EE5609 – Matrix Theory
Quiz 1

Aug 27, 2022

Important: Read the Instructions Carefully

- Any malpractice or misbehaviour guarantees a fail grade immediately, irrespective of your performance in future and past quizzes and assignments.
- This is a closed book, closed internet exam. Notes, calculators, laptops, phones, any electronic devices (except non-smart watches) are not allowed.
- Please avoid unnecessary breaks during the exam.
- Quiz 1 is an objective exam. Marks will be based only your final answers or choices, not on the proofs and derivations.
- Please use the first two pages of your answer sheet to provide only the final answers to all the questions. The rest of the pages can be used for rough work. Clearly indicate where the rough work begins.
- *Please answer the questions in the same order as they appear in this question paper.*
- In case you want to skip/avoid answering a question, you can write "SKIP" in the place of the answer.

  Note that "SKIP" is different from "NONE OF THE ABOVE" which could be a valid answer for some of the questions.
- Total marks for this quiz is 25, of which 9 marks are in Part A and 16 marks in Part B.
- *Notation:* Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

All the best!

Questions - Part A (9 marks)

Instructions: Write down only the final answers to the following questions. Do not include proofs or derivations.

1. (1 mark) Find $\boldsymbol{A}^*$ when

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix}.$$

2. (2 marks) Suppose $\boldsymbol{A}$ is a left-inverse of $\boldsymbol{B}$, and $\boldsymbol{C}$ is a right-inverse of $\boldsymbol{D}$, and the number of columns in $\boldsymbol{B}$ equals the number of rows in $\boldsymbol{C}$. Find a left-inverse of $\boldsymbol{BC}$.

3. (2 marks) What is the number of floating-point operations (or 'flops') required to multiply two 2×2 complex matrices?

4. (2 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that rotates the point $\begin{bmatrix} x \\ y \end{bmatrix}$ by θ radians counter-clockwise about the origin. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that maps a point $\begin{bmatrix} x \\ y \end{bmatrix}$ to the nearest point in the x-axis. What is the matrix corresponding to the function $g \circ f$?

5. (2 marks) Find all choices of α for which the following matrix is *not invertible*

$$\begin{bmatrix} 3 & 9 & 27 \\ 5 & 25 & 125 \\ \alpha & \alpha^2 & \alpha^3 \end{bmatrix}.$$

Questions - Part B (16 marks)

Instructions: Choose/Indicate **ALL** the correct options for the following questions. If none of the options are correct, you must write "NONE OF THE ABOVE" as your answer.

Example. If the correct options to a question are both (b) and (d), then the only correct answer to this question is: "(b) and (d)". Note that the answers "(b)", "(d)", or "(b), (d) and (a)" are all incorrect.

6. (4 marks) Consider the equation $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{x}$ is unknown, $\boldsymbol{A} \in \mathbb{F}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{F}^m$ are given.

   Choose all the options that are correct.

   (a) $m < n \Rightarrow$ infinitely many solutions.
   (b) $m < n \Rightarrow$ at least one solution.
   (c) $m > n \Rightarrow$ at the most one solution.
   (d) $m > n \Rightarrow$ no solution.

7. (4 marks) Suppose the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ are such that $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution.

   Which of the following are definitely true?

   (a) $\boldsymbol{Ax} = \boldsymbol{0}$ has exactly one solution.
   (b) $\boldsymbol{Ax} = \boldsymbol{c}$ has exactly one solution for every vector $\boldsymbol{c}$.
   (c) $\boldsymbol{A}$ has a left-inverse.
   (d) $\boldsymbol{A}$ has a right-inverse.

8. (4 marks) Suppose $\boldsymbol{A} \in \mathbb{F}^{m \times n}$, $\boldsymbol{B} \in \mathbb{F}^{n \times m}$ and $\boldsymbol{C} = \boldsymbol{AB}$ is invertible.

   Which of the following are definitely true?

   (a) The rows of $\boldsymbol{A}$ are linearly independent.
   (b) The columns of $\boldsymbol{A}$ are linearly independent.
   (c) $\boldsymbol{B}$ has a left-inverse.
   (d) The rows of $\boldsymbol{B}$ span $\mathbb{F}^m$.

9. (4 marks) Suppose $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix}$.

   Choose all the correct statements from below.

   (a) $\boldsymbol{D} = \boldsymbol{0}$ and $\boldsymbol{E} = \boldsymbol{0} \Rightarrow$ $\boldsymbol{A}$ has linearly dependent columns.
   (b) $\boldsymbol{D} = \boldsymbol{0}$ and $\boldsymbol{C} = \boldsymbol{0} \Rightarrow$ $\boldsymbol{A}$ has linearly dependent columns.
   (c) $\boldsymbol{B} = \boldsymbol{D} = \boldsymbol{0} \Rightarrow$ $\boldsymbol{A}$ has linearly dependent columns.
   (d) $\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{D} = \boldsymbol{E} \Rightarrow$ $\boldsymbol{A}$ has linearly dependent columns.

EE5609 – Matrix Theory
Quiz 2

Sep 24, 2022

Important: Please read all the instructions

- **Read each question carefully.**
- Any malpractice or misbehaviour guarantees a fail grade immediately, irrespective of your performance in future and past quizzes and assignments.
- This is a closed book, closed internet exam. Notes, calculators, laptops, phones, any electronic devices (except non-smart watches) are not allowed.
- Please avoid unnecessary breaks during the exam.
- Parts A and B of Quiz 2 contain objective questions. Marks for Parts A and B will be based only on your final answers or choices, not on the proofs and derivations.
- Please use the last 3 pages of your answer booklet for rough work. Clearly indicate where the rough work begins. No additional sheets will be provided.
- *Please try to answer the questions in the same order as they appear in this question paper.*
- In case you want to skip/avoid answering a question, you can write "SKIP" in the place of the answer.

  Note that "SKIP" is different from "NONE OF THE ABOVE" which could be a valid answer for some of the questions.
- Parts A, B, C contain questions worth 12+4+12 = 28 marks in all. However, the **maximum marks will be restricted to 25**.
- *Notation:* Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

All the best!

Part A (12 marks)

Instructions: Write down only the final answers to the following the questions. Do not include proofs or derivations.

Questions:

1. *On row reduction.*

   (a) (2 marks) Give a sufficient condition for a general 3×3 matrix $\boldsymbol{A}$ to admit LU factorization, i.e., $\boldsymbol{A} = \boldsymbol{LU}$ where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular. State this condition clearly, do not provide proof.

   For the following questions consider

   $$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

   Perform row reduction without using row interchanges.

   (b) (2 marks) What are the Gauss transforms used in stages 1 and 2: $\boldsymbol{M}_1, \boldsymbol{M}_2$?

   (c) (3 marks) What is $\boldsymbol{L}$?

   (d) (3 marks) What is $\boldsymbol{U}$?

   (e) (2 mark) What is the row reduced echelon form (RREF) of $\boldsymbol{A}$?

Part B (4 marks)

Instructions: Choose/Indicate **ALL** the correct options for the following questions. If none of the options are correct, you must write "NONE OF THE ABOVE" as your answer.

Example. If the correct options to a question are both (b) and (d), then the only correct answer to this question is: "(b) and (d)". Note that the answers "(b)", "(d)", or "(b), (d) and (a)" are all incorrect.

Questions:

2. (2 marks, *About column space and null space*). Choose all the correct correct statements from below.

   (a) $\boldsymbol{AB} = \boldsymbol{AC} \Rightarrow \mathrm{Nul}(\boldsymbol{A}) \supseteq \mathrm{Col}(\boldsymbol{B} - \boldsymbol{C})$.

   (b) $\boldsymbol{AB} = \boldsymbol{AC} \Rightarrow \mathrm{Nul}(\boldsymbol{A}) \subseteq \mathrm{Col}(\boldsymbol{B} - \boldsymbol{C})$.

3. (2 marks, *About matrices of rank* 1) Choose all the correct statements from below.

   (a) $\mathrm{rank}(\boldsymbol{A}) = 1 \Rightarrow$
   $\boldsymbol{A} = \boldsymbol{uv}^T$ for some vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

   (b) $\mathrm{rank}(\boldsymbol{A}) = 1 \Rightarrow$
   exactly one column or exactly one row of $\boldsymbol{A}$ is non-zero.

Part C (12 marks)

Instructions: For each question determine 'True' or 'False' and support your answer with a proof and/or counter-example. 'True' means that the statement is necessarily true given the scenario stated in the question. 'False' means that the statement in not always true.

Proofs must be brief and as short as possible. Concise proofs will be much appreciated.

You can use any result proved in the lectures by clearly stating it. Any new result must be proved before being used.

If your answer is 'True', write 'True' and give a supporting proof. If your answer is 'False', write 'False' and give either a proof or a counter-example (you must clearly explain/prove how the counter-example shows that the statement is false).

Questions:

4. (4 marks) `Scenario`: H is a subspace of $\mathbb{F}^m$, matrix $\boldsymbol{B} = [\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_n] \in \mathbb{F}^{m \times n}$, and none of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ belong to H.

   (a) $\boldsymbol{b}_i \neq \boldsymbol{0}$ for $i = 1, \ldots, n$.

   (b) $\mathrm{Col}(\boldsymbol{B}) \cap H = \{\boldsymbol{0}\}$.

5. (4 marks) `Scenario`: $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ is a complex matrix with rank equal to m, $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_m^T$ are its rows, and $\boldsymbol{v} \in \mathrm{Nul}(\boldsymbol{A})$.

   (a) $\boldsymbol{a}_i \neq \boldsymbol{0}$ for $i = 1, \ldots, m$.

   (b) $\boldsymbol{a}_i \neq \boldsymbol{v}$ for $i = 1, \ldots, m$.

6. (4 marks) `Scenario`: $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{F}^{m \times n}$.

   (a) $\mathrm{Nul}(\boldsymbol{A}) \cap \mathrm{Nul}(\boldsymbol{B}) = \mathrm{Nul}\left(\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix}\right)$.

   (b) $\mathrm{rank}\left(\begin{bmatrix} \boldsymbol{A} + \boldsymbol{B} \\ \boldsymbol{A} + 2\boldsymbol{B} \end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix}\right)$.

EE5609 – Matrix Theory
Quiz 3

Oct 27, 2022

Important: Please read all the instructions

- **Read each question carefully.**
- Any malpractice or misbehaviour guarantees a fail grade immediately, irrespective of your performance in future and past quizzes and assignments.
- This is a closed book, closed internet exam. Notes, calculators, laptops, phones, any electronic devices (except non-smart watches) are not allowed.
- Please avoid unnecessary breaks during the exam.
- Please use the last 1–2 pages of your answer booklet for rough work. Clearly indicate where the rough work begins. No additional sheets will be provided.
- *Please try to answer the questions in the same order as they appear in this question paper.*
- *Notation:* Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

All the best!

Part A (6 marks)

Instructions: Write down only the final answers to the following questions. Do not include proofs or derivations.

Questions:

1. *On Subspaces.*
   (a) (1 mark) How many subspaces S_1 and S_2 exist that simultaneously satisfy all the following conditions:
   - $\dim(S_1) \geq \dim(S_2)$,
   - $S_1 \oplus S_2 = \mathbb{R}^1$.

   (b) (1 mark) Suppose $S_1 = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix} : \alpha \in \mathbb{F} \right\}$. Clearly identify one subspace S_2 such that $S_1 \oplus S_2 = \mathbb{R}^2$.

2. *On permutations.*
   (a) (1 mark) Find the number of permutations π on $\{1, \ldots, n\}$ such that $\pi(i) > i$ for at least one value of i.

   (b) (1 marks) List all the even permutations on $\{1, 2, 3\}$. You must represent a permutation as a tuple $(\pi(1), \pi(2), \pi(3))$.

3. *On inner product and norm.*
   (a) (1 mark) Write the expression for the Cauchy-Schwarz inequality.

   (b) (1 mark) Write the expression for the triangle inequality.

Part B (19 marks)

Instructions: While solving the below problems you can directly state and use any result proved during the lectures. If you are using any other results/statements that have not been included in the lectures, you must prove them before using them in your solutions.

If you are explicitly asked to prove any statement that has been already proved in the lectures, please give the complete proof.

All proofs must use consistent notation, and logical and mathematically precise statements. All proofs must be complete. Partial proofs, proofs that work only for special cases, proof ideas etc. will not be awarded full marks.

Questions:

4. (3 marks) Consider a function $f : \mathbb{F}^m \times \mathbb{F}^m \to \mathbb{F}^n$, that is, for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^m$, $f(\boldsymbol{x}, \boldsymbol{y})$ is a vector in $\mathbb{F}^n$. Let f satisfy the following properties:
   - $f(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{0}$ for every $\boldsymbol{x} \in \mathbb{F}^m$, and
   - f is multilinear in its two input arguments.

   Show that $f(\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^m$.

5. (3 marks) What is the relation between the determinants of $\boldsymbol{A}$ and $\mathrm{adj}(\boldsymbol{A})$, where $\boldsymbol{A}$ is $n \times n$? Provide a proof.

6. (3 marks) Suppose $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are $n \times n$ matrices, and all the entries of both $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are integers. What values can $\det(\boldsymbol{A})$ take? Provide a proof.

7. (4 marks) Show that $\mathrm{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \mathrm{rank}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{B})$.

8. *(Proof of Cramer's rule)* Assume $\boldsymbol{A}$ is $n \times n$ invertible, and $\boldsymbol{b} \in \mathbb{F}^n$. Let $\boldsymbol{x}$ be such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Let j be a fixed integer between $1, \ldots, n$.
   (a) (3 marks) Consider the $n \times n$ identity matrix, and replace its j^{th} column by the vector $\boldsymbol{x}$. Call this matrix $\boldsymbol{M}$. Now show that $\det(\boldsymbol{M}) = x_j$, where x_j is the j^{th} entry of $\boldsymbol{x}$.

   (b) (2 marks) Consider the matrix $\boldsymbol{A}$, and replace its j^{th} column by $\boldsymbol{b}$. Call this matrix $\boldsymbol{D}$. Show that $\boldsymbol{D} = \boldsymbol{A}\boldsymbol{M}$.

   (c) (1 mark) Prove that
   $$x_j = \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{A})}.$$

EE5609 – Matrix Theory
Quiz 4

Nov 15, 2022

Important: Please read all the instructions

- **Read each question carefully.**
- Any malpractice or misbehaviour guarantees a fail grade immediately, irrespective of your performance in future and past quizzes and assignments.
- This is a closed book, closed internet exam. Notes, calculators, laptops, phones, any electronic devices (except non-smart watches) are not allowed.
- Please avoid unnecessary breaks during the exam.
- Please use the last 1–2 pages of your answer booklet for rough work. Clearly indicate where the rough work begins. No additional sheets will be provided.
- *Please try to answer the questions in the same order as they appear in this question paper.*
- *Notation:* Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
- *Instructions for answering the questions:* While solving the problems you can directly state and use any result proved during the lectures. If you are using any other results/statements that have not been included in the lectures, you must prove them before using them in your solutions.

  If you are explicitly asked to prove any statement that has been already proved in the lectures, please give the complete proof.

  All proofs must use consistent notation, and logical and mathematically precise statements. All proofs must be complete. Partial proofs, proofs that work only for special cases, proof ideas etc. will not be awarded full marks.

All the best!

Questions (25 marks)

1. *(Projection matrices and norm)*

   Let $\boldsymbol{P} \in \mathbb{C}^{n\times n}$ be a projection matrix such that $1 \le \operatorname{rank}(\boldsymbol{P}) \le n-1$ and $n \ge 2$.

   (a) (3 marks) Show that projection never increases the norm of a vector, i.e., show that

   $$\|\boldsymbol{P}\boldsymbol{x}\|_2 \le \|\boldsymbol{x}\|_2 \text{ for all } \boldsymbol{x} \in \mathbb{C}^n.$$

   (b) (3 marks) Find non-negative real numbers $\sigma_{\min}$ and $\sigma_{\max}$ such that

   $$\sigma_{\min}\|\boldsymbol{x}\|_2 \le \|\boldsymbol{P}\boldsymbol{x}\|_2 \le \sigma_{\max}\|\boldsymbol{x}\|_2 \text{ for all } \boldsymbol{x} \in \mathbb{C}^n.$$

   Support your answer with a proof.

   Important: Note that your answer should give the best possible values for $\sigma_{\min}$ and $\sigma_{\max}$. That is, there should exist a vector $\boldsymbol{w}$ such that $\|\boldsymbol{P}\boldsymbol{w}\|_2 = \sigma_{\min}\|\boldsymbol{w}\|_2$, and there should exist a vector $\boldsymbol{v}$ such that $\|\boldsymbol{P}\boldsymbol{v}\|_2 = \sigma_{\max}\|\boldsymbol{v}\|_2$. This implies that $\sigma_{\max} = \|\boldsymbol{P}\|_2$.

   (c) (3 marks) Identify the set of all vectors $\boldsymbol{x}$ for which $\|\boldsymbol{P}\boldsymbol{x}\|_2 = \sigma_{\min}\|\boldsymbol{x}\|_2$.
   Also, identify the set of all vectors $\boldsymbol{x}$ for which $\|\boldsymbol{P}\boldsymbol{x}\|_2 = \sigma_{\max}\|\boldsymbol{x}\|_2$.

2. *(Every linear isometry is a unitary transformation)*

   Assume that $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is a matrix satisfying the following property

   $$\|\boldsymbol{A}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2 \text{ for all } \boldsymbol{x} \in \mathbb{C}^n.$$

   Using the following steps we will eventually show that $\boldsymbol{A}$ is a unitary matrix.

   (a) (3 marks) Use the fact $\|\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y})\|_2 = \|\boldsymbol{x}+\boldsymbol{y}\|_2$ to show that the real parts of $\boldsymbol{x}^*\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{x}^*\boldsymbol{y}$ are equal.

   (b) (3 marks) Use the fact $\|\boldsymbol{A}(\boldsymbol{x}+i\boldsymbol{y})\|_2 = \|\boldsymbol{x}+i\boldsymbol{y}\|_2$ to show that the imaginary parts of $\boldsymbol{x}^*\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{x}^*\boldsymbol{y}$ are equal. Here $i = \sqrt{-1}$.

   (c) (3 marks) Parts (a) and (b) show that $\boldsymbol{A}$ preserves the inner product, i.e., at this point we have shown that

   $$\boldsymbol{x}^*\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{y} = (\boldsymbol{A}\boldsymbol{x})^*\boldsymbol{A}\boldsymbol{y} = \boldsymbol{x}^*\boldsymbol{y} \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n. \tag{1}$$

   Now, use the above equation (1) to prove that $\boldsymbol{A}$ is a unitary matrix.

3. *(Spectral decomposition of Householder reflection)*

   Suppose $\boldsymbol{H} \in \mathbb{C}^{n\times n}$ is a Householder reflection.

   (a) (5 marks) You must show that there exists a unitary matrix $\boldsymbol{U} \in \mathbb{C}^{n\times n}$ such that

   $$\boldsymbol{H} = \boldsymbol{U} \times \begin{bmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-1} \end{bmatrix} \times \boldsymbol{U}^*.$$

   `Remark`: This factorization is an example of *spectral decomposition*. The columns of $\boldsymbol{U}$ are the eigenvectors of $\boldsymbol{H}$.

   (b) (2 marks) What is the value of $\det(\boldsymbol{H})$? Why?

EE5609 – Matrix Theory, Quiz 5 (Nov 27, 2022)

Instructions

- Please provide answers in the same order as the questions appear.
- No breaks during the exam.
- No laptops, calculators, mobile devices etc. are allowed.

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- Hints are available with the invigilator for questions 7, 8 and 9. Accessing a hint will affect your marks. Please see questions 7, 8, 9 for details.
- To see a hint, please approach the invigilator *with your answer sheet* one person at a time.

Part A (20 marks)

Instructions: This section is objective. *Provide only the final answers, please do not write proofs or derivations.*

1. (2 marks) Choose all the correct statements.
   (a) The ranks of a matrix and its pseudoinverse are equal.
   (b) $\|\boldsymbol{A}\|_F^2$ equals the sum of the squares of all the singular values of $\boldsymbol{A}$.

2. (2 marks) Choose all the correct statements.
   (a) $\boldsymbol{A}$ is Hermitian $\Rightarrow$ every eigenvalue of $\boldsymbol{A}$ is real.
   (b) $\boldsymbol{A}$ is unitary $\Rightarrow$ every eigenvalue of $\boldsymbol{A}$ has unit magnitude.

3. (2 marks) Choose all the correct statements.
   (a) the geometric multiplicity of an eigen value is less than or equal to its algebraic multiplicity.
   (b) all eigen values of $\boldsymbol{A}$ are distinct $\Rightarrow \boldsymbol{A}$ is diagonalizable.

4. (2 marks) Find all three singular values of
$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

5. (2 marks) If the singular values of $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$, what is the value of $\|\boldsymbol{A}^{-1}\|_2$?

6. (2 marks) Suppose $\boldsymbol{A}$ is an $n \times n$ normal matrix (i.e., $\boldsymbol{A}$ is unitarily diagonalizable) with eigen values $\lambda_1, \ldots, \lambda_n$. What are the singular values of $\boldsymbol{A}$?

7. (4 marks; max marks is 2 if you use the hint. No negative marking.)
   Find the number of real eigenvalues of an $n \times n$ Givens rotation matrix.

8. (4 marks; max marks is 2 if you the use hint. No negative marking.)
   Find spec($\boldsymbol{A}$).
$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Part B (5 marks)

Instructions: You must provide proofs for the following questions. Each step in your answer must be supported by clear mathematical reasoning. Any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer. Any other result must be proved before using them in your answer. Clear and concise proofs will be much appreciated.

9. (5 marks; cost of hint: 1 mark will be subtracted from your overall score. This negative mark will be applied irrespective of whether your answer to this question is correct or not.)
   Let $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$, and let $\alpha + i\beta \in \text{spec}(\boldsymbol{A})$, where α and β are real and $\beta \neq 0$. In other words, $\alpha + i\beta$ is a non-real eigen value of a real 2×2 matrix $\boldsymbol{A}$. Here i is $\sqrt{-1}$.
   Show that there exists a 2×2 real invertible matrix $\boldsymbol{S} \in \mathbb{R}^{2\times 2}$ such that
$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$
