Finite difference

- A **finite difference** is a mathematical expression of the form $f(x + b) - f(x + a)$. If a finite difference is divided by $b - a$, one gets a difference quotient.
- The approximation of derivatives by finite differences plays a central role in finite difference methods for the numerical solution of differential equations, especially boundary value problems.
- the term "finite difference" is often taken as synonymous with finite difference approximations of derivatives, especially in the context of numerical methods.
- Finite difference approximations are finite difference quotients in the terminology employed above.

Forward difference

- Suppose that a function $y = f(x)$ is tabulated for the equally spaced arguments $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving the functional values $y_0, y_1, y_2, \ldots, y_n$.
- The constant difference between two consecutive values of x is called the **interval of differences** and is denoted by **h**.
- The operator Δ defined by

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0\\
\Delta y_1 &= y_2 - y_1\\
&\ldots\ldots\\
&\ldots\ldots\\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

is called **Forward difference** operator.

- Finite differences were introduced by Brook Taylor in 1715 and have also been studied as abstract self-standing mathematical objects in works by George Boole (1860), L. M. Milne-Thomson (1933), and Károly Jordan (1939).

Forward Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Example : If **f(x)** is known at the following data points

x_i	0	1	2	3	4
f_i	1	7	23	55	109

then find **f(0.5)** and **f(1.5)** using Newton's forward difference formula.

Solution :

Forward difference table

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

1.3 Backward Difference—

The first backward difference is denoted by ∇y_k and defined as—

$$\nabla y_k = y_k - y_{k-1}.$$

The symbol ∇ is the backward difference operator.

Second order backward difference

$$\nabla^2 y_k = \nabla(\nabla y_k) = \nabla y_k - \nabla y_{k-1}$$

In general

$$\nabla^n y_k = \nabla(\nabla^{n-1} y_k) = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1}$$

Now

$$\nabla^3 y_k = \nabla^2 y_k - \nabla^2 y_{k-1} = (\nabla y_k - \nabla y_{k-1}) - (\nabla y_{k-1} - \nabla y_{k-2})$$

$$\begin{aligned}
&= \nabla y_k - \nabla y_{k-1} - \nabla y_{k-1} + \nabla y_{k-2} \\
&= \nabla y_k - 2\nabla y_{k-1} + \nabla y_{k-2} \\
&= (y_k - y_{k-1}) - 2(y_{k-1} - y_{k-2}) - (y_{k-2} - y_{k-3}) \\
&= y_k - y_{k-1} - 2y_{k-1} + 2y_{k-2} - y_{k-2} + y_{k-3} \\
&= y_k - 3y_{k-1} + 3y_{k-2} + y_{k-3}
\end{aligned}$$

In general,

$$\nabla^n y_k = \sum_{i=0}^{n} (-1)^n \, {}^n c_i \, y_{k-i}$$

Backward Difference Table

Value of x	Value of y	1^{st} diff	2^{nd} diff	3^{rd} diff	4^{th} diff	5^{th} diff
x_0	y_0					
$x_1 = x_0 + h$	y_1	∇y_1				
$x_2 = x_0 + 2h$	y_2	∇y_2	$\nabla^2 y_2$			
$x_3 = x_0 + 3h$	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
$x_4 = x_0 + 4h$	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
$x_5 = x_0 + 5h$	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

Central Difference

The central difference for a function tabulated at equal intervals f_n is defined by

$$\delta(f_n) = \delta_n = \delta_n^1 = f_{n+1/2} - f_{n-1/2}.$$

3. **Central differences.** The central difference operator d is defined by the relations

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2},\ \ldots\ldots,\ y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}.$$

Similarly, high order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2$$

and so on.

These differences are shown as follows:

Central difference table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Shift operator, E

Let $y = f(x)$ be a function of x, and let x takes the consecutive values x, $x + h$, $x + 2h$, etc. We then define an operator having the property

$$E\,f(x) = f(x+h)$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on $f(x)$, we get

$$E^2 f(x) = E[E\,f(x)]$$
$$= E[f(x+h)] = f(x+2h)$$

Thus, in general, if we apply the operator 'E' n times on $f(x)$, we get

$$E^n f(x) = f(x+nh)$$

OR

$$E^n y_x = y_{x+nh}$$

$$Ey_0 = y_1, \qquad E^2 y_0 = y_2, \qquad E^4 y_0 = y_4, \qquad \ldots, \qquad E^2 y_2 = y_4$$

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly

$$E^{-n} f(x) = f(x-nh)$$

Average Operator, μ ;

it is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$
$$= \frac{1}{2}\left[y_{x+(h/2)} + y_{x-(h/2)} \right]$$

Relation Between Operator

1. $$\boxed{\Delta = \mathrm{E} - 1 \quad \text{or} \quad \mathrm{E} = 1 + \Delta.}$$

Proof. We know that,

$$\Delta y_x = y_{x+h} - y_x = \mathrm{E}y_x - y_x = (\mathrm{E} - 1)y_x$$

$\Rightarrow$ $$\Delta = \mathrm{E} - 1$$

or $$\mathrm{E} = 1 + \Delta$$

2. $$\boxed{\nabla = 1 - \mathrm{E}^{-1}}$$

Proof. $$\nabla y_x = y_x - y_{x-h} = y_x - \mathrm{E}^{-1} y_x$$

$\therefore$ $$\nabla = 1 - \mathrm{E}^{-1}$$

3. $$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Proof. $$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$$

$$= E^{1/2} y_x - E^{-1/2} y_x$$

$$= (E^{1/2} - E^{-1/2})\, y_x$$

$\therefore$ $$\delta = E^{1/2} - E^{-1/2}$$

4. $$\boxed{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$$

Proof. $$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right) = \frac{1}{2}(E^{1/2} + E^{-1/2})\, y_x$$

$\Rightarrow$ $$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

5. $$\boxed{\Delta = E\nabla = \nabla E = \delta E^{1/2}}$$

Proof. $$E(\nabla y_x) = E(y_x - y_{x-h}) = y_{x+h} - y_x = \Delta y_x$$

$\Rightarrow$ $$E\nabla = \Delta$$

$$\nabla(E\, y_x) = \nabla\, y_{x+h} = y_{x+h} - y_x = \Delta y_x$$

$\Rightarrow$ $$\nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+\frac{h}{2}} = y_{x+h} - y_x = \Delta y_x$$

$\Rightarrow$ $$\delta E^{1/2} = \Delta$$

6. $$\boxed{E = e^{hD}}$$

Proof. $$Ef(x) = f(x+h)$$

$$= f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \ldots.. \qquad \text{(By Taylor series)}$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots...$$

$$= \left[1 + hD + \frac{(hD)^2}{2!} + \ldots...\right] f(x) = e^{hD} f(x)$$

$\therefore$ $$E = e^{hD} \quad \text{or} \quad \Delta = e^{hD} - 1.$$

MISSING TERM TECHNIQUE

Suppose n values out of $(n + 1)$ values of $y = f(x)$ are given, the values of x being equidistant.

Let the unknown value be N. We construct the difference table.

Since only n values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $(n - 1)$ in x.

Equating to zero the n^{th} difference, we can get the value of N.

Example 4. *Find the missing values in the table:*

x:	45	50	55	60	65
y:	3	–	2	–	– 2.4.

Sol. The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	2		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3		$-0.4 - 2y_3$	
		$-2.4 - y_3$		
65	-2.4			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$$\therefore \qquad \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

$$\Rightarrow \qquad 3y_1 + y_3 = 9 \quad \text{and} \quad y_1 + 3y_3 = 3.6$$

Solving these, we get

$$y_1 = 2.925, \quad y_2 = 0.225.$$

Example 7. *Given, log 100 = 2, log 101 = 2.0043, log 103 = 2.0128, log 104 = 2.0170. Find log 102.*

Sol. Since four values are given, $\Delta^4 f(x) = 0$.

Let the missing value be y_2.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
100	2				
		.0043			
101	2.0043		$y_2 - 2.0086$		
		$y_2 - 2.0043$		$6.0257 - 3y_2$	
102	y_2		$4.0171 - 2y_2$		$6y_2 - 12.0514$
		$2.0128 - y_2$		$3y_2 - 6.0257$	
103	2.0128		$y_2 - 2.0086$		
		.0042			
104	2.0170				

Since $\qquad \Delta^4 y = 0$

$$\therefore \quad 6y_2 - 12.0514 = 0 \quad \Rightarrow \quad y_2 = 2.0086.$$

FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)$ ...... $(x-r+1)$ is denoted by $[x]^r$ and is called a factorial.

Particularly, $[x] = x$; $[x]^2 = x(x-1)$; $[x]^3 = x\,(x-1)(x-2)$, etc.

In case the interval of difference is h, then

$$[x]^n = x(x-h)\,(x-2h) \text{ ...... } (x - \overline{n-1}\,h)$$

Factorial notation helps in finding the successive differences of a polynomial directly by the simple rule of differentiation.

Example 1. *Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation and hence show that $\Delta^3 y = 12$.*

Sol. Let $\qquad y = A[x]^3 + B[x]^2 + c[x] + D$

Using the method of synthetic division, we divide by x, $x-1$, $x-2$ etc. successively, then

1	2	−3	3	−10 = D
		2	−1	
2	2	−1	2 = C	
		4		
3	2	3 = B		
	2 = A			

Hence, $\qquad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$$\therefore \quad \Delta y = 6[x]^2 + 6[x] + 2$$

$$\Delta^2 y = 12[x] + 6$$

$$\Delta^3 y = 12$$

which shows that the third differences of y are constant.

NEWTON’S FORMULAE FOR INTERPOLATION

Newton’s formula is used for constructing the interpolation polynomial. It makes use of divided differences. This result was first discovered by the Scottish mathematician James Gregory (1638–1675) a contemporary of Newton. Gregory and Newton did extensive work on methods of interpolation but now the formula is referred to as Newton’s interpolation formula. Newton has derived general forward and backward difference interpolation formulae.

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called **extrapolation**.

Forward Differences : The differences y1 – y0, y2 – y1, y3 – y2, ……, yn – yn–1 when denoted by dy0, dy1, dy2, ……, dyn–1 are respectively, called the first forward differences. Thus the first forward differences are :

NEWTON'S GREGORY FORWARD INTERPOLATION FORMULA :

$$f(a+hu) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\ldots+\frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n f(a)$$

This formula is particularly useful for interpolating the values of f(x) near the beginning of the set of values given. h is called the interval of difference and **u = (x – a) / h**, Here a is first term.

EXAMPLES

Example 1. *Find the value of sin 52° from the given table:*

θ°	45°	50°	55°	60°
sin θ	0.7071	0.7660	0.8192	0.8660

Sol. $a = 45°, h = 5, x = 52$

$\therefore \quad u = \frac{x-a}{h} = \frac{7}{5} = 1.4$

Difference table is:

$x°$	Differences			
	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	$\boxed{7071}$			
		$\boxed{589}$		
50°	7660		$\boxed{-57}$	
		532		$\boxed{-7}$
55°	8192		– 64	
		468		
60°	8660			

By forward difference formula,

$$f(a+hu) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow \quad 10^4 f(x) = 10^4 f(a) + 10^4 u\,\Delta f(a) + 10^4 \frac{u(u-1)}{2!}\Delta^2 f(a) + 10^4 \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow \quad 10^4 f(52) = 10^4 f(45) + (1.4)\,10^4\,\Delta f(45) + \frac{(1.4)(1.4-1)}{2!}10^4\,\Delta^2 f(45) + \frac{(1.4)(1.4-1)(1.4-2)}{3!}10^4\,\Delta^3 f(45)$$

$$= 7071 + (1.4)(589) + \frac{(1.4)(.4)}{2}(-57) + \frac{(1.4)(.4)(-.6)}{6}(-7)$$

$$= 7880$$

$\therefore \quad f(52) = .7880$. Hence, $\sin 52° = 0.7880$.

Example 2. *The population of a town in the decimal census was as given below. Estimate the population for the year 1895.*

Year x:	*1891*	*1901*	*1911*	*1921*	*1931*
Population y: *(in thousands)*	*46*	*66*	*81*	*93*	*101*

Sol. Here $a = 1891, h = 10,$ $a + hu = 1895$

$\Rightarrow$ $1891 + 10\,u = 1895$ $\Rightarrow$ $u = 0.4$

The difference table is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	$\boxed{46}$				
		$\boxed{20}$			
1901	66		$\boxed{-5}$		
		15		$\boxed{2}$	
1911	81		– 3		$\boxed{-3}$
		12		– 1	
1921	93		– 4		
		8			
1931	101				

Applying Newton's forward difference formula,

$$y(1895) = y(1891) + u\ \Delta y(1891) + \frac{u(u-1)}{2\,!}\ \Delta^2 y(1891)$$

$$+ \frac{u(u-1)(u-2)}{3\,!}\ \Delta^3 y(1891)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4\,!}\ \Delta^4 y(1891)$$

$$\Rightarrow \qquad y(1895) = 46 + (.4)(20) + \frac{(.4)(.4-1)}{2}\ (-5)$$

$$+ \frac{(.4)(.4-1)(.4-2)}{6}\ (2) + \frac{(.4)(.4-1)(.4-2)(.4-3)}{24}\ (-3)$$

$$\Rightarrow \qquad y(1895) = 54.8528 \text{ thousands}$$

Hence the population for the year 1895 is **54.8528 thousands** approximately.

Program

```
#include<stdio.h>
#define MAXN 100
#define ORDER 4

main()
{
   float ax[MAXN+1], ay [MAXN+1], diff[MAXN+1][ORDER+1], nr=1.0,
dr=1.0,x,p,h,yp;
   int n,i,j,k;
   printf("\nEnter the value of n:\n");
   scanf("%d",&n);
```

```
    printf("\nEnter the values in form x,y:\n");
    for (i=0;i<=n;i++)
        scanf("%f %f",&ax[i],&ay[i]);
    printf("\nEnter the value of x for which the value of y is wanted: \n");
    scanf("%f",&x);
    h=ax[1]-ax[0];

    //now making the difference table
    //calculating the 1st order of differences
    for (i=0;i<=n-1;i++)
        diff[i][1] = ay[i+1]-ay[i];

    //now calculating the second and higher order differences
    for (j=2;j<=ORDER;j++)
        for(i=0;i<=n-j;i++)
        diff[i][j] = diff[i+1][j-1] - diff[i][j-1];

    //now finding x0
    i=0;
    while (!(ax[i]>x))
        i++;

    //now ax[i] is x0 and ay[i] is y0
    i--;
    p = (x-ax[i])/h;
    yp = ay[i];

    //now carrying out interpolation
    for (k=1;k<=ORDER;k++)
    {
        nr *=p-k+1;
        dr *=k;
        yp +=(nr/dr)*diff[i][k];
    }
    printf("\nWhen x = %6.1f, corresponding y = %6.2f\n",x,yp);
}
```

NEWTON'S GREGORY BACKWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function of x which assumes the values $f(a), f(a + h)$, $f(a + 2h), ..., f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, ..., a + nh$ of the independent variable x.

Let $f(x)$ be a polynomial of the nth degree.

$$f(a + nh + uh) = f(a + nh) + u\,\nabla f(a + nh) + \frac{u(u+1)}{2!}\nabla^2 f(a + nh) + \ldots\ldots + \frac{u(u+1)\ldots\ldots(u+\overline{n-1})}{n!}\nabla^n f(a + nh)$$

EXAMPLES

Example 1. *The population of a town was as given. Estimate the population for the year 1925.*

Year (x):	*1891*	*1901*	*1911*	*1921*	*1931*
Population (y): *(in thousands)*	*46*	*66*	*81*	*93*	*101*

Sol. Here, $a + nh = 1931, h = 10, \quad a + nh + uh = 1925$

$$\therefore \quad u = \frac{1925 - 1931}{10} = -0.6$$

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		$\boxed{-3}$
		12		$\boxed{-1}$	
1921	93		$\boxed{-4}$		
		$\boxed{8}$			
1931	$\boxed{101}$				

Applying Newton's Backward difference formula, we get

$$y_{1925} = y_{1931} + u\nabla y_{1931} + \frac{u(u+1)}{2!}\nabla^2 y_{1931} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1931} + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1931}$$

$$= 101 + (-.6)(8) + \frac{(-.6)(.4)}{2!}(-4) + \frac{(-.6)(.4)(1.4)}{3!}(-1) + \frac{(-.6)(.4)(1.4)(2.4)}{4!}(-3)$$

$= 96.8368$ thousands.

Example 2. *The population of a town is as follows:*

Year:	*1921*	*1931*	*1941*	*1951*	*1961*	*1971*
Population: (in Lakhs)	*20*	*24*	*29*	*36*	*46*	*51*

Estimate the increase in population during the period 1955 to 1961.

Sol. Here, $a + nh = 1971$, $h = 10$, $a + nh + uh = 1955$

$\therefore$ $1971 + 10u = 1955 \Rightarrow u = -1.6$

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1921	20					
		4				
1931	24		1			
		5		1		
1941	29		2		0	
		7		1		$\boxed{-9}$
1951	36		3		$\boxed{-9}$	
		10		$\boxed{-8}$		
1961	46		$\boxed{-5}$			
		$\boxed{5}$				
1971	$\boxed{51}$					

Applying Newton's backward difference formula, we get

$$y_{1955} = y_{1971} + u\nabla y_{1971} + \frac{u(u+1)}{2!}\nabla^2 y_{1971} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1971}$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1971} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{1971}$$

$$= 51 + (-1.6)(5) + \frac{(-1.6)(-0.6)}{2!}(-5) + \frac{(-1.6)(-0.6)(0.4)}{6}(-8)$$

$$+ \frac{(-1.6)(-0.6)(.4)(1.4)}{24}(-9) + \frac{(-1.6)(-0.6)(0.4)(1.4)(2.4)}{120}(-9)$$

$= 39.789632$

$\therefore$ Increase in population during period 1955 to 1961 is

$= 46 - 39.789632 = 6.210368$ Lakhs

$= 621036.8$ Lakhs.

GAUSS' FORWARD DIFFERENCE FORMULA

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\,\Delta^2 f(-1) + \frac{(u+1)\,u\,(u-1)}{3!}\,\Delta^3 f(-1) + \frac{(u+1)\,u\,(u-1)(u-2)}{4!}\,\Delta^4 f(-2) + \ldots\ldots$$

This is called **Gauss' forward difference formula.**

NOTE *This formula is applicable when u lies between 0 and $\frac{1}{2}$.*

EXAMPLES

Example 1. *Apply a central difference formula to obtain f(32) given that:*

f(25) = 0.2707 *f(35) = 0.3386*

f(30) = 0.3027 *f(40) = 0.3794.*

Sol. Here $a + hu = 32$ and $h = 5$

Take origin at 30 $\therefore$ $a = 30$ then $u = 0.4$

The forward difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
– 1	25	.2707			
			.032		
0	30	$\boxed{.3027}$		$\boxed{.0039}$	
			$\boxed{.0359}$		$\boxed{.0010}$
1	35	.3386		.0049	
			.0408		
2	40	.3794			

Applying Gauss' forward difference formula, we have

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\,\Delta^2 f(-1) + \frac{(u+1)\,u\,(u-1)}{3!}\,\Delta^3 f(-1)$$

$$\therefore \quad f(.4) = .3027 + (.4)(.0359) + \frac{(.4)(.4-1)}{2!}\,(.0039) + \frac{(1.4)(.4)(.4-1)}{3!}\,(.0010)$$

$$= 0.316536.$$

Example 3. *The values of e^{-x} at $x = 1.72$ to $x = 1.76$ are given in the following table:*

x:	*1.72*	*1.73*	*1.74*	*1.75*	*1.76*
e^{-x}*:*	*0.17907*	*0.17728*	*0.17552*	*0.17377*	*0.17204*

Find the value of $e^{-1.7425}$ using Gauss' forward difference formula.

Sol. Here taking the origin at 1.74 and $h = 0.01$.

$$\therefore \qquad x = a + uh$$

$$\Rightarrow \qquad u = \frac{x-a}{h} = \frac{1.7425 - 1.7400}{0.01} = 0.25$$

The difference table is as follows:

u	x	$10^5 f(x)$	$10^5 \Delta f(x)$	$10^5 \Delta^2 f(x)$	$10^5 \Delta^3 f(x)$	$10^5 \Delta^4 f(x)$
– 2	1.72	17907				
			– 179			
– 1	1.73	17728		3		
			– 176		– 2	
0	1.74	$\boxed{17552}$		$\boxed{1}$		$\boxed{3}$
			$\boxed{-175}$			
1	1.75	17377		2	$\boxed{1}$	
			– 173			
2	1.76	17204				

Gauss's forward formula is

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$\therefore \quad 10^5 f(.25) = 17552 + (.25)(-175) + \frac{(.25)(-.75)}{2}(1) + \frac{(1.25)(.25)(-.75)}{6}(1) + \frac{(1.25)(.25)(-.75)(-1.75)}{24}(3)$$

$$= 17508.16846$$

$$\therefore \qquad f(0.25) = e^{-1.7425} = 0.1750816846.$$

GAUSS'S BACKWARD DIFFERENCE FORMULA

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \ldots\ldots \qquad (37)$$

This is known as **Gauss' backward difference formula.**

This formula is useful when u lies between $-\frac{1}{2}$ and 0.

EXAMPLES

Example 1. *Given that*

$$\sqrt{12500} = 111.803399,\ \sqrt{12510} = 111.848111$$

$$\sqrt{12520} = 111.892806,\ \sqrt{12530} = 111.937483$$

Show by Gauss's backward formula that $\sqrt{12516} = 111.8749301.$

Sol. Taking the origin at 12520

$$\therefore \qquad u = \frac{x-a}{h} = \frac{12516 - 12520}{10} = -\frac{4}{10} = -0.4$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \ldots \qquad (38)$$

The difference table is:

u	x	$10^6 f(x)$	$10^6 \Delta f(x)$	$10^6 \Delta^2 f(x)$	$10^6 \Delta^3 f(x)$
−2	12500	111803399			
			44712		
−1	12510	111848111		−17	
			$\boxed{44695}$		$\boxed{-1}$
0	12520	$\boxed{111892806}$		$\boxed{-18}$	
			44677		
1	12530	111937483			

From (38),

$$10^6 f(-.4) = 111892806 + (-.4)(44695) + \frac{(.6)(-.4)}{2!}(-18) + \frac{(.6)(-.4)(-1.4)}{3!}(-1)$$

$$= 111874930.1$$

$$\therefore \qquad f(-.4) = 111.8749301$$

Hence, $\sqrt{12516} = 111.8749301.$

STIRLING'S FORMULA

$$f(u) = f(0) + u\left\{\frac{\Delta f(0) + \Delta f(-1)}{2}\right\} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left\{\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right\} + \frac{u^2(u^2-1)}{4!}\Delta^4 f(-2) + \ldots\ldots \qquad (43)$$

Example 1. *Given:*

θ:	0°	5°	10°	15°	20°	25°	30°
tan θ:	0	0.0875	0.1763	0.2679	0.364	0.4663	0.5774

Find the value of tan 16° using Stirling formula.

Sol. Take origin at 15°

$$\therefore \qquad a = 15°,\ h = 5$$

$$a + hu = 16$$

$$\Rightarrow \qquad 15 + 5u = 16 \quad \Rightarrow \quad u = .2$$

The difference table is:

u	θ	$10^4 f(\theta)$	$10^4 \Delta f(\theta)$	$10^4 \Delta^2 f(\theta)$	$10^4 \Delta^3 f(\theta)$	$10^4 \Delta^4 f(\theta)$	$10^4 \Delta^5 f(\theta)$	$10^4 \Delta^6 f(\theta)$
– 3	0	0						
			875					
– 2	5	875		13				
			888		15			
– 1	10	1763		28		2		
			$\boxed{916}$		$\boxed{17}$		$\boxed{-2}$	
0	15	$\boxed{2679}$		$\boxed{45}$		$\boxed{0}$		$\boxed{11}$
			$\boxed{961}$		$\boxed{17}$		$\boxed{9}$	
1	20	3640		62		9		
			1023		26			
2	25	4663		88				
			1111					
3	30	5774						

Using Stirling's formula,

$$10^4 f(.2) = 2679 + (.2)\left(\frac{961+916}{2}\right) + \frac{(.2)^2}{2!}(45) + \frac{(1.2)(.2)(-.8)}{3!}\left(\frac{17+17}{2}\right) + \frac{(.2)^2\left\{(.2)^2 - 1\right\}}{4!}(0) + \frac{(2.2)(1.2)(.2)(-.8)(-1.8)}{5!}\left\{\frac{9+(-2)}{2}\right\} + \frac{(.2)^2\{(.2)^2-1\}\{(.2)^2-4\}}{6!}(11)$$

$$= 2866.980499$$

$$\therefore \qquad f(.2) = .2866980499$$

Hence $\qquad \tan 16° = 0.2866980499.$

BESSEL’S INTERPOLATION FORMULA

BESSEL'S INTERPOLATION FORMULA

$$f(u) = \left\{\frac{f(0)+f(1)}{2}\right\} + \left(u-\frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(-1)+\Delta^2 f(0)}{2}\right\} + \frac{(u-1)\left(u-\frac{1}{2}\right)u}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\left\{\frac{\Delta^4 f(-2)+\Delta^4 f(-1)}{2}\right\} + \ldots\ldots$$

EXAMPLES

Example 1. *Given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$ and $y_{32} = 40$ find y_{25} by Bessel's interpolation formula.*

Sol. Take origin at 24.

Here, $a = 24, \quad h = 4, \quad a + hu = 25$

$\therefore \quad 24 + 4u = 25 \quad \Rightarrow \quad u = .25$

The difference table is:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 1	20	24			
			8		
0	24	$\boxed{32}$		$\boxed{-5}$	
			$\boxed{3}$		$\boxed{7}$
1	28	$\boxed{35}$		$\boxed{2}$	
			5		
2	32	40			

Using Bessel's formula,

$$f(u) = \left\{\frac{f(0)+f(1)}{2}\right\} + \left(u-\frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{2}\left\{\frac{\Delta^2 f(-1)+\Delta^2 f(0)}{2}\right\} + \frac{(u-1)\left(u-\frac{1}{2}\right)u}{3!}\Delta^3 f(-1)$$

$$\Rightarrow \quad f(.25) = \left(\frac{32+35}{2}\right) + (.25 - .5)(3) + \frac{(.25)(.25-1)}{2}\left\{\frac{-5+2}{2}\right\} + \frac{(.25-1)(.25-.5)(.25)}{3!}(7)$$

$$= 32.9453125$$

Hence $\quad y_{25} = 32.9453125.$

DIVIDED DIFFERENCES

Lagrange's interpolation formula has the disadvantage that if another interpolation point were added, the interpolation coefficient will have to be recomputed.

We therefore seek an interpolation polynomial which has the property that a polynomial of higher degree may be derived from it by simply adding new terms.

Newton's general interpolation formula is one such formula and it employs divided differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ ...... be given points then the first divided difference for the arguments x_0, x_1 is defined by

$$\underset{x_1}{\Delta} y_0 = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $\quad [x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and so on.

The second divided difference for x_0, x_1, x_2 is defined as

$$\underset{x_1, x_2}{\Delta^2} y_0 = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

Third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \quad \text{and so on.}$$

EXAMPLES

Example 1. *Construct a divided difference table for the following:*

x:	*1*	*2*	*4*	*7*	*12*
f(x):	*22*	*30*	*82*	*106*	*216.*

Sol.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1} = 8$			
2	30		$\frac{26-8}{4-1} = 6$		
		$\frac{82-30}{4-2} = 26$		$\frac{-3.6-6}{7-1} = -1.6$	
4	82		$\frac{8-26}{7-2} = -3.6$		$\frac{0.535+1.6}{12-1} = 0.194$
		$\frac{106-82}{7-4} = 8$		$\frac{1.75+3.6}{12-2} = 0.535$	
7	106		$\frac{22-8}{12-4} = 1.75$		
		$\frac{216-106}{5} = 22$			
12	216				

Example 2. (*i*) *Find the third divided difference with arguments 2, 4, 9, 10 of the function* $f(x) = x^3 - 2x$.

(*ii*) *If* $f(x) = \dfrac{1}{x^2}$, *find the first divided differences* $f(a, b), f(a, b, c), f(a, b, c, d)$.

(*iii*) *If* $f(x) = g(x)\, h(x)$, *prove that*

$$f(x_1, x_2) = g(x_1)\, h(x_1, x_2) + g(x_1, x_2)\, h(x_2).$$

Sol. (*i*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\dfrac{56-4}{4-2} = 26$		
4	56		$\dfrac{131-26}{9-2} = 15$	
		$\dfrac{711-56}{9-4} = 131$		$\dfrac{23-15}{10-2} = 1$
9	711		$\dfrac{269-131}{10-4} = 23$	
		$\dfrac{980-711}{10-9} = 269$		
10	980			

Hence, the third divided difference is 1.

(*ii*)

x	$f(x) = \dfrac{1}{x^2}$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\dfrac{1}{a^2}$			
		$\dfrac{\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right)}{b-a} = \boxed{-\left(\dfrac{a+b}{a^2b^2}\right)}$		
b	$\dfrac{1}{b^2}$		$\boxed{\dfrac{ab+bc+ca}{a^2b^2c^2}}$	
		$-\left(\dfrac{b+c}{b^2c^2}\right)$		$\boxed{-\left(\dfrac{abc+acd+abd+bcd}{a^2b^2c^2d^2}\right)}$
c	$\dfrac{1}{c^2}$		$\dfrac{bc+cd+db}{b^2c^2d^2}$	
		$-\left(\dfrac{c+d}{c^2d^2}\right)$		
d	$\dfrac{1}{d^2}$			

From the above divided difference table, we observe that the first divided differences,

$$f(a, b) = -\left(\frac{a+b}{a^2b^2}\right)$$

$$f(a, b, c) = \frac{ab+bc+ca}{a^2b^2c^2}$$

and $$f(a, b, c, d) = -\left(\frac{abc+acd+abd+bcd}{a^2b^2c^2d^2}\right)$$

NEWTON'S GENERAL INTERPOLATION FORMULA

OR

NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

- Newton's Divided Difference formula was put forward to overcome a few limitations of Lagrange's formula.
- In Lagrange's formula, if another interpolation value were to be inserted, then the interpolation coefficients were to be calculated again. This is not the case in Divided Difference.
- In this tutorial, we're going to discuss a source code in C for Newton Divided Difference formula along with sample output.

The second divided difference is defined as: [x0, x1, x2] = ([x1, x2] – [x0, x1])/(x2-x0). This goes on in similar fashion for the third, fourth …. and nth divided differences. Based on these formulas, two basic properties of Newton's Divided Difference method can be outlined as given below:

- The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.
- The nth divided differences of a polynomial of the nth degree are constant.

Newton's divided difference formula can also be written as

$$\boxed{\begin{aligned} y = y_0 &+ (x - x_0)\,\Delta y_0 + (x - x_0)(x - x_1)\,\Delta^2 y_0 \\ &+ (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 y_0 \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 y_0 \\ &+ \ldots.. + (x - x_0)(x - x_1)\ldots..(x - x_{n-1})\,\Delta^n y_0 \end{aligned}}$$

Example 4. *Using Newton's divided difference formula, find a polynomial function satisfying the following data:*

x:	*– 4*	*– 1*	*0*	*2*	*5*
f(x):	*1245*	*33*	*5*	*9*	*1335*

Hence find f(1).

Sol. The divided difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
– 4	$\boxed{1245}$				
		$\boxed{-404}$			
– 1	33		$\boxed{94}$		
		– 28		$\boxed{-14}$	
0	5		10		$\boxed{3}$
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)\,94$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Hence, $f(1) = 3 - 5 + 6 - 14 + 5 = -5$.

Program

```
#include<stdio.h>
#include<conio.h>

void main()
{
   int x[10], y[10], p[10];
   int k,f,n,i,j=1,f1=1,f2=0;
   printf("\nEnter the number of observations:\n");
   scanf("%d", &n);

   printf("\nEnter the different values of x:\n");
   for (i=1;i<=n;i++)
      scanf("%d", &x[i]);

   printf("\nThe corresponding values of y are:\n");
   for (i=1;i<=n;i++)
      scanf("%d", &y[i]);

   f=y[1];
   printf("\nEnter the value of 'k' in f(k) you want to evaluate:\n");
   scanf("%d", &k);
```

```
do
{
   for (i=1;i<=n-1;i++)
   {
      p[i] = ((y[i+1]-y[i])/(x[i+j]-x[i]));
      y[i]=p[i];
   }
   f1=1;
   for(i=1;i<=j;i++)
      {
         f1*=(k-x[i]);
      }
   f2+=(y[1]*f1);
   n--;
   j++;
}

while(n!=1);
f+=f2;
printf("\nf(%d) = %d", k , f);
getch();
}
```

LAGRANGE'S INTERPOLATION FORMULA

Let $f(x0), f(x1)$,......, $f(xn)$ be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x0, x1, x2, ..., xn$. The polynomial $f(x)$ may be written as

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)} f(x_1) + \ldots\ldots + \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})} f(x_n) \tag{54}$$

EXAMPLES

Example 1. *Using Lagrange's interpolation formula, find y(10) from the following table:*

x	5	6	9	11
y	12	13	14	16

Sol. Here $x_0 = 5,\quad x_1 = 6,\quad x_2 = 9,\quad x_3 = 11$

$f(x_0) = 12,\quad f(x_1) = 13,\quad f(x_2) = 14,\quad f(x_3) = 16$

Lagrange's formula is

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12)$$

$$+ \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14)$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= -\frac{1}{2}(x-6)(x-9)(x-11) + \frac{13}{15}(x-5)(x-9)(x-11)$$

$$- \frac{7}{12}(x-5)(x-6)(x-11)$$

$$+ \frac{4}{15}(x-5)(x-6)(x-9)$$

Putting $x = 10$, we get

$$f(10) = -\frac{1}{2}(10-6)(10-9)(10-11) + \frac{13}{15}(10-5)(10-9)(10-11)$$

$$- \frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-9)$$

$$= 14.66666667$$

Hence,

$$y(10) = 14.66666667.$$

Lagrange Interpolation program

```
#include<stdio.h>
main()
{
    float x[100],y[100],a,s=1,t=1,k=0;
```

```
    int n,i,j,d=1;
    printf("\n\n Enter the number of the terms of the table: ");
    scanf("%d",&n);
    printf("\n\n Enter the respective values of the variables x and y: \n");
    for(i=0; i<n; i++)
    {
        scanf ("%f",&x[i]);
        scanf("%f",&y[i]);
    }
    printf("\n\n The table you entered is as follows :\n\n");
    for(i=0; i<n; i++)
    {
        printf("%0.3f\t%0.3f",x[i],y[i]);
        printf("\n");
    }
    while(d==1)
    {
        printf(" \n\n\n Enter the value of the x to find the respective value of y\n\n\n");
        scanf("%f",&a);
        for(i=0; i<n; i++)
        {
            s=1;
            t=1;
            for(j=0; j<n; j++)
            {
                if(j!=i)
                {
                    s=s*(a-x[j]);
                    t=t*(x[i]-x[j]);
                }
            }
            k=k+((s/t)*y[i]);
        }
        printf("\n\n The respective value of the variable y is: %f",k);
        printf("\n\n Do you want to continue?\n\n Press 1 to continue and any other key
to exit");
        scanf("%d",&d);
    }
}
```