EE5609 LECTURE-6

Matrix Multiplication

$$A \in \mathbb{F}^{m \times n} \qquad B \in \mathbb{F}^{n \times p}$$

$$C = AB \in \mathbb{F}^{m \times p}$$

$$c_{ij} = \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}.$$

In general: $AB = 0 \not\Rightarrow A = 0$ or $B = 0$.

In general, $AB = AC \not\Rightarrow B = C$.

<u>Remark:</u> cancellation law holds if A is invertible.

If A is invertible and if $AB = AC$

then $B = C$.

In general, $AB \neq BA$.

For instance: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Both $\underset{m \times n}{A}\,\underset{n \times \underline{m}}{B}$ and $\underset{n \times m}{B}\,\underset{m \times n}{A}$ are well defined only

when #rows B = #cols A

& #cols B = #rows A.

If we need AB & BA to be equal. then $m=n$, i.e. A & B must be square and of the same size.

<u>Definition:</u> We say that two square matrices A & B (of the same size) <u>commute</u> if

$$AB = BA$$

<u>Remark:</u> Matrix multiplication is <u>non-commutative</u> ($AB \neq BA$, in general).

Example: If $A, B \in \mathbb{F}^{n \times n}$ and are both diagonal.

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{n,n} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & & & 0 \\ & b_{22} & & \\ & & \ddots & \\ 0 & & & b_{n,n} \end{bmatrix}$$

$$AB = BA = \begin{bmatrix} a_{11}b_{11} & & & 0 \\ & a_{22}b_{22} & & \\ & & \ddots & \\ 0 & & & a_{n,n}b_{n,n} \end{bmatrix}$$

Two diagonal matrices of same size commute.

Example: $A \in \mathbb{F}^{n\times n}$ $\quad B = I_{n\times n}$

$AB = A, \quad BA = I \quad \therefore A \,\&\, B$ commute.

Scalar Matrix

Example: $A \in \mathbb{F}^{n\times n}$ $\quad B = \lambda I_{n\times n} = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}$

$\lambda \in \mathbb{F}.$

$AB = BA = \lambda A$

In matrix theory,

it is a fundamental and important question to identify matrices that commute with each other.

Diagonalization, eigen values, eigenvectors.

How do we multiply block matrices?

$$A = \begin{array}{c c} \left[\begin{array}{c|c|c} B & C & D \\ \hline E & F & G \end{array}\right] & \begin{array}{c} m_1 \\ m_2 \end{array} \\ \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} & \end{array} \qquad Z = \begin{array}{c c} \left[\begin{array}{c|c|c} H & L & P \\ \hline J & M & Q \\ \hline K & N & S \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \\ \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} & \end{array}$$

2×3 block matrix $\qquad\qquad$ 3×3 block matrix

$$AZ = \begin{bmatrix} BH + CJ + DK & BL + CM + DN & BP + CQ + DS \\ EH + FJ + GK & EL + FM + GN & EP + FQ + GS \end{bmatrix}$$

Properties of Matrix Multiplication :

1. $A(BC) = (AB)C \overset{\Delta}{=} ABC$

2. $A(B+C) = AB + AC$

   $(A+B)C = AC + BC$

3. $\alpha(AB) = (\alpha A)B = A(\alpha B) = \alpha AB$

5. Say A is $m \times n$:

$$I_{m\times m} A = A = A \cdot I_{n\times n}$$

Transpose and Hermitian Transpose

Definition: If $A \in \mathbb{F}^{m \times n}$ then $A^T \in \mathbb{F}^{n \times m}$

and A^T is defined as:

$$[A^T]_{i,j} = [A]_{j,i}$$

The rows of A correspond to the columns of A^T

The cols. of A " " " rows of A^T.

Example: $\mathbb{F} = \mathbb{C}$

$$A = \begin{bmatrix} 2+i & 3 & -i \\ 5 & 6 & 7+2i \end{bmatrix} \qquad A^T = \begin{bmatrix} 2+i & 5 \\ 3 & 6 \\ -i & 7+2i \end{bmatrix}$$

Properties:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $\boxed{(AB)^T = B^T A^T}$
4. If $\alpha \in \mathbb{F}$

   $(\alpha A)^T = \alpha (A^T)$

Can you prove this statement?

If $a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$

$$\overline{a+ib} = a - ib = a + i(-b)$$

$c_1, c_2 \in \mathbb{C}$, $\overline{c_1 c_2} = \overline{c_1} \cdot \overline{c_2}$

$$\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}.$$

Define: Hermitian transpose, Conjugate transpose, adjoint.

$$A_{m \times n} \longrightarrow A^H_{n \times m} \quad \left\{ \begin{array}{l} A^\dagger \leftarrow \text{dagger} \\ A^* \end{array} \right.$$

$$[A^H]_{i,j} = \overline{[A]_{j,i}}$$

$$A = \begin{bmatrix} 2+i & 3 & -i \\ 5 & 6 & 7+2i \end{bmatrix} \qquad A^H = \begin{bmatrix} 2-i & 5 \\ 3 & 6 \\ +i & 7-2i \end{bmatrix}$$

Remark: If A is a real matrix $\overline{[A]_{j,i}} = [A]_{j,i}$

$\therefore \quad A^H = A^T$

Properties:

1. $(A+B)^H = A^H + B^H$

2. $(AB)^H = B^H \cdot A^H$

3. $(A^H)^H = A$

4. $(A^H)^T = \overline{A}$

$[(A^H)^T]_{i,j} = \overline{[A]_{i,j}}$

5. $\alpha \in \mathbb{F}$,

$(\alpha A)^H = \overline{\alpha} A^H$

Symmetric and Hermitian Matrices

Definition: A square matrix A is symmetric if $A = A^T$.

Example: 1. $A = \begin{bmatrix} 3 & 5-i \\ 5-i & 4+i \end{bmatrix}$, $A^T = A$

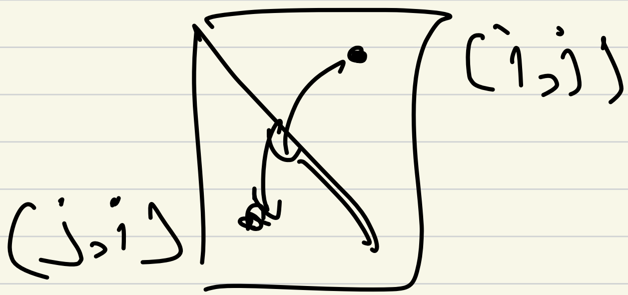
2. $A = \begin{bmatrix} 5 & 3 & -7 \\ 3 & 6 & 0 \\ -7 & 0 & 7 \end{bmatrix}$, $A^T = A$

Definition: A matrix A is skew-symmetric

if $A^T = -A$.

$[A^T]_{i,j} = [A]_{j,i}$, this must be equal to

$= [-A]_{i,j} = -[A]_{i,j}$



this implies:

$$[A]_{i,i} = -[A]_{i,i} \Rightarrow [A]_{i,i} = 0 \ \forall i.$$

If A is skew-symmetric, its main diagonal will be all zero.

Examples:

$$A = \begin{bmatrix} 0 & 2 & 5+i \\ -2 & 0 & 3 \\ -5-i & -3 & 0 \end{bmatrix}, \quad A^T = -A.$$

Claim: Every square matrix is the sum of a symmetric and a skew-symmetric matrix.

Proof:
$$A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T$$
$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

$$\left[\frac{1}{2}(A+A^T)\right]^T = \frac{1}{2}\left[(A+A^T)\right]^T$$

$$= \frac{1}{2}\left[A^T + (A^T)^T\right]$$

$$= \frac{1}{2}\left[A + A^T\right].$$

lll-ly $\left[\frac{1}{2}(A-A^T)\right]^T = -\frac{1}{2}(A-A^T).$

$\therefore$ A is a sum of a symmetric and a skew symmetric matrix.

Definition: A square matrix $A \in \mathbb{F}^{n \times n}$ is a Hermitian matrix if

$$A^H = A$$

A is skew-Hermitian if

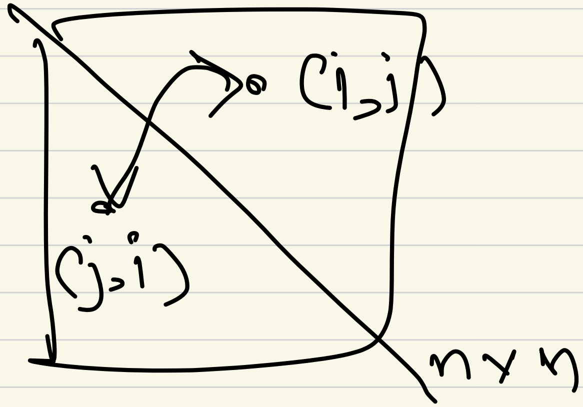
$$A^H = -A.$$

$$[A^H]_{i,j} = \overline{[A]_{j,i}}$$

If A is a Hermitian matrix: $A^H = A$.

$$[A^H]_{i,j} = [A]_{i,j}$$

$$= \boxed{\overline{[A]_{j,i}} = [A]_{i,j}}.$$



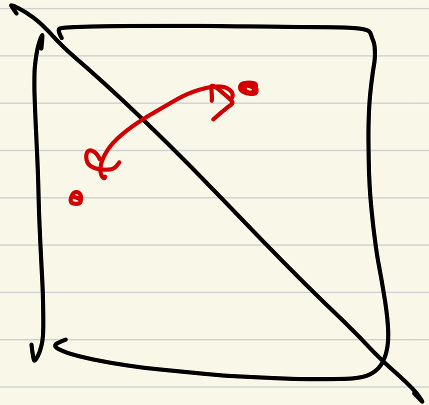
$$[A]_{i,i} = \overline{[A]_{i,i}} \Rightarrow [A]_{i,i} \text{ is real}$$

$$\forall\, i = 1, \dots, n.$$

Similarly,

for a skew-Hermitian matrix A:

$$[A]_{j,i} = -\overline{[A]_{i,j}}$$



For $i=j$:

$$[A]_{i,i} = -\overline{[A]_{i,i}}$$

$\Rightarrow$ $[A]_{i,i}$ is purely imaginary $\forall\, i$.

$$A = \begin{bmatrix} i & 5+3i \\ -5+3i & 2i \end{bmatrix}, \quad A^H = -A.$$

Claim: Any complex matrix is the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof:

$$A = \underbrace{\frac{1}{2}(A + A^H)}_{\text{Hermitian}} + \underbrace{\frac{1}{2}(A - A^H)}_{\text{Skew-Hermitian}}$$

Note:

For real matrices $A^T = A^H$

A is real and Hermitian

$\Longleftrightarrow$ A is real, symmetric

A is real, skew-Hermitian

$\Longleftrightarrow$ A is real, skew-Symmetric.