2 Tutorial Day 2

Preliminaries/Poruszające się pręty wydają się krótsze, a poruszające się zegary działają wolniej.

Two events $A = (x_1, y_1, z_1, ct_1)$ and $B = (x_2, y_2, z_2, ct_2)$ are called *simultaneous* with respect to an inertial frame K if $\Delta t = t_2 - t1 = 0$. Now consider a second frame K' is related to K by a boost. One can make use of Lorentz transformation to get

$$\Delta x' = \gamma(\Delta x - v\Delta t), \qquad \Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

Therefore for $\Delta t = 0$ implies:

$$\Delta t' = -\gamma\frac{v}{c^2}\Delta x \neq 0 \qquad if \qquad x_1 \neq x_2$$

showing us the effect known as **relativity of simultaneity** : simultaneity of spatially separated points is not an absolute concept.

Consider now a clock at rest in K' making successive 'ticks' at events(x', y', z', ct'_1) and(x', y', z', ct'_2). Then one can evaluate the time difference **according to K** as :

$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x') = \gamma\Delta t' \qquad if \qquad \Delta x' = 0$$

This gives us

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \geq \Delta t'$$

which is known as **time dilation-** a moving clock appears to slow down. Equivalently a stationary clock in K appears to run slow according to the moving observer K'

Now consider a rod of length $l = \Delta x$ at rest in K. Again using the inverse transformation we have

$$l = \Delta x = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x' \qquad if \qquad \Delta t' = 0$$

The length of the rod with respect to K' is determined by considering simultaneous moments $t'_1 = t'_2$ at the end points,

$$l' = \Delta x' = \frac{l}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}.l \leq l$$

This is interpreted as the lenght of a rod being contracted when viewed by a moving observer, an effect known as **Lorentz–FitzGerald contraction**.

2.1 Problems

- Short formula based problems:

  1. The average lifetime of a π meson in its own frame of reference is 26.0 ns. (called as proper lifetime.) If the π meson moves with speed 0.95c with respect to the Earth, what is its lifetime as measured by an observer at rest on Earth? What is the average distance it travels before decaying as measured by an observer at rest on Earth?
  2. A rod of length L_0 moves with speed v along the horizontal direction. The rod makes an angle θ_0 with respect to the x' axis. Determine the length of the rod as measured by a stationary observer. Determine the angle θ the rod makes with the x axis.
  3. As seen from Earth, two spaceships A and B are approaching along perpendicular directions. If A is observed by a stationary Earth observer to have velocity u_y=-0.90c and B to have velocity $u_x = +0.90c$, determine the speed of ship A as measured by the pilot of ship B.

- If two separate events occur at the same time in some inertial frame S, prove that there is no limit on the time separations assigned to these events in other frames, but that their space separation varies from infinity to a minimum that is measure in S . With what speed must an observer travel in order that two simultaneous events at opposite ends of a 10-metre room appear to differ in time by 100 years.

- A supernova is seen to explode on Andromeda galaxy, while it is on on the western horizon observers A and B are walking past each other, A at $5km/hr$ towards east and B at $5km/hr$ towards the west. Given that Andromeda is about a million light years away, calculate the difference in time attributed to the supernova event by A and B. Who says it happened earlier?

- There are two twins, twin A who stays on the earth and twin B travels away from earth at speed $0.5c$ on their common birthday. They decide to each blow out candles exactly four years from B's departure.

    1. Suppose when A blows candle we call that as an event P. What moment in $B's$ time corresponds to the event P that consists of A blowing his candle out. And what moment in $A's$ time corresponds to the event Q that consists of B blowing her candle out?
    2. According to A which happened earlier, P or Q? And according to B?
    3. How long will A have to wait before he *sees* his twin blowing her candle out?

Preliminaries

[Note : Latin indices run from 0,1,2,3]

Special relativity helps in bringing space and time on equal footing and then you generally begin by exploring what a *spacetime* interval is , i.e, $ds^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$. The dynamics of a particle is governed by something called a **4-vector**. It is nothing fancy, just that the three components will define the spatial part and one components will define the time part. So we have $x^\mu = (ct, -x, -y, -z)$ or $x^\mu = (-ct, x, y, z)$ depending on what is the signature of the metric $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ or

$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Once we have this, we con construct other useful $4-vectors$ like *4-velocity*, *4-acceleration*, *4-momentum* etc.

One important quantity we need is the time measure by the observer using his/her own watch , known as , **proper time** generally denoted as $\Delta\tau = \sqrt{1 - \frac{v^2}{c^2}}\Delta t = \frac{1}{\gamma}\Delta t$. With the help of this we can define **4-velocity** of particle as

$$V^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{d\tau} = \gamma(\mathbf{v}, c)$$

Unlike coordinate time t, proper time τ is a true scalar parameter independent of intertial frame; hence components of 4-velocty V^μ transform as something called a contravariant 4-vector

$$V'^\mu = L^\mu_\nu V^\nu$$

One can proceed to define **4-acceleration** as

$$A^\mu = \frac{dV^\mu}{d\tau} = \gamma(\frac{d\gamma}{dt}\mathbf{v} + \gamma\frac{d\mathbf{v}}{dt}, c\frac{d\gamma}{dt})$$

It is generally assumed that each particle has a constant scalar mass associated with it, known as, **rest mass** m_0. The **4-momentum** of a partciel is defined to be the 4-vector having components:

$$P^\mu = (\mathbf{p}, \frac{E}{c})$$

where $\mathbf{p} = mv = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = momentum$ and $E = mc^2 = \frac{m_0 c}{\sqrt{1-\frac{v^2}{c^2}}}$

Note : Relativistic mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ is a frame dependent quantity. It is only the rest mass m_0 of a particle which is invariant.

2.2 Problems

- Let K' be a frame with velocity v' relative to K in the $x-direction$

1. Show that for a particle having velocity u', acceleration a' in the $x'-direction$ relative to K', its acceleration in K is

$$a = \frac{a'}{[\gamma(1+\frac{vu'}{c^2})]^3}$$

[Hint : Try to differentiate the relativistic formula for addition : $u = \frac{u'+v}{1+\frac{v.u'}{c^2}}$].]

2. Suppose there is hidden rocket in the observatory. You decide to take it for a ride. You leave Earth at t=0 with constant acceleration g at every moment relative to your instantaneous rest frame. Show that your motion relative to Earth is given by

$$x = \frac{c^2}{g}\left(\sqrt{1+\frac{g^2}{c^2}t^2} - 1\right)$$

[Hint: In your instantaneous rest frame you have $u' = 0$ and $a' = g$, use this in equation you derived before. Once you get a differential equation in v and t this integral might be useful : $\int_0^x \frac{1}{(1-x^2)^{3/2}}\,dx = \frac{x}{\sqrt{1-x^2}}$. Then you will again get a differential equation in x and t. In other words acceleration $a = \frac{dv}{dt}$, I integrate it once to get velocity v and then again to get position x. If you get stuck in any other integrals please consult us.]

3. You ofcourse took your watch with you. So in terms of the time measure by you(proper time τ) show that

$$x = \frac{c^2}{g}\left(cosh(\frac{g}{c}\tau) - 1\right)$$

[Hint : Proper Time is $\tau = \int_0^t \frac{1}{\gamma}\,dt$, substitute the value of v you got from before and then get an expression for τ. Now with help of this this proper time you can see how your motion is.]

4. Suppose you continue on you path for 10 years, decelerate with $g = 9.8ms^{-2}$ for another 10 years to come to rest, and return to

the observatory the same way, taking 40 years in total. How much will you tutors on Earth have aged on your return? How far, in light years, will you have gone from Earth? [Hint : 1 year=3.15e7 sec. **Bylibyśmy bardzo, bardzo, bardzo starzy.**]

- A particle is in hyperbolic motion along a world-line whose equations is given by

$$x^2 - c^2t^2 = a \qquad\qquad y = z = 0$$

Show that $\gamma = \frac{\sqrt{a^2+c^2t^2}}{a}$ and that proper time starting from $t = 0$ along the earth is given by $\tau = \frac{a}{c} cosh^{-1} \frac{ct}{a}$. Evaluate the particle's 4-velocity V^μ and 4-acceleration A^μ. [Hint : Just differentiate the equation and obtain a equation in v and t. Then as before solve for proper time as $\tau = \int_0^t \frac{1}{\gamma}\, dt$. Remember 4-velocity is $V^\mu = \frac{dx^\mu}{dt} = \gamma(\mathbf{v}, c)$. Here you only have motion in $x - direction$ so you will get something like $(\gamma v, 0, 0, c)$. So you just need to substitute the value of v and γ which you have already obtained.]

- * A particle has momentum $\mathbf{p}$ and energy E in a frame K

  1. If K' is an inertial frame having velocity v relative to K, use the transformation law of the momentum 4-vector $P^\mu = (\mathbf{p}, \frac{E}{c})$ to show

  $$E' = \gamma(E - \mathbf{v}.\mathbf{p}) \qquad \mathbf{p}'_\perp = \mathbf{p}_\perp \qquad \mathbf{p}'_\parallel = \gamma(\mathbf{p}_\parallel - \frac{E}{c^2}\mathbf{v})$$

  where $\mathbf{p}_\perp$ and $\mathbf{p}_\parallel$ are the perpendicular and parallel component of $\mathbf{p}$ with respect to $\mathbf{v}$.

  2. If the particle is a photon, use the transformations to derive the aberration formula

  $$cos\theta' = \frac{cos\theta - v/c}{1 - cos\theta(v/c)}$$

  where θ is the angle between p and v.

* If time time permits.