Algorithms and Probability
Summary

April 6, 2020

Chapter 1

Graphentheorie

1.1 Basics and Definitions

Graph: A graph is a tuple (V,E), where V is a finite non empty set of vertices and E is a set of vertice pairs indicating the edges $V \subseteq E$
$(E \subseteq \binom{V}{2} := \{(x,y)|x,y \in V, x \neq y\})$

Complete (Vollständig): There is an edge between each pair of vertices (den. K_n)

Walk (Weg): A sequence of vertices $\langle v_1, v_2, \ldots, v_n\rangle$ if ∀i there exists and edge from v_i to v_{i+1}. The length of the walk is given by the number of steps, i.e n-1.

Path (Pfad): A walk which doesnt contain any vertice more than once (den. P_n)

Closed Walk (Zyklus): A walk in which $v_1 = v_n$ (den. C_n)

Cycle (Kreis): A closed walk with length of atleast three and the vertices $v_1, \ldots, v_{k-1}$ are pairwise distinct (in a directed graph it must have length of atleast two)

Loops (Schlingen): An edge from a vertice to itself

Multiple edges (Mehrfachkanten): When vertice pairs are connected by multiple edges

Multigraph (Multigraph): A Graph which contains loops and multiple edges (In this lecture we assume that a graph is not a multigraph unless stated otherwise)

Neighbourhood (Nachbarschaft): All outgoing and incoming edges to/from a vertice v denoted $N_G(v) := \{u \in V|\{v,u\} \in E\}$

Degree (Grad): Indicates the size of the neighbourhood $deg_G(v) := |N_G(v)|$

k-regular (k-regulär): If every vertice $v \in V$ has degree $deg(v) = k$
∘ **A complete graph K_n is n-1-regular**

Adjacent (Adjazent): Two vertices u and v if there is an edge u,v

Satz 1.2 For any Graph G=(V,E) we have $\sum_{v \in V} deg(v) = 2|E|$

Korollar 1.3 For any Graph G = (V,E) the number of vertices with uneven degree is even. (Direct Proof by splitting V into even and odd degree sets then use S1.2)

Subgraph (Teilgraph): A Graph H = (V_H, E_H) is a subgraph of a graph G =(V_G, E_G) if $V_H \subseteq V_G$ and $E_H \subseteq E_G$ denoted $H \subseteq G$

Induced Subgraph (Induzierte Teilgraph): If $E_H = E_G \cap \binom{V_H}{2}$ denoted $H = G[V_H]$. If there is an edge (u,v) in G and u,v are also vertices in H then there must be an edge (u,v) in H aswell

1.1.1 Connectivity and Trees

Connected (Zusammenhängend): if for any Vertices s,t ∈V there is a s-t path. A subgraph C ⊆G for which this trait is maximal is called a connected component (hence for all subgraphs $H \neq C$ with $C \subseteq H \subseteq G$ is not connected

Cycle Free (Kreisfrei): A Graph which doesn't contain a cycle

Tree (Baum): A Graph which is Cycle free and connected

Leaf (Blatt): T=(V,E) a tree and v∈V a vertice with deg(v) =1

Lemma 1.5: T = (V,E) a tree with $|V| \geq 2$, it follows:

a): T contains atleast 2 leafs. (Proof: If there was only one leaf
$2|E| = \sum_{v \in V} deg(v) \geq 1 + 2(|V| - 1)$ which is a contradiction to S1.6)

b): if v∈V is a leaf, the graph T-v is also a tree.

Satz 1.6 G =(V,E) a Graph with $|V| \geq 1$ vertices, the following is equivalent:

- G is a tree
- G is connected and cycle free
- G is connected and $|E| = |V| - 1$
- G is cycle free and $|E| = |V| - 1$
- for any x,y∈ V, G contains exactly one x-y path.

Forrest (Wald): W = (V,E) a graph which is cycle free, every component of a forrest is a tree

Lemma 1.7 A forrest G = (V,E) contains $|V| - |E|$ connected components (Proof by induction)

Directed Graph (Gerichteter Graph): A graph where the edges are represented by ordered pairs, i.e The directed graph D is given by the tuple (V,A) where V is the set of vertices and $A \subseteq VxV$ a set of directed edges. Compared to an undirected graph, between two vertices there can be two edges (x,y) and (y,x)

Out-Degree(Aus-Grad): $deg^+(v) := |\{(x, y) \in A | x = v\}|$

In-Degree(In-Grad): $deg^-(v) := |\{(x, y) \in A | y = v\}|$

Satz 1.8 For any directed graph D = (V,A) the following is true.
$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |A|$.

Acyclic (Azyklisch): A directed graph which deosn't contain a cycle (DAG). DAG's have a topological ordering.

Satz 1.9 For any DAG D=(V,A) we can find a topological ordering in $\mathcal{O}(|V| + |A|)$

Strongly Connected (start zusammenhängend): For a DAG D=(V,A), if for every pair of vertices u,v∈V a directed u-v-Path exists

Weakly Connected (schwach zusammenhängend): When the underlying graph(i.e ignoring the direction of the edges) is connected

1.1.2 Datastructures

The two main ways of storing graphs, is with Adjacency matrices and Adjacency lists.

1.2 Trees

1.3 Paths

Shortest Paths Given: A connected Graph G = (V,E), two vertices s,t∈V and a costfunction $c : E \to \mathbb{R}$
Goal: Find an s-t-path P in G with $\sum_{e \in P} c(e) = min$
Algorithms which can solve shortest path problems:

1. Dijkstras Algorithm
2. Floyd-Warshall
3. Bellman-Ford
4. Johnson's Algorithm

1.4 Connection

Definition 1.23 A Graph G = (V,E) is **k-connected (k-zusammenhängend)** if $|V| \geq k+1$ and for all subsets$X \subseteq V$ with $|X| < k$ the following is true: The Graph $G[V \backslash X]$ is connected. (Hence you would need to remove atleast k vertices to destroy the connectivity of the graph, the only exception is a complete graph which is by definition k-1 connected)

Definition 1.24 A Graph G = (V,E) is **k-edge-connected (k-kanten-zusammenhängend)**, if for all subsets $X \subseteq E$ with $|X| < k$ the following is true: $(V, E \backslash X)$ is connected. (Hence atleast k edges must be removed to detroy the connectivity of the Graph)

Satz 1.25 Menger G = (V,E) the following applies:

1. G is k-connected iff for all pairs of vertices u,v∈V, u≠v, atleast k internal-vertice disjoint u-v paths exist
2. G is k-edge-connected iff for all pairs of vertices u,v∈V, u≠v, atleast k edge-disjoint u-v-paths exist

1.4.1 Articulation vertice (Artikulationsknoten)

Articulation vertice (Artikulationsknoten): If a graph is connected but not 2-connected, then there exists a vertice v with the attribute that $G[V \backslash \{v\}]$ is not connected. Articulation vertices kann be detected using a modified DFS.

Forward Edge (Vorwärtskante): An edge starting from a vertice with a lower dfs number than the destination vertice

Backwards Edge(Rückwärtskante): An edge starting from a vertice with a higher dfs number than the destination vertice

we assign ∈V a number low[v]:= the smallest dfs-Number, that can be reached from the vertice v using any number of forward edges and atmost one backward edge. It follows for all v∈V: $low[v] \leq dfs[v]$

v is an Articulation vertice ⇔ v=s and s has degree of atleast 2 or v≠s and there exists a w ∈ V with {v,w} ∈ E(T) and low[w] ≥ dfs[v]

TODO: Implement DFS-Visit which finds the articulation vertices for a given graph

Satz 1.27 For a connected graph G = (V,E), implemented with an adjacency list Articulation vertices can be found in $\mathcal{O}(|E|)$

1.4.2 Bridges (Brücken)

Bridge (Brücke): An edge e∈E such that $(V, E \backslash \{e\})$ is not connected

From the definition of the bridge it follows that a spanning tree must contains all bridges of a graph and that the vertices at the end of the bridge are either Articulation vertices or vertices with degree 1.

An edge (v,w) of the depth-first-search tree is a bridge iff low[w] ¿ dfs[v]

Satz 1.28 For a connected graph G=(V,E) implemented with an adjacency list, articulation vertices and bridges can be found in $\mathcal{O}(|E|)$

1.5 Cycles

1.5.1 Eulerwalk (Eulertour)

Definition 1.29 A Eulerwalk in a graph G=(V,E) is a cycle which contains each edge exactly once. If G contains a Eulerwalk then deg(v) of all v∈V is even. (Proof by contradiction assume G has vertices of even degrees pick a starting node v and an arbitrary node u and show that the path cannot end in u arguing with the parity of the degree)

In a connected graph eulerian graph a Eulerwalk can be found in $\mathcal{O}(|E|)$

TODO: Implement an Algorithm which can find a Eulerwalk in $\mathcal{O}(|E|)$

1.5.2 Hamilton Cycles (Hamiltonkreise)

Defintion 1.31 A Hamilton Cycle in a graph G = (V,E) is a cycle in which all vertices of V are visited exactly once. If a graph contains a Hamilton Cycle it is called hamiltonian (hamiltonisch). Wether or not a graph contains a hamilton cycle is NP-complete.

Satz 1.33 The Algorithm HAMILTONKREIS to find a Hamilton cycle of a given graph G needs $\mathcal{O}(n * 2^n)$ memory and has a runtime of $\mathcal{O}(n^2 * 2^n)$, where $n = |V|$

TODO: Implement the Algorithm HAMILTONKREIS

1.5.3 Special Cases

Lattice (Gittergraph) An m x n lattice is hamiltonian if m or n is even (Proof using parity argument)

Lemma 1.35 $G = (A \uplus B, E)$ a bipartite Graph with $|A| \neq |B|$, then G cannot contain a Hamilton cycle

Hypercube (Hyperwürfel): The set of edges of a Hypercube H_d is $\{0,1\}^d$ hence the set of all 0-1 sequences of length d. Two vertices are connecten if their sequences differ at exactly one spot. A d-dimensional Hypercube contains a Hamilton cycle for all d≥2 (Proof by induction)

Satz 1.37(Dirac) If G = (V,E) is a graph with $|V| \geq 3$ vertices and each vertice has atleast $|V|/2$ neighbours,then G is hamiltonian.

1.5.4 The Travelling Salesman Problem

Given: A complete Graph K_n and a function $l : \binom{[n]}{2} \rightarrow \mathbb{N}$ which gives each edge a length

Goal: Find a Hamilton cycle C in K_n with $\sum_{e \in C'} l(e) = min\{\sum_{e \in C'} l(e)|$C' is a Hamilton cycle in $K_n\}$

For a graph G = (V,E) with V = [n] vertices we define a weight function l as $l(\{u,v\}) = \begin{cases} 0, \text{ if}\{u,v\} \in E \\ 1, \text{else} \end{cases}$
hence the length of a minimal Hamilton cycle in K_n when using l is 0 iff G contains a Hamiltoncycle, hence this allows us evaluate the efficiency of a solution. We define the optimum solution as $opt(K_n, l) := min\{\sum_{e \in C'} l(e)|$C' is a Hamilton cycle in $K_n\}$. An algorithm which always finds a Hamilton cycle with $\sum_{e \in C} l(e) \leq \alpha * opt(K_n, l)$ is known as an α-Approximation algorithm.

Satz 1.39 If there exists an α-Approximationalgorithm for an $\alpha > 1$ for the Traveling Salesman Problem with a runtime of $\mathcal{O}(f(n))$, then an algorithm exits for all graphs with n vertices which decides wether it is hamiltonian or not in the same time complexity

Metric Travelling Salesman Problem (Metrisches TSP)

Given: A complete Graph K_n and a function $l : \binom{[n]}{2} \rightarrow \mathbb{N}$ with the condition $l(\{x,z\}) \leq l(\{x,y\}) + l(\{y,z\})$ for all x,y,z $\in$ [n]

Goal: Find a Hamilton cycle C in K_n with $\sum_{e \in C'} l(e) = min\{\sum_{e \in C'} l(e)|$C' is a Hamilton cycle in $K_n\}$ The condition is called the triangle inequality (Dreiecksungleichung) and states that the direct connection between two vertices x and z cannot be longer than the detour over the vertice y.

Satz 1.40 The Metric Travelling Salesman Problem has a 2-Approximationsalgorithm with a runtime of $\mathcal{O}(n^2)$

1.6 Matchings

Matching A set of edges $M \subseteq E$ is called a Matching of a graph G = (V,E), if no vertice of the graph is incident to more than one edge from M i.e $e \cap f = \emptyset$ for all e,f∈ M with $e \neq f$. A vertice v is "covered" (überdeckt) if there is an edge $e \in M$ which contains v.

Perfect Matching If each vertice is covered by exactly one edge of the Matching M i.e $|M| = |V|/2$. Not all graphs contain a perfect matching for example a star graph.

Maximal matching (Inklusionsmaximal): G = (V,E), for a Matching M if $M \cup \{e\}$ is not a Matching for all edges $e \in E \backslash M$.

Maximum matching (Kardinalitätsmaximal): G = (V,E) for a Matching M if $|M| \geq |M'|$ for all Matchings M' in G

Example: A path consisting of three edges. Creating a Matching using the middle edge would creat a Maximal matching but not a Maximum matching. A Maximum and a Maximal Matching can be created by taking the two outer edges

1.6.1 Algorithms

GREEDY MATCHING This algorithm picks random edges from E and adds it to the matching at the same time it deletes all incident edges from E. The algorithm stops when $E = \emptyset$. This algorithm can find a Maximal matching in time $\mathcal{O}(|E|)$ for which the following applies: $|M_{Greedy}| \geq \frac{1}{2}|M_{max}|$ where M_{max} is a Maximum matching

Union of two Matchings: Let M_1 and M_2 be arbitrary Matchings. $G_M = (V; M_1 \cup M_2)$. Every vertice in G_M has degree of atmost 2, hence all components of the graph are paths and/or cycles (cycles having even length). If we assume $|M_1| < |M_2|$, then every cycle/path of even length will have the same amount of edges from M_1 as M_2. From our assumption we know that there must be a path P which contains more edges from M_2 than M_1 the two outter edges belonging to M_2. Hence we can create a new Matching M_1' using P which will contain one more edge. We achieve this by switching the edges in P i.e $M_1' := (M_1 \cup (P \cap M_2)) \backslash (P \cap M_1)$. We say P is a M_1-augmented path.

M-augmented Path (augmentierender Pfad): Let M be an arbitrary Matching, an M augmented path is a path where the last two edges of P are not covered by M and P consists of edges alternating between edges belonging to M and not belonging to M

AUGMENTED MATCHING This algorithm finds a Maximum matching. We start by finding a Matching which consists of only one arbitrary edge. As long as the matching is not a maximum matching we repeat the following: We take an augmented path and we increase the size of the Matching. We know that after $|V|/2 - 1$ times the matching will be maximum because a matching can't have more than $|V|/2$ edges. We can easily find augmented paths in a bipartite Graph using a modified BFS. The total runtime of our algorithm is $\mathcal{O}(|V| \cdot |E|)$

Satz 1.45 If n is even and $l : \binom{[n]}{2} \to \mathbb{N}$ a weight function of the complete graph K_n then we can find a minimal perfect Matching (i.e a matching where the sum of edge weights are minimal) in $\mathcal{O}(n^3)$

Satz 1.46 From S1.45 if follows that for the Metric Travelling Salesman Problem there is a 3/2-Approximationsalgorithm with a runtime of $\mathcal{O}(n^3)$

1.6.2 Der Satz von Hall

Bipartite A graph G=(V,E) is bipartite if we can partition the set of vertices V into two sets A and B such that all edges in E contain a vertice from A and a vertice from B (denoted: $G = (A \uplus B, E)$)

Satz von Hall/Heiratssatz: For a bipartite graph $G = (A \uplus B, E)$ there is a Matching M of cardinality $|M| = |A|$ iff $|N(X)| \geq |X|$ for all X$\subseteq$ A. From the satz von Hall it follows that a k-regular graph always has a perfect matching.

Satz 1.48 Let $G = (A \uplus B, E)$ be a k-regular bipartite graph. There exists an $M_1, \ldots, M_k$ such that $E = M_1 \uplus \cdots \uplus M_k$ and all $M_i, 1 \leq i \leq k$ are perfect matchings in G. The perfect matching can be found in O(—E—).

Satz 1.49 Let G=(V,E) a 2^k-regular bipartite graph. We can find a perfect matching in $\mathcal{O}(|E|)$

1.7 Colouring (Färbungen)

Vertex colouring (Knoten Färbung): The vertex colouring of a graph G = (V,E) with k colours is a mapping $c : V \to [k]$ such that the following holds: $c(u) \neq c(v)$ for all edges u,v$\in$ E

Chromatic number (chromatische Zahl): dentoted $x(G)$ is the minimal number of colours needed to color the vertices of G. A complete graph has the chromatic number n. Cycles of even length have chromatic number 2, uneven length have a chromatic number 3. Trees with atleast two vertices have a chromatic number 2. Graphs with chromatic number k are also called k-partite. To decide wether or not a graph G is bipartit can be done in $O(|E|)$ with a DFS of BFS.

Satz 1.53 A graph G=(V,E) is bipartite iff it does not contain a cycle of odd length as a subgraph

Satz 1.54 (Vierfarbensatz): Any map can be coloured using 4 colours.

GREEDY FARBUNG This algorithm calculates the colouring of a graph by picking vertices at random and giving it the lowest colour not used by its neighbours. There exists a order of vertices for which the GREEDY algorithm needs $x(G)$ colors.

Satz 1.55 Let G be a connected graph. For the number C(G) of colours needed by GREEDY FARBUNG to color the graph G the following applies: $x(G) \leq C(G) \leq \Delta(G) + 1$ ($\Delta(G) := max_{v \in V} deg(v)$, hence the max degree of a vertice in G). If the graph is saved in an adjacency list then we can find the colouring in $\mathcal{O}(|E|)$

Satz 1.59 (Satz von Brooks) Let G=(V,E) be a connected graph which is not complete nor a cycle with odd degree i.e $G \neq K_n$ and $G \neq C_{2n+1}$ then the following holds $x(G) \leq \Delta(G)$ and there exists and Algorithm which can colour the graph in $\mathcal{O}(|E|)$ with $\Delta(G)$ colours.

Satz 1.60 Let G=(V,E) be a graph and $k \in \mathbb{N}$ a natural number such that every induced subgraph of G has a vertice with degree atmost k. It follows that $x(G) \leq k + 1$ and we can find a (k+1)-colouring in $\mathcal{O}(|E|)$

Satz 1.61 (Mycielski-Konstruktion): For all k $\geq$2 there is a triangle free graph G_k with $x(G_k) \geq k$ (Proof by induction)

Satz 1.62 Every 3-colourable graph G=(V,E) can be coloured in $\mathcal{O}(|E|)$ with $\mathcal{O}(\sqrt{|V|})$ colours. Given a graph G=(V,E), is $x(G) \leq 3$ is NP-Complete.

Chapter 2

Probability Theory and Randomised Algorithms

2.1 Definitions and Notations

Definition 2.1 A **discrete Probabilityspace**(**diskreter Wahrscheinlichkeitsraum**) is defined by a Set of **outcomes** (**Ergebnismenge**) denoted $\Omega = \{\omega_1, \omega_2, \dots\}$ of **elementary outcomes**(**Elementarereignissen**). Each elementary outcome ω_i is assigned a (**elementary**)**probability**(**Elementar − Wahrscheinlichkeit**) denoted $Pr[\omega_i]$ where $0 \leq Pr[\omega_i] \leq 1$ and $\sum_{\omega \in \Omega} Pr[\omega] = 1$. A set $E \subseteq \Omega$ is called an **outcome**(**Ereignis**) The probability of Pr[E] of an outcome is defined by: $Pr[E] := \sum_{\omega \in E} Pr[\omega]$
If E is an outcome, we define $\bar{E} := \Omega \backslash E$ the **compliment outcome**(**Komplementarereignis**)

Finite probabilityspace (endlicher Wahrscheinlichkeitsraum): A Probability space $\Omega = \{\omega_1, \dots, \omega_n\}$ (Assumption for infinite probability spaces $\Omega = \mathbb{N}_0$)

Lemma 2.2 For outcomes A,B the following applies:

1. $Pr[\emptyset] = 0, Pr[\Omega] = 1$
2. $0 \leq Pr[A] \leq 1$
3. $Pr[\hat{A}] = 1 - Pr[A]$
4. if $A \subseteq B, Pr[A] \leq Pr[B]$
5. (Additionssatz) Wenn die Ereignisse $A_1, \dots, A_n$ paarweise disjunkt sind (also wenn für alle Paare $i \neq j$ gilt, dass $A_i \cap A_j = \emptyset$) so folgt:

$$Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} Pr[A_i]$$

Union Bound (Boolesche Ungleichung): Für beliebige Ereignisse $A_1, \dots, A_n$ gilt:

$$Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} Pr[A_i]$$

Siebformel, Prinzip Inklusion/Exklusion: Für Ereignisse $A_1, \dots, A_n (n \geq 2)$ gilt:

$$\begin{aligned} Pr\left[\bigcup_{i=1}^{n} A_i\right] = & \sum_{i=1}^{n} Pr[A_i] - \sum_{1 \leq i_1 \leq i_2 \leq n} Pr[A_{i_1} \cap A_{i_2}] + - \dots \\ & + (-1)^{l+1} \sum_{1 \leq i_1 < \dots < i_l \leq n} Pr[A_{i_1} \cap \dots \cap A_{i_l}] + - \dots \\ & + (-1)^{n+1} \cdot Pr[A_1 \cap \dots \cap A_n] \end{aligned}$$

Laplace Raum: endlicher Wahrscheinlichkeitsraum, in dem alle Elementarereignisse gleich wahrscheinlich sind. In einem Laplace-Raum gilt für jedes Ereignis E:

$$Pr[E] = \frac{|E|}{|\Omega|}$$

 # Möglichkeiten k Elemente aus einer n-elementigen Menge zu ziehen : $(n^{\underline{k}} := n(n-1)(n-2)\dots(n-k-1) = \frac{n!}{(n-k)!})$

Bedingte Wahrscheinlichkeit: A und B seien Ereignisse mit $Pr[B] > 0$. Die bedingte Wahrscheinlichkeit $Pr[A|B]$ (Die W'keit, dass Ereignis A eintrifft, wenn wir schon wissen dass Ereignis B eingetreten ist) von A gegeben B is t definiert durch:

$$Pr[A|B] := \frac{Pr[A \cap B]}{Pr[B]}$$

	geordnet	ungeordnet
mit Zurücklegen	n^k	$\binom{n+k-1}{k}$
ohne Zurücklegen	$n^{\underline{k}}$	$\binom{n}{k}$

(a)

Beispiel: k=2 Elemente aus S={1,2,3} ziehen (n=3)

	geordnet	ungeordnet
mit Zurücklegen	$(1,1),(1,2),(1,3)$ $(2,1),(2,2),(2,3)$ $(3,1),(3,2),(3,3)$	$\{1,1\}, \{1,2\}, \{1,3\}$ $\{2,2\}, \{2,3\}, \{3,3\}$
ohne Zurücklegen	$(1,2), (1,3), (2,1)$ $(2,3), (3,1), (3,2)$	$\{1,2\}, \{1,3\}, \{2,3\}$

(b)

Satz 2.12 Satz der totalen W'keit: Die Ereignisse $A_1, \dots, A_n$ seien paarweise disjunkt und es gelte $B \subseteq A_1 \cup \dots \cup A_n$ dann folgt:

$$Pr[B] = \sum_{i=1}^{n} Pr[A_i \cap B] = \sum_{i=1}^{n} Pr[B|A_i] \cdot Pr[A_i]$$

Satz 2.10 Multiplikationsatz: Seien die Ereignisse $A_1, \dots, A_n$ gegeben. Falls $Pr[A_1 \cap \dots \cap A_n] > 0$ ist, gilt:

$$Pr[A_1 \cap \dots \cap A_n] = Pr[A_1] \cdot Pr[A_2|A_1] \cdot Pr[A_3|A_1 \cap A_2] \dots Pr[A_n|A_1 \cap \dots \cap A_{n-1}]$$

Satz von Bayes: Die Ereigniss $A_1, \dots, A_n$ seien paarweise disjunkt. Ferner $B \subseteq A_1 \cup \dots \cup A_n$ ein Ereignis mit $Pr[B] > 0$ Dann gilt für ein beliebiges $i = 1, \dots, n$:

$$Pr[A_i|B] = \frac{Pr[A_i \cap B]}{Pr[B]} = \frac{Pr[B|A_i] \cdot Pr[A_i]}{\sum_{j=1}^{n} Pr[B|A_j] \cdot Pr[A_j]}$$

Unabhängigkeit Zwei Ereignisse: Die Ereignisse A und B heissen unabhängig, wenn gilt:

$$Pr[A \cap B] = Pr[A] \cdot Pr[B]$$

Unabhängigkeit: Die Ereignisse $A_1, \dots, A_n$ heissen unabhängig, wenn für alle Teilmengen $I \subseteq \{1, \dots, n\}$ mit $I = \{i_1, \dots, i_k\}$ gilt, dass

$$Pr[A_{1\,1}, \cap \dots \cap A_{i\,k}] = Pr[A_{i\,1}] \dots Pr[A_{i\,k}] \qquad (2.2)$$

Eine unendliche Familie von Ereignissen A_i mit $i \in \mathbb{N}$ heisst unabhängig, wenn (2.2) für jede endliche Teilmenge $I \subseteq \mathbb{N}$ erfüllt ist. Dies ist offensichtlich erfüllt, wenn die Ereignisse physikalisch unabhängig sind (e.g wenn jedes A_i einem unabhängigem Münzwurf entspricht) aber ist nicht unbedingt erforderlich.
Die Ereignisse $A_1, \dots, A_n$ sind genau dann unabhängig wenn für alle $(s_1, \dots, s_n) \in \{0,1\}^n$ gilt dass

$$Pr[{A_1}^{s_1} \cap \dots \cap {A_n}^{s_n}] = Pr[{A_1}^{s_1}] \dots Pr[{A_n}^{s_n}]$$

wobei ${A_i}^0 = \overline{A_i}$ und ${A_i}^1 = A_i$
Seien A, B und C unabhängige Ereignisee. Dann sind auch $A \cap B$ und C bzw. $A \cup B$ und C unabhängig.

2.2 Zufallsvariablen

Eine funktion welche jede element unser Wahrscheinlichkeitsraum ein Reellezahl zuordnet.

$$X : \Omega \to \mathbb{R}$$

Beispiel:
"X $\leq$ 5 steht für das Ereignis, dass die Zufallsvariable einen Wert kleiner gleich 5 annimmt also:
$\Rightarrow X \leq 5 \hat{=} \{\omega \in \Omega : X(\omega) \leq 5\}$

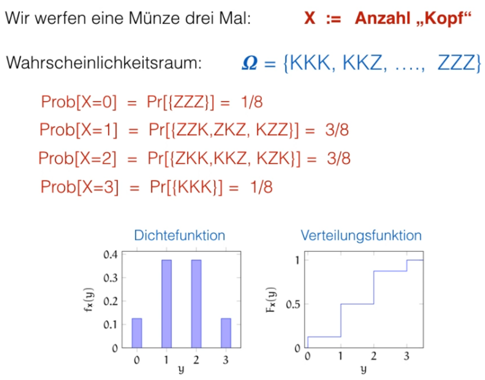
Dichtefunktion:

$$f_X : \mathbb{R} \to [0,1], \quad x \mapsto Pr[X = x]$$

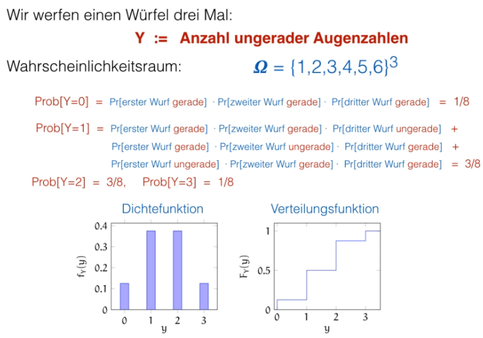
Verteilungsfunktion:

$$F_X : \mathbb{R} \to [0,1], \quad x \mapsto Pr[X \leq x] = \sum_{x' \in W_X : x' \leq x} Pr[X = x']$$

Beispiele:



(a)



(b)

Bernoulli-Verteilung:

$$X \sim \text{Bernoulli(p)}$$

$$f_X(x) = \begin{cases} p & \text{für } x = 1, \\ 1-p & \text{für } x = 0 \\ 0 & \text{sonst} \end{cases}$$

$$\mathbb{E}[X] = p$$

Binomial Verteilung:

$$X \sim \text{Bin(n,p)}$$

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, \ldots, n\} \\ 0, \quad sonst. \end{cases}$$

$\mathbb{E}[X] = np \qquad Var[X] = np(1-p)$ (gleichung für Varianz gilt nur wenn die X_i's unabhängig sind)

$Bin(n, \frac{\lambda}{n})$ konvergiert für $n \to \infty$ gegen Po(λ) Beispiel: Werfen einer Münze n mal, X = Anzahl Kopf

Negative Binomialverteilung:

$$X \sim \text{NegativeBinomial(n)}$$

$$f_X(k) = \begin{cases} \binom{k-1}{n-1} (1-p)^{k-n} p^n, & \text{für } k = 1, 2, \ldots \\ 0, \quad sonst \end{cases}$$

$$\mathbb{E}[X] = \frac{n}{p}$$

Beispiel: Warten auf den n-ten Erfolg

Geometrische Verteilung:

$$X \sim \text{Geo(p)}$$

$$f_X(i) = \begin{cases} p(1-p)^{i-1} & \text{für } i \in \mathbb{N} \\ 0 \quad sonst. \end{cases}$$

$$F_X(n) = 1 - (1-p)^n \quad \text{für alle n=1,2,...}$$

$$\mathbb{E}[X] = \frac{1}{p} \qquad Var[x] = \frac{1-p}{p^2}$$

Beispiel: Wiederholtes werfen einer Münze, X = # Würfe bis zum ersten Mal Kopf

Gedächtnislosigkeit: Ist X $\sim$ Geo(p), so gilt fpr alle s,t $\in \mathbb{N}$:

$$Pr[X \geq s + t | X > s] = Pr[X \geq t]$$

Beispiel: Wahrscheinlichkeit im ersten Wurf Kopf zu bekommen ist identisch zur Wahrscheinlichkeit nach 1000 Fehlversuchen im 1001ten Wurf Kopf zu bekommen.

Poisson-Verteilung:

$$X \sim Po(\lambda)$$

$$f_X(i) \begin{cases} \frac{e^{-\lambda}\lambda^i}{i!} & \text{für } i \in \mathbb{N}_0 \\ 0 \quad sonst \end{cases}$$

$$\mathbb{E}[X] = Var[X] = \lambda$$

Beispiel: Modellierung seltener Ereignisse e.g X:= # Herzinfarkte in der Schweiz in der nächsten Stunde

Erwartungswert: Den Zufallsvariablen X definieren wir den Erwartungswert $\mathbb{E}[X]$ durch:

$$\mathbb{E}[X] := \sum_{x \in W_X} x \cdot Pr[X = x]$$

sofern die Summe konvergiert. Ansonsten sagen wir, dass der Erwartungswert undefiniert ist.
Ist X eine Zufallsvariable, so gilt:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot Pr[\omega]$$

Sei X eine Zufallsvariable mit $W_x \subseteq \mathbb{N}_0$. Dann gilt

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} Pr[X \geq i]$$

Beobachtung: Für ein Ereignis $A \subseteq \Omega$ ist die zugehörige Indikatorvariable X_A definiert durch:

$$X_A(\omega) = \begin{cases} 1 & \text{falls } \omega \in A \\ 0 & \text{sonst.} \end{cases}$$

Für den Erwartungswert von X_A gilt: $\mathbb{E}[X_A] = Pr[A]$

- Schnitt: $A \cap B$ $\quad X_{A \cap B} = X_A \cdot X_B$
- Komplement: $\overline{A} := \Omega \backslash A$ $\quad X_{\overline{A}} = 1 - X_A$
- Vereinigung: $A \cup B$ $\quad X_{A \cup B}$

Beispiel:

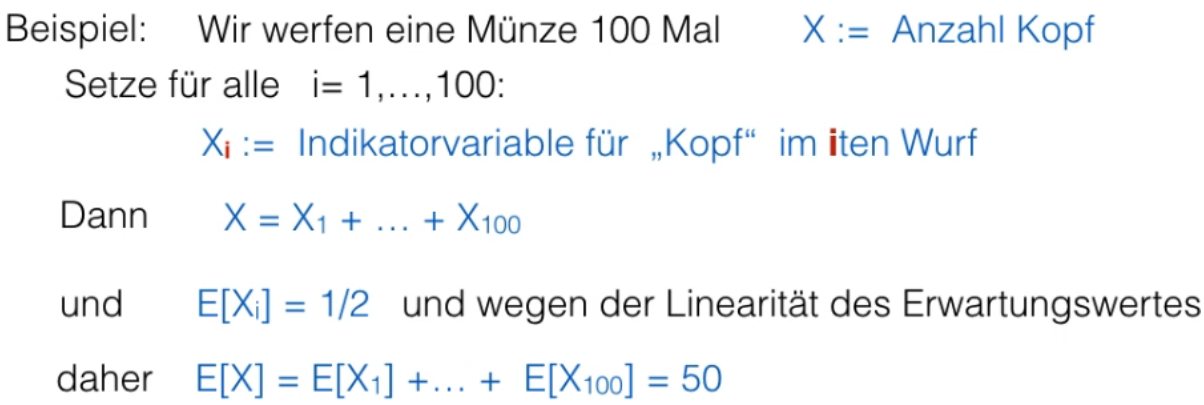


Figure 2.3

Linerarität des Erwartungswerts: Für Zufallsvariablen $X_1, \ldots, X_n$ und $X{=}a_1X_1{+}\cdots{+}a_nX_n{+}b$ mit $a_1, \ldots, a_n, b \in \mathbb{R}$ gilt:

$$\mathbb{E}[X] = a_1\mathbb{E}[X_1] + \cdots + a_n\mathbb{E}[X_n] + b$$

Stabiler Menge: Knoten, die nicht durch Kanten verbunden sind.

Satz Für jeden Graphen G=(V,E) mit $|V| = n$ und $|E| = m$ bestimmt der Algorithmus (gehe durch die Knotenmenge und entferne den Knoten und inzidente Kanten mit wahrscheinlichkeit 1-p, bei den übrig gebliebene kanten lösche ein Knoten) eine stabile Menge S mit

$$\mathbb{E}[S] \geq np - mp^2$$

Beweis:

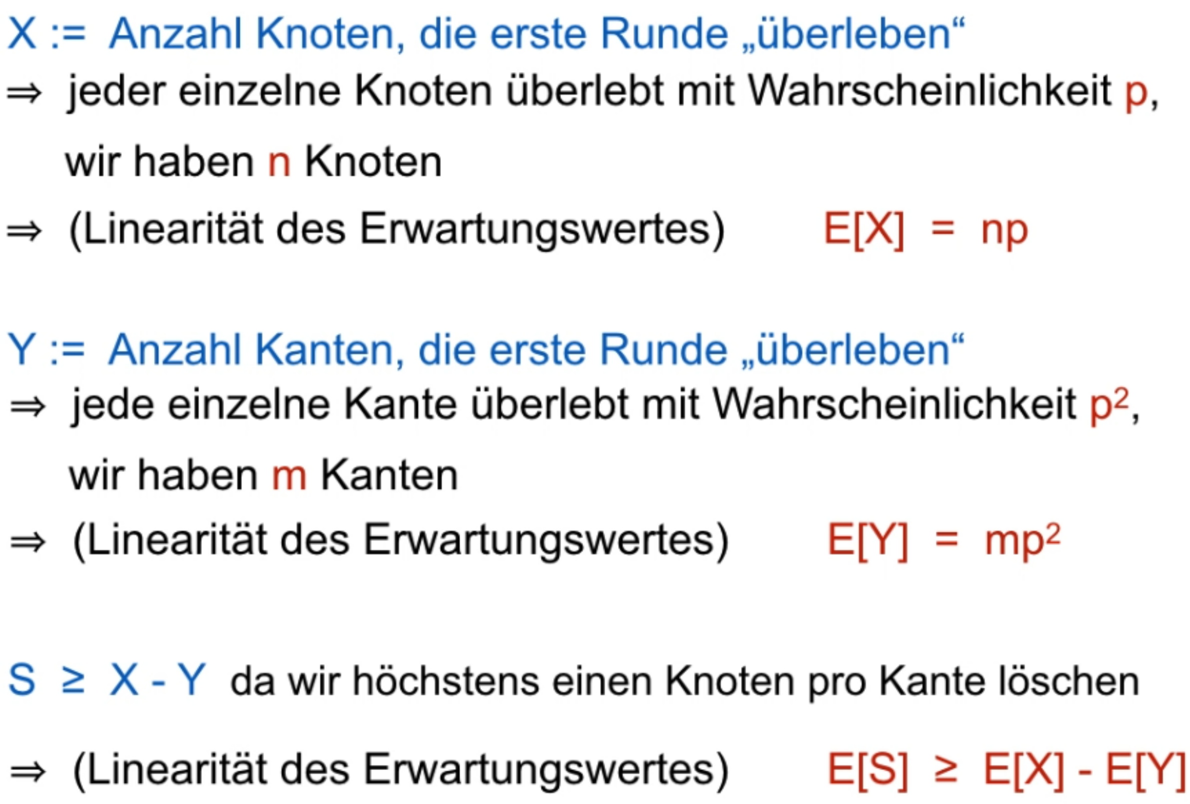
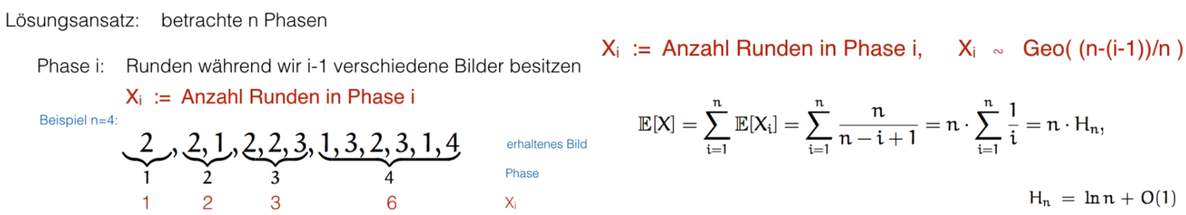


Figure 2.4

Coupon Collector:

Szenario: Es gibt n verschiedene Bilder in jeder Runde erhalten wir (gleichwahrscheinlich) eines der Bilder
X:= Anzahl Runden bis wir alle n Bilder besitzen Ziel: Berechne $\mathbb{E}[X]$



(a) (b)

⇒ die Laufzeit der Coupon Collector ist $\mathcal{O}(nlogn + n)$

Varianz: Für ein Zufallsvariable X mit $\mu = \mathbb{E}[X]$ definieren wir die Varianz $Var[X]$ durch:

$$Var[X] := \mathbb{E}[(X-\mu)^2] = \sum_{x \in W_X} (x-\mu)^2 \cdot Pr[X = x]$$

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$Var[a \cdot X + b] = a^2 \cdot Var[X] \qquad X \text{ beliebig, } a, b \in \mathbb{R}$$

(b verschwindet, da verschieben der werte kein einfluss auf die Abweichung vom Durchschnitt hat)

Standardabweichung:

$$\sigma := \sqrt{Var[X]}$$

Dichten: X,Y Zufallsvariablen:

Gemeinsame Dichte:

$$f_{X,Y}(x,y) := Pr[X = x, Y = y]$$

Randdichte:

$$f_X(x) = \sum_{y \in W_Y} f :_{X,Y} (x,y)$$

Unabhängigkeit: Zufallsvariablen $X_1, \dots, X_n$ heissen unabhängig genau dann, wenn für alle $(x_1, \dots, x_n) \in W_{X_1} \times \dots \times W_{X_n}$ gilt:

$$Pr[X_1 = x_1, \dots, X_n = x_n] = Pr[X_1 = x_1] \cdot \dots \cdot Pr[X_n = x_n]$$

Alternativ:

$$f_{X_1,\dots,X_n}(x_1,...,x_n) = f_{X_1}(x_1) \cdot ... \cdot f_{X_n}(x_n) \text{ für alle } (x_1,...,x_n) \in W_{X_1} \times ... \times W_{X_n}$$

Beispiel:

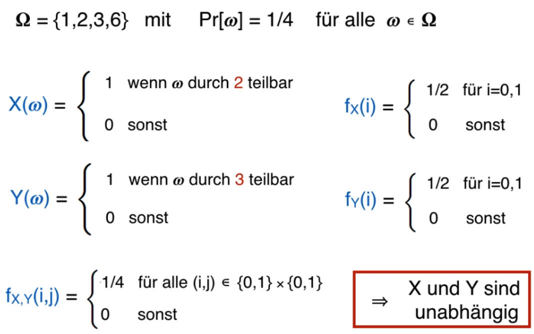


Figure 2.6

Für **zwei** Indikatorvariablen X und Y gilt:

$$\text{X und Y sind unabhängig} \iff f_{X,Y}(1,1) = f_X(1) \cdot f_Y(1)$$

Lemma: Sind $X_1, ..., X_n$ unabhängige Zufallsvariablen und $S_1, ..., S_n$ beliebige Mengen mit $S_i \subseteq W_{x_i}$, dann gilt:

$$Pr[X_1 \in S_1, ..., X_n \in S_n] = Pr[X_1 \in S_1] \dots Pr[X_n \in S_n]$$

Korollar: Sind $X_1, ..., X_n$ unabhängige Zufallsvariablen und ist $I = \{i_1, ..., i_k\} \subseteq [n]$, dann sind $X_{i_1}, ..., X_{i_k}$ ebenfalls unabhängig.
Satz: $f_1, ..., f_n$ seien reellwertige Funktionen ($f_i : \mathbb{R} \to \mathbb{R}$ für i = 1,...,n). Wenn die Zufallsvariablen $X_1, ..., X_n$ unabhängig sind dann gilt dies auch für $f_1(X_1), ..., f_n(X_n)$

Summe von Zufallsvariablen: Für zwei unabhängige Zufallsvariablen X und Y sei Z:= X + Y. Es gilt:

$$f_Z(z) = \sum_{x \in W_X} f_X(x) \cdot f_Y(z-x)$$

Es folgt:

$$Poisson(\lambda_1) + Poisson(\lambda_2) = Poisson(\lambda_1 + \lambda_2) \;\; Bon(n_1, p) + Bin(n_2, p) = Bin(n_1 + n_2, p)$$

falls die lambda's und n's unabhängig sind

Rechenregeln für Momente:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] \qquad \forall X, Y$$

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y] \qquad \forall \text{ X,Y unabhängig}$$

$$\text{Var[X + Y]} = \text{Var[X]} + \text{Var[Y]} \qquad \forall \text{ X,Y unabhängig}$$

$$Var[X \cdot Y] \neq Var[X] \cdot Var[Y] \qquad \text{muss ausgerechnet werden}$$

Multiplikativität des Erwartungswerts: Für unabhängige Zufallsvariablen $X_1, ..., X_n$ gilt:

$$\mathbb{E}[X_1 \cdot ... \cdot X_n] = \mathbb{E}[X_1] \cdot ... \cdot \mathbb{E}[X_n]$$

Satz: Für unabhängige Zufallsvariablen $X_1, ..., X_n$ und $X := X_1 + ... + X_n$ gilt:

$$Var[X] = Var[X_1] + ... + Var[X_n]$$

2.3 Abschätzen von Wahrscheinlichkeiten

Waldsche Identität: N und X seien zwei unabhängige Zufallsvariable, wobei für den Wertebereich von N gelte: $W_N \subseteq \mathbb{N}$
Weiter sei:

$$Z := \sum_{i=1}^{N} X_i$$

wobei $X_1, X_2, ...$ unabhängige Kopien von X seien. Dann gilt:

$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X]$$

Ungleichung von Markov: Sei X eine Zufallsvariable, die nur nicht negative Werte annimmt. Dann gilt für alle $t \in \mathbb{R}$ mit $t > 0$, dass

$$Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t} \qquad \forall X \geq 0, \forall t > 0$$

Ungleichung von Chebyshev: Sei X eine Zufallsvariable und $t \in \mathbb{R}$ mit $t > 0$ Dann gilt:

$$Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{Var[X]}{t^2} \qquad \forall X, \forall t > 0$$

insbesondere

$$Pr[X \geq \mathbb{E}[X] + t] \leq \frac{Var[X]}{t^2}$$