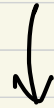


Lecture-9

Perform row reduction



RREF



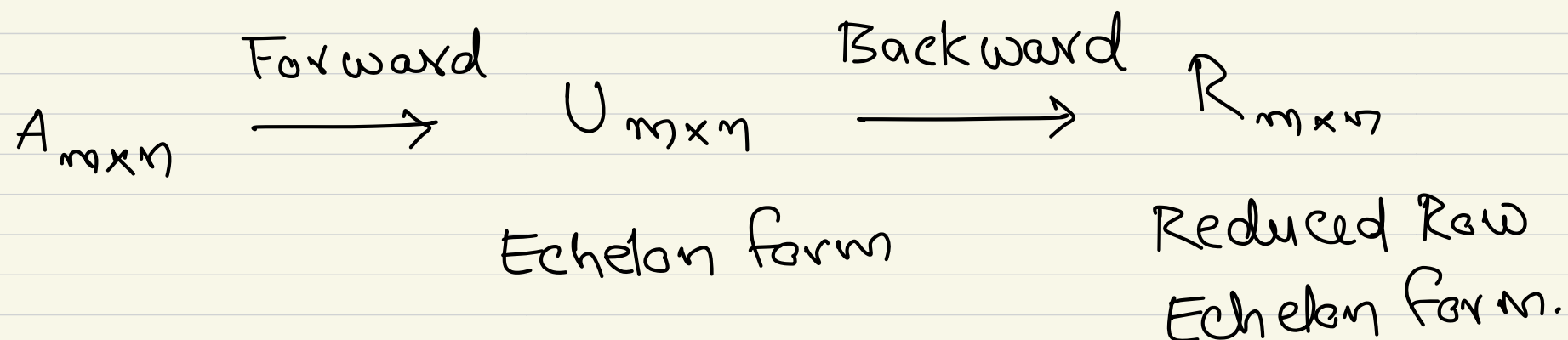
Relate #pivots to the existence of
inverse.



Method to compute the inverse

Recap:

Row Reduction



Each non-zero row of R will contain a pivot.

the pivot entries appear in distinct columns.

$$\# \text{ pivots} = \# \text{ non-zero rows in } R \leq m$$

$$\# \text{ pivots} \leq \# \text{ columns} = n$$

Thus, we have deduced that:

$$0 \leq \# \text{pivots} \leq \min\{m, n\}.$$

In particular, for a square $n \times n$, we have:

$$\# \text{pivots} \leq n.$$

What does $R_{n \times n}$ look like if $\# \text{pivots} = n$?

Example: $n = 3$. (3×3 matrix, RREF).

$R =$

1	0	0
0	1	0
0	0	1

Suppose R has
3 pivots.

In general, if R is an $n \times n$ RREF matrix
and if the number of pivots in R is equal to n
then,

$$R = I_{n \times n}.$$

Recalling some results from Practice Set 3.

①' If A, B are $n \times n$ matrices. Then:

Both A and AB \implies B is invertible.
are invertible

Contrapositive: $\text{Statement 1} \implies \text{Statement 2}.$

\curvearrowright $\underline{\text{Statement 2}} \implies \underline{\text{Statement 1}}$

B is not invertible \implies Either A or AB
is not invertible

② If A has a all-zero row or a all-zero column, then A is not invertible.

A has a all-zero row
or
 A has a all-zero column $\Rightarrow A$ is not invertible

A is invertible \Rightarrow A has no all-zero row
AND
 A has no all-zero column.

③ If $\exists \underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$ then
A is not invertible, where A is $n \times n$

$$\begin{aligned} \exists \underline{x} \neq \underline{0}, \quad & \implies A \text{ is not invertible.} \\ A\underline{x} &= \underline{0} \end{aligned}$$

$$\begin{aligned} \exists \underline{x} \neq \underline{0}, \quad & \implies A \text{ is not invertible.} \\ \underline{x}^T A &= \underline{0}^T \end{aligned}$$

④ $A_{n \times n}$ is invertible if and only if

A^T is invertible.

A is invertible $\iff A^T$ is invertible.

Contrapositive:

A is not invertible $\iff A^T$ is not invertible.

Relating the invertibility of a square matrix
to
the number of pivot entries.

Theorem: Suppose A is a $n \times n$ matrix.

A is invertible $\iff A$ has n pivots.

A is invertible $\iff \text{RREF}(A) = I_{n \times n}$

Proof:

Part 1 : Assume A has n pivots.

Suppose R is the RREF of A .

$$R = E_t E_{t-1} \cdots E_1 \cdot A.$$

where E_1, \dots, E_t are elementary matrices.

Since R is $n \times n$ RREF with n pivots, we

know that $R = I_{n \times n}$.

$$\therefore (E_t E_{t-1} \cdots E_1) A = I$$

\downarrow invertible \downarrow invertible
 invertible

$\Rightarrow A$ is invertible.

Part 2. Assume A is invertible.

Suppose $R = E_t \cdots E_1 A$ is the RREF of A .

Since A, E_1, \dots, E_t are invertible, we see that

R is invertible.

$\therefore R$ is an $n \times n$ invertible RREF matrix.

\Rightarrow none of the rows of R are all-zero.

\Rightarrow there is one pivot position per row of R .

$\Rightarrow R$ has n pivots

$\Rightarrow A$ has n pivots.

Computing the inverse of a matrix

Let us assume that A is an $n \times n$ invertible matrix.

$$\Rightarrow \text{RREF}(A) = I_{n \times n}, \quad A^{-1} \text{ exists.}$$

$$E_t E_{t-1} \cdots E_1 A = R = I$$

$$E_t E_{t-1} \cdots E_1 A \cdot A^{-1} = I \cdot A^{-1}$$

$$\Rightarrow A^{-1} = E_t E_{t-1} \cdots E_2 E_1.$$

Observation:

① The inverse of A is the product of the elementary matrices $E_t E_{t-1} \cdots E_1$ used in the row reduction of A .

② Since $A^{-1} = E_t \cdots E_2 E_1$,

$$A = E_1^{-1} E_2^{-1} \cdots E_{t-1}^{-1} E_t^{-1}$$

$\therefore A$ is a product of elementary matrices.

Augmented matrix $B = \begin{bmatrix} A & | & I \end{bmatrix}$
 $n \times n \quad n \times n$

$$B: n \times 2n$$

If we perform row reduction on B so that
the first half of B is reduced to I :

$$E_t E_{t-1} \cdots E_1 \cdot B = \begin{bmatrix} I & | & * \end{bmatrix}.$$

$$E_t \cdots E_1 B = E_t \cdots E_1 \cdot [A \quad I]$$

$$= [E_t \cdots E_1 \cdot A \quad E_t \cdots E_1 \cdot I].$$

$$= [I \quad E_t \cdots E_1].$$

→ inverse of A

$$= [I \quad A^{-1}].$$

Gauss-Jordan method / Gaussian elimination
method

for computing A^{-1} .

Complexity of this technique is $\Theta(n^3)$.

$f(n) = \# \text{ flops to compute inverse of}$
 $n \times n \text{ matrix.}$

$\lim_{n \rightarrow \infty} \frac{f(n)}{n^3}$ is a positive constant.

Remarks:

- * In practical applications, it is quite rare to actually compute the inverse of a matrix.

$$A \underline{x} = \underline{b}$$

↗ find this.
↘ given.

Given (invertible)

$$\Rightarrow A^{-1} A \underline{x} = A^{-1} \underline{b} \Rightarrow \underline{x} = A^{-1} \underline{b}.$$

① It is cheaper to use an alternative technique, that is based on

$$PA = LU \text{ factorization}$$

$$PA \underline{x} = \underline{Pb} \Rightarrow LU \underline{x} = \underline{Pb}$$

② This $PA = LU$ technique has smaller round-off errors, and has better numerical stability.

About the definition of invertibility.

Recall that :

A is invertible if $\exists B$ such that

$$AB = I$$

and $BA = I$.

We want to show that this can be relaxed to checking exactly one of these two above conditions.

In other words, we want to show that if

$$A, B \in \mathbb{F}^{n \times n} :$$

$$AB = I \implies \begin{array}{l} A \text{ and } B \text{ are invertible} \\ \text{and} \\ B = A^{-1}, A = B^{-1}. \end{array}$$

Claim: If $A, B \in \mathbb{F}^{n \times n}$ and $AB = I$

then B is invertible.

Proof: We will use proof by contradiction.

Assume that the result is not true, i.e.,

assume that B is Not invertible.

$\Rightarrow B^T$ is not invertible.

Consider performing row reduction on B^T ,
and say $R = \text{RREF}(B^T)$.

\Rightarrow For some integer $t \geq 0$:

$$E_t \cdots E_2 E_1 B^T = R$$

Since B^T is not invertible, $\# \text{pivots in } R < n$

\Rightarrow R has at least one all-zero row.

\Rightarrow the last row of R is all-zero.

\Rightarrow the last column of R^T is all-zero.

$$B^T = (E_t \cdots E_1)^{-1} R.$$

$$\therefore B = R^T (E_t \cdots E_1^{-1})^T$$

M , invertible

$$B = R^T M \quad \text{where } M \text{ is invertible}$$

$$R^T \underline{e}_n = \underline{0} \Rightarrow R^T M \cdot M^{-1} \underline{e}_n = \underline{0}$$

$$\Rightarrow B \cdot M^{-1} \underline{e}_n = \underline{0}$$

Since M^{-1} is invertible,

all its columns are non-zero

$$\Rightarrow M^{-1} \underline{e}_n \neq \underline{0}. \quad \text{--- (1)}$$

$$B [M^{-1} \underline{e}_n] = \underline{0}. \quad \text{--- (2)}$$

$M^{-1} \underline{e}_n$ is
the last
column of M^{-1} .

The contradiction:

$$\text{From (2): } AB [M^{-1} \underline{e}_n] = A \underline{0}$$

$$I [M^{-1} \underline{e}_n] = \underline{0} \Rightarrow M^{-1} \underline{e}_n = \underline{0}$$

See that this contradicts (1).

Theorem: If A, B are $n \times n$ and

$$AB = I \quad \text{---} \textcircled{*}.$$

$$\text{then } B = A^T, \quad A = B^T.$$

Proof: From the previous claim we know
that

$$AB = I \Rightarrow B \text{ has an inverse.}$$

$$\therefore B^{-1} \text{ exists.}$$

Multiplying with B^{-1} on both sides of $AB = I$

$$AB \cdot B^{-1} = I \cdot B^{-1}$$

$$\Rightarrow A \cdot I = B^{-1}$$

$$\Rightarrow A = B^{-1}. \quad (\text{this shows that } A \text{ is invertible as well}).$$

Taking inverse on both sides:

$$A^{-1} = (B^{-1})^{-1} = B.$$

$$\therefore A^{-1} = B \quad \text{and} \quad B^{-1} = A.$$