LA

Assignment 3 (K22 6007)

$$(u,v) = \frac{1}{2}u_1v_1 + 5u_2v_2$$

(u,v)

(v,v) , $v = (3,2)$

$$(v,v) = \frac{1}{2}(3)(3) + 5(2)(2)$$

$$= \frac{9}{2} + \frac{20 \times 2}{1 \times 2} = \frac{49}{2} \geq 0$$

$$\Rightarrow (u,v) \Rightarrow \frac{1}{2}u_1v_1 + 5u_2v_2$$

$u = (1,1)$, $v = (3,2)$

$$\Rightarrow \frac{1}{2}(1)(3) + 5(1)(2)$$

$$\Rightarrow \frac{3}{2} + 10 \Rightarrow \frac{23}{2} \Rightarrow 11.5$$

Ans.

(b) (kv, w)

$u = kv \Rightarrow 3(3, 2)$
$= (9, 6)$

$v = w = (0, -1)$

$(u, v) = \frac{1}{2} u_1 v_1 + 5 u_2 v_2$

$= \frac{1}{2}(9)(0) + 5(6)(-1)$

$= -30$

(c) $((u+v), w)$

$u = u+v \Rightarrow (1, 1) + (3, 2)$
$\Rightarrow (4, 3)$

$v = w = (0, -1)$

$u = (4, 3)$, $v = (0, -1)$

$(u+v, w) = \frac{1}{2}(4)(0) + 5(3)(-1)$

$= -15$ Ans.

d) $\|v\|$ $v = (3, 2)$

$$\|v\| = \sqrt{\frac{1}{2} v_1 v_1 + 5 v_2 v_2}$$

$$= \sqrt{\frac{1}{2}(3)(3) + 5(2)(2)}$$

$$= \sqrt{\frac{49}{2}}$$

e) $d(u, v)$ $u = (1, 1)$, $v = (3, 2)$

$$\Rightarrow \|\vec{u} - \vec{v}\| = (1, 1) - (3, 2)$$

$$\Rightarrow (-2, -1)$$

$$\Rightarrow \sqrt{\frac{1}{2}(-2)(-2) + 5(-1)(-1)}$$

$$\Rightarrow \sqrt{2 + 5} = \cancel{\sqrt{6}}\ \sqrt{7} \text{ Ans.}$$

(f) $\|u - kv\|$ $u = (1, 1)$

$$kv = 3(3, 2) = (9, 6)$$

$$u - kv \Rightarrow (1, 1) - (9, 6)$$

$$= (\underset{v_1}{-8}, \underset{v_2}{-5})$$

$$\| = \sqrt{\frac{1}{2} v_1 v_2 + 5 v_2 v_2} = \sqrt{\frac{1}{2}(-2)(-8) + 5(-5)(-5)}$$

$= \sqrt{157}$

Q2. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

(a)

$\Rightarrow \langle u, v \rangle = v^T A^T A u / Au.Av$

$v^T = [3, 2]$

$A^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$\langle u, v \rangle = [3, 2] \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

$= [3, 2] \begin{bmatrix} 1+4 & 0+(-2) \\ 0-2 & 0\ 1 \end{bmatrix}$

$= [3, 2] \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= [3, 2] \begin{bmatrix} 5-2 \\ -2+1 \end{bmatrix}$

$= [3, 2] \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\Rightarrow 9 - 2 = 7$

b) $U = KV \Rightarrow (3)(3, 2) \Rightarrow (9, 6)$

$V = W = (0, -1)$

$U = (9, 6)$, $V = (0, -1)$

$\langle U, V \rangle = AU \cdot AV$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9+0 \\ 18-6 \end{bmatrix} \cdot \begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow 0 + 13 = 13$ Ans.

(c) $(U + V, W)$

$U = U + V \Rightarrow (1, 1) + (3, 2) \Rightarrow (4, 3)$

$V = W = (0, -1)$

$U = (4, 3)$, $V = (0, -1)$

$\langle U, V \rangle = AU \cdot AV$

$$\langle u, v \rangle = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+0 \\ 8-3 \end{bmatrix} \cdot \begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow 0 + 5 = 5 \text{ Ans.}$$

(d) $\|v\| = Av \cdot Av$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow 9 + 16 = 25 \text{ Ans.}$$

$$\|v\| = \sqrt{25}$$

(e) $d(u,v) \Rightarrow \|u-v\|$

$u - v = (1,1) - (3,2)$

$x \Rightarrow (-2,-1)$

$\|u-v\| = \|x\| = Ax \cdot Ax$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 \\ -4+1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -4+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$= 4 + 9 = 13$ Ans.

$\|u-v\| = \sqrt{13}$

(f) $\|(u-kv)\|$

$u - kv = (1,1) - 3(3,2) = (-8,-5)$

$A(u-kv) \cdot A(u-kv) \Rightarrow$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -8 \\ -11 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ -11 \end{bmatrix} = 64 + 121 = 185$$

$\|u-kv\| = \sqrt{185}$ Ans.

Q3. $\|P\|$, $\|q\|$, $d(p,q)$ and $\cos\theta$

$p = -2 + x + 3x^2$, $q = 4 - 7x^2$

- $\|P\| = \sqrt{(-2)^2 + (1)^2 + (3)^2}$

$= \sqrt{4+1+9}$

$\|P\| = \sqrt{14}$

- $\|q\| = \sqrt{4^2 + (-7)^2}$

$= \sqrt{16+49}$

$= \sqrt{65}$

- $\langle p, q \rangle$

$\Rightarrow -8 + 1.0 + 3.-7$

$= -8 - 21 = -29$

Ans.

- $d(p,q) = \|p-q\|$

$p - q = (-2 + x + 3x^2) - (4 - 7x^2)$

$= -6 + x + 10x^2$

$\|p-q\| = \sqrt{(6)^2 + (1)^2 + (10)^2}$

$= \sqrt{36+1+100}$

$d(p,q) = \sqrt{137}$

4:

a. $w_1 = (0, 2, 0)$, $w_2 = (3, 0, 3)$,
$w_3 = (-4, 0, 4)$

Orthogonal:

- $\langle w_1, w_2 \rangle = (0,2,0) \cdot \cancel{(0,3} (3,0,3)$
  $\langle w_1, w_2 \rangle = 0$

- $\langle w_2, w_3 \rangle = (3, 0, 3) \cdot (-4, 0, 4)$
  $= -12 + 12 = 0$

- $\langle w_1, w_3 \rangle$
  $\Rightarrow (0, 2, 0) \cdot (-4, 0, 4)$
  $= 0$

$\langle w_1, w_2 \rangle$, $\langle w_2, w_3 \rangle$, $\langle w_1, w_3 \rangle = 0$
<orthogonal>

Orthonomal:

$v_1 = \frac{w}{\|w_1\|}$, $v_2 = \frac{w_2}{\|w_2\|}$, $v_3 = \frac{w_3}{\|w_3\|}$

$\frac{(0,2,0)}{2}$, $\frac{(3, 0, 3)}{3\sqrt{2}}$, $\frac{(-4,0,4)}{4\sqrt{2}}$

$\{v_1, v_2, v_3\} = \left\{(0,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right\}$

(b) $u = (1, 2, 4)$

→ linear Combination of orthogonal Basis

$v_1 = (0, 1, 0)$; $v_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$v_3 = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$u = (1, 2, 4)$

$$U_3 = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} \cdot v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \frac{\langle u, v_3 \rangle}{\|v_3\|^2} \cdot v_3$$

$$\Rightarrow \frac{(0,1,0)(1,2,4)}{1} \cdot (0,1,0) + \frac{(1,2,4)(1/\sqrt{2}, 0, 1/\sqrt{2})}{1} + \frac{(1,2,4)(-1/\sqrt{2}, 0, 1/\sqrt{2})}{1}$$

$$\Rightarrow \boxed{2(0,1,0) + 5/\sqrt{2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \frac{3}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$$

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$$= (0,2,0) + \left(\frac{5}{2}, 0, \frac{5}{2}\right) + \left(\frac{-3}{2}, 0, \frac{3}{2}\right)$$

$$\boxed{U = 1,\ 2,\ 4}$$

Q5 $\quad S = \{(1,1,1), (-1,1,0), (1,2,1)\}$

$$V_1 = U_1 = (1,1,1)$$

$$V_2 \Rightarrow U_2 - \frac{\langle U_2, V_1 \rangle}{\|V_1\|^2} \cdot V_1$$

$$\Rightarrow (-1,1,0) - \frac{(-1,1,0) \cdot (1,1,1)}{3} \cdot (1,1,1)$$

$$= (-1,1,0) - (0,0,0)$$

$$V_2 = -1, 1, 0$$

$$V_3 \Rightarrow U_3 - \frac{\langle U_3, V_1 \rangle}{\|V_1\|^2} \cdot V_1 - \frac{\langle U_3, V_2 \rangle}{\|V_2\|} \cdot V_2$$

$$\Rightarrow (1,2,1) - \frac{(1,2,1)(1,1,1)}{3} \cdot (1,1,1) - \frac{(1,2,1)(-1,1,0)}{2}(-1,1,0)$$

$$\Rightarrow (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$\Rightarrow (1,2,1) - (4/3, 4/3, 4/3) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$\Rightarrow \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

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Orthonormal:

$$w_1 = \frac{v}{||v_1||}, \quad w_2 = \frac{v_2}{||v_2||}, \quad w_3 = \frac{v_3}{||v_3||}$$

$$= \frac{(1,1,1)}{\sqrt{3}}, \quad = \frac{-1,1,0}{\sqrt{2}}, \quad \frac{1/6, 1/6, -1/3}{\sqrt{6}}$$

$$w_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$w_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right)$$

$$w_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Q6.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$u_1 = (1, 0, 1)$, $\quad u_2 = (2, 1, 4)$

~~$v_1 = (1$~~

$v_1 = (1, 0, 1)$

$$v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\|v_1\|^2} \cdot v_1$$

$$= (2, 1, 4) - \frac{(1,0,1)(2,1,4)}{2} \cdot (1, 0, 1)$$

$$= (2, 1, 4) - \frac{6}{2}(1, 0, 1)$$

$$= (2, 1, 4) - (3, 0, 3)$$

$v_2 = (-1, 1, 1)$

$$q_1 = w_1 = \frac{v_1}{\|v_1\|} \qquad q_2 = w_2 = \frac{v_2}{\|v_2\|}$$

$$= \frac{(1, 0, 1)}{\sqrt{2}} \qquad = \frac{(-1, 1, 1)}{\sqrt{3}}$$

$$q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \qquad q = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle \end{bmatrix}$$

$u_1 = (1, 0, 1)$, $u_2 = (2, 1, 4)$

$q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $q_2 = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$R = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$$A = QR$$

$$\Rightarrow \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

Q8.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$\lambda \begin{bmatrix} \lambda-2 & -1 \\ 0 & \lambda-3 \end{bmatrix} - 0 + 2 \begin{vmatrix} -1 & \lambda-2 \\ -1 & 0 \end{vmatrix}$$

$$\lambda \left[(\lambda-2)(\lambda-3) \right] + 2 \left[0 - (-1)(\lambda-2) \right]$$

$$\lambda \left[\lambda^2 - 3\lambda - 2\lambda + 6 \right] + 2 \left[\lambda - 2 \right]$$

$$\lambda \left[\lambda^2 - 5\lambda + 6 \right] + 2\lambda - 4$$

$$\lambda^3 - 5\lambda^2 + 6\lambda + 2\lambda - 4$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2$$

For $\lambda = 1$,

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}$$

$R_3 + R_1$
$R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2t$
$x_2 = t$
$x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \Rightarrow$$

$$\Rightarrow t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

$R_1/2$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$R_2+R_1,\ R_3+R_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 &= -t \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix}, \quad t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

P diagonalijes A

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{22} = P D^{22} P^{-1}$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1^{22} & 0 & 0 \\ 0 & 2^{22} & 0 \\ 0 & 0 & 2^{22} \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -4 \times 10^{6} & 0 \\ 1 & 0 & 41 \times 10^{6} \\ 1 & 41 \times 10^{6} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{22} = \begin{bmatrix} -4194302 & 0 & -8388606 \\ 4194303 & 4194304 & 4194303 \\ 4194303 & 0 & 8388607 \end{bmatrix}$$

Q7. $A = \begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}$

$\frac{R_1}{3}$ $\begin{bmatrix} 1 & -2 & -2/3 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}$

$R_2 - 5R_1 = \begin{bmatrix} 1 & -2 & -2/3 \\ 0 & 6 & 10/3 \\ 0 & 7 & 1/3 \\ 0 & 10 & 11/3 \end{bmatrix}$

$\frac{R_2}{6} = \begin{bmatrix} 1 & -2 & -2/3 \\ 0 & 1 & 5/9 \\ 0 & 7 & 1/3 \\ 0 & 10 & 11/3 \end{bmatrix}$

$R_1 + 2R_2$, $R_3 - 7R_2$, $R_4 - 10R_2$ $\begin{bmatrix} 1 & 0 & 4/9 \\ 0 & 1 & 5/9 \\ 0 & 0 & -32/9 \\ 0 & 0 & -17/9 \end{bmatrix}$

$-\frac{9}{32} R_3 = \begin{bmatrix} 1 & 0 & 4/9 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \\ 0 & 0 & -17/9 \end{bmatrix}$

$R_1 - \frac{4}{9} R_3 =$, $R_2 - \frac{5}{9} R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_4 + \frac{17}{9} R_3$

Column Space = 3

Row space = 3

Rank = 3

Nullity = 0