EE5609 – Matrix Theory 2023
Practice Set 3 (For Lectures 7 and 8)

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Solutions are not to be returned

Practice Set

Notation: Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

1. David Lay, Exercises 2.1:
   Questions 5–13, 15–16, 18–20.
2. David Lay, Exercises 2.2:
   Questions 11–14, 16, 18–20, 27, 28, 38.
3. David Lay, Exercises 1.2:
   Questions 1–5.
4. Argue that the inverse of a permutation matrix $\boldsymbol{P}$ is $\boldsymbol{P}^T$.
5. *Householder Reflection.* Let $\boldsymbol{u} \in \mathbb{C}^n$ be such that $\boldsymbol{u}^H\boldsymbol{u} = 1$. Define $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^H$. This matrix is called Householder reflection. Show that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}$.
6. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{A}$ be square $n \times n$ matrices where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are invertible. Argue that $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{XAY}$ is invertible.
7. Show that $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{A}^T$ is invertible. (Similar statement holds for $\boldsymbol{A}^H$ also).
8. In this question we will identify some conditions for a square matrix to be non-invertible.
   (a) Suppose one of the rows of $\boldsymbol{A}$ is all-zero, then show that $\boldsymbol{A}$ is not invertible.
   `Hint:` We must argue that $\boldsymbol{AB} \neq \boldsymbol{I}$ for any choice of matrix $\boldsymbol{B}$. One argument is as follows: since one of the rows of $\boldsymbol{A}$ is all-zero, can we say that one of the rows of $\boldsymbol{AB}$ is also all-zero? Observe that $\boldsymbol{I}$ does not have any all-zero row. Then can $\boldsymbol{AB}$ be equal to $\boldsymbol{I}$, no matter how we choose $\boldsymbol{B}$?
   (b) Suppose one of the columns of $\boldsymbol{A}$ is all-zero, then show that $\boldsymbol{A}$ is not invertible.
   (c) The above ideas can be generalized as follows. Suppose there exists a *non-zero* $\boldsymbol{x} \in \mathbb{F}^n$ such that $\boldsymbol{Ax} = \boldsymbol{0}$. Show that $\boldsymbol{A}$ is not invertible.
   `Hint:` We can use proof by contradiction. If $\boldsymbol{A}$ was invertible then we know that the function $\boldsymbol{x} \to \boldsymbol{Ax}$ is one-to-one and onto. Does the given matrix satisfy this property?
   (d) Similarly, show that if there exists $\boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{x}^T\boldsymbol{A} = \boldsymbol{0}^T$, then $\boldsymbol{A}$ is not invertible.
   `Hint:` First show that $\boldsymbol{A}^T$ is not invertible.
