Linear Algebra Midterm Note Sheet (Front)

Coverage: 1.1, 1.2, 2.1, 2.2, 2.3, 3.1 + Canvas Additional Theorems.

CORE TRIGGERS (write at top)

Row ops preserve solutions (Add'l Thm 1C). Use whenever you replace by echelon/RREF.
Free var => infinitely many solutions (Add'l Thm 1A). Count pivots vs vars.
Linearity: T(c1*u1+...+cr*ur)=c1*T(u1)+...+cr*T(ur) (Add'l Thm 3A).
Dependence preserved: if {ui} dependent then {T(ui)} dependent (Add'l Thm 3B).
One-to-one preserves independence (Add'l Thm 3C).
Onto: range(T)=R^n (Add'l Thm 3D). Matrix cues: 1-1 iff pivot in every column; Onto iff pivot in every row; rank + nullity = #cols.

1) How many solutions? Give examples

YES/NO: Unique? **Yes** iff pivot in every variable column. Infinite? **Yes** iff >=1 free var and consistent. None? **Yes** if a row [0 ... 0 **|** b!=0].
Justify: Thm 1C (row ops) + Thm 1A (free var => infinite).
Example: RREF gives z free => (x,y,z)=(1-2s, s, 0). Two solutions: s=0 and s=1.

2) Build RREF from a parametric solution

YES/NO: Is z free? **Yes** => put pivots in x,y and leave z free.
Justify: Parametric form <-> RREF reading (Sec 2.2) + Thm 1C.
Example: x=-1+s, y=2+s, z=s => rows [1 -1 0 **|** -3], [0 1 -1 **|** 2].

3) Linear independence fast tests

YES/NO: Are 5 vectors in R^4 independent? **No** (too many vectors in 4-D).
Justify: Max rank in R^4 is 4 (Secs 2.1-2.3).
YES/NO: Are 3 vectors in R^3 independent? **Yes** iff RREF of [v1 v2 v3] has a pivot in each column (so Ax=0 only has x=0).

4) Do these vectors span R^n?

YES/NO: Can fewer than n vectors span R^n? **No** (rank <= #vectors).
Yes iff rank = n (pivots in all rows). (Sec 2.3).

5) Make span grow strictly with each added vector (R^2 example)

Take u1=(1,0), u2=(2,0) (collinear) => span = {(t,0)}. Add u3=(0,1) => span = R^2.
Justify: Rank increases when adding a non-collinear vector.

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6) Extremal sets in R^4

Largest LI set that does NOT span: size 3. Reason: any 4 LI vectors in R^4 span.
Smallest LD set that DOES span: size 5. Reason: to be dependent and still span 4-D you need >=5 vectors.
Justify: Dimension and rank arguments (Sec 2.3).

7) Two distinct linear transformations with given values

YES/NO: Can two maps differ if images on a basis are fixed? **No**: basis images determine T uniquely (3A).
Strategy: Solve for basis images using linearity; any unconstrained basis vector gives degrees of freedom => infinitely many T.
Justify: Thm 3A (linearity).

8) One-to-one vs Onto (matrix cues)

1-1: pivot in each column <=> nullspace = {0}. Onto R^m: rank = m (pivot in each row).
Impossible: 1-1 R^3->R^2 (domain>codomain). Impossible: Onto R^4 from R^3 (domain **Justify:** Rank-nullity + Thm 3C/3D.

9) Row-equivalent?

YES/NO: Same RREF? **Yes** => row-equivalent. Pivot pattern differs? **No**.
Justify: Uniqueness of RREF; Thm 1C for solution-set preservation.

10) Can the same A give Ax=b with infinite solutions but Ay=c with a unique solution? (4x2)

No. If rank(A)<2 then any consistent RHS has infinite solutions; if rank(A)=2 then every RHS has at most one solution. One fixed A cannot produce both. **Justify:** free-variable logic (Sec 2.3).

11) Kernel constraints for T(x)=sum x_i*u_i

Require T([1,-1,1,-1])=0 with u1,u3,u4 independent: set u2 = u1+u3-u4 (works).
Require T([1,0,1,0])=0 and u1,u3,u4 independent: **impossible** since u1 = -u3 contradicts independence.
Justify: kernel = linear relations among the u_i.

12) Smallest n with 1-1 but not onto for T:R^3->R^n

n = 4. 1-1 => rank = 3; not onto requires codomain dim > 3. Example: T(x) = [I_3; 0]*x in R^4. **Justify:** rank/onto definition (3D).