$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right) \delta q \, dt$$

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0}$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0}$$

PROPERTIES:

1) $$L \to L' = L + \frac{df}{dt}$$

$$S' = \int_{t_1}^{t_2} \left(L + \frac{df}{dt} \right) dt = \underbrace{\int_{t_1}^{t_2} L}_{S} + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

2) Symmetries <-> conserved quantities

q_a $$\frac{\partial L}{\partial q_a} = 0$$

$$\cancel{\frac{\partial L}{\partial q_a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_a}\right) = 0 \qquad \frac{\partial L}{\partial \dot q_a} = \text{const}$$

3) homogeneity of time -> energy conservation.

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot q_i + \frac{\partial L}{\partial \dot q_i}\ddot q_i =$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\dot q_i + \frac{\partial L}{\partial \dot q_i}\ddot q_i =$$

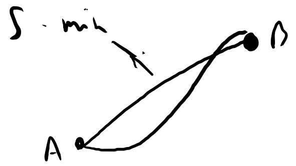
$$= \frac{d}{dt}\left(\left(\frac{\partial L}{\partial \dot q_i}\dot q_i\right)\right)$$

$$\frac{d}{dt}\Big(\overbrace{L - \frac{\partial L}{\partial \dot q_i}\dot q_i}^{\text{const.}}\Big) = 0$$

1. LAGRANGE EQUATIONS
2. ASSUMPTIONS OF GR
3. SCHWARZSCHILD METRIC
4. TRAJECTORIES

N positions $$\left.\begin{matrix} q_1 & q_2 & q_3 \dots & q_N \\ \dot q_1 & \dot q_2 & \dot q_3 & \dot q_N \end{matrix}\right\}$$ Generalised (canonical) coordinates

S - number of degrees of freedom



$q_i(t)$

$q^{(1)} = q(t_1)$ A

$q^{(2)} = q(t_2)$ B

$$S = \int_{t_1}^{t_2} L(q, \dot q, t)\, dt$$

HAMILTON'S (ACTION) PRINCIPLE:

S - action functional

replace $q(t)$ by $q(t) + \delta(q)$

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot q + \delta \dot q, t) = \int_{t_1}^{t_2} L(q, \dot q, t) =$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q \right) dt =$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q + \cancel{\frac{\partial L}{\partial \dot q}\delta q \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q \, dt$$

For a single particle

$L = \frac{1}{2}m\dot{x}^2 - U = T - U$

kinetic energy, potential

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\qquad$ $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$m\ddot{x} + \frac{\partial U}{\partial x} = 0$ $\qquad$ $m\ddot{x} = -\frac{\partial U}{\partial x}$

ASSUMPTIONS - GR

$m_g = m_i$

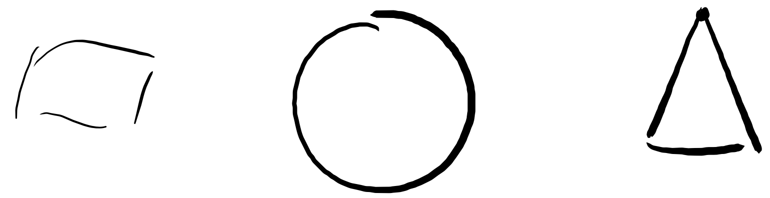
$m\frac{d^2\vec{r}}{dt^2} = \vec{F} + m\vec{g}$

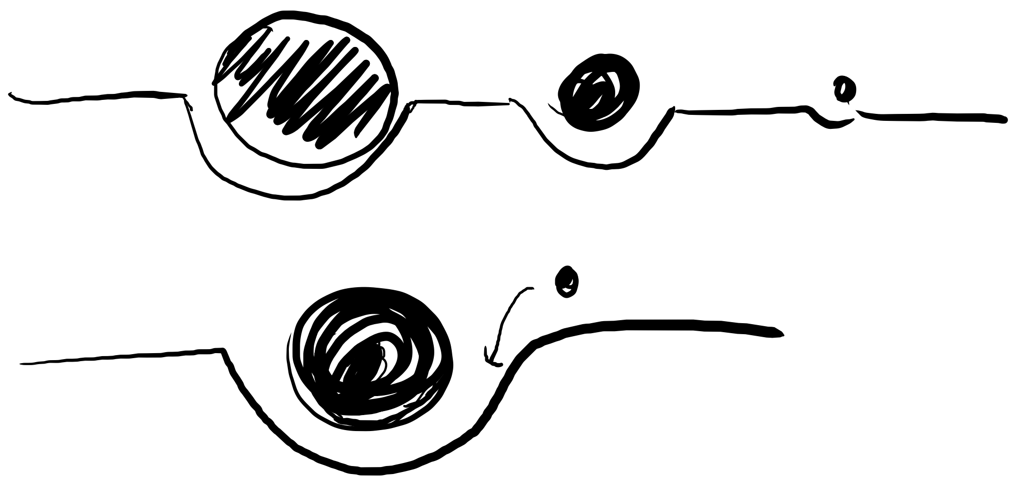
$\vec{r}' = \vec{r} - \frac{1}{2}\vec{g}t^2 \Rightarrow \vec{r} = \vec{r}' + \frac{1}{2}\vec{g}t^2$

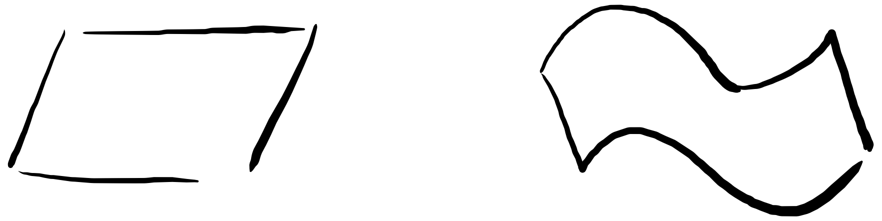
$m\frac{d^2}{dt^2}\left(\vec{r}' + \frac{1}{2}\vec{g}t^2\right) = \vec{F} + m\vec{g}$

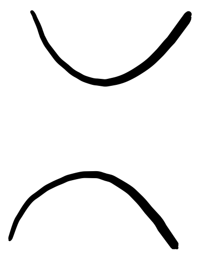
Assumptions:

1. Spacetime is a curved manifold with metric (M, g)





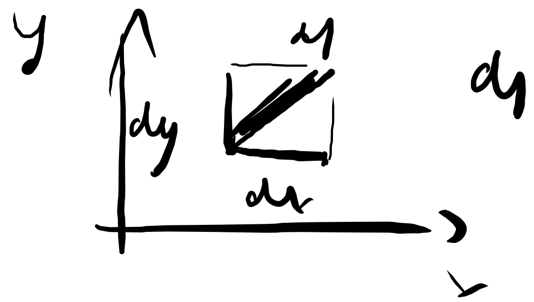




1. Flat space

$ds^2 = dx^2 + dy^2$

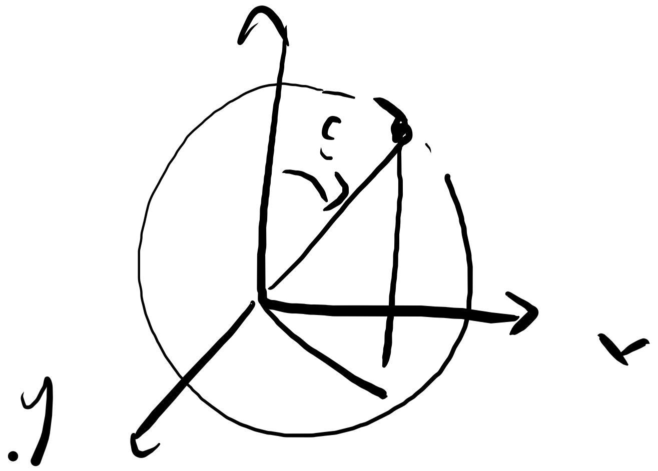
$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$ds^2 = dx^2 + dy^2$

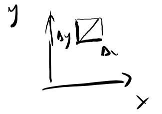
2. Sphere

$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$



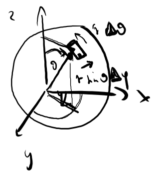
1. Flat space

$ds^2 = dx^2 + dy^2$



$\Delta s^2 = \Delta x^2 + \Delta y^2$

2.



$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

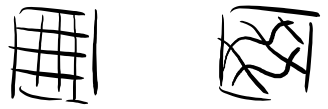
$\Delta s^2 = \Delta\theta^2 + \sin^2\theta\, \Delta\varphi^2$

4. $ds^2 = -dt^2 + \frac{1}{x^4} dx^2$

$(t, x) \to (t', x')$ $\qquad x' = \frac{1}{x}$

$dx' = -\frac{1}{x^2} dx$ $\qquad dt' = dt$

$ds^2 = -dt^2 + dx'^2$ $\qquad x \neq 0$



$ds^2 = dx^2 + dy^2$

$ds = \sqrt{(dx)^2 + (dy)^2}$

$ds = \sqrt{\frac{(dx)^2}{(ds)^2} + \frac{(dy)^2}{(ds)^2}}\, ds$

$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, ds$

$ds = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, ds$

$ds = \sqrt{\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{ds}\right)^2}\, ds$

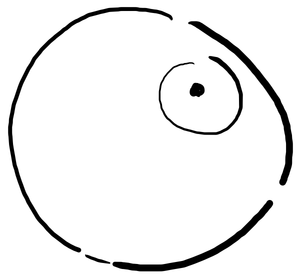
$(\Delta x, \Delta y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

$\Delta x^2 + \Delta y^2$

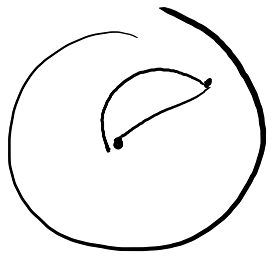
$ds^2 = 1 d\theta^2 + \sin^2\theta\, d\varphi^2$

$\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = g_{\mu\nu}$

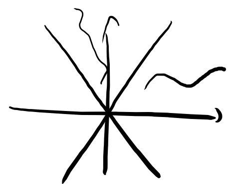
2. Locally we can special relativity



3. Einstein equations



4. Particles follow geodesics



time-like - particles with mass

null - photons

SPECIAL	GENERAL
flat-space	curved
Minkowski metric	general metric
$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$g_{\mu\nu}$