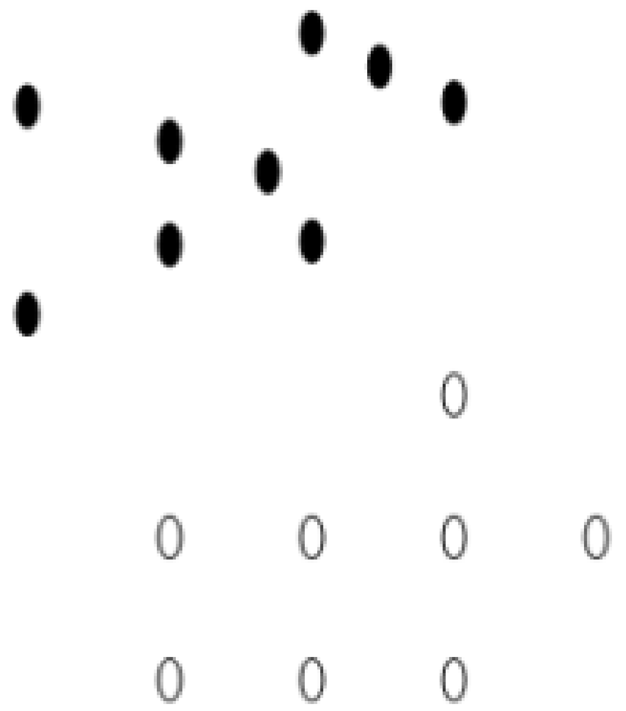
Lesson 6

Support vector machine (SVM)

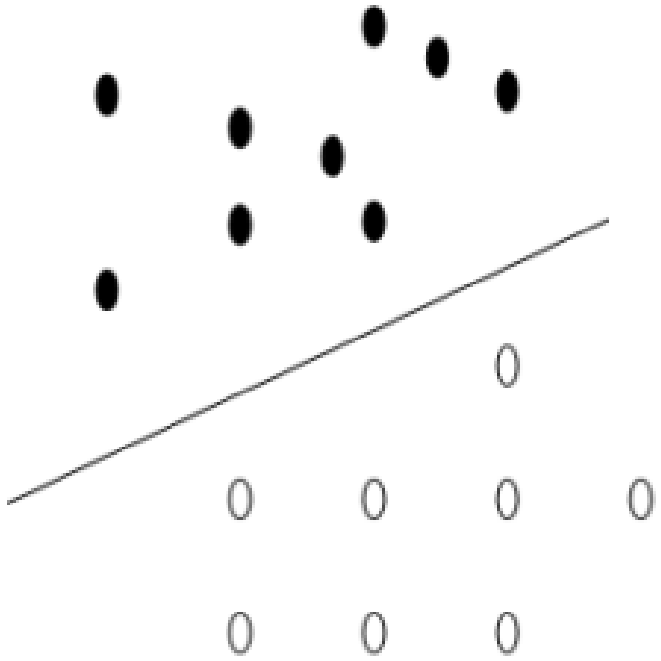
[
Support vector machine (SVM in short) is a Discriminant based classification method where the task is to find a decision boundary separating sample in one class from the other. it is a binary in nature, means it considers two classes.
SVM is a vase topic. In this lesion, we will focus only on introductory understanding of SVM.
]

Support vector machine



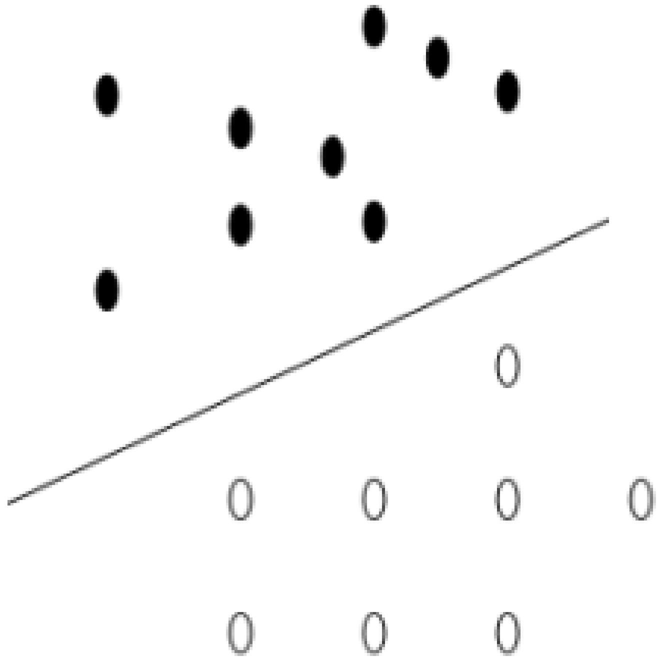
[
Let us say, we have samples in two different classes as shown in the figure. Black samples are in +ve class and white samples are in –ve class.
]

Support vector machine



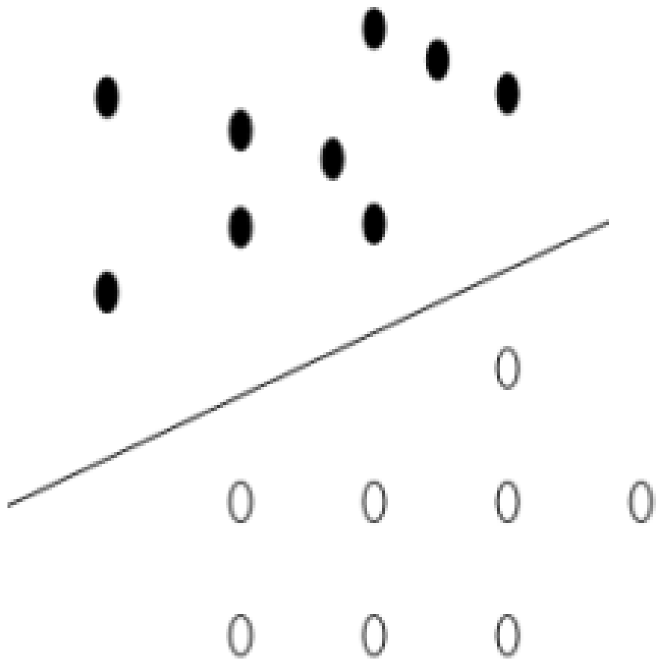
[
The task is to find best possible line separating samples in one class from the samples other class.
]

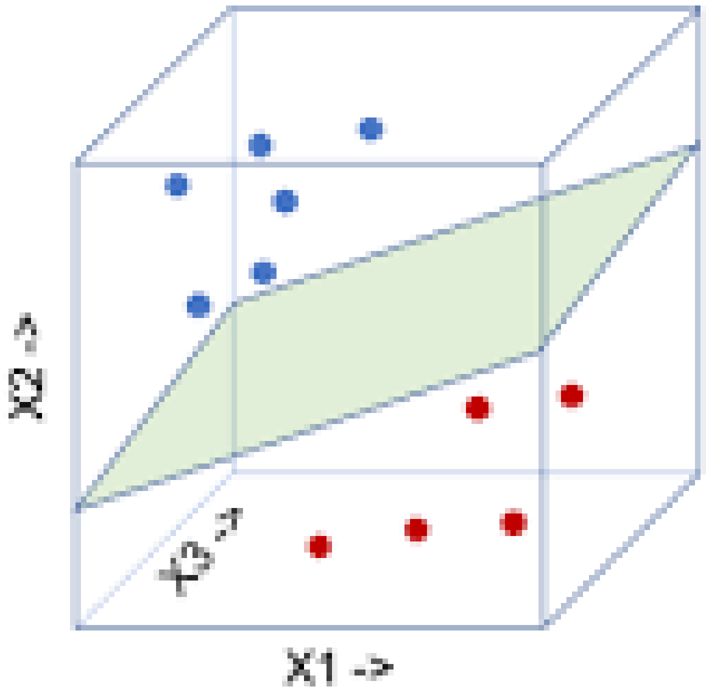
Support vector machine



[
In the 2D space, the decision boundary is a line,
]

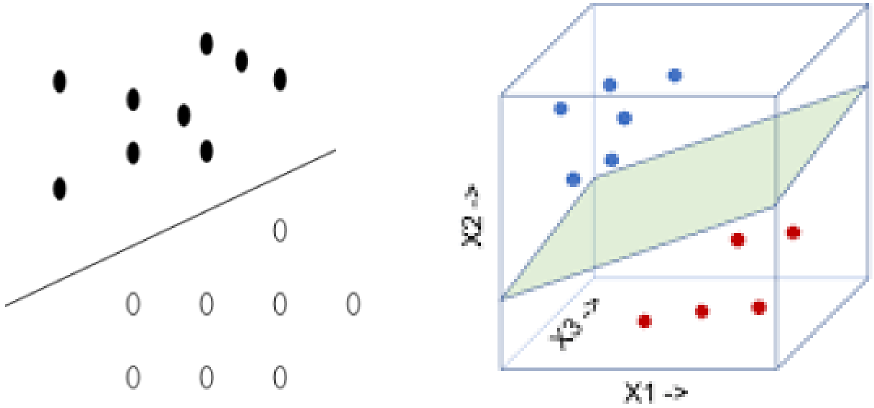
Support vector machine



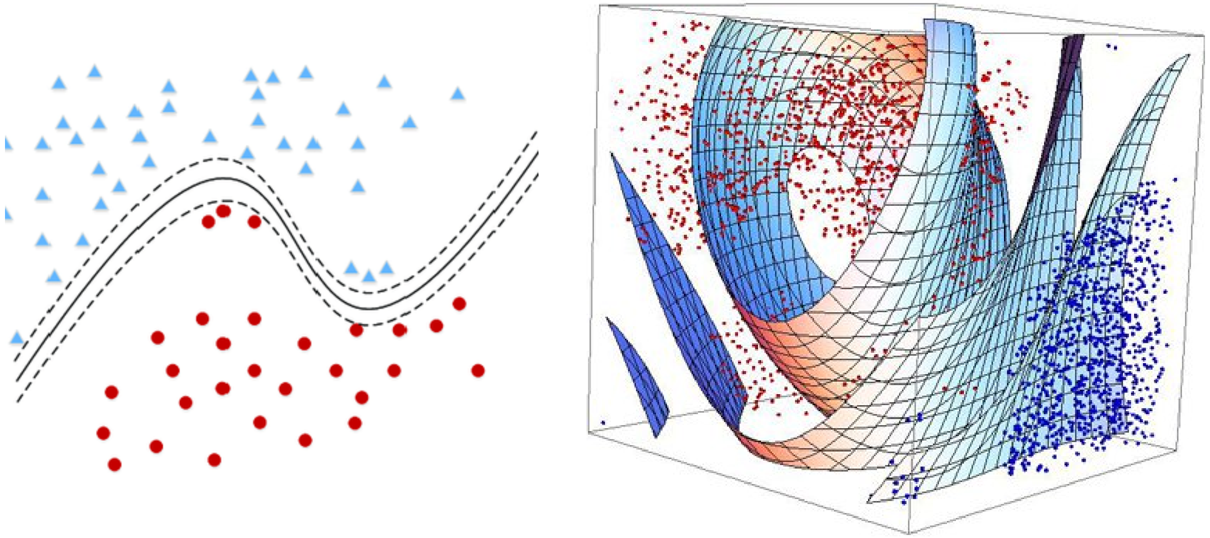


[
In the 3D space, the decision boundary is a hyperplane
]

Support vector machine – Linear or Non-linear



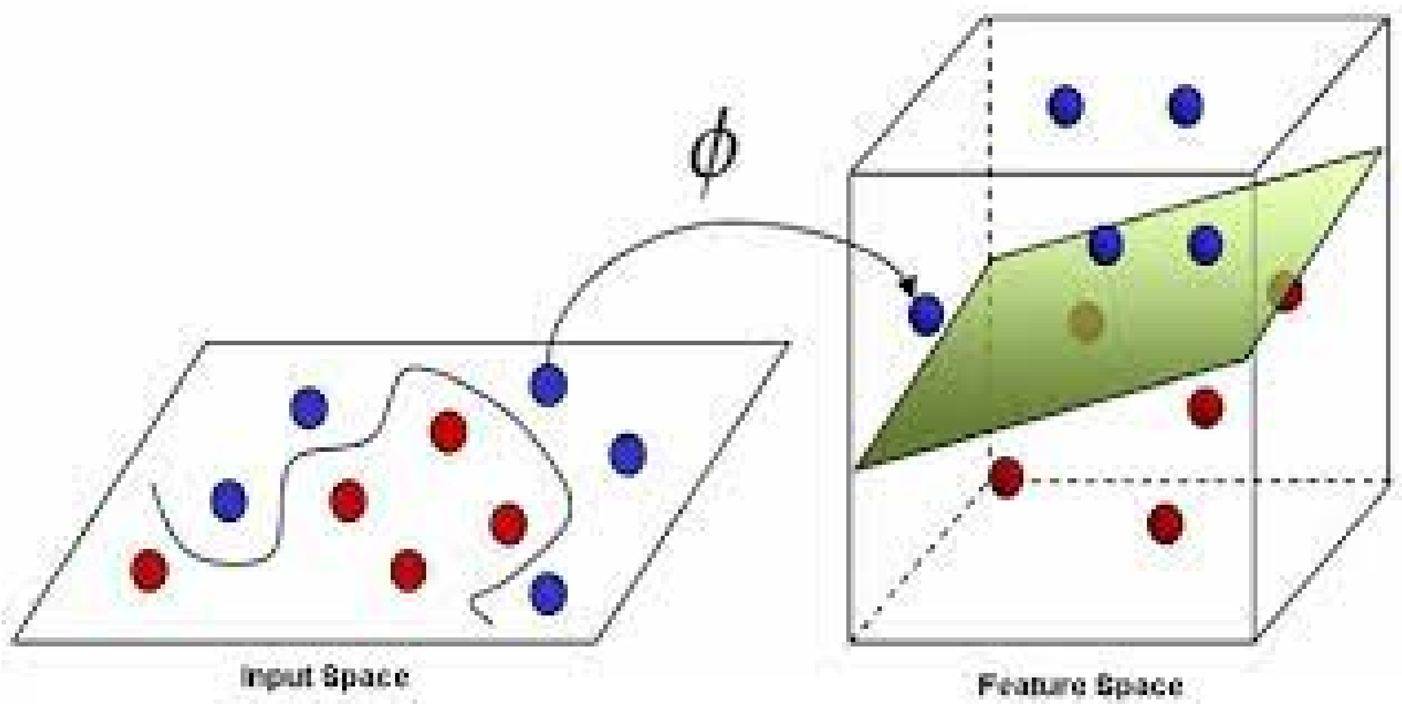
Linear SVM



Non-Linear SVM

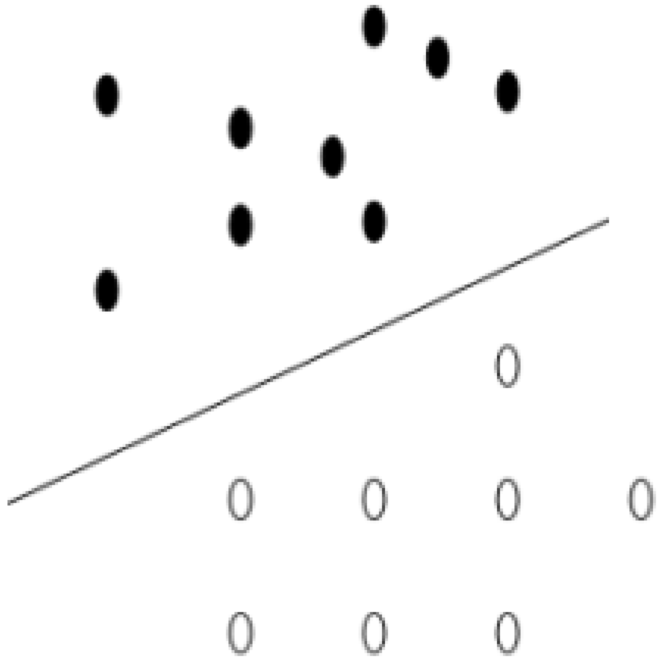
[
Depending on the nature of the samples, the decision boundary can be linear or non-linear, resulting to linear SVM or non-linear SVM.
]

Non-Linear to Linear



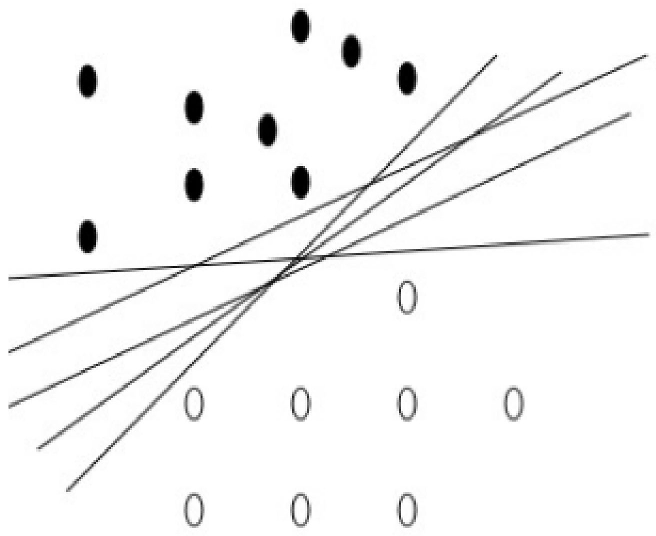
[
Finding a non-linear decision boundary is complex. Further, non-linear problems can be transformed to a linear problem. For example, ……. This lesson focusses on Linear SVM.
]

Linear Support vector machine



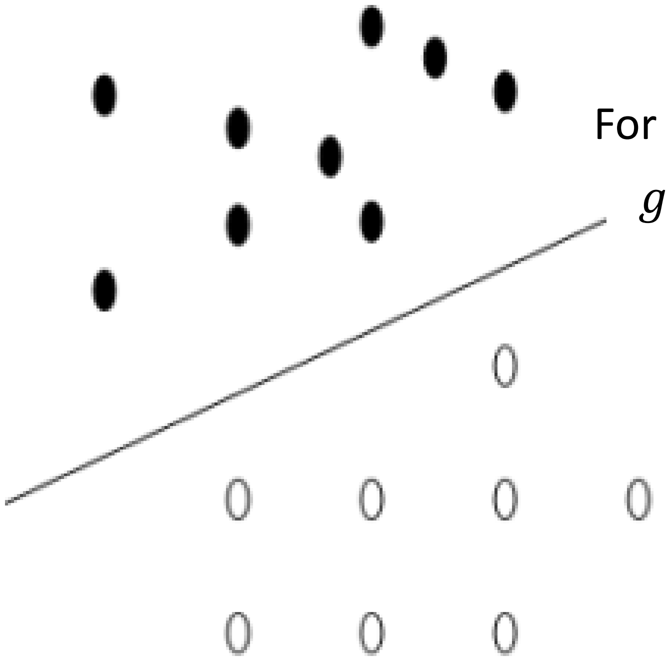
[
Given this example, As mentioned, the task is to find a separating line.
]

Linear Support vector machine



[
However, there are infinite number of lines which can separate the +ve samples from the –ve samples. Which one of them will be selected as the separating line. This lesson will explain how to choose the separating line.
]

Linear Support vector machine

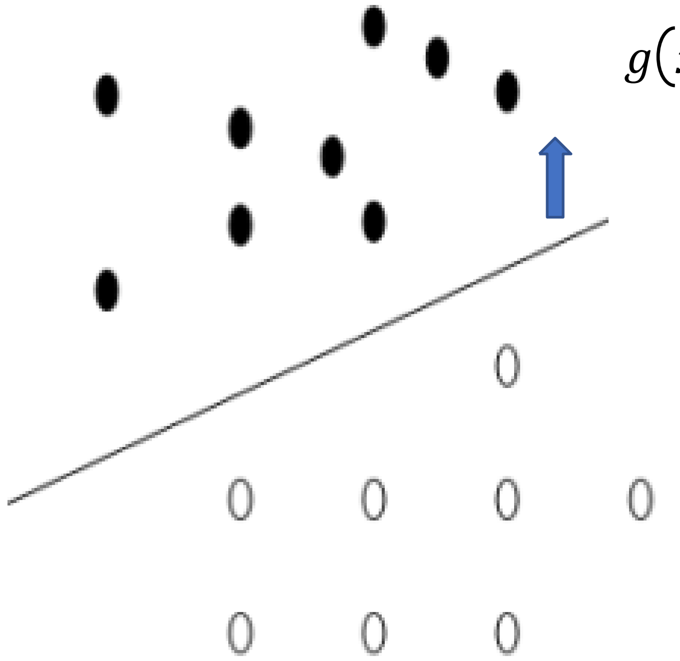


For all the points on the hyper plane, g(x) = 0

$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_0 = 0$$

[
To understand the process, let us begin with as separating hyperplane defined by the linear function g(x).
]

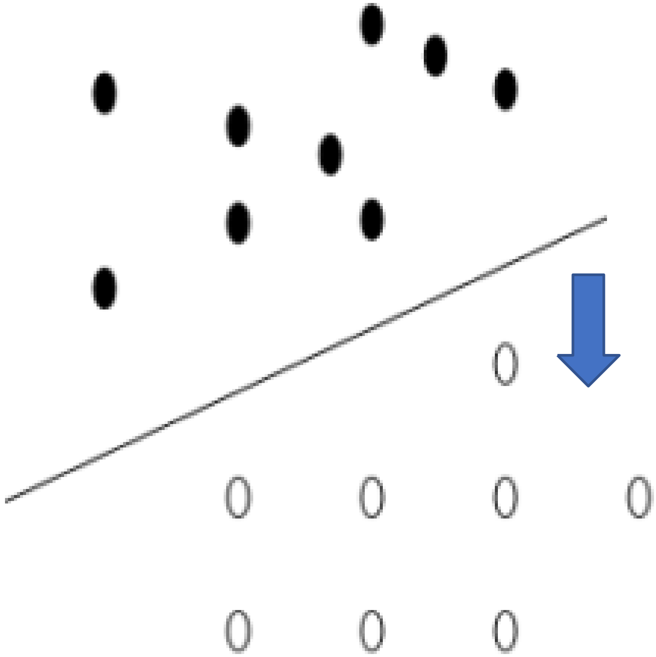
Linear Support vector machine



For all the points above the hyper plane, g(x) > 0

$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 > 0$$

Linear Support vector machine

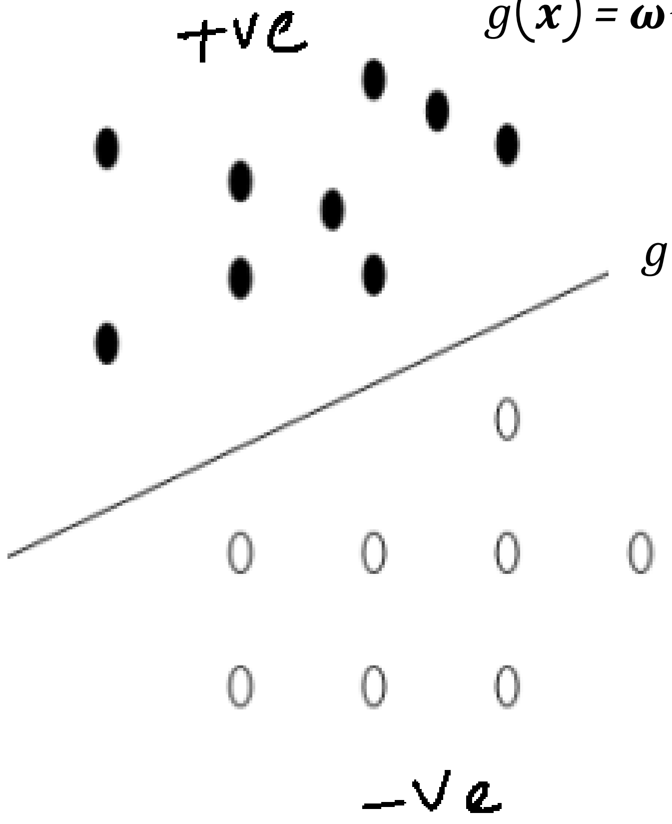


$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 < 0$

For all the points below the hyper plane, g(x) < 0

Linear Support vector machine

$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 > 0$$

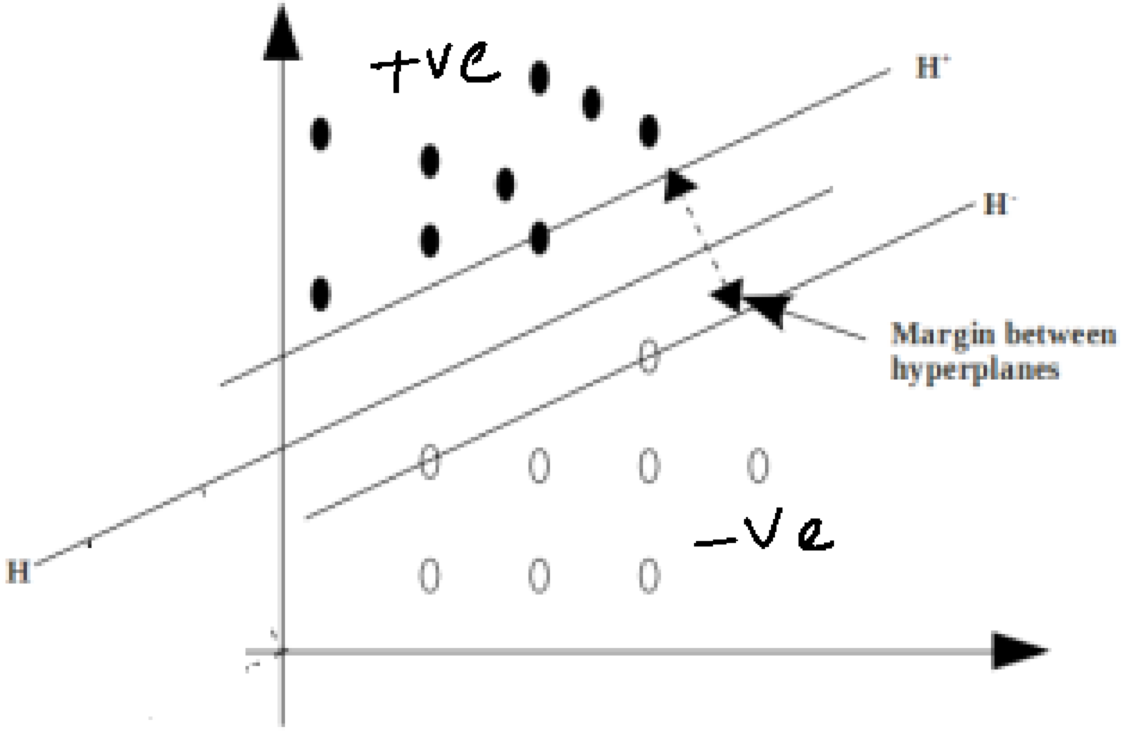


$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 = 0$$

$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 < 0$$

We choose the hyperplane that has the **maximum separating margin**

What is Separating Margin?



H is the separating hyperplane

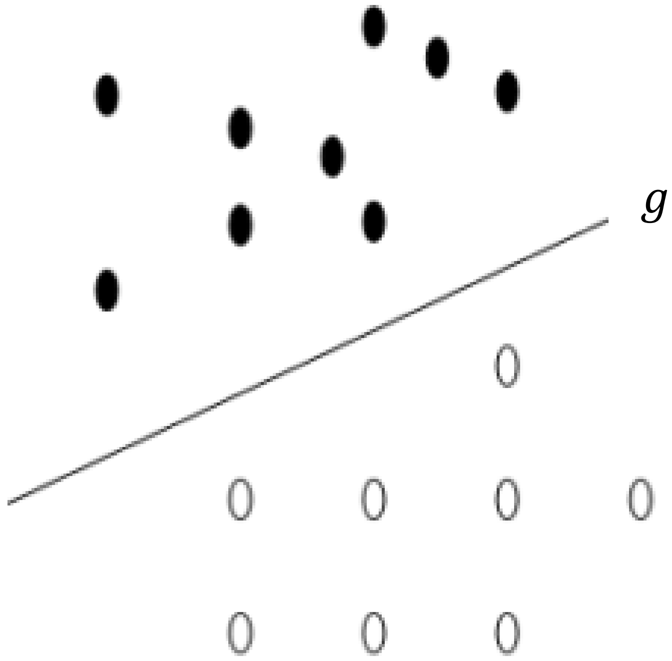
H^+ is the plane parallel to H and passing through the nearest +ve points to H

H^- is the plane parallel to H and passing through the nearest -ve points to H

Separating margin is the distance between H^+ and H^-

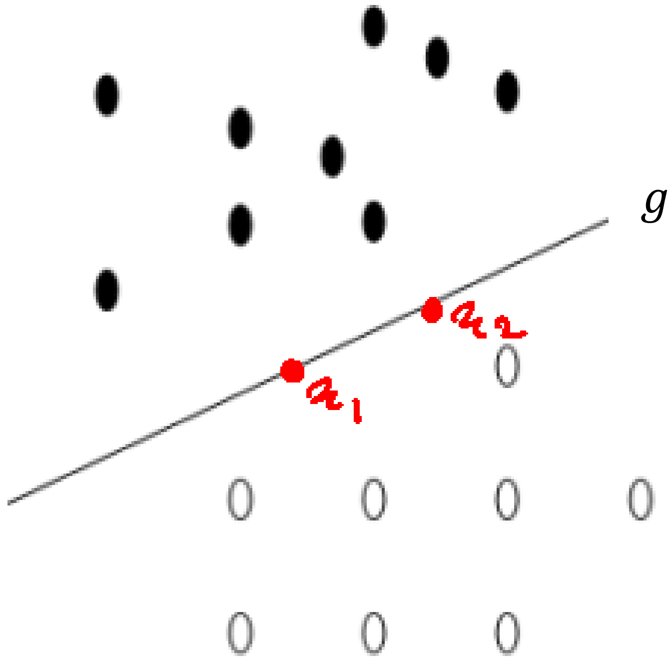
We choose H that has the maximum margin.

Finding Separating Hyperplane with maximum Margin?



$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 = 0$$

Finding Separating Hyperplane with maximum Margin?

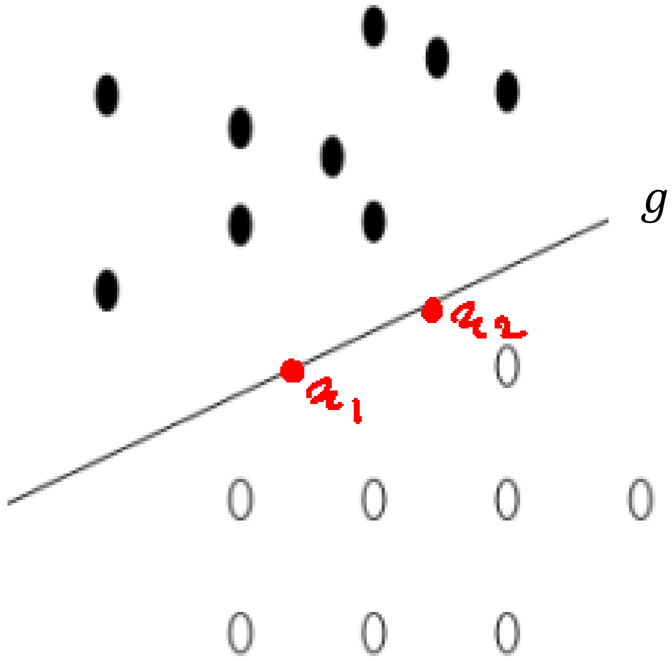


$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_0 = 0$$

Let us take two points x_1 and x_2 lying on $g(x)$

$$g(x_1) = g(x_2) = 0$$

Finding Separating Hyperplane with maximum Margin?



$g(\boldsymbol{x}) = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_0 = 0$

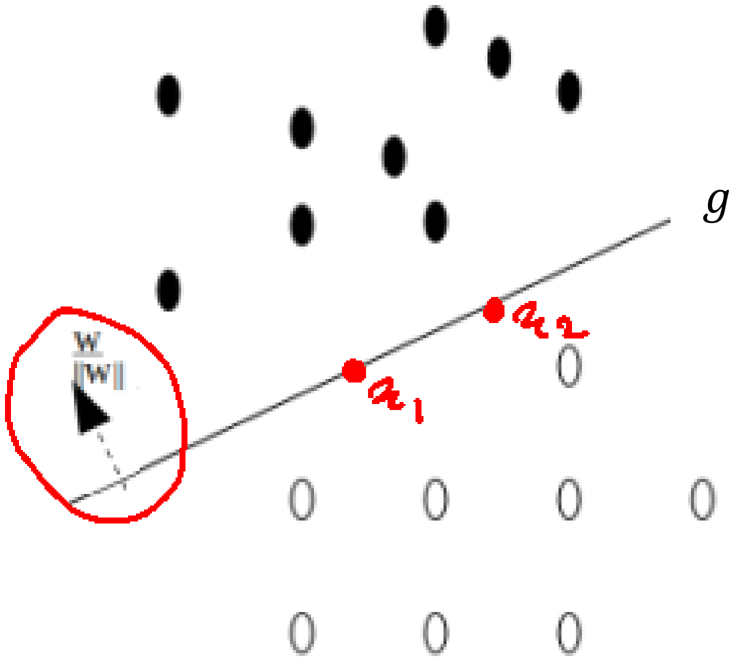
Let us take two points ***x₁*** and ***x₂*** lying on ***g(x)***

$$g(x_1) = g(x_2) = 0$$

$$\Rightarrow \boldsymbol{\omega}^T \boldsymbol{x}_1 + \omega_0 = \boldsymbol{\omega}^T \boldsymbol{x}_1 + \omega_0$$

$$\Rightarrow \boldsymbol{\omega}^T (\boldsymbol{x}_1 - \boldsymbol{x}_1) = 0$$

Finding Separating Hyperplane with maximum Margin?



$$g(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_0 = 0$$

Let us take two points $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ lying on $\boldsymbol{g(x)}$

$$g(x_1) = g(x_2)=0$$

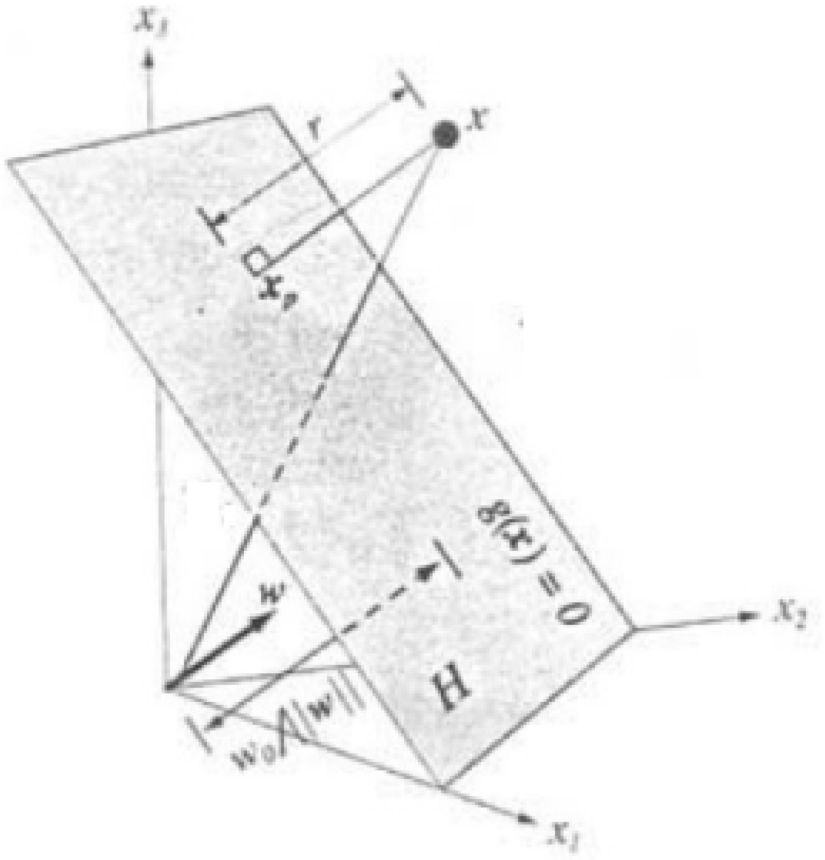
$$\Rightarrow \boldsymbol{\omega}^T\boldsymbol{x}_1 + \omega_0 = \boldsymbol{\omega}^T\boldsymbol{x}_1 + \omega_0$$

$$\Rightarrow \boldsymbol{\omega}^T(\boldsymbol{x}_1 - \boldsymbol{x}_1) = 0$$

It means $\boldsymbol{\omega}$ is orthogonal (90°/perpendicular) the vector ($\boldsymbol{x_1}$ - $\boldsymbol{x_2}$).

It means $\boldsymbol{\omega}$ is orthogonal (90°/perpendicular) to $\boldsymbol{g(x)}$.

Finding Separating Hyperplane with maximum Margin?



Let x be a point in space.

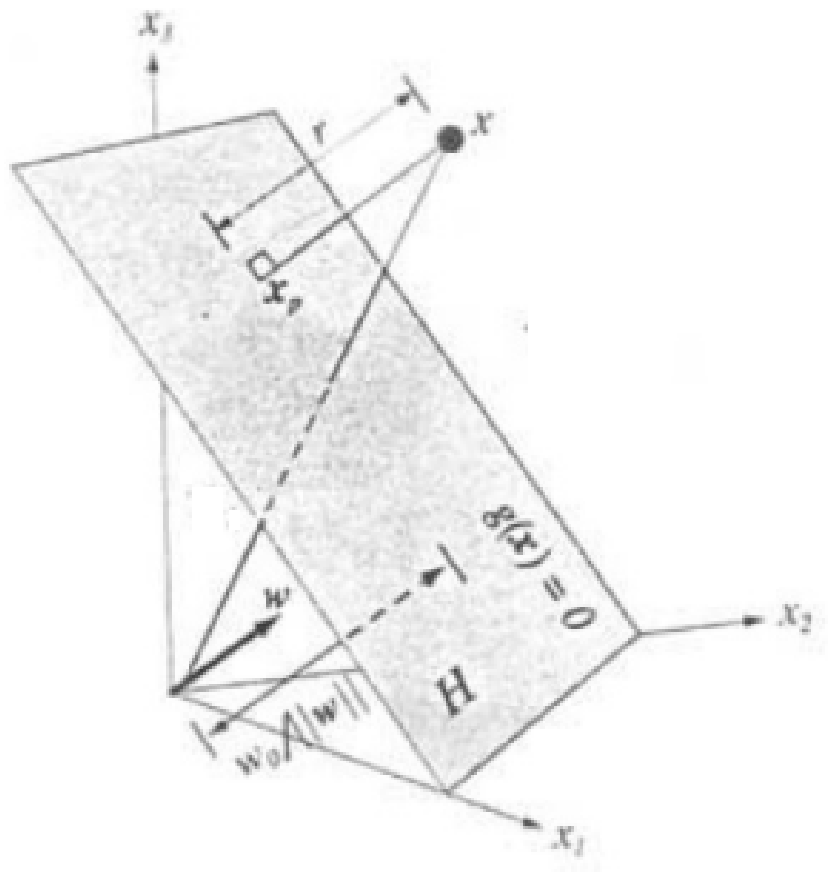
Let r be the distance of the point x from the hyperplane g(x),
and x_p be the corresponding projection point of x on g(x).

Now, vector $\boldsymbol{x}$ can be defined by the sum of the vector $\boldsymbol{x_p}$ and vector $\boldsymbol{r}$.

$$\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{r}$$

$$\Rightarrow \boldsymbol{x} = \boldsymbol{x_p} + r\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$$

Finding Separating Hyperplane with maximum Margin?



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$$\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{r}$$

$$\Rightarrow \boldsymbol{x} = \boldsymbol{x_p} + r\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$$

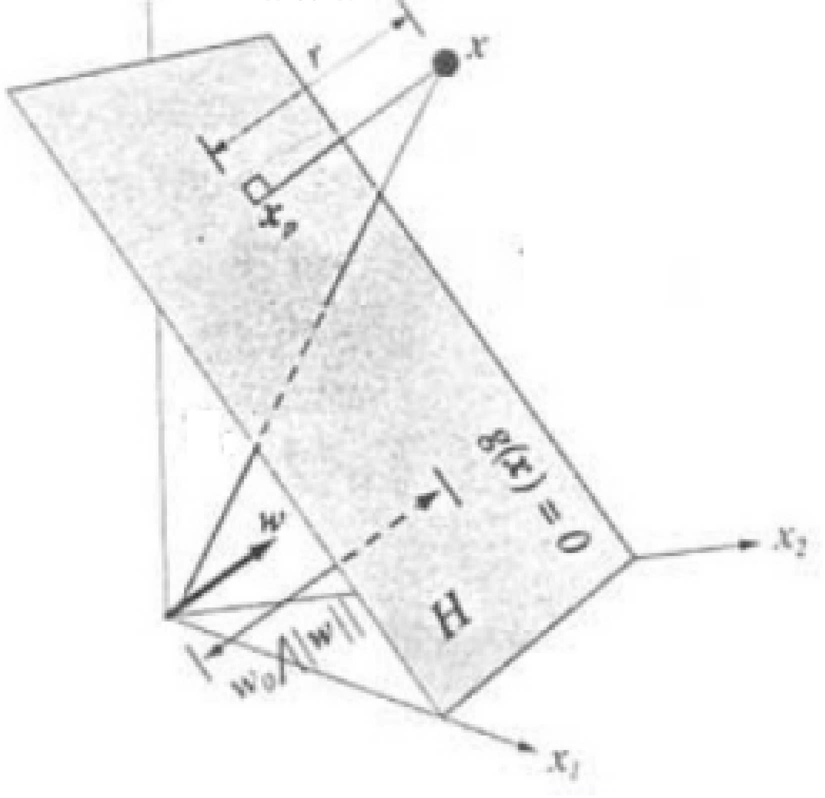
If you substitute $\boldsymbol{x}$ in g($\boldsymbol{x}$).

$$\Rightarrow \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x_p} + r\frac{\boldsymbol{\omega}^T\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} + \boldsymbol{\omega_0}$$

$$\Rightarrow \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x_p} + \boldsymbol{\omega_0} + r\frac{\boldsymbol{\omega}^T\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \qquad \Rightarrow \boldsymbol{g}(\boldsymbol{x}) = r\frac{\boldsymbol{\omega}^T\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \qquad \Rightarrow r = \frac{g(\boldsymbol{x})}{\|\boldsymbol{\omega}\|}$$

Finding Separating Hyperplane with maximum Margin?

$r = \frac{g(\boldsymbol{x})}{\|\boldsymbol{\omega}\|}$



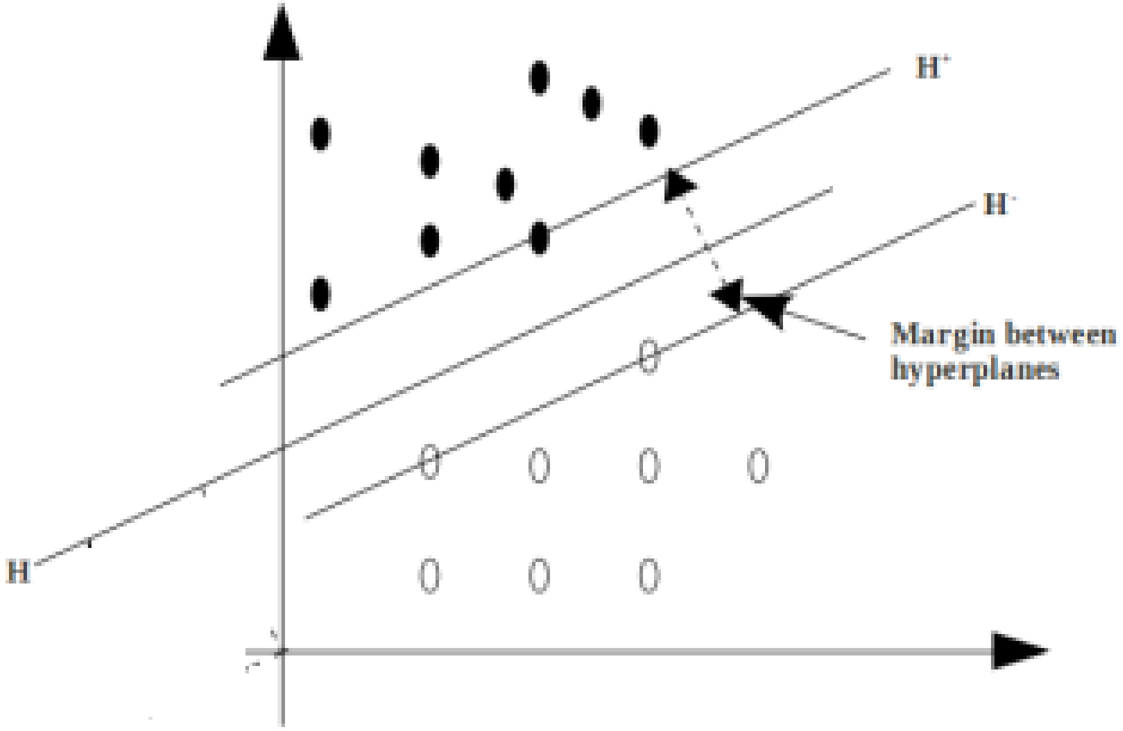
Distance of $\boldsymbol{x_p}$ from g(x)=0 is $r = \frac{g(\boldsymbol{x})}{\|\boldsymbol{\omega}\|}$

So, the distance of origin from g(x)=0 is

$$\Rightarrow r_{\mathbf{0}} = \frac{\boldsymbol{\omega}^T \mathbf{0} + \omega_0}{\|\boldsymbol{\omega}\|}$$

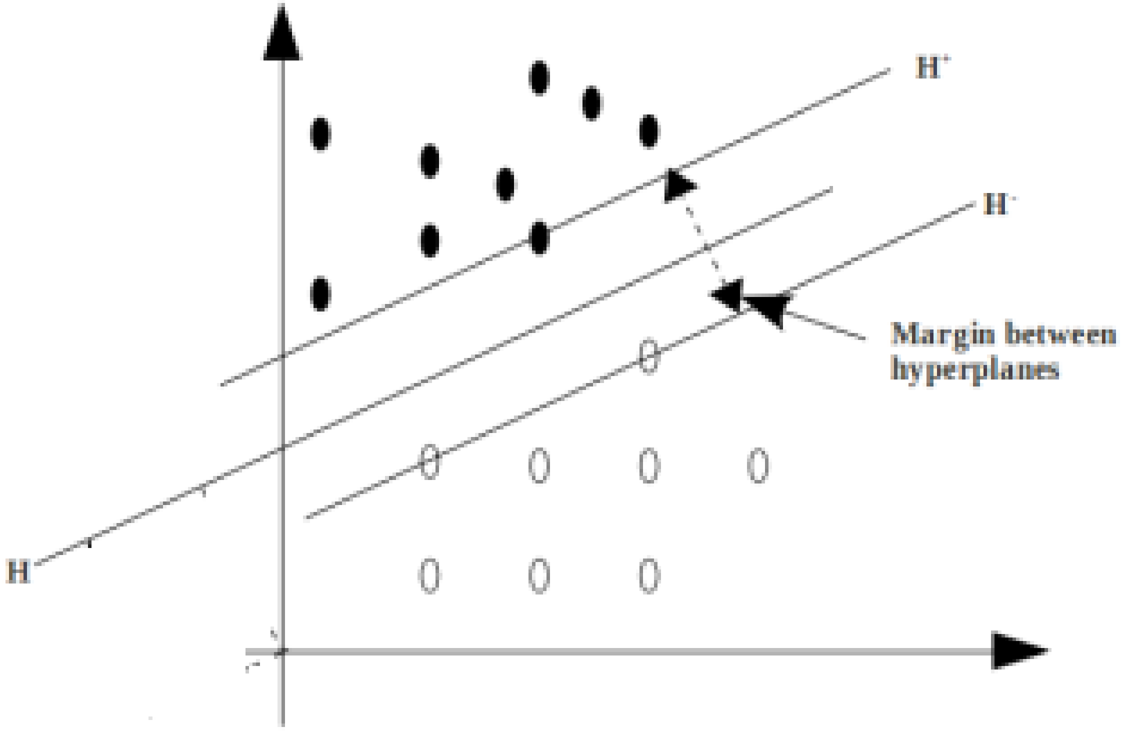
$$\Rightarrow r_{\mathbf{0}} = \frac{\omega_0}{\|\boldsymbol{\omega}\|}$$

Finding Separating Hyperplane with maximum Margin?



What is the Margin between H+ and H-?

Finding Separating Hyperplane with maximum Margin?

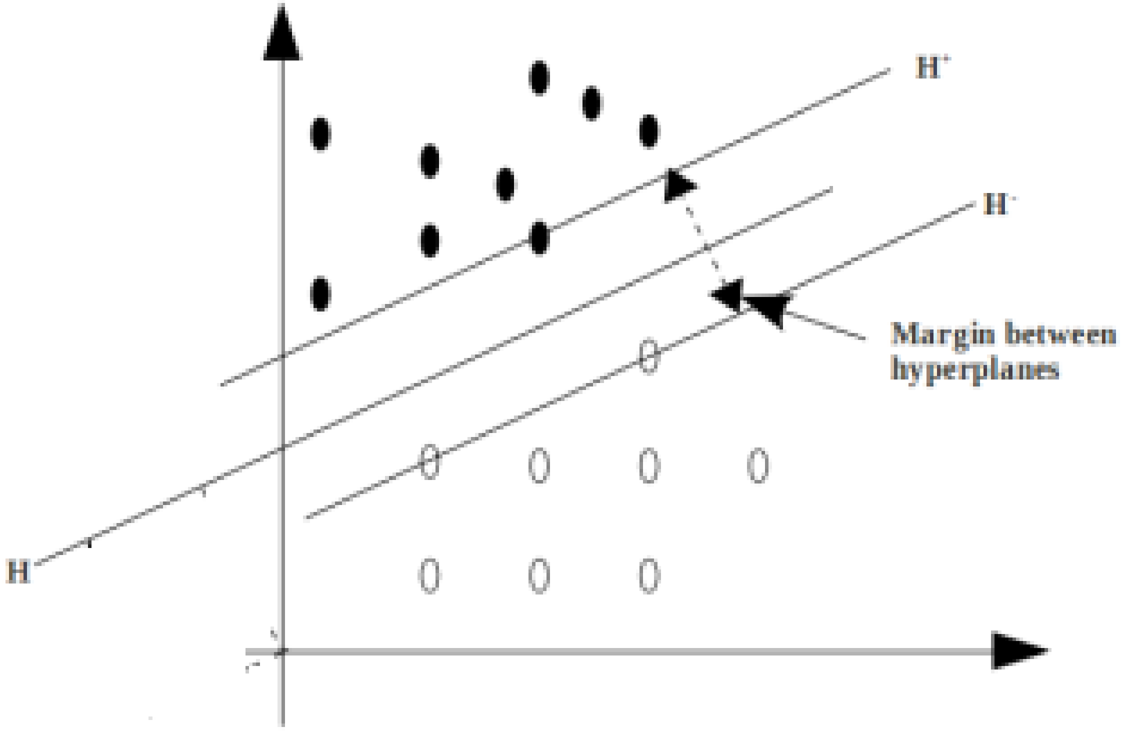


To define the expression for H+ and H-, let us make the following assumptions.

Given a datapoint $< \boldsymbol{x_i}, y_i >$ where $y_i \epsilon \{+ve, -ve\}$ is the class label of $\boldsymbol{x_i}$,

- let us replace **–ve by +1**, and **–ve by -1**.

Finding Separating Hyperplane with maximum Margin?



To define the expression for H+ and H-, let us make the following assumptions.

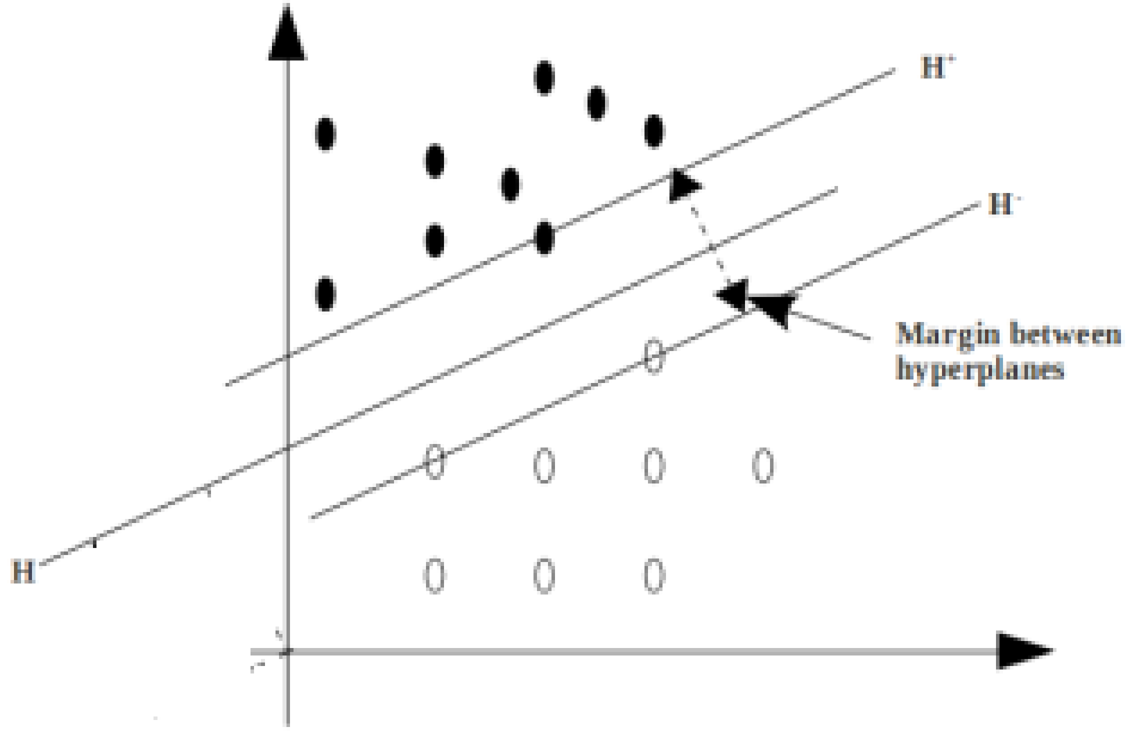
Given a datapoint < $\boldsymbol{x}_i, y_i$> where $y_i \epsilon \{+ve, -ve\}$ is the class label of $\boldsymbol{x}_i$,

- Let us replace **–ve by +1**, and **–ve by -1**.
- Now, each data point will satisfy the following

$$\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0 \geq 1, \ \forall\, y_i = +1$$

$$\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0 \leq -1, \ \forall\, y_i = -1$$

Finding Separating Hyperplane with maximum Margin?



To define the expression for H+ and H-, let us make the following assumptions.

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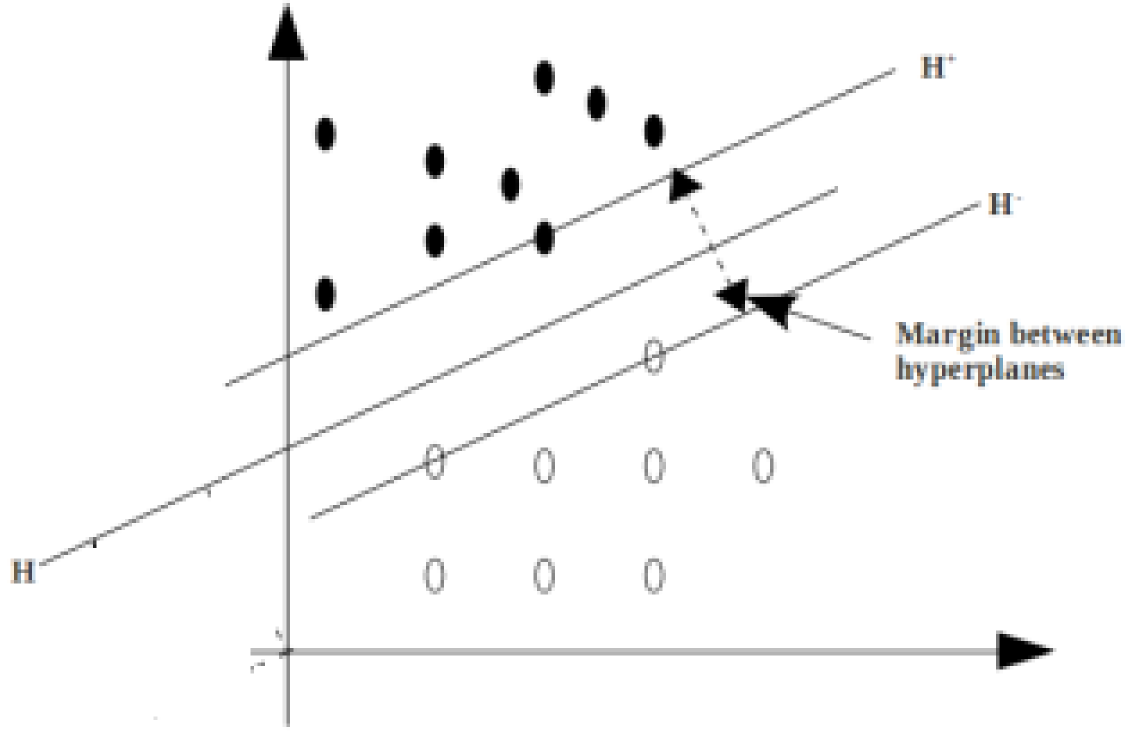
$$\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0 \geq 1, \ \forall\, y_i = +1$$

$$\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0 \leq -1, \ \forall\, y_i = -1$$

The above two expression can be merged to form a single expression

$$y_i(\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0) \geq 1, \ \forall\, \boldsymbol{x}_i$$

Finding Separating Hyperplane with maximum Margin?



To define the expression for H+ and H-, let us make the following assumptions.

Given a datapoint $< \boldsymbol{x}_i, y_i >$ where $y_i \epsilon \{+ve, -ve\}$ is the class label of $\boldsymbol{x}_i$,

- let us replace **–ve by +1**, and **–ve by -1**.
- Now, each data point will satisfy the following

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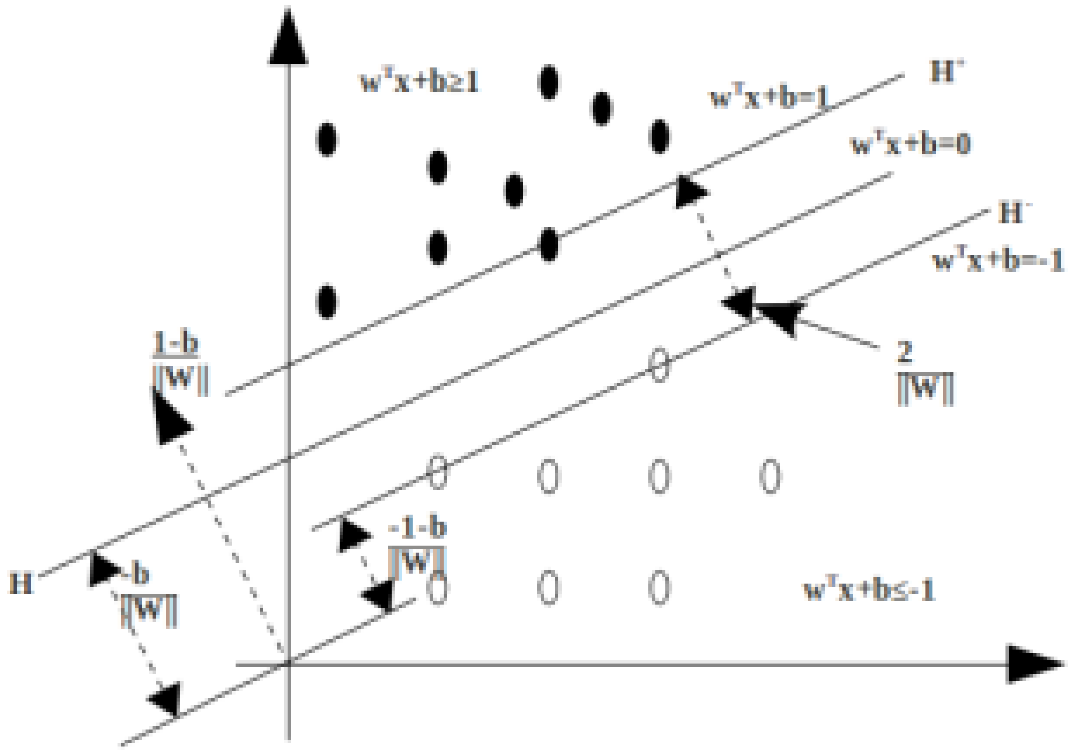
$$\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0 \leq -1, \ \forall \ y_i = -1$$

The above two expression can be merged to form a single expression

$$y_i(\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0) \geq 1, \ \forall \ \boldsymbol{x}_i$$

$$y_i(\boldsymbol{\omega}^T \boldsymbol{x}_i + \omega_0) - 1 \geq 0, \ \forall \ \boldsymbol{x}_i$$

Finding Separating Hyperplane with maximum Margin?

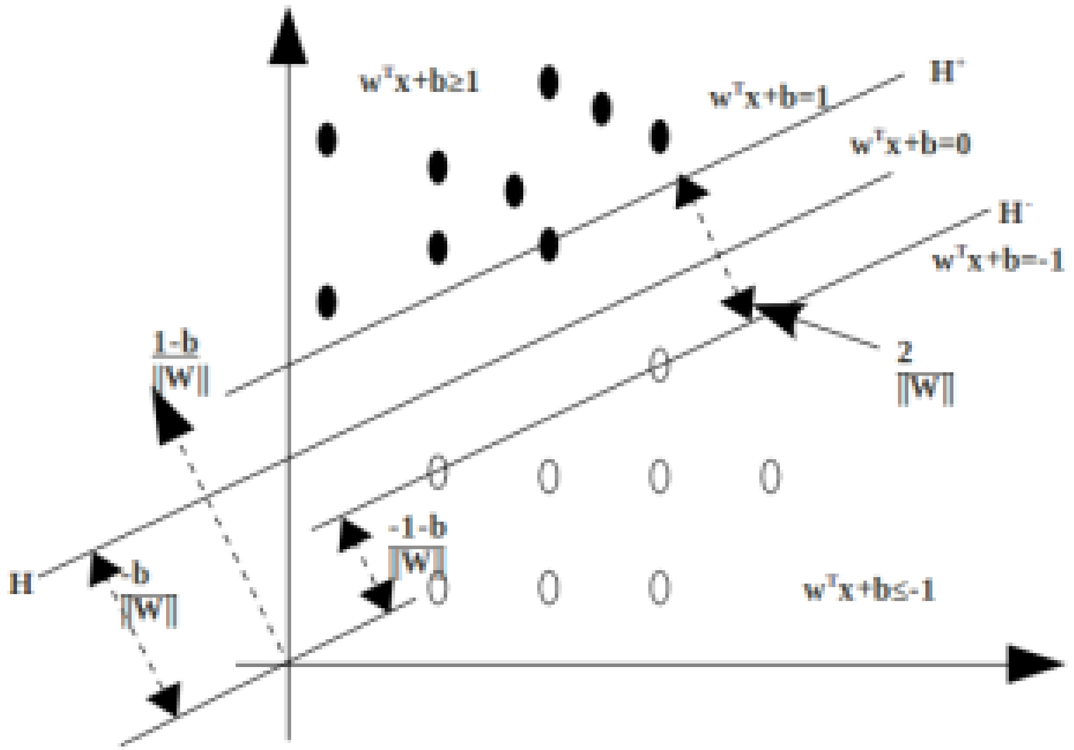


H: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = 0$

H+: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = 1$

H-: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = -1$

Finding Separating Hyperplane with maximum Margin?



H: $\boldsymbol{g(x)} = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_{\mathbf{0}} = 0$

H+: $\boldsymbol{g(x)} = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_{\mathbf{0}} = 1$

H-: $\boldsymbol{g(x)} = \boldsymbol{\omega}^T \boldsymbol{x} + \omega_{\mathbf{0}} = -1$

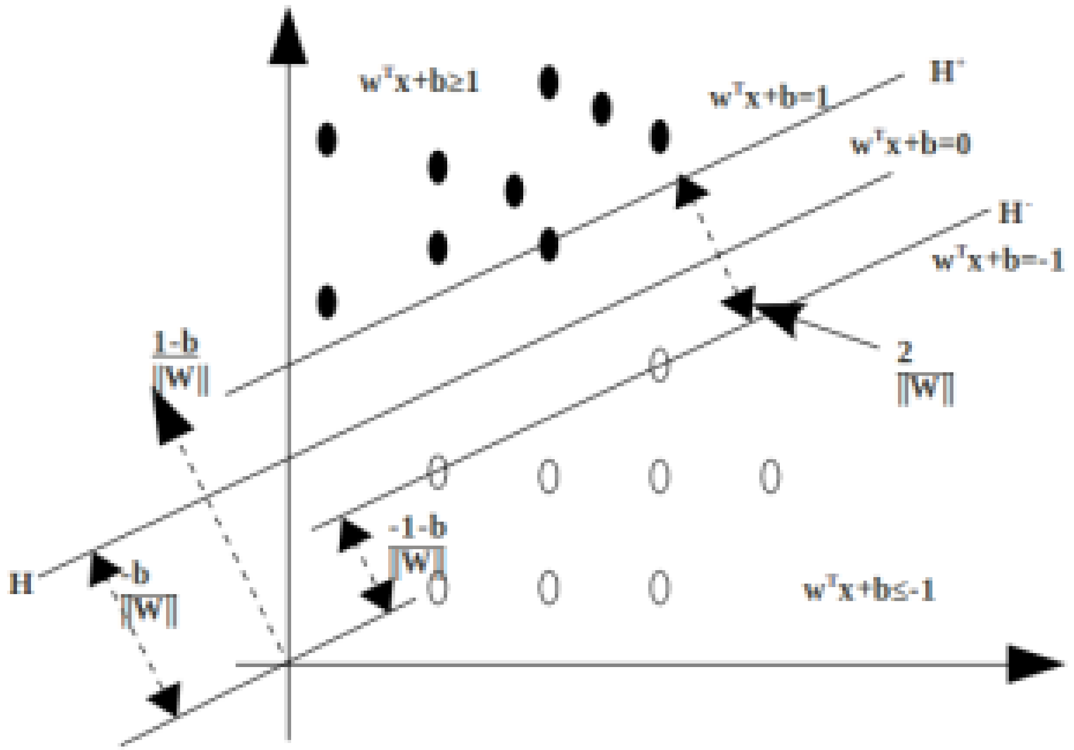
$$\text{Margin} = r_{H^+} - r_{H^-}$$

$$= \frac{\omega_0 - 1}{\|\boldsymbol{\omega}\|} - \frac{\omega_0 + 1}{\|\boldsymbol{\omega}\|}$$

$$= \frac{\omega_0 - 1 - \omega_0 - 1}{\|\boldsymbol{\omega}\|}$$

$$\Rightarrow Margin = \frac{-2}{\|\boldsymbol{\omega}\|}$$

Finding Separating Hyperplane with maximum Margin?



H: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = 0$

H+: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = 1$

H-: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\omega}^T\boldsymbol{x} + \omega_{\mathbf{0}} = -1$

Margin $= r_{\boldsymbol{H}+} - r_{\boldsymbol{H}-}$

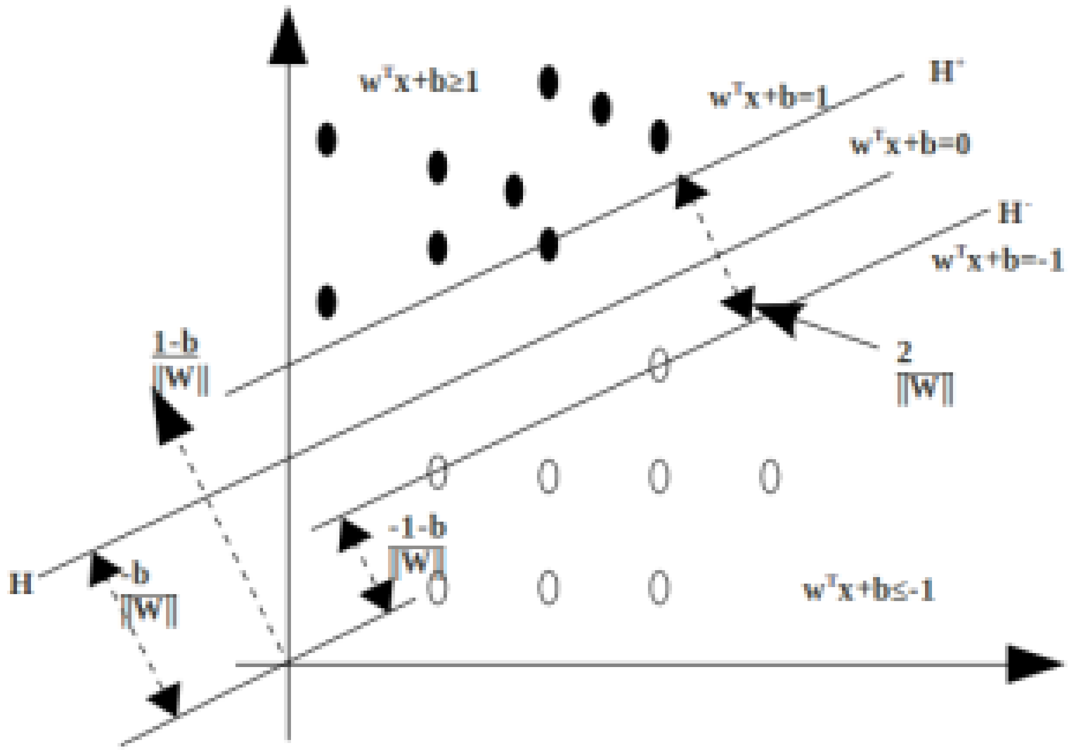
$$= \frac{\omega_0 - 1}{\|\boldsymbol{\omega}\|} - \frac{\omega_0 + 1}{\|\boldsymbol{\omega}\|}$$

$$= \frac{\omega_0 - 1 - \omega_0 - 1}{\|\boldsymbol{\omega}\|}$$

As we are interested in the absolute value, we consider the margin as $\frac{2}{\|\boldsymbol{\omega}\|}$

$$\Rightarrow Margin = \frac{-2}{\|\boldsymbol{\omega}\|}$$

Finding Separating Hyperplane with maximum Margin?

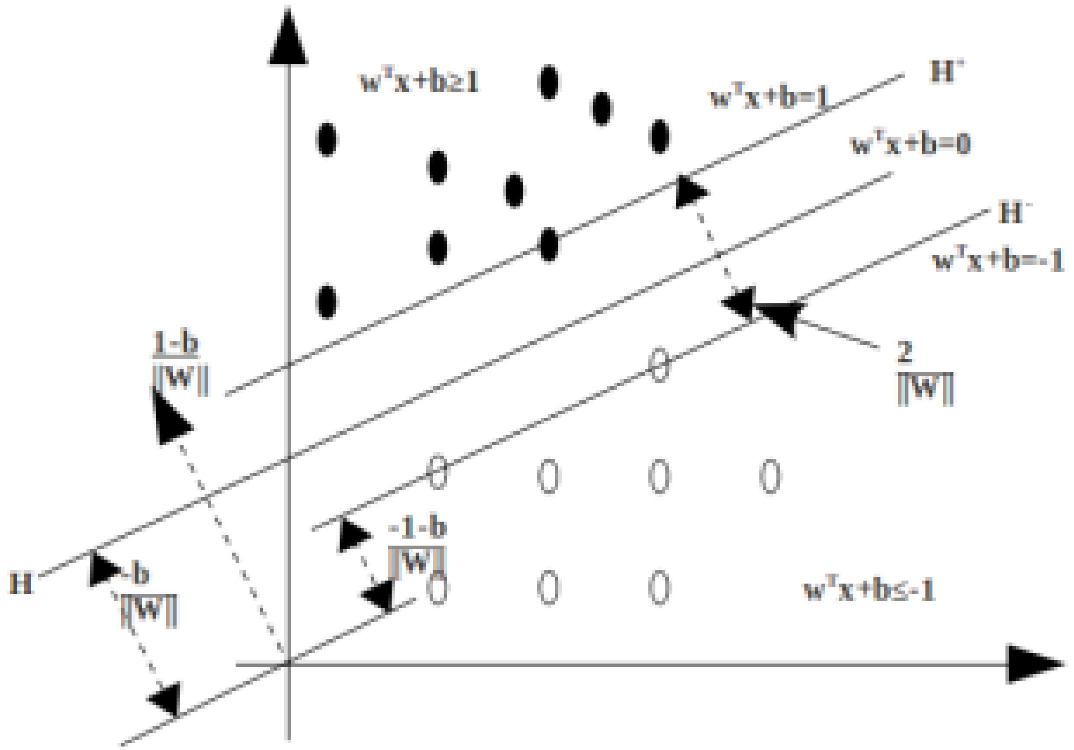


Task is to find the hyperplane (g(x)=0) that maximizes the margin $\frac{2}{\|\boldsymbol{\omega}\|}$

$$Margin = \frac{2}{\|\boldsymbol{\omega}\|}$$

$$\Rightarrow Margin = \frac{1}{\frac{\|\boldsymbol{\omega}\|}{2}}$$

Finding Separating Hyperplane with maximum Margin?



Task is to find the hyperplane (g(x)=0) that maximizes the margin $\frac{2}{\|\boldsymbol{\omega}\|}$

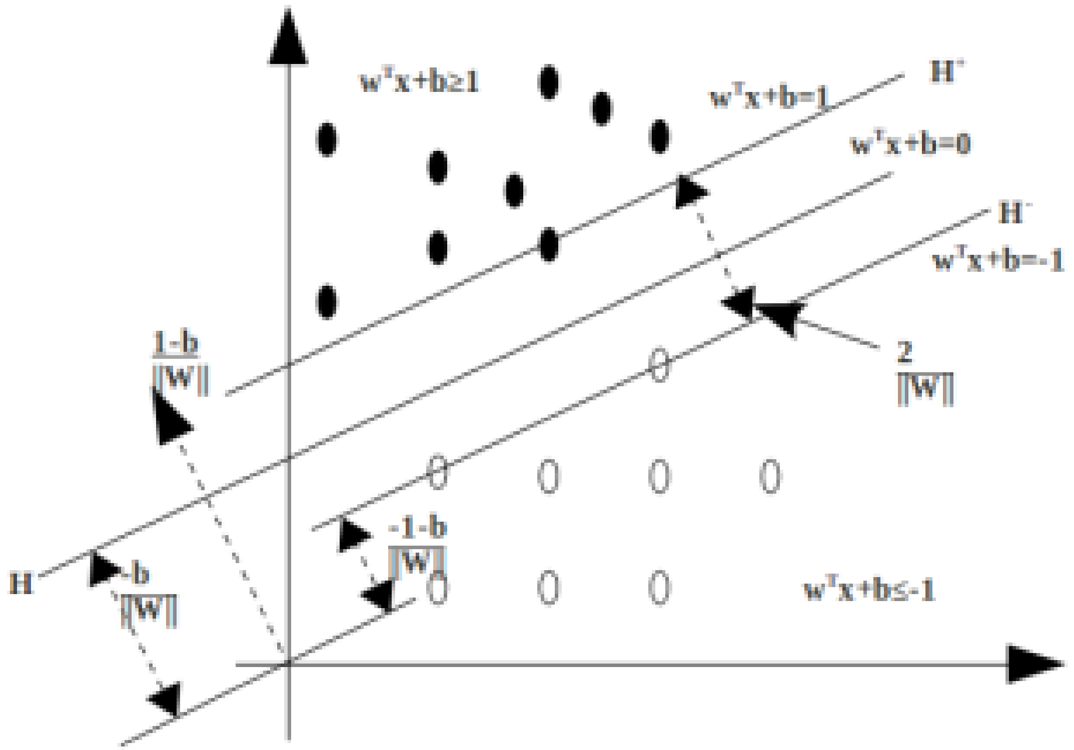
$$Margin = \frac{2}{\|\boldsymbol{\omega}\|}$$

$$\Rightarrow Margin = \frac{1}{\frac{\|\boldsymbol{\omega}\|}{2}}$$

Maximizing $\frac{2}{\|\boldsymbol{\omega}\|}$ is equivalent to minimizing $\frac{\|\boldsymbol{\omega}\|}{\mathbf{2}}$

Minimizing $\frac{\|\boldsymbol{\omega}\|}{\mathbf{2}}$ is equivalent to **minimizing** $\frac{\boldsymbol{\omega}^{T}\boldsymbol{\omega}}{\mathbf{2}}$

Finding Separating Hyperplane with maximum Margin?

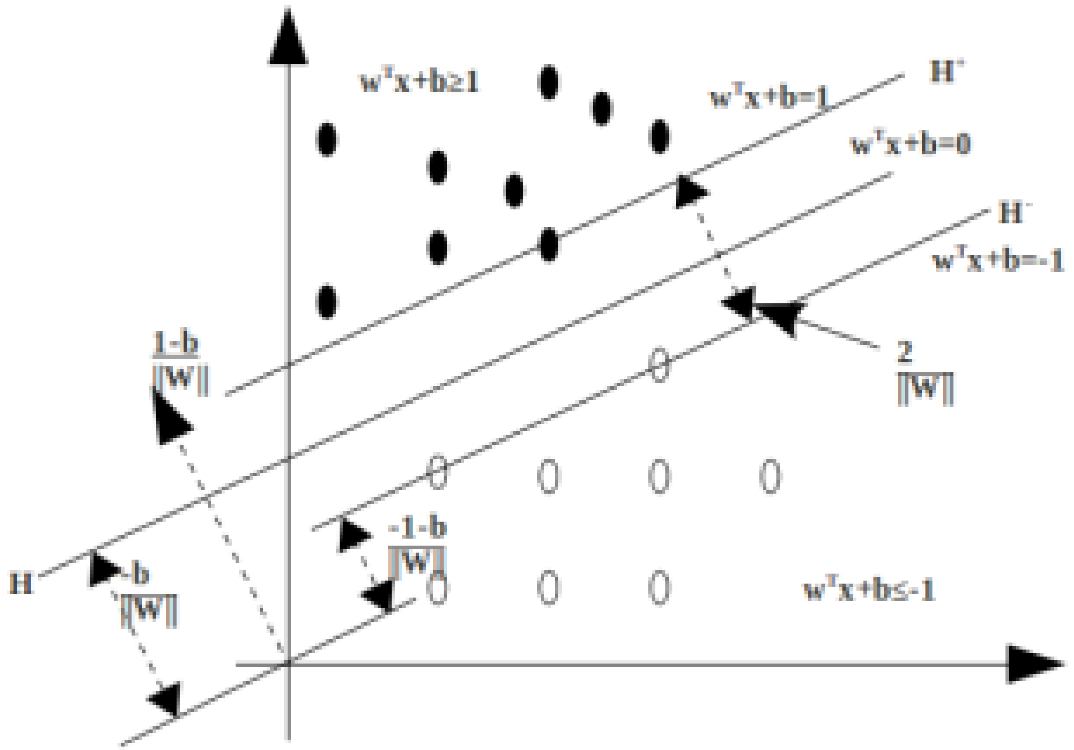


Now, we need to solve the following optimization

Minimize objective function $\frac{\omega^T \omega}{2}$

Subject to the constraint $y_i(\omega^T x_i + \omega_0) \geq 1, \ \forall \ x_i$

Finding Separating Hyperplane with maximum Margin?



Objective function is to **minimize** $\frac{\omega^T \omega}{2}$

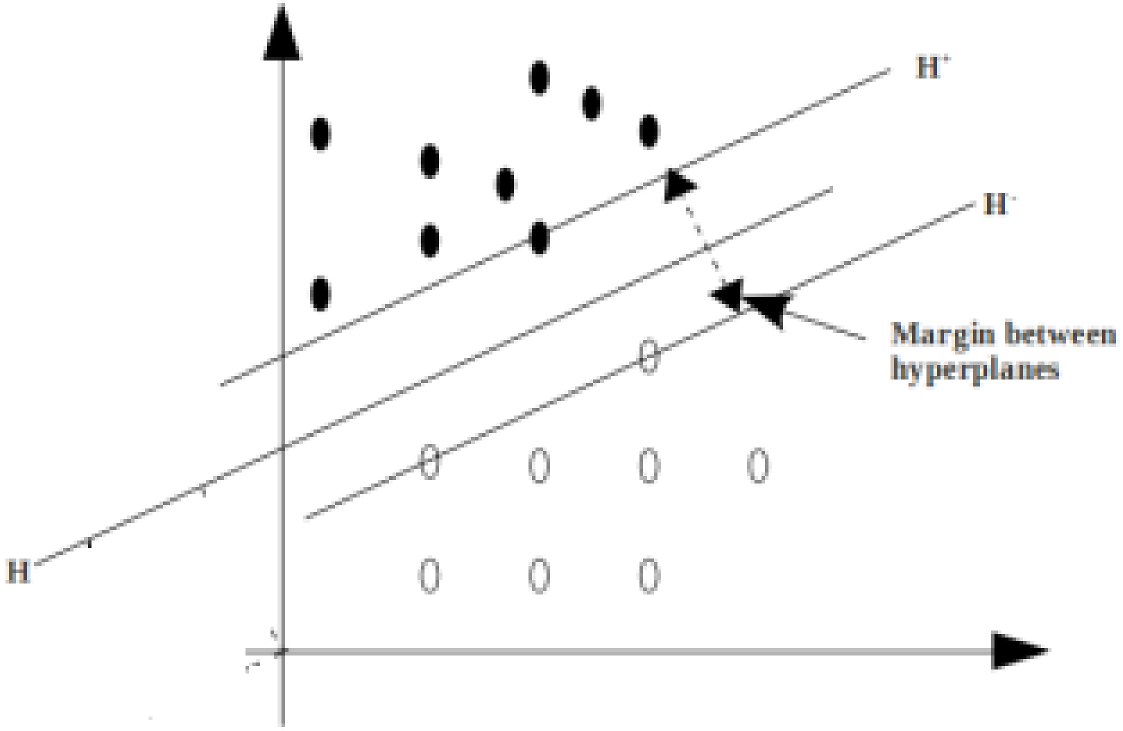
Subject to $y_i(\omega^T x_i + \omega_0) \geq 1, \ \forall \ x_i$

To find the parameters ω and where ω_0, solve the following optimization function

$$L_p = \frac{\omega^T \omega}{2} - \sum_{i=1}^{n} \lambda_i \left(y_i(\omega^T x_i + \omega_0) - 1 \right)$$

where λ_i are Lagrange multipliers

What are the support vectors?



$$L_p = \frac{\omega^T \omega}{2} - \sum_{i=1}^{n} \lambda_i \left(y_i (\omega^T x_i + \omega_0) - 1 \right)$$

[
In order to find the parameters, we need to solve this objective function.
]

Summary

- What is separating hyperplane?
- How to define separating hyperplane?
- What are Support Vector Machine?
- How to classify a new example using SVM