$$\frac{d^2x^i}{dt^2} = +\Gamma^i_{tt}\frac{dx^t}{dt}\frac{dx^t}{dt} + 2\Gamma^i_{tj}\frac{dx^t}{dt}\frac{dx^j}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

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$$= \cancel{\frac{dx^i}{d\tau}\frac{d\tau}{dt}} \qquad \frac{dx^a}{d\tau} \to \frac{dx^a}{dt}\frac{dt}{d\tau} \quad ②$$

$$\Gamma^\mu_{\rho\sigma} \to \Gamma^t_{\rho\sigma} - \Gamma^i_{\rho\sigma}$$

***

$$\to \Gamma^t_{ii} \qquad v^i\Gamma^t_{tt} \to \frac{dx^i}{dt}\Gamma^t_{tt}$$

$$\frac{dx^i}{c}$$

Lec 4: GW effect on Free Particle

Recap : TT-Gauge ⇒ specific Lorenz Gauge

$$(\partial_\mu \bar{h}^{\mu\nu} = 0)$$

$$\Downarrow$$

$$\left\{ \bar{h}_{\mu\nu} U^\mu = 0 \;;\; \bar{h}^\mu_\mu \right\}$$

U^μ → 4-vel of person observing GW

NOTE

* $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \to \bar{h}_{\mu\nu}\eta^{\mu\nu} = h_{\mu\nu}\eta^{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}h$

$\to \bar{h}^\mu_\mu = h^\mu_\mu - \frac{1}{2}\delta^\mu_\mu h \to \bar{h} = h - \frac{1}{2}\times 4 \times h$

$$\boxed{\bar{h} = -h}$$

$$\bar{h}^\mu_\mu = \bar{h} = 0$$

We know that in TT-gauge $\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu}} \to \boxed{\bar{h}^{TT}_{\mu\nu}} = \boxed{h^{TT}_{\mu\nu}}$

∴

*) Components of $A_{\mu\nu}$ ⇒ 16 components

↳ Sym of $A_{\mu\nu} = A_{\nu\mu}$ → -6 constraints

↳ Lorenz G : $A_{\mu\nu}k^\mu = 0$ → -4 constraints

↳ TT : $A_{\mu\nu}U^\mu = A^\mu_\mu = 0$ → -4 constraints

$$\Downarrow \quad \underline{2 \text{ dof}}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

*** Take for a wave in z-direct^n ← $U^\mu = (c,0,0,0)$; $k^\mu = [\frac{\omega}{c}, 0, 0, \frac{\omega}{c}]$

$$\begin{bmatrix} \cancel{0\;0\;0\;0} \\ \cancel{0\;A_\times\;A_+\;0} \\ 0\;\cancel{A_\times\;A_+}\;0 \\ 0\;0\;0\;0 \end{bmatrix}$$

I.M.P → A metric perturbatⁿ that has been put into TT-gauge will be written as $\boxed{h_{\mu\nu}^{TT}}$

$$h_{\mu\nu} \xrightarrow{TT} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & 0 & 0 \\ 0 & 0 & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_\times & 0 \\ 0 & A_\times & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{h_{tt} = h_{ti} = 0 \quad + \quad h = h^i_{\ i} = 0}$$

Why $\boxed{TT}$? → You can pick other gauge as well

⇓

This is convenient + completely fix all the local gauge freedom

⇓

(*) ∴ $h_{\mu\nu}^{TT}$ ⟹ contains only the physical inform about radiation

(*) In TT gauge ∃ a relatⁿ h & linearized R

{Recall $R^{\alpha\beta}_{\ \ \sigma\mu\nu} \simeq \partial\Gamma - \partial\Gamma \simeq \frac{1}{2}(\partial\partial h + \partial\partial h - \partial\partial h - \partial\partial h)$ }

(+)

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

"MATHS"

⇓

$$\boxed{R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}^{TT}}$$

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(*) In TT gauge $\Rightarrow$ GWs has 2 polarizat' components

Eg) Consider a GW propagating in z-direct

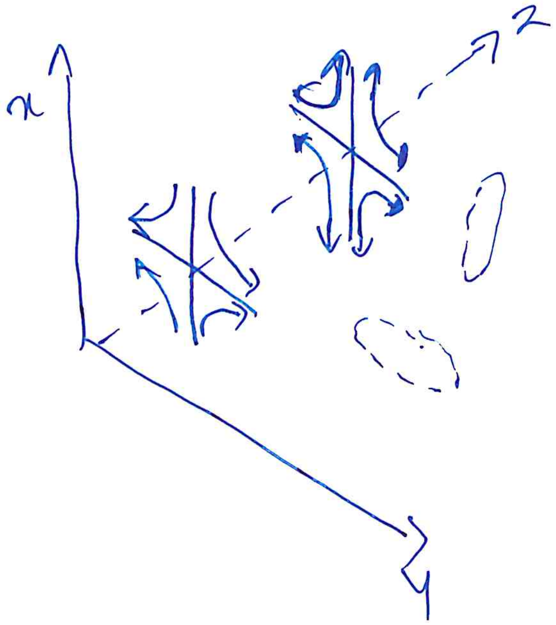
$$h_{ij}^{TT} = h_{ij}^{TT}(t-z)$$

~~Lorenz Gauge condit^n : $\partial_z h_{zj}^{TT} = 0$ implies $h_{zt}^{TT}(t-z)$ = constant~~

\# ~~This constant must be zero~~ ONLY NON-zero components of h_{ij}^{TT} are $\{h_{xx}^{TT}, h_{xy}^{TT}, h_{yx}^{TT}, h_{yy}^{TT}\}$

$\Downarrow$ symmetry + tracefree condition

$$\left.\begin{aligned} h_{xx}^{TT} = -h_{yy}^{TT} &\equiv h_+(t-z) \\ \text{\& } h_{xy}^{TT} = h_{yx}^{TT} &= h_\times(t-z) \end{aligned}\right\} \text{2 independent pol}^n \text{ of GW}$$



?) How does GW affect motion of single particle

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \Longrightarrow$$ At $\tau = 0$, particle at rest $[U^\alpha = 1, 0, 0, 0]$

$$\therefore \left(\frac{dU^\alpha}{d\tau}\right)_{\tau=0} = -\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\left[h^{TT}_{\beta 0,0} + h^{TT}_{0\beta,0} - h^{TT}_{00,\beta}\right]$$

$\downarrow$ TT-gauge all time-compon $\to 0$

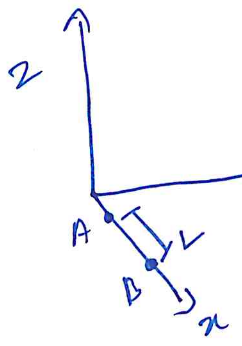
$$\therefore (dU^\alpha/d\tau)_{(\tau=0)} = 0$$

$\Downarrow$

Does this mean GW has no effect? $\Rightarrow$ NO! Its just in TT gauge, coordinate locatⁿ is w.r.t. unaffected by GW.

$\Downarrow$

Need co-ordinate invariant observables $\Rightarrow$



→ Consider two spatially-freely falling particles at $z = 0$, separated on x-axis by coordinate distance L_c

→ Consider a GW in TT-gauge that propagates down the z-axis $\boxed{h^{TT}_{ij}(t,z)}$

→ Proper-distance L_0 b/w two particles.

$$L_0 = \int_0^{L_c} dx\sqrt{g_{xx}} = \int_0^{L_c} dx\sqrt{1 + h^{TT}_{xx}(t, z=0)}$$

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$$\left\{ ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}^{TT}) dx^\mu dx^\nu \right.$$

$$\downarrow$$

$$\left\{ \left[(1 + h_{xx}^{TT}) dx^2 + (1 - h_{xx}^{TT}) dy^2 \right\} \right]$$

$$L_0 \simeq \int_0^{L_c} \left[1 + \frac{1}{2} h_{xx}^{TT}(t, z=0) \right] = L_c \left[1 + \frac{1}{2} h_{xx}^{TT}(t, z=0) \right]$$

$\therefore$ Proper distance oscillates with a fractional length change

L_0

$$\boxed{\frac{\delta L}{L} \simeq \frac{1}{2} h_{xx}^{TT}(t, z=0)}$$

Magnitude of $h \to$ "wave strain"

$\Downarrow$

IMP $\Rightarrow$ results in the accumulated phase measured in LIGO GW observatn

Effect of GWs on matter

$\Rightarrow$ Need to measure tidal force

$$\frac{D^2(\delta x^\alpha)}{D\tau^2} = R^\alpha{}_{\mu\nu k} \left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right) \delta x^k$$

$\downarrow$ under slow motion + weak gravity

$$\frac{d^2}{dt^2}(\delta x^i) = - R^i{}_{0j0}\, \delta x^j \quad \left[\text{To verify} \right]$$

{ Here we assume: $dt/d\tau \simeq c = 1$ (+) separatn b/w two particles are measured simultaneously $\delta x^0 = 0$ }

*) For a wave in z-dirn $\boxed{x^3 = z}$ we have $R^i{}_{0j0} = \dfrac{-\partial \Gamma^i{}_{0j}}{\partial x^0}$

$\downarrow$

{ use $\Gamma = \frac{1}{2}\eta\,[\partial h + \partial h - \partial h] + h_{0j} = 0$ }

to get $\Gamma^i{}_{0j} = \dfrac{1}{2}\dfrac{\partial h^i{}_j}{\partial x^0}$ [To verify]

$\Downarrow$

$$\frac{d^2}{dt^2}(\delta x^i) = -R^i{}_{0j0}\,\delta x^j \Rightarrow \frac{d^2(\delta x_i)}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2}\delta x^j$$

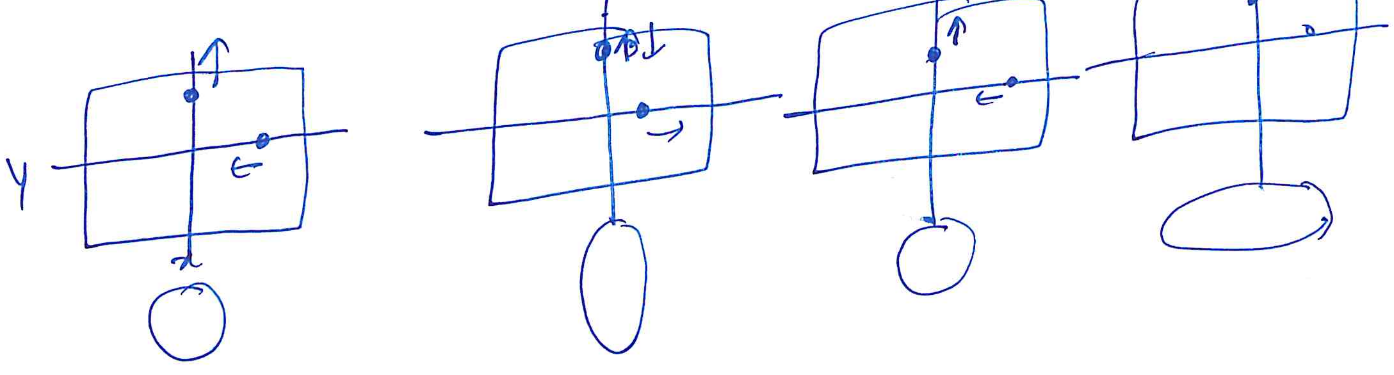
$\Downarrow$

It has soln: $\boxed{\delta x_i = \delta x_{i,0} + \frac{1}{2} h_{ij}\,\delta x^j}$

$\forall$ A_+ pol:

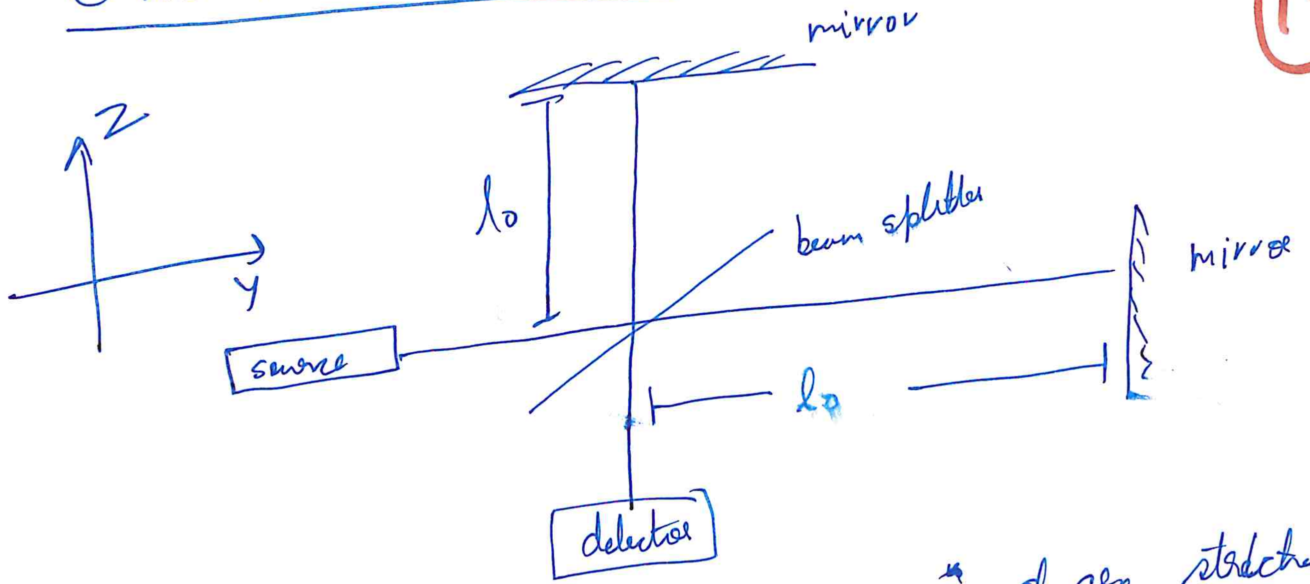
$$\delta x_1 = \delta x_{1,0} + \frac{1}{2} A_+ e^{ikt}\,\delta x^j$$

$$\delta x_2 = \delta x_{2,0} - \frac{1}{2} A_+ e^{ikt}\,\delta x^j$$



GWs + Michelson Interferometer

(17)



Issue: "Both the arms + λ of light $\not d$ are stretched by same amount" $\Rightarrow$ Interference pattern is affected by the TIME DELAY in the light propagat^n produced by GW.

\# For GW propagating in x direct^n with polarisat^n '+' in yz plane:

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{bmatrix};$$

$h_{yy} = -h_{zz} \simeq A_+ \cos\omega\left(t - \frac{x}{c}\right)$

$h_{yz} = h_{zy} = 0$

$\Downarrow$

Describe the interferometer in TT-gauge

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}^{TT})dx^\mu dx^\nu = -c^2dt^2 + dx^2 + (1+h_+)dy^2 + (1-h_+)dz^2$$

$\Downarrow$

assume $\lambda_{GW} >> l_0$

$\Downarrow$

GW perturbat^n (h_+) can be considered as constant as light crosses the arm.

*) A light ray follow in y-dire follows a null geodesic with

$$c^2 dt^2 = (1 + h_+) dy^2$$

$$dt^2 = c^{-1}\left(1 + \frac{h_+}{2}\right) dy + O(h^2)$$

*) Time to cross back & forth in y-arm

$$t(y) = \left(1 + \frac{h_+}{2}\right)\frac{2L_0}{c}$$

*) For light ray moving z-directⁿ: $t(z) = \left(1 - \frac{h_+}{2}\right)\frac{2l_0}{c}$

$$\Delta t = t(y) - t(z) = \frac{2l_0}{c} h_+ \Rightarrow$$ causes the interference fringes

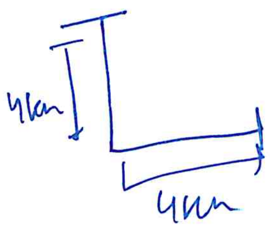
Most sensitive interferometer length $l_0 \simeq \frac{1}{4}\lambda_{GW}$

$\forall$ a GW freq ~ 100Hz, well within the sensitivity band of LIGO, $\lambda_{GW} \sim 3000$km

∴ optimal length $l_0 \sim 750$km

$\downarrow$

actual length (4km) ⊛ since light ray goes back and forth.



PROJECTOR Show the Polarisatⁿ_YT

how of rod ⊕