CSCI / MATH 2072U - Assignment 2

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QUESTION 1

To begin, since $f(x) = e^{-x} + \cos(x+1) - 1$ is a sum of functions, each of which being continuous and differentiable $\forall x \in \mathbb{R}$, it must be the case that $f(x)$ is also continuous and differentiable $\forall x \in \mathbb{R}$ including the interval $x \in [0,1]$.

(a) Let us first evaluate $f(x)$ as defined above at the lower and upper bounds of the interval $[0,1]$:

lower bound:	**upper bound:**
$f(0) = e^{-(0)} + \cos((0)+1) - 1$	$f(1) = e^{-(1)} + \cos((1)+1) - 1$
$= e^0 + \cos(1) - 1$	$= e^{-1} + \cos(2) - 1$
$= 1 + \cos(1) - 1$	$f(1) \approx -1.0483 < 0$
$f(0) \approx 0.5403 > 0$	

Now, because $f(x)$ is continuous over the interval $[0,1]$, and since $f(0) > 0 > f(1)$, then by the Intermediate Value Theorem, there must exist some $c \in [0,1]$ such that $f(c) = 0$. So we have established that the equation $f(x) = 0$ has at least one solution in the interval $[0,1]$.

(b) Next, we will use Rolle's Theorem to demonstrate that our discovery of the existence of at least one solution to the equation $f(x) = 0$ over the interval $[0,1]$ is actually unique.

Proof:

Suppose there exist two values $a, b \in [0,1]$ such that $f(a) = f(b) = 0$. Since we know that $f(x)$ is continuous and differentiable over the interval $[0,1]$, we can use Rolle's Theorem then to assert that there must exist some c such that $a \leq c \leq b$ and $f'(c) = 0$.

Computing the first derivative of $f(x)$:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(e^{-x} + \cos(x+1) - 1\right)$$
$$= \frac{d}{dx}e^{-x} + \frac{d}{dx}\cos(x+1) - \frac{d}{dx}(1) \tag{1}$$
$$= -e^{-x} - \sin(x+1)$$

Now setting $f'(x) = -e^{-x} - \sin(x+1) = 0$ we find:

$$-e^{-x} \quad < 0 \quad (\forall x \in [0,1])$$
$$-sin(x+1) \quad < 0 \quad (\forall x \in [0,1])$$
$$-e^{-x} + (-sin(x+1)) \quad < 0 \quad (\forall x \in [0,1])$$

$$\therefore -e^{-x} - sin(x+1) \quad < 0 \quad (\forall x \in [0,1]) \tag{2}$$

Meaning that there can exist no $c \in [0,1]$ such that $f'(c) = 0$. This contradicts our assumption that there exist values $a, b \in [0,1]$ such that $f(a) = f(b) = 0$ due the inapplicability of Rolle's Theorem over the interval $[0,1]$ for such values.

$\therefore$ we have shown by contradiction that the solution to the equation $f(x) = 0$ over the interval $[0,1]$ is unique.

(c) Equipped with the function definitions of both $f(x)$ and $f'(x)$ we can utilize the Newton-Raphson iteration formula to find the recurrence relation between iterates $x^{(k+1)}$ and $x^{(k)}$ for $k \in \{\mathbb{Z}|k \geq 0\}$.

We begin with the definition of $x^{(k+1)}$ in terms of $x^{(k)}$:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \tag{3}$$

Substituting the definitions of $f(x)$ and $f'(x)$ we arrive at our formula for $x^{(k+1)}$ in terms of $x^{(k)}$:

$$x^{(k+1)} = x^{(k)} - \frac{e^{-x^{(k)}} + \cos(x^{(k)} + 1) - 1}{-e^{-x^{(k)}} - \sin(x^{(k)} + 1)} \tag{4}$$

(g)) We can conclude based on the semi-log plots for the number of iterations vs error, the Newton-Raphson method was by far the optimal method for obtaining a solution approximation with the specified accuracy. Not only does this method require fewer iterations (as evident from the quadratic convergence of the Newton method vs the linear convergence of the bisection method) but it also requires fewer computations per iteration when compared with the bisection method.

QUESTION 2

(d) Although the Euler-Chebyshev Method did require one less iteration to converge, the amount of extra work required to dervive the recurrence relation involving both first and second order derivatives, as well as the extra computations required to compute the update step in the algorithm make this method not worth the effort in this case.

QUESTION 3

(b) By graphing $f(x) = 4x^4 - 8x^3 + 7x^2 - 3x + \frac{1}{2}$ and its derivative $f'(x) = 16x^3 - 24x^2 + 14x - 3$ on the same coordinate system, the reason for the non-quadratic, linear convergence of the Newton-Raphson approximation becomes immediately clear. As x approaches 0.5, we see that the function $f(x)$ approaches a root which turns out is actually the absolute minimum. More importantly, the absolute values of tangential slope, at points along the curve as x gets to be very close to 0.5, become orders of magnitude larger than the corresponding values of $f(x)$. This leads to an incredibly small update step $\delta x = -\frac{f(x^{(k)})}{f'(x^{(k)})}$ which results in a slower convergence rate. In other words, if the distance between the function $f(x)$ and the x-axis is much smaller then the magnitude of slope at the same point, then the Newton-Raphson iteration process will by design update to the next iterate by a very small amount since the tangent at the previous iterate will immediately intercept the x-axis leading to a small change in approximation.

This demonstrates the large degree of dependency of the Newton-Rapshon method on function behaviour. If we are to compare the convergence rate for this function to that of the function in question 1, we observe a much faster (quadratic) convergence rate due to a large update step resulting from a small difference between function value and slope value near the root.