Name		Grade
Surname		
Department		
Legi Nr.		
Date	04.11.2016	

1	2	3	4	5	Total
4	4	6	6	6	26

- **Keep only writing material and Legi on the table.**
- Keep mobile phones, tablets, smartwatches, etc. **turned off** in your bag.
- Fill in this cover sheet first.
- **Turn the cover sheet only when instructed to do so.**
- **Then write your name and Legi Nr. on each page.**
- **Read the rules on the next page carefully.**
- Do not write with red/green color or with pencil.
- **Make sure to hand in every sheet.**
- **Duration: 30 min.**
- Additional material: none.

Good luck!

Midterm exam

Num. CSE

D-INFK/D-MATH

HS 2016

Prof. R. Hiptmair

Rules:

- Motivation for the answers is **not** necessary. Remarks and computations have **no** influence on the total number of points.
- Wrong answers (for multiple choice problems) give negative points. The minimum number of points for each problem is 0.
- All notes outside the predefined boxes will not be considered.
- Each multiple choice box has one and only one correct answer.
- If required, write your solution in the predefined box:

  your text here

- Any unclear marking will be considered an error.

1. *Structured matrices* [4 P.]

   How many *real numbers* are required (at least) to fully determine a general:

   (a) symmetric tridiagonal matrix $\mathbf{T} \in \mathbb{R}^{n,n}$, $n \in \mathbb{N}$:

   ○ $3n-2$ ○ $2n$ ○ $2n-1$

   (b) matrix $\mathbf{M} \in \mathbb{R}^{m,n}$, $m,n \in \mathbb{N}$: $(M)_{ij} = 0$ if $i \neq j \wedge i > 1 \wedge j > 1$:

   ○ $\min\{2m+n-2,\, 2n+m-2\}$ ○ $m+n-1+\min\{m,n\}$
   ○ $2m+2n-\min\{m,n\}$ ○ $\min\{m,n\}+m+n$

Scratch space (not evaluated):

2. *Cancellation* [4 P.]

   Which of the following C++ expressions are affected by cancellation for the given ranges of the **`double`** variables `x` and `y`?

   (a) `z = x - std::sqrt(x*x - 1)`, given $x \in]1,2]$:

   ○ affected by cancellation ○ *not* affected by cancellation

   (b) `z = (std::sin(x) + y) / (x - y)`, given $-1 \leq x, y \leq 1$, $x \neq y$:

   ○ affected by cancellation ○ *not* affected by cancellation

   (c) `z = std::sqrt(1+1/x) - std::sqrt(1-1/x)`, given $x > 1$:

   ○ affected by cancellation ○ *not* affected by cancellation

   (d) `z = 1 / (std::log(x-1) - std::log(x+1))`, given $x \in]1,2[$:

   ○ affected by cancellation ○ *not* affected by cancellation

Scratch space (not evaluated):

3. *Singular value decomposition* [6 P.]

Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ be the *economical* singular value decomposition (SVD) of the matrix $\mathbf{A} \in \mathbb{R}^{m,n},\ m, n \in \mathbb{N},\ m \geq n$.

(a) Determine the dimensions of the matrices $\mathbf{U}$, $\mathbf{\Sigma}$ and $\mathbf{V}$ in the SVD:

- $\mathbf{U} \in \mathbb{R}^{\square,\square}$
- $\mathbf{\Sigma} \in \mathbb{R}^{\square,\square}$
- $\mathbf{V} \in \mathbb{R}^{\square,\square}$

(b) Let $\mathbf{I}$ denote the identity matrix of appropriate size. Decide which of the following expressions are true or false:

- $\mathbf{A} = \sum_{j=1}^{n} (\Sigma)_{j,j} (\mathbf{U})_{:,j} (\mathbf{V})_{:,j}^\top$

  ○ true ○ false

- $\mathbf{U}\mathbf{U}^\top = \mathbf{I}$

  ○ true ○ false

- $\mathbf{V}\mathbf{V}^\top = \mathbf{I}$

  ○ true ○ false

Scratch space (not evaluated):

4. *Compressed Row Storage (CRS)* [6 P.]

   Consider the following matrix:

$$\mathbf{A} := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 0 & 6 \\ 1 & 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 1 & 8 \\ 1 & 1 & 1 & 1 & 9 \end{bmatrix}$$

   The matrix is stored in CRS format as three vectors (see the hint on the next page):

```
1 std::vector<double> val;
2 std::vector<std::size_t> col_ind;
3 std::vector<std::size_t> row_ptr;
```

   Only nonzero entries are stored.

   (a) What is the length of these arrays (as true C++ expressions)?

```
val.size() ==                ;
col_ind.size() ==            ;
row_ptr.size() ==            ;
```

   (b) What is the content of these arrays (as comma-separated initializer lists, assuming C++ indexing)?

```
std::vector<double> val =
{
};
std::vector<std::size_t> col_ind =
{
};
std::vector<std::size_t> row_ptr =
{
```

```
};
```

HINT: The following relationship "defines" the CRS format (assuming C++ indices):

$$\texttt{val[k]} = a_{i,j} \Leftrightarrow \begin{cases} \texttt{col\_ind[k]} = j - 1 \\ \texttt{row\_ptr[i]} \leq k < \texttt{row\_ptr[i + 1]} \end{cases} \qquad 0 \leq k < \operatorname{nnz}(\mathbf{A}),$$

$$\texttt{row\_ptr[n]} = \operatorname{nnz}(\mathbf{A}).$$

Scratch space (not evaluated):

5. *Linear least squares problem* [6 P.]

We seek to determine the positions $x_i \in \mathbb{R}$, $x_1 < x_2 < x_3 < \cdots < x_n$, of $n > 2$ points on a line, given the measured distances $d_{ij} = x_i - x_j$, $1 \le i < j \le n$. This leads to an overdetermined linear system of equations:

$$\mathbf{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{=:\mathbf{x}} = \begin{bmatrix} d_{12} \\ d_{13} \\ \vdots \\ d_{1n} \\ d_{23} \\ \vdots \\ d_{2n} \\ d_{34} \\ \vdots \\ d_{n-1,n} \end{bmatrix} =: \mathbf{b}$$

with a suitable matrix $\mathbf{A} \in \mathbb{R}^{\frac{1}{2}n(n-1),n}$.

(a) What is the rank r of $\mathbf{A}$?

$r := \operatorname{rank}(\mathbf{A}) =$ ☐

(b) Let $\mathbf{M} \in \mathbb{R}^{k,k}$ be the coefficient matrix of the normal equations for $\mathbf{Ax} = \mathbf{b}$. What is k?

$k =$ ☐

(c) What is the rank of $\mathbf{M}$ as an expression of the rank r of $\mathbf{A}$?

$\operatorname{rank}(\mathbf{M}) =$ ☐

(d) The entries of $\mathbf{M}$ are given by:

$(M)_{ij} =$ ☐ , $1 \le i, j \le k$

Scratch space (not evaluated):