CSCI / MATH 2072U - Tutorial 2

Arham Naqvi

January 29, 2023

Exercise A

(a) Suppose we wanted to derive an iterative rule using the Newton-Raphson method for the following function:

$$f(x,a) = x^2 - a,\ (a \in \mathbb{R} \wedge \geq 0) \tag{1}$$

Before we begin our derivation, let us first define $f'(x,a)$ as it will be useful in future calculations:

$$f'(x,a) = 2x \tag{2}$$

Our first step is to take some initial value $x^{(k)}$ for $k = 0$, as a plausible estimate for x^* such that $f(x^*, a) = 0$. Evaluating (1) at $x^{(k)}$ yields the following:

$$f(x^{(k)}, a) = (x^{(k)})^2 - a \tag{3}$$

We can now define a point P along our curve $f(x,a)$ such that $P = (x^k, (x^{(k)})^2 - a)$.

Next, we extend a tangent line from P and define the x-intercept of this line as $(x^{(k+1)}, 0)$ where $x^{(k+1)}$ is the successive term after $x^{(k)}$ in our Newton-Raphson iteration. Using our tangent line at point P, its x-intercept and the defintion of slope, we can now derive a recurrence relation between $x^{(k+1)}$ and $x^{(k)}$ as follows:

$$\begin{aligned}
f'(x^{(k)}, a) &= \frac{0 - f(x^{(k)}, a)}{x^{(k+1)} - x^{(k)}} \\
(x^{(k+1)} - x^{(k)}) f'(x^{(k)}, a) &= -f(x^{(k)}, a) \\
x^{(k+1)} - x^{(k)} &= -\frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)} \\
x^{(k+1)} &= x^{(k)} - \frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)}
\end{aligned}$$

Substitutting our results from (2) and (3) into the recurrence relation above, we can find an iterative rule for calculating $x^{(k+1)}$:

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - \frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)} \\
&= x^{(k)} - \frac{(x^{(k)})^2 - a}{2(x^{(k)})} \\
&= \frac{2(x^{(k)})^2 - (x^{(k)})^2 + a}{2(x^{(k)})} \\
&= \frac{(x^{(k)})^2 + a}{2(x^{(k)})} \\
&= \left(\frac{1}{2}\right)\left(x^{(k)} + \frac{a}{x^{(k)}}\right)
\end{aligned}
$$

Here we arrive at the familiar definition for $x^{(k+1)}$ in terms of $\phi(x^{(k)})$, demonstrating that the iteration rule from the previous tutorial was actually an implementation of the Newton-Raphson method for solving equation (1):

$$
x^{(k+1)} = \phi(x^{(k)}) = \left(\frac{1}{2}\right)\left(x^{(k)} + \frac{a}{x^{(k)}}\right) \tag{4}
$$

(b) Here we will show that if $x^{(k+1)} = \phi(x^{(k)}) = x^{(k)}$, then $x^{(k)}$ is the solution to equation (1).

Suppose that $x^{(k+1)} = x^{(k)}$, we can see immediatley that $\delta(x)$, which is defined as the difference between successive approximates for x^* must be equal to zero since $\delta(x) = x^{(k+1)} - x^{(k)} = x^{(k)} - x^{(k)} = 0$.

Now consider the following:

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - \frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)} \\
x^{(k+1)} - x^{(k)} &= -\frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)} \\
0 &= -\frac{f(x^{(k)}, a)}{f'(x^{(k)}, a)} \\
0 &= f(x^{(k)}, a) \\
0 &= x^{(k)})^2 - a \\
\sqrt{a} &= x^{(k)}
\end{aligned}
$$

Hence, whenever an iterate ($x^{(k)}$) is equal to the sucessive iterate ($x^{(k+1)}$), the difference between the two iterates becomes zero ($\delta x = 0$). As evident from the algebraic manipulation above we find that whenever this is the case, it implies that the previous iterate must be the solution to the equation $x^{(k)} = \sqrt{a}$. It is also important to note that this finding holds true not only in the case of the initial guess where $k = 0$, but $\forall k (k \in \mathbb{Z} \ \wedge \ k \geq 0)$.