Lec 2 GWs: Lorenz Gauge

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Given

$$G_{\mu\nu} = \frac{1}{2}\left(\cancel{\partial^\alpha \partial_\mu h_{\alpha\nu}} + \cancel{\partial^\alpha \partial_\nu h_{\alpha\mu}} - \cancel{\eta_{\mu\nu}\partial^\alpha \partial^\beta h_{\alpha\beta}} - \partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}\right)$$

$\downarrow$

We can do change of coordinates

$\downarrow$

Then we can say metric + keeps the wave eq^n given $\bar{T}_{\mu\nu} = 0$

$$G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4} \xrightarrow[\text{vacuum}]{\text{Linearized gravity}} \boxed{\Box \bar{h}_{\mu\nu} = 0}$$

" gauge " $\approx$ " choice of coordinate system "

$\Downarrow$

So need a coordinate system where

$\# \to 0 \longrightarrow \partial_\beta \bar{h}^{\alpha\beta} = 0$ if we can find a coordinate system

$$\boxed{\partial_\mu \bar{h}^{\mu\nu} = \partial^\alpha \partial h_\alpha{}^\nu}$$

III $\to$ $\partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} = \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} = \partial_\alpha (\partial_\beta \bar{h}^{\alpha\beta}) \to 0$

$\therefore$ under " Lorenz Gauge "

I $\to$ $\partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} = \partial_\alpha \partial_\mu \bar{h}^\alpha{}_\nu = \partial_\alpha \partial_\mu \eta_{\beta\nu} \bar{h}^{\alpha\beta}$

$= \eta_{\beta\nu} \partial_\mu \partial_\alpha \bar{h}^{\alpha\beta} \to 0$

$\therefore$ under "L-G"

II $\to$ $\partial^\alpha \partial_\nu \bar{h}^{\mu\alpha} \xrightarrow{?} 0$

Need to proove Lorenz Gaugue Coordinate system

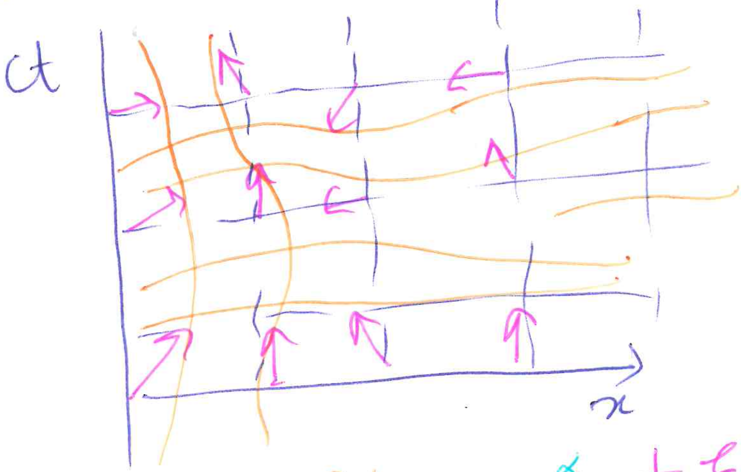
$\downarrow$

Need coordinate tran : $(ct, x, y, z) \to (\tilde{ct}, \tilde{x}, \tilde{y}, \tilde{z})$

$$x^\alpha \to \tilde{x}^\alpha$$

$\downarrow$

~~Conds~~ Consider: $\boxed{\tilde{x}^\alpha} = x^\alpha + \xi^\alpha$ ["displacement field"]



$\downarrow$

assume $\|\xi^\alpha\| \ll 1$

$$\left\| \frac{\partial \xi^\alpha}{\partial x^\beta} \right\| \ll 1$$

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha$$

$$x^\alpha = \tilde{x}^\alpha - \xi^\alpha$$

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial x^\alpha}{\partial x^\beta} + \frac{\partial \xi^\alpha}{\partial x^\beta}$$

$$= \delta^\alpha_\beta + \frac{\partial \xi^\alpha}{\partial x^\beta}$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \frac{\partial \tilde{x}^\alpha}{\partial \tilde{x}^\beta} - \frac{\partial \xi^\alpha}{\partial \tilde{x}^\beta}$$

$$= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial \tilde{x}^\beta}$$

$$= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\sigma} \boxed{\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}}$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\sigma}\left[\delta^\sigma_\beta - \frac{\partial \xi^\sigma}{\partial \tilde{x}^\beta}\right]$$ 7

$$= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\sigma}\delta^\sigma_\beta + \frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial \xi^\sigma}{\partial \tilde{x}^\beta}$$

$$\left[\ \left\|\frac{\partial \xi^\alpha}{\partial x^\beta}\right\| \ll 1,\ \left\|\ \frac{\partial \xi^\alpha}{}\right.\ \right]$$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\beta}$$

$$*)\quad \tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}$$

$$\tilde{g}_{\alpha\beta} = \left(\delta^\mu_\alpha - \frac{\partial \xi^\mu}{\partial x^\alpha}\right)\left(\delta^\nu_\beta - \frac{\partial \xi^\nu}{\partial x^\beta}\right)(\eta_{\mu\nu} + h_{\mu\nu})$$

$$\Downarrow$$

$$= \eta_{\alpha\beta} - \eta_{\alpha\nu}\xi^\nu{}_{,\beta} - \eta_{\mu\beta}\xi^\mu{}_{,\alpha} + h_{\alpha\beta}$$
$$\cancel{- h_{\alpha\nu}\xi^\nu{}_{,\beta} - h_{\mu\beta}\xi^\mu{}_{,\alpha}}$$

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

$$\Downarrow$$

$$\boxed{\tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}}$$

$\tilde{R}^{\rho}{}_{\sigma\mu\nu} \xrightarrow{\cdot} R^{\rho}{}_{\sigma\mu\nu}$

$\therefore$ Riemⁿ Tensor components don't change under a change of coordinates from small dispⁿ field ξ^{α} !

*) $x^{\alpha} \to \tilde{x}^{\alpha} \longrightarrow \bar{h}_{\mu\nu} \to \tilde{\bar{h}}_{\mu\nu}$

Goal is to change coordinates so that

$\Downarrow$ satisfy

$\partial_{\beta} \tilde{\bar{h}}^{\alpha\beta} = 0$

*) We have $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$

$$\tilde{\bar{h}}_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$$

$$= \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\tilde{h}_{\alpha\beta}$$

$$\downarrow$$

$$\tilde{\bar{h}}_{\mu\nu} = \left(h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\right) - \frac{1}{2}\eta_{\mu\nu}\left[\eta^{\alpha\beta}\left(h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}\right)\right]$$

$$\tilde{\bar{h}}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \frac{1}{2}\eta_{\mu\nu}\xi^{\beta}{}_{,\beta} + \frac{1}{2}\eta_{\mu\nu}\xi^{\beta}{}_{,\beta}$$

This is how our $\bar{h}$ components change using displace field ξ

$$= \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\sigma}{}_{,\sigma} \quad \text{— E(1)}$$

$\eta^{\mu\alpha}\eta^{\nu\beta}\tilde{\bar{h}}_{\mu\nu} = (\quad)$ $\Downarrow$ get $\tilde{\bar{h}}^{\alpha\beta}$

$$\overline{\tilde{h}^{\alpha\beta}} = \overline{h^{\alpha\beta}} - \eta^{\sigma\beta}\xi^{\alpha}{}_{,\sigma} - \cancel{\eta^{\mu\alpha}\xi^{\beta}{}_{,\mu}} \cancel{+ \eta^{\alpha\beta}\xi^{\sigma}{}_{,\sigma}}$$

$$\overline{\tilde{h}_{\alpha\beta}}{}_{,\beta} = \overline{h^{\alpha\beta}}{}_{,\beta} - \eta^{\sigma\beta}\xi^{\alpha}{}_{,\sigma\beta} - \cancel{\eta^{\mu\alpha}\xi^{\beta}{}_{,\mu\beta}} + \eta^{\alpha\beta}\xi^{\sigma}{}_{,\sigma\beta} \rightarrow \cancel{\eta^{\alpha\mu}\xi^{\beta}{}_{,\beta\mu}}$$

$$\downarrow$$

$$\overline{\tilde{h}^{\alpha\beta}}{}_{,\beta} = \overline{h^{\alpha\beta}}{}_{,\beta} - \eta^{\sigma\beta}\partial_\sigma\partial_\beta\xi^{\alpha}$$

$$\partial_\beta\overline{\tilde{h}^{\alpha\beta}} = \partial_\beta\overline{h^{\alpha\beta}} - \Box\xi^{\alpha}$$

$$\downarrow$$

So if we choose: ξ^α such that

$$\boxed{\partial_\beta\overline{h^{\alpha\beta}} = \Box\xi^{\alpha}}$$

$$\Downarrow$$

$$\boxed{\partial_\beta\overline{\tilde{h}^{\alpha\beta}} = 0}$$

satisfies the L-G condit^{n}

(!) Choice of ξ^α is not unique

Any X^α with $\Box X^\alpha = 0$ will also

$$\partial_\beta\overline{h^{\alpha\beta}} = \Box(\xi^\alpha + X^\alpha)$$
$$= \Box\xi^\alpha$$

"Lorenz - Gauge" → is a class of co-ordinate systems

$$\Downarrow$$

If $\partial_\beta\overline{h^{\alpha\beta}} = 0 \rightarrow$ (I), (II), (III) $\rightarrow 0$

Linearized Gravity
+
Lorenz Gauge : $\tilde{G}_{\mu\nu} = -\frac{1}{2}\Box \tilde{h}_{\mu\nu}$

$\downarrow$

Vaccum : $\tilde{T}_{\mu\nu} = 0$

$$\boxed{\Box \bar{\tilde{h}}_{\mu\nu} = 0}$$

$\Downarrow$

$$\therefore \quad \frac{1}{c^2}\frac{\partial^2 \bar{\tilde{h}}^{\mu\nu}}{\partial t^2} = \nabla^2 \bar{\tilde{h}}^{\mu\nu}$$

$\Downarrow$

$\therefore$ <u>GW travel at c</u>

Next Transverse-Traceless Gauge

$\Downarrow$

give  <u>2 GW pol</u>

"Gauge" $\rightarrow$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Different potentials result in same Faraday Tensor

EG. $A_\mu \rightarrow A_\mu + \partial_\mu f$

$\downarrow$

$F_{\mu\nu} \rightarrow F_{\mu\nu}$

$A_\mu + \partial_\mu f \rightarrow \boxed{F_{\mu\nu}}$

$A_\mu + \partial_\mu g \rightarrow \boxed{F_{\mu\nu}}$

$A_\mu + \partial_\mu h \rightarrow \boxed{F_{\mu\nu}}$

Freedom to pick $A^\mu \to$ "Gauge-Freedom"

Choosing a specific $A^\mu \to$ "Choosing a gauge" (9)

Lorenz-Gauge: $\boxed{\partial_\mu A^\mu = 0}$

EM

→ Choice of A_μ

→ Multiple $A_\mu \to$ same $F_{\mu\nu}$

→ Gauge T
$\partial A_\mu \to A_\mu + \partial_\mu f$

→ L.G: $\partial_\mu A^\mu = 0$

⇓

In E.M, L.G removes "spurious" dof in potential A^μ, that does not affect $\vec{E}$ & $\vec{B}$

Linearized GW

→ choice of coords x^α

→ Multiple $h_{\mu\nu}$ same $R^\rho{}_{\sigma\mu\nu}$

→ Gauge Tr.
$x^\alpha \to x^\alpha + \xi^\alpha$
$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$

→ L.G $\boxed{\partial_\beta \bar{h}^{\mu\nu} = 0}$

→ choice of coordinate system that cause metric to obey L.G

⇓

Remember it does not change the Riemann

⇓

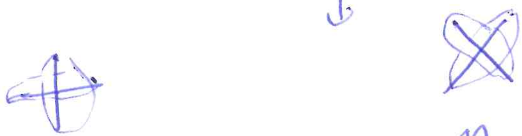
L.G eliminates unnecessary dof in $h_{\mu\nu}$ that don't effect curvature

Lec 3: GWs ⇒ TT-Gauge + Pol^n

Idea: Use TT-Gauge to show GWs have 2 pol

*) EM Waves are oscillat^n in $\vec{E}$ & $\vec{B}$ vector fields
⇓
Pola^n ⇒ direct^n of $\vec{E}$ vector

*) GWs are oscillat^n in metric perturbation $\bar{h}_{\mu\nu}$
Pol^n ⇒ ~~distance to of~~ direct^n of how distance change b/w points
↓



*) Linearized Gravity : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad ||h_{\mu\nu}<<1||$

$$\begin{bmatrix} h_{00} & h_{11} & \cdots \\ & \ddots & \\ \cdots & & h_{33} \end{bmatrix} \rightarrow 16 \text{ components}$$

↓ + a choice of coordinate

we get only 2 independent components
↓

$$\boxed{\text{TT-Gauge}}$$

*) We know in Lorenz gauge

$$\Box \bar{h}_{\mu\nu} \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta \bar{h}_{\mu\nu} = 0$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

→ Most straightforward sol^n ⇒ PLANE WAVES