概率论与数理统计 第6次作业

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1　泊松分布的和

由于$X_i \sim P(\lambda_i)$，我们有

$$P(X_i = k) = \frac{\lambda_i^k}{k!}e^{-\lambda_i}$$

则

$$\begin{aligned}
P(Y = k) &= \sum_{i=0}^{k} P(X_1 = i)P(X_2 = k - i) \\
&= \sum_{i=0}^{k} \frac{\lambda_1^i e^{-\lambda_1} \lambda_2^{k-i} e^{-\lambda^2}}{i!(k-i)!} \\
&= \sum_{i=0}^{k} \frac{\binom{k}{i}\lambda_1^i \lambda_2^{k-i}}{k!} e^{-\lambda_1 - \lambda_2} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{i=0}^{k} \binom{k}{i}\lambda_1^i \lambda_2^{k-i} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1 + \lambda_2)^k
\end{aligned}$$

这证明了$Y \sim P(\lambda_1 + \lambda_2)$.

2　二维随机变量换元

写出(X, Y)的PDF:

$$f(x, y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$$

2.1

Z的CDF（其中可以利用$f(x, y)$关于原点对陈的性质）

$$G(z) = P(Z \le z) = P(\frac{Y}{X} \le z) = 2P(\frac{Y}{X} \le z \wedge X > 0) = 2\int_0^{+\infty}(\int_{-\infty}^{zx} f(x, y)dy)dx$$

求导得PDF:

$$\begin{aligned}
g(z) = \frac{d}{dz}G(z) &= 2\int_0^{+\infty} xf(x, zx)dx \\
&= \frac{1}{\pi}\int_0^{+\infty} xe^{-\frac{1}{2}x^2(1+z^2)}dx \\
&= \frac{1}{\pi}\int_0^{+\infty} e^{-\frac{1}{2}x^2(1+z^2)}d(\frac{1}{2}x^2) \\
&= -\frac{1}{\pi(1+z^2)}e^{-\frac{1}{2}x^2(1+z^2)}|_0^{+\infty} \\
&= \frac{1}{\pi(1+z^2)}
\end{aligned}$$

2.2

这里要求$R > 0$.
$(R, \Theta) \to (X, Y)$的函数关系已知，则

$$g(r,\theta) = f(x(r,\theta), y(r,\theta))|\det \frac{\partial(x,y)}{\partial(r,\theta)}|$$

其中Jacobi矩阵为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

其行列式为

$$\det \frac{\partial(x,y)}{\partial(r,\theta)} = r(\cos^2\theta + \sin^2\theta) = r$$

故

$$g(r,\theta) = rf(r\cos\theta, r\sin\theta)$$
$$= \frac{r}{2\pi} e^{-\frac{1}{2}r^2}$$

此PDF与Θ无关，可以变量分离，故R, Θ独立.

2.3

$(X, Y) \to (U, V)$的映射关系是线性的. 其关系为

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

故

$$\det \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

于是

$$g(u,v) = \frac{1}{2} f(\frac{u+v}{2}, \frac{u-v}{2})$$
$$\frac{1}{4\pi} e^{-\frac{1}{4}(u^2+v^2)}$$

这可以变量分离，故U, V独立.

3 独立同分布变量的最值

先考虑Y的CDF，由于所有X_i独立且同分布，我们有

$$G(y) = P(Y \leq y) = P(\max\{X_1, ..., X_n\} \leq y)$$
$$= P(X_1 \leq y \wedge ... \wedge X_n \leq y)$$
$$= P(X_1 \leq y)...P(X_n \leq y)$$
$$= (F(y))^n$$

再考虑Z

$$H(z) = P(Z \leq z) = 1 - P(Z > z) = 1 - P(\min\{X_1, ..., X_n\} > z)$$
$$= 1 - P(X_1 > z \wedge ... \wedge X_n > z)$$
$$= 1 - P(X_1 > z)...P(X_n > z)$$
$$= 1 - (1 - F(z))^n$$

4 三种重要的分布

4.1 卡方分布

χ^2分布定义为n个独立的标准正态随机变量的平方和. 即，若

$$Q = \sum_{i=1}^{n} X_i^2, X_i \sim N(0,1)$$

则Q服从自由度为n的χ^2分布，记作$Q \sim \chi^2(n)$.

4.2 t分布

设两个独立的随机变量U, V，若$U \sim N(0,1), V \sim \chi^2(p)$，则$\frac{U}{\sqrt{V/p}}$服从自由度为$p$的$t$分布.

4.3 F分布

设独立随机变量$X \sim \chi^2(m), Y \sim \chi^2(n)$，则$\frac{X/m}{Y/n}$服从自由度为$m, n$的$F$分布.

5 投注

5.1

考虑A马，若其获胜，则全部1000元中50元由投注站抽走，剩余950元由投注A马的人赚取，赔率为

$$r_A = \frac{950-500}{500} = \frac{9}{10}$$

同理

$$r_B = \frac{950-300}{300} = \frac{13}{6}$$
$$r_C = \frac{950-200}{200} = \frac{15}{4}$$

5.2

$$p_A = \frac{1}{1+r_A} = \frac{10}{19}$$
$$p_B = \frac{1}{1+r_A} = \frac{6}{19}$$
$$p_C = \frac{1}{1+r_A} = \frac{4}{19}$$

5.3

$$p_A + p_B + p_C = \frac{20}{19} > 1$$

这说明投注站通过降低赔率提高了下注者对三匹马获胜的期望概率，从而保证能够获利.

6 数字特征判断题

6.1

不正确.
设$X, Y \sim U(0,1)$且X, Y独立，则

$$Var(XY) = E(X^2Y^2) - E^2(XY) = E(X^2)E(Y^2) - E^2(X)E^2(Y) = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

而

$$Var(X)Var(Y) = \frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$$

6.2

不正确.
设$X \sim B(0.4)$，则其中位数为0，而$E(X) = 0.4$.

7 相对期望的偏差的平方最小

左侧等于

$$\begin{aligned} E((X-c)^2) &= E(X^2 - 2cX + c^2) \\ &= E(X^2) - 2cE(X) + c^2 \end{aligned}$$

右侧等于

$$Var(X) = E(X^2) - E^2(X)$$

左右相减，有

$$E((X-c)^2) - Var(X) = E^2(X) - 2cE(X) + c^2 = (E(X) - c)^2 \geq 0$$

取等当且仅当$c = E(X)$.

8 相对中位数的偏差的绝对值最小

左侧等于

$$E(|X-c|) = \int_{-\infty}^{c} (c-x)f(x)dx + \int_{c}^{+\infty} (x-c)f(x)dx$$

$$= c\int_{-\infty}^{c} f(x)dx - \int_{-\infty}^{c} xf(x)dx + \int_{c}^{+\infty} xf(x)dx - c\int_{c}^{+\infty} f(x)dx$$

右侧等于

$$E(|X-m|) = \int_{-\infty}^{m} (m-x)f(x)dx + \int_{m}^{+\infty} (x-m)f(x)dx$$

$$= \int_{-\infty}^{m} mf(x)dx - \int_{-\infty}^{m} xf(x)dx + \int_{m}^{+\infty} xf(x)dx - \int_{m}^{+\infty} mf(x)dx$$

$$= \frac{1}{2}m - \int_{-\infty}^{m} xf(x)dx + \int_{m}^{+\infty} xf(x)dx - \frac{1}{2}m$$

$$= -\int_{-\infty}^{m} xf(x)dx + \int_{m}^{+\infty} xf(x)dx$$

作差

$$E(|X-c|) - E(|X-m|) = c\int_{-\infty}^{c} f(x)dx - c\int_{c}^{+\infty} f(x)dx + \int_{c}^{m} xf(x)dx + \int_{c}^{m} xf(x)dx$$

$$= c(\int_{-\infty}^{c} f(x)dx - \int_{c}^{+\infty} f(x)dx) + 2\int_{c}^{m} xf(x)dx$$

$$= c(\int_{-\infty}^{c} f(x)dx - \int_{-\infty}^{m} f(x)dx - \int_{c}^{+\infty} f(x)dx + \int_{m}^{+\infty} f(x)dx) + 2\int_{c}^{m} xf(x)dx$$

$$= -2c\int_{c}^{m} f(x)dx + 2\int_{c}^{m} xf(x)dx$$

由积分第一中值定理

$$\exists \xi \ between \ c \ and \ m \ s.t. \int_{c}^{m} xf(x)dx = \xi \int_{c}^{m} f(x)dx$$

实际上这里ξ对应的是X在c和m之间部分的期望.
于是差值化为

$$2(\xi - c)\int_{c}^{m} f(x)dx$$

若$c \geq m$，则积分和$\xi - c$都非正，差值非负.
若$c < m$，则积分和$\xi - c$都非负，差值非负.

9 对数正态分布的期望和方差

设$Y \sim N(\mu, \sigma^2)$，$\ln X = Y$，设Y的PDF为$f(y)$.

9.1 期望

$$
\begin{aligned}
E(X) = E(e^Y) &= \int_{-\infty}^{+\infty} e^y f(y) dy \\
&= \int_{-\infty}^{+\infty} \frac{e^y}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-\mu)^2}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{(y-\mu)^2 - 2\sigma^2 y}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{y^2 - 2(\mu+\sigma^2)y + \mu^2}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{(y-(\mu+\sigma^2))^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2}) dy \\
&= \frac{e^{\mu+\frac{1}{2}\sigma^2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-(\frac{y-(\mu+\sigma^2)}{\sqrt{2}\sigma})^2) d(\frac{y-(\mu+\sigma^2)}{\sqrt{2}\sigma}) \\
&= e^{\mu+\frac{1}{2}\sigma^2}
\end{aligned}
$$

9.2 方差

$$
\begin{aligned}
E(X^2) = E(e^{2Y}) &= \int_{-\infty}^{+\infty} e^{2y} f(y) dy \\
&= \int_{-\infty}^{+\infty} \frac{e^{2y}}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-\mu)^2}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{(y-\mu)^2 - 4\sigma^2 y}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{y^2 - 2(\mu+2\sigma^2)y + \mu^2}{2\sigma^2}) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp(-\frac{(y-(\mu+2\sigma^2))^2 - 4\mu\sigma^2 - 4\sigma^4}{2\sigma^2}) dy \\
&= \frac{e^{2\mu+2\sigma^2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-(\frac{y-(\mu+2\sigma^2)}{\sqrt{2}\sigma})^2) d(\frac{y-(\mu+2\sigma^2)}{\sqrt{2}\sigma}) \\
&= e^{2\mu+2\sigma^2}
\end{aligned}
$$

故

$$
Var(X) = E(X^2) - E^2(X) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}
$$

10 样本与随机变量的数字特征

10.1 样本均值的方差

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n}E(X_i) = \frac{1}{n}n\mu = \mu$$

$$E(\bar{X}^2) = \frac{1}{n^2}\sum_{i=1}^{n}E(X_i^2) + \frac{2}{n^2}\sum_{1\le i<j\le n}E(X_iX_j)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(Var(X_i) + E^2(X_i)) + \frac{2}{n^2}\sum_{1\le i<j\le n}E(X_i)E(X_j)$$

$$= \frac{1}{n^2}n(\sigma^2 + \mu^2) + \frac{2}{n^2}\frac{n(n-1)}{2}\mu^2$$

$$= \frac{1}{n}\sigma^2 + \mu^2$$

故

$$Var(\bar{X}) = E(\bar{X}^2) - E^2(\bar{X}) = \frac{1}{n}\sigma^2$$

10.2 样本方差的期望

$$E(S^2) = \frac{1}{n-1}\sum_{i=1}^{n}(E(X_i^2) - 2E(X_i\bar{X}) + E(\bar{X}^2))$$

$$= \frac{1}{n-1}(\sum_{i=1}^{n}(Var(X_i) + E^2(X_i)) - 2E(\bar{X}\sum_{i=1}^{n}X_i) + n(\frac{1}{n}\sigma^2 + \mu^2))$$

$$= \frac{1}{n-1}(n\sigma^2 + n\mu^2 - 2nE(\bar{X}^2) + \sigma^2 + n\mu^2)$$

$$= \frac{1}{n-1}(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2)$$

$$= \sigma^2$$

11 不相关性的等价描述

设$E(X) = \mu_1, E(Y) = \mu_2, Var(X) = \sigma_1^2, Var(Y) = \sigma_2^2$，则四个描述在$\sigma_1\sigma_2 \neq 0$时等价. 下面进行证明

11.1 $(1) \Leftrightarrow (2)$

$$Cov(X,Y) = 0$$

$$\Leftrightarrow \frac{Cov(X,Y)}{\sigma_1\sigma_2} = 0$$

$$\Leftrightarrow Corr(X,Y) = 0$$

这是X,Y不相关的定义.

11.2 (1) $\Leftrightarrow$ (3)

$$\begin{aligned}E(XY) &= E((X-\mu_1)(Y-\mu_2)+\mu_1 Y+\mu_2 X-\mu_1\mu_2)\\ &= E((X-\mu_1)(Y-\mu_2))+\mu_1 E(Y)+\mu_2 E(X)-\mu_1\mu_2\\ &= Cov(X,Y)+\mu_1\mu_2\end{aligned}$$

而

$$E(X)E(Y)=\mu_1\mu_2$$

故

$$\begin{gathered}E(XY)=E(X)E(Y)\\ \Leftrightarrow Cov(X,Y)+\mu_1\mu_2=\mu_1\mu_2\\ \Leftrightarrow Cov(X,Y)=0\end{gathered}$$

11.3 (3) $\Leftrightarrow$ (4)

$$\begin{gathered}Var(X+Y)=E((X+Y)^2)-E^2(X+Y)\\ =E(X^2+2XY+Y^2)-E^2(X)-2E(X)(Y)-E^2(Y)\\ =E(X^2)-E^2(X)+2E(XY)-2E(X)E(Y)+E(Y^2)-E^2(Y)\\ =Var(X)+Var(Y)-2(E(XY)-E(X)E(Y))\end{gathered}$$

于是

$$\begin{gathered}Var(X+Y)=Var(X)+Var(Y)\\ \Leftrightarrow E(XY)=E(X)E(Y)\end{gathered}$$

12 ρ是相关系数

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp{(-\frac{1}{2(1-\rho^2)}((\frac{x-\mu_1}{\sigma_1})^2+(\frac{y-\mu_2}{\sigma_2})^2-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}))}$$

$$\begin{gathered}Corr(X,Y)=E(\frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma-2})\\ =\iint_{\mathbb{R}^2}\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}f(x,y)dxdy\end{gathered}$$

令$\xi=\frac{x-\mu_1}{\sigma_1}$,$\eta=\frac{y-\mu_2}{\sigma_2}$，则

$$Corr(X,Y)=\iint_{\mathbb{R}^2}\frac{\xi\eta}{2\pi\sqrt{1-\rho^2}}\exp(-\frac{\xi^2+\eta^2-2\rho\xi\eta}{2(1-\rho^2)})d\xi d\eta$$

令

$$A=\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix}$$

则

$$\xi^2+\eta^2-2\rho\xi\eta=\begin{bmatrix}\xi & \eta\end{bmatrix}A\begin{bmatrix}\xi\\ \eta\end{bmatrix}$$

对A做Cholesky分解，有

$$L=\begin{bmatrix}\sqrt{1-\rho^2} & \\ -\rho & 1\end{bmatrix},A=L^TL$$

于是，令

$$\alpha=\sqrt{1-\rho^2}\xi,\ \beta=-\rho\xi+\eta$$

就有

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = L\begin{bmatrix}\xi\\ \eta\end{bmatrix}$$

故

$$\xi^2+\eta^2-2\rho\xi\eta=\alpha^2+\beta^2$$

由于

$$\det\frac{\partial(\xi,\eta)}{\partial(\alpha,\beta)}=\frac{1}{\det\frac{\partial(\alpha,\beta)}{\partial(\xi,\eta)}}=\frac{1}{\det L}=\frac{1}{\sqrt{1-\rho^2}}$$

且

$$\xi=\frac{\alpha}{\sqrt{1-\rho^2}},\ \eta=\frac{\rho\alpha}{\sqrt{1-\rho^2}}+\beta$$

我们有

$$\begin{aligned}
Corr(X,Y)&=\iint_{\mathbb{R}^2}\frac{\alpha(\rho\alpha+\sqrt{1-\rho^2}\beta)}{2\pi(1-\rho^2)^{\frac{3}{2}}}\exp(-\frac{\alpha^2+\beta^2}{2(1-\rho^2)})\frac{1}{\sqrt{1-\rho^2}}d\alpha d\beta\\
&=\int_{-\infty}^{+\infty}\frac{\alpha}{2\pi(1-\rho^2)^2}(\int_{-\infty}^{+\infty}(\rho\alpha+\sqrt{1-\rho^2}\beta)\exp(-\frac{\alpha^2+\beta^2}{2(1-\rho^2)})d\beta)d\alpha\\
&=\int_{-\infty}^{+\infty}\frac{\alpha\exp(-\frac{\alpha^2}{2(1-\rho^2)})}{\sqrt{2}\pi(1-\rho^2)^{\frac{3}{2}}}(\rho\alpha\int_{-\infty}^{+\infty}\exp(-(\frac{\beta}{\sqrt{(2(1-\rho^2))}})^2)d(\frac{\beta}{\sqrt{2(1-\rho^2)}})\\
&+\sqrt{2}(1-\rho^2)\int_{-\infty}^{+\infty}\frac{\beta}{\sqrt{2(1-\rho^2)}}\exp(-(\frac{\beta}{\sqrt{2(1-\rho^2)}})^2)d(\frac{\beta}{\sqrt{2(1-\rho^2)}}))d\alpha\\
&=\int_{-\infty}^{+\infty}\frac{\alpha\exp(-\frac{\alpha^2}{2(1-\rho^2)})}{\sqrt{2}\pi(1-\rho^2)^{\frac{3}{2}}}(\sqrt{\pi}\rho\alpha+0)d\alpha\\
&=\frac{\rho}{\sqrt{2\pi}(1-\rho^2)^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\alpha^2\exp(-\frac{\alpha^2}{2(1-\rho^2)})d\alpha\\
&=\frac{2\rho}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\frac{\alpha}{\sqrt{2(1-\rho^2)}})^2\exp(-(\frac{\alpha}{\sqrt{2(1-\rho^2)}})^2)d(\frac{\alpha}{\sqrt{2(1-\rho^2)}})\\
&=\rho
\end{aligned}$$

13 配对问题的期望与方差

用X_i表示第i个人拿到自己的帽子的情况，拿到为1，没拿到为0；X表示拿到自己帽子的人数，则

$$X=\sum_{i=1}^{n}X_i$$

$$E(X_i)=\frac{1}{n}$$

13.1 期望

$$E(X)=\sum_{i=1}^{n}E(X_i)=1$$

13.2 方差

$$E(X^2)=\sum_{i=1}^{n}E(X_i^2)+2\sum_{1\le i<j\le n}E(X_iX_j)$$

由于

$$E(X_i^2)=\frac{1}{n}$$
$$E(X_iX_j)=\frac{1}{n(n-1)}$$

故

$$E(X^2)=n\frac{1}{n}+2\tbinom{n}{2}\frac{1}{n(n-1)}=2$$

故

$$Var(X)=E(X^2)-E^2(X)=1$$

14 Cauchy-Schwarz不等式

14.1

先考虑$E(V^2)=0$时，此时$P(V=0)=1$，不等式两边都是0，且存在$c=0$时$P(V=cU)=1$，故成立.

再考虑$E(V^2)>0$时

$$E((U-tV)^2)=E(U^2)-2E(UV)t+E(V^2)t^2\ge 0$$

这是关于t的二次项系数为正的的二次函数，故

$$4E^2(UV)-4E(U^2)E(V^2)\le 0$$

即

$$E^2(UV)\le E(U^2)E(V^2)$$

此时等号成立等价于上述二次函数与t轴相切，即

$$\exists t_0\in\mathbb{R},\ E((U-t_0V)^2)=0$$

这意味着

$$P(U=t_0V)=1$$

若$t_0=0$，则$P(U=0)=1$，这相当于最开始讨论的情形，下面假设$t_0\neq 0$.
取$c=\frac{1}{t_0}$则有

$$P(V=cU)=1$$

14.2

设$E(X)=\mu_1,E(Y)=\mu_2,Var(X)=\sigma_1^2,Var(Y)=\sigma_2^2$，则

$$Corr(X,Y)^2=E^2(\frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma_2})$$
$$\le E((\frac{X-\mu_1}{\sigma_1})^2)E((\frac{Y-\mu_2}{\sigma_2})^2)=1$$

故$|Corr(X,Y)|\le 1$，且等号成立当且仅当

$$\exists c\ s.t.\ P(\frac{Y-\mu_2}{\sigma_2}=c\frac{X-\mu_1}{\sigma_1})=1$$

上述概率的条件等价于

$$Y=\frac{c\sigma_2}{\sigma_1}X-\frac{\mu_1\sigma_2}{\sigma_1}+\mu_2$$

取

$$a=\frac{c\sigma_2}{\sigma_1},\ b=-\frac{\mu_1\sigma_2}{\sigma_1}+\mu_2$$

即可.

15 独立同分布随机变量的进一步讨论

15.1

$$
\begin{aligned}
Cov(X_i - \bar{X}, \bar{X}) &= E((X_i - \bar{X})(\bar{X} - \mu)) \\
&= E(X_i\bar{X}) - E(\bar{X}^2) - \mu E(X_i) + \mu E(\bar{X}) \\
&= \frac{1}{n}\sum_{j=1}^{n} E(X_iX_j) - \frac{1}{n^2}(\sum_{j=1}^{n} E(X_j^2) + 2\sum_{1\le j<k\le n} E(X_jX_k)) - \mu^2 + \mu^2 \\
&= \frac{n-1}{n}\mu^2 + \frac{1}{n}(\mu^2+\sigma^2) - \frac{1}{n^2}n(\mu^2+\sigma^2) - \frac{2}{n^2}\frac{n(n-1)}{2}\mu^2 \\
&= 0
\end{aligned}
$$

15.2

不一定独立，反例如下：

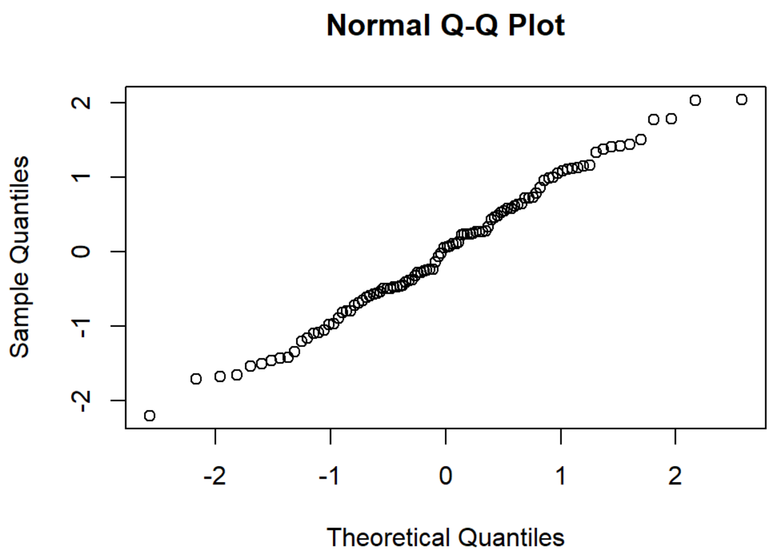
设$X_1, X_2 \sim B(\frac{1}{2})$，则

$$P(X_1 - \bar{X} = 0) = P(X_1 = X_2) = \frac{1}{2}$$

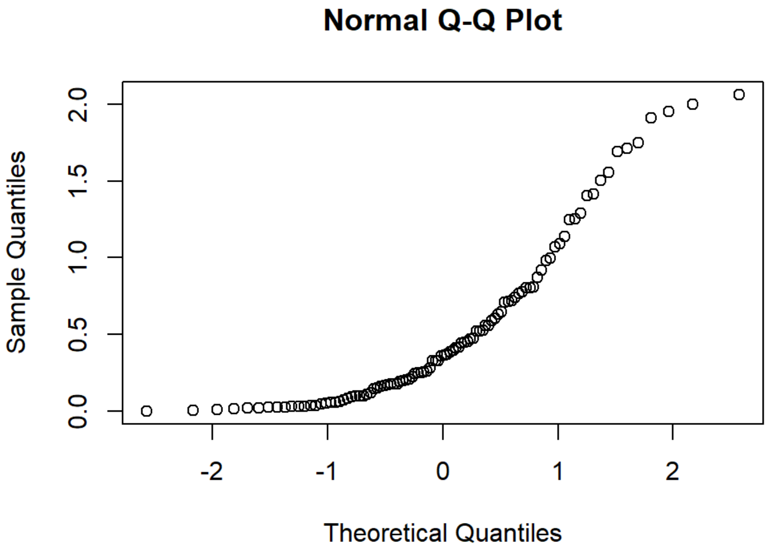
$$P(X_1 - \bar{X} = 0 | \bar{X} = \frac{1}{2}) = 0$$

16 计算机实验：Q-Q图验证正态性

16.1



16.2



16.3

可以，只要把Φ^{-1}改为指定分布的CDF的反函数即可.

16.4

可以，只要把两组数据都从小到大排列后一个作为x轴一个作为y轴作图即可，这可以使用R语言中的qqplot函数完成.

16.5 使用的代码

```
> x<-rnorm(100)
> qqnorm(x)
> y<-rexp(100,2)
> qqnorm(y)
```