EE5609 – Matrix Theory 2023
Practice Set 4

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Solutions are not to be returned

Practice Set

Notation: Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

1. David Lay, Exercises 2.2:

   Questions 33, 34.

2. David Lay, Exercises 2.3:

   Questions 27, 28, 30, 36, 37, 39.

3. Let $\boldsymbol{A}$ be $m \times n$. Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x} = \boldsymbol{c}$. Argue that the collection of all solutions is exactly equal to the set $\{\boldsymbol{c} + \boldsymbol{n} \ : \ \boldsymbol{n} \in \text{Null}(\boldsymbol{A})\}$.

4. *Linear Independence of Rows of* $\boldsymbol{R}$.

   (a) Suppose $\boldsymbol{R}$ is an $m \times n$ RREF matrix. If one of the rows of $\boldsymbol{R}$ is all-zero, does this matrix have linearly independent rows?

   (b) If all the rows of $\boldsymbol{R}$ are non-zero, does $\boldsymbol{R}$ have linearly independent rows? Use the fact that every pivot column of $\boldsymbol{R}$ is a standard basis vector.

   (c) Can we conclude that if $\boldsymbol{R}$ has m pivot entries then $\boldsymbol{R}$ has linearly independent rows?

5. *Linear Independence of Rows of* $\boldsymbol{A}$.

   (a) Let $\boldsymbol{A}$ be $m \times n$, let $\boldsymbol{T}$ be an $m \times m$ invertible matrix and let $\boldsymbol{B} = \boldsymbol{TA}$. Argue that $\boldsymbol{B}$ has linearly dependent rows if and only if $\boldsymbol{A}$ has linearly dependent rows.

   `Hint:` Use the fact that $\boldsymbol{T}$ and $\boldsymbol{T}^T$ are invertible. Hence, $\boldsymbol{x}^T\boldsymbol{T} = \boldsymbol{0}$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.

   (b) Suppose $\boldsymbol{R} = \text{RREF}(\boldsymbol{A})$. Can we say that $\boldsymbol{A}$ has linearly independent rows if and only if $\boldsymbol{R}$ has linearly rows?

   (c) Conclude that $\boldsymbol{A}$ has linearly independent rows if and only if number of pivots equals m.

   (d) Can we say that the left-nullspace of $\boldsymbol{A}$ is $\{\boldsymbol{0}\}$ if and only if the number of pivots equals m?

6. Let $\boldsymbol{A}$ be $m \times n$ and $\boldsymbol{R} = \text{RREF}(\boldsymbol{A})$. Use the fact that $\boldsymbol{A}$ and $\boldsymbol{R}$ have the same nullspace.

   (a) If the sum of the first two columns of $\boldsymbol{R}$ is all-zero, what is the sum of the first two columns of $\boldsymbol{A}$?

   (b) If the second and fourth columns of $\boldsymbol{R}$ are equal, can we say that the second and fourth columns of $\boldsymbol{A}$ are equal?

   (c) Argue that the last column of $\boldsymbol{R}$ is a linear combination of the other columns of $\boldsymbol{R}$ if and only if the last column of $\boldsymbol{A}$ is a linear combination of the other columns of $\boldsymbol{A}$.

   (d) Is it true that the last column of $\boldsymbol{R}$ is a linear combination of the other columns of $\boldsymbol{R}$ if and only if the last column of $\boldsymbol{R}$ is not a pivot column?

7. *Determining if* $\boldsymbol{b} \in \operatorname{Col}(\boldsymbol{A})$.

   Let $\boldsymbol{A}$ be $m \times n$ and $\boldsymbol{b}$ be of length m. Consider the *augmented matrix* $\boldsymbol{G} = [\boldsymbol{A}\ \boldsymbol{b}]$, which is of size $m \times (n+1)$. Let $\boldsymbol{R} = \operatorname{RREF}(\boldsymbol{G})$.

   (a) Argue that $\boldsymbol{b} \in \operatorname{Col}(\boldsymbol{A})$ if and only if the last column of $\boldsymbol{G}$ is a linear combination of the other columns of $\boldsymbol{G}$.

   (b) Use the previous question to show that $\boldsymbol{b} \in \operatorname{Col}(\boldsymbol{A})$ if and only if the last column of $\boldsymbol{R}$ is not a pivot column.

8. Let $\boldsymbol{A} = [\boldsymbol{B}\ \boldsymbol{C}]$ be a block matrix.

   (a) Argue that if $\boldsymbol{A}$ has linearly independent columns then both $\boldsymbol{B}$ and $\boldsymbol{C}$ have linearly independent columns.

   (b) What is the contrapositive of the above statement?

   (c) By giving a counter example, show that the converse of this statement is not true.

   (d) Argue that if $\boldsymbol{A}$ has linearly dependent rows then both $\boldsymbol{B}$ and $\boldsymbol{C}$ have linearly dependent rows.

   (e) What is the contrapositive of the above statement?

   (f) By giving a counter example, show that the converse of this statement is not true.

   (g) Can you make analogous observations for the case $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix}$.

9. Suppose $\boldsymbol{A}$ is $m_1 \times n$, $\boldsymbol{B}$ is $m_2 \times n$. Can you identify a matrix $\boldsymbol{C}$ (in terms of $\boldsymbol{A}$ and $\boldsymbol{B}$) such that $\operatorname{Null}(\boldsymbol{C}) = \operatorname{Null}(\boldsymbol{A}) \cap \operatorname{Null}(\boldsymbol{B})$?

10. If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix}$, is it true that $\operatorname{Row}(\boldsymbol{B}) \subset \operatorname{Row}(\boldsymbol{A})$ and $\operatorname{Row}(\boldsymbol{C}) \subset \operatorname{Row}(\boldsymbol{A})$?

11. If $\boldsymbol{A} = [\boldsymbol{B}\ \boldsymbol{C}]$, is it true that $\operatorname{Col}(\boldsymbol{B}) \subset \operatorname{Col}(\boldsymbol{A})$ and $\operatorname{Col}(\boldsymbol{C}) \subset \operatorname{Col}(\boldsymbol{A})$?

12. Determine if the following matrices have linearly independent columns or not. In each case you must give a proof to support your answer.

    (a) a square diagonal matrix where all the diagonal entries are non-zero.

    (b) a square upper triangular matrix where all the entries in the main diagonal are non-zero.

    (c) a "tall" upper triangular matrix where all the diagonal entries are non-zero.

    (d) an $m \times 2$ matrix of the form
    $$\begin{bmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \\ \vdots & \vdots \\ 1 & \alpha_m \end{bmatrix}$$
    where $\alpha_1, \ldots, \alpha_m$ are all distinct. Does your answer depend on the value of m?

    (e) the transpose of the above matrix. How does your answer depend on m?

    (f) an $m \times n$ matrix $\boldsymbol{A}$ with $\operatorname{nnz}(\boldsymbol{A}) < n$. (Recall that nnz is the number of non-zero entries in a matrix.)

    (g) the all-ones $m \times n$ matrix. How does your answer depend on m and n?

    (h) 2×2 matrix $\boldsymbol{A}$ with $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} \neq 0$.

    (i) the 2×2 real matrix that rotates a point in $\mathbb{R}^2$ by θ radians anti-clockwise.

13. *Product of matrices versus linear independence of columns.*

    (a) If $\boldsymbol{B}$ has linearly dependent columns, then what about $\boldsymbol{AB}$ (irrespective of what $\boldsymbol{A}$ is)?

(b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ have linearly independent columns then show that $\boldsymbol{AB}$ has linearly independent columns.

`Hint:` Use the column viewpoint for matrix-matrix multiplication.

(c) We will now want to show that the converse to the previous question is not true. In particular, show that there exist matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A}$ has linearly dependent columns, $\boldsymbol{B}$ has linearly independent columns and $\boldsymbol{AB}$ has linearly independent columns.

14. *Vandermonde matrices.*

Our intention is to show that square Vandermonde matrices are invertible. We will use a polynomial interpretation to proceed with the proof. We will rely on the following fundamental result about polynomials.

Theorem 1: Let $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots f_{n-1}x^{n-1}$ be any non-zero polynomial in the variable x and with coefficients $f_0, \ldots, f_{n-1} \in \mathbb{F}$. Then $f(x)$ has at the most $n-1$ distinct roots in $\mathbb{F}$.

This theorem can be restated as follows. Suppose $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ are distinct scalars, and suppose $f(x)$ is a polynomial such that $f(\alpha_1) = \cdots = f(\alpha_n) = 0$. Then, either $f(x)$ is the zero polynomial, i.e., $f(x) = 0$, or $f(x)$ has degree at least n.

Let us now use Theorem 1 to prove that square Vandermonde matrices are invertible.

(a) Suppose $f(x) = f_0 + f_1 x + \cdots + f_{n-1}x^{n-1}$. Note that this means the degree of $f(x)$ is at the most $(n-1)$. Let $\alpha \in \mathbb{F}$. Show that α is a root of $f(x)$ if and only if

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} = 0.$$

(b) Let $\alpha_1, \ldots, \alpha_n$ be distinct scalars. Show that all of them are roots of $f(x)$ if and only if

$$\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} = \mathbf{0}.$$

(c) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{bmatrix} \text{ and } \boldsymbol{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

Use Theorem 1 and the fact that the degree of $f(x)$ is at the most $(n-1)$ to show the following: $\boldsymbol{Af} = \mathbf{0}$ if and only if $\boldsymbol{f} = \mathbf{0}$.

(d) Argue that $\boldsymbol{A}$ is invertible if $\alpha_1, \ldots, \alpha_n$ are distinct.

(e) Prove the converse also, i.e., show that if any two of $\alpha_1, \ldots, \alpha_n$ are equal then $\boldsymbol{A}$ is not invertible.
`Hint:`Show that $\boldsymbol{A}$ has linearly dependent rows.

15. *The discrete Fourier transform (DFT) matrix.*

(Solve this after solving the previous question on Vandermonde matrices.) Let $\mathbb{F} = \mathbb{C}$, and let $\omega = \exp\left(-j\frac{2\pi}{n}\right)$ where $j = \sqrt{-1}$. The DFT matrix is defined as the $n \times n$ complex matrix $\boldsymbol{F}$ whose entry in row k and column ℓ is

$$[\boldsymbol{F}]_{k,\ell} = \frac{1}{\sqrt{n}}\omega^{(k-1)(\ell-1)}.$$

Prove the following:

(a) $\boldsymbol{F}$ is a symmetric matrix.

(b) $\sqrt{n}\boldsymbol{F}$ is a Vandermonde matrix.

(c) $\boldsymbol{F}$ is invertible.
