Lec 1: GWs ⇒ Introduct^n (LIGO, Wave Eq^n) 1

* Idea: Einstein Eq^n ⇒ GW wave eq^n

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \Rightarrow \frac{\partial^2 g_{\mu\nu}}{\partial x^2} + \frac{\partial^2 g_{\mu\nu}}{\partial y^2} + \frac{\partial^2 g_{\mu\nu}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$$

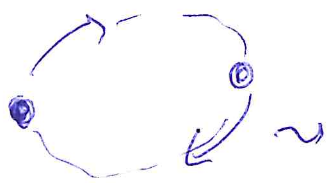
↳ GWs ⇒ disturbance in $g_{\mu\nu}$

⇒ Caused by change in $T_{\mu\nu}$

⇓

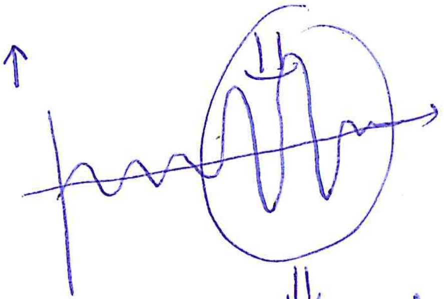
Eg) Two masses orbiting each other, "Merger"

Inspiral



⟶ Vel 1 ↑, Vel 2 ↑ ; GW ↑

Vel 1 : ↑
vel 2 : ↑
GW ↑



⇓

weak change in metric

* Wave Eq^n $\quad v^2 \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 A}{\partial t^2}$

$A(t,x)$ = wave amplitude

v = vel. wave

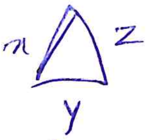
* E·M Wave Eq^n : $\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}$

$$\frac{\partial^2 E}{\partial (ct)^2} - \frac{\partial^2 E_i}{\partial x_i^2} \rightarrow \partial_0^2 E - \partial_1^2 E - \partial_2^2 E - \partial_3^2 E = 0$$

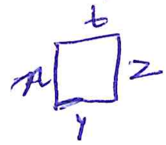
$$\eta^{\mu\nu}\partial_\mu \partial_\nu E = 0 \rightarrow \partial_\mu \partial^\mu E$$

$$\Box E = 0$$

$$\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad : \text{LAPLACIAN}$$



$$\Box \equiv \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$



*) GW Eq^n $\rightsquigarrow$ Rough Idea

$$\frac{\partial^2 g_{\mu\nu}}{\partial (ct)^2} - \frac{\partial^2 g_{\mu\nu}}{\partial x^2} - \frac{\partial^2 g_{\mu\nu}}{\partial y^2} - \frac{\partial^2 g_{\mu\nu}}{\partial z^2} = 0$$

$$\Downarrow$$

$$\Box g_{\mu\nu} = 0$$

Lec2 : GWs: Linearized Gravity : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

GOAL : Get L.H.S : $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

We need $\to$ $g^{\mu\nu}$ inverse

$\to$ $\Gamma^{\lambda}_{\mu\nu}$, $R^{\rho}{}_{\sigma\mu\nu}$, $R_{\mu\nu}$, R

"Linearized" $\to$ "weak" : $g_{\mu\nu} = \eta_{\mu\nu} + \underline{h_{\mu\nu}}$

(i) $||h_{\mu\nu}|| << 1 \Rightarrow h_{\alpha\beta} h_{\mu\nu} \approx 0$

(ii) $||\partial_\sigma h_{\mu\nu}|| << 1 \Rightarrow (\partial_\rho h_{\alpha\beta})(\partial_\sigma h_{\mu\nu}) \approx 0$

or

$h_{\alpha\beta}(\partial_\sigma h_{\mu\nu}) \approx 0$

*) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad ||h_{\mu\nu}|| << 1$

$$\boxed{g_{\mu\sigma} g^{\sigma\nu} = \delta^{\nu}_{\mu}}$$

L2

Can assume : $g^{\mu\sigma} = \eta^{\mu\sigma} + k^{\mu\sigma}$

$g_{\mu\sigma} g^{\sigma\nu} = \delta^{\nu}_{\mu}$

$\cancel{\delta^{\nu}_{\mu}} = (\eta_{\mu\sigma} + h_{\mu\sigma})(\eta^{\sigma\nu} + k^{\sigma\nu})$

$= \underbrace{\eta_{\mu\sigma}\eta^{\sigma\nu}}_{\cancel{\delta^{\nu}_{\mu}}} + \eta_{\mu\sigma}k^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} + h_{\mu\sigma}k^{\sigma\nu}$

$0 = \eta_{\mu\sigma}k^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} + \cancel{h_{\mu\sigma}k^{\sigma\nu}}$

$\because$ $\|h_{\mu\nu}\| \ll 1$
$\|k^{\mu\nu}\| \ll 1$

$\therefore$ $\left[h_{\mu\sigma}\eta^{\sigma\nu} = -\eta_{\mu\sigma}k^{\sigma\nu} \right] \times \eta_{\rho\nu}$

$h_{\mu\sigma}\underbrace{\eta^{\sigma\nu}\eta_{\rho\nu}} = -\eta_{\mu\sigma}\eta_{\rho\nu}k^{\sigma\nu}$

$h_{\mu\sigma}\delta^{\sigma}_{\rho} = -\eta_{\mu\sigma}\eta_{\rho\nu}k^{\sigma\nu}$

$h_{\mu\rho} = -\eta_{\mu\sigma}\eta_{\rho\nu}k^{\sigma\nu}$

$\boxed{h_{\mu\rho} = -k_{\mu\rho}}$ $\quad k^{\mu\rho} = -h^{\mu\rho}$

$\therefore$ $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$\left[h^{\mu\nu} = h_{\rho\sigma}\eta^{\rho\mu}\eta^{\sigma\nu} \right]$

raised indices
(not inverse!)

*) Raise index will $g \to \eta$

$$h^{\mu}{}_{\nu} = h_{00} g^{\mu\nu}$$

$$h^{\mu}{}_{\nu} = h_{00}(\eta^{\mu\nu} - h^{\mu\nu})$$

$$= h_{00}\eta^{\mu\nu} - \cancel{h_{00}h^{\mu\nu}} \to 0$$

We can raise by η

$\boxed{*}$ Connect^n Coefficient :

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right)$$

need $\partial_\rho g_{\mu\nu} = \partial_\rho(\cancel{\eta_{\mu\nu}} \to 0 + h_{\mu\nu}) = \partial_\rho h_{\mu\nu}$

$$\left\{ \Gamma \simeq \frac{1}{2}(\eta - h)(\partial h^{*} + \partial h^{\#} - \partial h^{\$}) \right.$$

$$= \frac{1}{2}\eta\left((\partial h)^{*} + (\partial h)^{\#} - (\partial h)^{\$}\right)$$

$$\simeq \cancel{- \frac{1}{2} h(\partial h^{*} + \partial h^{\#} - \partial h^{\$})} \to$$

$$\|\partial_\alpha h_{\mu\nu}\| \ll 1$$

$$\Downarrow$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\alpha}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right)$$

$\boxed{*}$ Riemann : $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu(\Gamma^{\rho}_{\nu\sigma}) - \partial_\nu(\Gamma^{\rho}_{\mu\sigma})$
$+ (\Gamma\Gamma)^{*} - (\Gamma\Gamma)^{\#}$

$$\Gamma\Gamma \sim \frac{1}{2}\eta.\eta \left[\cancel{\partial h \partial h + \partial h \partial h} \to 0 \right]$$

$$\therefore R^{\rho}{}_{\sigma\mu\nu} \cong \partial_\mu\left(\Gamma^{\rho}_{\sigma\nu}\right) - \partial_\nu\left(\Gamma^{\rho}_{\sigma\mu}\right)$$

$$= \partial_\mu\left\{\frac{1}{2}\eta^{\rho\alpha}\left(\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\alpha\sigma} - \partial_\alpha h_{\nu\sigma}\right)\right\}$$

$$- \partial_\nu\left[\ - - - - - - \ \right]$$

$$\downarrow$$

$$= \frac{1}{2}\eta^{\rho\alpha}\left(\partial_\mu\partial_\sigma h_{\alpha\nu} + \cancel{\partial_\mu\partial_\nu h_{\alpha\sigma}} - \partial_\mu\partial_\alpha h_{\nu\sigma} - \cancel{\partial_\nu\partial_\mu h_{\alpha\sigma}} + \partial_\nu\partial_\alpha h_{\mu\sigma} - \partial_\nu\partial_\alpha h_{\alpha\mu}\right)$$

$$\boxed{= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu\partial_\sigma h_{\alpha\nu} - \partial_\mu\partial_\sigma h_{\nu\sigma} - \partial_\nu\partial_\alpha h_{\alpha\mu} + \partial_\nu\partial_\alpha h_{\mu\sigma}\right)}$$

$\boxed{*}$ Ricci Tensor $R_{\sigma\nu} = R^{\mu}{}_{\sigma\mu\nu}$

$$\downarrow$$

$$R_{\sigma\nu} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h^{\mu}{}_{\nu} - \partial^{\mu}\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\sigma h^{\mu}{}_{\mu} + \partial_\nu\partial_\alpha h^{\alpha}{}_{\sigma}\right)$$

$$\downarrow \quad \Box \equiv \partial_\mu\partial^\mu, \quad h \equiv h^{\mu}{}_{\mu}$$

$$\boxed{R_{\sigma\nu} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h^{\mu}{}_{\nu} + \partial_\nu\partial_\alpha h^{\alpha}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h\right)}$$

(*) Ricci Scalar $\quad R = R^{\nu}{}_{\nu} = \eta^{\sigma\nu} R_{\sigma\nu}$

$$= \eta^{\sigma\nu} \frac{1}{2} \left(\partial_\mu \partial_\sigma h^{\mu}{}_{\nu} + \partial_\nu \partial_\alpha h^{\alpha}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu \partial_\sigma h \right)$$

$$\downarrow$$

$$= \frac{1}{2} \left(\partial_\mu \partial_\sigma h^{\mu\sigma} + \partial_\mu \partial_\sigma h^{\mu\sigma} - \Box h - \Box h \right)$$

$$\downarrow$$

$$\boxed{R = \partial_\mu \partial_\sigma h^{\mu\sigma} - \Box h}$$

$$\therefore \; G_{\mu\nu} = \frac{1}{2} \left(\partial_\alpha \partial_\mu h^{\alpha}{}_{\nu} + \partial_\nu \partial_\alpha h^{\alpha}{}_{\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h \right) - \frac{1}{2} \underbrace{g_{\mu\nu}}_{\eta_{\mu\nu}} \left(\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h \right)$$

$$= \frac{1}{2} \left(\partial_\alpha \partial_\mu h^{\alpha}{}_{\nu} + \partial_\nu \partial_\alpha h^{\alpha}{}_{\mu} - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \Box h \right)$$

$$\partial_\mu h^{\mu\nu} \rightarrow \partial_\mu \eta^{\mu\alpha} h_{\alpha}{}^{\nu} \rightarrow \partial^\alpha h_{\alpha}{}^{\nu}$$

$$\downarrow$$

$$= \frac{1}{2}\Big(\partial^\alpha \partial_\mu h_{\alpha\nu} + \partial^\alpha \partial_\nu h_{\mu\alpha} - \Box h_{\mu\nu}$$ ④

$$- \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \eta_{\mu\nu} \Box h \Big)$$

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$$

$\boxed{\text{Def}}$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} h$$

$$\partial^\alpha \partial_\mu h_{\alpha\nu} = \partial^\alpha \partial_\mu \left[\bar{h}_{\alpha\nu} + \frac{1}{2}\eta_{\alpha\nu} h \right]$$

$$= \partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \frac{1}{2} \partial^\alpha \partial_\mu \eta_{\alpha\nu} h$$

$$= \partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \frac{1}{2} \partial_\nu \partial_\mu h$$

Similarly : $\partial^\alpha \partial_\nu h_{\mu\alpha}$ ~~$= \frac{1}{2}\partial_\mu\partial_\nu h$~~

$$= \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} + \frac{1}{2} \partial_\mu \partial_\nu h$$

$$= \frac{1}{2}\Big[\partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} + \cancel{\partial_\mu \partial_\nu h} - \Box h_{\mu\nu} \cancel{- \partial_\mu \partial_\nu h} - \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \eta_{\mu\nu} \Box h \Big]$$

$$\eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} = \eta_{\mu\nu} \partial^\alpha \partial^\beta \left[\bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta} h \right]$$

$$= \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu} \partial^\alpha \partial_\alpha h$$

$$\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \partial^\alpha \partial_\alpha \frac{1}{2}\eta_{\mu\nu} h = \Box \bar{h}_{\mu\nu} + \partial^\alpha \partial_\alpha \frac{1}{2} \eta_{\mu\nu} h$$

$$= \frac{1}{2}\Big(\partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \partial^\alpha \partial_\nu \bar{h}_{\alpha\mu} - \Box \bar{h}_{\mu\nu} - \cancel{\frac{\partial^\alpha \partial_\alpha \eta_{\mu\nu} h}{2}}$$

$$- \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \cancel{\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial_\alpha h} \Big)$$

$$\cancel{= \frac{1}{2}\Big(\partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \partial^\alpha \partial_\nu \bar{h}_{\alpha\mu} - \Box \bar{h}_{\mu\nu}} + \cancel{\eta_{\mu\nu} \Box h}$$

$$\Downarrow$$

$$= \frac{1}{2}\Big(\partial^\alpha \partial_\mu \bar{h}_{\alpha\nu} + \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} - \overset{\Box \bar{h}_{\mu\nu}}{\cancel{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}}}$$

$$- \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} \Big)$$

$\therefore$ $\forall$ Linearized Gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \oplus \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\Downarrow$$

$$\frac{1}{2}\Big(\partial^\alpha (\partial_\mu \bar{h}_{\alpha\nu}) + \partial^\alpha (\partial_\nu \bar{h}_{\alpha\mu}) - \overset{\Box \bar{h}_{\mu\nu}}{\boxed{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}}} - \eta_{\mu\nu} \partial^\alpha (\partial^\beta \bar{h}_{\alpha\beta}) \Big)$$

$$= \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\downarrow \quad \text{"Lorenz Gauge"} \quad \partial_\beta \bar{h}^{\alpha\beta} = 0$$

$$\boxed{\Box \bar{h}^{\mu\nu} = 0}$$

E.F.E in weak field

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$ ⑤

$\Downarrow$

Inhomogeneous Wave eq^n $\rightarrow$ solved via Green's funct^n

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}\left[t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right]}{|\vec{x}-\vec{x}'|} d^3x'$$

$\downarrow$

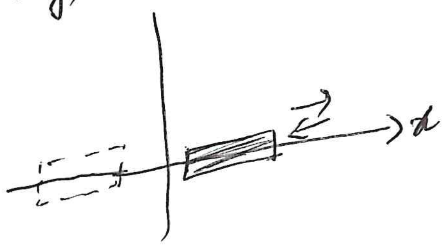
(i) Travel at c (ii) Value of $\frac{G}{c^4}$?

Eg) Thin oscillating rod along x-axis

$$T_{xx} = \mu \cos(\omega t)\delta(y)\delta(z)$$

$\mu \rightarrow$ mass-per unit len

$\omega \rightarrow$ oscillation freq.



$\downarrow$

$$\Box \bar{h}_{xx} = -\frac{16\pi G}{c^4}\mu \cos(\omega t)\delta(y)\delta(z)$$

solving [To check]

$\downarrow$

$$\bar{h}_{xx} = 4\frac{G}{c^4}\frac{\mu}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$$

$\frac{G}{c^4} \rightarrow$ weak

$\frac{1}{r} \rightarrow$ $1/r$ decay

c