Analysis II
Summary

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Chapter 1

Ordinary differential equations

1.1 Differential Equation:

An equation for a function f that relates the values of f at x, f(x) to the values of its derivatives at the same point x. We distinguish between the number of variables present in the function:

- **One variable:** Ordinary differential equations (ODE)
- **Several Variables:** Partial differential equations (PDE)

Examples:

- $f'(x) = f(x)$
- $f''(x) = -f(x)$

Notation: We write $y, y', y'', y^{(3)}, ...$ instead of $f(x), f'(x), f''(x), f^{(3)}(x)$

Order: The largest derivative present in the equation. Examples:

- $y' = 2xy$ order 1
- $y^{(3)} + 2xy'' + e^x y + 1 = 0$ order 3

The solution to an ODE is not unique in general. When given initial conditions then we can find unique solutions. E.g:

$$y' = x + 1$$
$$y = \frac{x^2}{2} + x + c$$

is a solution for any c. If we are also given y(0) = 1 then c= 1 is a unique solution.

1.2 Linear Differential equations

A linear ODE of order k on an interval $I \subset \mathbb{R}$ is an eqn of the form:

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where a(x) and b(x) are continuous functions from I to $\mathbb{C}$.
For a linear ODE the following hold:

- y and all its derivatives appear in order 1
- there are no products of the function y and its derivatives
- neither the function nor its derivatives are inside another function e.g $\sqrt{y}$, sin(y),...

If b = 0 then we say the equation is **homogeneous** otherwise **inhomogeneous**

Solving a linear ODE means finding all functions $f : I \to \mathbb{C}$ that are k times differentiable such that $\forall x \in I$ the function satisfies the differentiable equation.

Initial Condition A set of equations specifying the values of the derivatives at some initial point.

Theorem 2.2.3 Let $I \subset \mathbb{R}$ and open interval $k \geq 1$ and integer. Consider the linear ODE

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where coefs $a_i(x), b(x)$ are continous functions

1. Let S_0 be the set of solutions for b=0, then S_0 is a vector space of dimension k.
2. For any initial conditions, i.e for any choice of $x_0 \in I$ and $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$ there is a unique solution $f \in S$ such that $f_(x_0) = y_0, ...f^{(k)}(x_0) = y_k$
3. For an arbitrary b the set of solutions of the linear ODE is $S_b = \{f + f_p | f \in S_0\}$ where f_p is one **particular** solution. S_b is not a vector space.
4. For any initial condition there is a unique solution.

The linearity of the diff equation alsos implies a **superposition** principle. Suppose we have 2 different functions $b_1(x), b_2(x)$ on the RHS with solutions $f_1, f_2 : Df_1 = b_1, Df_2 = b_2$ then $f_1 + f_2$ solves $Df = b_1 + b_2$

Given a diff eqn and a possible solution we can always verify whether it is indeed a solution or not.

1.3 Linear differential equations of order 1

We consider y'+ay = b, where a,b are continous functions. 2 steps:

- Find solutions of the corresponding homogeneous equation y' + ay = 0.
- Find a particular solution $f_p : I \to \mathbb{C}$ such that $f_p + af_p = b$

If f is a solution then so is zf for any constant $z \in \mathbb{C}$

Homogeneous solution: $y' + ay = 0$
$\Rightarrow y' = -ay$
$\Rightarrow \frac{y'}{y} = a$
$\Rightarrow \int \frac{y'(x)}{y(x)} dx = -\int a(x)dx := A(x)$
$\Rightarrow ln|y(x)| = -A(x) + c$
$\Rightarrow y = z \cdot e^{-A(x)}$ for some constant z

Solution of inhomogeneous equation $y' + ay = b$
There are two methods to solve this:

- Educated guess: the LHS tries to imitate the RHS i.e if b(x) is a polynomial we guess that f_p is also a polynomial or if b is a trig function then we guess f_p is also a trig function
- Variation of constants: Assume
$$f_p = z(x)e^{-A(x)}$$
for some function $z : I \to \mathbb{C}$. We then put this into the equation and see what it forces z(x) to satisfy

The same particular solution can also be obtained by the method of **Integration factor** (IF). Given a ODE of the following form:

$$\frac{dy}{dx} + a(x)y = b(x)$$

one multiplies both sides of the equation by an IF of:

$$e^{\int a(x)dx}$$

$\Rightarrow$

$$\frac{dy}{dx}e^{\int a(x)dx} + a(x)ye^{a(x)dx} = b(x)e'\int a(x)dx$$

The left hand side simplifies to:

$$\frac{d}{dx}(ye^{\int a(x)dx}) := z(x)$$
$$\Rightarrow y = z(x)e^{-A(x)}$$

$\Rightarrow$

$$z'(x) = b(x)e^{\int a(x)dx} = b(x)e^{A(x)}$$

Example:

$$x\frac{dy}{dx} - 2y = x^2$$

Assume $x \neq 0$. We now put the equation in the above form.

$$\frac{dy}{dx} - \frac{2}{x}y = x$$

- $a(x) = \frac{-2}{x}$
- $b(x) = x$
- $A(x) = -2\int \frac{1}{x}dx = -2ln(x) = ln(x)^{-2}$
- $e^{A(x)} = e^{lnx^{-2}} = \frac{1}{x^2}$

$z'(x) = b(x)e^{A(x)} = x \cdot \frac{1}{x^2} = \frac{1}{x}$
$\Rightarrow z(x) = ln(x)$

- $y_h = ze^{-A(x)} = zx^2$
- $y_p = z(x)e^{-A(x)} = ln(x)x^2$

$\Rightarrow y = y_p + y_h = x^2ln(x) + zx^2$

1.4 Linear differential equations with constant coefficients

For a linear ODE with constant coefficients

$$y^{(k)} + a_{k-1}y^{k-1} + ... + a_0 y = 0$$

The Polynomial

$$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + ... + a_0$$

is called the **companion/charateristic polynomial** of the equation. The zeroes of $P(\lambda)$ are called the **eigenvalues**
Example:
$y'' - y = 0$
$\Rightarrow P(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$
$\Rightarrow$ 2 solutions: e^{-x}, e^x
Any solution of the equation are of the form

$$y(x) = z_1 e^{-x} + z_2 e^x$$

Theorem let $\lambda_1, ...\lambda_r$ be pairwise distince eigenvalues of $P(\lambda)$, characteristic polynomial of

$$(*) \quad y^k + a_{k-1}y^{k-1} + ... + a_0 y = 0$$

with corresponding multiplicities $m_1, ..., m_r$ Then the functions

$$f_{j,l} : \mathbb{R} \to \mathbb{C} \quad x \mapsto x^l e^{\lambda_j x}$$

for $1 \leq j \leq r, 0 \leq l < m_j$
form a system of solutions of the homogeneous D.E (*).

Example $y'' - 2y' + 1 = 0$
$\Rightarrow P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$
$\Rightarrow \lambda = 1$ has multiplicity of 2
$\Rightarrow$ the solutions are e^x, xe^x