Lecture-2

Multiplying a matrix with a column vector.

$$A \in \mathbb{F}^{m \times n}, \quad \underline{x} \in \mathbb{F}^{n}$$

$$A = [a_{i,j}] = [\underline{a}_1 \cdots \underline{a}_n]$$

$$\underline{y} = A\underline{x} = \sum_{j=1}^{n} x_j \underline{a}_j$$

Column interpretation of matrix-vector multiplication.

Row interpretation

$\underline{x} \in \mathbb{F}^n$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{x}^T = [x_1 \cdots x_n]$$

View $A \in \mathbb{F}^{m \times n}$ as a collection of row vectors.

$$A = \begin{bmatrix} \longleftarrow & \underline{\tilde{a}}_1^T & \longrightarrow \\ \longleftarrow & \underline{\tilde{a}}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \underline{\tilde{a}}_m^T & \longrightarrow \end{bmatrix}$$

$\underline{\tilde{a}}_i \in \mathbb{F}^n$

$$\underline{y} = A\underline{x}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \underline{\tilde{a}}_1^T \\ \vdots \\ \underline{\tilde{a}}_m^T \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$y_1 = \underline{\tilde{a}}_1^T \underline{x}$$

$$y_2 = \underline{\tilde{a}}_2^T \underline{x}$$

$$\vdots$$

$$y_m = \underline{\tilde{a}}_m^T \underline{x}$$

View

collec

$A =$

Examples.

① $\underline{O} \in \mathbb{F}^{m \times n}$

$\underline{O}\,\underline{x} = \underline{0}$

② $\underline{I} \in \mathbb{F}^{n \times n}$

$$\underline{I} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

$$\underline{I}\,\underline{x} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \underline{x}$$

(3) Suppose we want A

Such that

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \underline{y}$$

$$A_{3\times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(1)

(2)

a

h

) 0s.

there ?

4 Permuting the entries of a vector

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

$$A_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3

A

$A_{3\times 5} =$

An $n \times n$ matrix is called a

<u>permutation</u> <u>matrix</u> if each

row and each column has

exactly one 1 and $(n-1)$ 0s.

<u>Q</u>: How many such matrices are there?

<u>A</u>: $n!$

Suppose $f(x) = 3 + 2x + 7x^2$

and say $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$

$$y_1 = f(\alpha_1), \ldots, y_m = f(\alpha_m)$$

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ & \vdots & \\ 1 & \alpha_m & \alpha_m^2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

Properties.

(i) $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$

(ii) $(A+B)\underline{x} = A\underline{x} + B\underline{x}$

(iii) $A(\alpha\underline{x}) = \alpha(A\underline{x})$

$\alpha \in \mathbb{F}$

$= (\alpha A)x$

Multiplying a row vector

with a matrix

Say $\underline{x} \in \mathbb{F}^m$, $A \in \mathbb{F}^{m \times n}$

$$\underset{1 \times m}{\underline{x}^T} \cdot \underset{m \times n}{A} = \underset{1 \times n}{\underline{y}^T}, \quad \underline{y} \in \mathbb{F}^n$$

$$A = \begin{bmatrix} \underline{\tilde{a}}_1^T \\ \vdots \\ \underline{\tilde{a}}_m^T \end{bmatrix}$$

(i)

(ii) (

(iii) A

Row interpretation:

$$\underline{y}^T = \underline{x}^T A$$

$$= \begin{bmatrix} x_1 \cdots x_m \end{bmatrix} \begin{bmatrix} \tilde{\underline{a}}_1^T \\ \vdots \\ \tilde{\underline{a}}_m^T \end{bmatrix}$$

$$= x_1 \tilde{\underline{a}}_1^T + \cdots + x_m \tilde{\underline{a}}_m^T$$

Sa

$\underline{x}$

$1 \times m$

A

Column interpretation $A = \left[\underline{a}_1 \cdots \underline{a}_n\right]$

$$\underline{y}^T = \left[y_1 \cdots y_n\right] = \underline{x}^T A$$

$$= \underline{x}^T \cdot \left[\underline{a}_1 \;\; \underline{a}_2 \cdots \underline{a}_n\right]$$

$$y_1 = \underline{x}^T \underline{a}_1$$

$$= \left[x_1 \cdots x_m\right] \cdot \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$$

$$= \sum_{i=1}^{m} x_i a_{i,1}$$

$$y_j = \underline{x}^T \underline{a}_j$$

$$= \sum_{i=1}^{m} x_i a_{i,j}$$

$\ldots x_1 \quad x_4]$

Examples.

① $\begin{bmatrix} 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\underline{a}}_1^T \\ \tilde{\underline{a}}_2^T \\ \tilde{\underline{a}}_3^T \end{bmatrix}$

$$= 3\tilde{\underline{a}}_1^T - \tilde{\underline{a}}_2^T + \tilde{\underline{a}}_3^T$$

$+y^T A$

$+ \underline{x}^T B$

$= \alpha \cdot (\underline{x}^T A)$

2

$$[x_1 \cdots x_4] \cdot A = [x_1 \quad x_4]$$

$$A_{4\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Examp

1 $[3 \quad -1$

$= 3$

$$(\underline{x}^T + \underline{y}^T)A = \underline{x}^T A + \underline{y}^T A$$

$$\underline{x}^T(A+B) = \underline{x}^T A + \underline{x}^T B$$

$$\underline{x}^T(\alpha A) = (\alpha x)^T A = \alpha \cdot (\underline{x}^T A)$$

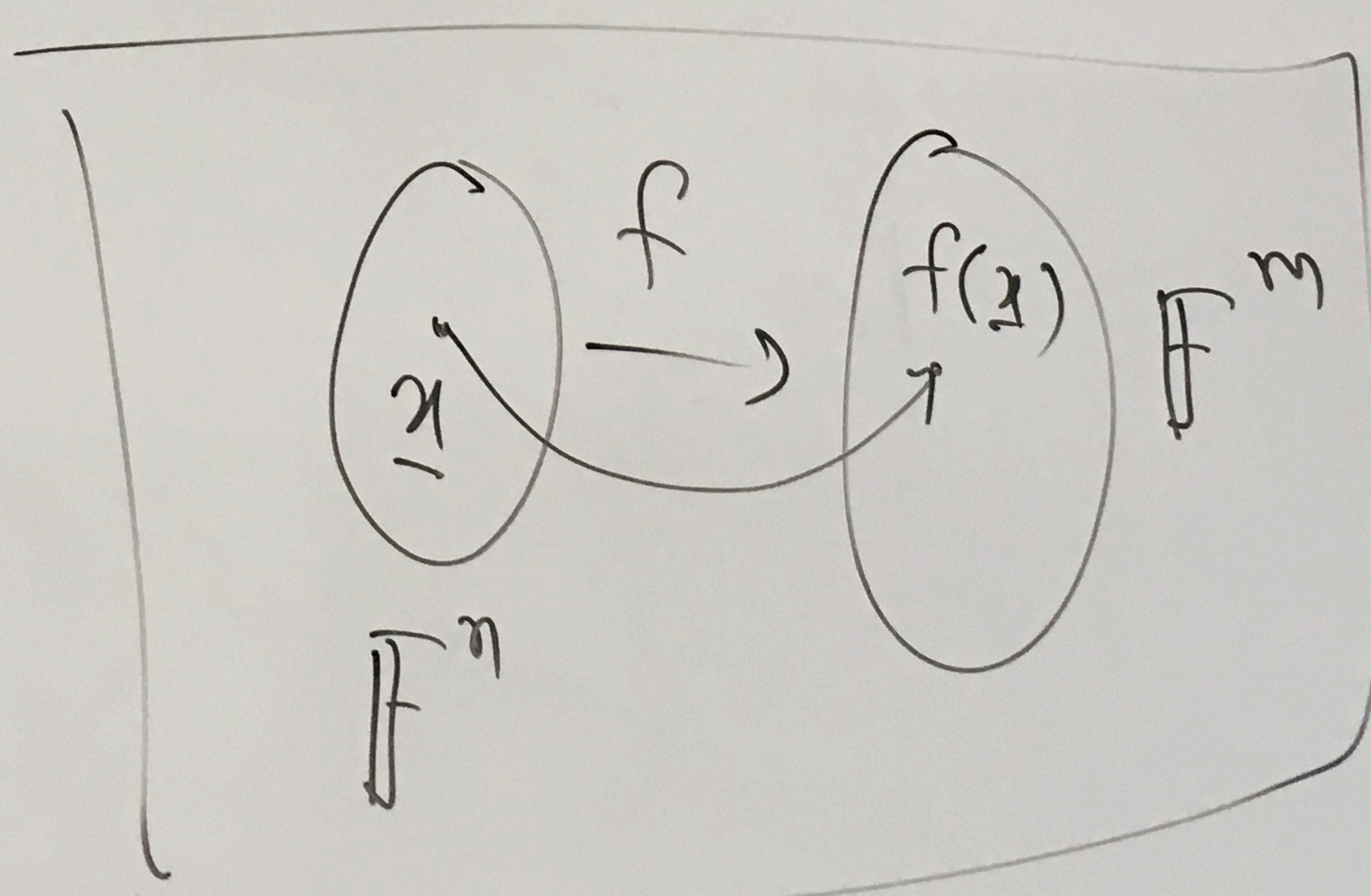
2

Linear Transformation/function.

(Section 1.8 of D. Lay)

Defn: A linear transformation or a linear function

$$f : \mathbb{F}^n \to \mathbb{F}^m$$



is.

x-axis
(dim 1)

is a map (or a rule) that

assigns a vector $f(\underline{x}) \in \mathbb{F}^m$

to every vector $\underline{x} \in \mathbb{F}^n$, such that

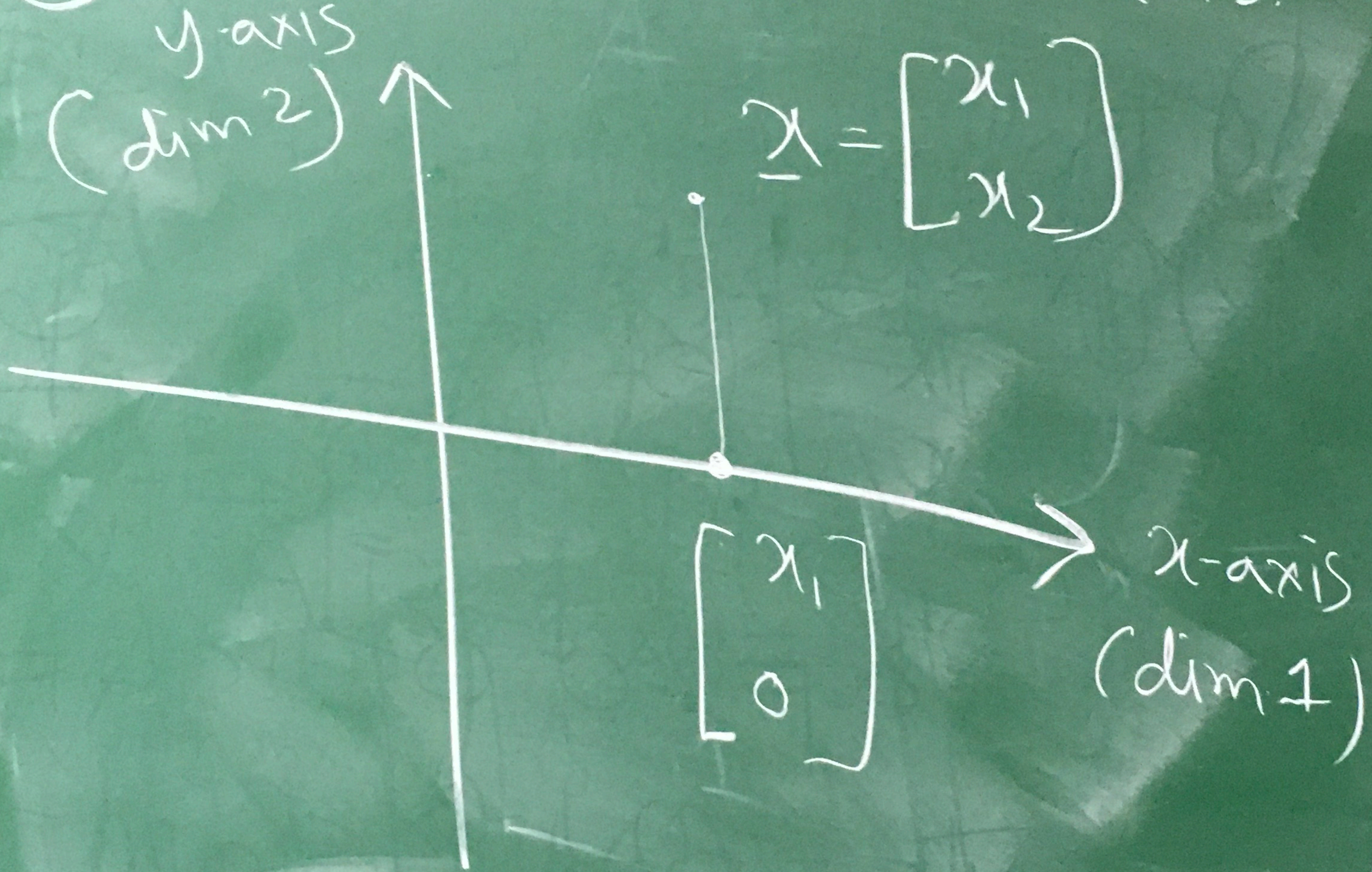
(i) $f(\underline{x} + \underline{y}) = f(\underline{x}) + f(\underline{y})$

(ii) $f(\alpha \underline{x}) = \alpha f(\underline{x})$

for all $\alpha \in \mathbb{F}$, $\underline{x}, \underline{y} \in \mathbb{F}^n$.

Examples

① Projection onto x-axis.



$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

is a map

assigns a

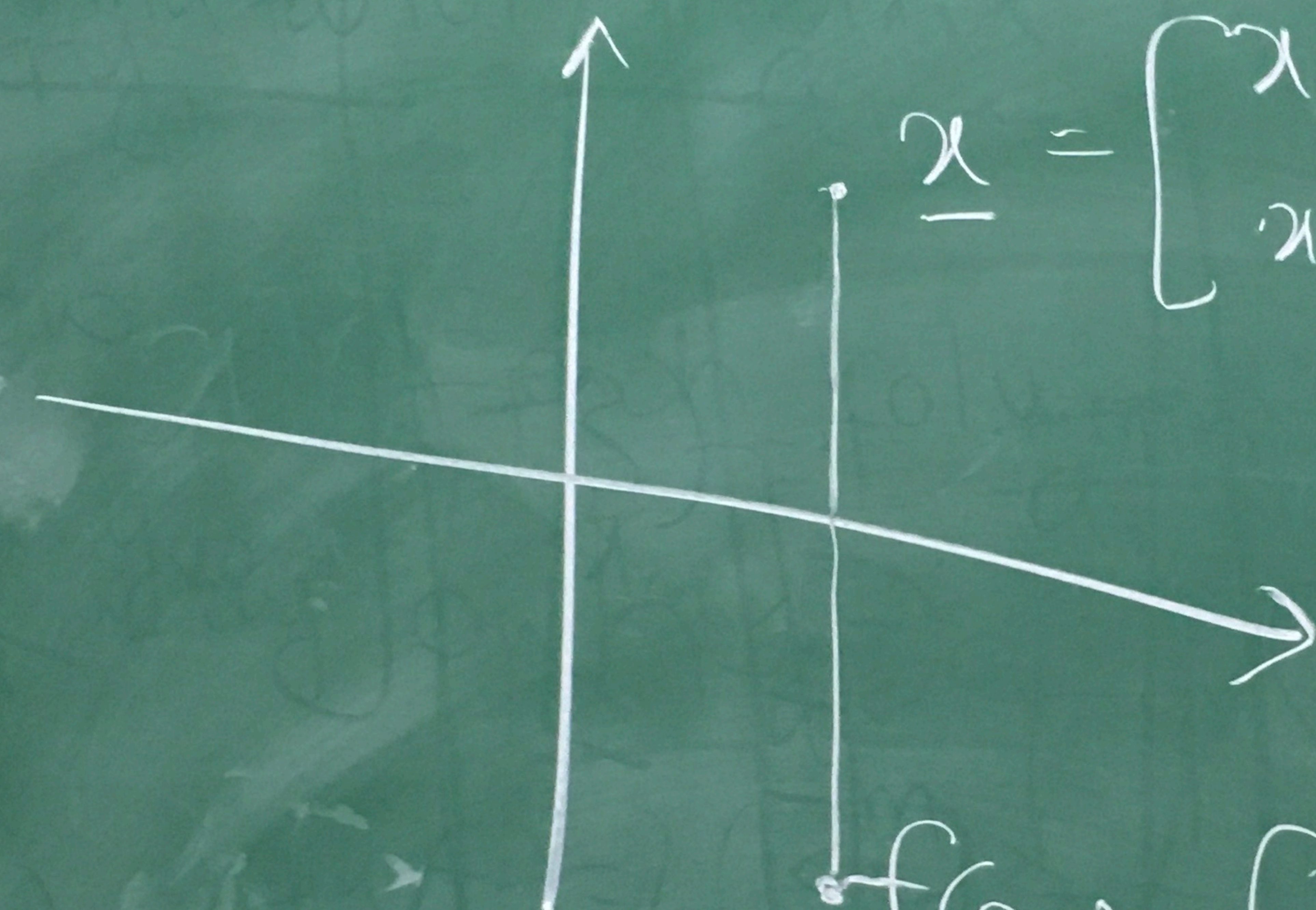
to every ve

(i) $f(\underline{x} +$

(ii) $f(\alpha \underline{x}$

for all

② Reflection about x-axis.



$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f(\underline{x}) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Exa

①

(di

f

(3) Circular convolution with $\underline{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = \underline{y} = f(\underline{x}) = \underline{c} \circledast \underline{x} \quad , \quad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$y_0 = \sum_{i=0}^{n-1} x_i \, c_{(0-i) \bmod n} \quad , \quad j = 0, \ldots, n-1$$

$\underline{\underline{n=3.}}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix}}_{} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

<u>circulant matrix.</u>

s.

x-axis

dim 1)

$$y_j = x_0 c_j + x_1 c_{j-1}$$

$$+ x_2 c_{j-2} + \cdots$$

$$= \sum_i x_i c_{(j-i) \bmod n}$$