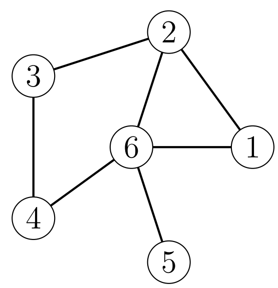
This homework is due Tuesday 19th September at 11:59pm. Answers without appropriate justification will receive no points.

1. For any undirected graph G, the **complement** of G, denoted G^C, is the graph consisting of the same set of nodes as G, such that two distinct nodes in G^C are adjacent in G^C if and only if they are *not* connected in G. That is, $V(G^C) = V(G)$, and for every $u, v \in V(G^C)$, $\{u, v\} \in E(G^C)$ if and only if $\{u, v\} \notin E(G)$.

   (a) Let G be the following graph. Draw G^C and write down its adjacency matrix.

   (b) Let G be any graph. If ρ is the density of G, what is the density for G^C in terms of ρ? Explain your answer.

2. Consider the following adjacency matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 3 & 1 & 1 & 0 \end{bmatrix}.$$

   Let G be the network with this adjacency matrix. Note that G is a weighted directed network.

   (a) Draw G. Note: to indicate the weight of an arc (directed edge), just write the weight next to the respective edge.

   (b) A *sink* node is defined as a node with in-arcs, but no out-arcs. Which nodes in G, if any, have this property?

   (c) The *in-strength* of a node i, denoted s_i^{in}, is the sum of the weights of all links from other vertices *to* i and the *out-strength* of a node i, denoted s_i^{out} is the sum of the weights of all links *from* i to other nodes. Find the in-strength and out-strength for each of the nodes in G.

(d) Convert this adjacency matrix to an undirected, unweighted network H by doing the following:

- Let $V(H) = V(G)$.
- Let $\{u, v\} \in E(H)$ if $(u, v) \in E(G)$ or if $(v, u) \in E(G)$ (or both!).

Write the adjacency matrix for H.

(e) Calculate the average degree of H.

(f) What is the density of H?

(g) Draw H^C.

3. For each of the following statements, determine whether it is true or false. Justify your answers fully, with a proof or counterexample* as necessary.

(a) *Definitions:* A dominating vertex in a graph with n vertices is a vertex v with $\deg(v) = n - 1$. A pendent vertex is a vertex which is only adjacent to one other vertex.
True or false? It is impossible for a graph to have two dominating vertices and a pendent vertex.

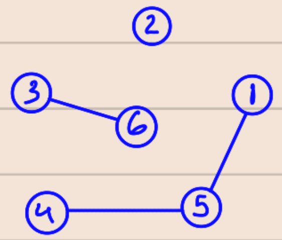
(b) **True or false?** It is impossible to draw a 4-regular graph on seven vertices.

(c) Let G be a directed network. Consider the *total in-degree* (i.e. the sum of the in-degrees of every node in the network), and the *total out-degree* (defined analogously). Which of the following statements is **true**?

- Total in-degree must be less than total out-degree
- Total in-degree must be greater than total out-degree
- Total in-degree must be equal to total out-degree
- None of these hold true in all instances

*A counterexample is an example that demonstrates something is false. For example, if I said "A network has to have at least one edge," you could say, "False. A network that's just an isolated node does not have any links." The nice thing about counterexamples is that you don't need to formally argue why something is false or provide a ton of examples; instead, you can just show one instance of the statement being false.

1.

a) $G^c =$ 

Adjacency Matrix A:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

b) If p is the density of G then the density of G^c must be $1-p$. This is because the amount of nodes remains constant in both graphs but $E(G^c)$ contains all those edges not in $E(G)$.

Let $n = |V(G)|$, we know also $n = |V(G^c)|$ and so $|E(K_n)|$ can be the denominator for both $p = \frac{|E(G)|}{|E(K_n)|}$ and $p^c = \frac{|E(G^c)|}{|E(K_n)|}$.

Thus if $E(G^c)$ are all those edges not in $E(G)$, then $E(K_n) = E(G) + E(G^c)$.

So $$\frac{|E(K_n)|}{|E(K_n)|} = \frac{|E(G) + E(G^c)|}{|E(K_n)|}$$

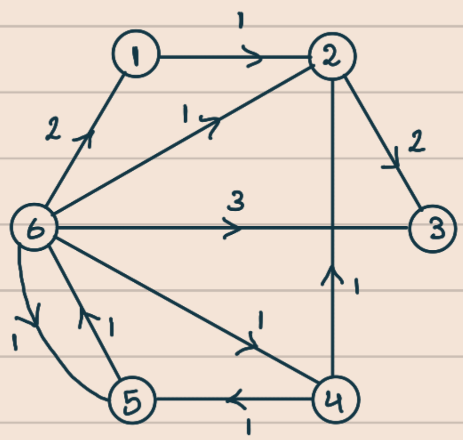
$$\frac{|E(K_n)|}{|E(K_n)|} = \frac{|E(G)|}{|E(K_n)|} + \frac{|E(G^c)|}{|E(K_n)|}$$

$$1 = p + p^c$$

$$p^c = 1 - p$$

$\therefore$ Density of G^c $= 1 - p$ $\square$

2.

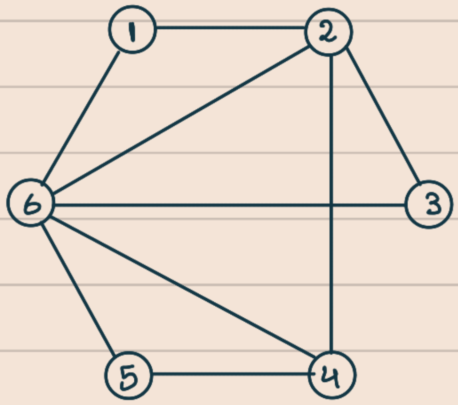
a) $G =$ 

b) Node 3 qualifies as a sink node.

c) $s_1^{in} = 2$ $s_2^{in} = 3$ $s_3^{in} = 5$ $s_4^{in} = 1$ $s_5^{in} = 2$ $s_6^{in} = 1$

$s_1^{out} = 1$ $s_2^{out} = 2$ $s_3^{out} = 0$ $s_4^{out} = 2$ $s_5^{out} = 1$ $s_6^{out} = 8$

d) $H =$



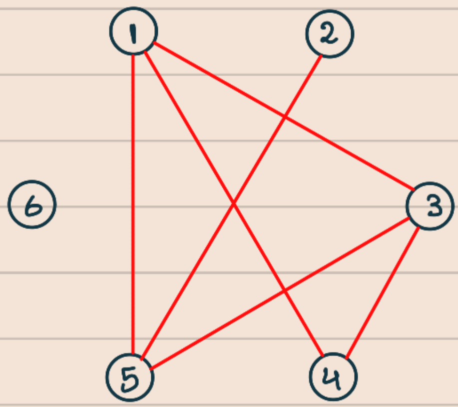
Adjacency Matrix for H:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 1 & 0 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

e) Average degree of H: $c = \frac{2m}{n} = \frac{2(9)}{6} = \frac{18}{6} = \boxed{3}$

f) Density of H: $p = \frac{2m}{n(n-1)} = \frac{2(9)}{6(5)} = \frac{18}{30} = 0.6$

g) H^c:



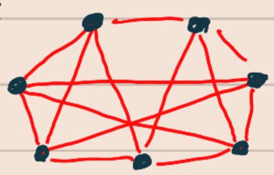
Adjacency Matrix of H^c:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 1 & 1 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

3.

a) True. If a graph has 2 dominant vertices then every other vertex in the graph must have at the very least 2 adjacent vertices, namely, the two dominant vertices.

b) False. A 4-regular graph means $k = 4$ and we know there are 7 vertices meaning the amount of edges will be $4(7)/2 = 28/2 = 14$ which is a whole number of edges and so such a graph can exist, here is an example:



c) Total in-degree must be equal to total out-degree because every directed arch counts simultaneously as an additional in-degree for one node and out-degree for another node. Thus, for each in-degree there is an out-degree and vice-versa resulting in counts always being equal for both attributes.