Lecture-5

Basic matrix, vector multiplication operations.

$\mathbb{F}$ = either $\mathbb{R}$ or $\mathbb{C}$

(i) $\underline{a}, \underline{b} \in \mathbb{F}^n$

$$\underline{a}^T \underline{b} = \underbrace{\boxed{a_1 \cdots\cdots a_n}}_{\underline{a}^T} \underbrace{\boxed{\begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix}}}_{\underline{b}} = \sum_{i=1}^{n} a_i b_i$$

Complexity: $2n-1$. flops

$\neq \underline{0}$

$\times - \underline{1}$ matrix

n matrix

$\underline{a}\,\underline{b}^T$.

$\mathbb{F}^n$

$\underline{y} \in \mathbb{F}^m$

(2) Outer Product

$\underline{a} \in \mathbb{F}^m \quad \underline{b} \in \mathbb{F}^n$

$$\underline{a}\,\underline{b}^T = \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}}_{\underline{a}} \underbrace{\begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}}_{\underline{b}^T}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & \cdots & & a_2 b_n \\ & & & \\ a_m b_1 & \cdots & & a_m b_n \end{bmatrix}$$

x_1

x_2

x_n

$+ a_{i,n} x_n$

$\approx 2mn$ flops

<u>Remark</u>: If $\underline{a}, \underline{b} \neq \underline{0}$

$\underline{a}\,\underline{b}^T$ is a rank-1 matrix

Every rank-1 $m \times n$ matrix

is of the form $\underline{a}\,\underline{b}^T$.

(3) $A \in \mathbb{F}^{m \times n}$, $\underline{x} \in \mathbb{F}^n$

$\underline{y} = A\underline{x}$, $\underline{y} \in \mathbb{F}^m$

(2) O

$\underline{a} \in \mathbb{F}$

$\underline{a}\,\underline{b}^T =$

$=$

$$\begin{bmatrix} y \\ \hline i \rightarrow \end{bmatrix} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$y_i = a_{i,1} x_1 + a_{i,2} x_2 + \cdots + a_{i,n} x_n$$

$$\forall i = 1, \ldots, m$$

Complexity: $m(2n-1) \approx 2mn$ flops

$$\underline{y} = \sum_{j=1}^{n} \underline{a}_j x_j$$

Remar

$\underline{a}$ $\underline{b}$

Every r

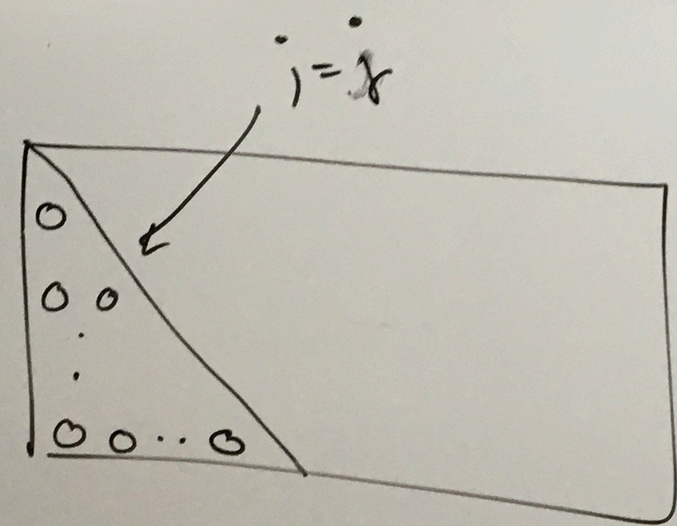
is of t

(3) $A \in \mathbb{R}$

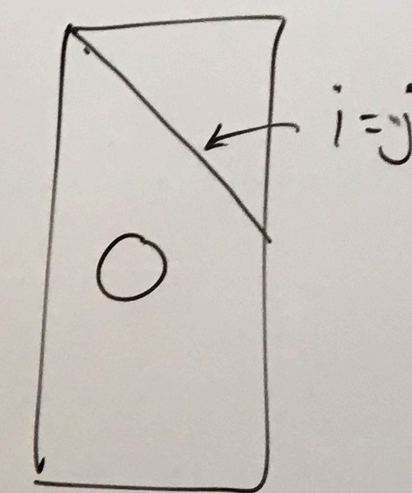
$\underline{y} =$

(3a) Suppose $A \in \mathbb{R}^{m \times n}$ is an upper triangular m

$A =$

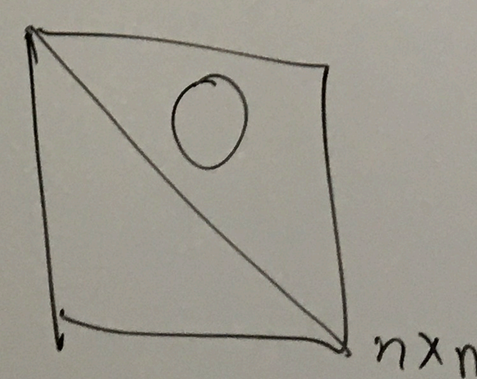


$A =$



If $i > j$: $A_{i,j} = 0$: A upper triangular

If $i < j$: $A_{i,j} = 0$: A lower triangular



if

$A_{ij} = 0$

$(i \neq j)$

flops

If A is lower triangular $n \times n$

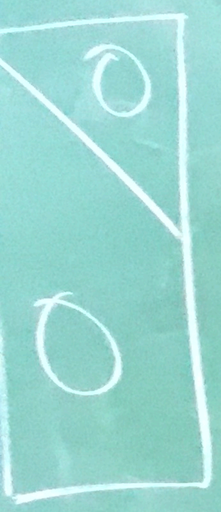
and $\underline{x} \in \mathbb{R}^n$

$$\underline{y} = A\underline{x}$$

$$\begin{bmatrix} \\ \underline{y} \\ \\ \end{bmatrix} = \begin{bmatrix} \ddots & & 0 \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \\ \underline{x} \\ \\ \end{bmatrix}$$

$$= \sum_{i=1}^{n} (2i-1) - n$$

$$= \cancel{2} \times \frac{n(n+1)}{\cancel{2}} - n = n^2$$



ces.

$$= \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

(3b) A matrix A

is <u>diagonal</u> if

for all $i \neq j$, $A_{i,j} = 0$

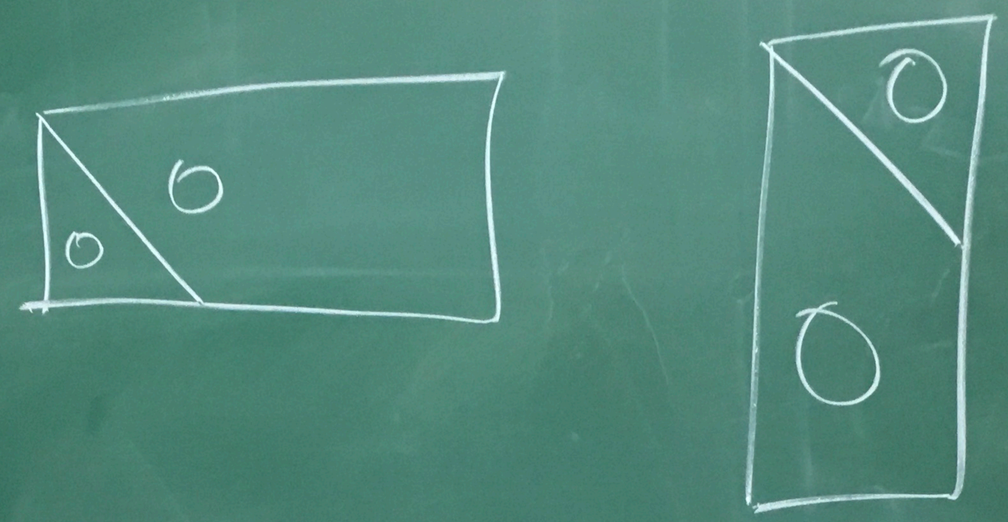
$(A_{i,j} = 0 \text{ if } i \neq j)$

$\underset{n \times n}{A} \cdot \underline{x} \rightarrow n$ <u>flops</u>.

If

a

<u>y</u>



Detour: Block Matrices.

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 5 & 11 & 16 \\ -1 & 3 & 6 & 5 & 0 \\ \hline 6 & 2 & 5 & 7 & 11 \end{array}\right] = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

B C D E

(3b

$n \times$

In a block matrix:

all the blocks in a row have the same number of rows, and

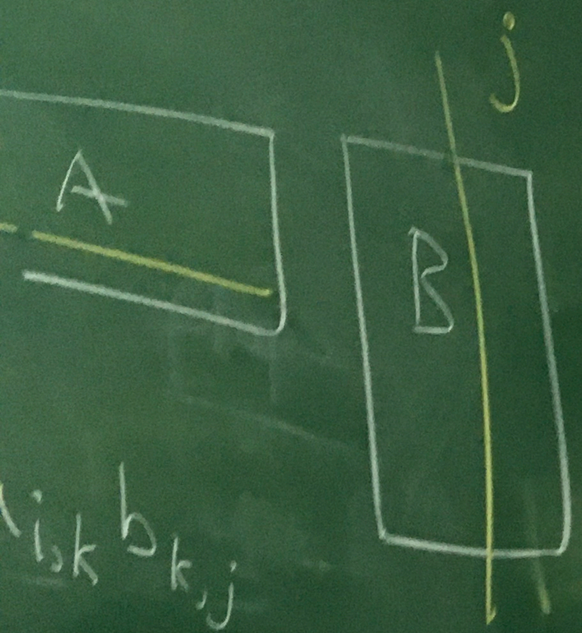
all the blocks in a column have the same number of columns.

$$A \cdot \underline{x} = \begin{array}{c} \left[\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right] \\ \begin{array}{cc} n_1 & n_2 \end{array} \end{array} \begin{array}{l} \updownarrow m_1 \\ \updownarrow m_2 \end{array} \left[\begin{array}{c} \underline{x}_1 \\ \hline \underline{x}_2 \end{array}\right] \begin{array}{l} \} n_1 \\ \} n_2 \end{array} = \begin{bmatrix} B\underline{x}_1 + C\underline{x}_2 \\ D\underline{x}_1 + E\underline{x}_2 \end{bmatrix}$$

trix multiplication

$B \in \mathbb{F}^{n \times p}$

$\in \mathbb{F}^{m \times p}$

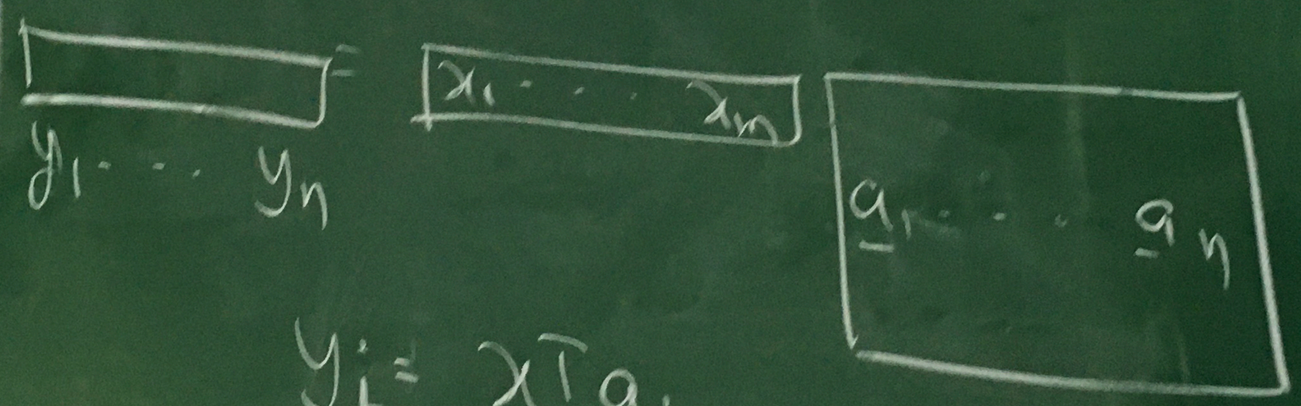


$a_{i,k} b_{k,j}$

④ Row vector × Matrix

$\underline{x} \in \mathbb{F}^m$, $A \in \mathbb{F}^{m \times n}$, $\underline{y} \in \mathbb{F}^n$

$\underline{y}^T = \underline{x}^T A$



$y_j = \underline{x}^T \underline{a}_j$, $j = 1, \ldots, n$

Complexity $= n \cdot (2m-1) \approx 2mn$ flops

$(\mathbb{F} = \mathbb{R})$

r1)

flops

Ab_p]

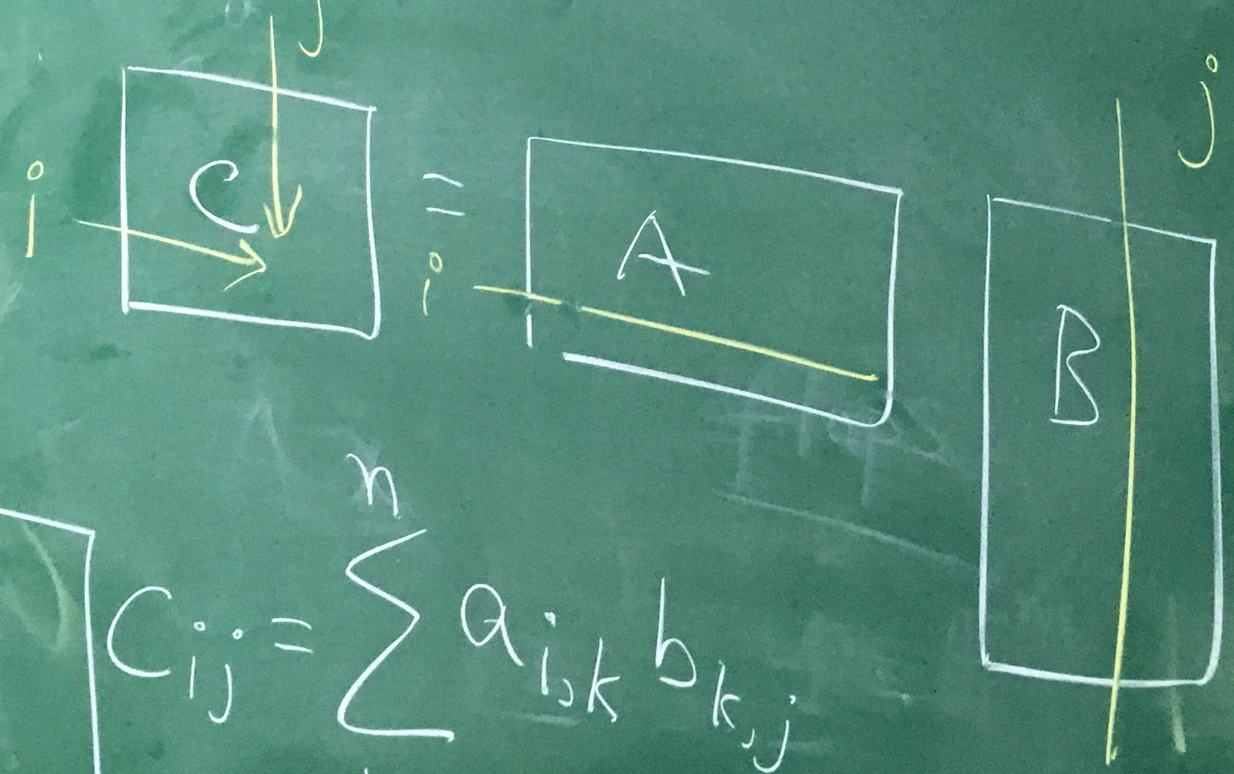
]

$B = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$

(5) Matrix-Matrix multiplication

$A \in \mathbb{F}^{m \times n} \qquad B \in \mathbb{F}^{n \times p}$

$C = AB \in \mathbb{F}^{m \times p}$

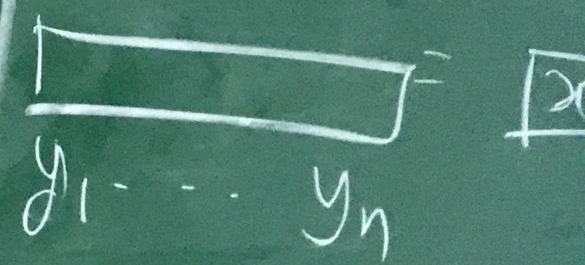


$$c_{ij} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

(4) Rou

$\underline{x} \in \mathbb{F}$

$\underline{y}^T =$



$y_j =$

Complexity

If $\mathbb{F} = \mathbb{R}$:

$$\text{complexity} = mp(2n-1)$$

$$\approx 2mnp \text{ flops}$$

(5a) $C = A \boxed{\underline{b}_1 \; \underline{b}_2 \cdots \underline{b}_p}$

$$= \left[A\underline{b}_1 \;\; A\underline{b}_2 \cdots A\underline{b}_p \right]$$

(5b) $C = \boxed{\begin{matrix} \leftarrow \tilde{\underline{a}}_1^T \rightarrow \\ \\ \leftarrow \tilde{\underline{a}}_m^T \rightarrow \end{matrix}} \; B = \begin{bmatrix} \tilde{\underline{a}}_1^T B \\ \vdots \\ \tilde{\underline{a}}_m^T B \end{bmatrix}$

(5) Mat

$A \in \mathbb{F}$

$C = A$

j

$i \; \boxed{C}$

$C_{ij} = \sum_{k=1}^{n}$

Examples:

(i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} \leftarrow & \tilde{\underline{b}}_1^T & \rightarrow \\ \leftarrow & \tilde{\underline{b}}_2^T & \rightarrow \\ \leftarrow & \tilde{\underline{b}}_3^T & \rightarrow \end{bmatrix}$

$AB = \begin{bmatrix} \tilde{\underline{b}}_2^T \\ \tilde{\underline{b}}_1^T \\ \tilde{\underline{b}}_3^T \end{bmatrix}$ → Permuted the rows of B.

(ii) $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$, $A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix}$

$C = AB = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

Ucation

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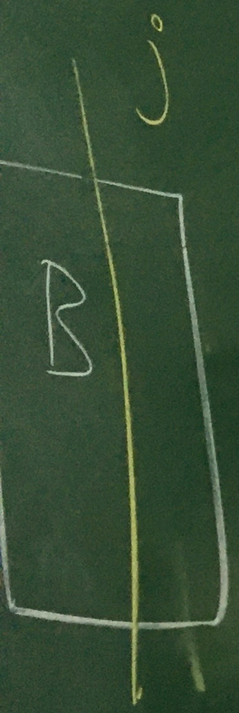

Remark:

In general:

$$AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

In general $AB = AC \not\Rightarrow B = C$

$BA = CA \not\Rightarrow B = C$

Examp

(i)

(ii)

C = A