概率论与数理统计 第15次作业

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1 线性回归(1)

1.1

考虑残差平方和

$$\mathrm{SSE} = \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

要使此值最小，求其偏导数:

$$\frac{\partial}{\partial \beta_1}\mathrm{SSE} = \sum_{i=1}^{n} 2x_i(\beta_1 x_i + \beta_0 - y_i)$$

$$\frac{\partial}{\partial \beta_0}\mathrm{SSE} = \sum_{i=1}^{n} 2(\beta_1 x_i + \beta_0 - y_i)$$

于是

$$\sum_{i=1}^{n} x_i(\hat{\beta}_1 x_i + \hat{\beta}_0 - y_i) = 0$$

$$\sum_{i=1}^{n} (\hat{\beta}_1 x_i + \hat{\beta}_0 - y_i) = 0$$

即

$$(\sum_{i=1}^{n} x_i^2)\hat{\beta}_1 + n\bar{x}\hat{\beta}_0 - \sum_{i=1}^{n} x_i y_i = 0$$

$$n\bar{x}\hat{\beta}_1 + n\hat{\beta}_0 - n\bar{y} = 0$$

由第二个方程即可知

$$\bar{y} = \hat{\beta}_1\bar{x} + \hat{\beta}_0$$

这即说明回归直线过$(\bar{x},\ \bar{y})$.

1.2

将上述第二个方程代入第一个方程，则

$$\frac{1}{n}(\sum_{i=1}^{n} x_i^2)\hat{\beta}_1 + \bar{x}\bar{y} - (\bar{x})^2\hat{\beta}_1 - \frac{1}{n}\sum_{i=1}^{n} x_i y_i = 0$$

即

$$\frac{1}{n}(\sum_{i=1}^{n}(x_i - \bar{x})^2)\hat{\beta}_1 = \frac{1}{n}(\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}))$$

即

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

这得到了线性回归的结果.

此时

$$\begin{aligned}\mathrm{Cov}(\hat\beta_0,\hat\beta_1)&=\mathrm{E}(\hat\beta_0\hat\beta_1)-\beta_0\beta_1\\&=\mathrm{E}((\bar y-\frac{S_{xy}}{S_{xx}}\bar x)\frac{S_{xy}}{S_{xx}})-\beta_0\beta_1\\&=\mathrm{E}(\sum_{i,j}(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)\frac{x_j-\bar x}{S_{xx}}y_iy_j)-\beta_0\beta_1\\&=\sum_{i,j}(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)\frac{x_j-\bar x}{S_{xx}}\mathrm{E}(y_iy_j)-\beta_0\beta_1\end{aligned}$$

由于

$$\mathrm{E}(y_i^2)=\sigma^2+(\beta_0+\beta_1x_i)^2$$
$$\mathrm{E}(y_iy_j)=(\beta_0+\beta_1x_i)(\beta_0+\beta_1x_j),\ i\neq j$$

故

$$\begin{aligned}\mathrm{Cov}(\hat\beta_0,\hat\beta_1)&=\sum_{i=1}^n(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)\frac{x_i-\bar x}{S_{xx}}\sigma^2+\sum_{i,j}(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)\frac{x_j-\bar x}{S_{xx}}(\beta_0+\beta_1x_i)(\beta_0+\beta_1x_j)-\beta_0\beta_1\\&=-\frac{\sigma^2\bar x}{S_{xx}}+\sum_{i,j}(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)\frac{x_j-\bar x}{S_{xx}}(\beta_0+\beta_1\bar x+\beta_1(x_i-\bar x))(\beta_0+\beta_1\bar x+\beta_1(x_j-\bar x))-\beta_0\beta_1\\&=-\frac{\sigma^2\bar x}{S_{xx}}+\beta_1\sum_{i=1}^n(\frac{1}{n}-\frac{x_i-\bar x}{S_{xx}}\bar x)(\beta_0+\beta_1\bar x+\beta_1(x_i-\bar x))-\beta_0\beta_1\\&=-\frac{\sigma^2\bar x}{S_{xx}}+\beta_0\beta_1+\beta_1^2\bar x-\beta_1^2\bar x-\beta_0\beta_1\\&=-\frac{\sigma^2\bar x}{S_{xx}}\end{aligned}$$

1.3

由于

$$\hat\beta_1=\frac{S_{xy}}{S_{xx}}=\frac{\sum\limits_{i=1}^n(x_i-\bar x)y_i}{S_{xx}}$$

可知

$$\mathrm{Var}(\hat\beta_1)=\frac{\sigma^2}{S_{xx}}$$

最小化此值需让S_{xx}尽可能大，由于

$$S_{xx}=\sum_{i=1}^n(x_i-\bar x)^2$$

于是在n固定的情况下，应尽可能取等量的-1和1作为输入.

1.4

$$\mathrm{SSE}=\sum_{i=1}^{n}(y_i-\beta_1 x_i)^2$$

令此值最小，即

$$\frac{d}{d\beta_1}\mathrm{SSE}=2\sum_{i=1}^{n}(y_i-\beta_1 x_i)=0$$

解得

$$\hat{\beta}_1=\frac{\bar{y}}{\bar{x}}$$

此时由于

$$y_0=\beta_1 x_0+\epsilon_0$$

知

$$\hat{y_0}-y_0=\frac{\bar{y}}{\bar{x}}x_0-\beta_1 x_0-\epsilon_0$$

知

$$\mathrm{Var}(\hat{y_0}-y_0)=\frac{x_0^2\sigma^2}{n\bar{x}^2}+\sigma^2$$

故

$$\frac{\hat{y_0}-y_0}{\sqrt{\frac{x_0^2}{n\bar{x}^2}+1}\sigma}\sim\mathrm{N}(0,1)$$

而令

$$\hat{\sigma}^2=\frac{\sum\limits_{i=1}^{n}(y_i-\frac{\bar{y}}{\bar{x}}x_i)}{n-1}$$

则有

$$(n-1)\hat{\sigma}^2\sim\chi^2(n-1)$$

于是

$$\frac{\hat{y_0}-y_0}{\sqrt{\frac{x_0^2}{n\bar{x}^2}+1}\hat{\sigma}}\sim\mathrm{t}(n-1)$$

故$1-\alpha$置信区间为

$$(\frac{\bar{y}}{\bar{x}}x_0-\hat{\sigma}\sqrt{\frac{x_0^2}{n\bar{x}^2}+1}t_{\frac{\alpha}{2}}(n-1),\ \frac{\bar{y}}{\bar{x}}x_0+\hat{\sigma}\sqrt{\frac{x_0^2}{n\bar{x}^2}+1}t_{\frac{\alpha}{2}}(n-1))$$

其中$\hat{\sigma}$见上述定义.

2 线性回归(2)

2.1

似然函数

$$L(\beta_0,\beta_1,\sigma)=\prod_{i=1}^{n}(\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(y_i-\beta_1 x_i-\beta_0)^2}{2\sigma^2}))$$

对数似然为

$$\ln L=-n\ln(\sqrt{2\pi}\sigma)-\sum_{i=1}^{n}\frac{(y_i-\beta_1 x_i-\beta_0)^2}{2\sigma^2}$$

则

$$\frac{\partial}{\partial\sigma}\ln L=-\frac{n}{\sigma}+\sum_{i=1}^{n}\frac{(y_i-\beta_1 x_i-\beta_0)^2}{\sigma^3}$$

$$\frac{\partial}{\partial \beta_0}\ln L = -\sum_{i=1}^{n}\frac{(y_i-\beta_1 x_i-\beta_0)}{\sigma^2}$$

$$\frac{\partial}{\partial \beta_1}\ln L = -\sum_{i=1}^{n}\frac{x_i(y_i-\beta_1 x_i-\beta_0)}{\sigma^2}$$

令上述三个偏导数为0，即得到极大似然估计：

$$(\sigma^*)^2 = \frac{\sum\limits_{i=1}^{n}(y_i-\hat{\beta}_1 x_i-\hat{\beta}_0)^2}{n}$$

$$\hat{\beta}_1\bar{x}+\hat{\beta}_0=\bar{y}$$

$$\sum_{i=1}^{n}x_i^2\hat{\beta}_1+n\bar{x}\hat{\beta}_0=\sum_{i=1}^{n}x_i y_i$$

解得

$$\hat{\beta}_1=\frac{S_{xy}}{S_{xx}}$$

$$\hat{\beta}_0=\bar{y}-\hat{\beta}_1\bar{x}$$

$$(\sigma^*)^2 = \frac{\sum\limits_{i=1}^{n}(y_i-\hat{\beta}_1 x_i-\hat{\beta}_0)^2}{n}=\frac{1}{n}\mathrm{SSE}$$

2.2

$$\mathrm{E}(\mathrm{SSE})=\sum_{i=1}^{n}\mathrm{E}((y_i-\hat{\beta}_1 x_i-\hat{\beta}_0)^2)$$

$$=\sum_{i=1}^{n}\mathrm{E}(y_i^2)+\sum_{i=1}^{n}\mathrm{E}(\hat{\beta}_1{}^2x_i^2)+\sum_{i=1}^{n}\mathrm{E}(\hat{\beta}_0{}^2)-2\sum_{i=1}^{n}\mathrm{E}(\hat{\beta}_1 x_i y_i)-2\sum_{i=1}^{n}\mathrm{E}(\hat{\beta}_0 y_i)+2\sum_{i=1}^{n}\mathrm{E}(\hat{\beta}_1\hat{\beta}_0 x_i)$$

$$=\sum_{i=1}^{n}(\sigma^2+(\beta_0+\beta_1 x_i)^2+\frac{\sigma^2}{S_{xx}}x_i^2+\beta_1^2x_i^2+(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}})\sigma^2+\beta_0^2)$$

$$-2\sigma^2-2\beta_1^2S_{xx}-2n\beta_0\beta_1\bar{x}-2n\beta_1^2\bar{x}^2-2\sigma^2-2n\beta_0^2-2n\beta_0\beta_1\bar{x}-2n\bar{x}^2\frac{\sigma^2}{S_{xx}}+2n\beta_0\beta_1\bar{x}$$

$$=n\sigma^2+n(\beta_0+\beta_1\bar{x})^2+\beta_1^2S_{xx}+\sigma^2+n\bar{x}^2\frac{\sigma^2}{S_{xx}}+\beta_1^2S_{xx}+n\bar{x}^2\beta_1^2+\sigma^2+n\bar{x}^2\frac{\sigma^2}{S_{xx}}+n\beta_0^2$$

$$-2\sigma^2-2\beta_1^2S_{xx}-2n\beta_0\beta_1\bar{x}-2n\beta_1^2\bar{x}^2-2\sigma^2-2n\beta_0^2-2n\beta_0\beta_1\bar{x}-2n\bar{x}^2\frac{\sigma^2}{S_{xx}}+2n\beta_0\beta_1\bar{x}$$

$$=(n-2)\sigma^2$$

即$\frac{\mathrm{SSE}}{n-2}$是σ^2的无偏估计.