Lecture-9

Perform row reduction

↓

RREF

↓

Relate #pivots to the existense of inverse.

↓

Method to compute the inverse

Recap:

Row Reduction

$$A_{m\times n} \xrightarrow{\text{Forward}} U_{m\times n} \xrightarrow{\text{Backward}} R_{m\times n}$$

	$U_{m\times n}$	$R_{m\times n}$
	Echelon form	Reduced Row Echelon form.

Each non-zero row of R will contain a pivot.

the pivot entries appear in distinct columns.

$\#\text{pivots} = \#\text{non-zero rows in } R \leq m$

$\#\text{pivots} \leq \#\text{columns} = n$

Thus, we have deduced that:

$$0 \leq \#\text{pivots} \leq \min\{m, n\}.$$

In particular, for a square $n \times n$, we have:

$$\#\text{pivots} \leq n.$$

What does $R_{n\times n}$ look like if $\#\text{pivots} = n$?

Example : $n = 3.$ (3×3 matrix, RREF).

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose R has 3 pivots.

In general, if R is an $n \times n$ RREF matrix and if the number of pivots in R is equal to n

then,

$$R = I_{n \times n}.$$

Recalling some results from Practice Set 3.

① If A, B are $n \times n$ matrices. Then:

Both A and AB are invertible $\implies$ B is invertible.

Contrapositive: Statement 1 $\implies$ Statement 2.

$$\overline{\text{Statement 2}} \implies \overline{\text{Statement 1}}$$

B is not invertible $\implies$ Either A or AB is not invertible

② If A has a all-zero row or a all-zero column, then A is not invertible.

A has a all-zero row

or

A has a all-zero column

$\Longrightarrow$ A is not invertible

A is invertible $\Longrightarrow$

A has no all-zero row

AND

A has no all-zero column.

③ If $\exists \underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$ then A is not invertible, where A is $n \times n$

$\exists \underline{x} \neq \underline{0}$, $\Longrightarrow$ A is not invertible.

$A\underline{x} = \underline{0}$

$\exists \underline{x} \neq \underline{0}$,

$\Rightarrow$, A is not invertible.

$\underline{x}^T A = \underline{0}^T$

④ $A_{n \times n}$ is invertible if and only if

A^T is invertible.

A is invertible $\Longleftrightarrow$ A^T is invertible.

Contrapositive :

A is not invertible $\Longleftrightarrow$ A^T is not invertible.

Relating the invertibility of a square matrix

to

the number of pivot entries.

Theorem: Suppose A is a $n \times n$ matrix.

A is invertible $\Longleftrightarrow$ A has n pivots.

A is invertible $\Longleftrightarrow$ $\mathrm{RREF}(A) = I_{n\times n}$

Proof:

Part 1 : Assume A has n pivots.

Suppose R is the RREF of A.

$$R = E_t E_{t-1} \cdots E_1 \cdot A.$$

where $E_1, \ldots, E_t$ are elementary matrices.

Since R is $n \times n$ RREF with n pivots, we know that $R = I_{n\times n}$.

$$\therefore \underbrace{(E_t E_{t-1} \cdots E_1)}_{\text{invertible}} A = \underbrace{I}_{\text{invertible}}$$

$\Longrightarrow$ A is invertible.

Part 2. Assume A is invertible.

Suppose $R = E_t \cdots E_1 A$ is the RREF of A.

Since $A, E_1, \ldots, E_t$ are invertible, we see that R is invertible.

$\therefore$ R is an $n \times n$ invertible RREF matrix.

$\Rightarrow$ none of the rows of R are all-zero.

$\Rightarrow$ there is one pivot position per row of R.

$\Rightarrow$ R has n pivots

$\Rightarrow$ A has n pivots.

Computing the inverse of a matrix

Let us assume that A is an $n \times n$ invertible matrix.

$\Rightarrow$ $\mathrm{RREF}(A) = I_{n\times n}$, A^{-1} exists.

$$E_t E_{t-1} \cdots E_1 A = R = I$$

$$E_t E_{t-1} \cdots E_1 A \cdot A^{-1} = I \cdot A^{-1}$$

$$\Rightarrow \quad A^{-1} = E_t E_{t-1} \cdots E_2 E_1 .$$

Observation:

① The inverse of A is the product of the elementary matrices $E_t E_{t-1} \cdots E_1$ used in the row reduction of A.

② Since $A^{-1} = E_t \cdots E_2 E_1$.

$$A = E_1^{-1} E_2^{-1} \cdots E_{t-1}^{-1} E_t^{-1}$$

$\therefore$ A is a product of elementary matrices.

Augmented matrix $B = [\underset{n\times n}{A} \mid \underset{n\times n}{I}]$

$B : n \times 2n$

If we perform row reduction on B so that the first half of B is reduced to I:

$$E_t E_{t-1} \cdots E_1 \cdot B = [I \mid *].$$

$$E_t \cdots E_1 B = E_t \cdots E_1 \cdot \begin{bmatrix} A & I \end{bmatrix}$$

$$= \begin{bmatrix} E_t \cdots E_1 \cdot A & E_t \cdots E_1 \cdot I \end{bmatrix}.$$

$$= \begin{bmatrix} I & \boxed{E_t \cdots E_1} \end{bmatrix}. \rightarrow \text{inverse of } A$$

$$= \begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

Gauss-Jordan method / Gaussian elimination method

for computing A^{-1}.

Complexity of this techique is $\Theta(n^3)$.

$f(n)$ = # flops to compute inverse of $n \times n$ matrix.

$$\lim_{n \to \infty} \frac{f(n)}{n^3}$$ is a positive constant.

Remarks:

* In practical applications, it is quite rare to actually compute the inverse of a matrix.

find this.

$$A\underline{x} = \underline{b}$$

given.

Given (invertible)

$$\Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}.$$

① It is cheaper to use an alternative technique, that is based on

$$PA = LU \text{ factorization}$$

$$PA\underline{x} = P\underline{b} \Rightarrow LU\underline{x} = P\underline{b}$$

② This $PA = LU$ technique has smaller round-off errors, and has better numerical stability.

About the definition of invertibility.

Recall that:

A is invertible if $\exists$ B such that

$$AB = I$$

and $BA = I$.

We want to show that this can be relaxed to checking exactly one of these two above conditions.

In other words, we want to show that if

$A, B \in \mathbb{F}^{n \times n}$:

$$AB = I \implies \text{A and B are invertible and } B = A^{-1},\ A = B^{-1}.$$

Claim: If $A, B \in \mathbb{F}^{n \times n}$ and $AB = I$

then B is invertible.

Proof: We will use proof by contradiction.

Assume that the result is not true, i.e.,

assume that B is NOT invertible.

$\Longrightarrow$ B^T is not invertible.

Consider performing row reduction on B^T,

and say $R = RREF(B^T)$.

$\Rightarrow$ For some integer $t \geqslant 0$:

$$E_t \cdots E_2 E_1 B^T = R$$

Since B^T is not invertible, # pivots in $R < n$

$\Rightarrow$ R has atleast one all-zero row.

$\Rightarrow$ the last row of R is all-zero.

$\Longrightarrow$ the last column of R^T is all-zero.

$$B^T = (E_t \cdots E_1)^{-1} R.$$

$$\therefore \quad B = R^T \underbrace{(E_t \cdots E_1^{-1})^T}_{M,\ \text{invertible}}$$

$B = R^T M$ where M is invertible

$$R^T \underline{e}_n = \underline{0} \quad \Longrightarrow \quad R^T M \cdot M^{-1} \underline{e}_n = \underline{0}$$

$$\Longrightarrow \quad B \cdot M^{-1} \underline{e}_n = \underline{0}$$

Since M^{-1} is invertible,

all its columns are non-zero

$$\Rightarrow \quad M^{-1}\underline{e}_n \neq \underline{0}. \quad \text{——} \; ①$$

$M^{-1}\underline{e}_n$ is the last column of M^{-1}.

$$B\left[M^{-1}\underline{e}_n\right] = \underline{0}. \quad \text{—} \; ②.$$

The contradiction:

From ②: $AB\left[M^{-1}\underline{e}_n\right] = A\underline{0}$

$$I\left[M^{-1}\underline{e}_n\right] = \underline{0} \Rightarrow M^{-1}\underline{e}_n = \underline{0}$$

See that this contradicts ①.

Theorem: If A, B are $n \times n$ and

$$AB = I \quad \text{———} \quad \circledast.$$

then $B = A^{-1}$, $A = B^{-1}$.

Proof: From the previous claim we know that

$$AB = I \implies B \text{ has an inverse.}$$

$\therefore B^{-1}$ exists.

Multiplying with B^{-1} on both sides of $AB = I$

$$AB \cdot B^{-1} = I \cdot B^{-1}$$

$$\Rightarrow \quad A \cdot I = B^{-1}$$

$$\Rightarrow \quad A = B^{-1}.$$ (this shows that A is invertible as well).

Taking inverse on both sides:

$$A^{-1} = (B^{-1})^{-1} = B.$$

$$\therefore \quad A^{-1} = B \quad \text{and} \quad B^{-1} = A.$$