Lecture-3.

Recall:

A function $f : \mathbb{F}^n \to \mathbb{F}^m$ is linear

if $f(\underline{x}+\underline{y}) = f(\underline{x}) + f(\underline{y})$

& $f(\alpha \underline{x}) = \alpha f(\underline{x})$

$\forall\, \underline{x}, \underline{y} \in \mathbb{F}^n$ & $\forall \alpha \in \mathbb{F}$.

If f is linear

$$f(\alpha \underline{x} + \beta \underline{y})$$

$$= f(\alpha \underline{x}) + f(\beta \underline{y})$$

$$= \alpha f(\underline{x}) + \beta(\underline{y})$$

$$x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If f is linear,

$$f(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \alpha_3 \underline{x}_3)$$

$$= f(\alpha_1 \underline{x}_1) + f(\alpha_2 \underline{x}_2 + \alpha_3 \underline{x}_3)$$

$$= \alpha_1 f(\underline{x}_1) + \alpha_2 f(\underline{x}_2) + \alpha_3 f(\underline{x}_3).$$

In general:

$$f\left(\sum_{i=1}^{p} \alpha_i \underline{x}_i\right) = \sum_{i=1}^{p} \alpha_i f(\underline{x}_i).$$

$$f : \mathbb{F}^n \to \mathbb{F}^m$$

$$\underline{x} \to f(\underline{x})$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If f

$f(\alpha_1 \underline{x}$

$= f(\alpha_1 \underline{x}$

$= \alpha_1 f(\underline{x}_1$

In general:

$f(\sum_{i=1}^{p} \alpha_i$

Denote the standard basis vectors.

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The collection $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ is the standard basis of $\mathbb{F}^n$

$), \quad f(\underline{e}_j) \in \mathbb{F}^m$

r function

aposing

$$f(\underline{x}) = \begin{bmatrix} f(\underline{e}_1) & \cdots & f(\underline{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\downarrow$

Denote this matrix as A.

Note that A is $m \times n$.

We have shown that

$$f(\underline{x}) = A\underline{x}$$

for all $\underline{x}$.

ctor onto

Remark: The matrix A corresponding to a linear function f is obtained by juxtaposing the vectors

$$f(\underline{e}_1), f(\underline{e}_2), \ldots, f(\underline{e}_n).$$

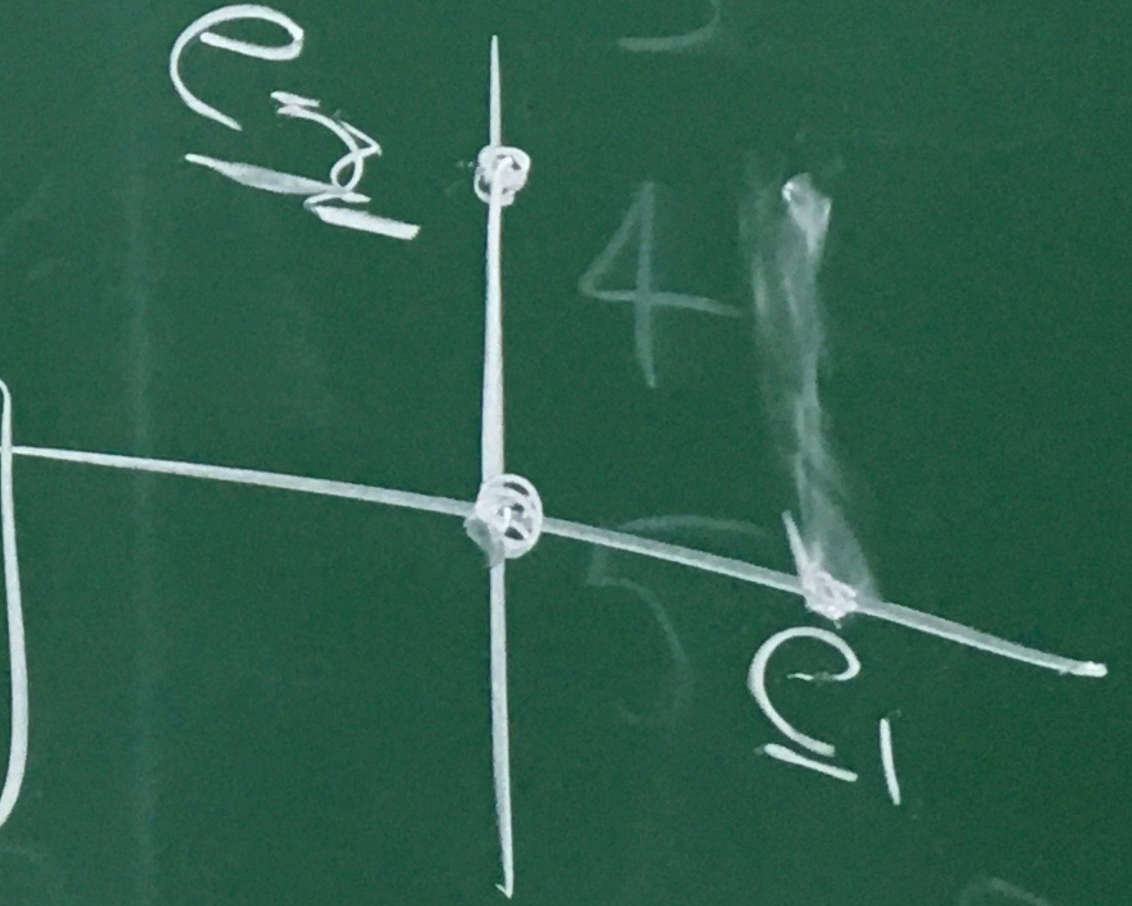
We

Ex: Say $\mathbb{F} = \mathbb{R}$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$n = m = 2$

projects a 2-dim vector onto x-axis.



$$A = \begin{bmatrix} f(\underline{e}_1) & f(\underline{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix representing $f+g$:

$$\begin{aligned}(f+g)(\underline{x}) &= f(\underline{x}) + g(\underline{x}) \\ &= A\underline{x} + B\underline{x} \\ &= (A+B)\underline{x}, \quad \forall\, \underline{x} \in \mathbb{F}^n.\end{aligned}$$

Similarly: Say $f : \mathbb{F}^n \to \mathbb{F}^m$ is linear, and say $\alpha \in \mathbb{F}$ is a scalar.

Define the function $\alpha f : \mathbb{F}^n \to \mathbb{F}^m$ as

$$(\alpha f)(\underline{x}) = \alpha \cdot f(\underline{x})$$

Easy to show that αf is linear.

The matrix corresp. to αf is αA.

Suppose that

$$f : \mathbb{F}^n \to \mathbb{F}^m \qquad (A)$$

and $$g : \mathbb{F}^p \to \mathbb{F}^n \qquad (B)$$

are linear functions represented by $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$, respectively.

Consider the composition of f & g.

$$\mathbb{F}^p \xrightarrow{g} \mathbb{F}^n \xrightarrow{f} \mathbb{F}^m$$

$$\underline{x} \longrightarrow g(\underline{x}) \longrightarrow f\big(g(\underline{x})\big)$$

Define

$$f \circ g(\underline{x}) \triangleq f\big(g(\underline{x})\big)$$

$$f \circ g : \mathbb{F}^p \to \mathbb{F}^m$$

Suppos

f

and g :

are lin

$A \in \mathbb{F}^n$

Conside

$$(f \circ g)(\underline{x} + \underline{y})$$

$$= f\left(g(\underline{x} + \underline{y})\right)$$

$$= f\left(g(\underline{x}) + g(\underline{y})\right)$$

$$= f(g(\underline{x})) + f(g(\underline{y}))$$

$$= f \circ g(\underline{x}) + f \circ g(\underline{y})$$

$$\text{|||}^{ly} \quad f \circ g(\alpha \underline{x}) = \alpha\, f \circ g(\underline{x})$$

Matrix for $f \circ g$:

$$= \left[A\underline{b}_1 \quad A\underline{b}_2 \quad \cdots \quad A\underline{b}_p \right]$$

$$C = \left[f\circ g(\underline{e}_1) \quad f\circ g(\underline{e}_2) \quad \cdots \quad f\circ g(\underline{e}_p) \right]$$

$$B = \left[\underline{b}_1 \; \cdots \; \underline{b}_p \right]$$

$$f\circ g(\underline{e}_i) = f(g(\underline{e}_i))$$

$$= f(B\underline{e}_i)$$

$$= f(\underline{b}_i)$$

$$= A\underline{b}_i$$