6351 project 3

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Two-Link System Analysis

For this project the noise and disturbance are introduced as:

$$M(\phi)\ddot{\phi} + C(\phi, \dot{\phi}) + G(\phi) + \tau_d(\phi, \dot{\phi}) = \tau$$

$$\phi_m = \phi + \omega_\phi$$

$$\dot{\phi_m} = \dot{\phi} + \omega_{\dot{\phi}}$$

$$\ddot{\phi_m} = \ddot{\phi} + \omega_{\ddot{\phi}}$$

$$\tau_m = \tau + \omega_\tau$$

$\phi_m, \dot{\phi}_m, \ddot{\phi}_m$ are the measure values and $\omega_\tau \triangleq \begin{bmatrix} \omega_{\tau 1} \\ \omega_{\tau 2} \end{bmatrix}$, $\omega_\phi \triangleq \begin{bmatrix} \omega_{\phi 1} \\ \omega_{\phi 2} \end{bmatrix}$, $\omega_{\dot{\phi}} \triangleq \begin{bmatrix} \omega_{\dot{\phi} 1} \\ \omega_{\dot{\phi} 2} \end{bmatrix}$,and $\omega_{\ddot{\phi}} \triangleq \begin{bmatrix} \omega_{\ddot{\phi} 1} \\ \omega_{\ddot{\phi} 2} \end{bmatrix}$ are the bounded disturbance and noise values with known bounds given and $\tau_d(\phi, \dot{\phi}) \in \mathbb{R}$ is a function of the sate and could be a force of an object the system picked up, unknown friction at the joints, air resistance, any combination of these, pr various other types of unstructured dynamics. There are a few ways to approach this problem but we will leverage the known structure for the inertia, centripetal Coriolis, and gravity matrices but use various basis to approximate $\tau_d(\phi, \dot{\phi}) \in \mathbb{R}$, the single-layer neural networks form shows as follow:

$$\tau_d(\phi, \dot{\phi}) = W^\top \sigma(\phi, \dot{\phi}) + \epsilon(\phi, \dot{\phi})$$

where $W \in \mathbb{R}^{L \times 2}$, $\sigma(\phi, \dot{\phi}) \in \mathbb{R}^L$ and $\epsilon(\phi, \dot{\phi})$, and the two-layer neural networks for shows as follow:

$$\tau_d(\phi, \dot{\phi}) = W^\top \sigma(\Phi(\xi)) + \epsilon(\phi, \dot{\phi})$$

where since the output is dimension 2, $W \in \mathbb{R}^{L \times 2}$,$\sigma(\Phi(\xi)) \in \mathbb{R}^L$, $\Phi(\xi) \in \mathbb{R}^l$ is $\Phi(\xi) = V^\top \xi$,$V \in \mathbb{R}^{5 \times l}$, $\xi \in \mathbb{R}^5$ is $\xi = \begin{bmatrix} \phi \\ \dot{\phi} \\ 1 \end{bmatrix}$, and $\epsilon(\phi, \dot{\phi}) \in \mathbb{R}^2$

But in order to simplify the analysis we still use the expression form without noise and interference, and add all the noise and interference in the final implementation of the system will be fine, so that

$$M(\phi)\ddot{\phi} + C(\phi, \dot{\phi}) + G(\phi) + \tau_d(\phi, \dot{\phi}) = \tau$$

where $\phi(t)$, $\dot{\phi}(t)$, $\ddot{\phi}(t)$, $\tau(t) \in \mathbb{R}$ are the angle, angular velocity, angular acceleration, and input torque of the arm joints, the inertia terms $M(\phi)$,the Coriolis and centripetal terms $C(\phi, \dot{\phi})$, and the gravity terms $G(\phi)$ are defined as

$$M(\phi) \triangleq \begin{bmatrix} m_1 + l_1^2 + m_2(l_1^2 + 2l_1l_2c_2 + l_2^2) & m_2(l_1l_2c_2 + l_2^2) \\ m_2(l_1l_2c_2 + l_2^2) & m_2l_2^2 \end{bmatrix}$$

$$C(\phi, \dot{\phi}) \triangleq \begin{bmatrix} -2m_2l_1l_2s_2\dot{\phi}_1\dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2l_1l_2s_2\dot{\phi}_2^2 \\ m_2l_1l_2s_2\dot{\phi}_2^2 \end{bmatrix}$$

$$G(\phi) \triangleq \begin{bmatrix} (m_1 + m_2gl_1c_1 + m_2gl_2c_{12}) \\ m_2gl_2c_{12} \end{bmatrix}$$

where the $m_i, l_i \in \mathbb{R}_{>0}$ are the *unknown* constant mass and length of link $i, g \in \mathbb{R}_{>0}$ is gravity,

$$c_i = cos(\phi_i)$$

$$s_i = sin(\phi_i)$$

$$c_{12} = cos(\phi_1 + \phi_2)$$

$$s_{12} = sin(\phi_1 + \phi_2)$$

$ErrorDevelopment$: Assume we want to track the desired angles, angular velocities, and angular accelerations $\phi_d(t), \dot{\phi}_d(t), \ddot{\phi}_d(t) \in \mathbb{R}^2$. Then let the tracking error be defined as

$$e \triangleq \phi_d - \phi$$

$$\dot{e} = \dot{\phi}_d - \dot{\phi}$$

$$\ddot{e} = \ddot{\phi}_d - \ddot{\phi}$$

and let the filtered tracking error be defined as

$$r = \dot{e} + \alpha e$$

$$\dot{r} = \ddot{e} + \alpha \dot{e}$$

where $\alpha \in \mathbb{R}^{2x2}$ is a positive definite design gain matrix (usually a diagonal). Now we want to substitute our dynamics to multiply both sides by $M(\phi)$

$$M(\phi)\dot{r} = M(\phi)\ddot{e} + M(\phi)\alpha\dot{e}$$

$$M(\phi)\dot{r} = M(\phi)(\ddot{\phi}_d - \ddot{\phi}) + M(\phi)\alpha\dot{e}$$

$$M(\phi)\dot{r} = M(\phi)\ddot{\phi}_d - M(\phi)\ddot{\phi} + M(\phi)\alpha\dot{e}$$

where since

$$M(\phi)\ddot{\phi} + C(\phi,\dot{\phi}) + G(\phi) + \tau_d(\phi,\dot{\phi}) = \tau$$
$$\Rightarrow M(\phi)\ddot{\phi} = -C(\phi,\dot{\phi}) - G(\phi) + \tau - \tau_d(\phi,\dot{\phi})$$

we get

$$M(\phi)\dot{r} = M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi,\dot{\phi}) + G(\phi) - \tau + \tau_d(\phi,\dot{\phi})$$

From previous lectures, the above dynamics are linear in the unknown parameters and we were able to develop a gradient update law for the unknown parameters

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 \\ m_2 l_1 l_2 \\ m_2 l_2^2 \\ (m_1 + m_2) l_1 \\ m_2 l_2 \end{bmatrix}$$

because we have

$$Y\theta = M(\phi)\varphi + C(\phi,\dot{\phi}) + G(\phi) + \frac{1}{2}\dot{M}(\phi,\dot{\phi})r$$

where $\varphi \triangleq (\ddot{\phi}_d + \alpha\dot{e})$

$$Y \triangleq Y_M(\phi,\varphi) + Y_C(\phi,\dot{\phi}) + Y_G(\phi) + \frac{1}{2}Y_{\dot{M}}(\phi,r)$$

$$Y_M(\phi,\varphi) \triangleq \begin{bmatrix} \varphi_1 & (2c_2\varphi_1 + c_2\varphi_2) & \varphi_2 & 0 & 0 \\ 0 & c_2\varphi_1 & (\varphi_1 + \varphi_2) & 0 & 0 \end{bmatrix}$$

$$Y_C(\phi,\dot{\phi}) \triangleq \begin{bmatrix} 0 & -(2s_2\dot{\phi}_1\dot{\phi}_2 + s_2\dot{\phi}_2^2) & 0 & 0 & 0 \\ 0 & s_2\dot{\phi}_1^2 & 0 & 0 & 0 \end{bmatrix}$$

$$Y_G(\phi) \triangleq \begin{bmatrix} 0 & 0 & 0 & gc_1 & gc_{12} \\ 0 & 0 & 0 & 0 & gc_{12} \end{bmatrix}$$

$$Y_{\dot{M}}(\phi) \triangleq \begin{bmatrix} 0 & -(2s_2\dot{\phi}_2 r_1 + s_2\dot{\phi}_2 r_2) & 0 & 0 & 0 \\ 0 & -s_2\dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix}$$

Controller Design

Robust Control Design

$$M(\phi)\dot{r} = Y(\phi,\dot{\phi},\ddot{\phi},\ddot{\phi}_d,\dot{e},r)\theta + \tau_d(\phi,\dot{\phi}) - \frac{1}{2}\dot{M}(\phi,\dot{\phi})r - \tau$$

let

$$\zeta \triangleq \begin{bmatrix} e \\ r \\ \widetilde{\theta} \end{bmatrix}$$

and we define our Lyapunov candidate as

$$V(\zeta,t) \triangleq \frac{1}{2}e^\top e + \frac{1}{2}r^\top.M(\phi)r + \frac{1}{2}\widetilde{\theta}\Gamma_\theta^{-1}\widetilde{\theta}$$

then

$$\dot{V}(\zeta,t) = e^\top \dot{e} + \frac{1}{2}r^\top \dot{M}(\phi)r + r^\top M(\phi)\dot{r} + \widetilde{\theta}\Gamma_\theta^{-1}\dot{\widetilde{\theta}}$$

we substitute what we got before in

$$\dot{V}(\zeta,t) = e^\top(r-\alpha e) + \frac{1}{2}r^\top \dot{M}(\phi)r + r^\top(Y\theta + \tau_d - \frac{1}{2}\dot{M}r - \tau) - \widetilde{\theta}\Gamma_\theta^{-1}\dot{\widehat{\theta}}$$

$$= -e^\top \alpha e + r^\top e + r^\top(Y\theta + \tau_d - \tau) - \widetilde{\theta}\Gamma_\theta^{-1}\dot{\widehat{\theta}}$$

so we design the input as

$$\tau = Y\widehat{\theta} + e + \beta_r r + \beta_{\tau_d} sgn(r)$$

we can get

$$\dot{V}(\zeta,t) = -e^\top \alpha e - r_r^\top r - r^\top \beta_{\tau_d} sgn(r) + r^\top \tau_d + r^\top Y\widetilde{\theta} - \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}}$$

now we need to bound

$$-r^\top \beta_{\tau_d} sgn(r) + r^\top \tau_d$$

as

$$-\underline{\beta_{\tau_d}}||r|| + r^\top \overline{\tau_d} \leq 0$$

so that

$$-\underline{\beta_{\tau_d}} > \overline{\tau_d}$$

then we design

$$\dot{\widehat{\theta}} = proj(\Gamma_\theta Y^\top r)$$

finally we can have

$$\dot{V}(\zeta,t) \leq -\alpha||e||^2 - \beta_r||r||^2$$

then we can show asymptotic tracking using Barbalats Lemma.

Single-Layer Design

$$M(\phi)\dot{r} = Y\theta + W^\top \sigma + \epsilon - \frac{1}{2}\dot{M}(\phi,\dot{\phi})r - \tau$$

let

$$\zeta \triangleq \begin{bmatrix} e \\ r \\ \widetilde{\theta} \\ vec(\widetilde{W}) \end{bmatrix}$$

let

$$vec(\widetilde{W}) \triangleq \begin{bmatrix} \widetilde{W}_{11} \\ \vdots \\ \widetilde{W}_{L1} \\ \widetilde{W}_{12} \\ \vdots \\ \widetilde{W}_{L2} \end{bmatrix}$$

and

$$V(\zeta,t) \triangleq \frac{1}{2}e^\top e + \frac{1}{2}r^\top M(\phi)r + \frac{1}{2}\widetilde{\theta}\Gamma_\theta^{-1}\widetilde{\theta} + \frac{1}{2}tr(\widetilde{W}^\top \Gamma_W^{-1}\widetilde{W})$$

taking the time derivative

$$\dot{V}(\zeta,t) = e^\top \dot{e} + \frac{1}{2}r^\top \dot{M}(\phi)r + r^\top M(\phi)\dot{r} + \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widetilde{\theta}} + tr(\widetilde{W}^\top \Gamma_W^{-1}\dot{\widetilde{W}})$$

then we can use substitute what we got from before

$$\dot{V}(\zeta,t) = e^\top(r-\alpha e)+\frac{1}{2}r^\top \dot{M}(\phi)r+r^\top(Y\theta+W^\top\sigma+\epsilon-\tau-\frac{1}{2}\dot{M}r)-\widetilde{\theta}\Gamma_\theta^{-1}\dot{\widehat{\theta}}+tr(-\widetilde{W}\Gamma_W^{-1}\dot{\widehat{W}})$$

$$\dot{V}(\zeta,t) = -e^\top \alpha e + r^\top e + r^\top(Y\theta + W^\top + \epsilon - \tau) - \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}} + tr(-\widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}})$$

Now we can design the input

$$\tau = Y\widehat{\theta} + \widehat{W}^\top \sigma + e + \beta_r r + \beta_\epsilon sgn(r)$$

which yields

$$\dot{V}(\zeta,t) = -e^\top\alpha e+r^\top e+r^\top(Y\theta+W^\top\sigma+\epsilon-(Y\widehat{\theta}+\widehat{W}^\top\sigma+e+\beta_r r+\beta_\epsilon sgn(r)))-\widetilde{\theta}^\top\Gamma_\theta^{-1}\dot{\widehat{\theta}}+tr(-\widetilde{W}^\top\Gamma_W^{-1}\dot{\widehat{W}})$$

$$\dot{V}(\zeta,t) = -e^\top\alpha e-r^\top\beta_r r+r^\top Y\widetilde{\theta}+r^\top\widetilde{W}^\top\sigma+r^\top\epsilon-r^\top\beta_\epsilon sgn(r)-\widetilde{\theta}\Gamma_\theta^{-1}\dot{\widehat{\theta}}+tr(-\widetilde{W}^\top\Gamma_W^{-1}\dot{\widehat{W}})$$

First we will examine gradient-based estimators for this systems. Similar to before we can design $\dot{\widehat{\theta}}$

$$\dot{\widehat{\theta}} = \Gamma_\theta Y^\top r$$

We can use a trick for this which says for two vectors $a, b \in \mathbb{R}^n$

$$tr(ba^\top) = a^\top b$$

which help us

$$\underbrace{r^\top}_{a^\top}\underbrace{W^\top\sigma}_{b} = tr\left(\underbrace{W^\top\sigma}_{b}\underbrace{r^\top}_{a^\top}\right)$$

so we can move this into the trace term $tr(= \widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}})$ and get

$$tr(-\widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}} + \widetilde{W}^\top \sigma r^\top)$$

and we can design $\dot{\widehat{W}}$

$$\dot{\widehat{W}} = \Gamma_W \sigma r^\top$$

With these designs we can get

$$\dot{V}(\zeta,t) = -e^\top \alpha e - r^\top \beta_r r + r^\top \epsilon - r^\top \beta_\epsilon sgn(r)$$

and we select $\beta_\epsilon > \bar{\epsilon}$ where $||\epsilon(\phi(t),\dot{\phi}(t),\ddot{\phi}(t))|| \leq \bar{\epsilon}$ then

$$\dot{V}(\zeta,t) \leq -\underline{\alpha}||e||^2 - \underline{\beta}_r ||r||^2$$

and we can show asymptotic tracking.

Two-Layer Design

$$M(\phi)\dot{r} = Y\theta + W^\top \sigma + \epsilon - \frac{1}{2}\dot{M}(\phi,\dot{\phi})r - \tau$$

Now let's substitute $\sigma = \widehat{\sigma} + \widehat{\sigma'}\widetilde{\Phi} + \epsilon_\sigma$

$$M(\phi)\dot{r} = Y\theta + W^\top(\widehat{\sigma} + \widehat{\sigma'}\widetilde{\Phi} + \epsilon_\sigma) + \epsilon - \frac{1}{2}\dot{M}(\phi,\dot{\phi})r - \tau$$

Now lets add and subtract $\widehat{W}\frac{\partial\sigma}{\partial\Phi}\widetilde{\Phi}$

$$M(\phi)\dot{r} = Y\theta + W^\top(\widehat{\sigma} + \widehat{\sigma'}\widetilde{\Phi} + \epsilon_\sigma) + \epsilon - \frac{1}{2}\dot{M}(\phi,\dot{\phi})r - \tau \pm \widehat{W}\frac{\partial\sigma}{\partial\Phi}\widetilde{\Phi}$$

$$M\dot{r} = Y\theta + W^\top\widehat{\sigma} + \underbrace{\widetilde{W}^\top\widehat{\sigma'}\widetilde{\Phi} + W^\top\epsilon_\sigma + \epsilon}_{\delta} - \tau - \frac{1}{2}\dot{M}r + \widehat{W}^\top\widehat{\sigma'}\widetilde{\Phi}$$

With $\delta \triangleq \widetilde{W}^\top\widehat{\sigma'}\widetilde{\Phi} + W^\top\epsilon_\sigma + \epsilon$

$$M\dot{r} = Y\theta + W^\top\widehat{\sigma} + \delta - \tau - \frac{1}{2}\dot{M}r + \widehat{W}^\top\widehat{\sigma'}\widetilde{V}^\top\xi$$

Let let

$$\zeta \triangleq \begin{bmatrix} e \\ r \\ \widetilde{\theta} \\ vec(\widetilde{W}) \\ vec(\widetilde{V}) \end{bmatrix}$$

let

$$vec(\widetilde{W}) \triangleq \begin{bmatrix} \widetilde{W}_{11} \\ \vdots \\ \widetilde{W}_{L1} \\ \widetilde{W}_{12} \\ \vdots \\ \widetilde{W}_{L2} \end{bmatrix}$$

$$vec(\widetilde{V}) \triangleq \begin{bmatrix} \widetilde{V}_{11} \\ \vdots \\ \widetilde{V}_{1l} \\ \widetilde{V}_{51} \\ \vdots \\ \widetilde{V}_{5l} \end{bmatrix}$$

and

$$V(\zeta,t) \triangleq \frac{1}{2}e^\top e + \frac{1}{2}r^\top M(\phi)r + \frac{1}{2}\widetilde{\theta}\Gamma_\theta^{-1}\widetilde{\theta} + \frac{1}{2}tr(\widetilde{W}^\top\Gamma_W^{-1}\widetilde{W}) + \frac{1}{2}tr(\widetilde{V}^\top\Gamma_V^{-1}\widetilde{V})$$

taking the time derivative

$$\dot{V}(\zeta,t) = e^\top\dot{e} + \frac{1}{2}r^\top\dot{M}(\phi)r + r^\top M(\phi)\dot{r} + \widetilde{\theta}^\top\Gamma_\theta^{-1}\dot{\widetilde{\theta}} + tr(\widetilde{W}^\top\Gamma_W^{-1}\dot{\widetilde{W}}) + tr(\widetilde{V}^\top\Gamma_V^{-1}\dot{\widetilde{V}})$$

then we can use substitute what we got from before

$$\dot{V}(\zeta,t) = e^\top(r-\alpha e) + \frac{1}{2}r^\top\dot{M}(\phi)r + r^\top(Y\theta + W^\top\widehat{\sigma} + \delta - \tau - \frac{1}{2}\dot{M}r + \widehat{W}^\top\widehat{\sigma}'\widetilde{V}^\top\xi)$$
$$-\widetilde{\theta}\Gamma_\theta^{-1}\dot{\widehat{\theta}} + tr(-\widetilde{W}\Gamma_W^{-1}\dot{\widehat{W}}) + tr(-\widetilde{V}\Gamma_V^{-1}\dot{\widehat{V}})$$

$$\dot{V}(\zeta,t) = -e^\top\alpha e + r^\top e + r^\top(Y\theta + W^\top\widehat{\sigma} + \delta - \tau - \frac{1}{2}\dot{M}r + \widehat{W}^\top\widehat{\sigma}'\widetilde{V}^\top\xi)$$
$$-\widetilde{\theta}^\top\Gamma_\theta^{-1}\dot{\widehat{\theta}} + tr(-\widetilde{W}^\top\Gamma_W^{-1}\dot{\widehat{W}}) + tr(-\widetilde{V}\Gamma_V^{-1}\dot{\widehat{V}})$$

Now we can design the input

$$\tau = Y\widehat{\theta} + \widehat{W}^\top\widehat{\sigma} + e + \beta_r r + \beta_\delta sgn(r)$$

which yields

$$\dot{V}(\zeta,t) = -e^\top\alpha e + r^\top e + r^\top(Y\theta + W^\top\widehat{\sigma} + \delta - (Y\widehat{\theta} + \widehat{W}^\top\widehat{\sigma} + e + \beta_r r + \beta_\delta sgn(r)) - \frac{1}{2}\dot{M}r + \widehat{W}^\top\widehat{\sigma}'\widetilde{V}^\top\xi)$$
$$-\widetilde{\theta}^\top\Gamma_\theta^{-1}\dot{\widehat{\theta}} + tr(-\widetilde{W}^\top\Gamma_W^{-1}\dot{\widehat{W}}) + tr(-\widetilde{V}\Gamma_V^{-1}\dot{\widehat{V}})$$

$$\dot{V}(\zeta,t) = -e^\top\alpha e + r^\top e + r^\top Y\widetilde{\theta} + r^\top\widetilde{W}^\top\widehat{\sigma} + r^\top\delta - r^\top e - r^\top\beta_r r - r^\top\beta_\delta sgn(r) + r^\top\widehat{W}^\top\widehat{\sigma}'\widetilde{V}^\top\xi$$
$$-\widetilde{\theta}^\top\Gamma_\theta^{-1}\dot{\widehat{\theta}} + tr(-\widetilde{W}^\top\Gamma_W^{-1}\dot{\widehat{W}}) + tr(-\widetilde{V}\Gamma_V^{-1}\dot{\widehat{V}})$$

$$\dot{V}(\zeta,t) = -e^\top \alpha e - r^\top \beta_r r$$
$$-r^\top \beta_\delta sgn(r) + r^\top \delta$$
$$+r^\top Y\widetilde{\theta} - \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}}$$
$$+r^\top \widetilde{W}^\top \widetilde{\sigma} - tr(\widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}})$$
$$+r^\top \widehat{W}^\top \widehat{\sigma}'\widetilde{V}^\top \xi - tr(\widetilde{V}^\top \Gamma_V^{-1}\dot{\widehat{V}})$$

Now we can bound

$$-r^\top \beta_\delta sgn(r) + r^\top \delta$$

as

$$-\underline{\beta_\delta}||r|| + \overline{\delta}||r|| \leq 0$$
$$\underline{\beta_\delta} \geq \overline{\delta}$$

Yielding

$$\dot{V}(\zeta,t) = -e^\top \alpha e - r^\top \beta_r r$$
$$+r^\top Y\widetilde{\theta} - \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}}$$
$$+r^\top \widetilde{W}^\top \widetilde{\sigma} - tr(\widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}})$$
$$+r^\top \widehat{W}^\top \widehat{\sigma}'\widetilde{V}^\top \xi - tr(\widetilde{V}^\top \Gamma_V^{-1}\dot{\widehat{V}})$$

Now we can design $\dot{\widehat{\theta}}$ using

$$r^\top Y\widetilde{\theta} - \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}} = 0$$

$$\widetilde{\theta}^\top Y^\top r = \widetilde{\theta}^\top \Gamma_\theta^{-1}\dot{\widehat{\theta}}$$

$$\dot{\widehat{\theta}} = proj(\Gamma_\theta Y^\top r)$$

yielding

$$\widetilde{\theta}^\top Y^\top r = \widetilde{\theta}^\top \Gamma_\theta^{-1} Y^\top r$$
$$\widetilde{\theta}^\top Y^\top r = \widetilde{\theta}^\top Y^\top r$$

$$\dot{V}(\zeta,t) = -e^\top \alpha e - r^\top \beta_r r$$
$$+r^\top \widetilde{W}^\top \widetilde{\sigma} - tr(\widetilde{W}^\top \Gamma_W^{-1}\dot{\widehat{W}})$$
$$+r^\top \widehat{W}^\top \widehat{\sigma}'\widetilde{V}^\top \xi - tr(\widetilde{V}^\top \Gamma_V^{-1}\dot{\widehat{V}})$$

Now we can design $\dot{\widehat{W}}$ using the trace trick for two vectors $a, b \in \mathbb{R}^n$

$$tr(ba^\top) = a^\top b$$

$$r^\top \widetilde{W}^\top \widetilde{\sigma} + tr(-\widetilde{W}^\top \Gamma_W^{-1} \dot{\widehat{W}}) = 0$$

$$r^\top \widetilde{W}^\top \widehat{\sigma} = tr(\widetilde{W}^\top \Gamma_W^{-1} \dot{\widehat{W}})$$

$$\underbrace{r^\top}_{a^\top} \underbrace{\widetilde{W}^\top \widehat{\sigma}}_{b} = tr\left(\underbrace{\widetilde{W}^\top \widehat{\sigma}}_{b} \underbrace{r^\top}_{a^\top} \right)$$

$$tr(\widetilde{W}^\top \widetilde{\sigma} r^\top) = tr(\widetilde{W}^\top \Gamma_W^{-1} \dot{\widehat{W}})$$

$$\dot{\widehat{W}} = proj(\Gamma_W \widehat{\sigma} r^\top)$$

implying

$$tr(\widetilde{W}^\top \widetilde{\sigma} r^\top) = tr(\widetilde{W}^\top \Gamma_W^{-1} \Gamma_W \widehat{\sigma} r^\top)$$

$$tr(\widetilde{W}^\top \widetilde{\sigma} r^\top) = tr(\widetilde{W}^\top \widetilde{\sigma} r^\top)$$

yielding

$$\dot{V}(\zeta, t) = -e^\top \alpha e - r^\top \beta_r r$$

$$+ r^\top \widehat{W}^\top \widehat{\sigma}' \widetilde{V}^\top \xi - tr(\widetilde{V}^\top \Gamma_V^{-1} \dot{\widehat{V}})$$

Last we can design $\dot{\widehat{V}}$

$$r^\top \widehat{W}^\top \widehat{\sigma}' \widetilde{V}^\top \xi + tr(-\widetilde{V}^\top \Gamma_V^{-1} \dot{\widehat{V}}) = 0$$

$$r^\top \widehat{W}^\top \widehat{\sigma}' \widetilde{V}^\top \xi = tr(\widetilde{V}^\top \Gamma_V^{-1} \dot{\widehat{V}})$$

$$\underbrace{r^\top \widehat{W}^\top \widehat{\sigma}'}_{a^\top} \underbrace{\widetilde{V}^\top \xi}_{b} = tr\left(\underbrace{\widetilde{V}^\top \xi}_{b} \underbrace{r^\top \widehat{W}^\top \widehat{\sigma}'}_{a^\top} \right)$$

$$tr(\widetilde{V}^\top \xi r^\top \widehat{W}^\top \widehat{\sigma}') = tr(\widetilde{V}^\top \Gamma_V^{-1} \dot{\widehat{V}})$$

$$\dot{\widehat{V}} = proj(\Gamma_V \xi r^\top \widehat{W}^\top \widehat{\sigma}')$$

implying

$$tr(\widetilde{V}^\top \xi r^\top \widehat{W}^\top \widehat{\sigma}') = tr(\widetilde{V}^\top \Gamma_V^{-1} \Gamma_V \xi r^\top \widehat{W}^\top \widehat{\sigma}')$$

$$tr(\widetilde{V}^\top \xi r^\top \widehat{W}^\top \widehat{\sigma}') = tr(\widetilde{V}^\top \xi r^\top \widehat{W}^\top \widehat{\sigma}')$$

yielding

$$\dot{V}(\zeta, t) = -e^\top \alpha e - r^\top \beta_r r$$

And we can bound these as

$$\dot{V}(\zeta, t) \leq -\underline{\alpha} ||e||^2 - \underline{\beta_r} ||r||^2$$

then we can show asymptotic tracking using Barbalats Lemma

Results and Discussion

Plots of Robust Control

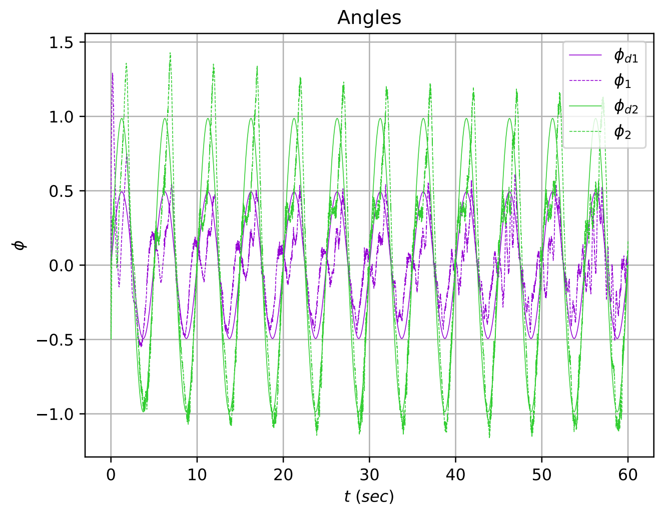

Figure 1: Angles and Angular Velocities Plot of Robust Control

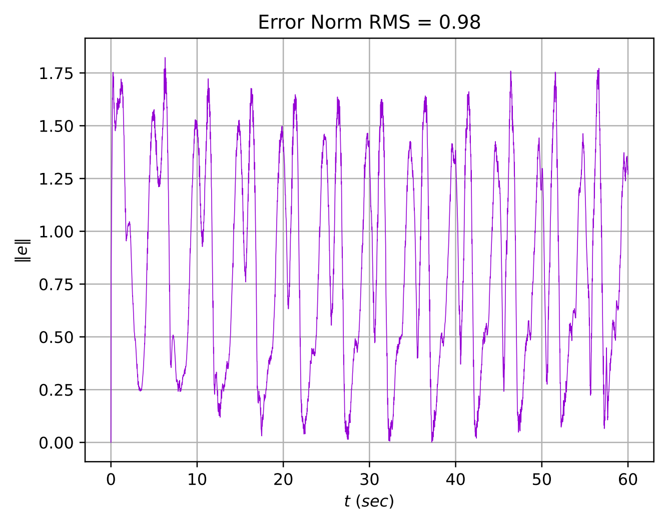

Figure 2: Tracking Error Plot of Robust Control

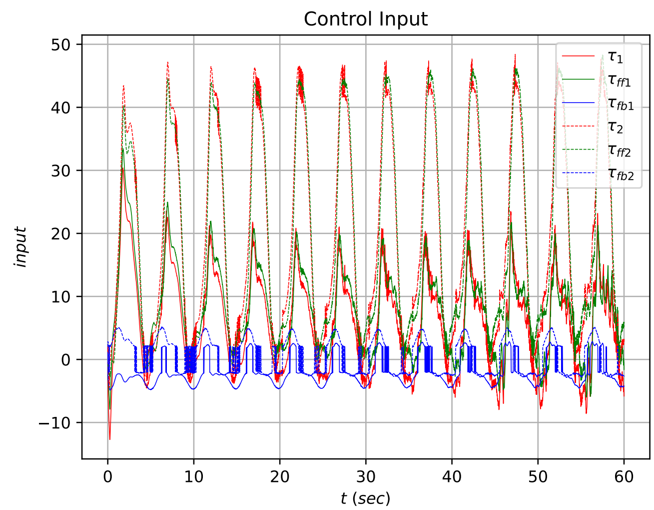

Figure 3: Input Plot of Robust Control

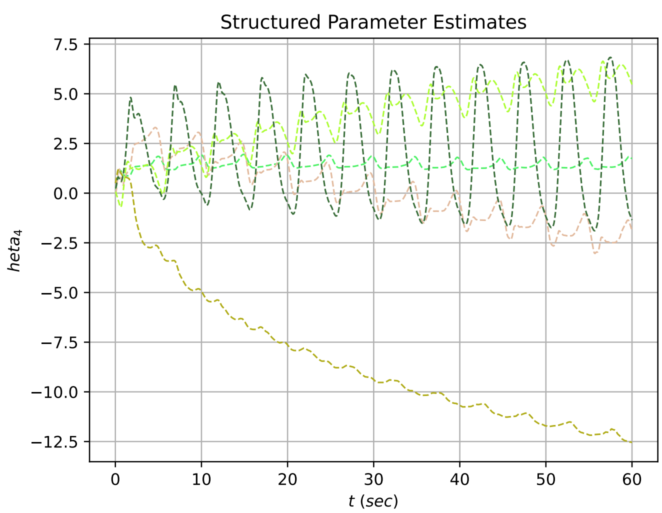

Figure 4: Structured Parameter Estimates Plot of Robust Control

Plots of Single-Layer

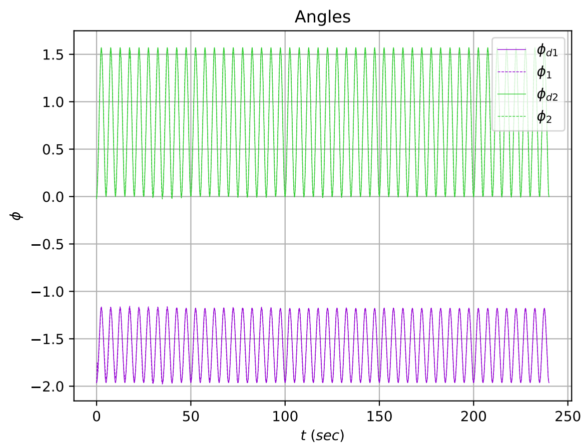

Figure 5: Angles and Angular Velocities Plot of Single-Layer

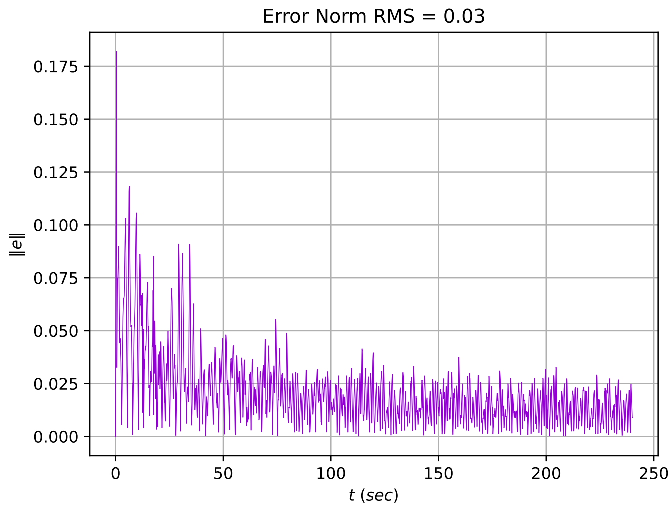

Figure 6: Tracking Error Plot of Single-Layer

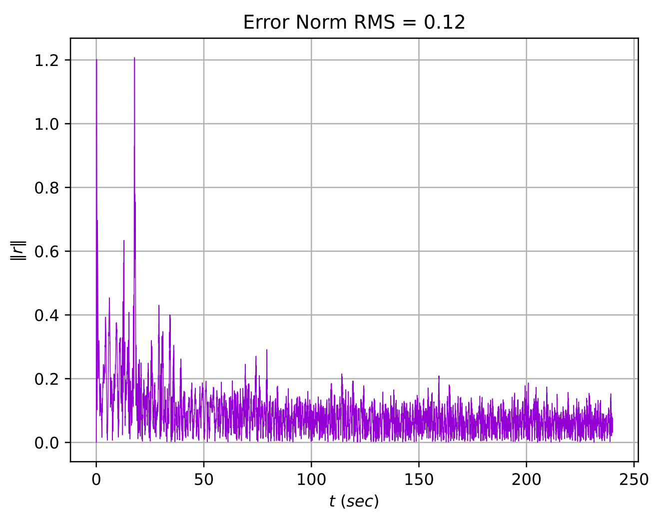

Figure 7: Filtered Tracking Error Plot of Single-Layer

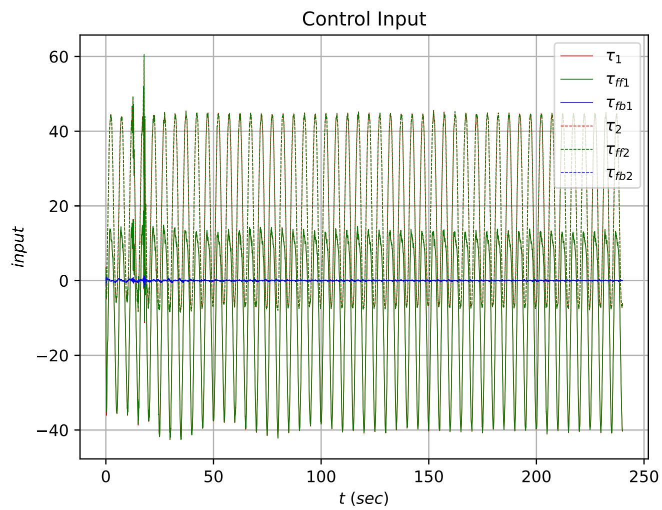

Figure 8: Input Plot of Single-Layer

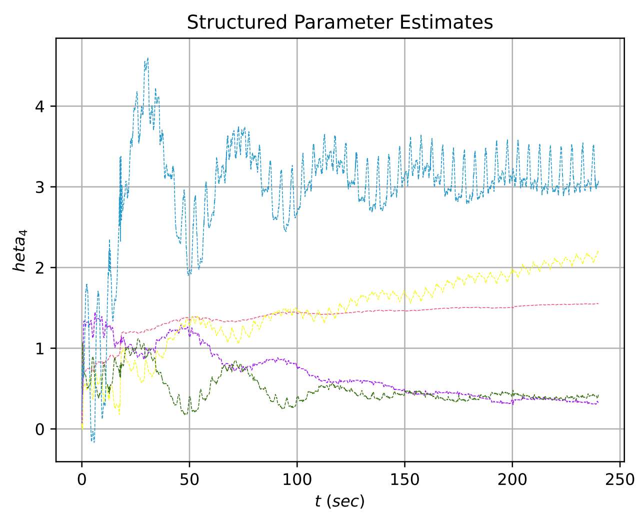

Figure 9: Structured Parameter Estimates Plot of Single-Layer

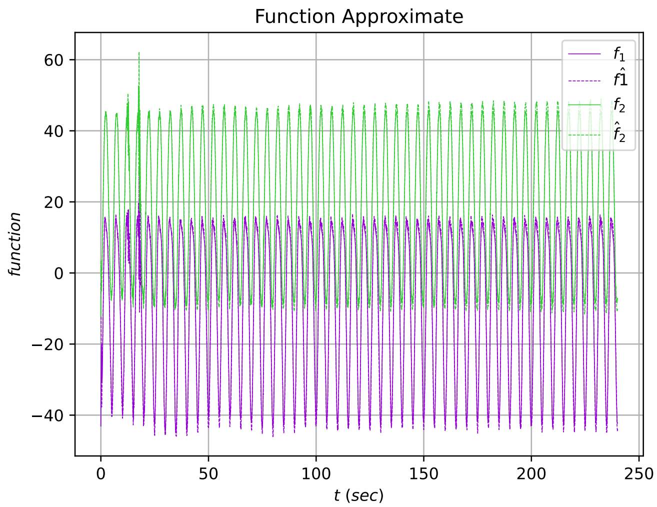

Figure 10: Function Estimation Plot of Single-Layer

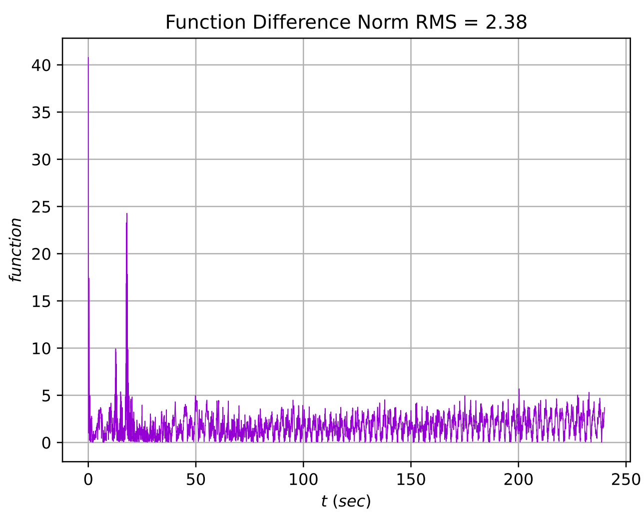

Figure 11: Function Approximation Error Plot of Single-Layer

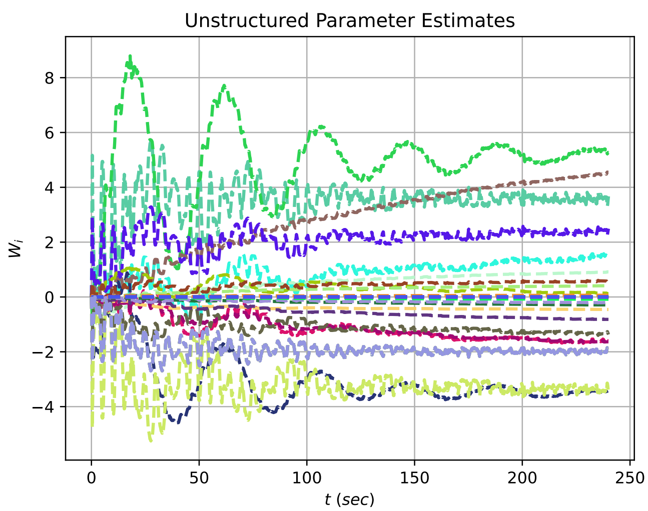

Figure 12: Unstructured Parameter Estimates Plot of Single-Layer

Plots of Two-Layer

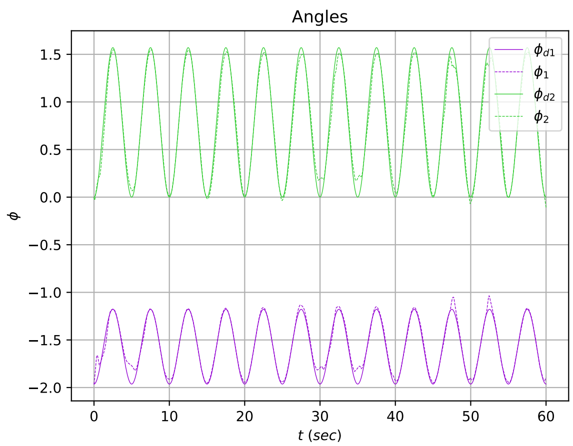

Figure 13: Angles and Angular Velocities Plot of Two-Layer

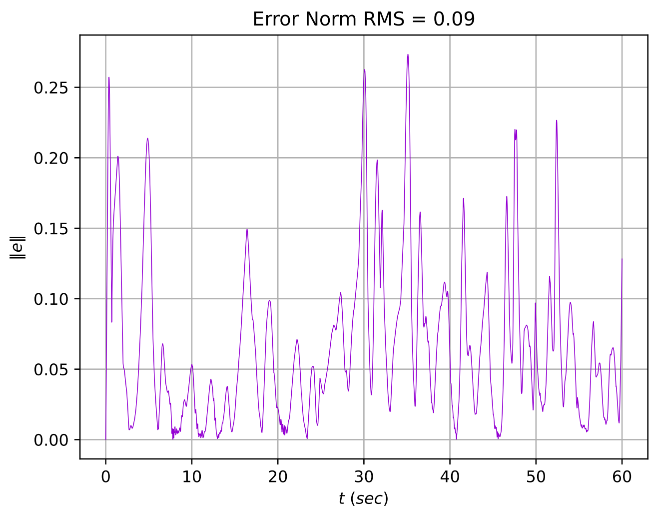

Figure 14: Tracking Error Plot of Two-Layer

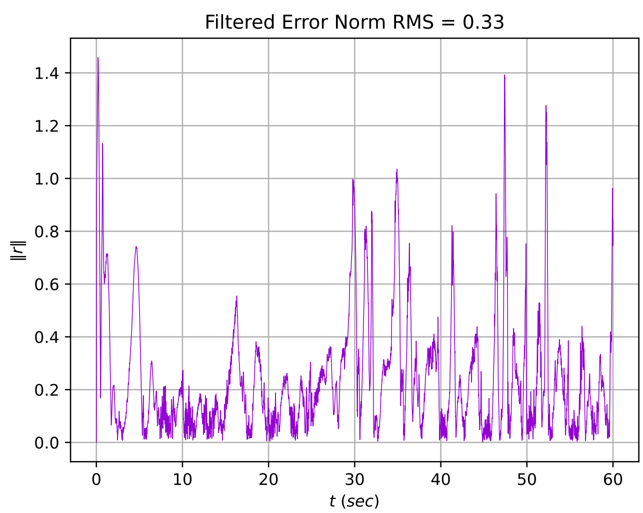

Figure 15: Filtered Tracking Error Plot of Two-Layer

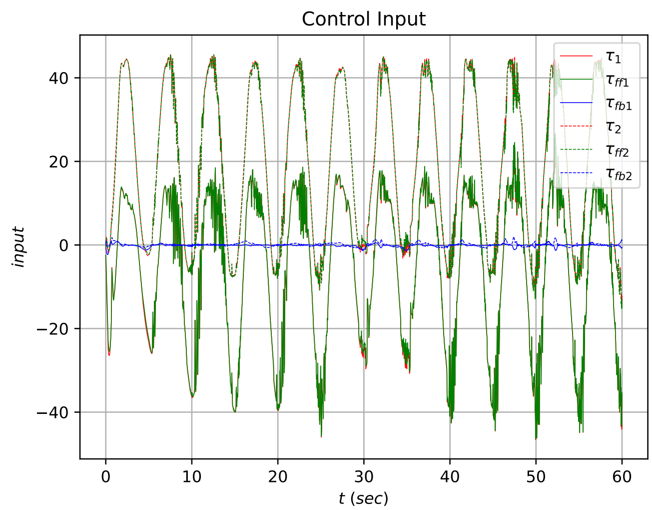

Figure 16: Input Plot of Two-Layer

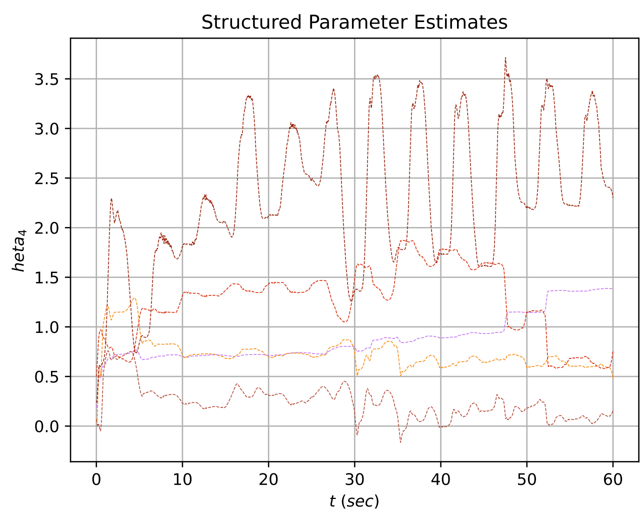

Figure 17: Structured Parameter Estimates Plot of Two-Layer

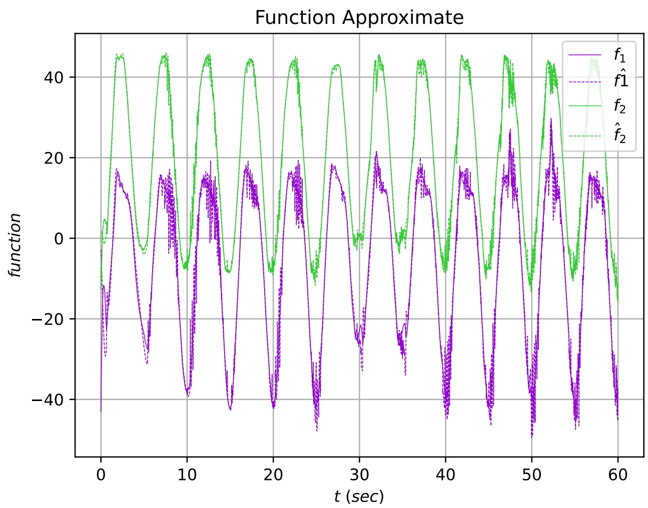

Figure 18: Function Estimation Plot of Two-Layer

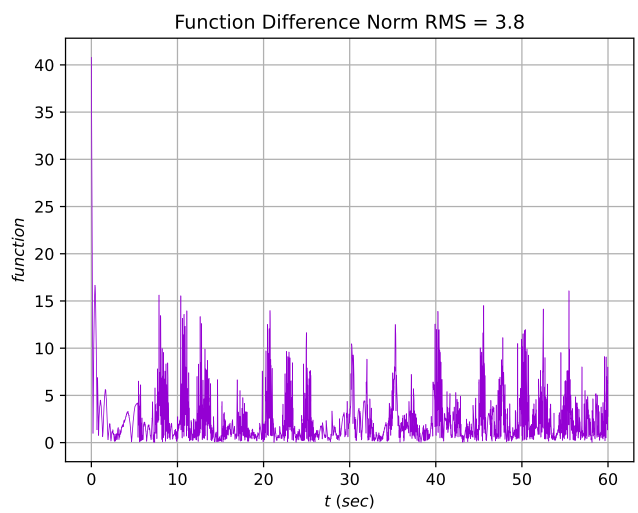

Figure 19: Function Approximation Error Plot of Two-Layer

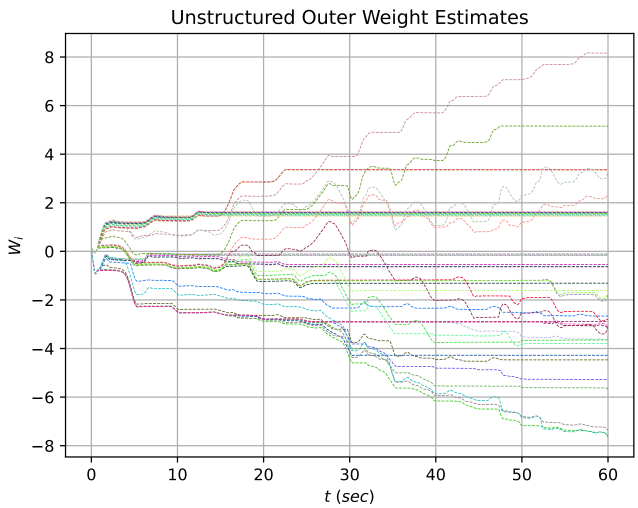

Figure 20: Unstructured Parameter Estimates Plot of Two-Layer

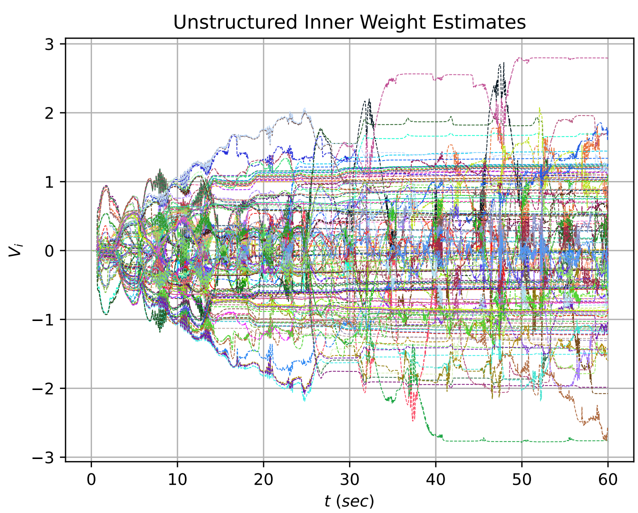

Figure 21: Unstructured Outer Weight Estimates Plot of Two-Layer

Discussion

(a) For single-layer, at the beginning I tried gaussian normal distribution as the basis function, for some reason the best function estimation error that I can get is around 15 which I think is unacceptable. Then, I also tried the Fourier and Polynomial as the basis function. After several tuning, I chose the polynomial basis function with a gain(1.2) when deriving the getsigma function, and gains that I chose shows as follow:

$$\alpha = 3 \quad \beta_r = 1.5 \quad \beta_\epsilon = 0.001 \quad \Gamma_\theta = 0.75 \quad \Gamma_w = 8 \quad l = 20$$

For two-layer, luckily I tried gaussian normal distribution as the basis function, and I got some good results with gains that I chose shows as follow:

$$\alpha = 3 \quad \beta_r = 1.5 \quad \beta_\epsilon = 0.001 \quad \Gamma_\theta = 0.5 \quad \Gamma_w = 2 \quad \Gamma_v = 20 \quad l = 30$$

(b)Based on all the plots that I got, either single or two layer works pretty good. RMS tracking error of single-layer equals 0.03 and two-layer has an error = 0.09 and for RMS filtered tracking error I got 0.12 and 0.33 for single and two layer respectively. Relatively speaking the single layer did a better job in this project.

(c)Speaking of function approximation, I got 2.38 for single-layer and 3.8 for two-layer, up to this point single-layer still works better approximating the unknown function, but both of them did a good job. In terms of principle, the two-layer structure should work better than the single-layer, the reason why it doesn't for this project might relate to tuning, I think with better tuning method we can get better results.

(d) Base on the control plots, total controls rely on feedback portion a little bit more at the very beginning then turn into feedforward portion(feedforward portion became the majority of time). This pattern works for both single-layer and two-layer.