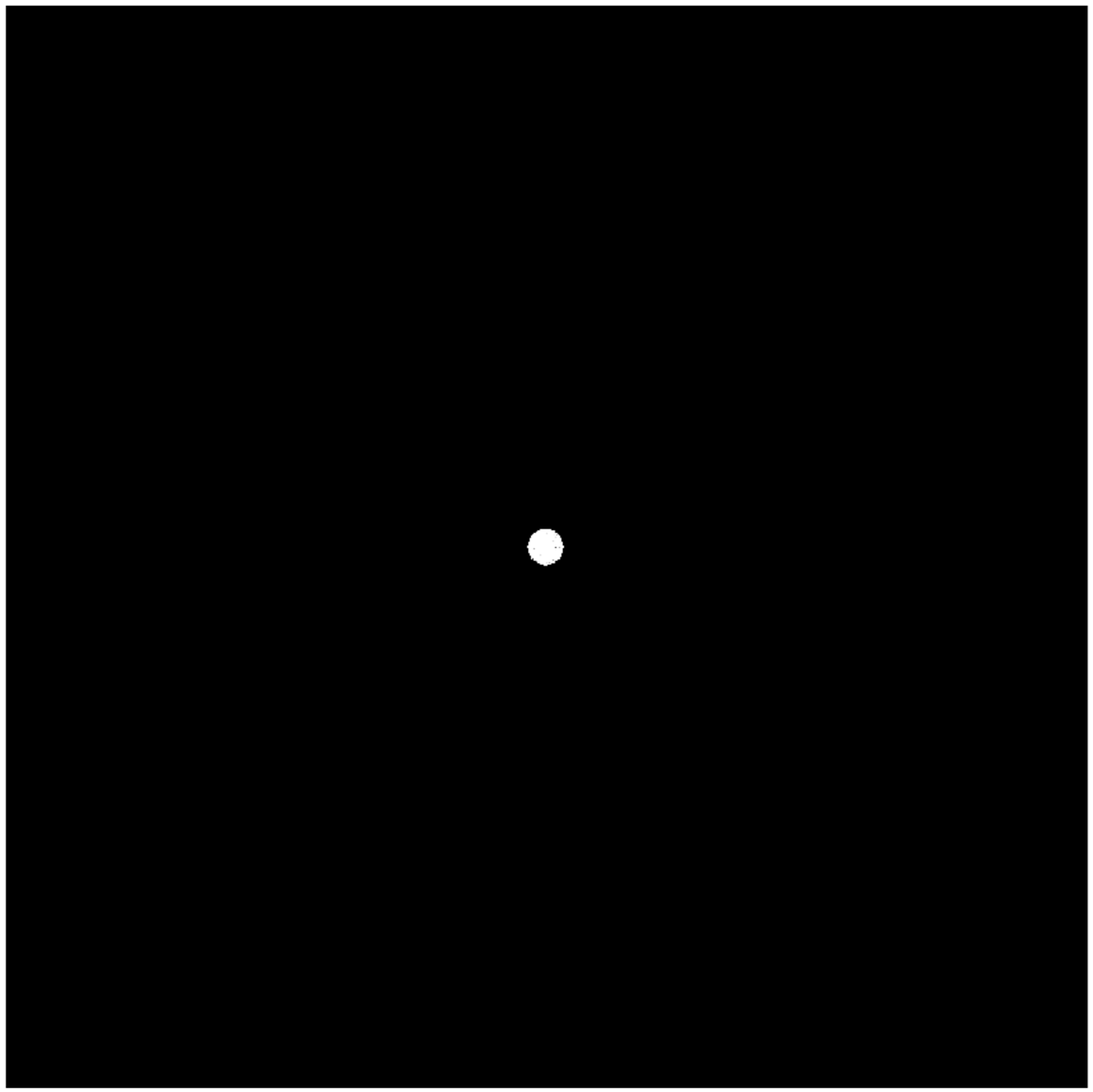
Spatial domain image



Frequency Domain Image

Low frequencies only

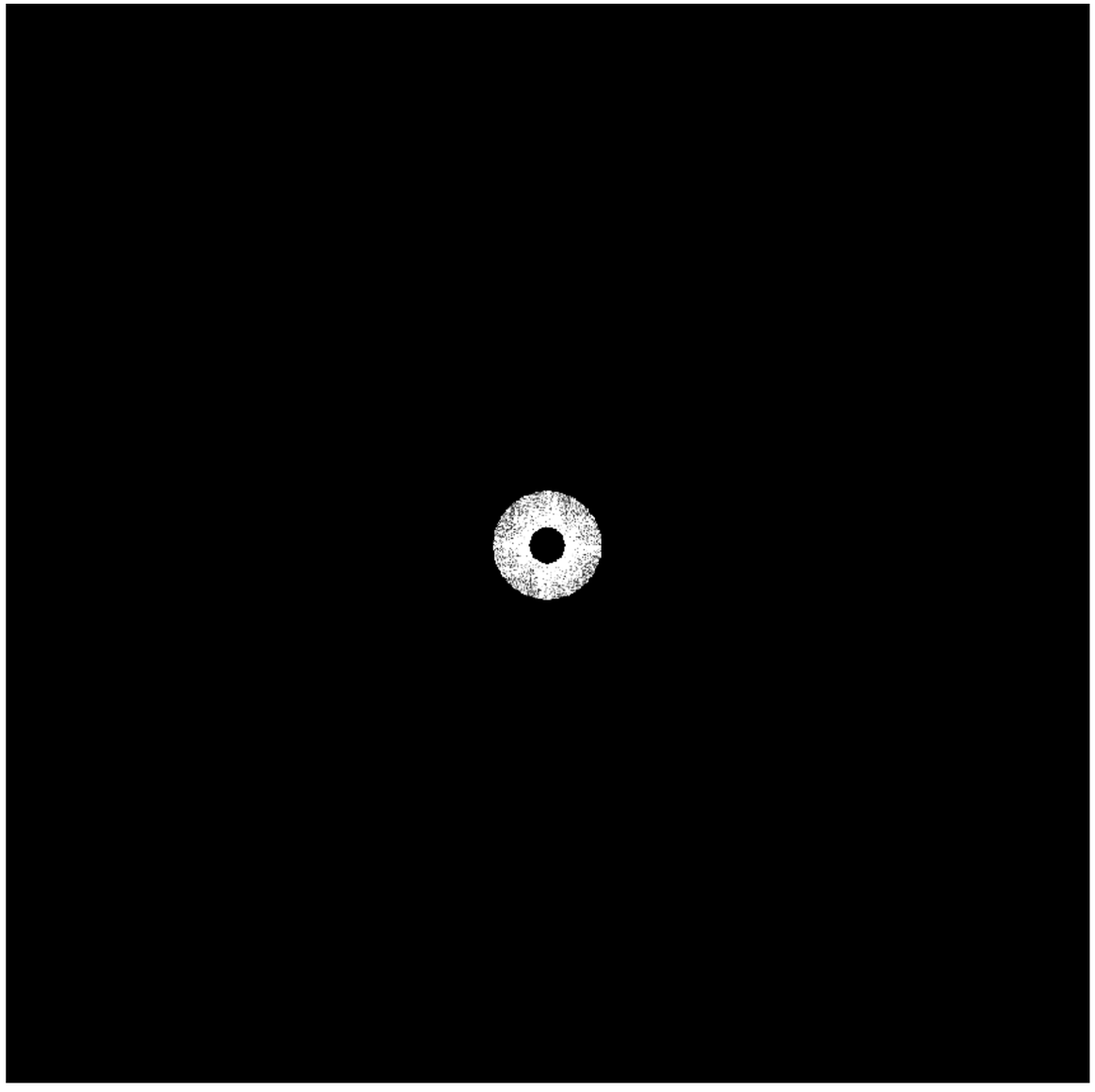
Spatial domain result



Spectrum (after low-pass filter)
All frequencies above cutoff have
0 magnitude

Mid-range frequencies

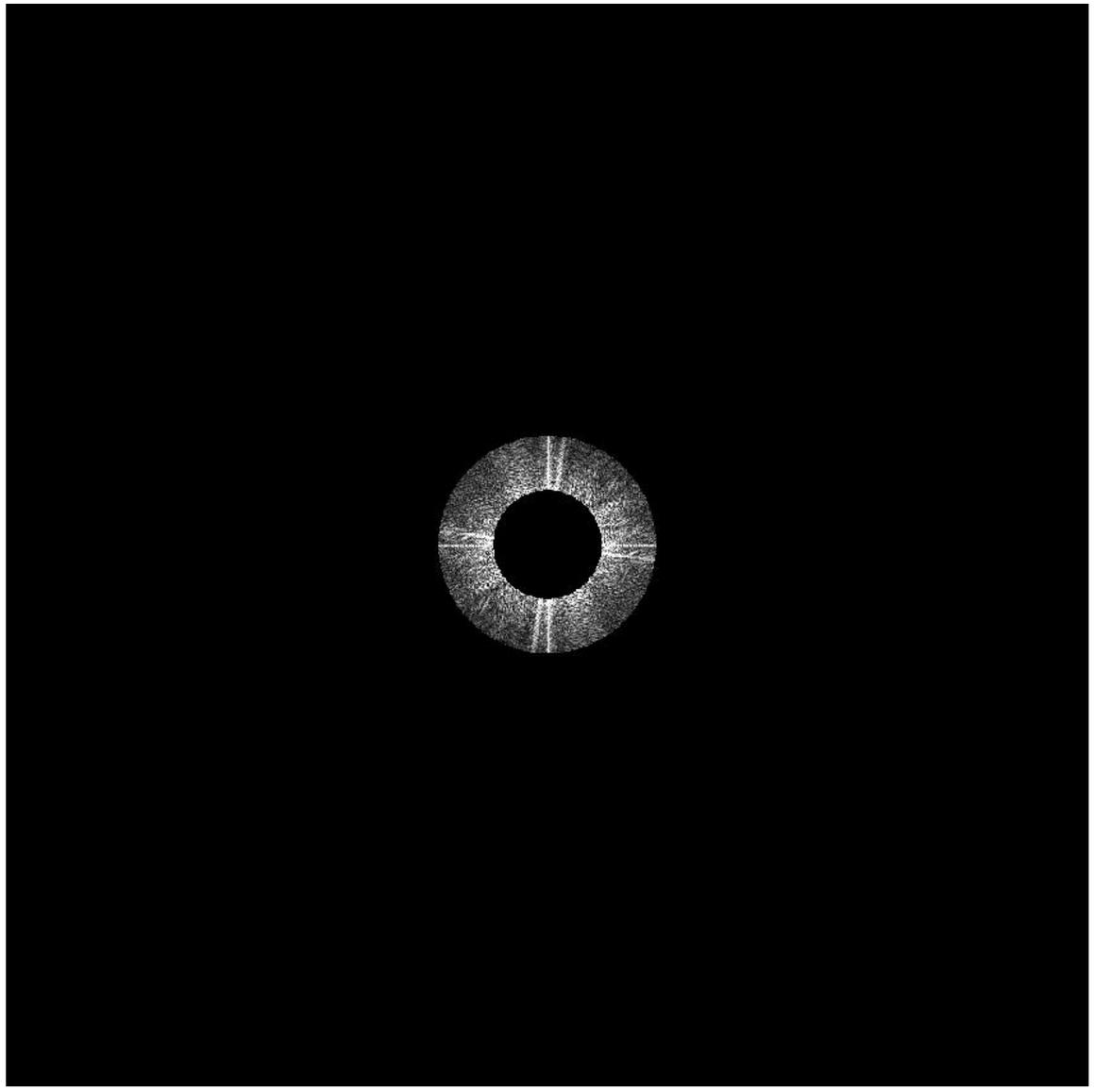
Spatial domain result



Spectrum (after band-pass filter)

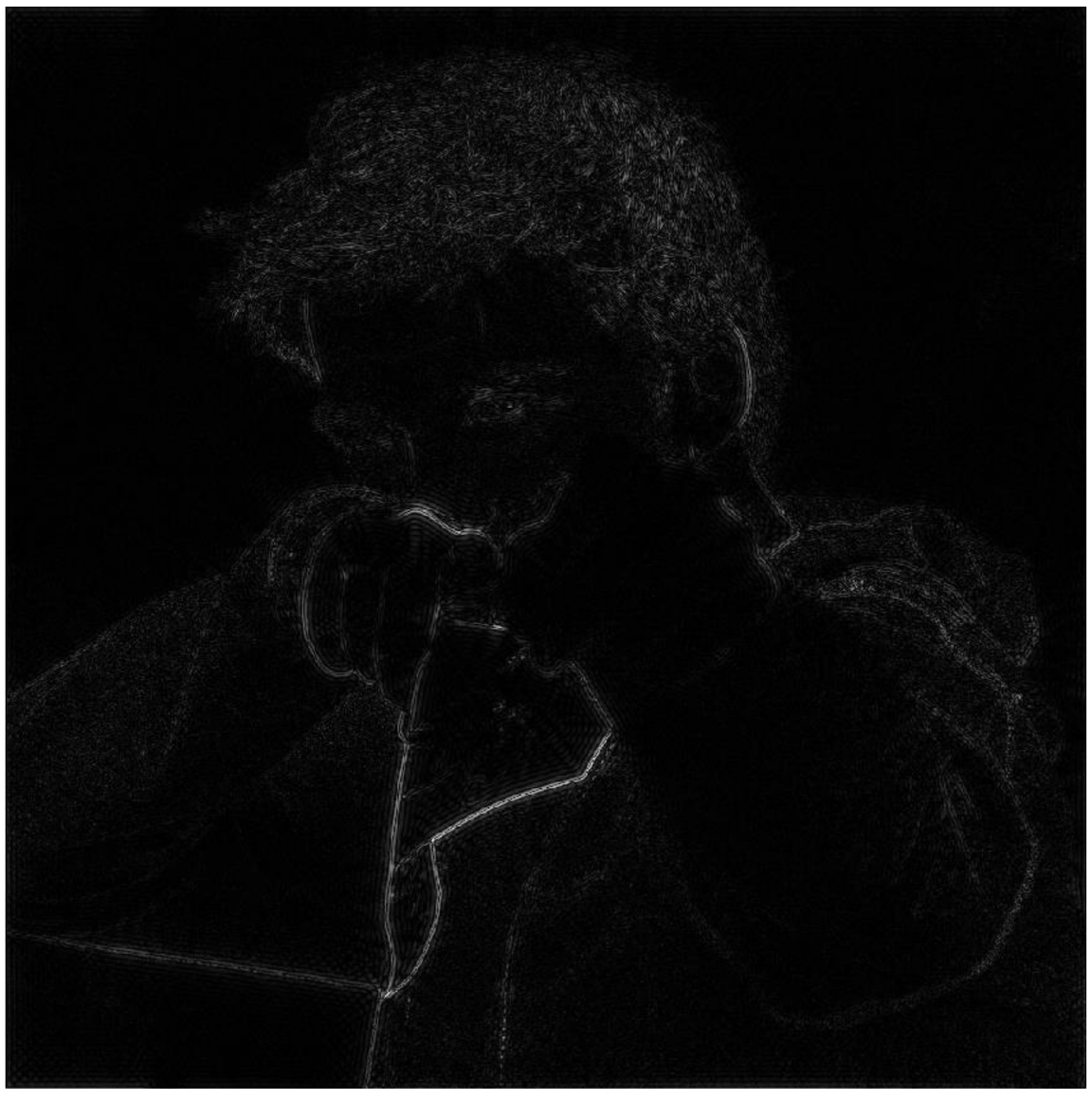
Mid-range frequencies

Spatial domain result

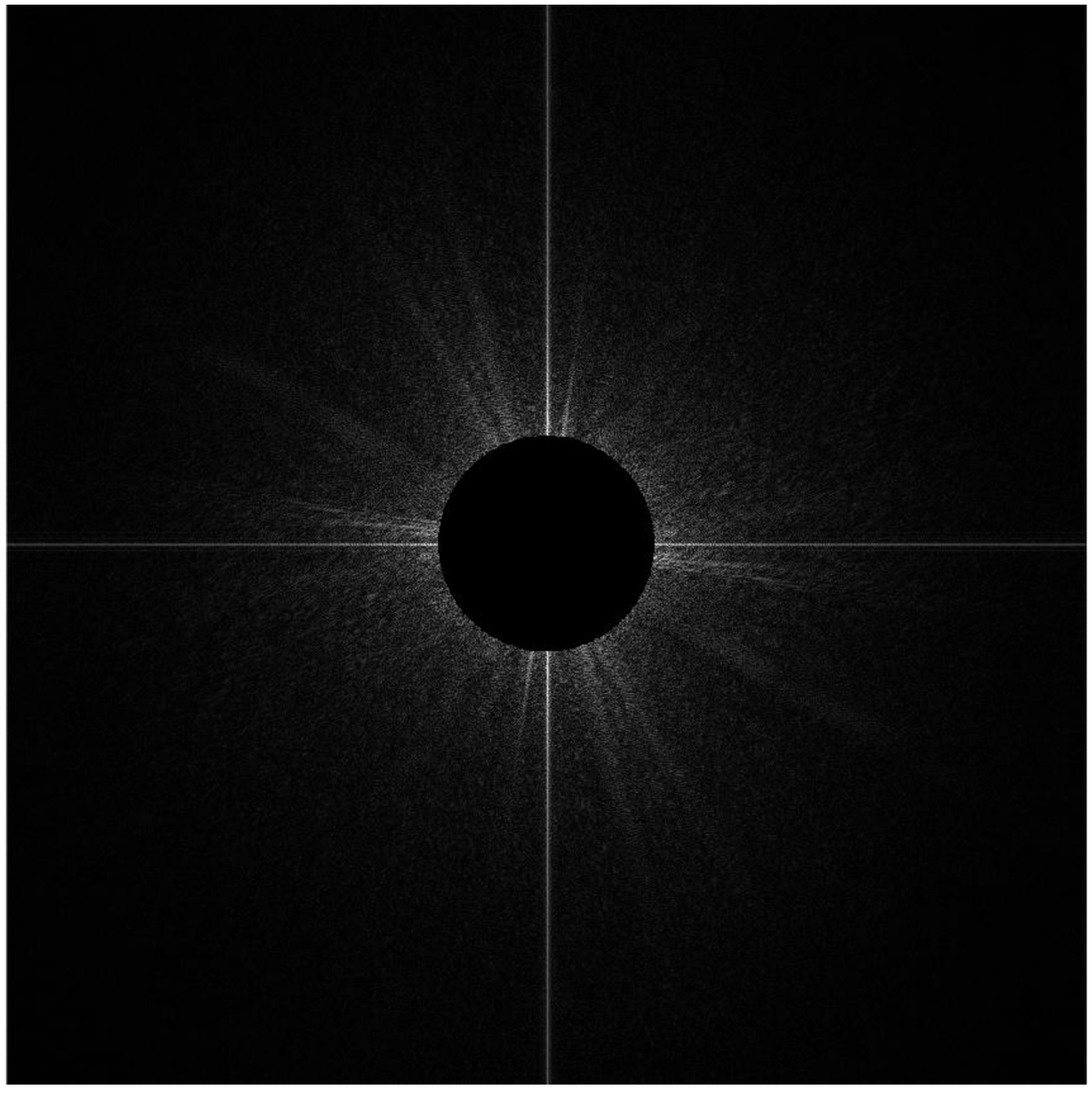


Spectrum (after band-pass filter)

High frequencies (edges)

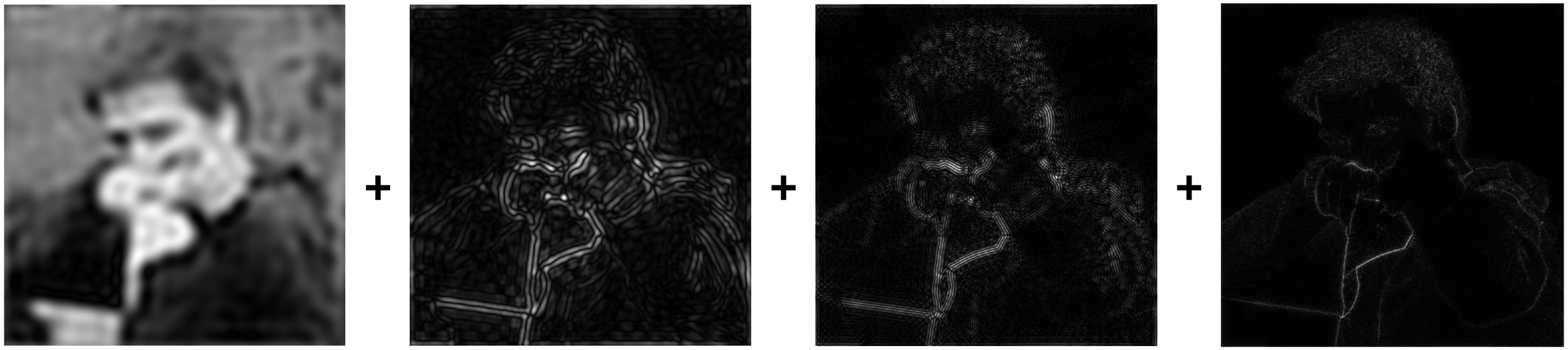


Spatial domain result
(strongest edges)

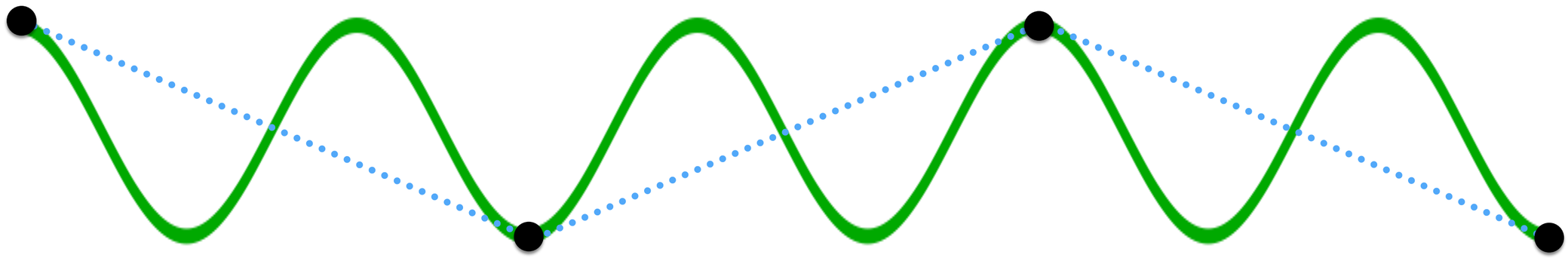


Spectrum (after high-pass filter)
All frequencies below threshold
have 0 magnitude

An image as a sum of its frequency components

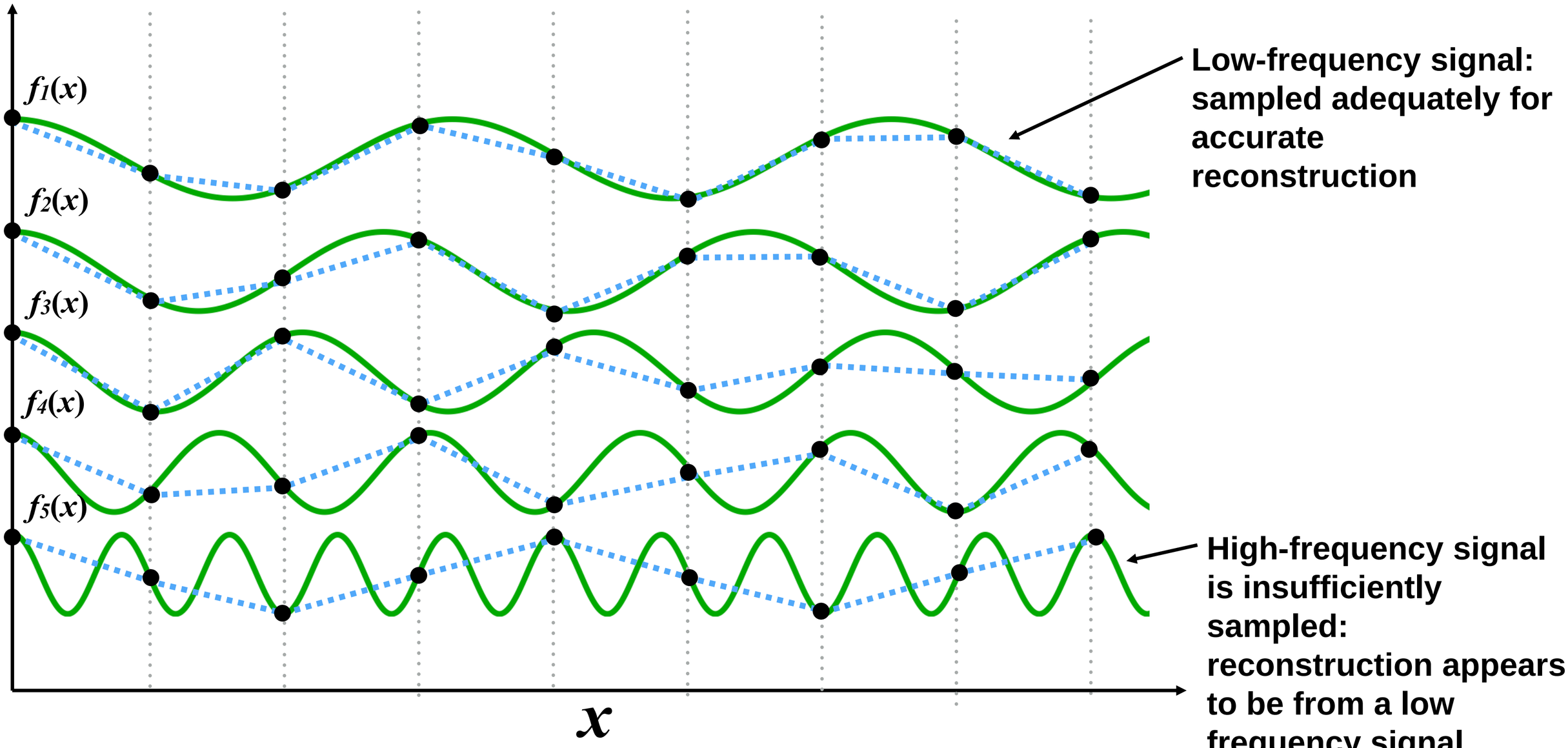


Back to 1D example: Sampling rate, high-frequency signals & aliasing



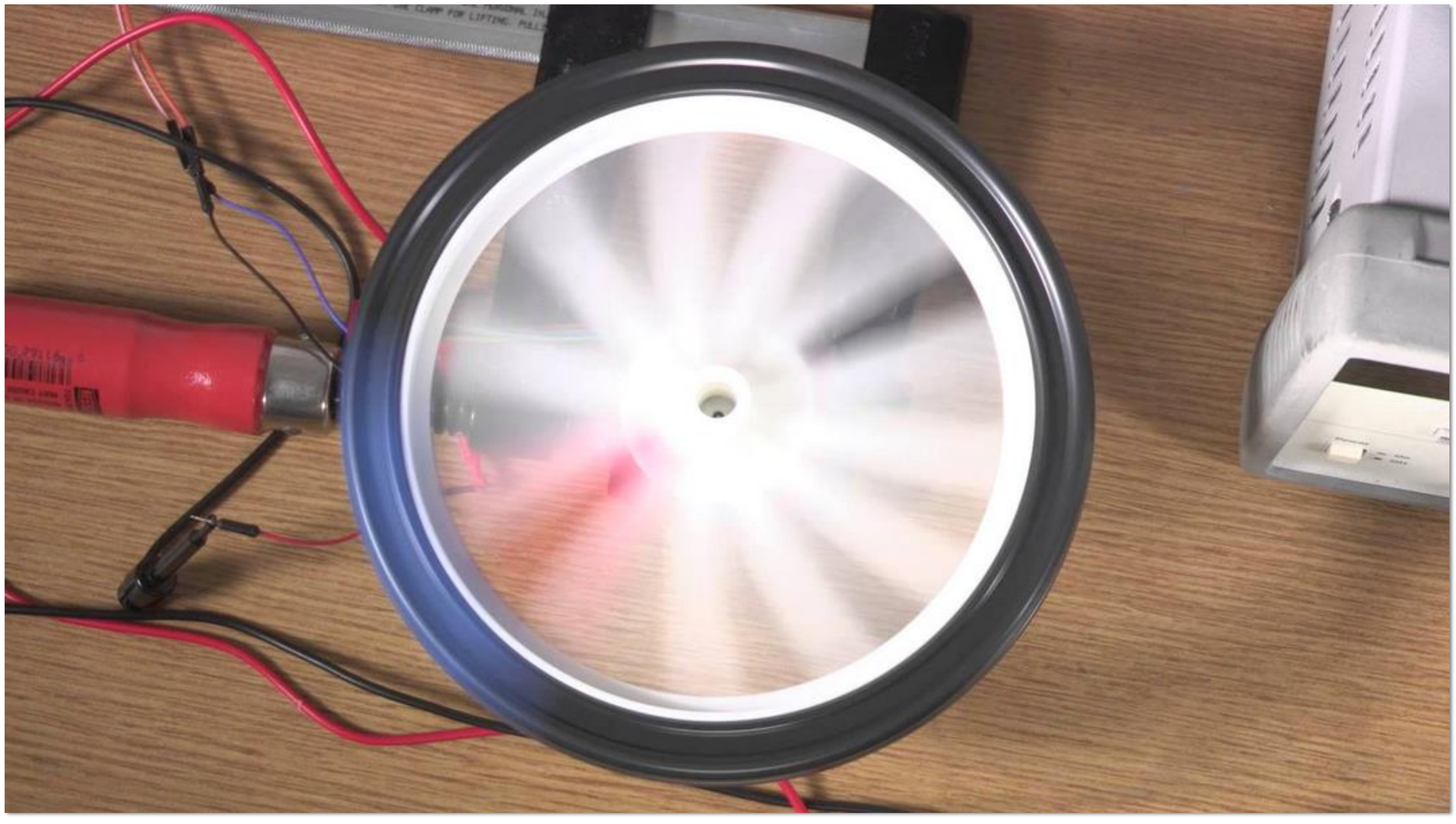
“Aliasing”: high frequencies in the original signal masquerade as low frequencies after reconstruction (due to undersampling)

Back to 1D example: Sampling rate, high-frequency signals & aliasing



“Aliasing”: high frequencies in the original signal masquerade as low frequencies after reconstruction (due to undersampling)

Temporal aliasing: wagon wheel effect



Camera’s frame rate (temporal sampling rate) is too low for rapidly spinning wheel.

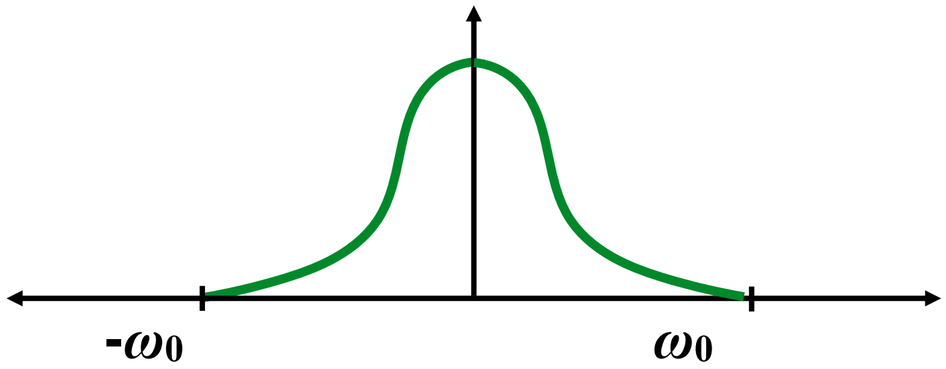
https://www.youtube.com/watch?v=VNftf5qLpiA

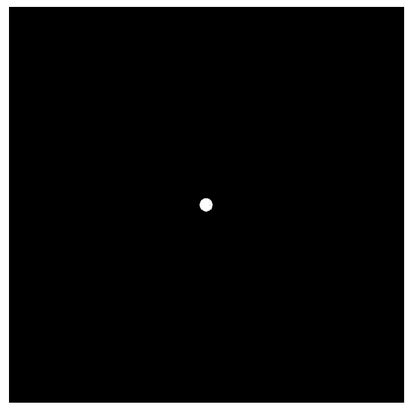
Sampling rate, high-frequency signals & aliasing

- So, how densely should you be sampling?

Nyquist-Shannon theorem

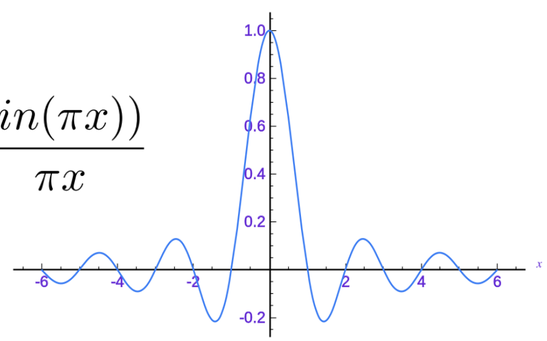
- Consider a band-limited signal: has no frequencies above ω_0
  - 1D: consider low-pass filtered audio signal
  - 2D: recall the blurred image example from a few slides ago





- The signal can be perfectly reconstructed if sampled with period $T > 1 / 2\omega_0$
- And reconstruction is performed using a normalized sinc (ideal reconstruction filter with infinite extent)

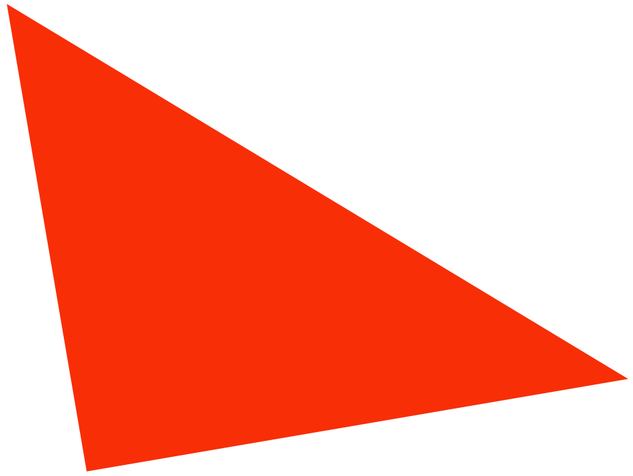
$$sinc(x) = \frac{sin(\pi x))}{\pi x}$$



Challenges of sampling-based approaches in graphics

- **Signals are often not band-limited in computer graphics. Why?**

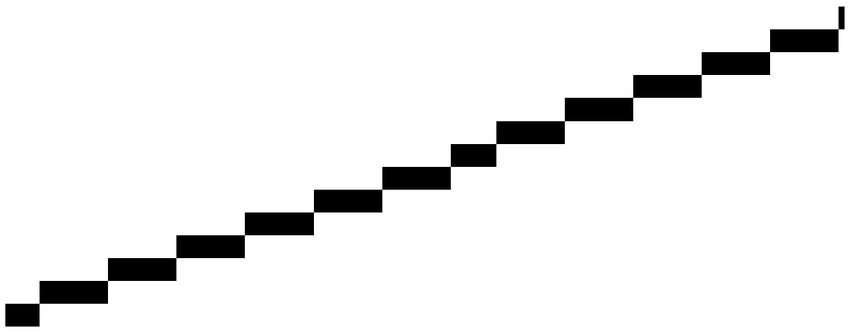
Hint:

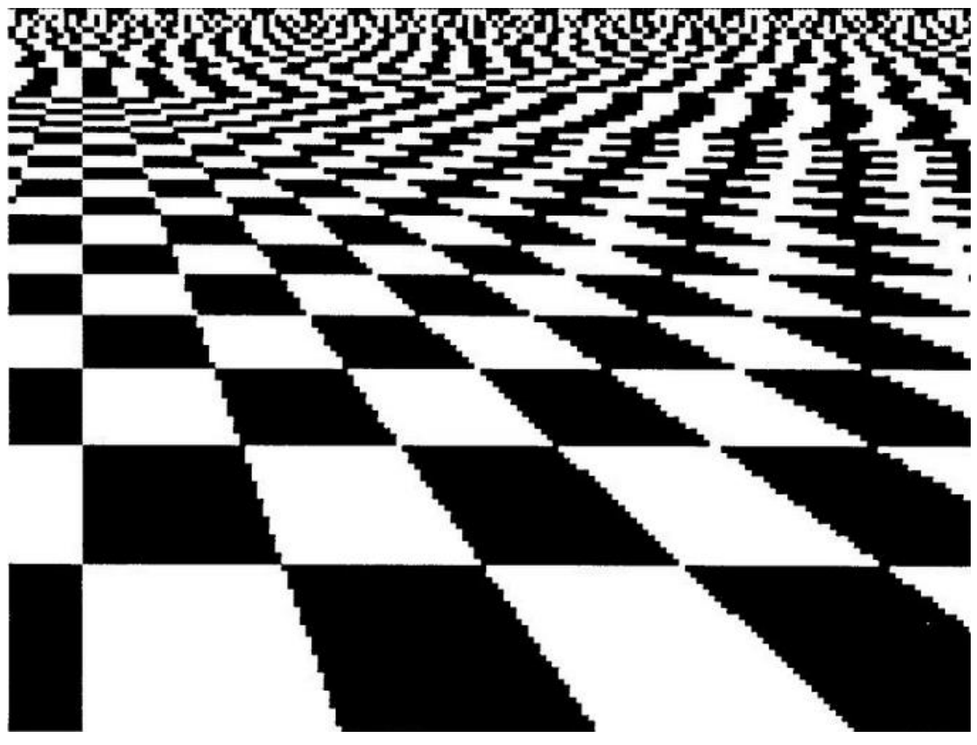


- **Also, infinite extent of “ideal” reconstruction filter (sinc) is impractical for efficient implementations. Why?**

Aliasing artifacts in images

- **Undersampling high-frequency signals and the use of non-ideal resampling filters yields image artifacts**
  - **“Jaggies” in a single image**
  - **“Roping” or “shimmering” of images when animated**
  - **Moiré patterns in high-frequency areas of images**





Sampling a zone plate: $\sin(x^2 + y^2)$

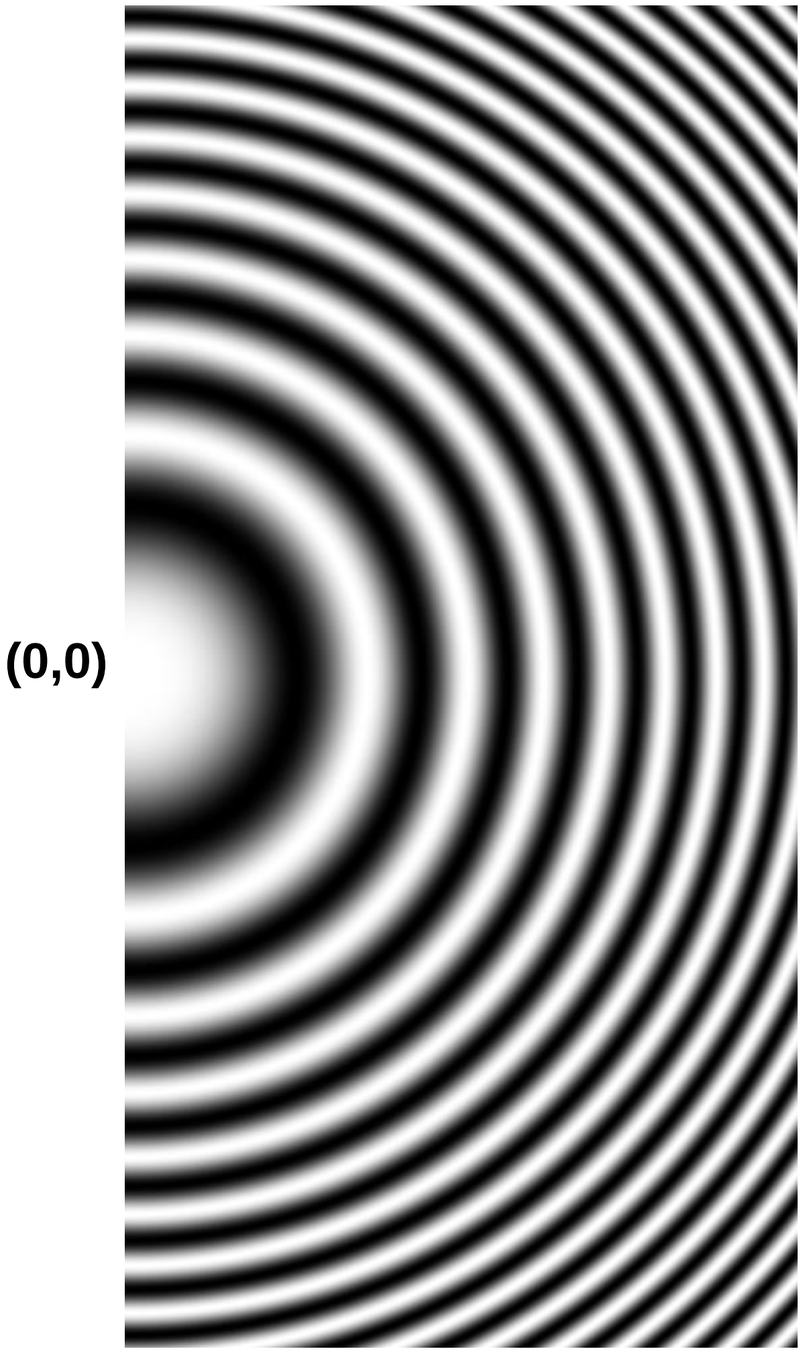
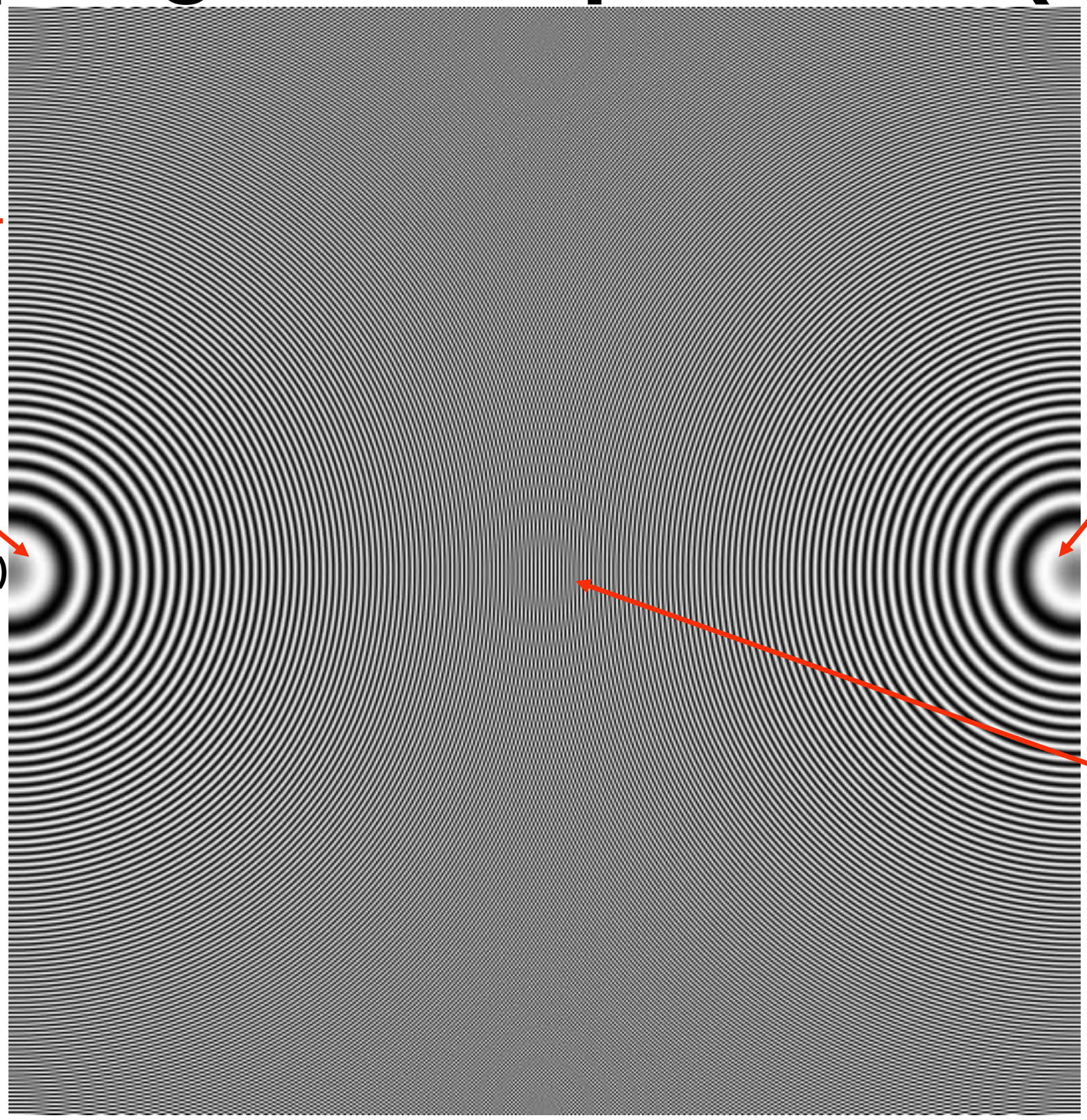


Figure credit: Pat Hanrahan and Bryce Summers

Sampling a zone plate: $\sin(x^2 + y^2)$



Rings in center-left:
Actual signal (low frequency oscillation)

(0,0)

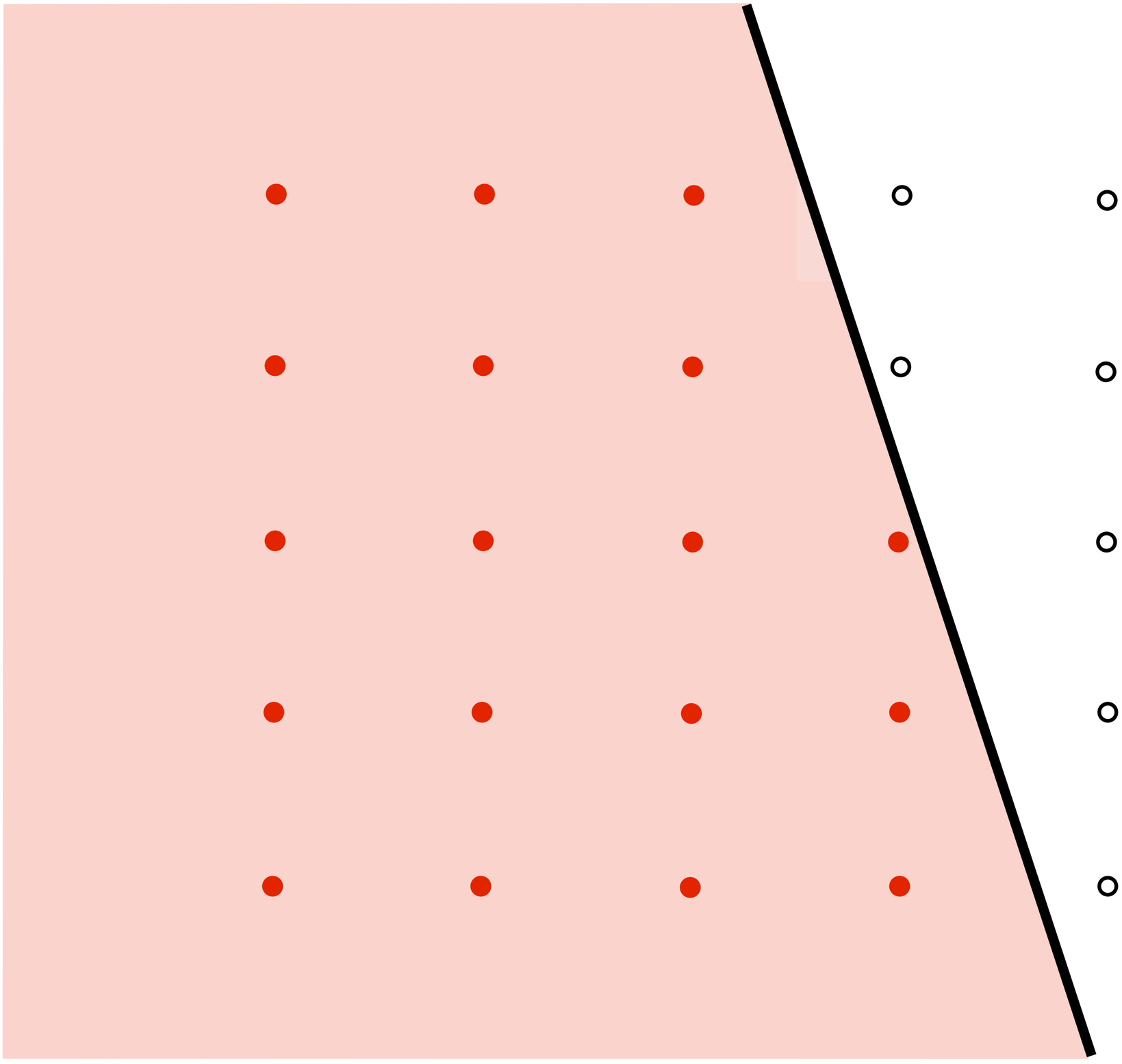
Rings on right: aliasing from undersampling high frequency oscillation and then resampling back to slide resolution

Middle: (interaction between actual signal and aliased resconstruction)

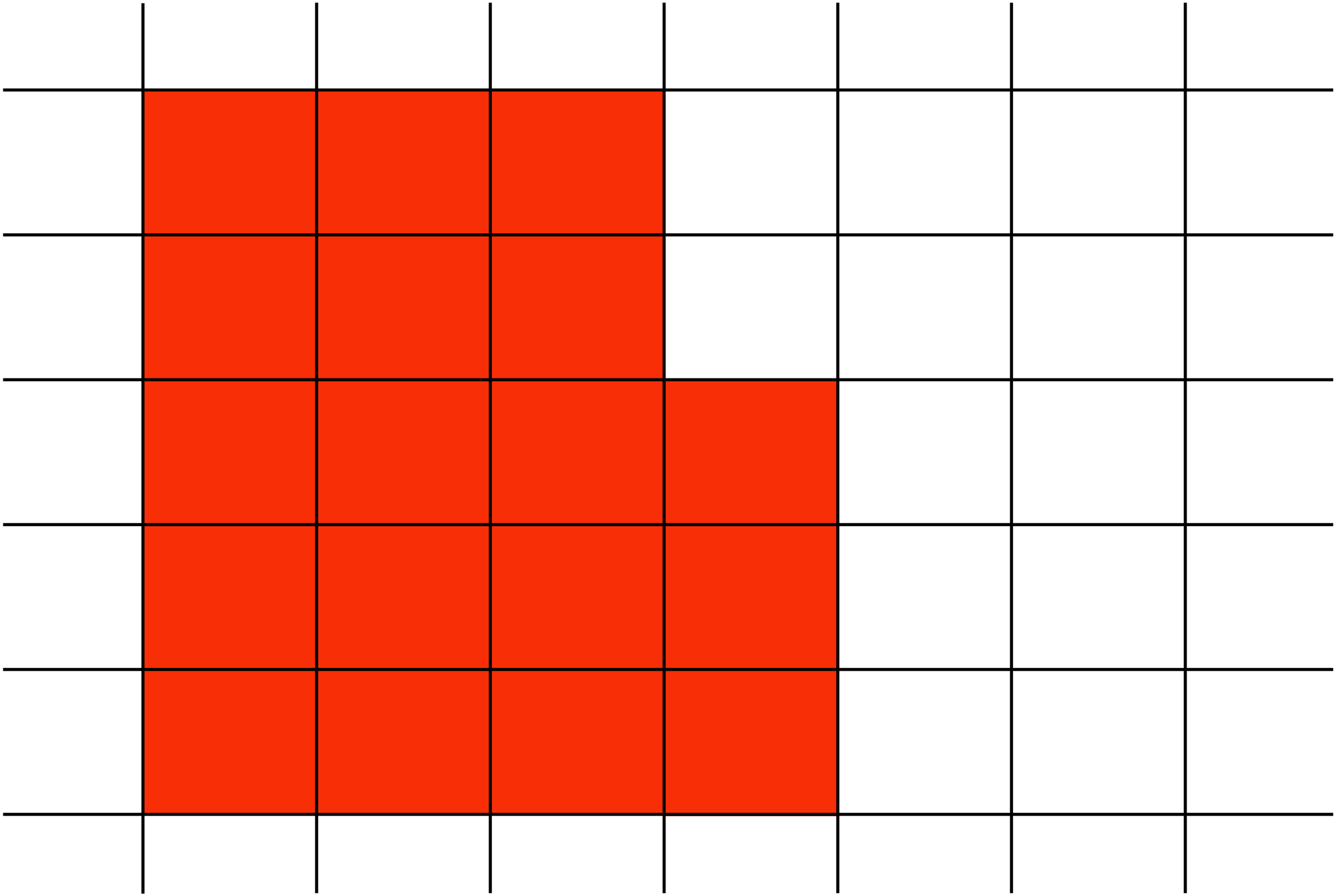
Figure credit: Pat Hanrahan and Bryce Summers

Recall: the real coverage signal

Initial coverage sampling rate (1 sample per pixel)

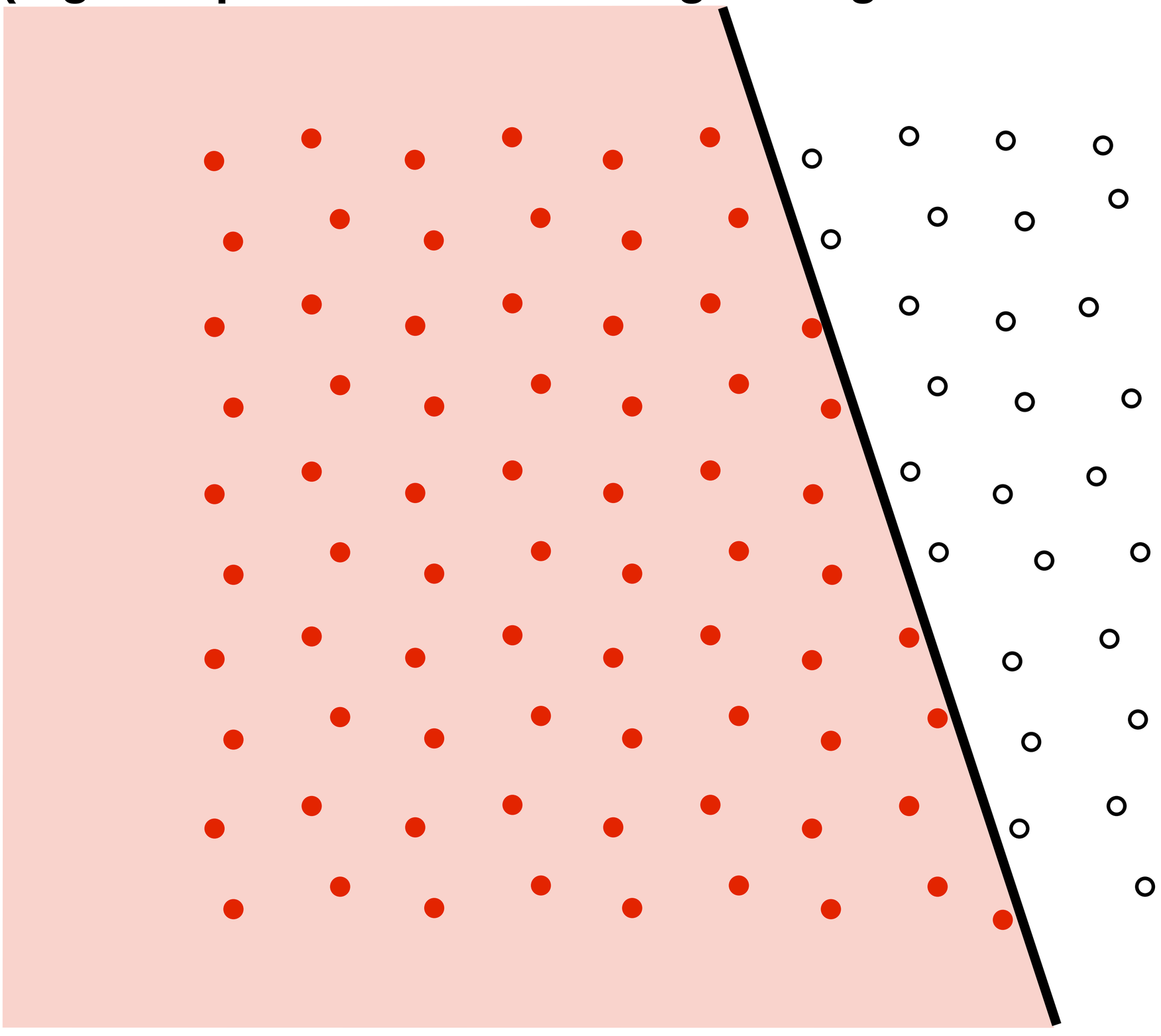


We see this on the screen



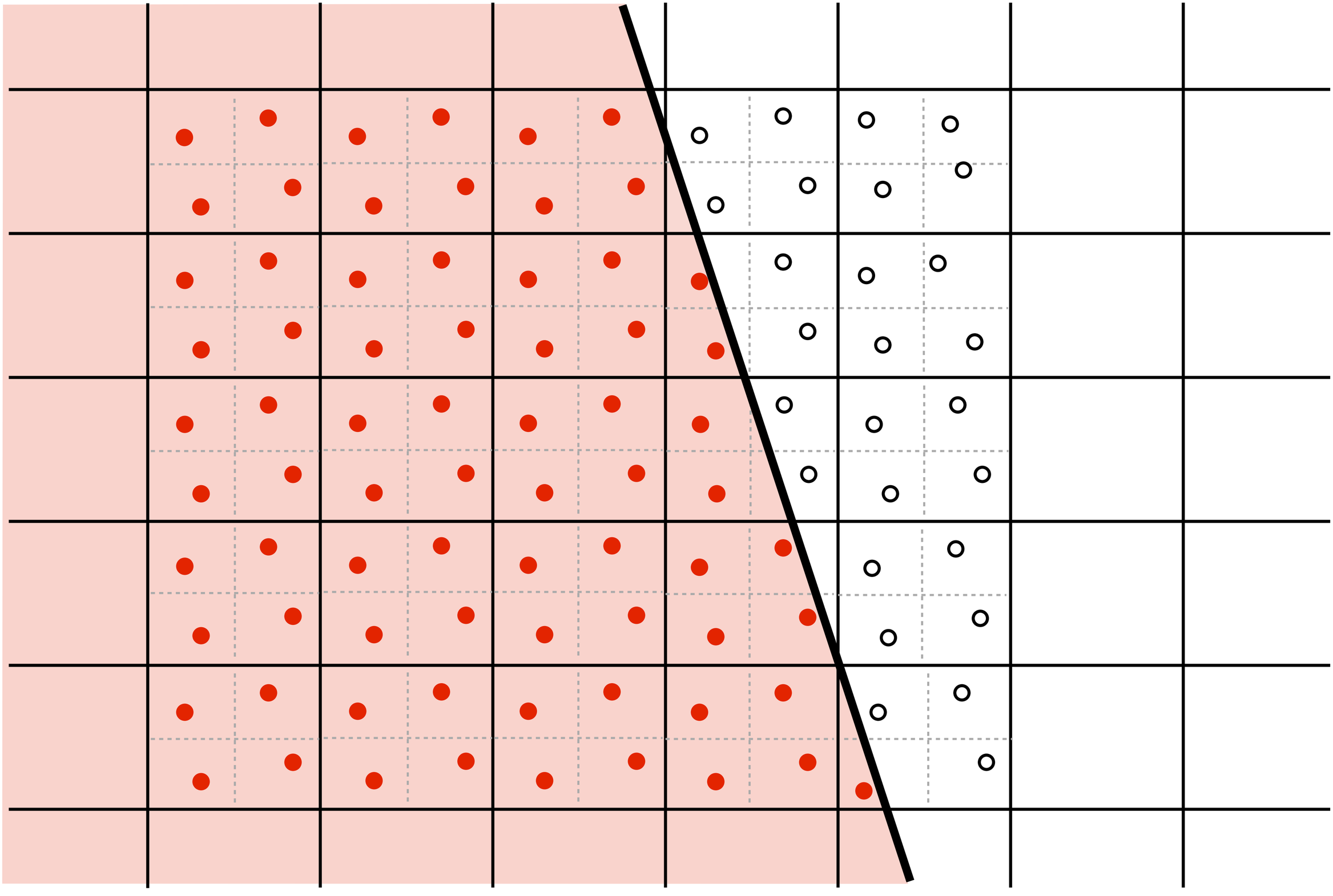
Increase density of sampling coverage signal

(high frequencies exist in original signal because of triangle edges)



Supersampling

Example: stratified sampling using four samples per pixel

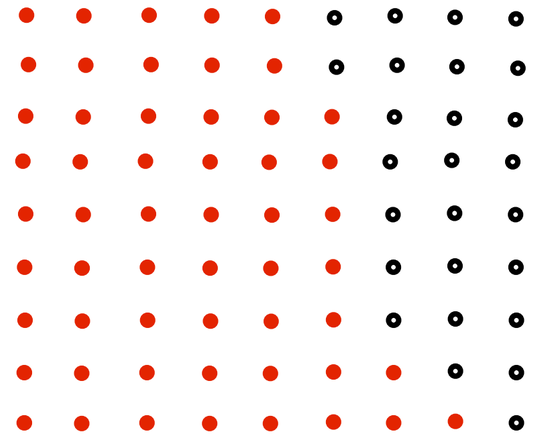


Ok, but now we have more samples than pixels!

Resampling

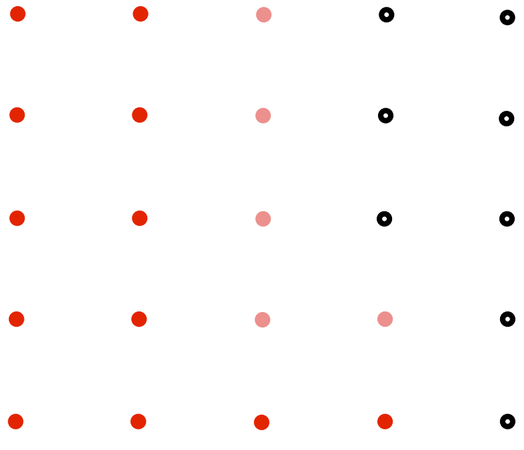
Converting from one discrete sampled representation to another

Original signal
(high frequency edge)



Dense sampling of
initial signal

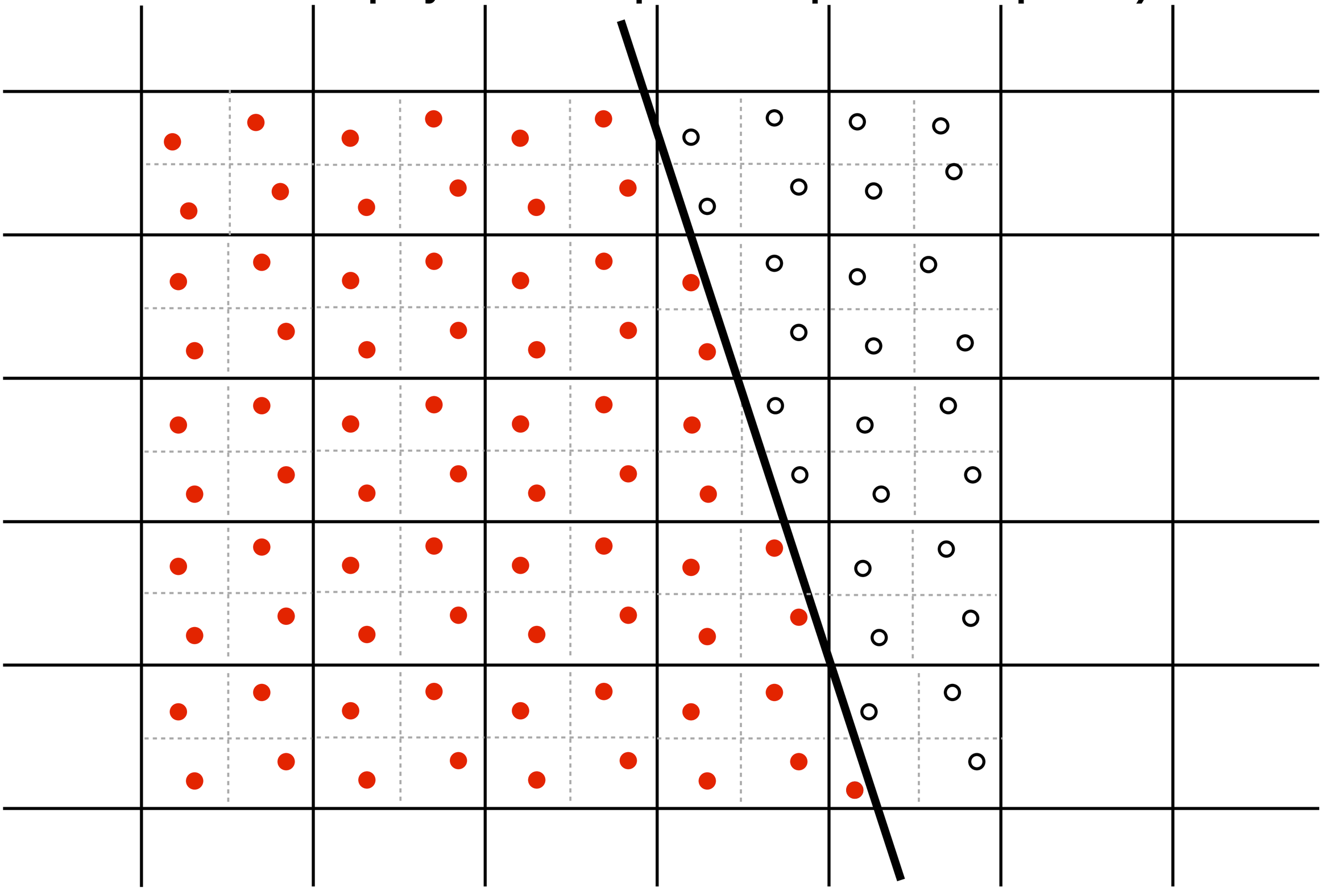
Reconstructed signal
(lacks high frequencies)



Coarsely sampled
reconstructed signal

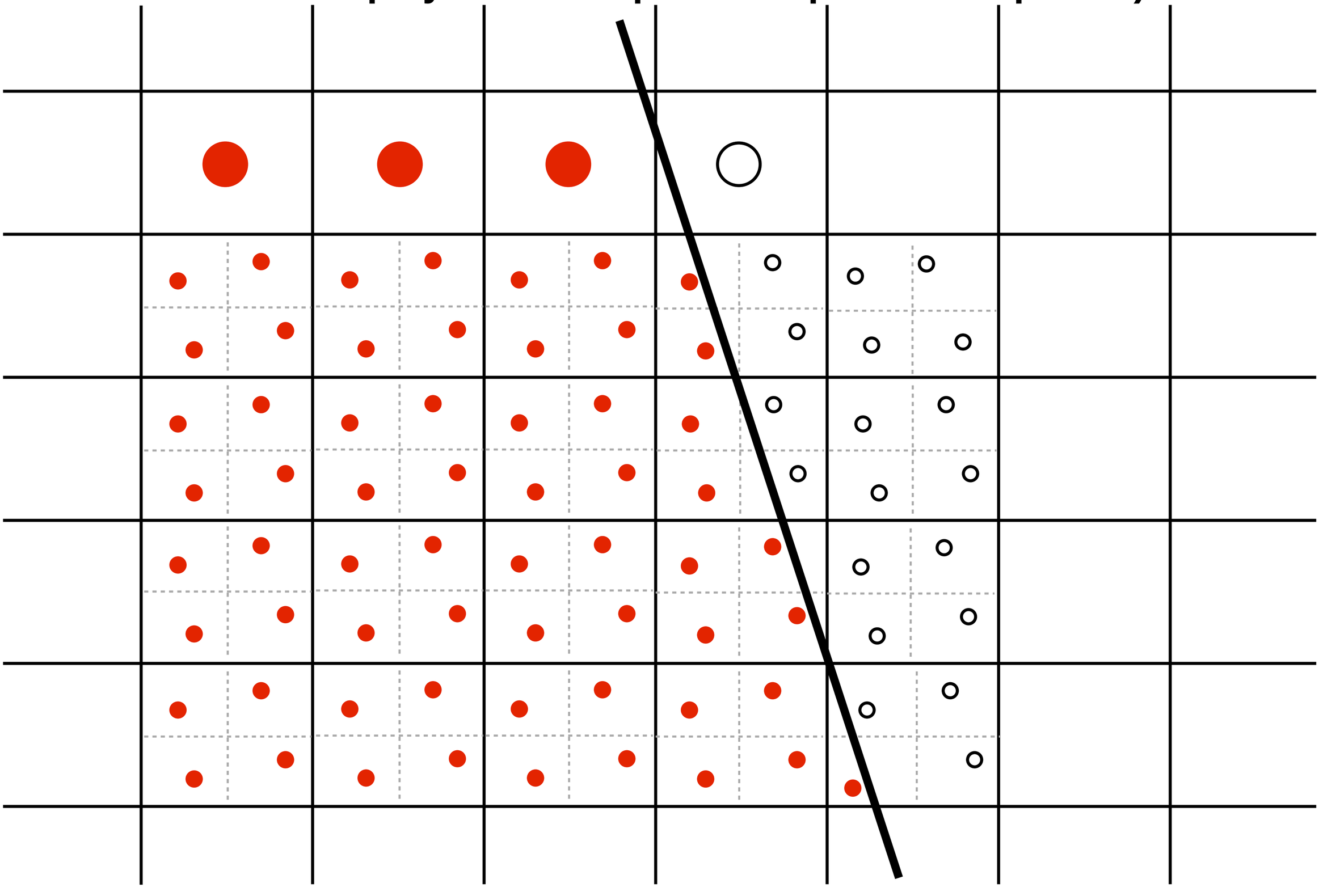
Resample to display's pixel resolution

(Because a screen displays one sample value per screen pixel...)

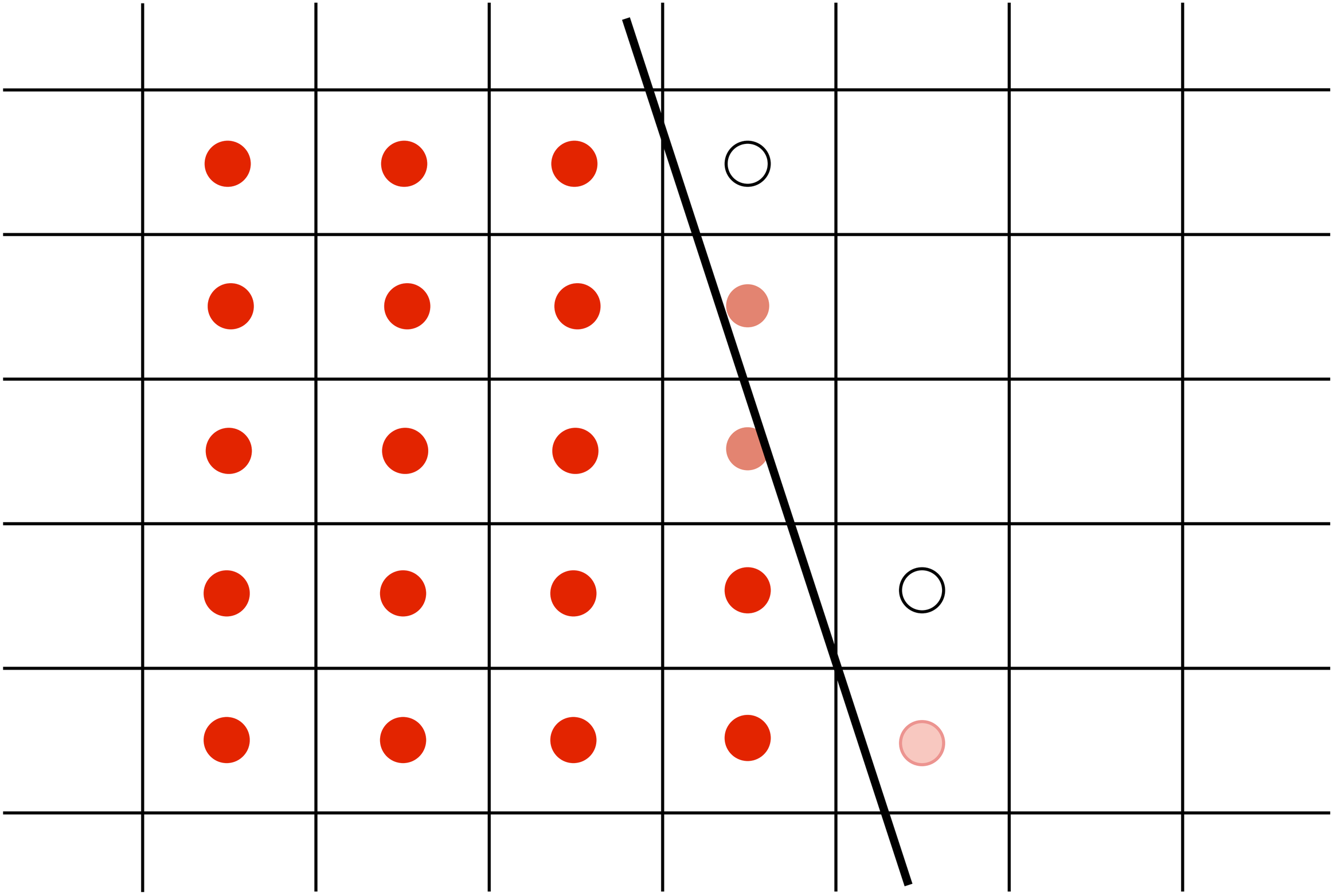


Resample to display's pixel resolution

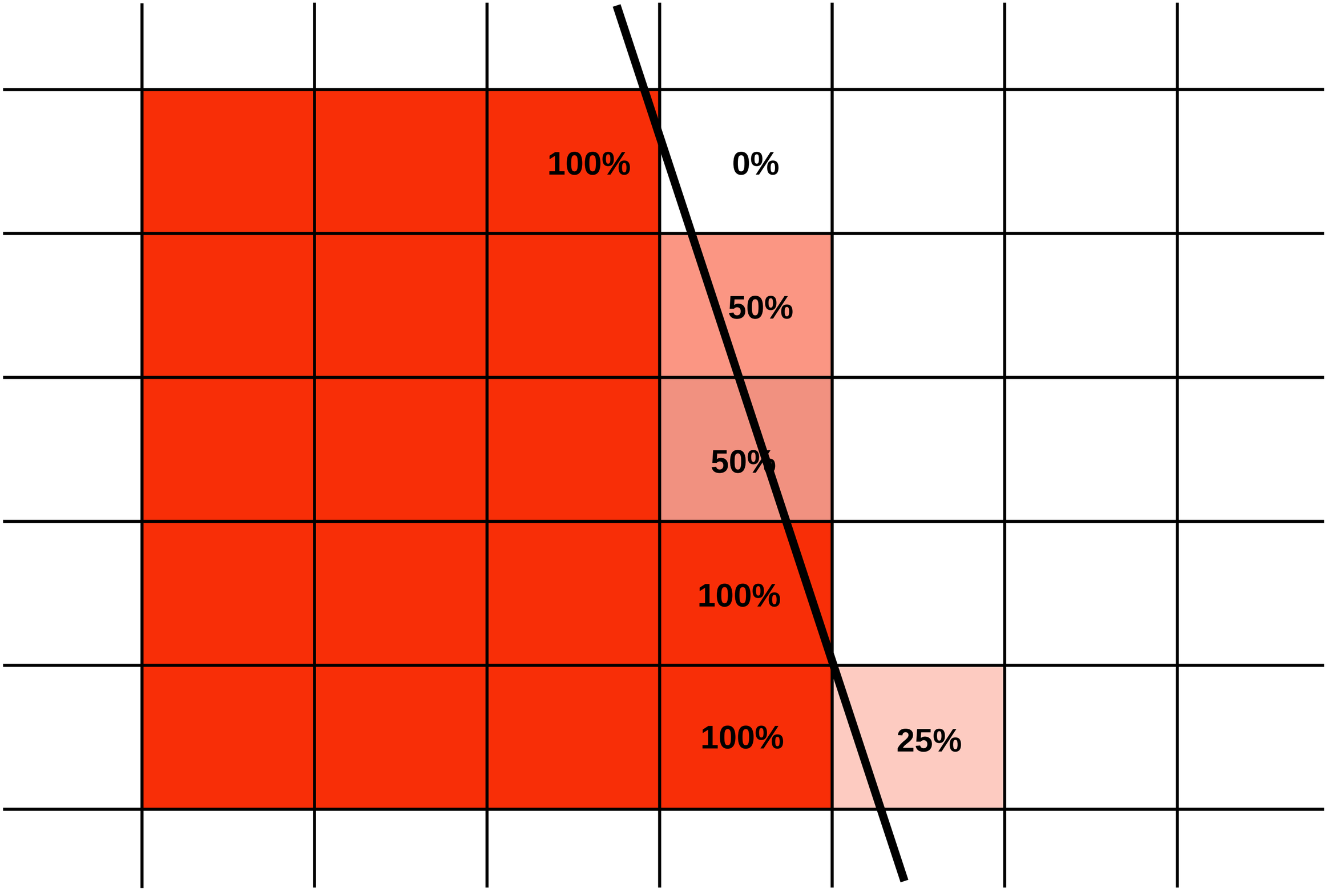
(Because a screen displays one sample value per screen pixel...)



Resample to display's pixel resolution

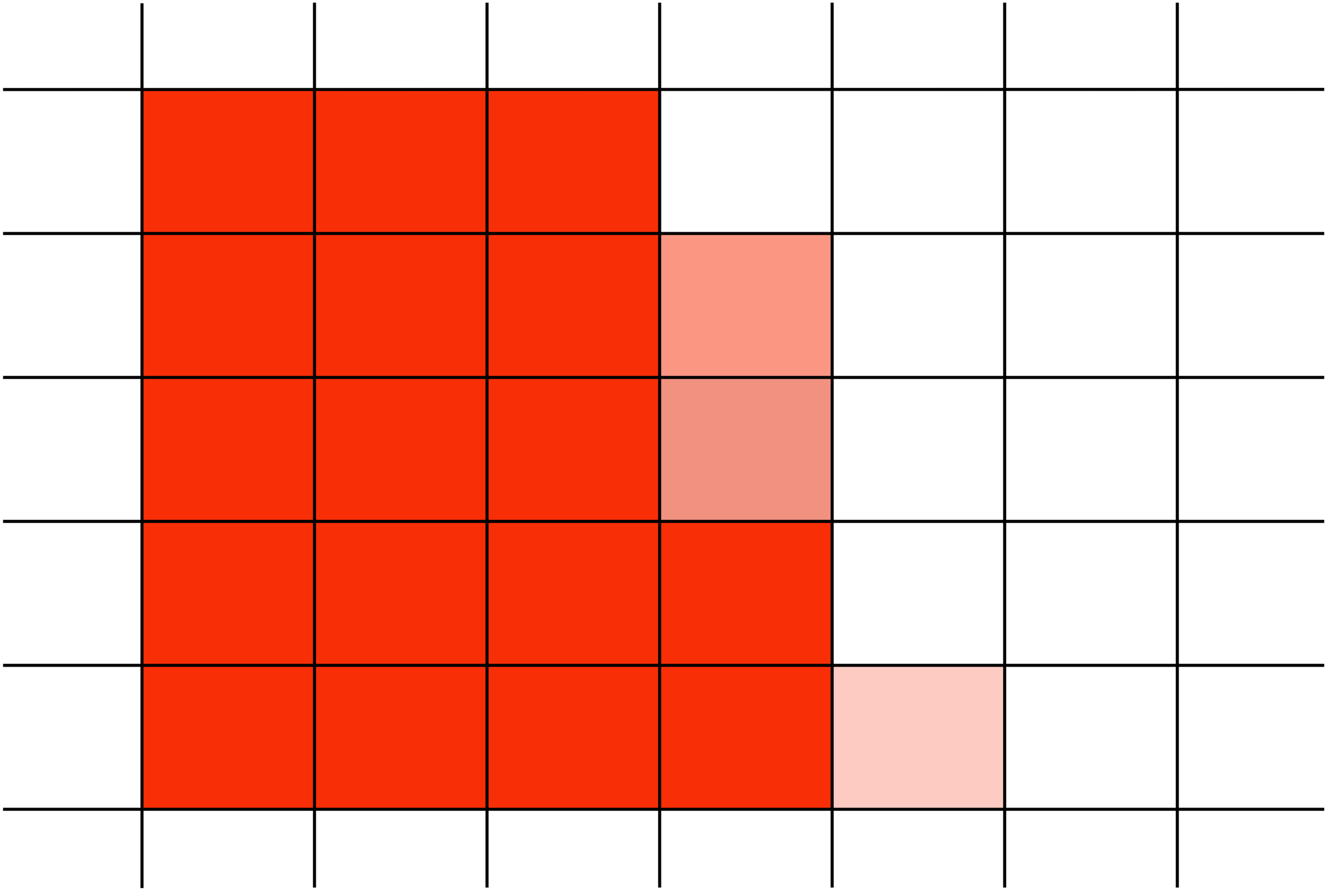


Displayed result (note anti-aliased edges)



Recall: the real coverage signal

Displayed result (note anti-aliased edges)



Pretty much as well as we can do without an “infinite resolution display”

Sampling triangle coverage (evaluating coverage(x,y) for a triangle)

Point-in-triangle test

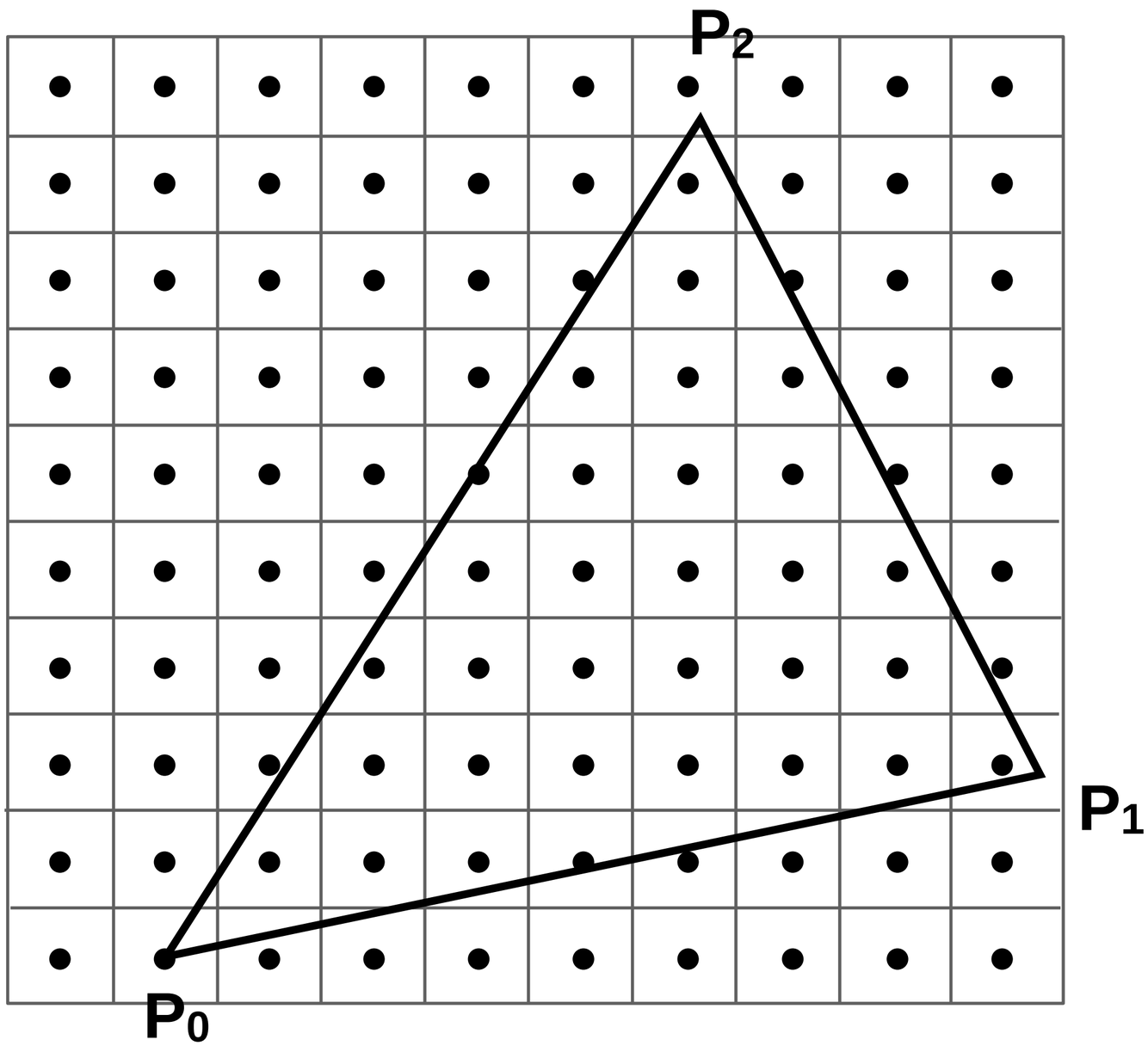
Compute triangle edge equations from projected positions of vertices

$$P_i = (X_i, Y_i)$$

$$dX_i = X_{i+1} - X_i$$
$$dY_i = Y_{i+1} - Y_i$$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i\, x + B_i\, y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



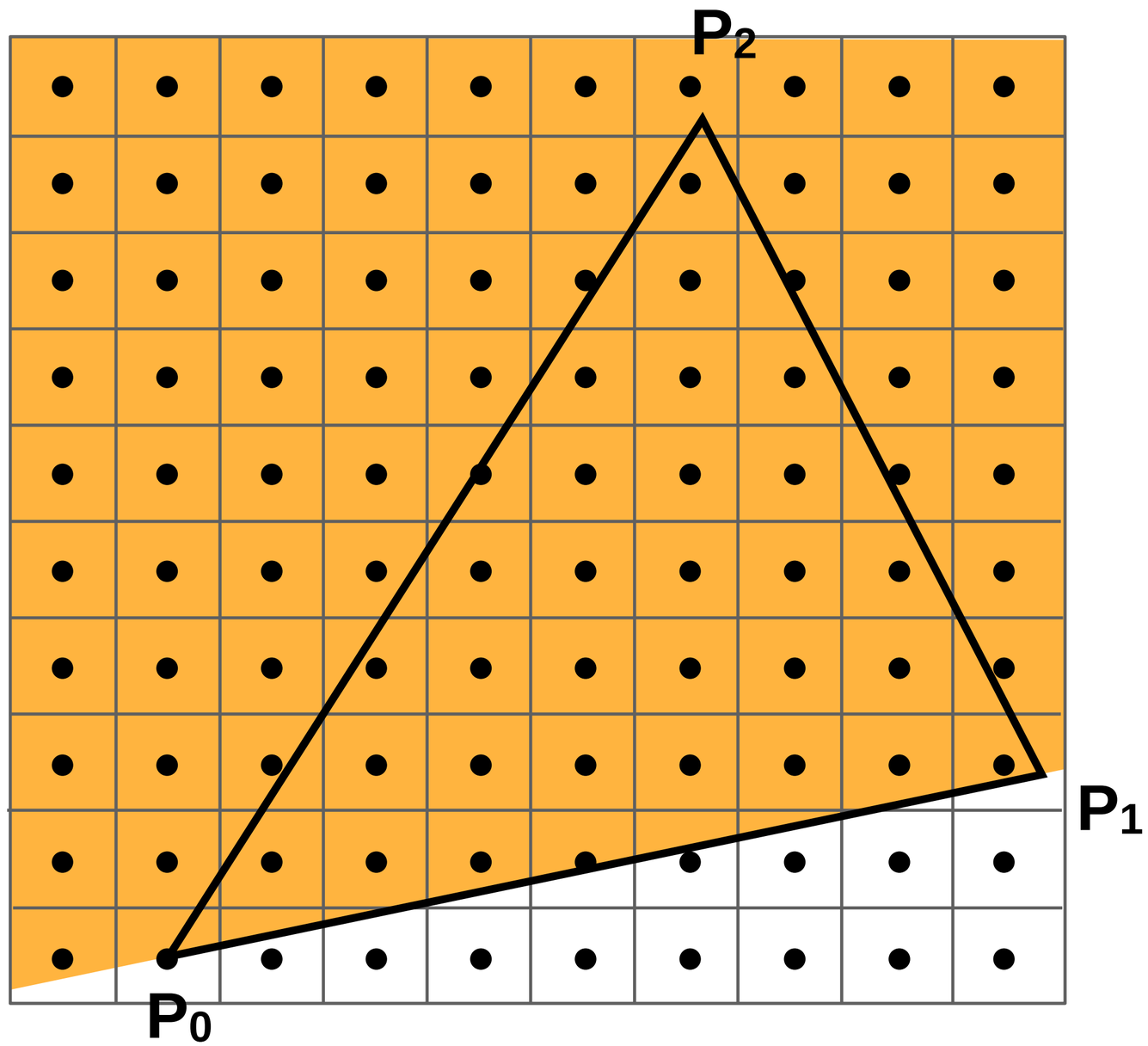
Point-in-triangle test

$$P_i = (X_i, Y_i)$$

$$dX_i = X_{i+1} - X_i$$
$$dY_i = Y_{i+1} - Y_i$$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i\, x + B_i\, y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



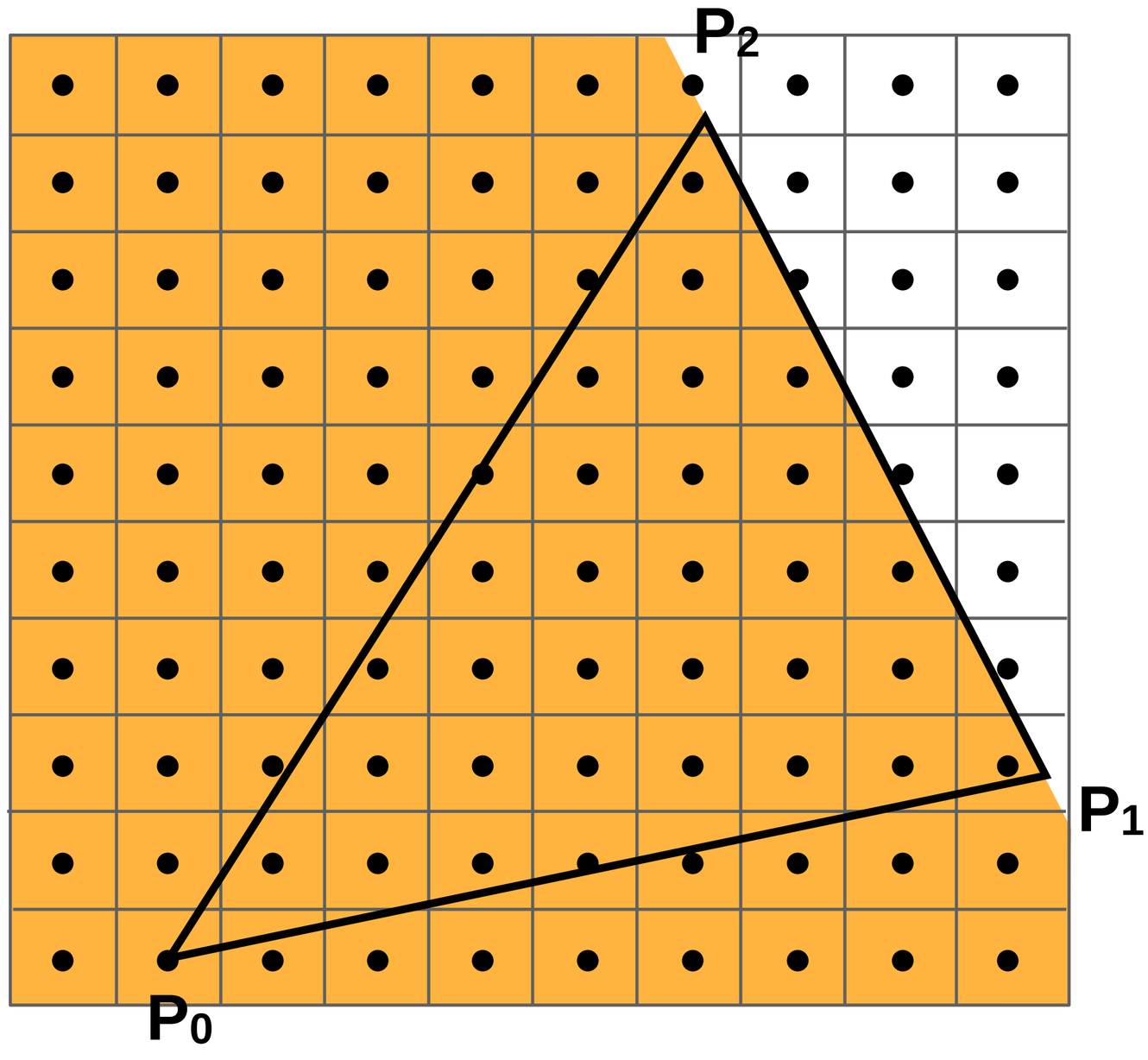
Point-in-triangle test

$$P_i = (X_i, Y_i)$$

$$dX_i = X_{i+1} - X_i$$
$$dY_i = Y_{i+1} - Y_i$$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i\, x + B_i\, y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



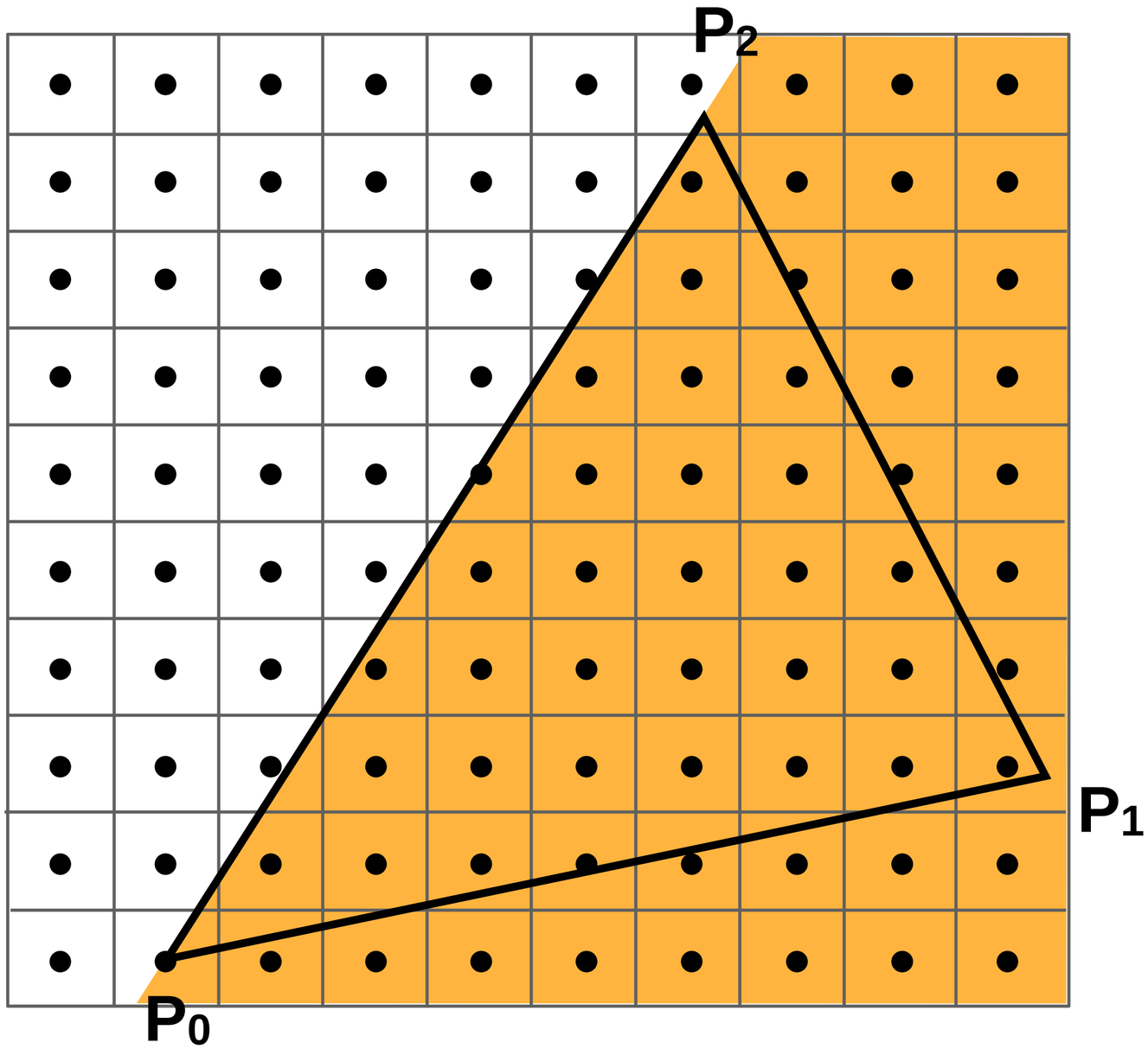
Point-in-triangle test

$P_i = (X_i, Y_i)$

$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$
$= A_i\, x + B_i\, y + C_i$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge

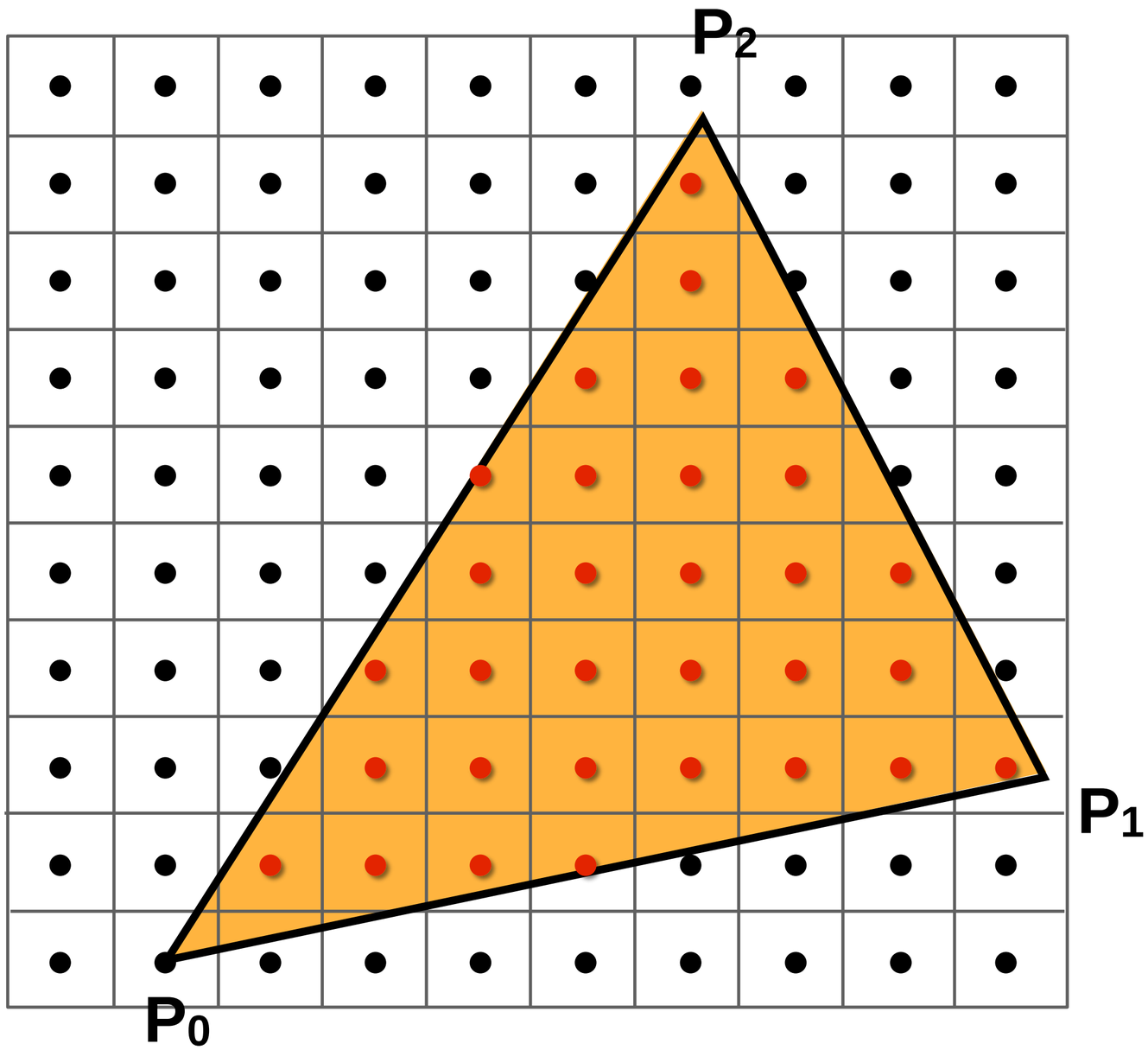


Point-in-triangle test

Sample point $s = (sx, sy)$ **is inside the triangle if it is "inside" all three edges.**

$$inside(sx, sy) =$$
$$E_0(sx, sy) < 0 \;\&\&$$
$$E_1(sx, sy) < 0 \;\&\&$$
$$E_2(sx, sy) < 0;$$

Note: actual implementation of $inside(sx,sy)$ **involves ≤ checks based on the triangle coverage edge rules (see earlier slides)**

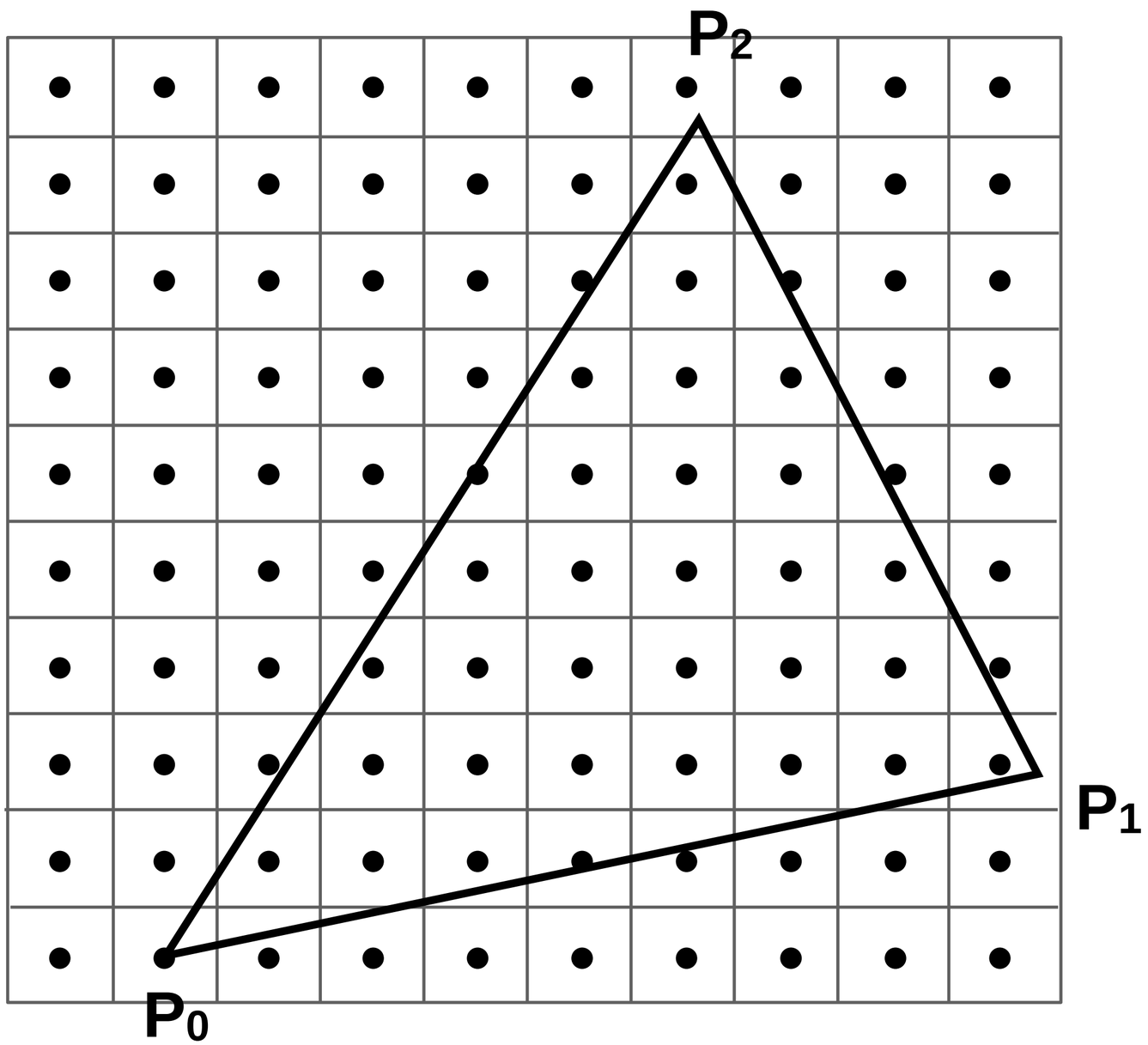


Sample points inside triangle are highlighted red.

Point-in-triangle test

Which points should we test?

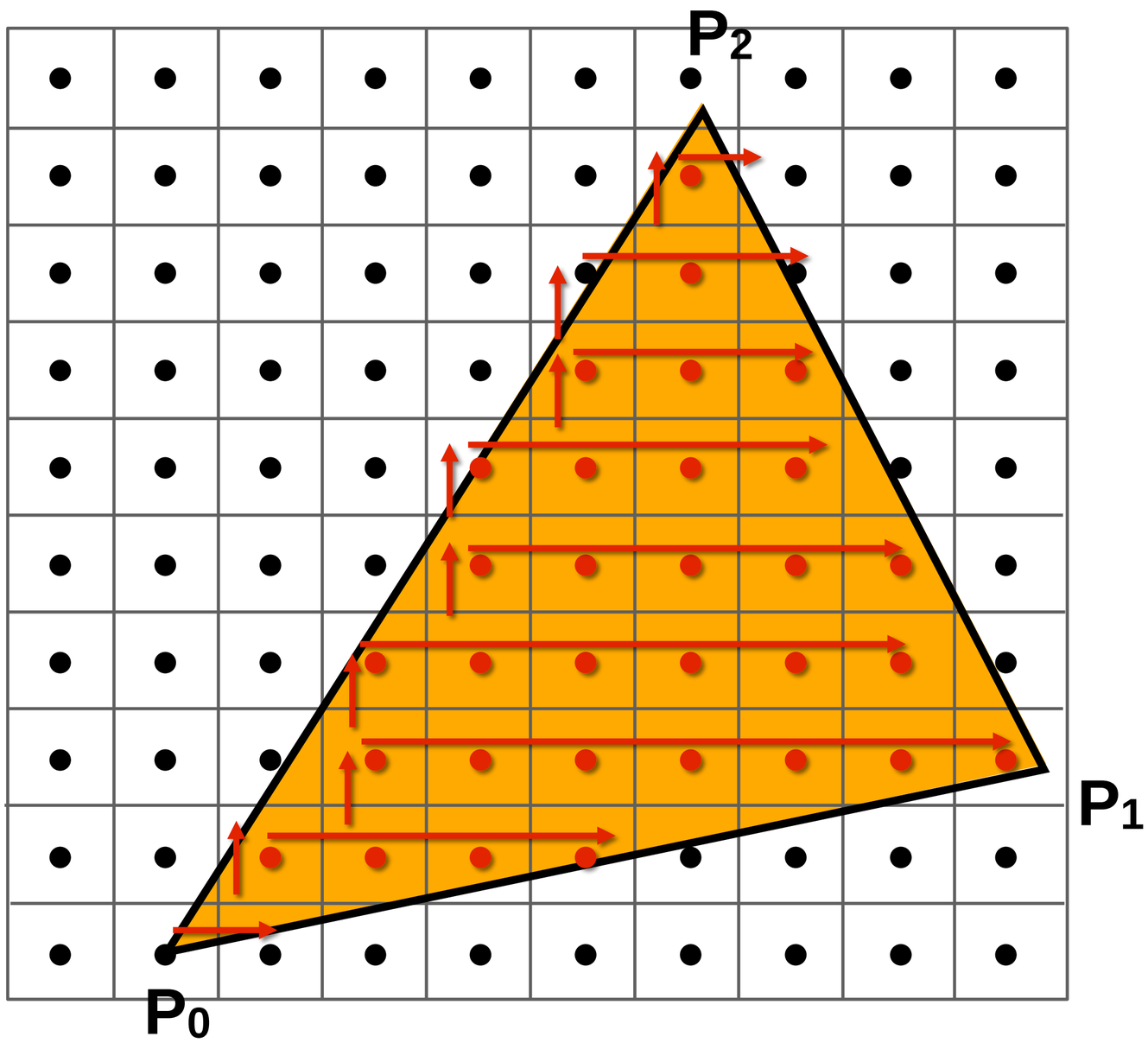
- **All of them?**
- **Points within bounding box?**



Incremental triangle traversal

Rather than testing all points on screen, traverse them incrementally

Many traversal orders are possible: backtrack, zig-zag, Hilbert/Morton curves (locality maximizing)



All modern GPUs have special-purpose hardware for efficiently performing point-in-triangle tests

Modern approach: tiled triangle traversal

Traverse triangle in blocks

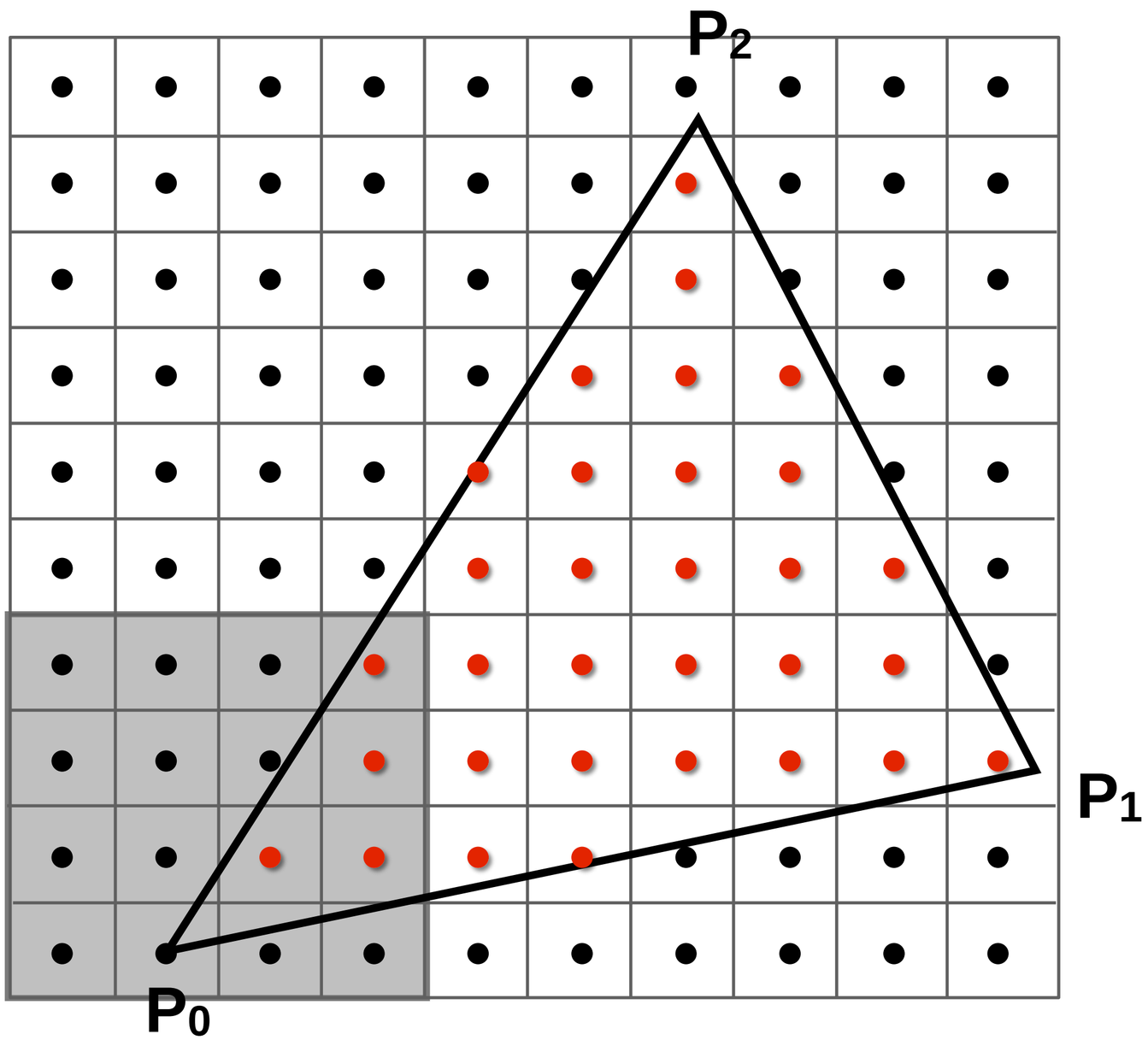
Test all samples in block against triangle in parallel

Advantages:

- **Simplicity of wide parallel execution overcomes cost of extra point-in-triangle tests (most triangles cover many samples, especially when super-sampling coverage)**

- **Can skip sample testing work: entire block not in triangle (“early out”), entire block entirely within triangle (“early in”)**

- **Additional advantages related to accelerating occlusion computations (not discussed today)**



All modern GPUs have special-purpose hardware for efficiently performing point-in-triangle tests

Summary

- **We formulated computing triangle-screen coverage as a sampling problem**
  - **Triangle-screen coverage is a 2D signal**
  - **Undersampling and the use of simple (non-ideal) reconstruction filters may yield aliasing**
  - **In today's example, we reduced aliasing via supersampling**

- **Image formation on a display**
  - **When samples are 1-to-1 with display pixels, sample values are handed directly to display**
  - **When "supersampling", resample densely sampled signal down to display resolution**

- **Sampling screen coverage of a projected triangle:**
  - **Performed via three point-inside-edge tests**
  - **Real-world implementation challenge: balance conflicting goals of avoiding unnecessary point-in-triangle tests and maintaining parallelism in algorithm implementation**