Cold trapped ion for Quantum Computing

Yuan Yujia, 20703331, Phys 467

May 5, 2020

Contents

1 Abstract

While quantum computer reveals significant speedup in parallel calculation, and can theoretically brings unimaginable power to some specific algorithms, the difficulty of implementing it in reality restricts scientists to exploit its great power. Many proposals of implementations of quantum computer are present in journals all over the world these years, from which cold ion trap is a promising one due to long coherent time and high efficiency in producing and measuring states. This paper first explains the design of linear trap which can hold a string of ions as qubits for quantum computing, then shows how the state of bits can be controlled on demand by laser, which constructs universal quantum gate set. Finally it demonstrates the functionality of ion trap quantum computer through some recent results.

2 Introduction

Ion Trap can realize quantum computer by manipulating the internal and motional states of ions stored in a linear trap. It satisfies five criteria necessary for any practicable quantum computer, known as Divincenzo's criteria[1]. They can be characterized as:

1. Scalable array of quantum bits
2. Long coherent time so states can stay long enough
3. Initializing qubits in the state with lowest energy
4. There exists universal quantum gate for modifying states
5. The state of a qubit be easily read out

Experimentalists have suggested many proposals for scalable array of quantum bits. One of which is Linear Paul Trap (LPT) system first designed by Raizen in 1992[2]:

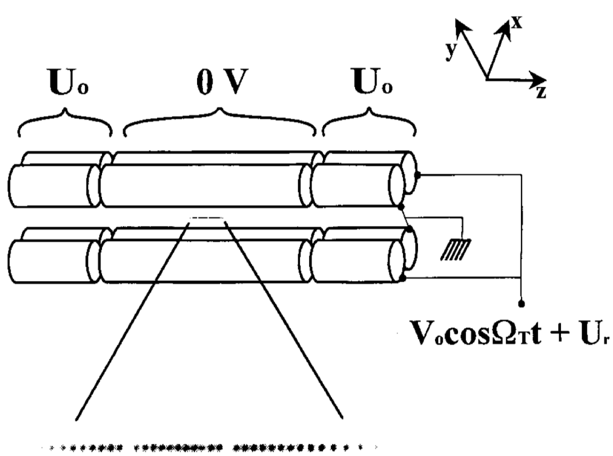

Figure 1: Linear Paul Trap[3]

Ions normally lives in 3-dimensional euclidean space. However, while being trapped in the LPT, they only moves relatively freely to +x or -x directions (it is +z and -z in the above picture). For other two dimensions, they can only perform micro motion which is vibrational perturbation. Thus the LPT can be treated as a one-dimensional system after micro motion is compensated by procedures such as DC voltage control[21]. In the LPT, potential energy can be approximate as quadratic. That is, if we define the symmetrical center of LPT as 0, then $V(x) \sim x^2$. Thus ions in the trap with +1 charge such as $^{171}Yb^+$[3] is usually treated as an N-objects harmonic oscillator confined to one dimension, which is x in this paper. The collective ions behave harmonic motion with N normal modes in the space. Close analogy can be drawn between this and a classical spring-masses system that consists masses connected to each other and to the boundaries of the space by springs. The state of this collective motion is called motional state.

Due to repulsive Coulomb forces originating from their positive charge, individual ions are separated from each other. Each ion has $|0\rangle$ and $|1\rangle$ states. Therefore, it can represent a qubit, which can entangle to other qubits or motional state. Motional state and internal states will be further explained in section 3.

According to Divincenzo's criteria, before "turning on" the quantum computer, initializing qubits to lowest energy states is necessary, which means both motional and internal states should be in their ground states. These can be achieved by cooling techniques and optical pumping respectively. Atoms can be cooled down to the limit of $T = \hbar\gamma/2k_B$ by doppler cooling[4, 5]. Here T is the temperature and γ is broad natural line width whose dimension is rad/s. Sideband Cooling mechanism further suppresses the temperature to lower than 100mK, at which the motional state of ions comprises mainly the ground state component[6]. We call this state $|0\rangle_m$. Then optical pumping[7] is used to prepare ions in $|0\rangle_n$, one of the possible two internal states for individual ion with lower energy. In another word, we can initialize the system to the lowest energy state in the form of $|00000\rangle$ or $|0000\rangle_n \otimes |0\rangle_m$, assuming here there are four ions. The $|0\rangle_n$ will be further explained in the next section, sometimes with different notations for simplicity.

With previous work done by predecessors of quantum computer, that is shown in the last paragraph, cold ion trap quantum computers can be proved achievable if three more conditions are satisfied: long coherent time, universal quantum gate and easy-read-out are possible. This paper will first introduce how LPT can work as a quantum computer by referring to some of its physical properties as well as mathematical deductions. This will be followed by the implementation of quantum gates in LPT, especially CNOT gate because it is not possible for any classical systems. Finally it will show the high accuracy of measuring bit states and long coherent time indicated by experiments done before, which are some of its most intriguing advantages outperforming other implementations of quantum computer.

3 Internal and motional states

3.1 Internal state

Hyperfine structure is the small split and shifts of energy levels due to the interaction between the angular momentum of nucleus and the spin of electron[9]. It can be calculated by[10]

$$\Delta E = \frac{A}{2}[\hat{F}(\hat{F}+1) - \hat{I}(\hat{I}+1) - \hat{J}(\hat{J}+1)]$$

where A is hyperfine structure constant, $F = \pm 1, 0$, J is the intrinsic spin number and I is nuclear spin.

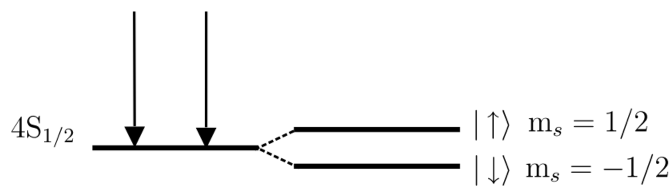

Figure 2: Splitting of S orbital due to hyperfine effect[11]

As shown in Figure 2, due to hyperfine splitting, $4S_{1/2}$ orbit will be split into two energy states which are very close to each other: the magnitude of difference of the split is several orders smaller than that of fine structure, which is usually several orders smaller than that of optical transition. Ions with one outer shell electron such as $^{171}Yb^{+}$ will have different hyperfine energy levels. Both of the two states are ground states since the phrase "ground state" refers to the whole $4S_{1/2}$ orbit, regardless of the hyperfine or fine structure splitting[11]. We call the state with higher energy as $|1\rangle_n$ while that with lower energy as $|0\rangle_n$. Since hyperfine states are about the atomic structure of individual ions, they are also called internal states. More than one factor can cause difference in energy levels: different orbit, fine splitting, Zeeman effect and so on. The reason why hyperfine splitting is more widely preferred is that it is stable compared to other ones, and is not susceptible from external magnetic field which may vary by a great amount[3].

Technically ^{88}Sr+, ^{40}Ca+, ^{138}Ba+ are all candidates for providing with hyperfine states. However, ^{171}Yb+ is selected here because its Rabi frequency of energy level transition[3] is

$$\omega_0 \simeq 12.6 \text{ GHz}.$$

Thus the wavelength of photons required for the ion to do Rabi oscillation is in the microwave range, which easy to manipulate with high accuracy. Applying microwave at Rabi frequency to state $|0\rangle_n$ or $|1\rangle_n$ will precess the states and flip the probability of two states back and forth, which is the effect of quantum gates on the internal states. As a result, 1-qubit gate can be achieved by directing laser beam to single ion. More details can be found [13].

Since each ion has a linear combination of $|0\rangle_n$ and $|1\rangle_n$ states, the ensemble of N ions will have 2^N states in total. They can be expressed as a summation[8]:

$$|\psi\rangle = \sum_{i=0}^{2^N-1} C_i |i\rangle_n = \sum_{x=\{0,1\}^N} C_x |x\rangle_n = C_0 |000...0\rangle + C_1 |000...1\rangle + \cdots$$

The energy term for the state is:

$$\hat{H}_n = \hbar\omega_{eg} \sum_{i=1}^{N} \sigma_i$$

where $\hbar\omega_{eg}$ is energy splitting between two hyperfine levels, and σ_i is $1/2(|e\rangle\langle e| - |g\rangle\langle g|)$ for the ith ion.

3.2 Motional states

Ions trapped in the Linear Paul Trap not only have internal states respectively, they also present in motional states as a collective. In LPT, normal modes of motional states are eigenstates of a one-dimensional harmonic oscillator, with the raising, lowering and level operators are defined as:

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$
$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$
$$\hat{n} |n\rangle = \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$$

The hamiltonian for harmonic oscillator with respect to those operators are:

$$\hat{H}_m = \hbar\omega_v \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega_v \left(\hat{n} + \frac{1}{2}\right)$$

Since the motional states we use for quantum computing are pure normal modes instead of superposition modes, the following formula satisfies:

$$\hat{H}_m |\psi\rangle = E_m |\psi\rangle$$

Because motional states in the trap are shared by all ions, they can be treated as a bus which transport information from ion to ion. Hence, entangling gates are possible to create with interaction of motional states and internal states. Sometimes we define the sum of energy of uncoupled internal states and motional state as $\hat{H}_0$:

$$\hat{H}_0 = \hat{H}_n + \hat{H}_m$$

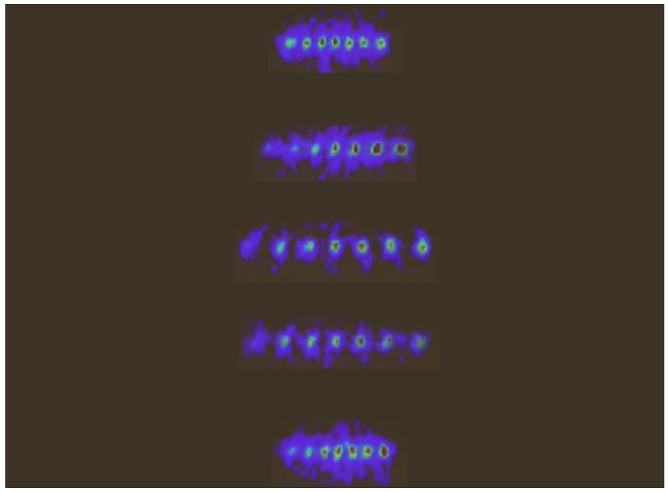

Figure 3: One normal mode of motional state[11]

3.3 Interaction Hamiltonian

Arbitrary one-qubit quantum gates can be achieved by directing laser beams onto individual ions in the linear trap, as illustrated in section 3.1. So CNOT is the last gate required to build a universal set[14]. But CNOT requires interaction between internal and motional states. Therefore, laser used to couple the two bears another term of hamiltonian—interactive hamiltonian[16]:

$$\hat{H}_{\text{int}} = -\hat{d} \cdot \hat{E}$$

where $\hat{d}$ is the dipole operator and $\hat{E}$ is the quantized electric field operator. This term resembles the one we are familiar with, the Zeeman effect hamiltonian under magnetic field[15]:

$$\hat{H}_{\text{int}} = -\hat{\mu} \cdot \hat{B}$$

To expand $\hat{H}_{\text{int}}$[16], we have:

$$\hat{d} = \hat{z} \cdot \langle g|\hat{d}|e\rangle \bigg(|g\rangle\langle e| + |e\rangle\langle g| \bigg) = \hat{z} \cdot d_{ge}(\hat{\sigma}^{\dagger} + \hat{\sigma})$$

where $|g\rangle \equiv |0\rangle_n$, $|e\rangle \equiv |1\rangle_n$ and σ is the lowering operator of the ion energy level. Furthermore, the quantized electric field is assumed to be plane wave which propagates in x direction and is polarized along z direction[3]. It can be expanded as following:

$$\hat{E}(\hat{r}, t) = \hat{z} \cdot E_0 \cos(kx - \omega t + \phi) = \hat{z} \cdot \frac{E_0}{2} \bigg(e^{i(kx-\omega t+\phi)} + e^{-i(kx-\omega t+\phi)} \bigg)$$

where ω is the laser frequency. E_0 is static electric field, k is wavevector which has only one dimension here, and ϕ is laser phase. Rabi frequency for the interactive hamiltonian is defined as $-E_0 d_{ge}/\hbar$[16], thus we have:

$$
\begin{aligned}
\hat{H}_{\text{int}} &= -\frac{E_0 \cdot d_{ge}}{2}\left(e^{i(kx-\omega t+\phi)} + e^{-i(kx-\omega t+\phi)}\right)\left(\hat{\sigma}^\dagger + \hat{\sigma}\right) \\
&= \frac{\hbar\Omega}{2}\left(\left(\hat{\sigma}^\dagger + \hat{\sigma}\right) \cdot e^{i(kx-\omega t+\phi)} + c.c\right)
\end{aligned}
$$

Now the interaction takes place. For harmonic oscillators,

$$
\begin{cases}
x = x_0(a + a^\dagger) \\
x_0 = \sqrt{\frac{\hbar}{2m\omega}}
\end{cases}
$$

Replacing x in $\hat{H}_{int}$ with raising and lowering operators, and substituting kx_0 with η (Lamb-Dicke parameter)[18]:

$$
\hat{H}_{\text{int}}(t) = \frac{\hbar\Omega}{2}\left(\left(\hat{\sigma}^\dagger + \hat{\sigma}\right) \cdot e^{i(\eta(\hat{a}^\dagger + \hat{a}) - \omega t + \phi)} + c.c\right)
$$

For the system to work properly, $\eta(2n+1) \ll 1$[17]. For n is not a large number in our system, we have to ensure $\eta \ll 1$. This limit is important for quantum computing.

Total energy of the system is $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}(t)$. Both internal and motional states evolve (change with time) under the effect of $\hat{H}_0$, while the time dependent hamiltonian also changes with time. So it is more convenient to represent everything in a rotation frame, a frame itself rotates at the rate of states evolving under $\hat{H}_0$, and thus only the effect of $\hat{H}_{\text{int}}$ will be left. By doing this, experimentalists can concentrate on doing Rabi flipping between interactive modes between internal and motional states. To so so, we define the state evolving unitary matrix due to $\hat{H}_0$ to be

$$
\hat{U}(t) = e^{-i\hat{H}_0 t/\hbar}
$$

The hamiltonian under new frame[19]:

$$
\hat{H}_R = \hat{U}^\dagger \hat{H} \hat{U} - i\hbar \hat{U}^\dagger \frac{d}{dt}\hat{U}
$$

and the state under new frame is

$$
|\psi\rangle_R = \hat{U}(t) \cdot |\psi\rangle
$$

Before doing the transformation, we first simplify the total hamiltonian by applying Rotation-Wave-Approximation to interaction hamiltonian[16]. The atomic raising and lowering operator in the rotation frame will become:

$$
\hat{U}(t)^\dagger(\hat{\sigma}^\dagger + \hat{\sigma})\hat{U}(t) = \hat{\sigma}^\dagger \cdot e^{i\omega_0 t} + \hat{\sigma} \cdot e^{-i\omega_0 t}
$$

Because $\omega_0 \simeq \omega$, the summation of those two will be very large, causing the exponential term rotate too fast to have any obvious effect to the system; while the terms containing the difference of those two will be reserved. Thus the hamiltonian in lab frame will be:

$$
\hat{H} = \hat{H}_0 + \hat{H}_{int} = \hbar\omega_{eg}\sum_{i=1}^{N}\sigma_i + \hbar\omega_v\left(\hat{N} + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\hat{\sigma}^\dagger \cdot e^{i\left(\eta(\hat{a}^\dagger + \hat{a}) - \omega t + \phi\right)} + c.c.
$$

We can use the result under rotation frame calculated by Christopher Monroe[3]:

$$\hat{H}_R = \hbar\Omega\hat{\sigma}^\dagger e^{i[\eta(\hat{a}e^{i\omega t}+\hat{a}^\dagger e^{-i\omega t})-\delta t+\phi]} + c.c$$

Here $\delta = \omega - \omega_0$, where ω_0 is the frequency from $\hat{H}_0$

4 Quantum Gates

4.1 Carrier gate

We look at how single ion interact with motional state in this section first, because generalize it to all qubits is easy. If $\delta = 0$, or the laser frequency exactly equals to the internal Rabi frequency, then due to Lamb-Dicky-Limit, there is negligible time dependence. If $\phi = 0$, then it is an internal flip gate, which will cause the internal state to do Rabi oscillation.

$$\hat{H}_R = \hbar\Omega\big(\hat{\sigma}^\dagger + \hat{\sigma}\big)$$

Unitary matrix for state evolution under this hamiltonian is:

$$U_C = e^{-i\Omega(\hat{\sigma}^\dagger+\hat{\sigma})t}$$

If we choose $t = \pi/2\Omega$, we can use formula[14] $e^{i\hat{A}x} = \cos(x)\hat{I} + i\sin(x)\hat{A}$ to simplify the expression:

$$U_C = e^{-i\pi/2(\hat{\sigma}^\dagger+\hat{\sigma})} = e^{-i\pi/2\hat{\sigma}_x} = i\cdot\hat{\sigma}_x$$

So basically it is effectively flipping the internal state of an ion with phase factor i.

4.2 1^{st} Sideband hamiltonian

By choosing[3] $\delta = -\omega$, $\phi = -\pi/2$ the hamiltonian becomes

$$\hat{H}_R = \hbar\Omega\hat{\sigma}^\dagger e^{i[\eta(\hat{a}e^{i\omega t}+\hat{a}^\dagger e^{-i\omega t})]}\cdot e^{-i\pi/2-i\omega t} + c.c$$

Since η is small, we can do Maclaurin expansion to $\text{Exp}\{i[\eta(\hat{a}e^{i\omega t}+\hat{a}^\dagger e^{-i\omega t})]\}$, and only take up to the linear term:

$$\hat{H}_R = -i\hbar\Omega\hat{\sigma}^\dagger\cdot\left(1 + i\eta(\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}) + O(\eta^2)\right)\cdot e^{-i\omega t} + c.c.$$

Laser used here has a high frequency(ω). So we do RWA again, and only reserve the terms that rotate slowly. Thus we have:

$$\hat{H}_R = \hbar\Omega(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma})$$

The physical meaning of this expression is that the internal state of an ion in the trap interacts with motional state, and keeps the total number of photons in the system not change while doing Rabi oscillation. This is called the 1^{st} red sideband hamiltonian.

When $\delta = \omega$ and keep $\phi = -\pi/2$, we can do the same thing as section 4.2 and get the 1^{st} blueband gate:

$$\hat{H}_R = \hbar\Omega(\hat{a}^\dagger\hat{\sigma}^\dagger + \hat{a}\hat{\sigma})$$

4.3 CNOT

CNOT gate is defined as[14]:

$$\widehat{\text{CNOT}}\big(\,|x\rangle \otimes |y\rangle\,\big) = |x\rangle \otimes |y \oplus x\rangle$$

It cannot be achieved classically. Instead, CNOT gate is "purely quantum"[14]. So that is one of the reasons why scientists show great interest to it. With preliminary knowledge from section 3 and 4, CNOT gate is not difficult to achieve. Doing the same as for carrier gate gives the unitary evolution operator for 1^{st} red sideband:

$$\hat{U}_n^k = e^{-i\Omega(\hat{a}^\dagger\hat{\sigma}^\dagger + \hat{a}\hat{\sigma})t}$$

N is the number of the single ion(bit) being operated on. K is the mode of the operation, which decides the time of applying the unitary evolution to the state: $t = (k\pi)/(2\Omega)$. Thus[8]

$$\hat{U}_n^k = \text{Exp}[-ik\frac{\pi}{2}(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma})]$$

For our purpose we treat motional state as a two-state system, which means it can only be 0 or 1 state. The above unitary operator thus will not have any effect on $|g0\rangle$ or $|e1\rangle$ for nth ion. Unitary operations can be further specified, here we abbreviate $|x\rangle_n \otimes |m\rangle$ as $|xm\rangle$:

$$\begin{aligned}
\hat{U}_n^1\,|g1\rangle &= -i\,|e0\rangle \\
\hat{U}_n^1\,|e0\rangle &= -i\,|g1\rangle \\
\hat{U}_n^2\,|g1\rangle &= -\,|g1\rangle \\
\hat{U}_n^2\,|e0\rangle &= -\,|e0\rangle
\end{aligned}$$

Cirac and Zoller found[8] if $\hat{U}_m^1 \cdot \hat{U}_n^2 \cdot \hat{U}_m^1$ is implemented sequentially on the state, then $|g\rangle_m \otimes |g\rangle_n \otimes |0\rangle$, $|g\rangle_m \otimes |e\rangle_n \otimes |0\rangle$ or $|e\rangle_m \otimes |g\rangle_n \otimes |0\rangle$ will not change at all, while $|e\rangle_m \otimes |e\rangle_n \otimes |0\rangle$ will keep the state unchanged but with a leading negative phase factor: $-\,|e\rangle_m \otimes |e\rangle_n \otimes |0\rangle$.

This result resembles CNOT operation, the only difference is that CNOT changes state instead of phase: $|x\rangle \otimes |y\rangle \to |x\rangle \otimes |y \oplus x\rangle$. To do so, we can choose to initialize the system in Hadmard basis $|\pm\rangle = (|g\rangle \pm |e\rangle)/\sqrt{2}$[8]. It is not difficult to show that under the same unitary operations, we can get the following equations with only phase difference which is trivial:

$$\begin{aligned}
|g\rangle_m\,|\pm\rangle_n\,|0\rangle &\Rightarrow |g\rangle_m\,|\pm\rangle_n\,|0\rangle \\
|g\rangle_m\,|\pm\rangle_n\,|0\rangle &\Rightarrow |g\rangle_m\,|\mp\rangle_n\,|0\rangle
\end{aligned}$$

This is exactly what CNOT is doing. 3-bit gates are also achievable since we already have the universal quantum set and all gates can be approximated to arbitrary precision[14].

5 Measurement

One of the most important features for a quantum computer is the capability of producing correct result of measurement, the output. Experimentalists uses "cycling optical transition" to perform the measurement[12]. In addition to the hyperfine energy levels of S orbit used to represent qubits, P orbit is also used here. Laser with frequency ω is continuously shine onto an ion, where $\hbar\omega$ is the energy difference between the hyperfine state with higher energy and the P orbit. Thus if the qubit is in $|e\rangle$, then it will absorb the photon and be excited to P orbit. But optical transition is ephemeral[3], and in about 1ns, the electron returns to ground state by scattering one photon with frequency ω. Thus there is a small possibility for the camera(powermeter or photon counter) above the laser to detect photons. Since the electron will return to the original state, it can be excited again, which forms a loop. If there are, for example, 10^8 cycles, which takes less than a second[13], there is almost 100% chance for the camera to detect the scattered photon.

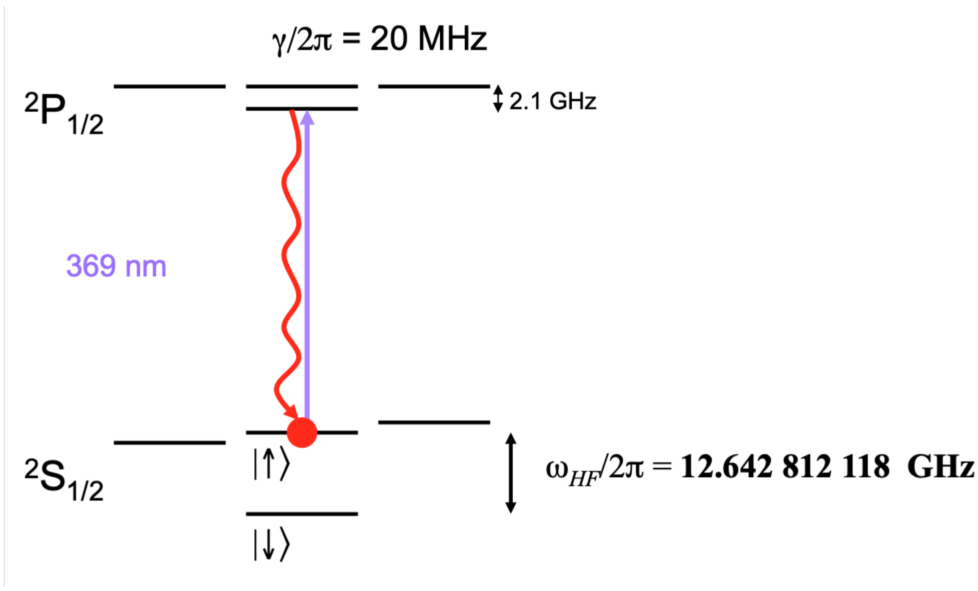

Figure 4: Optical transition for measurement[11]

On the contrary, if the electron is in the lower energy hyperfine state while being measured, it will not interact with photons from the same laser because the detuning is large. As a result, light from the laser will just pass by the ion with almost no scattering or reflection. Thus the camera above the laser will receive close-to-zero count of photons, if photons from the environment are insulated well by the system. So counting the number of photons and comparing it to the threshold will tell the internal state of qubits with high accuracy. The accuracy is greater than 99.9% as stated in [13].

6 Conclusion and outlook

In the beginning of the page, five criteria necessary for a system to achieve quantum computer are listed. In which, "scalable array of quantum bits" and "preparing the system in the lowest energy state" are previously proved by scientists. This review paper go through the way of establishing universal quantum gates with laser. It also explains the method of easily read out states of qubits with high accuracy. Early to the year 1995, Cirac and Zoller had ran an experiment: they trapped eight Ba^+ ions in the Linear Paul Trap and interact them with lasers to perform Quantum Fourier Transformation, which is the core of factoring integers[8]. Their quantum computer could work with surprising only 5% error rate. The decoherent time is about 6 seconds, longer than enough for their algorithm to output final result, which only took 35 milliseconds. Recently professor Christopher Monroe at University of Maryland successfully did individual control to 53 ions and allowed measurement in a single shot to reach nearly 100% efficiency[20]. Those tremendous achievements push cold ion trap to be one of the most promising systems for building universal quantum computer and attract more and more scientists to work on it.

References

[1] Divincenzo, David P., "The Physical Implementation of Quantum Computer". Fortschritte Der Physik, vol. 48, no. 9-11, 2000, pp. 771-783.

[2] Raizen, M. G., et al. "Ionic Crystals in a Linear Paul Trap." Physical Review A, vol. 45, no. 9(1992)

[3] Wineland, D.j., et al. "Experimental Issues in Coherent Quantum-State Manipulation of Trapped Atomic Ions." Journal of Research of the National Institute of Standards and Technology, vol. 103, no. 3, 1998

[4] Janik, G., et al. "Doppler-Free Optical Spectroscopy on the Ba^+ Mono-Ion Oscillator." Journal of the Optical Society of America B, vol. 2, no. 8, Jan. 1985, p. 1251.

[5] Letokhov, V. S.; Minogin, V. G.; Pavlik, B. D. (1977). "Cooling and capture of atoms and molecules by a resonant light field". Soviet Physics JETP. 45: 698.

[6] Teufel, J. D., et al. "Sideband Cooling Micromechanical Motion to the Quantum Ground State." Proceedings of the International Quantum Electronics Conference and Conference on Lasers and Electro-Optics Pacific Rim 2011, 2011

[7] W. Happer, "Optical Pumping", Rev. Mod. Phys. 44, 169 (1970)

[8] Cirac, J. I., and P. Zoller. "Quantum Computations with Cold Trapped Ions." Phys. Rev. Lett. 74, 4091 (1995).

[9] Hyperfine Structure, quantummechanics.ucsd.edu/ph130a/130_notes/node34.html.

[10] Woodgate, Gordon K. (1983). Elementary Atomic Structure. ISBN 978-0-19-851156-4. Retrieved 2009-03-03.

[11] Hui, Jonathan. "How to Build a Quantum Computer with Trapped Ions?" Medium, 2019

[12] Christopher Monroe, "Ion Trapping", 2012, https://www.youtube.com/watch?v=aOm50nPd5kQ.

[13] Olmschenk, S., et al. "Manipulation and Detection of a Trapped Yb^+ Hyperfine Qubit." Physical Review A, vol. 76, no. 5, 2007

[14] Kaye, Phillip, et al. An Introduction to Quantum Computing. Oxford University Press, 2010.

[15] McIntyre, David H., and Corinne A. Manogue. Quantum Mechanics: a Paradigms Approach. Pearson, 2012.

[16] Daniel A. Steck, "Quantum optics notes", https://steck.us/academic.html

[17] Bergquist, James C, et al. "Trapped Ions and Laser Cooling." 1985.

[18] Labaziewicz, Jaroslaw. ”High Fidelity Quantum Gates with Ions in Cryogenic Microfabricated Ion Traps”

[19] Ludwig E., ”Transport Quantum Logic Gates for Trapped Ions”, ETH Zurich, 2015

[20] Trapped Ion Quantum Information, 2019, iontrap.umd.edu/

[21] Timothy A. M, ”Quantum information processing with trapped ion chains”, 2014