Midterm Theorem Quick Finder (Back)

Sec 1.1: Echelon Systems / Consistency

Thm: Consistent echelon system has infinitely many solutions iff there is a free variable (Canvas Add'l Thm 1A).

Use for: After RREF, to decide 0/1/inf solutions quickly. Count free vars.

Trigger words: how many solutions, consistent system, free variable.

Sec 1.2: Elementary Row Operations & RREF

Thm: Elementary row ops preserve the solution set (Canvas Add'l Thm 1C).

Thm: If A to B by deleting column j, the reduced A0 to B0 similarly (Canvas Add'l Thm 1D).

Facts: Each matrix has a unique RREF. Two matrices are row-equivalent iff they have the same RREF.

Use for: Replace A by RREF to analyze consistency, pivots, rank, nullspace; decide row-equivalence.

Recipe: Count Solutions
RREF of [A|b]: - Row [0 ... 0 | nonzero] -> No solution. - Pivot in all variable columns -> Unique solution. - Otherwise -> Free var(s) -> Infinitely many.

Sec 2.1: Linear Combinations & Span

Def: Span({v1,...,vk}) is all linear combinations of the vectors.

Facts: In R^m you need at least m vectors to possibly span R^m; zero vector never helps rank.

Use for: Span questions; construct examples/non-examples; growing span dimension.

Sec 2.2: Linear Independence / Dependence

Def: {v1,...,vk} independent iff the only solution to c1*v1+...+ck*vk=0 is all ci=0.

Quick tests: If k>m in R^m then dependent. If RREF of [v1 ... vk] has a pivot in every column then independent.

Use for: Max size of LI set, min size of spanning LD set, construct counterexamples.

Sec 2.3: Rank, Nullity, and Systems

Def: rank(A) = number of pivot columns; nullity(A) = #cols - rank(A).

Matrix cues: Columns span R^m iff rank(A)=m (pivot in each row).

Use for: Span of columns, existence/uniqueness, designing A to be 1-1 or onto when viewed as T(x)=Ax.

Decision Box: 1-1 vs Onto for T(x)=Ax
Domain R^n, Codomain R^m - 1-1 iff pivot in each column (nullspace = {0}). - Onto R^m iff rank = m (pivot in each row). - Impossible combos: * 1-1 when n>m. * Onto when n<m.

Sec 3.1: Linear Transformations

Add'l Thm 3A (Linearity): T(c1*u1+...+cr*ur)=c1*T(u1)+...+cr*T(ur).

Use for: Compute T(x) from values on basis/generators; show two T must agree if they agree on a basis.

Add'l Thm 3B: If {u1,...,ur} are linearly dependent, then {T(u1),...,T(ur)} are linearly dependent.

Use for: If images are independent, conclude preimages were independent (contrapositive).

Add'l Thm 3C: If T is one-to-one and {u1,...,ur} are independent, then {T(u1),...,T(ur)} are independent.

Use for: Rule out 1-1 when images are dependent; preserve independence.

Add'l Thm 3D (Onto): T is onto R^n iff range(T)=R^n (equals the codomain). For matrices: rank = #rows.

Use for: Decide onto by rank; construct 1-1 but not onto maps by choosing m>n and rank=n.

Unifying Row-Equivalence Theorem (Practical Form)

A row-equivalent to B iff RREF(A) = RREF(B). Left-multiplication by an invertible matrix preserves column relations (dependence/independence).

Use for: Decide if two matrices are equivalent; reason about column independence via the pivot structure in RREF.

When to Cite What (Problem Triggers)

Counting solutions or free variables -> Add'l Thm 1A + RREF (Sec 1.2).

Replacing a matrix by RREF -> Add'l Thm 1C.

Span R^m or rank tests -> Sec 2.3 rank=m.

Too many vectors to be independent -> Sec 2.2 size test.

Images under a linear map; building T from basis -> Add'l Thm 3A.

Show independence/dependence is preserved or impossible under T -> Add'l Thm 3B/3C.

Onto/Not onto design -> Add'l Thm 3D + rank.

Linear Algebra Midterm Note Sheet (Front)

Coverage: 1.1, 1.2, 2.1, 2.2, 2.3, 3.1 + Canvas Additional Theorems.

CORE TRIGGERS (write at top)

Row ops preserve solutions (Add'l Thm 1C). Use whenever you replace by echelon/RREF.
Free var => infinitely many solutions (Add'l Thm 1A). Count pivots vs vars.
Linearity: T(c1*u1+...+cr*ur)=c1*T(u1)+...+cr*T(ur) (Add'l Thm 3A).
Dependence preserved: if {ui} dependent then {T(ui)} dependent (Add'l Thm 3B).
One-to-one preserves independence (Add'l Thm 3C).
Onto: range(T)=R^n (Add'l Thm 3D). Matrix cues: 1-1 iff pivot in every column; Onto iff pivot in every row; rank + nullity = #cols.

1) How many solutions? Give examples

YES/NO: Unique? **Yes** iff pivot in every variable column. Infinite? **Yes** iff >=1 free var and consistent. None? **Yes** if a row [0 ... 0 | b!=0].
Justify: Thm 1C (row ops) + Thm 1A (free var => infinite).
Example: RREF gives z free => (x,y,z)=(1-2s, s, 0). Two solutions: s=0 and s=1.

2) Build RREF from a parametric solution

YES/NO: Is z free? **Yes** => put pivots in x,y and leave z free.
Justify: Parametric form <-> RREF reading (Sec 2.2) + Thm 1C.
Example: x=-1+s, y=2+s, z=s => rows [1 -1 0 | -3], [0 1 -1 | 2].

3) Linear independence fast tests

YES/NO: Are 5 vectors in R^4 independent? **No** (too many vectors in 4-D).
Justify: Max rank in R^4 is 4 (Secs 2.1-2.3).
YES/NO: Are 3 vectors in R^3 independent? **Yes** iff RREF of [v1 v2 v3] has a pivot in each column (so Ax=0 only has x=0).

4) Do these vectors span R^n?

YES/NO: Can fewer than n vectors span R^n? **No** (rank <= #vectors).
Yes iff rank = n (pivots in all rows). (Sec 2.3).

5) Make span grow strictly with each added vector (R^2 example)

Take u1=(1,0), u2=(2,0) (collinear) => span = {(t,0)}. Add u3=(0,1) => span = R^2.
Justify: Rank increases when adding a non-collinear vector.

Linear Algebra Midterm Note Sheet (Back)

6) Extremal sets in R^4

Largest LI set that does NOT span: size 3. Reason: any 4 LI vectors in R^4 span.
Smallest LD set that DOES span: size 5. Reason: to be dependent and still span 4-D you need >=5 vectors.
Justify: Dimension and rank arguments (Sec 2.3).

7) Two distinct linear transformations with given values

YES/NO: Can two maps differ if images on a basis are fixed? **No**: basis images determine T uniquely (3A).
Strategy: Solve for basis images using linearity; any unconstrained basis vector gives degrees of freedom => infinitely many T.
Justify: Thm 3A (linearity).

8) One-to-one vs Onto (matrix cues)

1-1: pivot in each column <=> nullspace = {0}. Onto R^m: rank = m (pivot in each row).
Impossible: 1-1 R^3->R^2 (domain>codomain). Impossible: Onto R^4 from R^3 (domain **Justify:** Rank-nullity + Thm 3C/3D.

9) Row-equivalent?

YES/NO: Same RREF? **Yes** => row-equivalent. Pivot pattern differs? **No**.
Justify: Uniqueness of RREF; Thm 1C for solution-set preservation.

10) Can the same A give Ax=b with infinite solutions but Ay=c with a unique solution? (4x2)

No. If rank(A)<2 then any consistent RHS has infinite solutions; if rank(A)=2 then every RHS has at most one solution. One fixed A cannot produce both. **Justify:** free-variable logic (Sec 2.3).

11) Kernel constraints for T(x)=sum x_i*u_i

Require T([1,-1,1,-1])=0 with u1,u3,u4 independent: set u2 = u1+u3-u4 (works).
Require T([1,0,1,0])=0 and u1,u3,u4 independent: **impossible** since u1 = -u3 contradicts independence.
Justify: kernel = linear relations among the u_i.

12) Smallest n with 1-1 but not onto for T:R^3->R^n

n = 4. 1-1 => rank = 3; not onto requires codomain dim > 3. Example: T(x) = [I_3; 0]*x in R^4. **Justify:** rank/onto definition (3D).