# Homework 1



## Written assignment

**Question 1.**

*Evaluate the general solutions of the below ODEs:*

*(a)* $m\ddot{x} + kx = 0$

*(b)* $m\ddot{x} + \lambda\dot{x} + kx = 0$

*(c)* $m\ddot{x} + \lambda\dot{x} + kx = F_0 \cos \omega_f t$

(a)

Rewrite the ODE

$$\ddot{x} = -\frac{k}{m}x$$

then

$$x = A \sin\left(\sqrt{\frac{k}{m}} + \phi\right),$$

where $A$ is the amplitude and $\phi$ is the phase.

(b)

Rewrite the ODE

$$\ddot{x} + \frac{\lambda}{m}\dot{x} + \frac{k}{m}x = 0$$

Let $\frac{d}{dt} \equiv D$, $\gamma \equiv \frac{\lambda}{2m}$ and $\omega_0 \equiv \frac{k}{m}$, then

$$(D^2 + 2\gamma D + \omega_0^2)x = 0$$

$$D = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \equiv -\gamma \pm \omega$$

Therefore,

$$[D - (-\gamma + \omega)][D - (-\gamma - \omega)]x = 0$$

$$x = e^{-\gamma t}(A_1 e^{-\omega t} + A_2 e^{\omega t})$$

There are three scenarios we need to consider. If

(1) $\omega$ is imaginary ($\gamma^2 < \omega_0^2$), then

$$x = e^{-\gamma t}\left(A_1 \sin(\sqrt{\omega_0^2 - \gamma^2}t) + A_2 \cos(\sqrt{\omega_0^2 - \gamma^2}t)\right) = Ae^{-\gamma t}\cos\left(\sqrt{\omega_0^2 - \gamma^2}t + \phi\right)$$

(2) $\omega = 0$ ($\gamma^2 = \omega_0^2$), then

$$x = A_1 e^{-\gamma t} + A_2 t e^{-\gamma t}$$

(3) $\omega$ is real. ($\gamma^2 > \omega_0^2$) Then

$$x = e^{-\gamma t}(A_1 e^{-\omega t} + A_2 e^{\omega t})$$

where $A_i$ are the coefficients in the complex plane.
(c)
This ODE is not homogeneous. The general solution of such ODE $x = x_h + x_p$. $x_h$ is the homogeneous solution that I showed in part (b). Here we focus on the particular solution $x_p$.
Let $F_0 \cos(\omega_f t) = \Re[F_0 \exp i\omega_f t]$, $x(t) = \Re[X(t)]$, $\dot{x} = \Re[\dot{X}]$ and $\ddot{x} = \Re[\ddot{X}]$.
The ODE can be rewritten in the complex plane

$$m\ddot{X} + \lambda \dot{X} + kX = F_0 \exp(i\omega_f t)$$

Let $X_p = A \exp(i\omega_f t)$.

$$(-m\omega_f^2 + i\omega_f \lambda + k) A \exp(i\omega_f t) = F_0 \exp(i\omega_f t)$$

$$A = \frac{F_0/m}{(\omega_0^2 - \omega_f^2) + 2i\gamma\omega_f}$$

We convert that back.

$$x_p = \Re[X_p] = |A| \cos(\omega_f t)$$

where

$$|A| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2}}$$

As $t \to \infty$, $x \to x_p$.

## Programming assignment

**Question 2.**

*Reuse your IVP solver* ***solver.py*** *and perform simulations of a damped oscillator (following the definition we used in the lecture). Make a plot for $x$ and $\dot{x}$ versus $t$ and a phase diagram (in polar coordinate) with the following initial conditions:*

*(a)* $A = 1$ cm, $\omega_0 = 1$ rad s$^{-1}$, $\gamma = 0.2$ s$^{-1}$, and $\phi = -\pi/2$ rad,

*(b)* $A = 1$ cm, $\omega_0 = 1$ rad s$^{-1}$, $\gamma = 1.0$ s$^{-1}$, and $\phi = -\pi/2$ rad,

*(c)* $A = 1$ cm, $\omega_0 = 1$ rad s$^{-1}$, $\gamma = 1.2$ s$^{-1}$, and $\phi = -\pi/2$ rad,

From the given conditions, in the underdamped case, $x(0) = 0$, $\dot{x}(0) = 1$. According to the theory, the critical damping parameter $\gamma_c = 1$ s$^{-1}$. The three provided initial conditions are underdamped, critically damped, and overdamped respectively. Therefore, the behaviors are expected to be different.

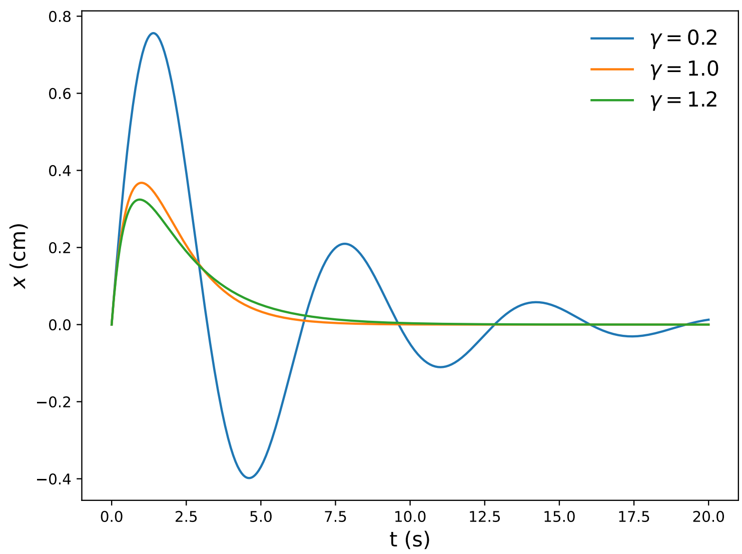


Figure 1: x-t plot of three different initial conditions.

It is clear in Figure 1 that the damping parameter $\gamma$ affects the curve. The critically damped curve rapidly falls to position $x = 0$ while the positions are still moving in the other two scenarios.

From Figure 2, the reason why the time when the position falls to zero in the overdamped scenario is also longer than in the critically damped scenario. The damping parameter is too large that the velocity of the overdamped particle is smaller than that of the critically damped particle.

In Figure 3, the phase diagram of the three different initial conditions is very interesting. The critically damped scenario clearly shows how it is critical. The kinetic energy falls to zero the fastest.

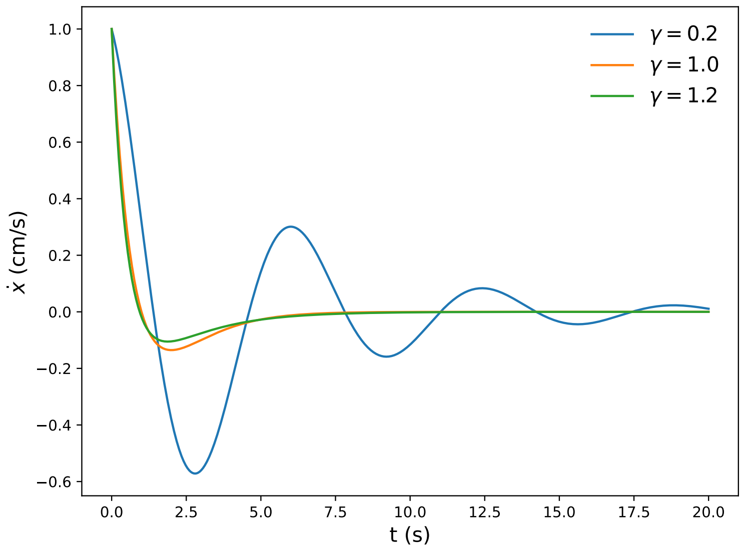


Figure 2: v-t plot of three different initial conditions.

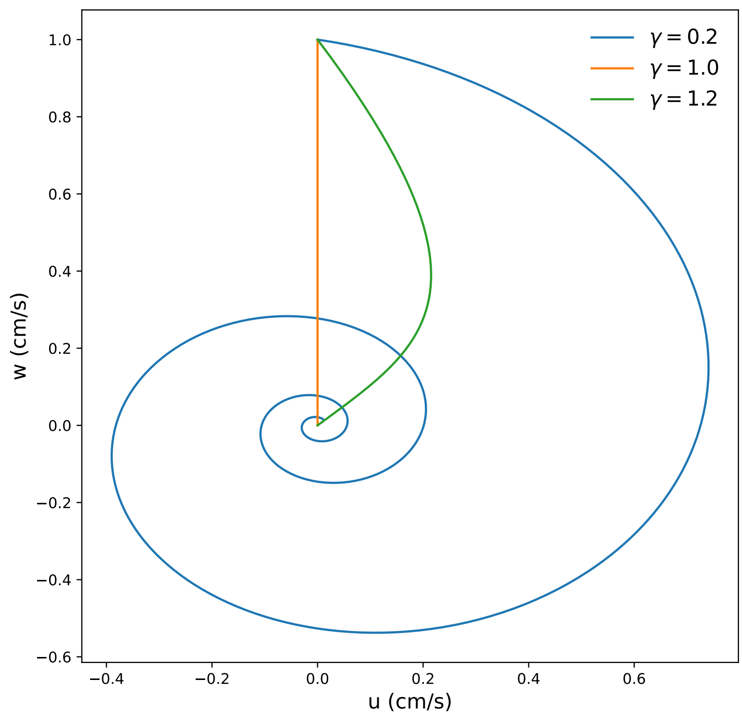


Figure 3: Phase diagram of three different initial conditions.

**Question 3.**

*Following the programming problem 1(a), make two plots of the total energy and energy loss rate versus time for the damped oscillator.*

The total energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

However, since $k$ and $m$ are not defined, we can divide the energy by mass and write down the total energy per unit mass

$$e = \frac{1}{2}v^2 + \frac{1}{2}\omega_0^2 x^2$$

Figure 4 shows how the energy evolves for the initial condition in 1(a).

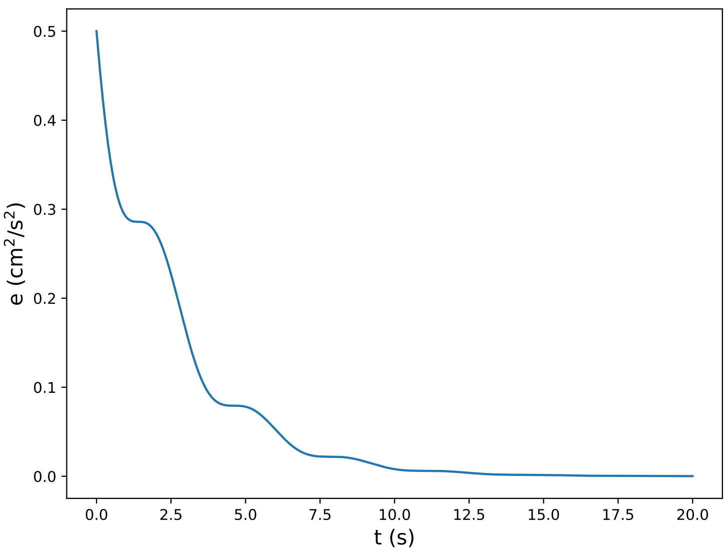


Figure 4: Energy per unit mass versus time plot.

By differentiating the energy by time, the energy loss rate can be shown in Figure 5. The periodic behaviors in the plots are obvious. Comparing Figure 5 to Figure 1 and Figure 2, the energy loss rate goes to zero when the particle is at the peaks of $x - t$ plot when the velocity also at zero, which matches the theoretical expectations.

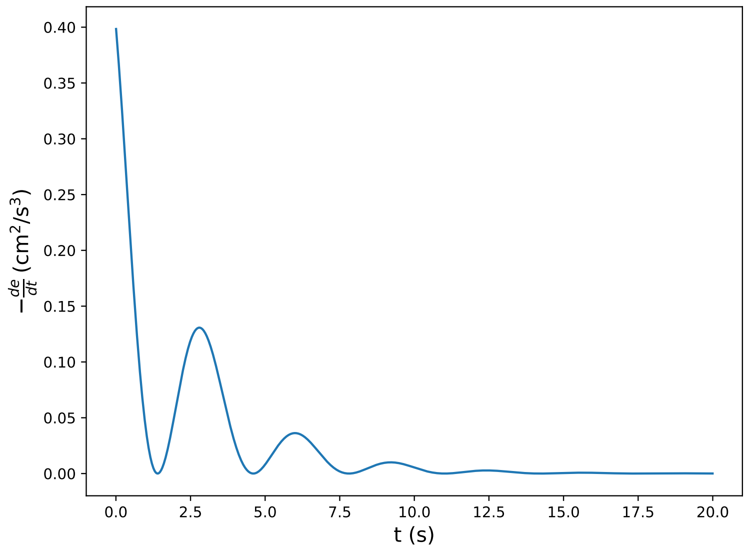


Figure 5: Energy loss rate versus time plot.

**Question 4.**

*Resonance: Now add a sinusoidal driving force ($F = F_0 \cos(\omega_f t)$ in your damped oscillator with $F_0 = 0.5$. Vary $\omega_f$ from 0.5 to 1.5 with an interval of 0.05. Rerun your simulation up to $t_{\max} = 50$. Measure the average amplitude of your oscillator (define $D = \langle |x(t)| \rangle$ between $40 < t < 50$). Make plots of $D$ versus $\omega_f$ with (a) $\lambda = 0.01$, (b) $\lambda = 0.1$, (c) $\lambda = 0.3$. Draw these three plots on the same figure. Do you find resonance? Are these resonance frequencies consistent with the analytical values?*

In these cases, they are all under-damped. Cases (b) and (c) are stable between $40 < t < 50$ but the damping parameter in case (a) is too small that the oscillator is not yet stable between the averaging time period. However, the resonance frequency is still visible in Figure 6.

Except for case (c), the resonance frequencies are obvious and correspond to the analytical value. In case (c), the damping parameter is big so that the resonance frequency is not obvious but it is still close to the analytical value.

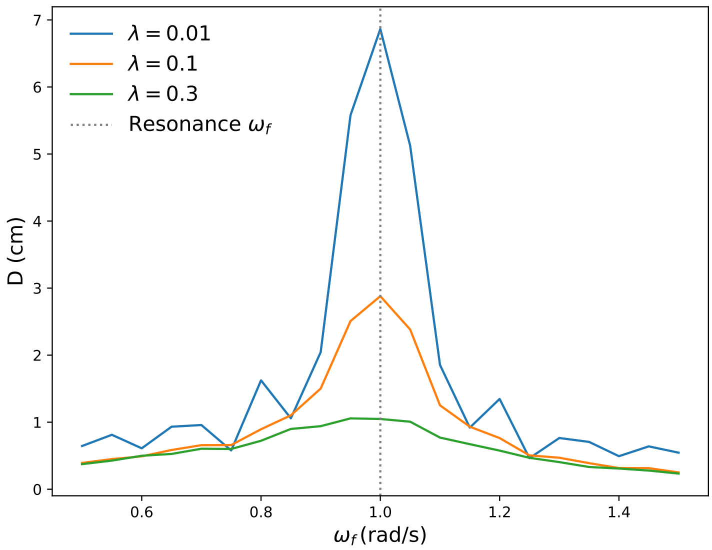


Figure 6: $D$ versus $\omega$ plot. The analytical value of resonance frequency is labeled with the gray dashed line.

**Question 5.**

*A hanging mass-spring system can be analogized to RLC circuit systems. The gravitational force ($F = mg$) is analogous to the emf. The damping parameter has the electrical analog resistance ($R$). The displacement has the electrical analog charge ($q$), ...etc.*

*Consider the series RLC circuit shown in the below Figure driven by an alternating emf of value $E_0 \sin \omega t$.*

*(a) Show that the RLC system can be described by an ODE system,*

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = E_0 \sin \omega t$$

*(b) Use the same IVP solver we developed in the class to numerically solve the system with initial conditions: $L = C = E0 = 1$, $R = 0.8$, and $\omega = 0.7$. Make plots of the current and the voltage $V_L$ across the inductor as functions of time.*

*(c) Redo the problem by varying $\omega$ from 0.3 to 1.5 with an interval 0.1. Do you see any special $\omega$? What are the meaning of these frequencies?*

(a) In an RLC circuit, the resistor ($R$), inductor ($L$), and capacitor ($C$) are all in series with the voltage source. From the Kirchhoff's voltage law (KVL),

$$V_R + V_L + V_C = V(t),$$

where $V_R$, $V_L$ and $V_C$ are the voltages across $R$, $L$ and $C$ respectively and $V(t)$ is a time-varying voltage source.

Substitute with $V_R = IR$, $V_L = L\frac{dI}{dt}$ and $V_C = \frac{1}{C}\int_0^t Id\tau$, the equation becomes

$$IR + L\frac{dI}{dt} + \frac{1}{C}\int_0^t Id\tau = V(t)$$

With the relation $I = \frac{q}{t}$ and $V(t) = E_0 \sin\omega t$, the given ODE system is obtained

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = E_0 \sin\omega t$$

(b)

Figure 7 shows the current and the voltage across the inductor versus time. It is clear that the voltage across the inductor and the current have a phase difference.

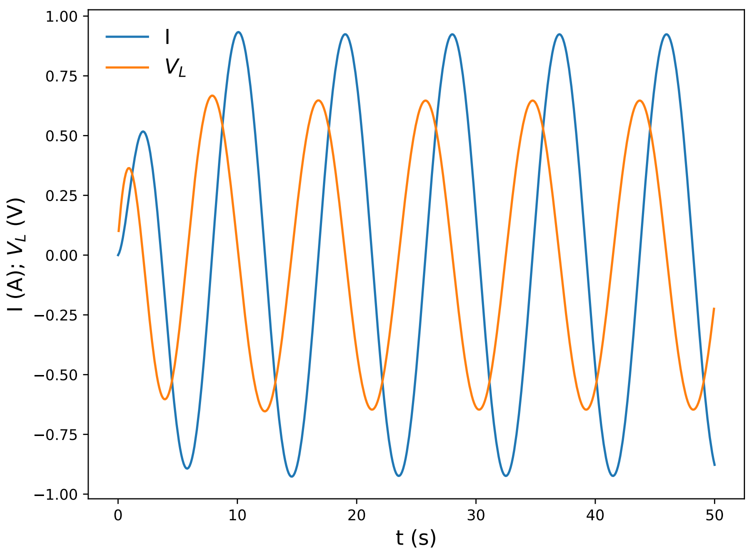


Figure 7: The current and the voltage across the inductor time evolution plot.

(c)

In Figure 8, there are two special frequencies, where the peaks of current and $V_L$ appear respectively. It is predicted in the theory that the resonance frequency of $V_L$ (where the $V_L$ goes to peak) will be larger than that of the current $I$, which can also be observed in the figure.

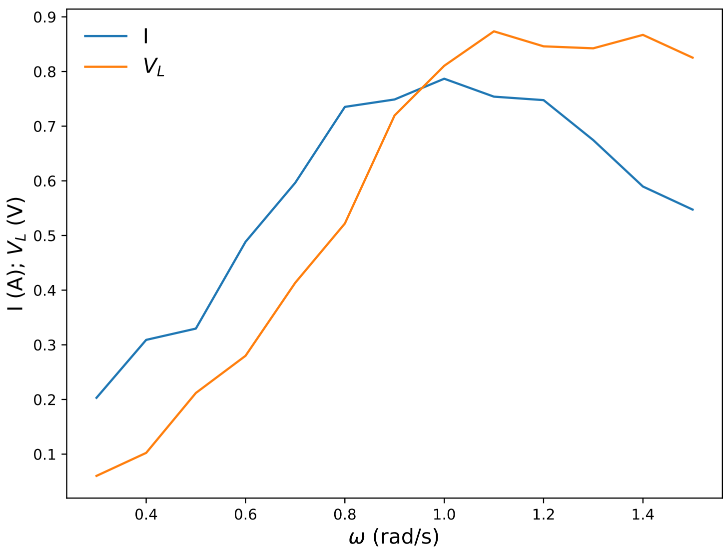


Figure 8: The average current and voltage across the inductor versus the frequency $\omega$.