Obserwatorium Astronomiczne
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Relativistic Astrophysics

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Abstract

The goal would be to apply our knowledge of special relativity to problems and grasp the conceptual framework of Einstein Special Theory of Relativity. We would tackle problems involving Length contraction , Time dilation, paradoxes in special relativity and how objects appear rotated when moving close to speed of light. We will use *Mathematica* to visualize special relativity and use this software to understand spacetime diagrams better.

Introduction

If you know a bit of Special Relativity good. If you have forgotten and don't know then *bardzo dobrze*. This is because you now have the opportunity to learn the fundamentals from the starting and in a manner which suits you. During the lectures and tutorials the only request is to ask questions. As by asking questions you have a chance to learn something new and we have a chance to learn something better. If you get any 'stupid' physics questions please ask them too(*Mówią, że nie ma głupich pytań, tylko głupie odpowiedzi*).The tutorials include preliminaries to help you remember few stuff and also help with the problems.

1 Tutorial Day 1

Preliminaries/Zabawa z matrycami

The Lorentz group. Before Einstein there already existed the transformations relating two frames moving with respect to each other. They were the non-relativistic Galilean Transformations. The Galilean transformations do not preserve the light cone at the origin :

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = c^2 \Delta t^2$$

where $\Delta x = x_2 - x_1$ and c is speed of light.

The correct transformations that achieve this important property preserve the metric of **Minkowski space**

$$\begin{aligned} \Delta s^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \\ &= \Delta \mathbf{x}^T G \Delta \mathbf{x} \end{aligned} \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \quad \Delta\mathbf{x} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ c\Delta t \end{pmatrix} \quad G = [g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The transformations that one wants in order to get an invariant spacetime interval is :

$$\mathbf{x'} = \mathbf{Lx} + \mathbf{a}$$

The invariance law we seek $\Delta s'^2 = \Delta s^2$ implies :

$$\Delta\mathbf{x}'^T G \Delta\mathbf{x}' = \Delta\mathbf{x}^T L^T G L \Delta\mathbf{x} = \Delta\mathbf{x}^T G \Delta\mathbf{x}$$

As this equation holds for arbitary $\Delta\mathbf{x}$, the 4×4 matrix L must satisfy the equation

$$G = L^T G L \tag{2}$$

Such linear transformations, having $\mathbf{a} = 0$ are called **Lorentz transformations** while the general transformations for arbitrary $\mathbf{a}$ are called **Poincaré transformations**. The corresponding groups are known as **Lorentz group** and **Poincaré group**. One of the essence of special relativity is that all laws of physics are Poincaré invariant.

1.1 Problems

- i) Show that

$$[L'^{\mu}_{\nu}] = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\gamma & \gamma \\ \alpha & \beta & -\gamma & 1+\gamma \end{pmatrix}$$

  where $\gamma = \frac{1}{2}(\alpha^2 + \beta^2)$ is a Lorentz transformation for all values of α and β[Hint : Use Eqn 2].

Preliminaries/Dobre wspomnienia z transformacji Lorentza

We know that if a frame S' is moving with velocity v in positive x-direction, then the two frames are related to each other by these Lorentz transformations :

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t - \frac{vx}{c^2})$$

And the inverse transformations are obtained by $v \longrightarrow -v$ as :

$$x = \gamma(x' + vt') \qquad y = y' \qquad z = z' \qquad t' = \gamma(t + \frac{vx}{c^2})$$

where in both the above equations we have $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ known as *Lorentz factor*. Also the parameter v has the role of a relative velocity between two frames as the spatial origin please note$(x' = 0, y' = 0, z' = 0)$ in the primed frame satisfies the equation $x = vt$ in unprimed frame. This relative velocity v is always less than c acting as a limiting velocity for material particles.(See figure 1)

Now thinking a bit component wise(składnik wektora). Suppose there is a particle with velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to a frame K, and $\mathbf{u'} = (u'_x, u'_y, u'_z)$ with respect to K'. Then :

$$u_x = \frac{dx}{dt}, \qquad u'_x = \frac{dx'}{dt'}, \qquad u_y = \frac{dy}{dt} \qquad ....$$

Via the Lorentz transformations one can arrive at

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \qquad u'_y = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})}, \qquad u'_z = \frac{u_z}{\gamma(1 - \frac{u_x v}{c^2})} \qquad (3)$$

These are called the **relativistic law of transformation of velocities**(please let us know if you have not seen this before). One can go to inverse part(i.e unprimed) by doing the same as before $v \longrightarrow -v$

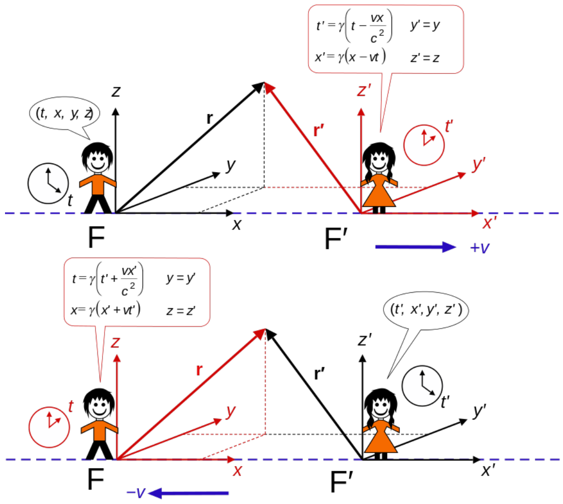

Figure 1: A cartoon image showing Lorentz transformation[Credit : University Physics with Modern Physics, 14th Edition Hugh D Young ,Roger A. Freedman]

1.2 Problems

- ii) Consider one Lorentz transformation with velocity v_1. Now you apply another successive Lorentz transformation with a different velocity say $v2$. Both are applied in x-direction. What is the final resulting velocity you get? Is it the same as applying a Lorentz transformation of $v = v_1 + v_2$?

- iii) Use Eqn(3) and show that velocity in an arbitary direction is **invariant** under boosts.(Recall that if you have a rod right now, it has some length. If you rotate the axes and measure it, ofcourse the components will differ but the thing which will be constant is its distance from the origin,i.e, $x^2+y^2+z^2 = r^2$, you have to use something similar)

- iv) Now consider a particle which is moving in $x - y$ plane. We have $u_x = ucos\theta$, $u_y = usin\theta$, $u_z = 0$ and $u'_x = u'cos\theta'$, $u'_y = u'cos\theta'$ and $u'_z = 0$. If $u' = c$ then u is ?(Here you have light moving in the prime frame, hence the velocity of light you get in unprimed frame should be the same)

- v) Using the inverse form of Eq(3) attain a relation between θ and θ'

which is the angles the light beam subtends with x and x' direction respectively. The formula you will obtain is known as **relativistic aberration of light**. Explore what you get for $\frac{v}{c} << 1$(you will get a Newtonian formula for aberration of light).

Preliminaries/Poruszające się pręty wydają się krótsze, a poruszające się zegary działają wolniej.

Two events $A = (x_1, y_1, z_1, ct_1)$ and $B = (x_2, y_2, z_2, ct_2)$ are called *simultaneous* with respect to an inertial frame K if $\Delta t = t_2 - t1 = 0$. Now consider a second frame K' is related to K by a boost. One can make use of Lorentz transformation to get

$$\Delta x' = \gamma(\Delta x - v\Delta t), \qquad \Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

Therefor for $\Delta t = 0$ implies:

$$\Delta t' = -\gamma \frac{v}{c^2}\Delta x \neq 0 \qquad if \qquad x_1 \neq x_2$$

showing us the effect known as **relativity of simultaneity** : simultaneity of spatially separated points is not an absolute concept.

Consider now a clock at rest in K' making successive 'ticks' at events(x', y', z', ct'_1) and(x', y', z', ct'_2). Then one can evaluate the time difference **according to K** as :

$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x') = \gamma \Delta t' \qquad if \qquad \Delta x' = 0$$

This gives us

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \geq \Delta t'$$

which is known as **time dilation-** a moving clock appears to slow down. Equivalently a stationary clock in K appears to run slow according to the moving observer K'

Now consider a rod of length $l = \Delta x$ at rest in K. Again using the inverse transformation we have

$$l = \Delta x = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x' \qquad if \qquad \Delta t' = 0$$

The length of the rod with respect to K' is determined by considering simultaneous moments $t'_1 = t'_2$ at the end points,

$$l' = \Delta x' = \frac{l}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}.l \leq l$$

This is interpreted as the lenght of a rod being contracted when viewed by a moving observer, an effect known as **Lorentz–FitzGerald contraction**.

1.3 Problems

- If two intersecting light beams appear to be making a non-zero angle ϕ in one frame K, show that there always exist a frame K' whose motion relative to K is in the plane of the beams such that beams appear to be directed in opposite directions

- A source of light emits photons uniformly in all directions in its own rest frame

  1. If the source moves with velocity v with respect to an inertial frame K, show the 'head light' effect where half the photons seem to be emitted in a forward cone whose semi-angle is given by $cos\theta = v/c$
  2. In the films of *Star Wars* genre, star fields are usually seen to be swept backwards around a rocket as it accelerates towards the speed of light. What would such a rocketeer really see as his velocity v approaches c.

- If two separate events occur at the same time in some inertial frame S, prove that there is no limit on the time separations assigned to these events in other frames, but that their space separation varies from infinity to a minimum that is measure in S . With what speed must an observer travel in order that two simultaneous events at opposite ends of a 10-metre room appear to differ in time by 100 years.

- A supernova is seen to explode on Andromeda galaxy, while it is on on the western horizon observers A and B are walking past each other, A at $5km/hr$ towards east and B at $5km/hr$ towards the west. Given that Andromeda is about a million light years away, calculate the difference in time attributed to the supernova event by A and B. Who says it happened earlier?

- There are two twins, twin A who stays on the earth and twin B travels away from earth at speed $0.5c$ on their common birthday. They decide to each blow out candles exactly four years from B's departure.

  1. Suppose when A blows candle we call that as an event P. What moment in $B's$ time corresponds to the event P that consists of A blowing his candle out. And what moment in $A's$ time corresponds to the event Q that consists of B blowing her candle out?
  2. According to A which happened earlier, P or Q? And according to B?
  3. How long will A have to wait before he *sees* his twin blowing her candle out?