

# Exact solutions for optimal execution of portfolios transactions and the Riccati equation

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## Abstract

We propose two methods to obtain exact solutions for the Almgren-Chriss model about optimal execution of portfolio transactions. In the first method we rewrite the Almgren-Chriss equation and find two exact solutions. In the second method, employing a general reparametrized time, we show that the Almgren-Chriss equation can be reduced to some known equations which can be exactly solved in different cases. For this last case we obtain a quantity conserved. In addition, we show that in both methods the Almgren-Chriss equation is equivalent to a Riccati equation.

## 1 Introduction

In all financial phenomena there are many random variables, this is the reason why it is so difficult to construct mathematical models that provide realistic predictions in markets. Frequently, when in a model all financial variables are considered, the model can not be tractable and then it is not

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useful. Meanwhile, when not all important variables are taken into account, the model may not yield realistic predictions. Then, the challenge to construct a financial model is to consider all important variables, but without sacrificing model tractability. It is worth mentioning that in some cases it is possible to construct simple models that provide realistic predictions. An example is given by the Black-Scholes equation for the european option price [Black and Scholes (1973), Merton (1973)]. Amazingly, this equation can be exactly solved. Now, an important problem in finance is to obtain an optimal execution of portfolios transactions. In this regard, Merton proposed a model with constant volatility that can be analytically solved [Merton (1969), Merton (1971)]. The Merton's model has been extended for more realistic situations [Magill and Constantinides (1976)], [Dumas and Luciano (1991)], [Liu and Loewenstein (2002)]. A particular case of optimal execution problem is given when an investor wants to buy (or sell) a large amount of options with the investment strategy which provides the major profit. In this case rapid buying may increasing stock price while order slicing may add to the uncertainty in the stoke price. For this last problem, using the market impact  $\eta$ , the volatility  $\sigma$  and supposing that the risk aversion  $\lambda$  is a constant, Almgren and Chriss showed that if an investor at time  $t$  has  $x_0$  orders and at the final time  $T$  must have zero orders, the optimal cost for trading is given by the optimal value of the functional [Almgren-Chriss (1999), Almgren-Chriss (2001)]

$$C = \int_t^T ds \left( \eta(s) \left( \frac{dx}{ds} \right)^2 + \lambda \sigma^2(s) x^2(s) \right). \quad (1)$$

From calculus of variations, it is well known that a functional as  $C$  reaches its optimal value when is evaluated in the function  $x(s)$  which satisfies the Euler-Lagrange equation [Elsgoc (2007)]. For the cost (1) the Euler-Lagrange is given by

$$\eta(s) \frac{d^2x}{ds^2} + \frac{d\eta(s)}{ds} \frac{dx}{ds} = \lambda \sigma^2(s) x(s), \quad (2)$$

which must be solved with the boundary conditions

$$x(t) = x_0, \quad x(T) = 0. \quad (3)$$

The case when  $\eta$  and  $\sigma$  are constants can be exactly solved. However, when  $\eta$  and  $\sigma$  are not constants, is not easy to find solutions to the equation (2).

Now, it is worth pointing out that some systems have a "natural time", for example the affine parameter to study geodesic curves [Eisenhart (2005)]. In fact, in some financial models the "natural time" is a stochastic time [Gema, Madan and Yor (2001)]. Frequently, the equation of motions are more friendly when are rewritten with the "natural time". In this respect, Almgren and Chriss shown that using the time parameter

$$d\hat{s} = \sigma^2(s)ds \quad (4)$$

and imposing the condition

$$\eta(\hat{s})\sigma^2(\hat{s}) = \text{constant} \quad (5)$$

the equation (2) can be exactly solved for some realistic cases. Some extensions of the Almgren-Chriss model can be seen in [Obizhaeva and Wang (2013)], [Schied and Schöneborn (2009)], [Gatheral and Schied (2011)], [Gatheral and Schied (2013)].

It is clear that to obtain optimal investment strategies in the Almgren-Chriss model is important to understand what kind of solutions this model has. With this in mind, in this paper we propose two methods to obtain exact solutions for the Almgren-Chriss equation. In the first method we rewrite the Almgren-Chriss equation and find two exact solutions for this equation. Furthermore, using a Riccati equation, we show that the Almgren-Chriss can be solved. In this sense we can say that the Almgren-Chriss equation is equivalent to a Riccati equation. In the second method we take the time as function of a general parameter,  $\tau$ , namely we take  $s = s(\tau)$ . Moreover, using a particular time parameter we find an exact solution for the Almgren-Chriss equation. Furthermore, we show that using a "special time" the Almgren-Chriss equation can be reduced a known equation which can be exactly solved in different cases. In addition we show that using this "special time" the Riccati equation equivalent to the Almgren-Chriss equation can be simplified.

This paper is organized as follow: in the section 2 we provide the first method and obtain two exact solutions for the Almgren-Chriss equation, in addition we show that the Almgren-Chriss equation is equivalent to a Riccati equation; in the section 3 we provide the second method and obtain an exact solution for the Almgren-Chriss equation; in the section 4 we give a summary.

## 2 First case

In order to obtain some exact solution for the Almgren-Chriss equation (2) we propose

$$x(s) = \frac{u(s)}{\sqrt{\eta(s)}}, \quad (6)$$

in this case the cost (1) becomes

$$\begin{aligned} C &= \int_t^T ds \left( \left( \frac{du}{ds} \right)^2 + \left( \frac{\frac{d^2\eta(s)}{ds^2}}{2\eta(s)} + \lambda \frac{\sigma^2(s)}{\eta(s)} - \frac{\left( \frac{d\eta(s)}{ds} \right)^2}{4\eta^2(s)} \right) u^2(s) \right) \\ &\quad - \left( \frac{u^2(s)}{2} \frac{d\eta(s)}{\eta(s)} \right) \Big|_t^T. \end{aligned} \quad (7)$$

From this expression we obtain the following Euler-Lagrange equation

$$\frac{d^2u(s)}{ds^2} = \left( \frac{\frac{d^2\eta(s)}{ds^2}}{2\eta(s)} + \lambda \frac{\sigma^2(s)}{\eta(s)} - \frac{\left( \frac{d\eta(s)}{ds} \right)^2}{4\eta^2(s)} \right) u(s), \quad (8)$$

which should be solved with the boundary condition

$$u(t) = x_0 \sqrt{\eta(t)}, \quad u(T) = 0. \quad (9)$$

Now, by an integration by parts, the cost (7) can be written as

$$\begin{aligned} C &= - \int_t^T ds u(s) \left[ \frac{d^2u}{ds^2} - \left( \frac{\frac{d^2\eta(s)}{ds^2}}{2\eta(s)} + \lambda \frac{\sigma^2(s)}{\eta(s)} - \frac{\left( \frac{d\eta(s)}{ds} \right)^2}{4\eta^2(s)} \right) u(s) \right] \\ &\quad + \left[ u(s) \frac{du(s)}{ds} - \left( \frac{u^2(s)}{2} \frac{d\eta(s)}{\eta(s)} \right) \right] \Big|_t^T. \end{aligned} \quad (10)$$

Then, if  $u(s)$  satisfies the equation of motion (8) and the boundary conditions (9), we obtain

$$C = - \left[ u(s) \frac{du(s)}{ds} - \left( \frac{u^2(s)}{2} \frac{d\eta(s)}{\eta(s)} \right) \right] \Big|_{s=t}. \quad (11)$$

## 2.1 Exact solution 1

Let us consider the case

$$\eta(s) = \eta_0 \gamma^2 (\cosh as)^2, \quad (12)$$

$$\sigma(s) = \sigma_0 \gamma \cosh as, \quad (13)$$

where  $a, \eta_0, \gamma$  and  $\sigma_0$  are constants. Substituting these functions in the equation (8), we get

$$\frac{d^2 u(s)}{ds^2} = \left( a^2 + \frac{\lambda \sigma_0^2}{\eta_0} \right) u(s), \quad (14)$$

which is solved by the function

$$u(s) = \frac{\sqrt{\eta_0} \gamma x_0 (\cosh at)}{\sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t)} \sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - s). \quad (15)$$

Furthermore, using the equation (6), we have

$$x(s) = x_0 \frac{(\cosh at)}{\sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t)} \frac{\sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - s)}{(\cosh as)} \quad (16)$$

and the cost (11) is given by

$$\begin{aligned} C &= \frac{\eta_0 \gamma^2 x_0^2 \cosh^2(at)}{\sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t)} \left( \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} \cosh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t) \right. \\ &\quad \left. + a \left( \frac{\sinh as}{\cosh as} \right) \sinh \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t) \right) \\ &= \eta_0 \gamma^2 x_0^2 \cosh^2(at) \left( \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} \coth \sqrt{a^2 + \frac{\lambda \sigma_0^2}{\eta_0}} (T - t) + a \tanh as \right). \end{aligned} \quad (17)$$

## 2.2 Exact solution 2

Now, considering the case

$$\eta(s) = \eta_0 e^{\zeta_0 s}, \quad (18)$$

$$\sigma(s) = \sigma_0 e^{\frac{\zeta_0}{2} s}, \quad \zeta_0 = \text{constant}, \quad (19)$$

the equation (8) becomes

$$\frac{d^2u(s)}{ds^2} = \left( \frac{\zeta_0^2}{4} + \lambda \frac{\sigma_0^2}{\eta_0} \right) u(s), \quad (20)$$

which has the solution

$$u(s) = x_0 \sqrt{\eta_0} e^{\frac{\zeta_0}{2}t} \frac{\sinh \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}}(T-s)}{\sinh \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}}(T-t)}. \quad (21)$$

In addition, using the equation (6), we obtain

$$x(s) = x_0 e^{-\frac{\zeta_0}{2}(s-t)} \frac{\sinh \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}}(T-s)}{\sinh \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}}(T-t)}. \quad (22)$$

Substituting this last expression in the equation (11), we obtain the following cost

$$C = \eta_0 x_0^2 e^{\zeta_0 t} \left( \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}} \coth \sqrt{\frac{\zeta_0^2}{4} + \frac{\lambda\sigma_0^2}{\eta_0}}(T-t) + \frac{\zeta_0}{2} \right). \quad (23)$$

### 2.3 General case and Riccati equation

In general, given the functions  $\sigma(s)$  and  $\eta(s)$ , to obtain a solution for the Almgren-Chriss equation is a difficult task. However, the problem can be reduced to solve a known equation. In fact, due that  $\sigma$  and  $\eta$  are positive functions, we can propose

$$\eta(s) = \eta_0 e^{2\zeta_1(s)}, \quad \sigma(s) = \sigma_0 e^{\zeta_2(s)}. \quad (24)$$

Using these expressions in the equation (8), we obtain

$$\frac{d^2u(s)}{ds^2} = \left( \frac{d^2\zeta_1(s)}{ds^2} + \left( \frac{d\zeta_1(s)}{ds} \right)^2 + \frac{\lambda\sigma_0^2}{\eta_0} e^{2(\zeta_2(s)-\zeta_1(s))} \right) u(s). \quad (25)$$

This equation is solved by

$$u(s) = \frac{\sqrt{\eta(t)}}{e^{g(t)} \int_t^T dz e^{-2g(z)}} e^{g(s)} \int_s^T dz e^{-2g(z)}, \quad (26)$$

where the function  $g(s)$  must solve the equation

$$\frac{d^2g(s)}{ds^2} + \left( \frac{dg(s)}{ds} \right)^2 = \left( \frac{d^2\zeta_1(s)}{ds^2} + \left( \frac{d\zeta_1(s)}{ds} \right)^2 + \frac{\lambda\sigma_0^2}{\eta_0} e^{2(\zeta_2(s) - \zeta_1(s))} \right). \quad (27)$$

Notice that in this case the equation (26) implies

$$x(s) = x_0 \frac{\sqrt{\eta(t)}}{e^{g(t)} \int_t^T dz e^{-2g(z)}} \frac{e^{g(s)}}{\sqrt{\eta(s)}} \int_s^T dz e^{-2g(z)}. \quad (28)$$

Furthermore, if we take

$$G(s) = \frac{dg(s)}{ds} \quad (29)$$

the equation (27) becomes

$$\frac{dG(s)}{ds} + G^2(s) = \omega^2(s), \quad (30)$$

here

$$\omega^2(s) = \left( \frac{d^2\zeta_1(s)}{ds^2} + \left( \frac{d\zeta_1(s)}{ds} \right)^2 + \frac{\lambda\sigma_0^2}{\eta_0} e^{2(\zeta_2(s) - \zeta_1(s))} \right). \quad (31)$$

We can see that the expression the expression (30) is a Riccati equation [Arfken (2005)]. Then, if the Riccati equation (30) can be solved, the equation (25) can be solved too. There is not a general method to solve the Riccati equation, however some solutions for this equation are known.

### 3 Time reparametrization

Some systems have a natural time, for instance relaxation time in thermodynamic, mean lifetime in particle physics and proper time in relativistic physics. Frequently, using the natural time the equations of motion are more kindly.

In this section we propose a general time parameter and study different cases that can be exactly solved. First, let us take the general parametrization

$$s = s(\tau). \quad (32)$$

Using this parametrization, the cost (1) becomes

$$C = \int_{\tau_0}^{\tau_F} d\tau \left( \frac{\eta(\tau)}{\frac{ds(\tau)}{d\tau}} \left( \frac{dx}{d\tau} \right)^2 + \lambda \sigma^2(\tau) \frac{ds(\tau)}{d\tau} x^2(\tau) \right), \quad (33)$$

which implies the Euler-Lagrange equation

$$\frac{d^2x(\tau)}{d\tau^2} + \frac{dx(\tau)}{d\tau} \left( \frac{\frac{d\eta(\tau)}{d\tau}}{\eta(\tau)} - \frac{\frac{d^2s(\tau)}{d\tau^2}}{\frac{ds(\tau)}{d\tau}} \right) = \frac{\lambda \sigma^2(\tau)}{\eta(\tau)} \left( \frac{ds(\tau)}{d\tau} \right)^2 x(\tau). \quad (34)$$

The boundary conditions for this last equation are given by

$$x(\tau_0) = x_0, \quad x(\tau_F) = 0. \quad (35)$$

Notice that the cost (33) can be written as

$$\begin{aligned} C &= - \int_{\tau_0}^{\tau_F} d\tau \frac{\eta(\tau)x(\tau)}{\frac{ds(\tau)}{d\tau}} \left[ \frac{d^2x(\tau)}{d\tau^2} + \left( \frac{\frac{d\eta(\tau)}{d\tau}}{\eta(\tau)} - \frac{\frac{d^2s(\tau)}{d\tau^2}}{\frac{ds(\tau)}{d\tau}} \right) \frac{dx(\tau)}{d\tau} \right. \\ &\quad \left. - \frac{\lambda \sigma^2(\tau)}{\eta(\tau)} \left( \frac{ds(\tau)}{d\tau} \right)^2 x(\tau) \right] + \frac{\eta(\tau)x(\tau)}{\frac{ds(\tau)}{d\tau}} \frac{dx(\tau)}{d\tau} \Big|_{\tau_0}^{\tau_F}. \end{aligned} \quad (36)$$

Then, when  $x(\tau)$  satisfies the equation (34) and the boundary conditions (35), we obtain the cost

$$C = - \frac{\eta(\tau)x(\tau)}{\frac{ds(\tau)}{d\tau}} \frac{dx(\tau)}{d\tau} \Big|_{\tau=\tau_0}. \quad (37)$$

We can see that taking the Almgren-Chriss parameter (4), that is

$$\frac{ds}{d\tau} = \frac{1}{\sigma^2(\tau)}, \quad (38)$$

we obtain the cost

$$C = \int_{\tau_0}^{\tau_F} d\tau \left( \eta(\tau)\sigma^2(\tau) \left( \frac{dx}{d\tau} \right)^2 + \lambda x^2(\tau) \right), \quad (39)$$

and the Euler-Lagrange equation

$$\frac{d^2x(\tau)}{d\tau^2} + \frac{dx(\tau)}{d\tau} \frac{d \ln (\eta(\tau)\sigma^2(\tau))}{d\tau} = \frac{\lambda}{\eta(\tau)\sigma^2(\tau)} x(\tau). \quad (40)$$

Almgren and Chriss shown that this equation is tractable when  $\eta(\tau)\sigma^2(\tau)$  is a constant [Almgren-Chriss (1999), Almgren-Chriss (2001)].

In the following subsections we introduce two additional useful time parameters.

### 3.1 First parameter

Due that  $\sigma$  and  $\eta$  are positive quantities, we can propose

$$\eta(\tau) = \eta_0 e^{\zeta_1(\tau)}, \quad \sigma(\tau) = \sigma_0 e^{\zeta_2(\tau)}. \quad (41)$$

Then, the equation (34) can be written as

$$\frac{d^2x(\tau)}{d\tau^2} + \frac{dx(\tau)}{d\tau} \left( \frac{d\zeta_1(\tau)}{d\tau} - \frac{\frac{d^2s(\tau)}{d\tau^2}}{\frac{ds(\tau)}{d\tau}} \right) = \frac{\lambda\sigma_0^2}{\eta_0} e^{2\zeta_2(\tau)-\zeta_1(\tau)} \left( \frac{ds(\tau)}{d\tau} \right)^2 x(\tau). \quad (42)$$

Furthermore, using the parameter

$$\frac{ds(\tau)}{d\tau} = e^{\frac{\zeta_1(\tau)-2\zeta_2(\tau)}{2}}, \quad (43)$$

we obtain

$$\frac{d^2x(\tau)}{d\tau^2} + \frac{dx(\tau)}{d\tau} \left( \frac{1}{2} \frac{d\zeta_1(\tau)}{d\tau} + \frac{d\zeta_2(\tau)}{d\tau} \right) = \frac{\lambda\sigma_0^2}{\eta_0} x(\tau). \quad (44)$$

We can see that when

$$\frac{1}{2} \frac{d\zeta_1(\tau)}{d\tau} + \frac{d\zeta_2(\tau)}{d\tau} = \alpha, \quad \alpha = \text{constant} \quad (45)$$

namely

$$\zeta_2(\tau) = \alpha\tau + \beta - \frac{\zeta_1(\tau)}{2}, \quad \beta = \text{constant}, \quad (46)$$

the equation (44) becomes

$$\frac{d^2x(\tau)}{d\tau^2} + \alpha \frac{dx(\tau)}{d\tau} = \frac{\lambda\sigma_0^2}{\eta_0} x(\tau). \quad (47)$$

Notice that using the functions (41), the condition (46) can be written as

$$\eta(\tau)\sigma^2(\tau) = Ae^{2\alpha\tau}, \quad (48)$$

where  $A$  is a constant. Furthermore, the solution for the equation (47) is given by

$$x(\tau) = x_0 \frac{e^{-\frac{\alpha}{2}(\tau-\tau_0)} \sinh \frac{1}{2}\sqrt{\alpha^2 + \frac{4\lambda\sigma_0^2}{\eta_0}}(\tau_F - \tau)}{\sinh \frac{1}{2}\sqrt{\alpha^2 + \frac{4\lambda\sigma_0^2}{\eta_0}}(\tau_F - \tau_0)}, \quad (49)$$

while the cost is

$$C = \frac{x_0^2}{2}\eta_0 e^{\alpha\tau_0 + \beta} \left( \alpha + \frac{1}{2}\sqrt{\alpha^2 + \frac{4\lambda\sigma_0^2}{\eta_0}} \coth \frac{1}{2}\sqrt{\alpha^2 + \frac{4\lambda\sigma_0^2}{\eta_0}}(\tau_F - \tau_0) \right). \quad (50)$$

### 3.2 Second parameter

Now, if we take the parameter

$$\frac{ds}{d\tau} = \eta(\tau), \quad (51)$$

the cost (33) becomes

$$C = \int_{\tau_0}^{\tau_F} d\tau \left( \left( \frac{dx}{d\tau} \right)^2 + \lambda\sigma^2(\tau)\eta(\tau)x^2(\tau) \right). \quad (52)$$

This functional implies the Euler-Lagrange equation

$$\frac{d^2x(\tau)}{d\tau^2} = \lambda\sigma^2(\tau)\eta(\tau)x(\tau). \quad (53)$$

Moreover, when  $x(\tau)$  satisfies the equation (53) and the boundary conditions (35), the cost (37) becomes

$$C = -x(\tau) \frac{dx(\tau)}{d\tau} \Big|_{\tau=\tau_0}. \quad (54)$$

Notice that the original equation (2) depends on both functions  $\sigma(s)$  and  $\eta(s)$ . While the equation (53) only depends on the product  $\lambda\sigma^2(\tau)\eta(\tau)$ . In

addition, we can see that the equation (53) is simpler than the original equation (2) and the equation (40).

The equation (53) has some interesting properties. For example, if the equation

$$\frac{d^2\rho(\tau)}{d\tau^2} + \lambda\sigma^2(\tau)\eta(\tau)\rho(\tau) - \frac{1}{\rho^3(\tau)} = 0 \quad (55)$$

is satisfied, then the function

$$I = \frac{1}{2} \left[ \left( \rho(\tau) \frac{dx(\tau)}{d\tau} - x(\tau) \frac{d\rho(\tau)}{d\tau} \right)^2 - \left( \frac{x(\tau)}{\rho(\tau)} \right)^2 \right], \quad (56)$$

is a conserved quantity. In particular, when

$$\lambda\sigma^2(\tau)\eta(\tau) = \lambda\sigma_0^2\eta_0 = \text{constant}, \quad (57)$$

we obtain the conserved quantity

$$I = \frac{1}{2} \left[ \frac{1}{\sqrt{\lambda\sigma_0^2\eta_0}} \left( \frac{dx(\tau)}{d\tau} \right)^2 - \sqrt{\lambda\sigma_0^2\eta_0} x^2(\tau) \right], \quad (58)$$

which can be interpreted as the energy for a particle with mass  $1/\sqrt{\lambda\sigma_0^2\eta_0}$  under the repulsive force  $\sqrt{\lambda\sigma_0^2\eta_0}x(\tau)$ , see [Goldstein (1980)]. For the case (57), we have

$$x(\tau) = x_0 \frac{\sinh \sqrt{\lambda\sigma_0^2\eta_0}(\tau_F - \tau)}{\sinh \sqrt{\lambda\sigma_0^2\eta_0}(\tau_F - \tau_0)}, \quad (59)$$

and the cost

$$C = x_0^2 \sqrt{\lambda\sigma_0^2\eta_0} \frac{\cosh \sqrt{\lambda\sigma_0^2\eta_0}(\tau_F - \tau_0)}{\sinh \sqrt{\lambda\sigma_0^2\eta_0}(\tau_F - \tau_0)}. \quad (60)$$

In addition, it is possible to obtain solutions for the equation (53) for other functions  $\lambda\sigma^2(\tau)\eta(\tau)$ . For instance, when

$$\lambda\sigma^2(\tau)\eta(\tau) = \left( \alpha + \frac{\beta}{\tau + b} + \frac{\gamma}{(\tau + b)^2} \right) \quad (61)$$

where

$$\sqrt{1 + 4\gamma} \pm \frac{\beta}{\alpha} \quad (62)$$

is an integer, the equation (53) is completely solved [Ermakov (2008)]. Other cases where the equation (53) is completely solved can be seen in [Ermakov (2008)].

### 3.3 Riccati equation

Furthermore, in order to solve the equation (53), we can propose the function

$$x(\tau) = \frac{x_0}{e^{f(\tau_0)} \int_{\tau_0}^{\tau_F} dz e^{-2f(z)}} e^{f(\tau)} \int_{\tau}^{\tau_F} dz e^{-2f(z)}, \quad (63)$$

where the function  $f(\tau)$  must satisfy the equation

$$\frac{d^2 f(\tau)}{d\tau^2} + \left( \frac{df(\tau)}{d\tau} \right)^2 = \lambda \sigma^2(\tau) \eta(\tau). \quad (64)$$

Thus, if we find a solution for this last equation, we can solve the equation (53). In addition, using the equation (71) we obtain the cost

$$C = x_0^2 \left( \frac{e^{-2f(\tau_0)}}{\int_{\tau_0}^{\tau_F} dz e^{-2f(z)}} - \left. \frac{df(\tau)}{d\tau} \right|_{\tau=\tau_0} \right). \quad (65)$$

Notice that if we take

$$F(\tau) = \frac{df(\tau)}{d\tau}, \quad (66)$$

the equation (64) can be written as

$$\frac{dF(\tau)}{d\tau} + F^2(\tau) = \lambda \sigma^2(\tau) \eta(\tau), \quad (67)$$

which is a Riccati equation [Arfken (2005)]. Then, if the Riccati equation (67) can be solved, the equation (53) can be solved too.

For instance, when

$$\lambda \sigma^2(\tau) \eta(\tau) = \lambda \sigma_0^2 \eta_0 (1 + \lambda \sigma_0^2 \eta_0 \tau^2) \quad (68)$$

we obtain

$$F(\tau) = \lambda\sigma_0^2\eta_0\tau, \quad (69)$$

which implies

$$f(\tau) = \lambda\sigma_0^2\eta_0\frac{\tau^2}{2} + B, \quad B = \text{constant}. \quad (70)$$

In this case, we have

$$x(\tau) = \frac{x_0}{\int_{\tau_0}^{\tau_F} dz e^{-\lambda\sigma_0^2\eta_0 z^2}} e^{\frac{\lambda\sigma_0^2\eta_0}{2}(\tau^2 - \tau_0^2)} \int_{\tau}^{\tau_F} dz e^{-\lambda\sigma_0^2\eta_0 z^2}, \quad (71)$$

and the cost is given by

$$C = x_0^2 \left( \frac{e^{-\lambda\sigma_0^2\eta_0\tau_0^2}}{\int_{\tau_0}^{\tau_F} dz e^{-\lambda\sigma_0^2\eta_0 z^2}} - \lambda\sigma_0^2\eta_0\tau_0 \right). \quad (72)$$

## 4 Summary

In this paper two methods to obtain exact solutions for the Almgren-Chriss equation were proposed. In the first method the Almgren-Chriss equation was rewritten and two exact solutions for this equation were found. Furthermore, using a Riccati equation, we show that the Almgren-Chriss can be solved. In this sense we can say that the Almgren-Chriss equation is equivalent to a Riccati equation. In the second method the Almgren-Chriss equation was reparametrized on the time. Moreover, using a particular parameter "time" we found an exact solution for the Almgren-Chriss equation. Furthermore, we show that using a special reparametrization the Almgren-Chriss equation can be reduced to a known equation which can be exactly solved for different cases. For this last case we obtained a conserved quantity.

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