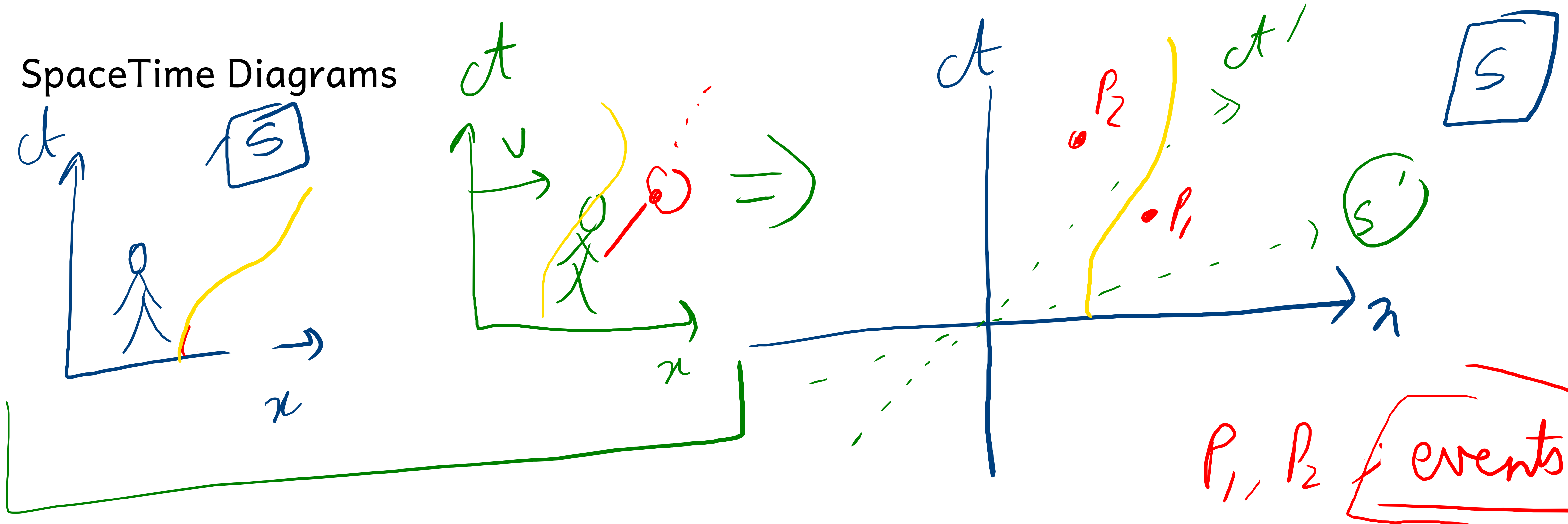
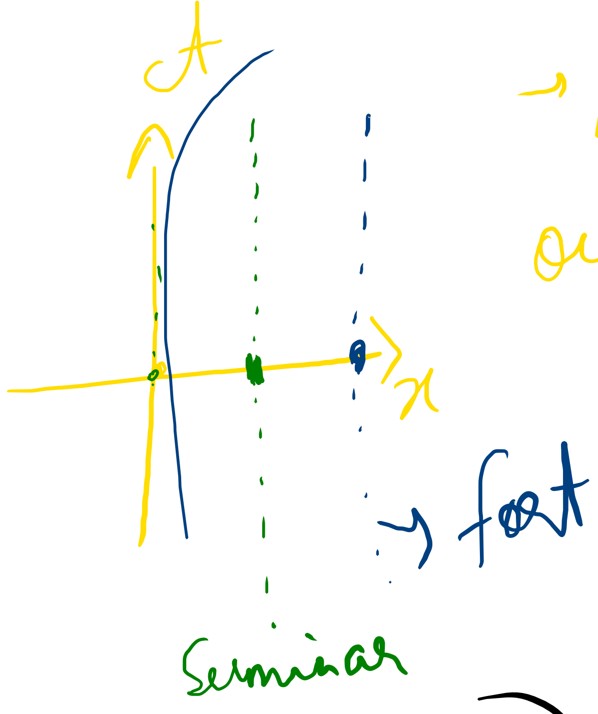
SpaceTime Diagrams

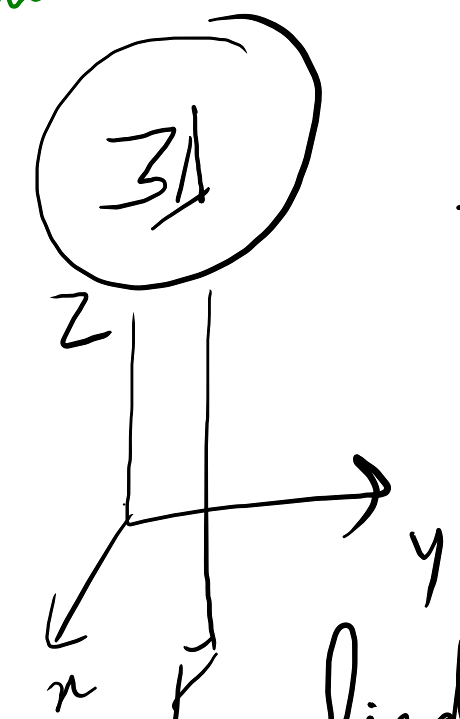


P_1, P_2 ; $\boxed{\text{events}}$

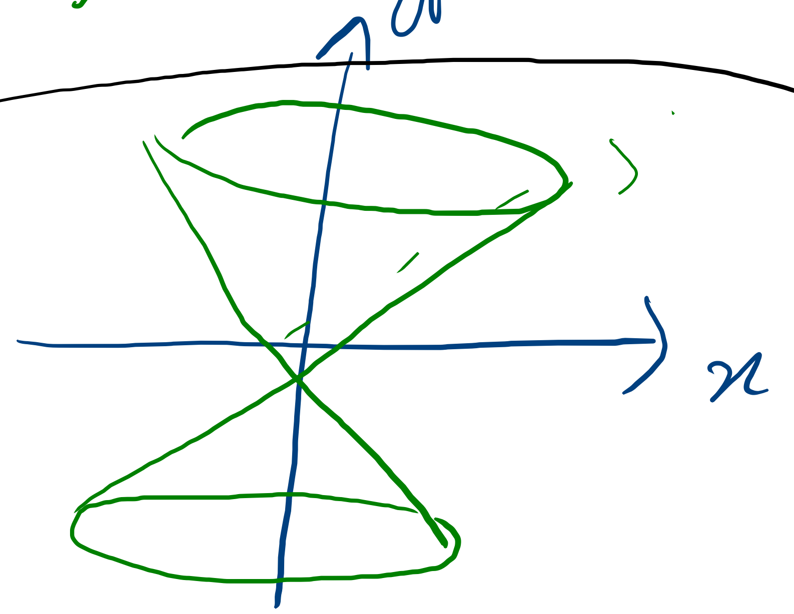


→ A particle moves in a ~~spacetime~~, it traces out a curve k/a → $\boxed{\text{WORLDLINE}}$

→ Light ray travels at 45°



Euclidean space (time absolute)



Minkowski spacetime

↓

S.R.T

space & time → $\boxed{\text{spacetime}}$

The Geometry of Spacetime : [3D] → [4D] → Lorentz Trans

[!] Via Lorentz Transf (backbone of S.R.T)
Time is relative, space is relative and simultaneity is also relative

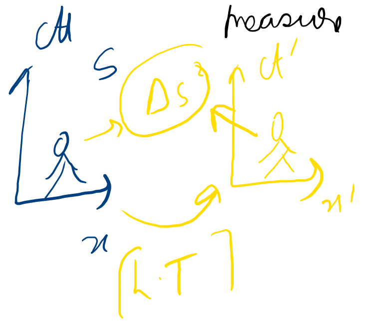
(?) What is left sacred

THE INVARIANT INTERVAL ⇒ Consider a spacetime with just coordinate x.

In S, two events P_1 and P_2 have coordinates (ct_1, x_1) (ct_2, x_2)

The events are separated by $\Delta t = t_1 - t_2$, $\Delta x = x_1 - x_2$

We can define an invariant interval $(\Delta s)^2$ as a measure of 'distance' b/w two points P_1 & P_2



$$\boxed{(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2}$$

The value of this has to be agreed by all observers!

$$\Delta x = \gamma[\Delta x' + v\Delta t'] \qquad \Delta t = \gamma\left[\Delta t' + \frac{v}{c^2}\Delta x'\right]$$

$$(\Delta s)^2 = c^2\left[\gamma^2\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)^2\right] - \gamma^2[\Delta x' + v\Delta t']^2$$

$$= c^2\gamma^2\left[(\Delta t')^2 + \frac{v^2\Delta x'^2}{c^4} + \frac{2\Delta x'\Delta t' v}{c^2}\right] - \gamma^2\left[\Delta x'^2 + v^2\Delta t'^2 + 2\Delta x'\Delta t' v\right]$$

$$= c^2\gamma^2(\Delta t')^2 + \frac{\gamma^2 v^2\Delta x'^2}{c^2} + \cancel{2\Delta x'\Delta t' v\gamma^2} - \gamma^2\Delta x'^2 - \gamma^2 v^2\Delta t'^2 \cancel{- 2\gamma^2\Delta x'\Delta t' v}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (c^2 - v^2)\gamma^2(\Delta t')^2 - \Delta x'^2\gamma^2\left[1 - \frac{v^2}{c^2}\right]$$

$$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}} \qquad \boxed{\Delta s^2 = c^2\Delta t'^2 - \Delta x'^2}$$

With 3D : $\boxed{\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$ → Minkowski Space

dim : 1 + 3

1 → temporal (t)

3 → spatial (x, y, z)

Euclidean

$x^2 + y^2 + z^2 = r^2$

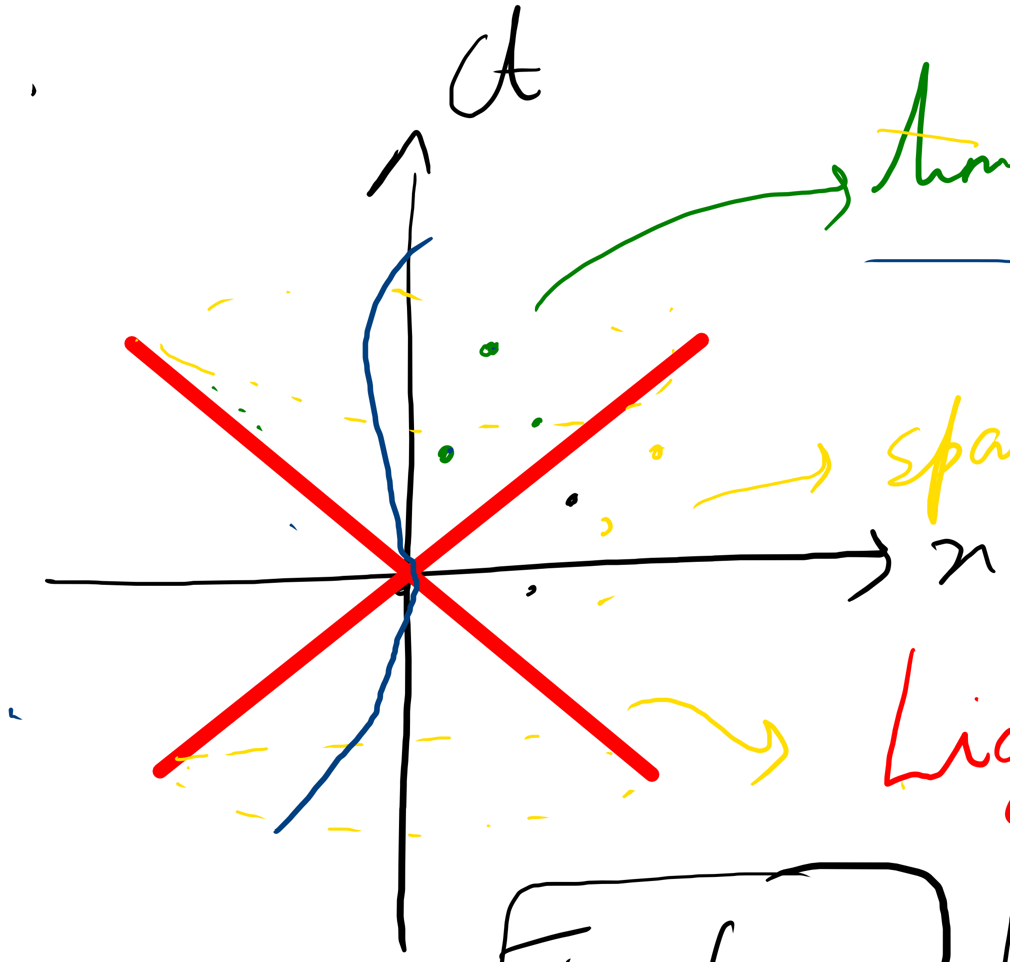
> 0

This spacetime interval provides an observer-independent observable b/w two events.

$\Delta s^2 > 0$ → timelike events

$\Delta s^2 < 0$ → spacelike event

$\Delta s^2 = 0$ → lightlike even



timelike event

spacelike events

Light cone

Tachyons $v > c$

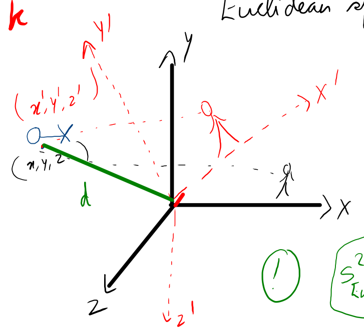
↳ These causality

A Rotational Analogy : Consider 3-dimensions

Euclidean space : $\mathbf{X} = (x, y, z)$ ← Position of helicopter

$\mathbf{X} = x\hat{i} + y\hat{j} + z\hat{k}$

↳ 3-vector



For a rotated frame $\mathbf{X}' = (x', y', z')$

$\mathbf{X}' = R\,\mathbf{X}$

↳ rotation matrix

(1) $S^2_{\text{Euclidean}} = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$

Essence of S.R.T = extrapolation of ↑

↳ rotations in space and time

4-vectors

Lorentz Group ⇒ If 'you' sit at the origin in fixed frame S, then the coordinates of events can be written as a 4-vector :

position 4-vector

$$\boxed{X^{\mu} =} \left[\underbrace{ct}_{0^{th} \to \text{time}}, \underbrace{x, y, z}_{i^{th}\text{ component} \to \text{space}}\right] \qquad \mu = 0, 1, 2, 3$$

Δs^2

The invariant interval can be written as inner product

$$X \cdot X \equiv X^T \eta X = X^{\mu} \eta_{\mu\nu} X^{\nu}$$

$$= [ct\ x\ y\ z] \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times4} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\boxed{X \cdot X = ct^2 - x^2 - y^2 - z^2}$$

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \text{Minkowski Metric}$$

∴ Lorentz transformation → can be thought of as a 4×4 matrix

$$x' = \gamma[x - vt]$$

$$c\,t' = \gamma\left[t - \frac{vx}{c^2}\right]c$$

$$x' = \gamma x - \gamma v t$$

$$ct' = ct - \frac{\gamma v x}{c}$$

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{bmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}$$

$\Lambda \to$ Lorentz matrix : it takes your one-4 vector to another from $S \to S'$

$$\boxed{X' = \Lambda X} \to X^{\mu'} = \Lambda^{\mu}{}_{\nu} X^{\mu}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

x

L

$$\boxed{\Lambda^T \eta \Lambda = \eta} \to \text{Matrix eq}^n \Rightarrow \text{Solutions}$$

$\Lambda \to 4 \times 4 \to 16$ elements [6 independent]

$\boxed{\text{2 class of sdns}}$

Λ

$\to$ 3 boosts ($y, x' \to t'$)

$\to$ 3 rotation ($x' \to y'$)

$$\Lambda = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right]$$

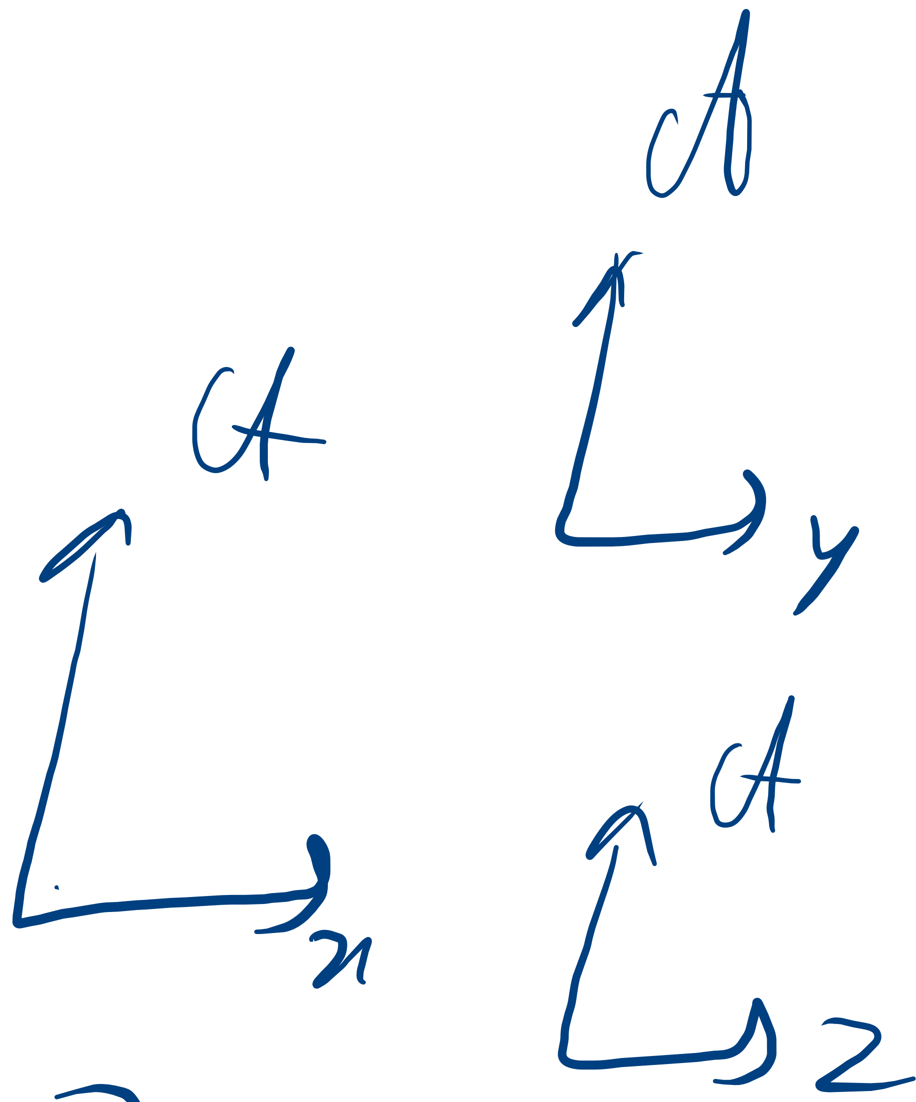
Rotations

3 spatial rot

$$\Lambda = \left[\begin{array}{cc|cc} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Boost in x-directi

$\boxed{\text{3 boosts}}$



x^μ

$\downarrow$

$U^\mu \rightarrow p^\mu$

$\downarrow$?

$\boxed{a^\mu}$

Proper time:

$\Delta s^2 = (c\Delta t')^2 - (\Delta x')^2 > 0$

$\Delta s^2 = c^2 \Delta t'^2$

the invariant time interval is proportional to the time experienced by the particle

$\boxed{\Delta \tau = \frac{\Delta s}{c}}$

worldline

→ With proper-time you can parametrize the trajectory of particle

$x(\tau),\ t(\tau)$

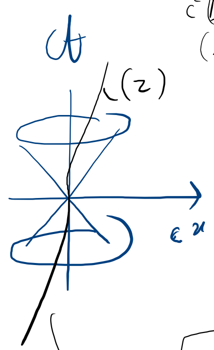
$c^2\Delta t'^2 - \Delta x'^2 = (c\Delta t)^2 - (\Delta x)^2$

$c^2(\Delta \tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2$

$(\Delta \tau)^2 = (\Delta t)^2 - \frac{(\Delta x)^2}{c^2}$

$\Delta \tau^2 = (\Delta t)^2 \left[1 - \frac{(\Delta x)^2}{(\Delta t)^2}\frac{1}{c^2}\right]$

$\Delta \tau = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$ $v = \frac{\Delta x}{\Delta t}$ speed of particle



$\Delta \to d$

$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} \rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\boxed{T = \int d\tau = \int \frac{dt}{\gamma}}$ $\boxed{\frac{dt}{d\tau} = \gamma}$

$\vec{A} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

4-velocity vector:

$X(\tau) = \begin{bmatrix} ct(\tau) \\ x(\tau) \end{bmatrix}$

$\vec{v} = \frac{dx}{dt}$

$U = \frac{dX}{d\tau} = \begin{bmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \end{bmatrix} = \begin{bmatrix} c\gamma \\ \frac{dx}{dt}\frac{dt}{d\tau} \end{bmatrix} = \begin{bmatrix} c\gamma \\ \vec{v}\gamma \end{bmatrix}$

$\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z$

$\boxed{U = \gamma \begin{bmatrix} c \\ \vec{v} \end{bmatrix}} \rightarrow \boxed{P = m_0 U} = \begin{bmatrix} m_0 c\gamma \\ m_0 \vec{v}\gamma \end{bmatrix}$ → 4-momentum

$\vec{p} = m_0\vec{v}\gamma$

$\boxed{m_0 \rightarrow \text{rest mass}}$

$v \ll 1 \rightarrow \gamma \rightarrow 1$

$\vec{p} = m\vec{v}$

4-vector → $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$K.E = \frac{1}{2}mv^2$

$P = \begin{bmatrix} m_0 c\gamma \\ m_0 \vec{v}\gamma \end{bmatrix}$

✓ momentum

4-momentum vector

$p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix}$

$\frac{E}{c} = m_0 c\gamma$

$E = m_0 c^2 \gamma$

$\frac{v}{c} \ll 1$

$\gamma \rightarrow 1$

$E = m_0 c^2$

$\downarrow$

$E = mc^2$

$m_0 c\gamma \rightarrow \frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\rightarrow m_0 c\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$\rightarrow \frac{m_0 c^2}{c}\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right]$

$\rightarrow \frac{1}{c}\left[m_0 c^2 + \frac{1}{2}m_0 v^2\right]$

$\downarrow$

non-relativistic K.E