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State Transfer and Analysis of Quantum Systems on the Bloch Sphere

There is a one-to-one correspondence between the state of a single qubit and the point on the Bloch sphere, based on which the action of a control field can be analyzed clearly. The transfer between arbitrary states can be decomposed on the Bloch sphere into three rotations around the x-axis, y-axis, and z-axis.

2.1 Analysis of a Two-level Quantum System State

The Bloch vector (Bloch, 1946) provides the representation of the quantum state of a two-level system in terms of real observables, and allows the identification of quantum states with points in a closed ball in three-dimensional Euclidean space. In quantum information theory, for instance, the states of a single qubit can be identified with points on the surface of the Bloch sphere (when the state is pure) or points inside the sphere (when the state is mixed). The unitary operations can be interpreted as rotations of the Bloch sphere. The decoherence processes as the linear or affine contractions of the Bloch sphere (Bloch, 1946; Lindblad, 1976). In this section, we focus on the description of the different states of a qubit and the trajectories of the control actions on/in the Bloch sphere.

2.1.1 Pure State Expression on the Bloch Sphere

The simplest quantum mechanical system is the quantum-bit, or qubit. A qubit system is a two-state system that can be described by a vector in two-dimensional complex Hilbert space. The favorite qubit models are the spin of a spin-1/2 particle, nucleus spin in magnetic fields, the horizontal and vertical polarizations of a photon, or the ground and the first excited states of an electron in an atom. In general, qubits are denoted in Dirac's bra-ket notation. $|0\rangle$ represents a qubit in the zero state, pronounced "ket zero." The two basis vectors $|0\rangle$ and $|1\rangle$ correspond to the possible states a classical bit can take. However, in contrast to the state 0 or 1 of a

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classical bit, the state of a single qubit is $|0\rangle$ or $|1\rangle$ or their superposition. According to quantum mechanics, the evolution of the closed system obeys the Schrödinger equation:

$$i\hbar|\dot{\psi}\rangle = H|\psi\rangle \tag{2.1}$$

where H is the system Hamiltonian.

According to the superposition principle of quantum states, the pure state of a single qubit can be presented as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.2}$$

where α and β are complex coefficients of eigenstates.

The qubit described by $|\psi\rangle$ in Equation 2.2 is in a coherent superposition of $|0\rangle$ and $|1\rangle$. The values $|\alpha|^2$ and $|\beta|^2$ give the probabilities of measuring $|0\rangle$ and $|1\rangle$ states, respectively. They satisfy the probability completeness:

$$\alpha^2 + \beta^2 = 1 \tag{2.3}$$

which is the fundamental difference distinguishing quantum bits from classical ones. Using Equation 2.3, the qubit description can be rewritten as:

$$|\psi(\theta, \phi)\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right) \tag{2.4}$$

Since $e^{i\gamma}$ is just an arbitrary phase factor that does not have any observable effect, Equation 2.4 has an equivalent form:

$$|\psi\,(\theta, \varphi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle = U(t)|\psi(0)\rangle \tag{2.5}$$

where θ and ϕ are $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, $U(t)$ is the state transfer matrix (or evolution operator) of the system and $|\psi(0)\rangle$ is the initial state of the system.

The parameters θ and ϕ in Equation 2.5 define a point on a three-dimensional sphere. When it is used to represent a qubit, this sphere is known as the Bloch sphere, as shown in Figure 2.1.

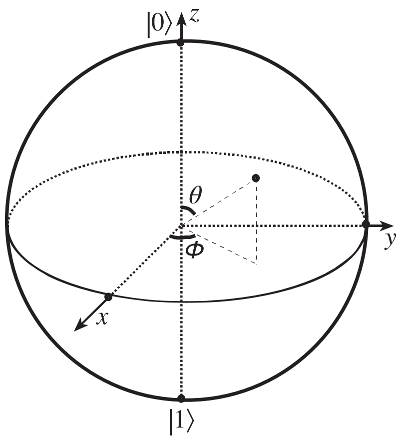

Figure 2.1 Presentation of qubit on the Bloch sphere

A qubit can be any point on the surface of this sphere. It can therefore hold an infinite amount of information. However, this actually is not the case because a qubit collapses to either $|0\rangle$ or $|1\rangle$ after being measured. Measurements are possible in something other than the computational basis. In terms of the Bloch sphere, the usual measurement operator described really measures zero for "up" and one for "down," but it could just as easily measure "left" and "right" (along the y-axis). Usually, though, such a measurement is made by rotating the y-axis to the z-axis. The measuring in the up–down direction is equivalent to measure the Z operator, and the measuring in the left–right direction is equivalent to measure the Y operator. It is also possible to measure a more complicated operator by using a basis with multiple qubits.

The standard observables for a qubit are the Pauli operators (or matrices). They are Hermitian and thus represent observables. Defining $|0\rangle = (1 \quad 0)^T$ and $|1\rangle = (0 \quad 1)^T$, we can write the Pauli matrices as:

$$X \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1| \tag{2.6a}$$

$$Y \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1| \tag{2.6b}$$

$$Z \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{2.6c}$$

The eigenvalues of each Pauli operator are $+1$ and -1. The standard basis ket $|0\rangle$ and $|1\rangle$ are the eigenvectors of Z operator. To emphasize the connection with the Z operator, one sometimes denotes $|0\rangle$ by $|\uparrow\rangle$ and $|1\rangle$ by $|\downarrow\rangle$. Basis vectors span the space and therefore one may write the eigenvectors of X and Y as linear combinations of $|0\rangle$ and $|1\rangle$. For example, the eigenvectors of X are $|x_+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|x_-\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$; the subscripts of $+$ and $-$ denote the eigenvalue signature. Similarly, the eigenvectors of Y may be written as $|y_+\rangle = (1/\sqrt{2})(|0\rangle + i|1\rangle)$ and $|y_-\rangle = (1/\sqrt{2})(|0\rangle - i|1\rangle)$. They are shown in the Bloch sphere in Figure 2.2.

The X, Y, and Z bases have the property that if a qubit is in an eigenstate of one basis and is projected onto another basis, the probability of finding it to be either up or down is 1/2. Such

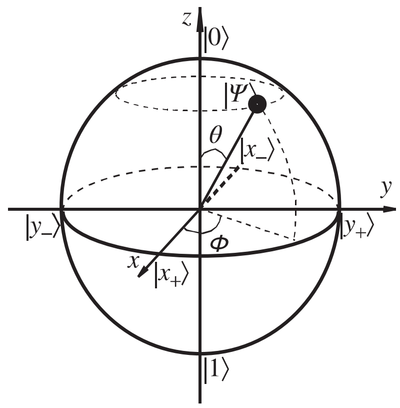

Figure 2.2 The Bloch sphere representation of a qubit. The angle θ is measured from the z-axis and the angle ϕ from the x-axis in x–y plane

a set of basis is termed mutually unbiased. X is also called the bit flip operator as it turns a $|0\rangle$ into a $|1\rangle$ and vice versa. Z is called the phase flip operator as it switches between the states $\alpha|0\rangle + \beta|1\rangle$ and $\alpha|0\rangle - \beta|1\rangle$.

2.1.2 Mixed States in the Bloch Sphere

In the pure state case we think of the state as being isolated and not affected by the entire universe. In a more realistic situation we do not have total knowledge about the physical system. Maybe we have restricted access to the information, such that the best we can say about the system is that it may be in the pure state A or B with certain probabilities. What we can do in this situation is to represent the state with a statistical ensemble that gives us probabilistic information. We say that the system is in a mixed state and we call the representation of it in the quantum mechanics a density operator.

In the density operator formalism, we describe quantum states by operators on the system's Hilbert space instead of unit vectors on it. For any quantum state vector $|\psi\rangle$, the corresponding density operator ρ is the projection operator $|\psi\rangle\langle\psi|$. As linear operators may be, and often are, represented by matrices, density operators are also called density matrices.

A mixed state has the definite probabilities of being in some mixture of pure states $|\psi_i\rangle$, with corresponding probabilities p_i that sum to unity, therefore one may write the density operator in more general terms, which incorporate both mixed and pure states, as a weighted sum of pure states $|\psi_i\rangle$ with probabilities p_i as their weighting factors:

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \tag{2.7}$$

The density matrix ρ of a two-state system is a positive semi-definite Hermitian 2×2 matrix having unit trace. It can always be a given expression in terms of the three trace-free Pauli matrices σ_i, $(i = x, y, z)$, which are generators of $su(2)$, and $I/\sqrt{2}$ (Mandilara and Clark1, 2005):

$$\rho = (1/2)(1 + \sigma \cdot r) \tag{2.8}$$

in which $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the standard Pauli vector and $r = (r_x, r_y, r_z)$ is the Bloch vector with $|r| = 1$ if and only if ρ is a pure state, and $|r| < 1$ for mixed states.

Because $1/2 \leq Tr(\rho^2) \leq 1$, two limiting values 1/2 and 1 correspond to maximally mixed states and pure states, respectively. In matrix form with respect to $|0\rangle$, the $|1\rangle$ basis density matrix ρ reads (Mosseri and Dandoloff, 2001):

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} \tag{2.9}$$

To find out the range of values for (r_x, r_y, r_z) one can use the fact that $r_x^2 + r_y^2 + r_z^2 = Tr(\rho^2) \leq 1$, where the equation holds for pure states only. Consequently, one can identify the density operator through the points in the position of the radius of the Bloch sphere, in which pure states are points on the surface while non-pure states are points inside the sphere, as shown in Figure 2.3.

A projection measurement of the system described by a mixed state will generate a pure state, defined by a point on the Bloch sphere. When $|r| = 0$ the system is said to be maximally mixed and we have minimal information about it. Performing a measurement on such a state

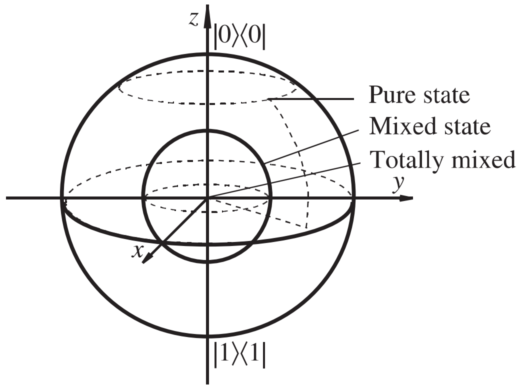

Figure 2.3 Pure states and mixed states of the Bloch sphere, in which the state $|0\rangle\langle 0|$ is found at the north pole and $|1\rangle\langle 1|$ at the south pole. The totally mixed state is in the center of the Bloch sphere

will generate a pure state, defined by any point on the Bloch sphere with probability 1/2. A maximally mixed state is symmetric under the rotation.

For one qubit system, a general qubit is described by:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix} \tag{2.10}$$

A general qubit may have a point anywhere on the Bloch sphere, with uniform probability. The diagonal elements of the density operator determine the populations of the energy eigenstates, while the off-diagonal elements determine the coherences between energy eigenstates. However, the presentation of a density operator does not tell us exactly what kind of pure state is. For example, we can have a superposition state of $|0\rangle$ and $|1\rangle$, and each has probability 1/2. The density operator is then $(1/2)(|0\rangle\langle 0| + |1\rangle\langle 1|) = I/2$ (I is the unit matrix) or:

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{2.11}$$

In this case the information is zero, which represents the fact that we know nothing about where a general qubit point is on the Bloch sphere. Because the same density operator can be made by mixing the states $|+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|-\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$ in equal amounts, or the three states $|0\rangle$, $|+\rangle$, and $(1/2)|0\rangle - (\sqrt{3}/2)|1\rangle$ in amounts $p_1 = (1/2) - \sqrt{3}/6$, $p_2 = \sqrt{3}/(3 + \sqrt{3})$, and $p_3 = 1 - \sqrt{3}/3$, respectively, we call these different collections of pure states with corresponding probabilities $\{p_i, |\psi_i\rangle\}$ ensembles. Sometimes they are also called realizations and we say that an ensemble realizes a mixed state. Hence, when we describe a state of an ensemble by its density operator, we discard the information about which ensemble the mixed state is made from. The density operator still describes the mixed state as well as can be done, as states from different ensembles having the same density operator are experimentally indistinguishable. This inability to distinguish them makes the density operator a perhaps more intuitive representation of a state than the state vector. As shown above, two vectors differing by a phase factor represent the same state, but when the vector is put together with its dual vector to form a pure state density operator, the phase factor vanishes because of the complex conjugation. Hence, all state vectors representing the same state are represented by the same density operator.

The evolution of a closed quantum system is always the unitary evolution. Unitary evolution preserves the spectrum of the quantum state (i.e., the eigenvalues of the density matrix). All density matrices that have the same eigenvalues form a set of unitarily equivalent states (e.g., the set of all pure states).

2.1.3 Control Trajectory on the Bloch Sphere

A given physical system is characterized by a state vector $|\psi(t)\rangle$ whose dynamics are governed by the Schrodinger equation as:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle \tag{2.12}$$

It can be shown from the Schrodinger equation (Equation 2.12) that the density operator satisfies the following differential equation:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho] \tag{2.13}$$

which is called the quantum Liouville equation.

A master equation is a differential equation that describes a quantum system in contact with its surroundings. Under certain conditions, one can derive the Lindblad equation, which is a master equation, as:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}\rho \tag{2.14}$$

where the generator $\mathcal{L}$ of the semigroup represents a super-operator. The explicit form of this matrix can be derived using rigorous master equation formalism. The first term of the time evolution describes the standard coherent two-level dynamics and the last term accounts for the gain and the damping mechanism. This term has the form of the Liouville super-operator, which can be written in the Lindblad form (Lindblad, 1976):

$$\mathcal{L}\rho = \sum[F_i^\dagger F_i\rho + \rho F_i^\dagger F_i - 2F_i\rho F_i^\dagger] \tag{2.15}$$

where F_i and F_i^+ form a collection of generalized atomic creation and annihilation operators characteristic for a particular problem. The Lindblad form of the master equation guarantees that the interaction with the damping reservoir preserves the positivity of the density operator.

In realistic setups, the qubit is usually exposed to a noisy channel leading to the decoherence of quantum information. Because of the decoherence there is no such thing as a perfect qubit or a perfect quantum logical gate (Mahler and Weberru, 1998). If a qubit is exposed to a noisy channel one can observe a loss of coherence and it is still an open question whether or not there is an intrinsic lower limit to the decoherence rate for an arbitrary qubit. The quantum or the stochastic noise is an undesirable but unavoidable property of quantum logical gates. The issue that the stochastic noise of a qubit can dramatically change logical operations has become one of the central themes of error correction and noise stabilization in quantum computational schemes. The control of the decoherence of a qubit is a central problem of various models of quantum computation, where an incoming qubit is transmitted through a noisy channel. Because of the noise the statistical properties of such a qubit can be changed, for example pure states become mixed states. For a class of such noisy channels, one could ask the question of a possible optimalization or minimization of the decoherence effect of a qubit. One fundamental

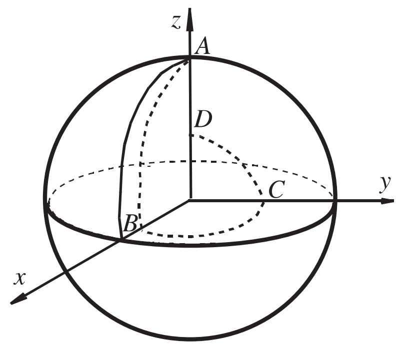

Figure 2.4 The evolution of a state in the case of three magnetic fields

goal is to seek those noises that minimize a fidelity or an entropy of such a channel. In order to perform such a task, a general understanding of a wide class of noises acting on the qubit is required. This problem has attracted a lot of attention and has been studied in the framework of quantum information theory. For this system, unitary evolution is on a spherical shell within the Bloch sphere. The dissipation causes more general motion within the Bloch sphere. This motion corresponds to the action of a certain semi-group (Alicki and Lendi, 1987). When two magnetic external control fields act on the system, this control will let the coordinate system be chosen so that the Bloch vector of the qubit points along the positive z-axis with unit length at time $t = 0$. Then we switch on a magnetic field directed along the y-axis until the Bloch vector has rotated $\pi/2$ radians. The vector is now found in the equatorial plane of the Bloch sphere, pointing in the x-direction. In the same manner, switch on the magnetic field in the z-direction and rotate the vector by $\pi/2$ radians. The magnetic field is assumed to be equal to the strength in the two directions.

For three control magnetic fields acting on the system, the procedure is analogous to the former case with two magnetic fields. We first operate with a magnetic field in the y-direction and then in the z-direction. What differs is that we also operate with a third magnetic field in the direction of the positive x-axis. This last field forces the Bloch vector to rotate to a final position, on the positive z-axis, which is the same direction as we started with.

In the pure and unitary cases this evolution describes a closed path on the Bloch sphere. In the decoherence case the evolution is non-cyclic: we end up with a Bloch vector in the same direction as we started with, but shorter. The trajectories of system under action of control are as shown in Figure 2.4, in which the thick line represents the evolution in the pure case and the dashed line is the decoherence case (note that the unitary case is a cyclic, while the decoherence case is a non-cyclic).

In conclusion, we have presented the pure and mixed states of a two-level system in detail from the aspect of the Bloch sphere in this section. The trajectories of the control action with two and three magnetic field actions in the Bloch sphere have been also analyzed. This should help in understanding the properties that exist in the states of quantum systems.

2.2 State Transfer of Quantum Systems on the Bloch Sphere

We know from Section 2.1 that the Bloch sphere provides a significant geometric description (Nielsen and Chuang, 2000) for spin-1/2 quantum systems, and gives an intuitionistic understanding to quantum states and their evolutions. It maps the state of a single qubit as a point on a three-dimensional sphere, and a unitary evolution of the state as a rotation operation of the point around an axis. The coherent vector representation or the Bloch equation (Allen and Eberly, 1987; Alicki and Lendi, 1987) is under this comprehension. In mathematics, the unitary state transfer matrices of an N-level quantum system are made up of a unitary group $SU(N)$, whose generator, namely Lie algebra $g = su(N)$, forms the orthogonal basis set of the system density matrices. The coherent vectors are the adjoint representation of $su(2)$ elements on a $so(3)$ basis (Altafini, 2003). The Bloch sphere description takes an important role in the domains of quantum information, quantum computation, and quantum control. A lot of investigations are carried out under this description, such as the study of the stochastic decoherence of qubits (Wodkiewicz, 2001), the decomposition of the unitary operator (Cong, 2006), the analysis of the controllability of coherent control (Altafini, 2003), and the problem of preserving the coherence of quantum systems (Lidar and Schneider, 2005). Initially the Bloch sphere representation is restricted to a single qubit; how the geometric notions of the Bloch description can be extended to two qubits (Fano, 1957) and even higher dimensions (Byrd and Khaneja, 2003; Kimura, 2003; Kimura and Kossakowski, 2004) are attractive issues. Rossen Dandoloff *et al.* developed the Bloch sphere representation of entangled states of two qubits (Mosseri and Dandoloff, 2001) and Mandilara *et al.* discussed the evolution tracking of a system induced by local operation on one of two qubits (Mandilara, Clark, and Byrd, 2005).

In a single qubit situation, as will be shown in this section, it is feasible to design control fields to steer an initial state to a specific target state under the Bloch sphere description of a quantum system. In such a way a clear understanding can be made of the relation between the actual action of control fields and its mathematical representation, for example the invariable magnetic field along the z-axis corresponds to the rotation operator around the z-axis σ_z, and the rotation magnetic field in the x–y plane corresponds to the rotation operator around the x-axis σ_x.

From the Bloch sphere shown in Figure 2.1 one can see that the sphere coordinates of a point on the Bloch sphere can be presented with right-angle coordinates as follows:

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases} \tag{2.16}$$

The states of a single qubit could have an intuitionistic illustration using the Bloch sphere, and any unitary evolution of the state of a single qubit can be decomposed as rotations of a point on the Bloch sphere. Based on this intuitionistic representation of a quantum state, the state evolution trajectory can be observed on the Bloch sphere, the effect of parameters can be illustrated, and control fields can be designed to steer states to the arbitrary target states of a single qubit under different conditions.

2.2.1 *Control of a Single Spin-1/2 Particle*

Taking a spin-1/2 particle in an invariable magnetic field B_0 along the z-axis as the controlled system, the control magnetic fields on the x–y plane are:

$$\begin{cases} B_x = A\cos(\omega t + \varphi) \\ B_y = -A\sin(\omega t + \varphi) \end{cases} \tag{2.17}$$

In Equation 2.1, the system Hamiltonian H is composed of the free Hamiltonian H_0 and the control Hamiltonian H_c:

$$H = H_0 + H_c \tag{2.18}$$

and:

$$H_0 = -\frac{\hbar}{2}\omega_0\sigma_z, H_c = -\frac{\hbar\Omega}{2}(e^{-i(\omega t+\varphi)}I^- + e^{i(\omega t+\varphi)}I^+) \tag{2.19}$$

in which $\omega_0 = \gamma B_0$ is the eigenfrequency of the qubit in the outside magnetic field, γ is the magnetic ratio of spin particle, and $\Omega = \gamma A$ is the Rabi frequency of the particle, which is a real number: $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $I^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $I^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Hereby the matrix form of the control system Hamiltonian H is:

$$H = H_0 + H_c = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{i(\omega t+\phi)} \\ \Omega e^{-i(\omega t+\phi)} & -\omega_0 \end{pmatrix} \tag{2.20}$$

In a resonant situation therefore, namely the frequency of the control fields is equal to the eigen-frequency of the system, $\omega = \omega_0$, the state transfer matrix of system $U(t)$ can be deduced according to the Schrödinger equation as:

$$U(t) = \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i(\omega_0 t+2\phi)/2} \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\Omega t}{2}\right) & i\sin\left(\frac{\Omega t}{2}\right) \\ i\sin\left(\frac{\Omega t}{2}\right) & \cos\left(\frac{\Omega t}{2}\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{2.21}$$

It can be seen from Equation 2.21 that the state transfer matrix $U(t)$ comprises three matrices: the first matrix is $\begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i(\omega_0 t+2\phi)/2} \end{pmatrix}$, the second term is $\begin{pmatrix} \cos\left(\frac{\Omega t}{2}\right) & i\sin\left(\frac{\Omega t}{2}\right) \\ i\sin\left(\frac{\Omega t}{2}\right) & \cos\left(\frac{\Omega t}{2}\right) \end{pmatrix}$, and the third item is $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$. The first matrix makes the point rotate around the z-axis by an angle of $(\omega_0 t + \phi)$, and an additional global phase factor; the third matrix makes a rotation around the z axis by an angle of ϕ, while the second matrix makes the point on the Bloch sphere rotate around the x-axis by an angle of Ωt.

It can be also seen from Equation 2.21 that to steer the state transfer is equivalent to controlling the three parameters Ω, ϕ, and t. The different selection of the parameters leads to different state transfer matrices, although from the same initial state to the same given target state. For example, the two trajectories on the Bloch sphere shown in Figure 2.5 have the same initial state a and the same target state d. In Figure 2.5a, the control fields make the state rotate an initial phase of ϕ from a to b around the z-axis under the action of the third matrix in Equation 2.21. Then the second matrix of the control makes the state rotate an angle of Ωt from b to c around the x-axis, and c is rotated to d under the action of the first matrix. In Figure 2.5b the initial phase of the control field is 0, so the action of the third matrix can be ignored, and a is rotated to e around the x-axis by the second matrix and the first matrix rotates e to d. The thick lines in Figure 2.5b are the actual trajectories of the compound actions of three parts of control field while the thin lines denote the states transfer trajectories decomposed according to Equation 2.21.

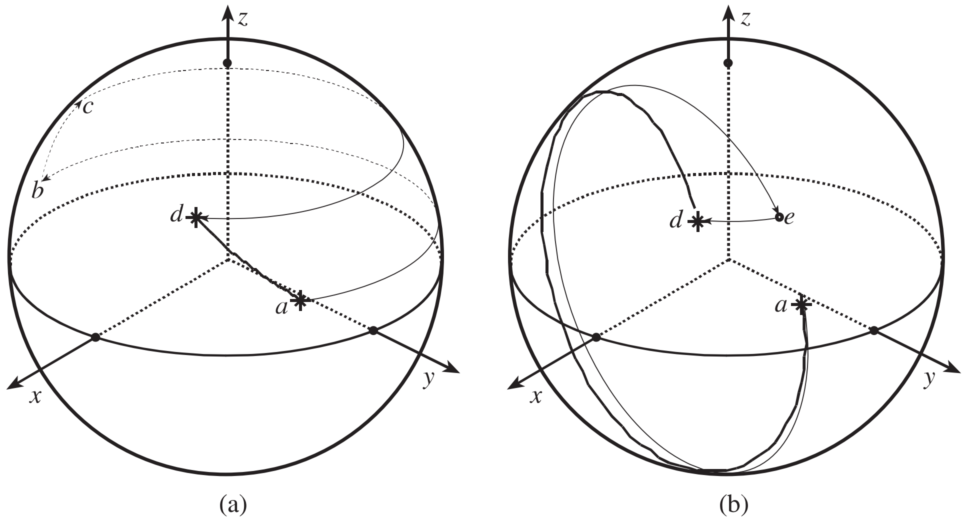

Figure 2.5 Different trajectories of states transfer on the Bloch sphere: (a) state transfer with an initial phase of ϕ and (b) state transfer with zero initial phase

For a given initial state and prescribed target state, one always desires the transfer trajectory to be optimal under the action of control, which means the control either makes the shortest time or the minimal trajectory. We will investigate these two different situations in the following subsections.

2.2.2 *Situation with the Minimum Ωt of Control Fields*

In this situation the minimum rotation angle around the x-axis Ωt is desired. For a given initial point $a\ (\theta_1, \phi_1, 1)$ and a prescribed target point $d\ (\theta_2, \phi_2, 1)$ on the Bloch sphere, the minimum value of Ωt is equal to $\Delta\theta = |\theta_2 - \theta_1|$ when points a and d are rotated to points a' and d' on the plane $x = 0$, as shown in Figure 2.6, and the minimum rotation angle from point a' to point d' is $\Delta\theta$, which occurs when $\theta_1 > \theta_2$. If $\theta_1 \leq \theta_2$, then points a' and d' should be on the right side with positive y-coordinates. In order to get the minimum Ωt, the rotation of ϕ should be from a to a', and the rotation of $\omega_0 t + \phi$ should be from d' to d. In the other case in which the control field has the opposite direction, the rotation around the z-axis should be in the opposite direction. In this way the required time of steering for the state transfer can be decreased.

The parameters of a control field with a minimum Ωt must satisfy the following conditions:

$$\begin{cases} \Omega t = \Delta\theta \\ \phi = \frac{3\pi}{2} - \phi_1, \theta_1 > \theta_2; \quad \phi = \frac{\pi}{2} - \phi_1, \theta_1 \leq \theta_2 \\ \omega_0 t = \bmod\left(\phi_1 - \phi_2, 2\pi\right) \end{cases} \tag{2.22}$$

If there is upper limit on Ω, for example, $\Omega \leq \Omega_{\max}$, then one has $t \geq t_{\min} = \Delta\theta/\Omega_{\max}$ according to Equation 2.22, thus we get:

$$t = \frac{\bmod(\phi_1 - \phi_2, 2\pi) + 2k\pi}{\omega_0} \tag{2.23}$$

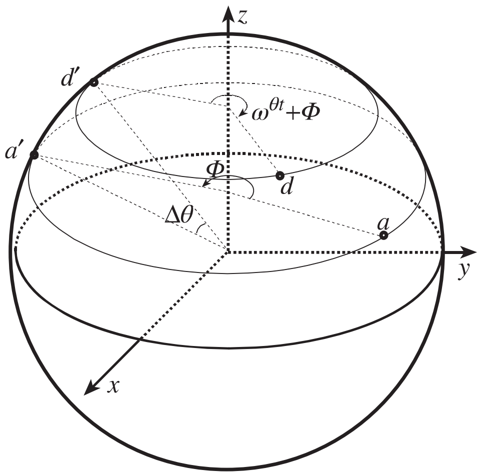

Figure 2.6 The parameters with a minimum Ωt

where k can be determined by the following equation:

$$k = ceil\left(\frac{\omega_0 t_{\min} - \bmod\left(\phi_1 - \phi_2, 2\pi\right)}{2\pi}\right) \tag{2.24}$$

in which the function $ceil(\bullet)$ takes the integer of $(\bullet)$ toward positive infinitude; it is 0 only when $\omega_0 t_{\min} \leq \bmod(\phi_1 - \phi_2, 2\pi)$.

If there is a restriction on finish time, for example $t \leq t_{\max}$, it can also be shown that:

$$\Omega \geq \Omega_{\min} = \Delta\theta / t_{\max} \tag{2.25}$$

In general situations there are restrictions on both Ω and t, so the following equations must be satisfied according to the analysis above:

$$\begin{cases} t_{\min} \leq t \leq t_{\max} \\ \Omega_{\min} \leq \Omega \leq \Omega_{\max} \end{cases} \tag{2.26}$$

Hence in the situation with the minimum Ωt, Equations 2.22 and 2.26 should be satisfied in general.

It should be emphasized that neither the time nor the length of the trajectory on the Bloch sphere is optimized in this situation, although the control field has the minimum Ωt. In addition, the uniqueness of the trajectory under the control design is determined by the uniqueness of the coordinates of the initial state and the target state. If the point is on the pole of the Bloch sphere, viz. on the z-axis, the trajectory will not be unique for the arbitrary value of ϕ_1 or ϕ_2, which is a special case.

2.2.3 Situation with a Fixed Time T

Usually the control operations to quantum systems should be fast enough because of the existence of the decoherence of systems in actually applications, for example the control action should be finished in a fixed time T, but without the request of minimum Ωt. Now we will study how to obtain the parameters Ω and ϕ in such a situation.

Because the eigen-frequency ω_0 is a constant when the control field is invariable, the point on the Bloch sphere is rotated by an angle of $\omega_0 T$ for a given T. If d is rotated toward the opposite direction to the point d', it is equivalent to the problem of obtaining the control parameters Ω and ϕ to make a transfer from the point a to the point d' but without considering the action of ω_0. The action of the initial phase ϕ is to rotate the state from points a and d' by an angle of ϕ around the z-axis to the points a' and d'' on the same y–z plane, so the point a will be rotated to the point a' under the action of the third matrix of Equation 2.21 and the point d'' will be rotated to d' after the rotation around the x-axis under the action of the first matrix (without considering ω_0). So Ω can be obtained if the angle on the y–z plane $\psi = \mathrm{mod}(\arg(y_{a'} + iz_{a'}) - \arg(y_{d''} + iz_{d''}), 2\pi)$ is known. The decomposition of state transfer is shown in Figure 2.7.

Thus the initial phase ϕ must satisfy the following equation:

$$\tan\phi = \frac{\sin\theta_1 \cos\phi_1 - \sin\theta_2 \cos(\phi_2 + \omega_0 T)}{\sin\theta_1 \sin\phi_1 - \sin\theta_2 \sin(\phi_2 + \omega_0 T)} \tag{2.27}$$

therefore we can obtain the initial phase ϕ from Equation 2.27 and Ω can be determined by the angle ψ of points a' and d'' on the y–z plane:

$$\Omega = \psi / T \tag{2.28}$$

According to the connection of the sphere coordinates and the right-hand coordinates, one can get:

$$\tan(\psi) = \tan(\Omega T) = \frac{\cos\theta_1 \sin\theta_2 \sin(\phi_2 + \omega_0 T + \phi) - \sin\theta_1 \cos\theta_2 \sin(\phi_1 + \phi)}{\sin\theta_1 \sin\theta_2 \sin(\phi_1 + \phi)\sin(\phi_2 + \omega_0 T + \phi) + \cos\theta_1 \cos\theta_2} \tag{2.29}$$

In the situation that the initial state or the target state is an eigenstate of the Hamiltonian, one has $\psi = \Delta\theta$. If $(\theta_1 + \theta_2)/2 = \pi/2$, then $\psi = \pi$. Ω is in inverse proportion to T according

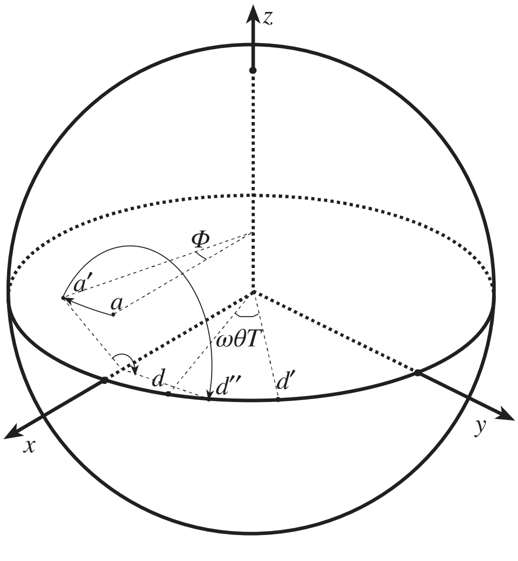

Figure 2.7 Decomposition of state transfer for a given time T

to Equation 2.28. For general situations the relation of Ω and T can be found according to Equations 2.27–2.29 to be:

$$\Omega = \begin{cases} \arctan\left(f\left(T\right)\right)/T, & \psi \in (0, \frac{\pi}{2}] \\ (\pi - \arctan(f(T)))/T, & \psi \in (\frac{\pi}{2}, \pi] \\ (\pi + \arctan(f(T)))/T, & \psi \in (\pi, \frac{3\pi}{2}] \\ (2\pi - \arctan(f(T)))/T, & \psi \in (\frac{3\pi}{2}, 2\pi] \end{cases} \tag{2.30}$$

where $f(T)$ is defined as:

$$f(T) = \frac{\sqrt{\sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1 + \omega_0 T)}\left[\left(-\cot\theta_1 - \cot\theta_2\right)\cos(\phi_2 - \phi_1 + \omega_0 T) + \frac{\cos\theta_1}{\sin^2\theta_1} + \frac{\cos\theta_2}{\sin^2\theta_2}\right]}{\cos^2(\phi_2 - \phi_1 + \omega_0 T) + \left(2\cot\theta_1\cot\theta_2 - \frac{\sin^2\theta_1 + \sin^2\theta_2}{\sin\theta_1\sin\theta_2}\right)\cos(\phi_2 - \phi_1 + \omega_0 T) + \left(1 - \frac{\left(\sin^2\theta_1 + \sin^2\theta_2\right)\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}\right)} \tag{2.31}$$

Equations 2.27, 2.30, and 2.31 give the conditions that the parameters need to satisfy for a given T. If there is a restriction on Ω, then the valid range of T can be obtained by Equations 2.30 and 2.31.

The following numerical simulations verify the effectiveness of the control fields designed for both situations discussed above.

2.2.4 *Numerical Simulations and Results Analyses*

Numerical system simulation experiments are done based on the previous analysis. The constant ω_0 is set to be a relatively small value 5 in the simulations in order to observe the trajectory of the state transfer on the Bloch sphere more clearly. Because γ is a constant related to the individual spin particle, only $\gamma B_x = \Omega\cos(w_0 t + \phi)$ will be given in figures below for convenience.

1) Numerical simulations for the minimum Ωt

The example used in the numerical simulation is to steer a state transfer from initial state $|0\rangle$ to target state $|1\rangle$. In this case $\Delta\theta = \pi$, and the values of ϕ_1 and ϕ_2 are arbitrary because the projections of the two states on the x–y plane are both in the origin. In the simulation experiments, $\phi_1 = \phi_2 = \pi/2$. Then one can get $t = 1.2566$ and $k = 1$ according to the third condition of Equations 2.22 and 2.24, respectively. Hence $\Omega = 2.5$ and $\phi = 0$ can be obtained according to the first condition of Equation 2.22. The control field and the trajectory of state transfer on the Bloch sphere are shown in Figure 2.8, from which one can see that the transfer from state $|0\rangle$ to $|1\rangle$ is successful under the action of the control field, but the trajectory of the state transfer goes a circuit around the z-axis, which is mostly caused by the values of ϕ_1 and ϕ_2. In fact, in the present case, the state transfer will always be successful as long as $\Omega t = \pi$, and when Ω tends to ∞, t tends to 0, and the length of the trajectory will tend to the minimum, which is the situation with semicircle.

The situation of a state transfer from the superposition state $(|0\rangle + |1\rangle)/\sqrt{2}$ to the state $0.8|0\rangle + 0.6|1\rangle$ is different from that in the case from $|0\rangle$ to $|1\rangle$, as shown in Figure 2.9, in which $\Delta\theta \approx 0.284$, and ϕ_1 and ϕ_2 are unique values: $\phi_1 = \phi_2 = 0$. In such a case the trajectory of the state transfer is also unique, and it goes a circuit around the z-axis so that makes

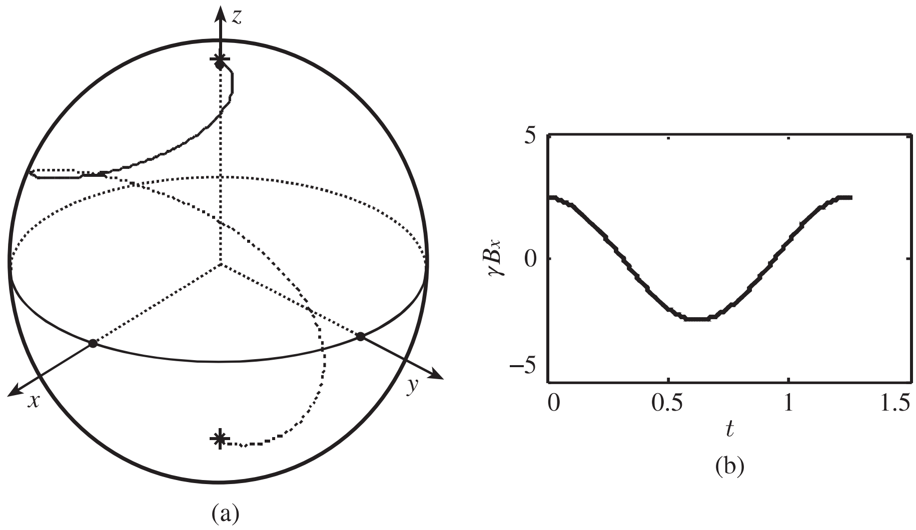

Figure 2.8 State transfer trajectory with a minimum $\Omega t = \pi$: (a) state transfer trajectory from $|0\rangle$ to $|1\rangle$ on the Bloch sphere and (b) control field

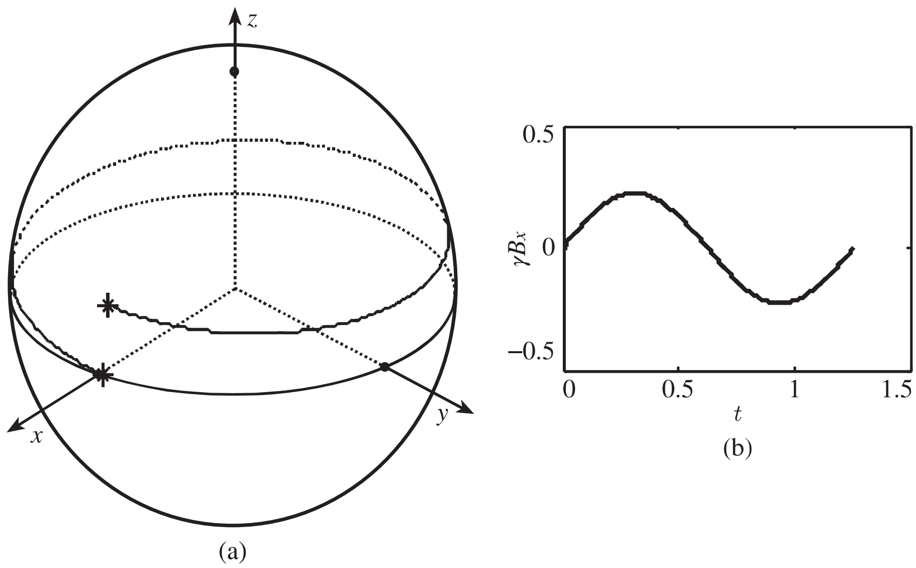

Figure 2.9 State transfer trajectory with a minimum $\Omega t \approx 0.284$: (a) state transfer trajectory from $(|0\rangle + |1\rangle)/\sqrt{2}$ to $0.8|0\rangle + 0.6|1\rangle$ and (b) control field

the trajectory and the time longer, which is caused by the conditions of $\phi_1 = \phi_2$ and $k = 1$ according to Equation 2.24, and is inescapable under the request of a minimum Ωt, viz. the trajectory has to go a circuit around the z-axis under a small value of $\Omega t_{\min} = \Delta\theta \approx 0.284$.

2) Numerical simulations for a given T

We will use the same example of steering a state transfer from $|0\rangle$ to $|1\rangle$. It is unrealistic to obtain the control parameters according to Equations 2.27, 2.30, and 2.31 for $\sin\theta_1 = \sin\theta_2 = 0$. It can be shown by the analysis from which Equations 2.27, 2.30, and

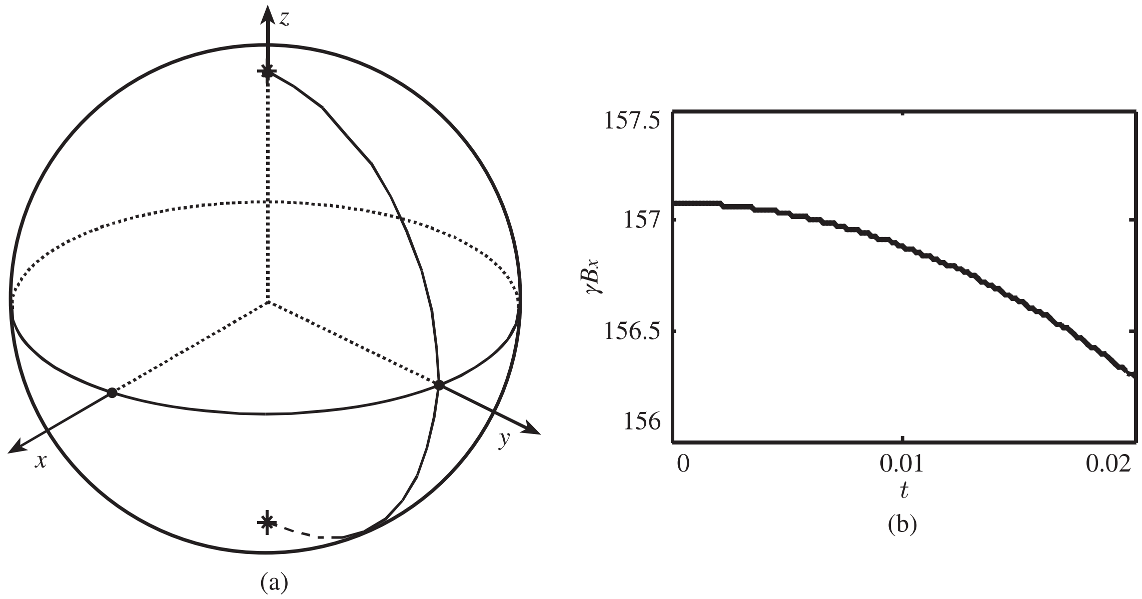

Figure 2.10 State transfer trajectory with a given time $T = 0.02$: (a) state transfer trajectory from $|0\rangle$ to $|1\rangle$ on the Bloch sphere and (b) control field

2.31 are derived that the time T determines the point d' obtained by a rotation of the target state around the z-axis by an angle of $\omega_0 T$ toward the opposite direction, and the points a and d' should be rotated by an angle of ϕ to points a' and d'' on the same y–z plane. Because the rotations around the z-axis do not change the position of the initial or target states on the Bloch sphere, which are already on the same y–z plane in this situation, the value of ϕ can be arbitrary. Here we set $\phi = 0$. Furthermore, the angle of rotation around the x-axis, viz. ψ, is equal to the difference between θ_1 and θ_2, $\psi = \Omega T = \pi$. Then $\Omega = 50\pi = 157.08$ can be obtained for a given $T = 0.02$. The control field and the trajectory of the state transfer are shown in Figure 2.10. In such a case, the trajectory is also the shortest and needs a larger control value.

For the state evolution from $(|0\rangle + |1\rangle)/\sqrt{2}$ to $0.8|0\rangle + 0.6|1\rangle$ when $T = 0.02$, we can get $\theta_1 = \pi/2$, $\theta_2 \approx 1.287$, and $\phi_1 = \phi_2 = 0$ according to Equation 2.5, and we can obtain $\phi = -0.4372$ after substituting the values of θ_1, θ_2, ϕ_1, and ϕ_2 in Equation 2.27, and $\Omega = 36.125$ in Equations 2.30 and 2.31. The control field and the track of the state evolution are shown in Figure 2.11. The trajectories in Figures 2.10 and 2.11 are much shorter compared with those in Figures 2.8 and 2.9. Because the state transfer under the control field design can be finished at a given time T, the trajectory will go around the z-axis by an angle of $\omega_0 T = \gamma B_0 T$. If the time T is small, then the trajectory will be short, and does not need to go a circuit around the z-axis, as shown in Figures 2.8 and 2.9, despite $\phi_1 = \phi_2$. On the other hand, the values of control fields $A = \Omega/\gamma$ will be larger according to Equation 2.28 as T becomes smaller, and Equation 2.31 is a limit function with periods $\tau = 1/\omega_0$, thus T determines mostly the values of the control field.

In fact there is usually a restriction on Ω, and the minimum time $T_{\min}$ needed to complete the operation can be determined by Equation 2.30 for given initial and target states. The value of $T_{\min}$ can be estimated by graphics since it is difficult to calculate precisely. Figure 2.12 shows

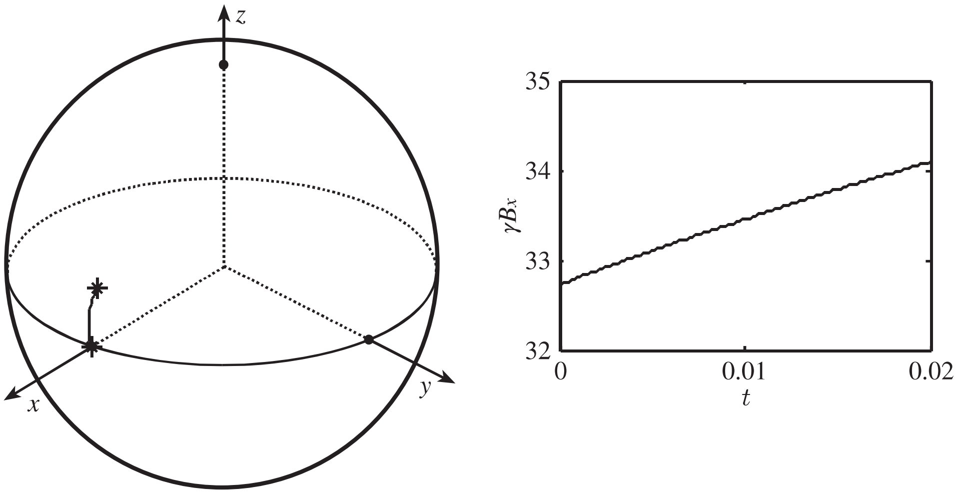

Figure 2.11 State transfer trajectory with a given time $T = 0.02$: (a) state transfer trajectory from $(|0\rangle + |1\rangle)/\sqrt{2}$ to $0.8|0\rangle + 0.6|1\rangle$ and (b) control field

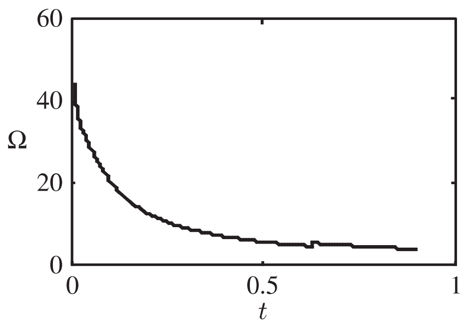

Figure 2.12 Values of Ω with T change

the variation of Ω with T in the situation with the initial state $(|0\rangle + |1\rangle)/\sqrt{2}$ and the target state $0.8|0\rangle + 0.6|1\rangle$, in which $\phi_1 = \phi_2 = 0$, $\theta_1 = \pi/2$, $\sin\theta_2 = 0.92$ *and* $\cos\theta_2 = 0.28$.

In fact it can be seen from Figure 2.11a that the intensities of the magnetic fields can be smaller than $A = \Omega/\gamma$ in practice because of the initial phase. For instance, in Figure 2.10 the initial phase of the control field is $\phi = \pi/4 - \omega_0 T/2 = 0.7354$ and the intensities of the magnetic fields during the whole evolution are $|A'| \leq A\cos(\phi)$ or $|A'| \leq 0.7416A$.

The numerical simulations demonstrate that control fields designed under different requirements can complete various kinds of transfer tasks between specific states, and the trajectory of a particular state transfer on the Bloch sphere is not unique. In practice the parameter Ω should be relatively small, and the operation should be fast enough to weaken the effect of decoherence. However, small control amplitude and fast operation time requirements are the parameters Ω and T, and should be selected carefully when designing the control fields. In general situations, for example, there is only the restriction on the maximum Ω, and the parameters can be selected according to Equations 2.22 and 2.26. If there is a requirement on finish time T, it is convenient to obtain parameters from Equations 2.27 and 2.30.

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