Piecewise Interpolation, Least-Squares and Quadrature

Due: Monday, April 10, 11:59pm.

Push your assignment solutions to the appropriate GitHub Classroom repository. Submit the following for this assignment:

- A PDF file, typeset in LaTeX, with answers to questions 1(d) and 3(e), along with the corresponding .tex file. Also, incorporate the figures from 1(a), 1(b), 1(c), 2(b), 3(b) and 3(d) into your LaTeX document.
- A file called `CompTrapA4.py` containing the Python function `CompTrap` and a file called `CompSimpsonA4.py` containing the Python function `CompSimpson` for questions 3(a) and 3(c), respectively. Use the templates provided in the assignment repository.
- Python scripts called `Quest1.py` for question 1, `Quest2.py` for question 2, and `Quest3b.py` and `Quest3d.py`, for questions 3(b) and 3(d), respectively. Use the templates provided in the assignment repository.
- Six plots. One for each of questions 1(a), 1(b), 1(c), 2(b), 3(b) and 3(d).

Question 1 **30 marks**

Consider the function

$$f(x) = \tan^{-1}(3x - 2), \qquad x \in [-3, 3]$$

Define interpolating nodes to be 13 equally space points (including the endpoints) between $x_0 = -3$ and $x_{12} = 3$, i.e. the points $x_0 = -3.0$, $x_1 = -2.5$, $x_2 = -2.0$, $\ldots$, $x_{11} = 2.5$, $x_{12} = 3.0$.

Write a script called `Quest1.py` that

(a) creates a figure with two subplots. The first subplot should plot the interpolating polynomial $P_{12}(x)$ that is equal to the function $f(x)$ at the interpolating nodes. It should also plot the corresponding data points, and the function $f(x)$. Use a legend to identify what you are plotting. The second subplot should plot the error $E = |f(x) - P_{12}(x)|$ on the interval $x \in [-3, 3]$ for the interpolating polynomial $P_{12}(x)$. Include the figure in your LaTeX document;

(b) creates a second figure with two subplots. The first subplot of this figure should plot the piecewise lineaer interpolant $P_{lin}(x)$ that is equal to the function $f(x)$ at the interpolating nodes. It should also plot the corresponding data points, and the function $f(x)$. Use a legend to identify what you are plotting. The second subplot of this figure should plot the error $E = |f(x) - P_{lin}(x)|$ on the interval $x \in [-3, 3]$ for the interpolant $P_{lin}(x)$. Include the figure in your LaTeX document;

(c) creates a third figure with two subplots. The first subplot of this figure should plot the cubic-spline interpolant $P_{cub}(x)$ that is equal to the function $f(x)$ at the interpolating nodes. It should also plot the corresponding data points, and the function $f(x)$. Use a legend to identify what you are plotting. The second subplot of this figure should plot the error $E = |f(x) - P_{cub}(x)|$ on the interval $x \in [-3, 3]$ for the interpolant $P_{cub}(x)$. Include the figure in your LaTeX document.

(d) Discuss your results. Include observations of how the size of the error varies with location for each interpolant. What do you think leads to these observations? Which interpolant is the best? Include both aesthetic and numerical reasoning in your argument. Include your response in your LaTeX document.

Question 2 **30 marks**

The following noisy data are given:

k	0	1	2	3	4	5	6	7	8	9	10
x_k	0	0.50	1.10	1.30	1.90	2.20	2.60	3.10	3.80	4.10	5.00
y_k	0.98	1.03	1.29	1.12	1.44	1.49	1.55	1.87	2.06	2.13	2.29

Write a Python script called `Quest2.py` that:

(a) sets up the linear system for the coefficients of the linear least-squares approximation and solves it;

(b) produces a figure that plots the data, along with the linear least-squares approximation to the data. Include a legend that labels the data and the linear function by its explicit functional form (i.e. in the legend label the line by $y = a_0 + a_1 x$, where a_0 and a_1 are replaced by the values of the coefficients you found in part (a)). Include the figure in your LaTeX document.

Question 3 **30 marks**

Consider the integral:

$$I = \int_a^b f(x)\,\mathrm{d}x \tag{1}$$

and the function

$$f(x) = e^{\sin(x)+1} \tag{2}$$

(**a**) In a file called `CompTrapA5.py`, write a function called `CompTrap`, that inputs a Python function f, and the variables a, b, and n, and outputs the composite trapezoidal rule approximation of the integral I (equation 1) on n subintervals of equal size. Use the template provided in the assignment repository.

(**b**) In a file called `Quest3b.py`, implement your composite trapezoidal rule function to approximate the integral I, with $f(x)$ defined as in equation (2), with limits of integration $a = 0$ and $b = 2$, and with n subintervals of equal size, where $n = 2^k$, $k = 1, \dots, 10$. Let us call these results I_k. Print out I_k for each k. Also in `Quest3b.py`, include code that plots $|I_k - I_{k-1}|$ versus h on a logarithmic scale (i.e. a log-log plot), where h is the length of each subinterval. Use the template for `Quest3b.py` provided in the assignment repository, and include the figure in your LaTeX document.

(**c**) In a file called `CompSimpsonA5.py`, write a function called `CompSimpson`, that inputs a Python function f and the variables a, b, and n, and outputs the composite Simpson's rule approximation of the integral I (equation 1) on n subintervals of equal size. Use the template provided in the assignment repository.

(**d**) In a file called `Quest3d.py`, implement your composite Simpson's rule function to approximate the integral I, with $f(x)$ defined as in equation (2), with limits of integration $a = 0$ and $b = 2$, and with n subintervals of equal size, where $n = 2^k$, $k = 1, \dots, 10$. Let us call these results I_k. Print out I_k for each k. Also in `Quest3d.py`, include code that plots $|I_k - I_{k-1}|$ versus h on a logarithmic scale (i.e. a log-log plot), where h is the length of each subinterval. Use the template for `Quest3d.py` provided in the assignment repository, and include the figure in your LaTeX document.

(**e**) Explain the figures from parts (b) and (d) in terms of the error analysis for the trapezoidal and Simpson's rules. Include your response in your LaTeX document.