Tutorial Day 3

1 Classical mechanics - a short repetition

1.1 Lagrange equations

We consider the motion of material points. By a matetial point we mean an object of neglectible size (example: a planet moving around a star). Assume that there is a system of N points with positions: $q_1, \; q_2, \quad ..., \; q_N$ and momenta: $\dot{q}_1, \; \dot{q}_2, ..., \; \dot{q}_N$. These quantities depend on time and are usually called generalized (canonical) coordinates. Such system is also characterised by a number of degrees of freedom s. Usually $s = 3N$.

The motion is characterised by Hamilton's principle: the evolution $q(t)$ of a system described by generalized coordinates between two specified states $q^{(1)} = q(t_1)$ and $q^{(2)} = q(t_2)$ at two specified times t_1 and t_2 is a stationary point (a point where the variation is zero) of the action functional

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \; ,$$

where $L(q, \dot{q}, t)$ is the Lagrangian function. If we replace $q(t)$ by $q(t) + \delta q(t)$, S will increase. Let $\delta q(t)$ be the variation of $q(t)$. Boundary conditions tell us that

$$\delta q(t_1) = \delta q(t_2) = 0$$

The action S changes according to:

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \; ,$$

so we can write Hamilton's principle as

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \; = 0$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

Since $\delta \dot{q} = \frac{d}{dt} q$, integrating the second term by parts we get

$$\delta S = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0$$

First term vanishes since $\delta q(t_1) = \delta q(t_2) = 0$. Then we obtain Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

If we have several degrees of freedom, then this can be generalized to

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

1.2 Langrange function - properties

- Notice that we can add to L a derivative of f with respect to t

  $$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

  since

  $$S = \int_{t_1}^{t_2} L'(q, \dot{q}, t) = \int_{t_1}^{t_2} L(q, \dot{q}, t) + \int_{t_1}^{t_2} \frac{df}{dt} dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

  The last terms dissapear in variation.

- Symmetries of the Lagrange function are connected to conserved quantities. If L does not depend on a particular q_i, then

  $$\frac{\partial L}{\partial \dot{q}_i} = const.$$

  Examples: the homogenity of space implies momentum conservation; isotropy of space implies angular momentum conservation

- A symmetry of special kind is connected to the homogenity of time. It implies energy conservation

  $$\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = const.$$

1.3 Langrange function for a system of particles - application

Usually L can be expressed as

$$L = T - U,$$

where T is the kinetic energy

$$T = \sum_i \frac{m_i v_i^2}{2}$$

and U is the potential energy.

Example: for a single particle in a potential U we have

$$L = \frac{mv^2}{2} - U$$

$$\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial r}$$

$$m \frac{dv}{dt} = -\frac{\partial U}{\partial r}$$

In this way we have obtained Newton's equations.

1.4 Problems

1. Consider a simple pendulum of length l and mass m, operating in a gravitational field. Let its angle with the vertical be $\theta(t)$.

   a) write the Lagrange function

   b) derive the equations of motion

   c) solve the equations of motion for small oscillations around the equilibrium position

2.* Consider a pendulum made of a spring with a mass m on the end. The spring is arranged to lie in a straight line. The equilibrium length of the spring is l. Let the spring have length $l(t)+x(t)$, and let its angle with the vertical be $\theta(t)$. Assuming that the motion takes place in a vertical plane, find the equations of motion for x and $\theta(t)$

2 Assumptions of general relativity

2.1 Assumptions of general relativity

Assumptions (very simplified!)

1. Spacetime is a 4-dimensional differential manifold with metric (M, g)

2. The metric resembles Lorentz metric at any point (locally we can apply special relativity rules)

3. The metric satisfies Einstein equations

4. Particles follow geodesics (in case of photons the geodesics are null, in case of particles with mass there are timelike; geodesic - 'shortest' possible line)

A few rules which allow us to go from special to general relativity:

special relativity	general relativity
flat spacetime	curvature
Minkowski metric $\eta_{\mu\nu}$	general metric $g_{\mu\nu}$
$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$	$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$
derivative	absolute derivative
partial derivative	covariant derivative
....	....

Instead of using Einstein equations, we'll use Lagrange equations (note that particles follow geodesics). First we write the Lagrange function

$$L = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

and then use Lagrange equations. For simplicity we can take

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

We can also use the normalization condition (assumption 4): for particles with mass we have $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, for photons $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$.

Example: for a particle in Minkowski space ($c = 1$, $dim = 2$) we have

$$ds^2 = \eta_{\mu\nu}x^\mu x^\nu$$

$$ds^2 = dt^2 - dx^2$$

$$L = \dot{t}^2 - \dot{x}^2$$

Lagrange equations:

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{t}} = \frac{d}{dt}(2\dot{t}) = 2\ddot{t}, \quad \frac{\partial L}{\partial t} = 0, \quad \Rightarrow \quad \ddot{t} = 0$$

We proceed similarly for x. Then

$$t = As + B, \quad x = Cs + D$$

We also apply a normalization condition

$$\dot{t}^2 - \dot{x}^2 = 1$$

Note that if $v = \frac{dx}{dt}$, then $v = \frac{\dot{x}}{\dot{t}} = \frac{C}{A}$ and

$$A^2 - C^2 = A^2 - (Av)^2 = A^2(1 - v^2) = 1$$

$$A = \pm\frac{1}{1 - v^2}, \quad C = \pm\frac{v}{1 - v^2}$$

$$t = \pm\frac{1}{\sqrt{1 - v^2}}s + B, \quad x = \pm\frac{v}{\sqrt{1 - v^2}}s + D$$

3 Schwarzschild spacetime

3.1 Metric and properties

The geometry of empty space around a spherically symmetric object (e.g a star or a black hole) is described by the metric (from now on the equations will be labelled for clarity)

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \qquad (1)$$

The coordinate ranges are $t \in (-\infty, \infty)$, $r \in (0, \infty)/\{2M\}$, $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$. The parameter M is interpreted as mass. The units are geometrised. The surface $r = 2M$ is the event horizon.

Notice that there are two singularities: $r = 0$ and $r = 2M$. The second one can be avoided by a coordinate transformation, while for the first one it is not possible.

3.2 Problems

3. Find the metric of a flat space in spherical coordinates

4. Derive the geodesic equations on a unit sphere with metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ from Lagrange equations.

4 Geodesics in Schwarzschild spacetime

4.1 The equations of motion

We start by writing the Lagrange function

$$L = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2, \tag{2}$$

and the equations of motion

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0, \tag{3a}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0, \tag{3b}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0, \tag{3c}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0. \tag{3d}$$

Due to spherical symmetry we can set $\theta = \pi/2$. Then the induced metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2 d\phi^2. \tag{4}$$

From (3a) and (3d) we find

$$\frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)\dot{t}^2\right] = 0, \tag{5}$$

$$\frac{d}{d\tau}\left(r^2\dot{\phi}^2\right) = 0. \tag{6}$$

Therefore, we get two constants of motion

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E, \tag{7}$$

$$r^2\dot{\phi} = L. \tag{8}$$

The parameter E has an interpretation os energy per unit mass, while L angular momentum per unit mass (like in classical mechanics: conserved quantities are related to symmetries; L does not depend on time and does not change after rotation). Then the four-velocities $\dot{x}^t$ i $\dot{x}^\phi$ expressed by E and L are:

$$\dot{t} = \frac{E}{1-\frac{2M}{r}}, \tag{9}$$

$$\dot{\phi} = \frac{L}{r^2}. \tag{10}$$

Equation (3b) is quite complicated. Because of that, to simplify the problem, we will use normalization condition

$$\left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1. \tag{11}$$

Inserting (11), (9) and (10) into this formula we get

$$\frac{E^2}{1-\frac{2M}{r}} - \frac{\dot{r}^2}{1-\frac{2M}{r}} - \frac{L^2}{r^2} = 1. \tag{12}$$

Assuming $r \neq 2M$, after algebraic simplifications we have

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right) = E^2. \tag{13}$$

We rewrite the above equation to get

$$\frac{E^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right)-1\right) \tag{14}$$

and define effective potential

$$V_{ef} = \frac{1}{2}\left(\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right)-1\right). \tag{15}$$

Our goal is to find the trajectory, i.e. describe how r depends on ϕ. To achieve that, we replace $\dot{r}$ by $\frac{dr}{d\phi}$ applying (10), and use the result in equation (13):

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{L}{r^2}\frac{dr}{d\phi}, \tag{16}$$

$$\frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right),$$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(E^2-1\right)\frac{r^4}{L^2} + \frac{2Mr^3}{L^2} - r^2 + 2Mr. \tag{17}$$

Now we change variables in (17) from r to u:

$$u = \frac{1}{r}, \tag{18}$$

then

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi},$$

$$r^4\left(\frac{du}{d\phi}\right)^2 = \left(E^2 - 1\right)\frac{r^4}{L^2} + \frac{2Mr^3}{L^2} - r^2 + 2Mr,$$

and finally

$$\left(\frac{du}{d\phi}\right)^2 = 2Mu^3 - u^2 + \frac{2M}{L^2}u - \frac{1-E^2}{L^2}. \tag{19}$$

The variables t and τ can be obtained using:

$$\frac{d\tau}{d\phi} = \frac{1}{Lu^2}, \tag{20}$$

$$\frac{dt}{d\phi} = \frac{E}{Lu^2\left(1-2Mu\right)}. \tag{21}$$

4.2 Bounded trajectories

Trajectories turn out to be bounded if $E^2 < 1$ and unbounded if $E^2 \geqslant 1$. Now we consider the first case $E^2 < 1$.

To sum up, the trajectories are decsribed by

$$\left(\frac{du}{d\phi}\right)^2 = f(u), \tag{22}$$

where

$$f(u) = 2Mu^3 - u^2 + \frac{2M}{L^2}u - \frac{1-E^2}{L^2}. \tag{23}$$

The solutions depend on the roots of $f(u) = 0$. Function f is a polynomial of third degree, so there are two possibilities: it can have either three real roots u_1, u_2, u_3, or one real root u_1 and two complex roots such that $u_2^* = u_3$:

$$f(u) = 2M(u-u_1)(u-u_2)(u-u_3). \tag{24}$$

From (24) and (23):

$$u_1u_2u_3 = \frac{1-E^2}{2ML^2}, \tag{25}$$

$$u_1 + u_2 + u_3 = \frac{1}{2M}. \tag{26}$$

Assuming $1-E^2 > 0$ we find that $f(0) < 0$ for $u = 0$ and $f \to \infty$ for $u \to \infty$. Then $f(u) = 0$ has at least one real root. Possibilities (1):

(a) $u_1 < u_2 < u_3$

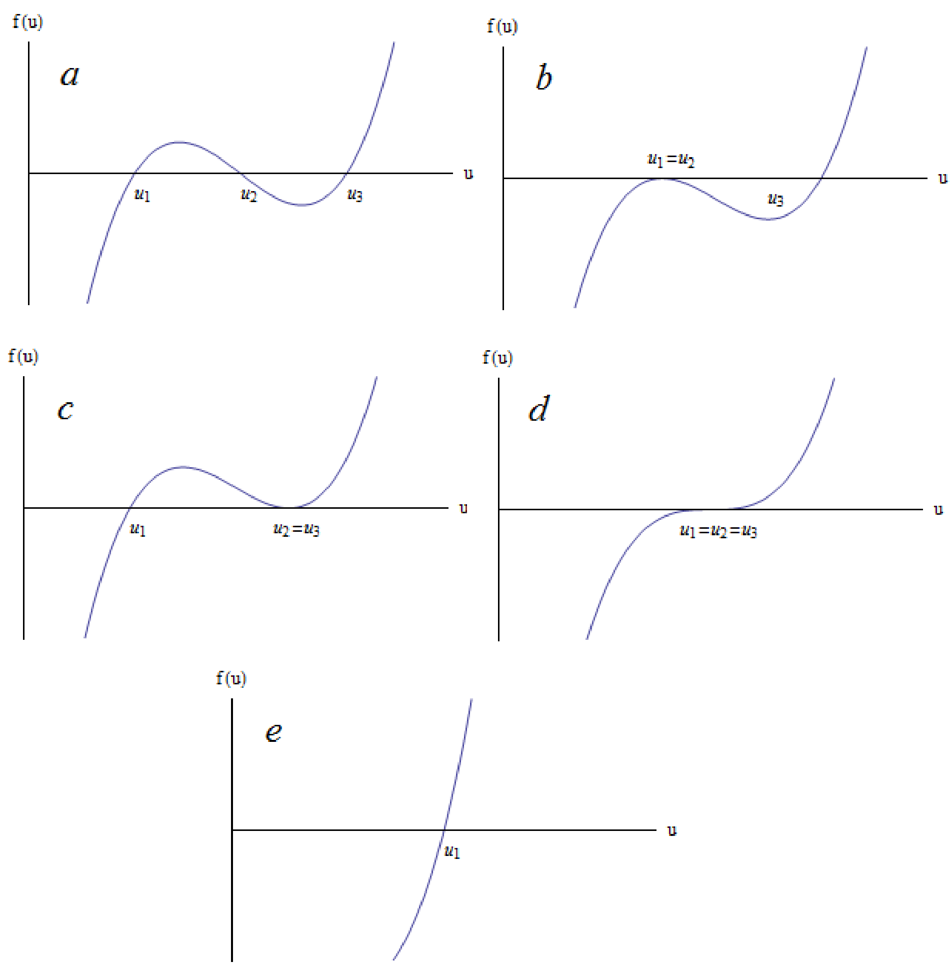
All three roots are real. There are two classes of orbits. The first class includes orbits which oscillate between u_1 and u_2 and correspond to keplerian orbits in Newtonian limit). The second class includes orbits which fall into black hole from u_3.

(b) $u_1 = u_2 < u_3$
First class orbits are round and stable. Second class orbits fall into the black hole.

(c) $u_1 < u_2 = u_3$
First class orbits fall into singularity in $r = 0$ starting from $u = u_1$ to u_3, spiraling around infinite times. Second class orbits fall into singularity, starting from u_3

(d) $u_1 = u_2 = u_3$
Orbits become round and unstable

(e) $u_1 \in R$, $u_2 = u_3 \in Z$
Orbits fall into singularity from some point



Rysunek 1: Possible solutions of $f(u) = 0$, where f is given by(23).

4.2.1 First class orbits

The first class embraces orbits from cases (a), (b), (c) and (d) when all three solutions $f(u)=0$ are real and positive. Then we introduce notation

$$u_1 = \frac{1}{l}\left(1-e\right), \tag{27a}$$

$$u_2 = \frac{1}{l}\left(1+e\right), \tag{27b}$$

$$u_3 = \frac{1}{2M} - \frac{2}{l}. \tag{27c}$$

where u_3 was derived from (26). Parameter l is a positive constant. Parameter e is interpreted as the excentricity of the orbit. The variable u must be real and positive, so e must satisfy condition $0 \leqslant e < 1$. We introduce notation

$$\mu = \frac{M}{l}. \tag{28}$$

The solutions of $f(u)=0$ satisfy (1)

$$\frac{1}{l}\left(1-e\right) \leqslant \frac{1}{l}\left(1+e\right) \leqslant \frac{1}{2M} - \frac{2}{l}. \tag{29}$$

Then

$$\mu \leqslant \frac{1}{2\left(3+e\right)}, \tag{30}$$

$$1 - 6\mu - 2\mu e \geqslant 0. \tag{31}$$

Inserting (27) into equation (24) and comparing the result with (23) we can express parameters E, L by e, l:

$$f(u) = 2M\left(u - \frac{1}{l}\left(1-e\right)\right)\left(u - \frac{1}{l}\left(1+e\right)\right)\left(u - \left(\frac{1}{2M} - \frac{2}{l}\right)\right) =$$

$$= 2Mu^3 - u^2 + \left(\frac{2}{l} - \frac{6M}{l^2} - \frac{2e^2M}{l^2}\right)u + \left(-\frac{1}{l^2} + \frac{e^2}{l^2} + \frac{4M}{l^3} - \frac{4e^2M}{l^3}\right),$$

$$\frac{M}{L^2} = \frac{1}{lM}\left[l - \mu\left(3+e^2\right)\right], \tag{32}$$

$$\frac{1-E^2}{L^2} = \frac{1}{l^2}\left(1-4\mu\right)\left(1-e^2\right). \tag{33}$$

Inequalities (30) i (31) guarantee that the right sides of these equations are positive. Now we solve (22). We change variables

$$u = \frac{1}{l}\left(1 + e\cos\chi\right), \tag{34}$$

The solution can be written using an elliptic integral defined as

$$F(\psi, k) = \int_0^{\psi} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}}. \tag{35}$$

and takes the form

$$\phi = \frac{2}{(1 - 6\mu + 2\mu e)^{\frac{1}{2}}} F\left(\frac{1}{2}\pi - \frac{1}{2}\chi, k\right), \tag{36}$$

where

$$k^2 = \frac{4\mu e}{1 - 6\mu + 2\mu e} \tag{37}$$

Here the starting point is $\phi = 0$ in the aphelium, where $\chi = \pi$, which follows from (34).

Case $e = 0$

Corresponds to the case (b). The orbit is round with

$$r \stackrel{\text{def}}{=} r_c = l. \tag{38}$$

This leads to expressions for L i E:

$$\frac{1}{L^2} = \frac{1}{lM} (1 - 3\mu), \tag{39}$$

$$\frac{E^2}{L^2} = \frac{1}{lM} (2\mu - 1)^2. \tag{40}$$

From these equations we get

$$r_c = \frac{L^2}{2M} \left(1 \pm \sqrt{1 - \frac{12M^2}{L^2}}\right). \tag{41}$$

A circular orbit exists only if $L/M > 2\sqrt{3}$. The orbit with larger r_c is stable, since the effective potential achieves a minimal value. For smaller r_c the orbit is unstable.

Case $2\mu(3 + e) = 1$

In this case the radii at aphelium and perihelium are equal

$$r_p = \frac{l}{1 + e} = 2M\frac{3 + e}{1 + e}, \tag{42}$$

$$r_{ap} = 2M\frac{3 + e}{1 - e}. \tag{43}$$

Maximal and minimal values of e lead to a constraint

$$4M \leqslant r_p < 6M. \tag{44}$$

Then the solution is

$$\phi = -\frac{1}{\sqrt{\mu e}} \log\left(\tan\frac{\chi}{4}\right). \tag{45}$$

It shows that $\phi \to \infty$ when $\chi \to 0$ as the orbit approaches perihelium. It spirals infinitely without reaching r_p.

4.3 Second class orbits

The second class obits also appear in cases (a), (b), (c), (d). Trajectories start at u_3^{-1}, then fall into singularity at $r = 0$. The aphelium lies outside the horizon, which follows from (26):

$$u_1 + u_2 > 0 \Rightarrow u_3 < \frac{1}{2M}. \tag{46}$$

In this case to solve (22) we introduce

$$u = \left(\frac{1}{2M} - \frac{2}{l}\right) + \left(\frac{1}{2M} - \frac{3+e}{l}\right)\tan^2\frac{\xi}{2}. \tag{47}$$

Then at the aphelium and the singularity the value of ξ is

$$u = u_3 = \frac{1}{2M} - \frac{2}{l},\ \xi = 0, \tag{48}$$

$$u \to \infty,\ \xi \to \pi. \tag{49}$$

The solution can be expressed by elliptic integrals

$$\phi = \frac{2}{(1 - 6\mu + 2\mu e)^{1/2}} F\left(\frac{\xi}{2}, k\right). \tag{50}$$

At the aphelium $\phi = 0$ and $\xi = 0$. At the singularity ϕ takes a finite value:

$$\phi_0 = \frac{2}{(1 - 6\mu + 2\mu e)^{1/2}} K(k), \tag{51}$$

where $K(k)$ is an elliptic integral

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\gamma}{\sqrt{1 - k^2\sin^2\gamma}}. \tag{52}$$

Case $e = 0$
If $e = 0$, $k = 0$. Then

$$\xi = (1 - 6\mu)^{1/2}(\phi - \phi_0). \tag{53}$$

From (47) we find that this corresponds to the value of u

$$u = \frac{1}{l} + \left(\frac{1}{2M} - \frac{3}{l}\right)\left(1 + \tan^2\frac{\xi}{2}\right) = \frac{1}{l} + \left(\frac{1}{2M} - \frac{3}{l}\right)\sec^2\left(\frac{1}{2}(1 - 6\mu)^{1/2}(\phi - \phi_0)\right). \tag{54}$$

These orbits are not round, but reach singularity at ϕ:

$$\phi - \phi_0 = \frac{\pi}{(1 - 6\mu)^{1/2}}. \tag{55}$$

Because of that we consider the $e = 0$ i $\mu = 1/6$ independently. Then all three roots u_1, u_2, u_3 take the value $1/6M$. This leads to

$$\left(\frac{du}{d\phi}\right)^2 = 2M\left(u - \frac{1}{6M}\right)^3, \tag{56}$$

$$u = \frac{1}{6M} + \frac{2}{M(\phi - \phi_0)}. \tag{57}$$

This orbit falls into singularity, spiralling around infinite times.

Case $2\mu(3 + e) = 1$

In this case using (47) is not appriopriate since the coefficient near $\tan^2(\xi/2)$ vanishes. We introduce another variable

$$u = \frac{1}{l}\left(1 + e + 2e\tan^2\frac{\xi}{2}\right). \tag{58}$$

Then u_1, u_2, u_3 are:

$$u_1 = \frac{1-e}{l}, \tag{59}$$

$$u_2 = u_3 = \frac{1+e}{l}. \tag{60}$$

From (58) it follows that the orbit reaches singularity for $\xi = \pi$. For u_2 and u_3 $\xi = 0$. The solution is

$$\phi = -\frac{1}{\sqrt{\mu e}}\log\left(\tan\frac{\xi}{4}\right). \tag{61}$$

As we approach the singularity $\xi \to \pi$ i $\phi \to 0$. At the aphelium $\xi \to \infty$ and $\phi \to 0$. The orbit approaches aphelium assymptotically, spiraling around the central mass infinite times. It's similar to the first class orbits, but in that case this value corresponds to perihelium.

4.4 Problems

5. Find the first class orbits by solving equation

$$\left(\frac{du}{d\phi}\right)^2 = f(u),$$

where

$$f(u) = 2Mu^3 - u^2 + \frac{2M}{L^2}u - \frac{1-E^2}{L^2}.$$

Use parametrization

$$f(u) = 2M(u - u_1)(u - u_2)(u - u_3),$$

$$u_1 = \frac{1}{l}(1 - e), \quad u_2 = \frac{1}{l}(1 + e), \quad u_3 = \frac{1}{2M} - \frac{2}{l}$$

and introduce a new variable

$$u = \frac{1}{l}(1 + e\cos\chi) \ .$$

Pay attention to the special cases $e = 0$ and $2\frac{M}{l}(3+e) = 1$.

6. Find the second class orbits (similarly to the first class orbits)

   a) in the case $2\frac{M}{l}(3+e) \neq 1$, introducing a new variable

   $$u = \left(\frac{1}{2M} - \frac{2}{l}\right) + \left(\frac{1}{2M} - \frac{3+e}{l}\right)\tan^2\frac{\xi}{2}.$$

   Pay attention to the special cases $e = 0$.

   b) in the special case $2\frac{M}{l}(3+e) = 1$, introducing a new variable

   $$u = \frac{1}{l}\left(1 + e + 2e\tan^2\frac{\xi}{2}\right).$$

4.5 Mathematica problems

Print the the first class orbits, following the steps below. The goal is to show analytical and numerical solutions on one plot and compare them

1. Set the Lagrange function L according to (4) for $\theta = \frac{\pi}{2}$).

2. Calculate $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$, $\frac{\partial L}{\partial \dot{\phi}}$, $\frac{\partial L}{\partial \dot{t}}$

3. Set the three equations of motion using (7) and (8) and (3b)

4. Introduce notation e, l and μ. Express L and E in terms of e, l, M, E (equations (32) and (33). Choose some partiular values of M, e, l as an example, e.g. $l = 2$, $e = 0.2$, $M = 3/14$.

5. Plot the analytical solution (36) using the ParametricPlot command. Hint: the cartesian coordinates can be expressed as $(u\cos\phi, u\sin\phi)$, where u is given by (34)

6. Uuse NDSolve command to solve the equations obtained in step 3. directly for initial conditions $r(0) = r_0$, $\phi(0) = 0$, $t(0) = 0$. Note that there might be problems with the acurracy of the solution. To fix them use Method→"ImplicitRungeKutta" and MaxSteps→ ∞ options. Choose r_0 so that the numerical solutions can be easily compared to analytical ones (e.g. r_0 close to aphelium). Also the initial condition $r'(0)$ is needed. To find it, solve the condition (11) for $\dot{r}$

7. Compare the analytical solutions obtained in step 5. with numerical solutions obtained in 6. Plot the circles corresponding to the event horizon, aphelium and perihelium(the orbit must lie between the last two)

8. Repeat the calculations for different values of parameters l and e.