Newton and fixed point iteration

Question 1

(**a**) Show that the following equation

$$f(x) = \cos(x) - x^3 - 2x = 0 \tag{1}$$

has a unique solution in $[0, 1]$. **2 marks**

(**b**) The function $g(x) = f(x) + x$ has a fixed point where $f(x) = 0$. Explain why we **can not** use g to approximate the solution using fixed point iteration. **2 marks**

(**c**) Show that the function $h(x) = \cos(x)/(x^2+2)$ also has a fixed point if $f(x) = 0$. Explain why we **can** use h to approximate the solution using fixed point iteration from any initial point in $[0, 1]$. **2 marks**

(**d**) How many iterations do you need to compute an approximate solution with an absolute error of 10^{-2}? (The answer depends on your initial point!) **2 marks**

(**e**) Use fixed point iteration with function h to find an approximate solution with an absolute error of 10^{-2}. At each iteration, list the current point, the difference between the current point and the next point and the value of f at the current point. **3 marks**

(**f**) Use the Newton-Raphson method with initial point $x_0 = 1$ to find an approximate solution with a residual smaller than 0.01. At each iteration, list the current point, the difference between the current point and the next point and the value of f at the current point. **3 marks**

Question 2

Suppose we have the system of equations

$$\begin{aligned} f_1(x, y) &= 3x^2y^2 - x - 1 = 0 \\ f_2(x, y) &= x^2y - 5y^2 - 1 = 0 \end{aligned} \tag{2}$$

(**a**) Compute the first three Newton-Raphson iterations of the initial point $(x_0, y_0) = (2, 0.5)$ and compute the residue for each iterate. **3 marks**

(**b**) Find functions $g_1(x, y)$ and $g_2(x, y)$ such that a solution of Eqs. (2) coincides with a solution of the system $g_1(x, y) = x$ and $g_2(x, y) = y$ and such that your g_1 and g_2 are suitable for fixed point iteration. **4 marks**

Hint: solve $f_1(x, y)$ for y and $f_2(x, y)$ for x. Show at least graphically that the conditions for fixed point iteration are satisfied.

(**c**) Compute the first three iterates of the initial point of (**a**) and estimate the number of iterations needed to achieve the same accuracy obtained after three Newton-Raphson iterations. **4 marks**

Question 3

In Lecture 1 I mentioned that the following two statements are equivalent:

1. $|g(x) - g(y)| \leq \lambda|x - y|$ for all $x, y \in [a, b]$ and $\lambda < 1$;
2. $|g'(x)| \leq \lambda < 1$ for all $x \in [a, b]$.

where $g \in C^1[a, b]$. Prove this. **5 marks**