

Lecture-8

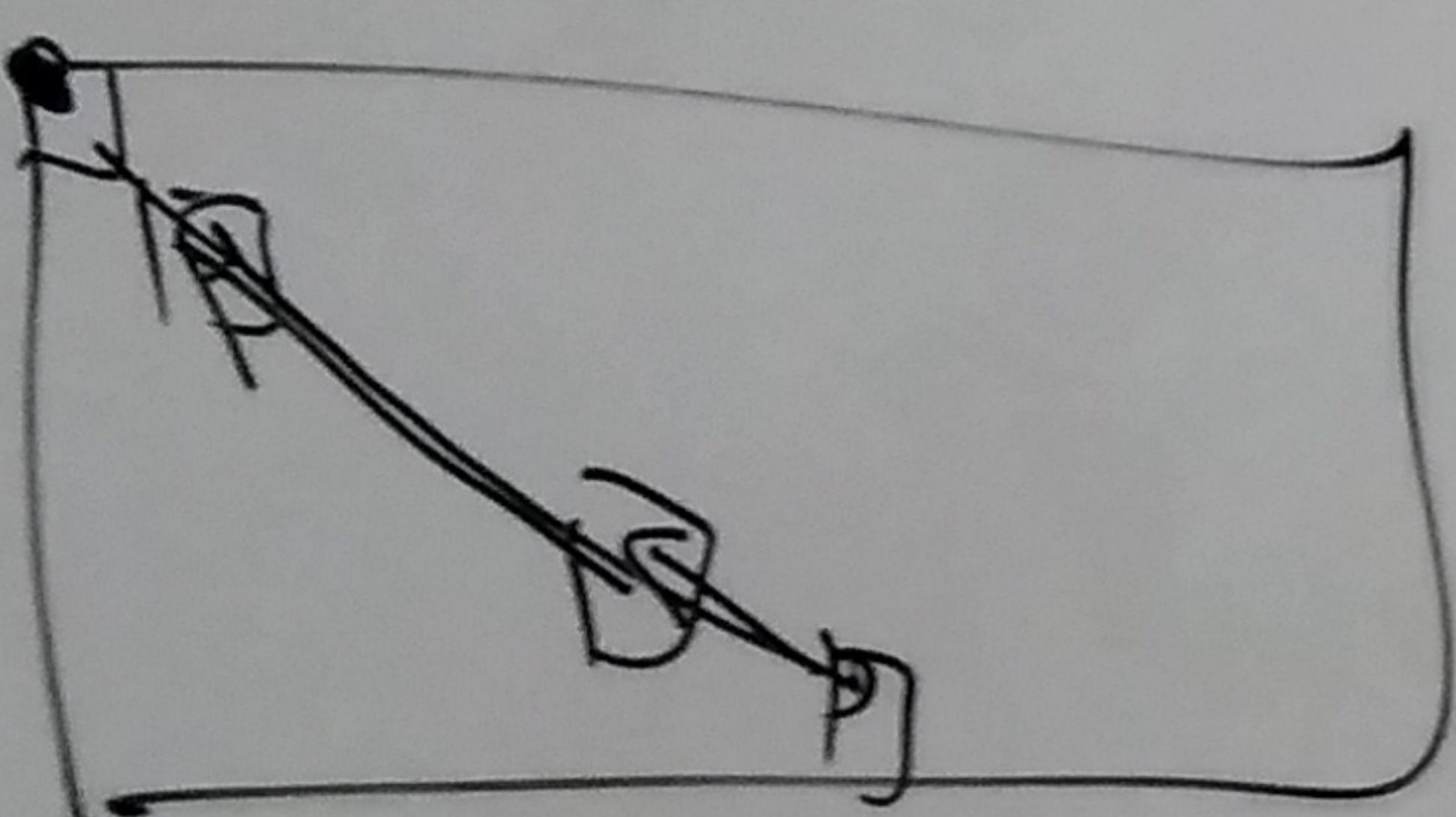
Row-Reduction - (Section 1.2 of Lay)

$$A \xrightarrow{\text{Forward Phase}} U \xrightarrow{\text{Backward Phase}} R = \text{RREF}(A)$$

$m \times n$

Row Echelon Form

Reduced Row Echelon Form
(RREF)



All the steps involve only elementary row operations.

Definition: Row Echelon Form

We say that a matrix \mathbf{U}
is in row echelon form if

(i) all-zero rows, if any,
should appear at the bottom
of \mathbf{U} .

(ii) the leading entry of row
 $(i+1)$ should be to the
right of leading entry
of row i .

Leading entry of a

row is the first non-zero
entry in that row

$$\begin{array}{|c|c|c|c|c|} \hline & 0 & 0 & 5 & 6 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

De

We

is

(1) a

sho

of

(4) th

(1 +
y 19

$$(2) U = \begin{matrix} 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -2 & 3 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

(iii) all entries below
any leading entry
should be 0.

Examples:

(1)

$$U = \begin{matrix} 6 & -11 & 3 & \sqrt{2} & 5 \\ 0 & 0 & 7 & \sqrt{8} & -13 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$(3) U = 0_{m \times n}$$

$$(4) U = I_{n \times n}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Definition:

We say that a matrix R is in RREF form if it satisfies: all the 3 conditions of echelon form, and

- (iv) every leading entry equals 1.
- (v) all entries above & below a leading entry should be 0.

$$\text{row2} \leftarrow \text{row2} - \text{row1}$$

$$\rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 2 & -4 & 4 \\ 0 & 3 & -6 & 6 \end{bmatrix}$$

$$= E_2 E_1 A, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we do "partial pivoting"

Swap row2, row3

$$\rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 3 & -6 & 6 \\ 0 & 2 & -4 & 4 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1 A ; E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Row Reduction (Example)

Forward Phase

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Swap row3 & row1

$$\rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \end{bmatrix}$$

Backward Phase

$$\rightarrow \text{row}2 \leftarrow \text{row}2 \times \frac{1}{3}$$

$$\rightarrow \begin{bmatrix} 3 & -1 & 12 & -9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \text{row}1 \leftarrow \text{row}1 + 9 \times \text{row}2$$

$$E_6 = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 \\ 0 & 1 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\text{row}3 \leftarrow \text{row}3 - \frac{2}{3} \text{row}2$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$$

$$U = E_7 \cdots E_1 \cdot A$$

$\rightarrow y_0 \leftarrow k_3 \times \text{row } 1$

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = (E_7 E_6 \dots E_2 E_1) A$$

$$E_7 = \begin{bmatrix} k_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & 5 & 7 \\ 0 & 3 & 6 \\ 0 & 2 & 9 \end{bmatrix}$$

$A \rightarrow \dots$

$$\begin{array}{|c|c|c|c|c|} \hline & 2 & -1 & 11 & 5 \\ \hline & 0 & 0 & 0 & 7 \\ \hline & 0 & 0 & 6 & 11 \\ \hline \end{array}$$

2. Create zeroes in all entries below (i, j)

using row replacement with (i, j) entry as pivot.

$$\begin{array}{l} j \leftarrow j+1 \\ i \leftarrow i+1 \end{array}$$

Backward Phase

Start with the last pivot.

Forward Phase:

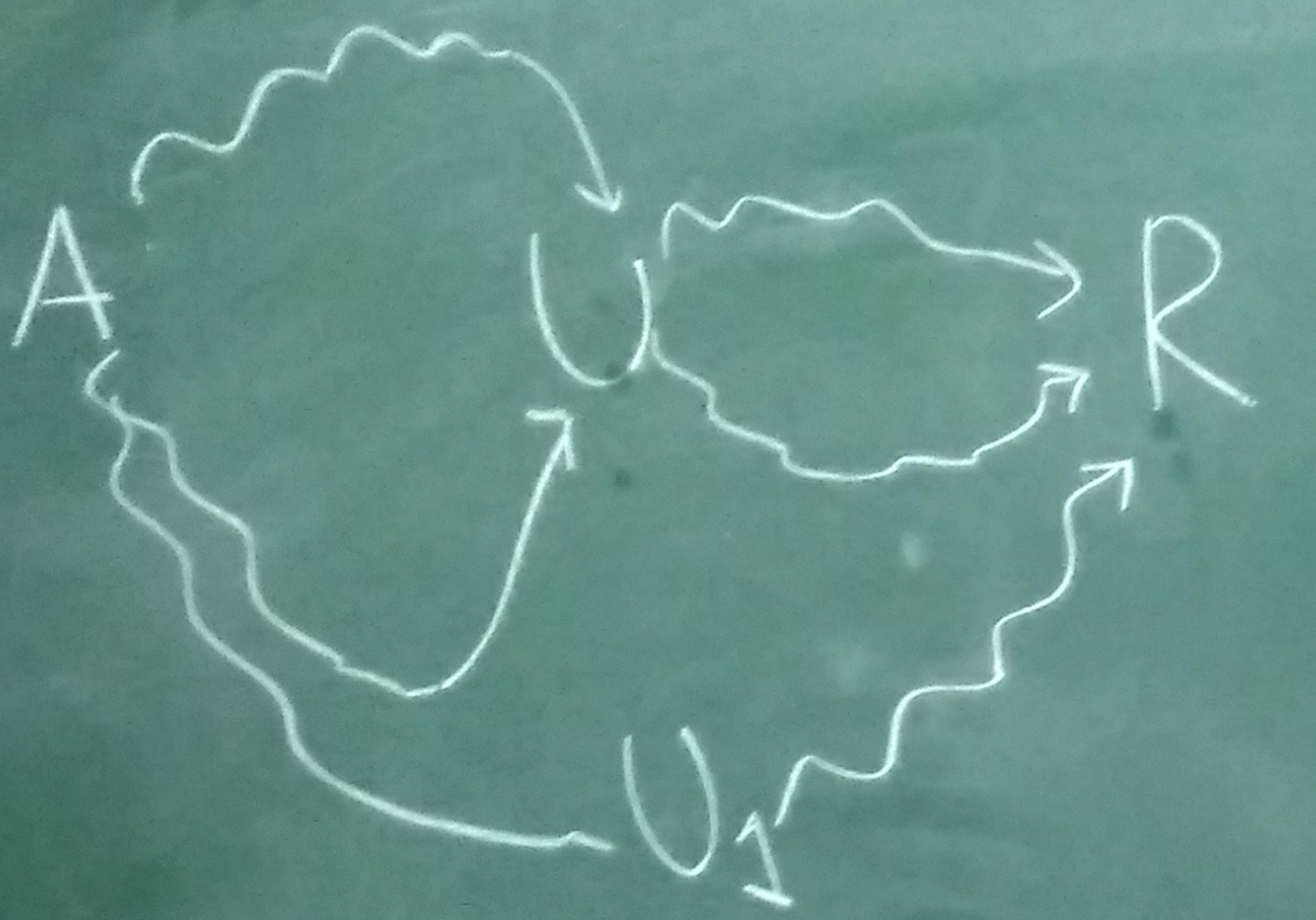
Let $(i, j) = (1, 1)$.

1. Create a non-zero entry at (i, j) by swapping row i with the rows below it.

(A) If not possible

$j \leftarrow j+1$ (move right)
& go to Step 1.

(B) If possible: (we have a pivot!)
go to Step 2.



for a given A ,

$$RREF(A) = R$$

is unique.

U is NOT unique.

1. Make the pivot entry
equal to 1.
(use row scaling)

2. Create zeroes above the
pivot using row
replacements.

3. Move to the previous
pivot, go to Step 1.