

Dynamic Shannon Coding

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Abstract—We present a new algorithm for dynamic prefix-free coding, based on Shannon coding. We give a simple analysis and prove a better upper bound on the length of the encoding produced than the corresponding bound for dynamic Huffman coding. We show how our algorithm can be modified for efficient length-restricted coding, alphabetic coding and coding with unequal letter costs.

Index Terms—Data compression, length-restricted codes, alphabetic codes, codes with unequal letter costs.

I. INTRODUCTION

Prefix-free coding is a well-studied problem in data compression and combinatorial optimization. For this problem, we are given a string $S = s_1 \dots s_m$ drawn from an alphabet of size n and must encode each character by a self-delimiting binary codeword. Our goal is to minimize the length of the entire encoding of S . For static prefix-free coding, we are given all of S before we start encoding and must encode every occurrence of the same character by the same codeword. The assignment of codewords to characters is recorded as a preface to the encoding. For dynamic prefix-free coding, we are given S character by character and must encode each character before receiving the next one. We can use a different codeword for different occurrences of the same character, we do not need a preface to the encoding and the assignment of codewords to characters cannot depend on the suffix of S not yet encoded.

The best-known algorithms for static coding are by Shannon [1] and Huffman [2]. Shannon's algorithm uses at most $(H + 1)m + O(n \log n)$ bits to encode S , where

$$H = \sum_{a \in S} \frac{\#_a(S)}{m} \log \left(\frac{m}{\#_a(S)} \right)$$

is the empirical entropy of S and $\#_a(S)$ is the number of occurrences of the character a in S . By \log we mean \log_2 . Shannon proved a lower bound of Hm bits for all coding algorithms, whether or not they are prefix-free. Huffman's algorithm produces an encoding that, excluding the preface, has minimum length. The total length is $(H+r)m+O(n \log n)$ bits, where $0 \leq r < 1$ is a function of the character frequencies in S [3].

Both algorithms assign codewords to characters by constructing a *code-tree*, that is, a binary tree whose left and right edges are labelled by 0's and 1's, respectively, and whose leaves are labelled by the distinct characters in S . The codeword assigned to a character a in S is the sequence of edge labels on the path from the root to the leaf labelled a . Shannon's algorithm builds a code-tree in which, for $a \in S$,

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the leaf labelled a is of depth at most $\lceil \log(m/\#_a(S)) \rceil$. Huffman's algorithm builds a Huffman tree for the frequencies of the characters in S . A *Huffman tree* for a sequence of weights w_1, \dots, w_n is a binary tree whose leaves, in some order, have weights w_1, \dots, w_n and that, among all such trees, minimizes the weighted external path length. To build a Huffman tree for w_1, \dots, w_n , we start with n trees, each consisting of just a root. At each step, we make the two roots with smallest weights, w_i and w_j , into the children of a new root with weight $w_i + w_j$.

A *minimax tree* for a sequence of weights w_1, \dots, w_n is a binary tree whose leaves, in some order, have weights w_1, \dots, w_n and that, among all such trees, minimizes the maximum sum of any leaf's weight and depth. Golumbic [4] gave an algorithm, similar to Huffman's, for constructing a minimax tree. The difference is that, when we make the two roots with smallest weights, w_i and w_j , into the children of a new root, that new root has weight $\max(w_i, w_j) + 1$ instead of $w_i + w_j$. Notice that, if there exists a binary tree whose leaves, in some order, have depths d_1, \dots, d_n , then a minimax tree T for $-d_1, \dots, -d_n$ is such a tree and, more generally, the depth of each node in T is bounded above by the negative of its weight. So we can construct a code-tree for Shannon's algorithm by running Golumbic's algorithm, starting with roots labelled by the distinct characters in S , with the root labelled a having weight $-\lceil \log(m/\#_a(S)) \rceil$.

Both Shannon's algorithm and Huffman's algorithm have three phases: a first pass over S to count the occurrences of each distinct character, an assignment of codewords to the distinct characters in S (recorded as a preface to the encoding) and a second pass over S to encode each character in S using the assigned codeword. The first phase takes $O(m)$ time, the second $O(n \log n)$ time and the third $O((H + 1)m)$ time.

For any static algorithm \mathcal{A} , there is a simple dynamic algorithm that recomputes the code-tree from scratch after reading each character. Specifically, for $i = 1 \dots m$:

- 1) We keep a running count of the number of occurrences of each distinct character in the current prefix $s_1 \dots s_{i-1}$ of S .
- 2) We compute the assignment of codewords to characters that would result from applying \mathcal{A} to $\perp s_1 \dots s_{i-1}$, where \perp is a special character not in the alphabet.
- 3) If s_i occurs in $s_1 \dots s_{i-1}$, then we encode s_i as the codeword c_i assigned to that character.
- 4) If s_i does not occur in $s_1 \dots s_{i-1}$, then we encode s_i as the concatenation c_i of the codeword assigned to \perp and the binary representation of s_i 's index in the alphabet.

We can later decode character by character. That is, we can recover $s_1 \dots s_i$ as soon as we have received $c_1 \dots c_i$. To see why, assume that we have recovered $s_1 \dots s_{i-1}$. Then we can compute the assignment of codewords to characters

that \mathcal{A} used to encode s_i . Since \mathcal{A} is prefix-free, c_i is the only codeword in this assignment that is a prefix of $c_i \cdots c_m$. Thus, we can recover s_i as soon as c_i has been received. This takes the same amount of time as encoding s_i .

Faller [5] and Gallager [6] independently gave a dynamic coding algorithm based on Huffman's algorithm. Their algorithm is similar to, but much faster than, the simple dynamic algorithm obtained by adapting Huffman's algorithm as described above. After encoding each character of S , their algorithm merely updates the Huffman tree rather than rebuilding it from scratch. Knuth [7] implemented their algorithm so that it uses time proportional to the length of the encoding produced. For this reason, it is sometimes known as Faller-Gallager-Knuth coding; however, it is most often called *dynamic Huffman coding*. Milidiú, Laber, and Pessoa [8] showed that this version of dynamic Huffman coding uses fewer than $2m$ more bits to encode S than Huffman's algorithm. Vitter [9] gave an improved version that he showed uses fewer than m more bits than Huffman's algorithm. These results imply Knuth's and Vitter's versions use at most $(H + 2 + r)m + O(n \log n)$ and $(H + 1 + r)m + O(n \log n)$ bits to encode S , but it is not clear whether these bounds are tight. Both algorithms use $O((H + 1)m)$ time.

In this paper, we present a new dynamic algorithm, *dynamic Shannon coding*. In Section II, we show that the simple dynamic algorithm obtained by adapting Shannon's algorithm as described above, uses at most $(H + 1)m + O(n \log m)$ bits and $O(mn \log n)$ time to encode S . Section III contains our main result, an improved version of dynamic Shannon coding that uses at most $(H + 1)m + O(n \log m)$ bits to encode S and only $O((H + 1)m + n \log^2 m)$ time. The relationship between Shannon's algorithm and this algorithm is similar to that between Huffman's algorithm and dynamic Huffman coding, but our algorithm is much simpler to analyze than dynamic Huffman coding.

In Section IV, we show that dynamic Shannon coding can be applied to three related problems. We give algorithms for dynamic length-restricted coding, dynamic alphabetic coding and dynamic coding with unequal letter costs. Our algorithms have better bounds on the length of the encoding produced than were previously known. For length-restricted coding, no codeword can exceed a given length. For alphabetic coding, the lexicographic order of the codewords must be the same as that of the characters.

Throughout, we make the common simplifying assumption that $m \geq n$. Our model of computation is the unit-cost word RAM with $\Omega(\log m)$ -bit words. In this model, ignoring space required for the input and output, all the algorithms mentioned in this paper use $O(|\{a : a \in S\}|)$ words, that is, space proportional to the number of distinct characters in S .

II. ANALYSIS OF SIMPLE DYNAMIC SHANNON CODING

In this section, we analyze the simple dynamic algorithm obtained by repeating Shannon's algorithm after each character of the string $\perp s_1 \cdots s_m$, as described in the introduction. Since the second phase of Shannon's algorithm, assigning codewords to characters, takes $O(n \log n)$ time, this simple algorithm uses

$O(mn \log n)$ time to encode S . The rest of this section shows this algorithm uses at most $(H + 1)m + O(n \log m)$ bits to encode S .

For $1 \leq i \leq m$ and each distinct character a that occurs in $\perp s_1 \cdots s_{i-1}$, Shannon's algorithm on $\perp s_1 \cdots s_{i-1}$ assigns to a a codeword of length at most $\lceil \log(i/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$. This fact is key to our analysis.

Let R be the set of indices i such that s_i is a repetition of a character in $s_1 \cdots s_{i-1}$. That is, $R = \{i : 1 \leq i \leq m, s_i \in \{s_1, \dots, s_{i-1}\}\}$. Our analysis depends on the following technical lemma.

Lemma 1:

$$\sum_{i \in R} \log \left(\frac{i}{\#_{s_i}(s_1 \cdots s_{i-1})} \right) \leq Hm + O(n \log m).$$

Proof: Let

$$L = \sum_{i \in R} \log \left(\frac{i}{\#_{s_i}(s_1 \cdots s_{i-1})} \right).$$

Notice that $\sum_{i \in R} \log i < \sum_{i=1}^m \log i = \log(m!)$. Also, for $i \in R$, if s_i is the j th occurrence of a in S , for some $j \geq 2$, then $\log \#_{s_i}(s_1 \cdots s_{i-1}) = \log(j - 1)$. Thus,

$$\begin{aligned} L &= \sum_{i \in R} \log i - \sum_{i \in R} \log \#_{s_i}(s_1 \cdots s_{i-1}) \\ &< \log(m!) - \sum_{a \in S} \sum_{j=2}^{\#_a(S)} \log(j - 1) \\ &= \log(m!) - \sum_{a \in S} \log(\#_a(S)!) + \sum_{a \in S} \log \#_a(S). \end{aligned}$$

There are at most n distinct characters in S and each occurs at most m times, so $\sum_{a \in S} \log \#_a(S) \in O(n \log m)$. By Stirling's Formula,

$$x \log x - x \ln 2 < \log(x!) \leq x \log x - x \ln 2 + O(\log x).$$

Thus,

$$\begin{aligned} L &< m \log m - m \ln 2 - \\ &\quad \sum_{a \in S} (\#_a(S) \log \#_a(S) - \#_a(S) \ln 2) + O(n \log m). \end{aligned}$$

Since $\sum_{a \in S} \#_a(S) = m$,

$$L < \sum_{a \in S} \#_a(S) \log \left(\frac{m}{\#_a(S)} \right) + O(n \log m).$$

By definition, this is $Hm + O(n \log m)$. ■

As an aside, we note $\sum_{a \in S} \log \#_a(S) \in o((H + 1)m)$; to see why, compare corresponding terms in $\sum_{a \in S} \log \#_a(S)$ and the expansion

$$(H + 1)m = \sum_{a \in S} \#_a(S) \left(\log \left(\frac{m}{\#_a(S)} \right) + 1 \right).$$

Using Lemma 1, it is easy to bound the number of bits that simple dynamic Shannon coding uses to encode S .

Theorem 2: Simple dynamic Shannon coding uses at most $(H + 1)m + O(n \log m)$ bits to encode S .

Proof: If s_i is the first occurrence of that character in S (i.e., $i \in \{1, \dots, m\} - R$), then the algorithm encodes s_i as the

codeword for \perp , which is at most $\lceil \log m \rceil$ bits, followed by the binary representation of s_i 's index in the alphabet, which is $\lceil \log n \rceil$ bits. Since there are at most n such characters, the algorithm encodes them all using $O(n \log m)$ bits.

Now, consider the remaining characters in S , that is, those characters whose indices are in R . In total, the algorithm encodes these using at most

$$\begin{aligned} & \sum_{i \in R} \left\lceil \log \left(\frac{i}{\#_{s_i}(\perp s_1 \cdots s_{i-1})} \right) \right\rceil \\ & < m + \sum_{i \in R} \log \left(\frac{i}{\#_{s_i}(s_1 \cdots s_{i-1})} \right) \end{aligned}$$

bits. By Lemma 1, this is at most $(H+1)m + O(n \log m)$.

Therefore, in total, this algorithm uses at most $(H+1)m + O(n \log m)$ bits to encode S . \blacksquare

III. DYNAMIC SHANNON CODING

This section explains how to improve simple dynamic Shannon coding so that it uses at most $(H+1)m + O(n \log m)$ bits and $O((H+1)m + n \log^2 m)$ time to encode the string $S = s_1 \cdots s_m$. The main ideas for this algorithm are using a dynamic minimax tree to store the code-tree, introducing “slack” in the weights and using background processing to keep the weights updated.

Gagie [10] showed that Faller’s, Gallager’s and Knuth’s techniques for making Huffman trees dynamic can be used to make minimax trees dynamic. A *dynamic minimax tree* T supports the following operations:

- given a pointer to a node v , return v ’s parent, left child, and right child (if they exist);
- given a pointer to a leaf v , return v ’s weight;
- given a pointer to a leaf v , increment v ’s weight;
- given a pointer to a leaf v , decrement v ’s weight;
- and, given a pointer to a leaf v , insert a new leaf with the same weight as v .

In Gagie’s implementation, if the depth of each node is bounded above by the negative of its weight, then each operation on a leaf with weight $-d_i$ takes $O(d_i)$ time. Next, we will show how to use this data structure for fast dynamic Shannon coding.

We maintain the invariant that, after we encode $s_1 \cdots s_{i-1}$, T has one leaf labelled a for each distinct character a in $\perp s_1 \cdots s_{i-1}$ and this leaf has weight between $-\lceil \log((i+n)/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$ and $-\lceil \log(\max(i,n)/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$. Notice that applying Shannon’s algorithm to $\perp s_1 \cdots s_{i-1}$ results in a code-tree in which, for $a \in \perp s_1 \cdots s_{i-1}$, the leaf labelled a is of depth at most $\lceil \log(i/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$. It follows that the depth of each node in T is bounded above by the negative of its weight.

Notice that, instead of having just i in the numerator, as we would for simple dynamic Shannon coding, we have at most $i+n$. Thus, this algorithm may assign slightly longer codewords to some characters. We allow this “slack” so that, after we encode each character, we only need to update the weights of at most two leaves. In the analysis, we will show

that the extra n only affects low-order terms in the bound on the length of the encoding.

After we encode s_i , we ensure that T contains one leaf labelled s_i and this leaf has weight $-\lceil \log((i+1+n)/\#_{s_i}(\perp s_1 \cdots s_i)) \rceil$. First, if s_i is the first occurrence of that distinct character in S (i.e., $i \in \{1, \dots, m\} - R$), then we insert a new leaf labelled s_i into T with the same weight as the leaf labelled \perp . Next, we update the weight of the leaf labelled s_i . We consider this processing to be in the foreground.

In the background, we use a queue to cycle through the distinct characters that have occurred in the current prefix. For each character that we encode in the foreground, we process one character in the background. When we dequeue a character a , if we have encoded precisely $s_1 \cdots s_i$, then we update the weight of the leaf labelled a to be $-\lceil \log((i+1+n)/\#_a(\perp s_1 \cdots s_i)) \rceil$, unless it has this weight already. Since there are always at most $n+1$ distinct characters in the current prefix (\perp and the n characters in the alphabet), this maintains the following invariant: For $1 \leq i \leq m$ and $a \in \perp s_1 \cdots s_{i-1}$, immediately after we encode $s_1 \cdots s_{i-1}$, the leaf labelled a has weight between $-\lceil \log((i+n)/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$ and $-\lceil \log(\max(i,n)/\#_a(\perp s_1 \cdots s_{i-1})) \rceil$. Notice that $\max(i,n) < i+n \leq 2\max(i,n)$ and $\#_a(s_1 \cdots s_{i-1}) \leq \#_a(\perp s_1 \cdots s_i) \leq 2\#_a(s_1 \cdots s_{i-1}) + 1$. Also, if s_i is the first occurrence of that distinct character in S , then $\#_{s_i}(\perp s_1 \cdots s_i) = \#_\perp(\perp s_1 \cdots s_{i-1})$. It follows that, whenever we update a weight, we use at most one increment or decrement.

Our analysis of this algorithm is similar to that in Section II, with two differences. First, we show that weakening the bound on codeword lengths does not significantly affect the bound on the length of the encoding. Second, we show that our algorithm only takes $O((H+1)m + n \log^2 m)$ time. Our analysis depends on the following technical lemma.

Lemma 3: Suppose $I \subseteq \mathbb{Z}^+$ and $|I| \geq n$. Then

$$\sum_{i \in I} \log \left(\frac{i+n}{x_i} \right) \leq \sum_{i \in I} \log \left(\frac{i}{x_i} \right) + n \log(\max I + n).$$

Proof: Let

$$L = \sum_{i \in I} \log \left(\frac{i+n}{x_i} \right) = \sum_{i \in I} \log \left(\frac{i}{x_i} \right) + \sum_{i \in I} \log \left(\frac{i+n}{i} \right).$$

Let $i_1, \dots, i_{|I|}$ be the elements of I , with $0 < i_1 < \dots < i_{|I|}$. Then $i_j + n \leq i_{j+n}$, so

$$\begin{aligned} & \sum_{i \in I} \log \left(\frac{i+n}{i} \right) \\ &= \log \left(\frac{\left(\prod_{j=1}^{|I|-n} (i_j + n) \right) \left(\prod_{j=|I|-n+1}^{|I|} (i_j + n) \right)}{\left(\prod_{j=1}^n i_j \right) \left(\prod_{j=n+1}^{|I|} i_j \right)} \right) \\ &\leq \log \left(\frac{\left(\prod_{j=1}^{|I|-n} i_{j+n} \right) (\max I + n)^n}{1 \cdot \prod_{j=1}^{|I|-n} i_{j+n}} \right) \\ &= n \log(\max I + n). \end{aligned}$$

Therefore,

$$L \leq \sum_{i \in I} \log \left(\frac{i}{x_i} \right) + n \log(\max I + n).$$

■

Using Lemmas 1 and 3, it is easy to bound the number of bits and the time dynamic Shannon coding uses to encode S , as follows.

Theorem 4: Dynamic Shannon coding uses at most $(H + 1)m + O(n \log m)$ bits and $O((H + 1)m + n \log^2 m)$ time.

Proof: First, we consider the length of the encoding produced. Notice that the algorithm encodes S using at most

$$\begin{aligned} & \sum_{i \in R} \left\lceil \log \left(\frac{i+n}{\#_{s_i}(\perp s_1 \cdots s_{i-1})} \right) \right\rceil + O(n \log m) \\ & \leq m + \sum_{i \in R} \log \left(\frac{i+n}{\#_{s_i}(s_1 \cdots s_{i-1})} \right) + O(n \log m) \end{aligned}$$

bits. By Lemmas 1 and 3, this is at most $(H + 1)m + O(n \log m)$.

Now, we consider how long this algorithm takes. We will prove separate bounds on the processing done in the foreground and in the background.

If s_i is the first occurrence of that character in S (i.e., $i \in \{1, \dots, m\} - R$), then we perform three operations in the foreground when we encode s_i : we output the codeword for \perp , which is at most $\lceil \log(i+n) \rceil$ bits; we output the index of s_i in the alphabet, which is $\lceil \log n \rceil$ bits; and we insert a new leaf labelled s_i and update its weight to be $-\lceil \log(i+1+n) \rceil$. In total, these take $O(\log(i+n)) \subseteq O(\log m)$ time. Since there are at most n such characters, the algorithm encodes them all using $O(n \log m)$ time.

For $i \in R$, we perform at most two operations in the foreground when we encode s_i : we output the codeword for s_i , which is of length at most $\lceil \log((i+n)/\#_{s_i}(s_1 \cdots s_{i-1})) \rceil$; and, if necessary, we increment the weight of the leaf labelled s_i . In total, these take $O(\log((i+n)/\#_{s_i}(s_1 \cdots s_{i-1})))$ time.

For $1 \leq i \leq m$, we perform at most two operations in the background when we encode s_i : we dequeue a character a ; if necessary, decrement the weight of the leaf labelled a ; and re-enqueue a . These take $O(1)$ time if we do not decrement the weight of the leaf labelled a and $O(\log m)$ time if we do.

Suppose s_i is the first occurrence of that distinct character in S . Then the leaf v labelled s_i is inserted into T with weight $-\lceil \log(i+n) \rceil$. Also, v 's weight is never less than $-\lceil \log(m+1+n) \rceil$. Since decrementing v 's weight from w to $w-1$ or incrementing v 's weight from $w-1$ to w both take $O(-w)$ time, we spend the same amount of time decrementing v 's weight in the background as we do incrementing it in the foreground, except possibly for the time to decrease v 's weight from $-\lceil \log(i+n) \rceil$ to $-\lceil \log(m+1+n) \rceil$. Thus, we spend $O(\log^2 m)$ more time decrementing v 's weight than we do incrementing it. Since there are at most n distinct characters in S , in total, this algorithm takes

$$\sum_{i \in R} O \left(\log \left(\frac{i+n}{\#_{s_i}(s_1 \cdots s_{i-1})} \right) \right) + O(n \log^2 m)$$

time. It follows from Lemmas 1 and 3 that this is $O((H + 1)m + n \log^2 m)$. ■

IV. VARIATIONS ON DYNAMIC SHANNON CODING

In this section, we show how to implement efficiently variations of dynamic Shannon coding for dynamic length-restricted coding, dynamic alphabetic coding and dynamic coding with unequal letter costs. Abrahams [11] surveys static algorithms for these and similar problems, but there has been relatively little work on dynamic algorithms for these problems.

We use dynamic minimax trees for length-restricted dynamic Shannon coding. For alphabetic dynamic Shannon coding, we dynamize Melhorn's version of Shannon's algorithm. For dynamic Shannon coding with unequal letter costs, we dynamize Krause's version.

A. Length-Restricted Dynamic Shannon Coding

For length-restricted coding, we are given a bound and cannot use a codeword whose length exceeds this bound. Length-restricted coding is useful, for example, for ensuring that each codeword fits in one machine word. Liddell and Moffat [12] gave a length-restricted dynamic coding algorithm that works well in practice, but it is quite complicated and they did not prove bounds on the length of the encoding it produces. We show how to length-restrict dynamic Shannon coding without significantly increasing the bound on the length of the encoding produced.

Theorem 5: For any fixed integer $\ell \geq 1$, dynamic Shannon coding can be adapted so that it uses at most $2\lceil \log n \rceil + \ell$ bits to encode the first occurrence of each distinct character in S , at most $\lceil \log n \rceil + \ell$ bits to encode each remaining character in S , at most $(H + 1 + \frac{1}{(2^\ell-1)\ln 2})m + O(n \log m)$ bits in total, and $O((H + 1)m + n \log^2 m)$ time.

Proof: We modify the algorithm presented in Section III by removing the leaf labelled \perp after all of the characters in the alphabet have occurred in S , and changing how we calculate weights for the dynamic minimax tree. Whenever we would use a weight of the form $-\lceil \log x \rceil$, we smooth it by instead using

$$\begin{aligned} & - \left\lceil \log \left(\frac{2^\ell}{(2^\ell-1)/x + 1/n} \right) \right\rceil \\ & \geq - \min \left(\left\lceil \log \left(\frac{2^\ell x}{2^\ell-1} \right) \right\rceil, \lceil \log n \rceil + \ell \right). \end{aligned}$$

With these modifications, no leaf in the minimax tree is ever of depth greater than $\lceil \log n \rceil + \ell$. Since

$$\begin{aligned} \left\lceil \log \left(\frac{2^\ell x}{2^\ell-1} \right) \right\rceil & < \log x + 1 + \frac{\log \left(1 + \frac{1}{2^\ell-1} \right)^{2^\ell-1}}{2^\ell-1} \\ & < \log x + 1 + \frac{1}{(2^\ell-1)\ln 2}, \end{aligned}$$

essentially the same analysis as for Theorem 4 shows this algorithm uses at most $(H + 1 + \frac{1}{(2^\ell-1)\ln 2})m + O(n \log m)$ bits in total, and $O((H + 1)m + n \log^2 m)$ time. ■

It is straightforward to prove a similar theorem in which the number of bits used to encode s_i with $i \in R$ is bounded above by $\lceil \log(|\{a : a \in S\}| + 1) \rceil + \ell + 1$ instead of $\lceil \log n \rceil + \ell$. That is, we can make the bound in terms of the number of distinct characters in S instead of the size of the alphabet. To do this, we modify the algorithm again so that it stores a counter n_i of the number of distinct characters that have occurred in the current prefix. Whenever we would use n in a formula to calculate a weight, we use $2(n_i + 1)$ instead.

B. Alphabetic Dynamic Shannon Coding

For alphabetic coding, the lexicographic order of the codewords must always be the same as the lexicographic order of the characters to which they are assigned. Alphabetic coding is useful, for example, because we can compare encoded strings without decoding them. Although there is an alphabetic version of minimax trees [13], it cannot be efficiently dynamized [10]. Mehlhorn [14] generalized Shannon's algorithm to obtain an algorithm for alphabetic coding. In this section, we dynamize Mehlhorn's algorithm.

Theorem 6 (Mehlhorn, 1977): There exists an alphabetic prefix-free code such that, for each character a in the alphabet, the codeword for a is of length $\lceil \log((m+n)/\#_a(S)) \rceil + 1$.

Proof: Let a_1, \dots, a_n be the characters in the alphabet in lexicographic order. For $1 \leq i \leq n$, let

$$f(a_i) = \frac{\#_{a_i}(S) + 1}{2(m+n)} + \sum_{j=1}^{i-1} \frac{\#_{a_j}(S) + 1}{m+n} < 1.$$

For $1 \leq i \neq i' \leq n$, notice that $|f(a_i) - f(a_{i'})| \geq \frac{\#_{a_i}(S)+1}{2(m+n)}$. Therefore, the first $\lceil \log\left(\frac{m+n}{\#_{a_i}(S)+1}\right) \rceil + 1$ bits of the binary representation of $f(a_i)$ suffice to distinguish it. Let this sequence of bits be the codeword for a_i . ■

Repeating Mehlhorn's algorithm after each character of S , as described in the introduction, is a simple algorithm for alphabetic dynamic Shannon coding. Notice that we always assign a codeword to every character in the alphabet; thus, we do not need to prepend \perp to the current prefix of S . This algorithm uses at most $(H+2)m + O(n \log m)$ bits and $O(mn)$ time to encode S .

To make this algorithm more efficient, after encoding each character of S , instead of computing an entire code-tree, we only compute the codeword for the next character in S . We use an augmented splay tree [15] to compute the necessary partial sums.

Theorem 7: Alphabetic dynamic Shannon coding uses $(H+2)m + O(n \log m)$ bits and $O((H+1)m)$ time.

Proof: We keep an augmented splay tree T and maintain the invariant that, after encoding $s_1 \dots s_{i-1}$, there is a node v_a in T for each distinct character a in $s_1 \dots s_{i-1}$. The node v_a 's key is a ; it stores a 's frequency in $s_1 \dots s_{i-1}$ and the sum of the frequencies of the characters in v_a 's subtree in T .

To encode s_i , we use T to compute the partial sum

$$\frac{\#_{s_i}(s_1 \dots s_{i-1})}{2} + \sum_{a_j < s_i} \#_{a_j}(s_1 \dots s_{i-1}),$$

where $a_j < s_i$ means that a_j is lexicographically less than s_i . From this, we compute the codeword for s_i , that is, the first $\lceil \log\left(\frac{i-1+n}{\#_{s_i}(s_1 \dots s_{i-1})+1}\right) \rceil + 1$ bits of the binary representation of

$$\frac{\#_{s_i}(s_1 \dots s_{i-1}) + 1}{2(i-1+n)} + \sum_{a_j < s_i} \frac{\#_{a_j}(s_1 \dots s_{i-1}) + 1}{i-1+n}.$$

If s_i is the first occurrence of that character in S (i.e., $i \in \{1, \dots, m\} - R$), then we insert a node v_{s_i} into T . In both cases, we update the information stored at the ancestors of v_{s_i} and splay v_{s_i} to the root.

Essentially the same analysis as for Theorem 4 shows this algorithm uses at most $(H+2)m + O(n \log m)$ bits. By the Static Optimality theorem [15], it uses $O((H+1)m)$ time. ■

C. Dynamic Shannon Coding with Unequal Letter Costs

It may be that one code letter costs more than another. For example, sending a dash by telegraph takes longer than sending a dot. Shannon [1] proved a lower bound of $Hm \ln(2)/C$ for all algorithms, whether prefix-free or not, where the channel capacity C is the largest real root of $e^{-\text{cost}(0) \cdot x} + e^{-\text{cost}(1) \cdot x} = 1$ and $e \approx 2.71$ is the base of the natural logarithm. Krause [16] generalized Shannon's algorithm for the case with unequal positive letter costs. In this section, we dynamize Krause's algorithm.

Theorem 8 (Krause, 1962): Suppose $\text{cost}(0)$ and $\text{cost}(1)$ are constants with $0 < \text{cost}(0) \leq \text{cost}(1)$. Then there exists a prefix-free code such that, for each character a in the alphabet, the codeword for a has cost less than $\frac{\ln(m/\#_a(S))}{C} + \text{cost}(1)$.

Proof: Let a_1, \dots, a_k be the characters in S in non-increasing order by frequency. For $1 \leq i \leq k$, let

$$f(a_i) = \sum_{j=1}^{i-1} \frac{\#_{a_j}(S)}{m} < 1.$$

Let $b(a_i)$ be the following binary string, where $x_0 = 0$ and $y_0 = 1$: For $j \geq 1$, if $f(a_i)$ is in the first $e^{-\text{cost}(0) \cdot C}$ fraction of the interval $[x_{j-1}, y_{j-1}]$, then the j th bit of $b(a_i)$ is 0 and x_j and y_j are such that $[x_j, y_j]$ is the first $e^{-\text{cost}(0) \cdot C}$ fraction of $[x_{j-1}, y_{j-1}]$. Otherwise, the j th bit of $b(a_i)$ is 1 and x_j and y_j are such that $[x_j, y_j]$ is the last $e^{-\text{cost}(1) \cdot C}$ fraction of $[x_{j-1}, y_{j-1}]$. Notice that the cost to encode the j th bit of $b(a_i)$ is exactly $\frac{\ln((y_{j-1}-x_{j-1})/(y_j-x_j))}{C}$; it follows that the total cost to encode the first j bits of $b(a_i)$ is $\frac{\ln(1/(y_j-x_j))}{C}$.

For $1 \leq i \neq i' \leq k$, notice that $|f(a_i) - f(a_{i'})| \geq \#_{a_i}(S)/m$. Therefore, if $y_j - x_j < \#_{a_i}(S)/m$, then the first j bits of $b(a_i)$ suffice to distinguish it. So the shortest prefix of $b(a_i)$ that suffices to distinguish $b(a_i)$ has cost less than $\frac{\ln(e^{\text{cost}(1) \cdot C} m / \#_a(S))}{C} = \frac{\ln(m/\#_a(S))}{C} + \text{cost}(1)$. Let this sequence of bits be the codeword for a_i . ■

Repeating Krause's algorithm after each character of S , as described in the introduction, is a simple algorithm for dynamic Shannon coding with unequal letter costs. This algorithm produces an encoding of S with cost at most $(\frac{H \ln 2}{C} + \text{cost}(1))m + O(n \log m)$ in $O(mn)$ time.

As in Subsection IV-B, we can make this simple algorithm more efficient by only computing the codewords we need.

However, instead of lexicographic order, we want to keep the characters in non-increasing order by frequency in the current prefix. We use a data structure for dynamic cumulative probability tables [17], due to Moffat. This data structure stores a list of characters in non-increasing order by frequency and supports the following operations:

- given a character a , return a 's frequency;
- given a character a , return the total frequency of all characters before a in the list;
- given a character a , increment a 's frequency; and,
- given an integer k , return the last character a in the list such that the total frequency of all characters before a is at most k .

If a 's frequency is a p fraction of the total frequency of all characters in the list, then an operation that is given a or returns a takes $O(\log(1/p))$ time.

Dynamizing Krause's algorithm using Moffat's data structure gives the following theorem, much as dynamizing Mehlhorn's algorithm with an augmented splay tree gave Theorem 7. We omit the proof because it is very similar.

Theorem 9: Suppose $\text{cost}(0)$ and $\text{cost}(1)$ are constants with $0 < \text{cost}(0) \leq \text{cost}(1)$. Then dynamic Shannon coding produces an encoding of S with cost at most $(\frac{H \ln 2}{C} + \text{cost}(1))m + O(n \log m)$ in $O((H+1)m)$ time.

If $\text{cost}(0) = \text{cost}(1) = 1$, then $C = 1$ and Theorem 9 is the same as Theorem 4. We considered this special case first because it is the only one in which we know how to efficiently maintain the code-tree, which may be useful for some applications.

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REFERENCES

- [1] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, 1948.
- [2] D. A. Huffman, "A method for the construction of minimum redundancy codes," *Proceedings of the IERE*, vol. 40, pp. 1098–1101, 1952.
- [3] R. De Prisco and A. De Santis, "On the redundancy achieved by Huffman codes," *Information Sciences*, vol. 88, pp. 131–148, 1996.
- [4] M. Golumbic, "Combinatorial merging," *IEEE Transactions on Computers*, vol. 24, pp. 1164–1167, 1976.
- [5] N. Faller, "An adaptive system for data compression," in *Proceedings of the 7th Asilomar Conference on Circuits, Systems, and Computers*, 1973, pp. 593–597.
- [6] R. G. Gallager, "Variations on a theme by Huffman," *IEEE Transactions on Information Theory*, vol. 24, pp. 668–674, 1978.
- [7] D. E. Knuth, "Dynamic Huffman coding," *Journal of Algorithms*, vol. 6, pp. 163–180, 1985.
- [8] R. L. Milidiú, E. S. Laber, and A. A. Pessoa, "Bounding the compression loss of the FGK algorithm," *Journal of Algorithms*, vol. 32, pp. 195–211, 1999.
- [9] J. S. Vitter, "Design and analysis of dynamic Huffman codes," *Journal of the ACM*, vol. 34, pp. 825–845, 1987.
- [10] T. Gagie, "Dynamic length-restricted coding," Master's thesis, University of Toronto, 2003.
- [11] J. Abrahams, "Code and parse trees for lossless source encoding," *Communications in Information and Systems*, vol. 1, pp. 113–146, 2001.
- [12] M. Liddell and A. Moffat, "Length-restricted coding in static and dynamic frameworks," in *Proceedings of the IEEE Data Compression Conference*, 2001, pp. 133–142.
- [13] D. G. Kirkpatrick and M. M. Klawe, "Alphabetic minimax trees," *SIAM Journal on Computing*, vol. 14, pp. 514–526, 1985.
- [14] K. Mehlhorn, "A best possible bound for the weighted path length of binary search trees," *SIAM Journal on Computing*, vol. 6, pp. 235–239, 1977.
- [15] D. D. Sleator and R. E. Tarjan, "Self-adjusting binary search trees," *Journal of the ACM*, vol. 32, pp. 652–686, 1985.
- [16] R. M. Krause, "Channels which transmit letters of unequal duration," *Information and Control*, vol. 5, pp. 13–24, 1962.
- [17] A. Moffat, "An improved data structure for cumulative probability tables," *Software—Practice and Experience*, vol. 29, pp. 647–659, 1999.