Tutorial Day 3

1 Classical mechanics - a short repetition

1.1 Lagrange equations

We consider the motion of material points. By a matetial point we mean an object of neglectible size (example: a planet moving around a star). Assume that there is a system of N points with positions: $q_1,\ q_2,\ \ ...,\ q_N$ and momenta: $\dot{q}_1,\ \dot{q}_2, ...,\ \dot{q}_N$. These quantities depend on time and are usually called generalized (canonical) coordinates. Such system is also characterised by a number of degrees of freedom s. Usually $s = 3N$.

The motion is characterised by Hamilton's principle: the evolution $q(t)$ of a system described by generalized coordinates between two specified states $q^{(1)} = q(t_1)$ and $q^{(2)} = q(t_2)$ at two specified times t_1 and t_2 is a stationary point (a point where the variation is zero) of the action functional

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \ ,$$

where $L(q, \dot{q}, t)$ is the Lagrangian function. If we replace $q(t)$ by $q(t) + \delta q(t)$, S will increase. Let $\delta q(t)$ be the variation of $q(t)$. Boundary conditions tell us that

$$\delta q(t_1) = \delta q(t_2) = 0$$

The action S changes according to:

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \ ,$$

so we can write Hamilton's principle as

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \ = 0$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

Since $\delta \dot{q} = \frac{d}{dt} q$, integrating the second term by parts we get

$$\delta S = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0$$

First term vanishes since $\delta q(t_1) = \delta q(t_2) = 0$. Then we obtain Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

If we have several degrees of freedom, then this can be generalized to

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

1.2 Langrange function - properties

- Notice that we can add to L a derivative of f with respect to t

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

  since

$$S = \int_{t_1}^{t_2} L'(q, \dot{q}, t) = \int_{t_1}^{t_2} L(q, \dot{q}, t) + \int_{t_1}^{t_2} \frac{df}{dt} dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

  The last terms dissapear in variation.

- Symmetries of the Lagrange function are connected to conserved quantities. If L does not depend on a particular q_i, then

$$\frac{\partial L}{\partial \dot{q}_i} = const.$$

  Examples: the homogenity of space implies momentum conservation; isotropy of space implies angular momentum conservation

- A symmetry of special kind is connected to the homogenity of time. It implies energy conservation

$$\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = const.$$

1.3 Langrange function for a system of particles - application

Usually L can be expressed as

$$L = T - U,$$

where T is the kinetic energy

$$T = \sum_i \frac{m_i v_i^2}{2}$$

and U is the potential energy.

Example: for a single particle in a potential U we have

$$L = \frac{mv^2}{2} - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial r}$$

$$m \frac{dv}{dt} = -\frac{\partial U}{\partial r}$$

In this way we have obtained Newton's equations.

1.4 Problems

1. Consider a simple pendulum of length l and mass m, operating in a gravitational field. Let its angle with the vertical be $\theta(t)$.

   a) write the Lagrange function

   b) derive the equations of motion

   c) solve the equations of motion for small oscillations around the equilibrium position

2.* Consider a pendulum made of a spring with a mass m on the end. The spring is arranged to lie in a straight line. The equilibrium length of the spring is l. Let the spring have length $l(t)+x(t)$, and let its angle with the vertical be $\theta(t)$. Assuming that the motion takes place in a vertical plane, find the equations of motion for x and $\theta(t)$

2 Assumptions of general relativity

2.1 Assumptions of general relativity

Assumptions (very simplified!)

1. Spacetime is a 4-dimensional differential manifold with metric (M, g)
2. The metric resembles Lorentz metric at any point (locally we can apply special relativity rules)
3. The metric satisfies Einstein equations
4. Particles follow geodesics (in case of photons the geodesics are null, in case of particles with mass there are timelike; geodesic - 'shortest' possible line)

A few rules which allow us to go from special to general relativity:

special relativity	general relativity
flat spacetime	curvature
Minkowski metric $\eta_{\mu\nu}$	general metric $g_{\mu\nu}$
$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$
derivative	absolute derivative
partial derivative	covariant derivative
....	....

Instead of using Einstein equations, we'll use Lagrange equations (note that particles follow geodesics). First we write the Lagrange function

$$L = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

and then use Lagrange equations. For simplicity we can take

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

We can also use the normalization condition (assumption 4): for particles with mass we have $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, for photons $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$.

Example: for a particle in Minkowski space ($c = 1$, $dim = 2$) we have

$$ds^2 = \eta_{\mu\nu}x^\mu x^\nu$$

$$ds^2 = dt^2 - dx^2$$

$$L = \dot{t}^2 - \dot{x}^2$$

Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{t}} = \frac{d}{dt}(2\dot{t}) = 2\ddot{t}, \quad \frac{\partial L}{\partial t} = 0, \quad \Rightarrow \quad \ddot{t} = 0$$

We proceed similarly for x. Then

$$t = As + B, \quad x = Cs + D$$

We also apply a normalization condition

$$\dot{t}^2 - \dot{x}^2 = 1$$

Note that if $v = \frac{dx}{dt}$, then $v = \frac{\dot{x}}{\dot{t}} = \frac{C}{A}$ and

$$A^2 - C^2 = A^2 - (Av)^2 = A^2(1 - v^2) = 1$$

$$A = \pm\frac{1}{1 - v^2}, \quad C = \pm\frac{v}{1 - v^2}$$

$$t = \pm\frac{1}{\sqrt{1 - v^2}}s + B, \quad x = \pm\frac{v}{\sqrt{1 - v^2}}s + D$$