Lecture-7

(Section 2.2 of Lay)

Definition: We say that a matrix $A \in \mathbb{F}^{n \times n}$ is invertible or non-singular if

$$\exists\, B \in \mathbb{F}^{n \times n} : AB = BA = I$$

The opposite: non-invertible or singular.

Inverse

↓

Elementary matrices

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Row Reduction

Linear dep., Span

Defn. If A is invertible, the unique matrix B that satisfies $AB = BA = I$ is called the inverse of A, and is denoted as A^{-1}.

Examples:

① $I^{-1} = I$

$(I \cdot I = I \cdot I = I)$

Claim. If A is invertible then there exists a unique matrix B such that

$$AB = BA = I.$$

Proof: Suppose B and C are such that

$$AB = BA = I,$$
$$AC = CA = I.$$

$$B = B \cdot I = B(AC) = (BA) \cdot C$$
$$= I \cdot C = C.$$

$\therefore B = C.$

(3) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(4) $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$A^{-1} = A$

(2) D is a diagonal matrix with non-zero diagonal entries

$D = \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix}$

$\alpha_i \neq 0 \ \forall i$

$D^{-1} = \begin{bmatrix} 1/\alpha_1 & & & 0 \\ & 1/\alpha_2 & & \\ & & \ddots & \\ 0 & & & 1/\alpha_n \end{bmatrix}$

D

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an

Exam

(1)

Claim: If A is invertible, then so are A^T and A^H.

Proof:

$$A^T \cdot (A^{-1})^T = [A^{-1} \cdot A]^T = I^T = I$$

$$\|^{ly} \quad (A^{-1})^T \cdot A^T = [A \cdot A^{-1}]^T = I$$

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

Similarly, we can show that: $\boxed{(A^H)^{-1} = (A^{-1})^H}$

Claim: Product of invertible matrices is invertible.

Proof: Suppose A, B are invertible.

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Remark.

In general, if

$A_1, A_2, \ldots, A_k$ are invertible

Then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$$

We know that A^{-1}, B^{-1} exist.

Consider the product AB.

$$\begin{aligned} AB\,B^{-1}A^{-1} &= A(BB^{-1})A^{-1} \\ &= (A \cdot I) \cdot A^{-1} \\ &= A\,A^{-1} \\ &= I \end{aligned}$$

$|||^{ly}$ $(B^{-1}A^{-1})AB = I$.

$$\therefore (AB)^{-1} = B^{-1}A^{-1}$$

Remark:

Suppose A is invertible:

$\Rightarrow A^{-1}$ exists

$$A^{-1} \cdot A = I = A \cdot A^{-1}$$

$$(A^{-1})^{-1} = A.$$

Remark:

Suppose A is invertible.

$$(\underbrace{A \cdot A \cdots A}_{k \text{ times}})^{-1} \quad A^{k}$$

$$= (\underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}}) \quad A^{-k}$$

$$(A^{k})^{-1} = A^{-k}$$

Remark

In ge

A_1, ,

Then

$(A_1 A_2$

$= A_k$

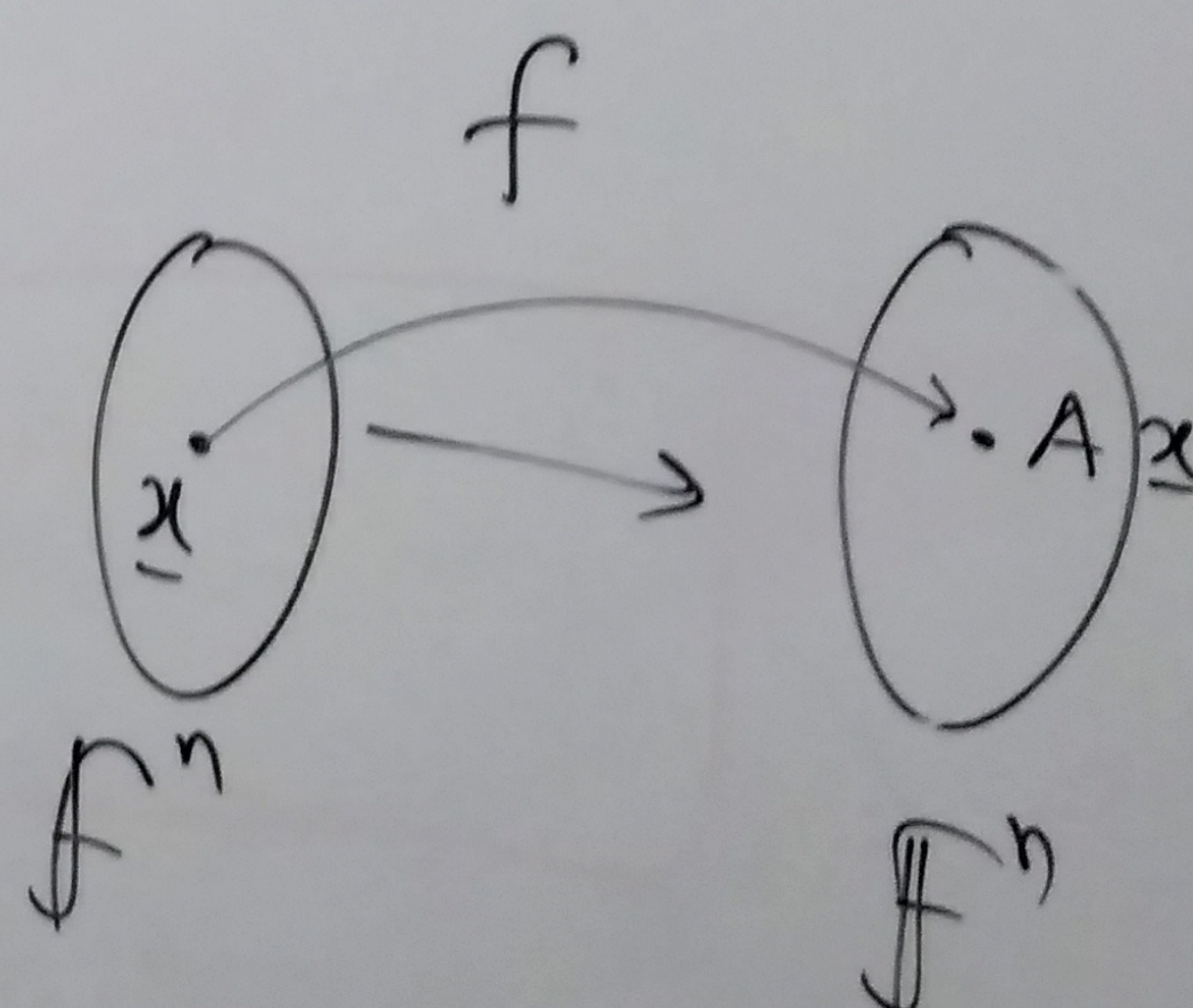
Claim: Suppose $A \in \mathbb{F}^{n \times n}$ is invertible. Consider the linear function $f : \mathbb{F}^n \to \mathbb{F}^n$ given by

$$f(\underline{x}) = A\underline{x}, \quad \forall\, \underline{x} \in \mathbb{F}^n.$$

is invertible (f is onto and one-to-one).

Proof: Note that A^{-1} exists.

(a) To show that f is onto:



Pick any $\underline{y} \in \mathbb{F}^n$. We need to show that $\exists\, \underline{x} \in \mathbb{F}^n$ such that $A\underline{x} = \underline{y}$. Choosing $\underline{x} = A^{-1}\underline{y}$ we see that

$$A\underline{x} = A \cdot A^{-1}\underline{y} = \underline{y}.$$

Elementary Matrices

(m×m matrices)

1. Type-I.

$$E_i = \begin{array}{c c} & \begin{array}{ccccc} & i & & j & \end{array} \\ i & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & \cdots & \boxed{\alpha} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \\ & \end{array} \quad i \neq j, \quad \alpha \neq 0$$

$$= I + \begin{array}{c c} & \begin{array}{c} j \end{array} \\ i & \begin{bmatrix} & \alpha \\ & \end{bmatrix} \end{array}$$

$$= I + \alpha\, \underline{e}_j \cdot \underline{e}_i^T$$

(b) To show that f is one-to-one, i.e. no two points in the domain have the same image

Suppose $\underline{x}$ and $\underline{x}'$ are such that

$$A\underline{x} = A\underline{x}'$$

Multiply A^{-1} on both sides.

$$A^{-1}A\underline{x} = A^{-1}A\underline{x}'$$

$$\Rightarrow \underline{x} = \underline{x}'$$

$E^{-1}EA =$

$$i \rightarrow \begin{bmatrix} \underset{\sim}{a}_1^T \\ \underset{\sim}{a}_2^T \\ \vdots \\ \underset{\sim}{a}_i^T + \alpha\, \underset{\sim}{a}_j^T \\ \vdots \\ \underset{\sim}{a}_m^T \end{bmatrix}$$

$$E^{-1} = I - \alpha\, \underset{\sim}{e}_j \underset{\sim}{e}_i^T \,?$$

Consider $\underset{m \times m}{E}\ \underset{m \times n}{A}$

$$EA = \begin{bmatrix} 1 & & \\ & 1 & \alpha \\ & & 1 \end{bmatrix} A$$

i^{th} row of EA =

i^{th} row of A +

$\alpha \times j^{th}$ row of A

All other rows of EA

are same as that of A

El

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E

Consider:

$$\left(I + \alpha \underline{e}_j \underline{e}_i^T\right) \cdot \left(I - \alpha \underline{e}_j \underline{e}_i^T\right)$$

$$= I - \alpha \underline{e}_j \underline{e}_i^T + \alpha \underline{e}_j \underline{e}_i^T - \alpha^2 \underbrace{\underline{e}_j}_{m\times 1} \underbrace{\underline{e}_i^T}_{1\times m} \underbrace{\underline{e}_j}_{m\times 1} \underbrace{\underline{e}_i^T}_{1\times m}$$

$$= I.$$

Note that $\underline{e}_i^T \underline{e}_j = 0$.

|||ly: $\left(I - \alpha \underline{e}_j \underline{e}_i^T\right)\left(I + \alpha \underline{e}_j \underline{e}_i^T\right) = I$

$$\therefore E^{-1} = \left(I + \alpha \underline{e}_j \underline{e}_i^T\right)^{-1} = I - \alpha \underline{e}_j \underline{e}_i^T.$$

Here, $E^{-1} = E$

③ Type-III

Multiply a row of A with $\alpha \neq 0$.

Suppose, we want to scale ith row of A

$$E = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & \cdots & \alpha & 0 \\ & & & & \\ 0 & 0 & \cdots & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \end{array}$$

② Type-II

Swap two rows of A.

$i \neq j$, Swap ith row with jth row

$$E = \begin{bmatrix} \underline{e}_1^T \\ \vdots \\ \underline{e}_j^T \\ \underline{e}_i^T \\ \underline{e}_m^T \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \leftarrow j \\ \\ \end{array}$$

e_i^T ?

$$\begin{array}{cc} & \begin{matrix} & & i \\ & & \downarrow \end{matrix} \\ E^{-1} = \; i \rightarrow & \begin{bmatrix} 1 & & \\ & 1 \ddots & 0 \\ 0 & & \alpha^{-1} \\ & & & 1 \end{bmatrix} \end{array}$$

Here, $E^{-1} = E$

3 Type-III.

Multiply a row of A with $\alpha \neq 0$.

Suppose, we want to scale ith row of A

$$\begin{array}{cc} E = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & \cdots & \alpha & 0 \\ 0 & 0 & \cdots & & 1 \end{bmatrix} & \begin{matrix} \\ \\ \leftarrow i \\ \\ \end{matrix} \end{array}$$