EE5609 – Matrix Theory 2023
Practice Set 2

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Solutions are not to be returned

Practice Set

Notation: Bold lower case letters will denote column vectors, and bold upper case letters are matrices. The symbol $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

1. *Complexity of basic matrix operations.*

   For this question assume the scalar field $\mathbb{F} = \mathbb{R}$. You can find the solutions to some of these questions in Boyd & Vandenberghe's textbook, but try to solve these problems independently.

   (a) What is the complexity of
      i. adding two $m \times n$ matrices
      ii. adding a dense $m \times n$ and a sparse $m \times n$ matrix.
      iii. adding two $n \times n$ upper triangular matrices
      iv. adding an $n \times n$ upper triangular and an $n \times n$ lower triangular matrix
      v. adding two skew-symmetric matrices
      vi. multiplying two $n \times n$ upper triangular matrices.

   (b) What is the complexity of multiplying two dense matrices of sizes $m \times n$ and $n \times p$, respectively.

   (c) Consider the task of multiplying three matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ of sizes $m \times n$, $n \times p$ and $p \times s$, respectively. Does the order in which we multiply, i.e., $\boldsymbol{A}(\boldsymbol{BC})$ or $(\boldsymbol{AB})\boldsymbol{C}$, affect the complexity of computing $\boldsymbol{ABC}$?
   For example, consider three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$. In which order would you compute the product $\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{c}$?

2. Repeat parts (a) and (b) of the previous question for $\mathbb{F} = \mathbb{C}$.
   `Hint:` What are the complexities of adding and multiplying two complex numbers represented in the cartesian form?

3. Suppose $\boldsymbol{A} \in \mathbb{F}^{n \times n}$ is such that $\boldsymbol{AB} = \boldsymbol{BA}$ for all $\boldsymbol{B} \in \mathbb{F}^{n \times n}$, then what is $\boldsymbol{A}$? More precisely, can you identify the following set

$$\{\boldsymbol{A} \in \mathbb{F}^{n \times n} \ : \ \boldsymbol{AB} = \boldsymbol{BA} \text{ for all } \boldsymbol{B} \in \mathbb{F}^{n \times n}\}.$$

4. How many bytes are required to store an $n \times n$ Hermitian matrix (assume $\mathbb{F} = \mathbb{C}$), when the real and imaginary parts use the type `Float64` ?

5. *Structural Properties.*

   Can you prove the following statements? You must be able to give rigorous proofs with precise mathematical statements. Showing or exhibiting an example for which the statement holds can not be considered a proof.

   (a) The product of two $n \times n$ upper triangular matrices is upper triangular.

(b) Product of two $n \times n$ lower triangular matrices is lower triangular.

(c) If a matrix is both upper triangular and lower triangular, then this is a diagonal matrix.

(d) For any $m \times n$ matrix $\boldsymbol{A}$, the matrix $\boldsymbol{A}^H\boldsymbol{A}$ is a Hermitian matrix.

(e) For any $m \times n$ matrix $\boldsymbol{A}$, the matrix $\boldsymbol{A}^T\boldsymbol{A}$ is a symmetric matrix.

(f) If $\boldsymbol{A}$ is a Hermitian matrix, then $i\boldsymbol{A}$ is a skew-Hermitian matrix, where $i = \sqrt{-1}$. Similarly, if $\boldsymbol{A}$ is skew-Hermitian, then $i\boldsymbol{A}$ is Hermitian.

(g) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices, then $\alpha\boldsymbol{A} + \beta\boldsymbol{B}$ is symmetric, where $\alpha, \beta \in \mathbb{F}$.

(h) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitian matrices, then $\alpha\boldsymbol{A} + \beta\boldsymbol{B}$ is Hermitian, where $\alpha, \beta \in \mathbb{R}$.
`Question:` Does this statement hold if we allow $\alpha, \beta \in \mathbb{C}$?

(i) If a matrix is both symmetric and skew-symmetric, then this is an all-zero matrix.

(j) If a matrix is both Hermitian and skew-Hermitian, then this is an all-zero matrix.

(k) If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is an $n \times n$ Hermitian matrix and $\boldsymbol{x} \in \mathbb{C}^n$, then $\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}$ is a real number.

(l) If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is skew-Hermitian and $\boldsymbol{x} \in \mathbb{C}^n$, then $\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}$ is a purely imaginary number.

(m) If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is skew-symmetric and $\boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$.

(n) Diagonal matrices commute, that is if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ diagonal matrices, then $\boldsymbol{AB} = \boldsymbol{BA}$.

(o) The product of two $n \times n$ permutation matrices is a permutation matrix.
`Hint:` You can use the fact the rows (or columns) of a permutation matrix are precisely the rows (or columns) of the identity matrix enumerated in a possibly different order.

(p) The transpose of a permutation matrix is also a permutation matrix.

(q) The product of a permutation matrix and its transpose is equal to the identity matrix.
`Hint:` Use the fact that each row (or a column) of a permutation matrix contains exactly one 1 and all other elements are zeros.

6. We saw in Lecture 6 that if $\boldsymbol{A}$ is any $n \times n$ matrix, then it can be expressed as the sum of a symmetric and a skew-symmetric matrix. Show that this decomposition is unique, i.e., for a given matrix $\boldsymbol{A}$, there exists a unique symmetric matrix $\boldsymbol{B}$ and a unique skew-symmetric matrix $\boldsymbol{C}$ such that $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$.

   `Proof Idea`. Assume that there are two such decompositions, i.e., $\boldsymbol{A} = \boldsymbol{B}_1 + \boldsymbol{C}_1$, and $\boldsymbol{A} = \boldsymbol{B}_2 + \boldsymbol{C}_2$, where $\boldsymbol{B}_1, \boldsymbol{B}_2$ are symmetric and $\boldsymbol{C}_1, \boldsymbol{C}_2$ are skew-symmetric. Use some of the statements from the previous question to show that $\boldsymbol{B}_1 = \boldsymbol{B}_2$ and $\boldsymbol{C}_1 = \boldsymbol{C}_2$.

   `Remark`. It is also true that for any matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix. Can you prove this statement as well?

7. *Circulant Matrices.* Recall the definition of a circulant matrix: we say that an $n \times n$ matrix $\boldsymbol{A}$ is circulant, if each row of the matrix is the circular shift (towards the right) of the previous row in the matrix.

   Can you prove the following: if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ circulant, then $\boldsymbol{A}+\boldsymbol{B}$ and $\boldsymbol{AB}$ are also $n \times n$ circulant.

   Also, show that circulant matrices commute, i.e., $\boldsymbol{AB} = \boldsymbol{BA}$.

   `Remark.` Circulant matrices capture circular convolution operation in digital signal processing. These matrices have very nice structure: they commute with each other and are closed under addition and multiplication. This structure leads to the property that all circulant matrices share the same set of eigen vectors: these eigen vectors are the columns of the Discrete Fourier Transform (DFT) matrix.
