Foundation of Machine Learning

Revisiting Prerequisite of Maths : Calculus

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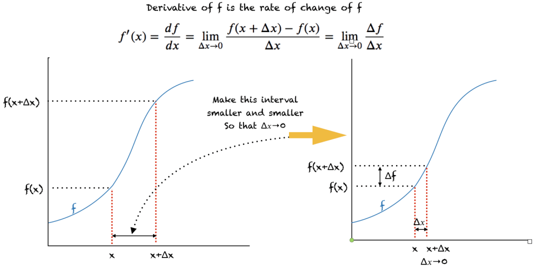
Derivatives

The slope of the secent line through two points on the graph of f is given by :

$$\frac{\delta f(x)}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

As we reduce δx we reach a limiting case where we get the slope at a particular point.



The derivative of f points in the direction of steepest ascent of f.

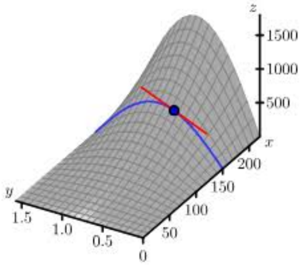
Partial Derivatives

Lets consider $f : \mathbb{R}^n \to \mathbb{R}$. So finding partial derivative comes into picture when you would like to find the derivative of the function w.r.t. each element/dimension of input $\mathbf{x}$.

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \cdots, x_n) - f(\mathbf{x})}{h}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = \lim_{h \to 0} \frac{f(x_1, x_2, \cdots, x_n + h) - f(\mathbf{x})}{h}$$



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Gradients

Gradients are derivatives of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ w.r.t input vector x. It is exactly equal to the collection of all the partial derivatives in a particular order.

$$\nabla_x f = grad\ f = \frac{df}{dx} = [\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n}] \in \mathbb{R}^{1\times n}$$

For this session we consider gradients to be Row Vectors. But its not uncommon to represent gradients as column vectors.

Chain Rule:

$$\frac{\partial}{\partial x}(g \circ f)(x) = \frac{\partial g}{\partial f}\frac{\partial f}{\partial x}$$

In this case also its the same as it was for uni-variate case.

Jacobian

Let's make derivatives more general. Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with input as $x = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ and output of it is given by a vector function

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} \in \mathbb{R}^m$$

Then the derivative of f w.r.t. x is given by the collection of all possible partial derivatives arranged as a matrix $J \in \mathbb{R}^{m \times n}$ in a certain order.

$$J = \frac{df(x)}{dx} = \begin{bmatrix} \nabla_x f_1 \\ \nabla_x f_2 \\ \vdots \\ \nabla_x f_m \end{bmatrix} \textit{ where } \nabla_x f_i \in \mathbb{R}^n$$

Example

Consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with input as $x(t) = [x_1(t), x_2(t)]^T$ which is itself a function $x : \mathbb{R} \rightarrow \mathbb{R}^2$. We need to find $\frac{df}{dt}$.

Example

Going beyond 2-Dimension Jacobian

Consider a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}^{p}$. A function that takes matrix as input and outputs a vector.
Let the input be $A \in \mathbb{R}^{m\times n}$ and the output be $f(A) = [f_1(A), \cdots, f_p(A)]^T$
Then to find the derivative, its again the collection of all possible partial derivatives presented in a particular order.

$$\frac{df}{dA} = \begin{bmatrix} \frac{df_1(A)}{dA} \\ \frac{df_2(A)}{dA} \\ \vdots \\ \frac{df_p}{dA} \end{bmatrix}, \frac{df_i}{dA} \in \mathbb{R}^{1\times(m\times n)}$$

Similarly, one can find dimension of the derivatives as well as the particular ordering even for higher dimensions like $\mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$,
$\mathbb{R}^{m\times n\times k} \to \mathbb{R}^{p\times q}$and so on.

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Some of the most seen functions in ML

- $\frac{\partial x^T a}{\partial x} = a^T$
- $\frac{\partial a^T x}{\partial x} = a^T$
- $\frac{\partial a^T X b}{\partial X} = ab^T$
- $\frac{\partial x^T B x}{\partial x} = x^T(B + B^T)$
- $\frac{\partial (x-As)^T W (x-As)}{\partial s} = -2(x - As)^T WA$ for symmetric W
- $\frac{\partial a^T X^{-1} b}{\partial X} = -(X^{-1})^T ab^T (X^{-1})^T$