IBM Qiskit Workshop
CHSH Inequality

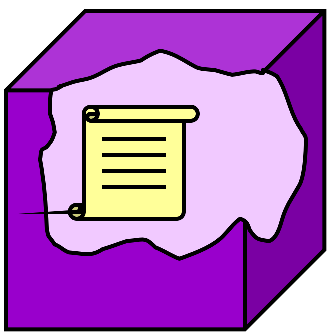
Speaker: Lavish Kumar Aidasani

CHSH Inequality and its violation

Part I: physicist's view:

Can a quantum state have ***pre-determined*** outcomes for each possible measurement that can be applied to it?

qubit:



where the "manuscript" is something like this:

if $\{|0\rangle,|1\rangle\}$ measurement then output **0**

if $\{|+\rangle,|-\rangle\}$ measurement then output **1**

if **...** (etc)

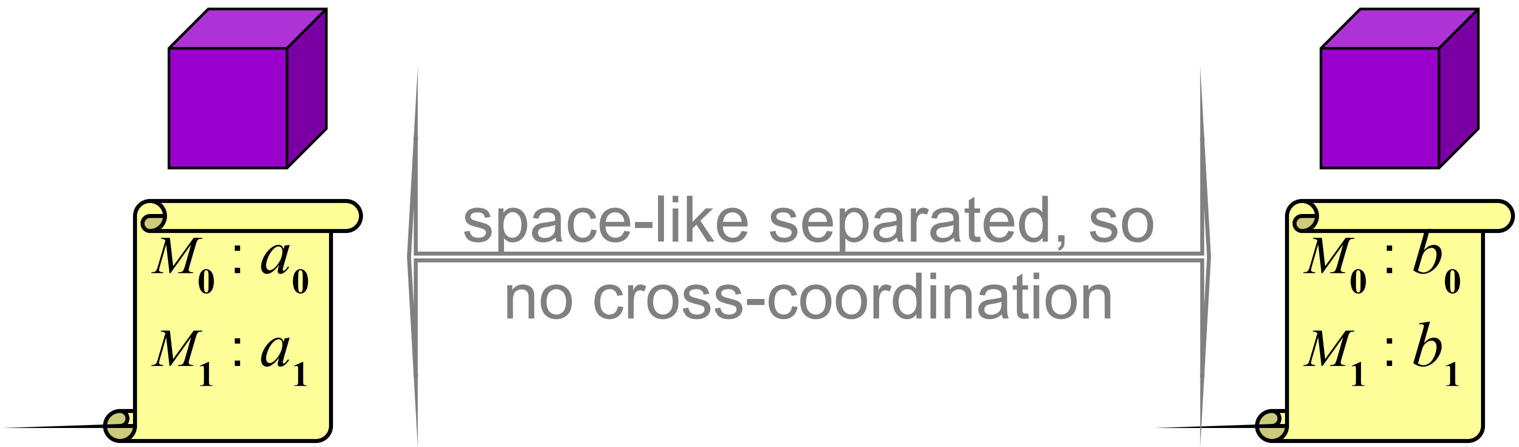
called ***hidden variables***

table could be implicitly given by some formula

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]

CHSH Inequality

Imagine a two-qubit system, where one of two measurements, called M_0 and M_1, will be applied to each qubit:



Define:

$$A_0 = (-1)^{a_0}$$
$$A_1 = (-1)^{a_1}$$
$$B_0 = (-1)^{b_0}$$
$$B_1 = (-1)^{b_1}$$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 (B_0 + B_1) + A_1 (B_0 - B_1) \leq 2$

one is ±2 and the other is 0

Nonlocal games

Mathematical abstractions of *games* are both important and useful.

The CHSH game is an example of a *nonlocal game.*

Set-up

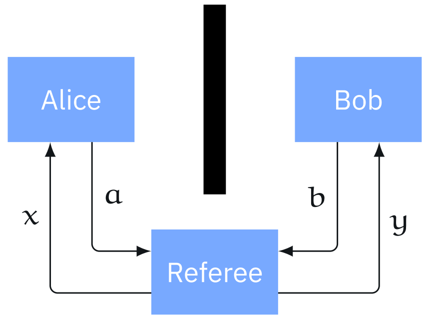
The players are Alice and Bob, who *cooperate as a team.*

The game is run by a *referee.*

Alice and Bob can prepare for the game however they choose...

...but once the game starts they are *forbidden from communicating.*

No communication between Alice and Bob



Nonlocal games

Mathematical abstractions of *games* are both important and useful.

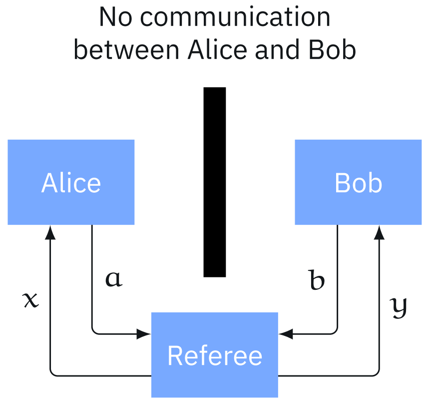
The CHSH game is an example of a *nonlocal game.*

The referee

The referee uses *randomness* to select the questions x and y.

The referee determines whether a pair of answers (a, b) *wins or loses* for the questions pair (x, y) according to some fixed rule.

(A precise description of the referee defines an instance of a nonlocal game.)



Nonlocal games

CHSH game referee

1. The questions and answers are all bits:

$$x, y, a, b \in \{0, 1\}$$

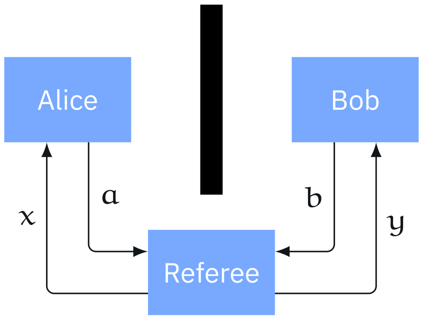
2. The questions x and y are chosen *uniformly at random.*

3. A pair of answers (a, b) *wins* for (x, y) if

$$a \oplus b = x \wedge y$$

and *loses otherwise.*

(x, y)	winning condition
$(0, 0)$	$a = b$
$(0, 1)$	$a = b$
$(1, 0)$	$a = b$
$(1, 1)$	$a \neq b$



Deterministic strategies

No deterministic strategy can win every time.

$$\begin{aligned} a(0) \oplus b(0) &= 0 \\ a(0) \oplus b(1) &= 0 \\ a(1) \oplus b(0) &= 0 \\ a(1) \oplus b(1) &= 1 \end{aligned}$$

It follows that no deterministic strategy can with with probability greater than $3/4$.

Nonlocal games

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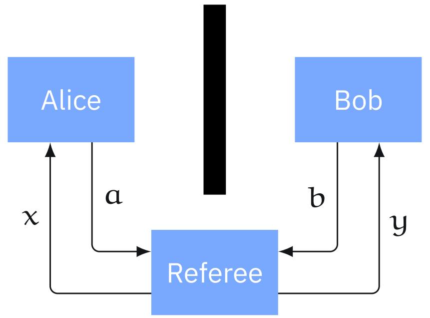

Probabilistic strategies

Every probabilistic strategy can be viewed as a *random choice* of a *deterministic* strategy.

It follows that no probabilistic strategy can win with probability greater than $3/4$.

Nonlocal games

CHSH game referee

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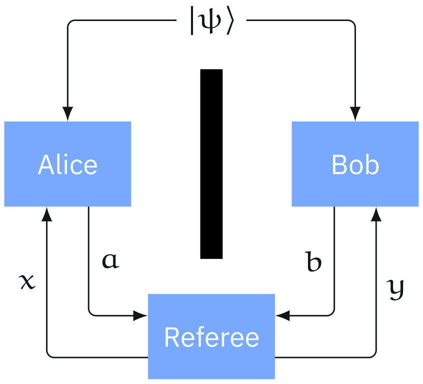
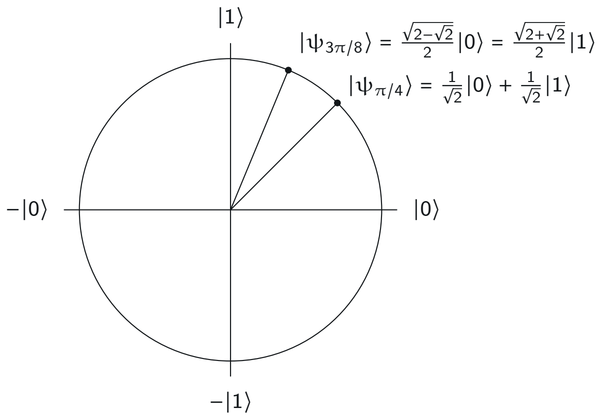


Can a *quantum strategy* do better?

CHSH game strategy

For each angle θ (measured in radians), define a unit vector

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$



θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$\frac{\pi}{8}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\frac{\pi}{2}$	0	1

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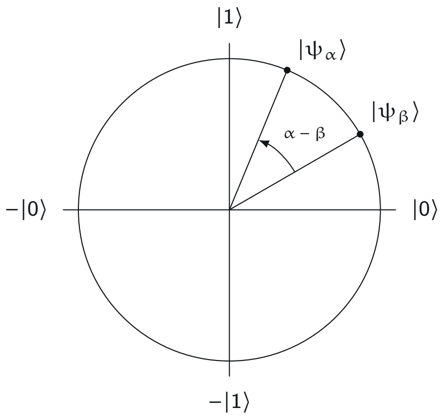

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$\frac{3\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\frac{\pi}{2}$	0	1

By one of the *angle addition formulas* we have

$$\langle\psi_\alpha|\psi_\beta\rangle = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$$

$$\langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{\sqrt{2}} = \frac{\cos(\alpha - \beta)}{\sqrt{2}}$$

CHSH game strategy

For each angle θ (measured in radians), define a unit vector

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

Also define a unitary matrix

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}|$$

Alice and Bob's strategy

Alice and Bob share an e-bit (A, B).

Alice's actions

Alice applies an operation to A as follows:

$$\begin{cases} U_0 & \text{if } x = 0 \\ U_{\pi/4} & \text{if } x = 1 \end{cases}$$

She then measures A and sends the result to the referee.

Bob's actions

Bob applies an operation to B as follows:

$$\begin{cases} U_{\pi/8} & \text{if } y = 0 \\ U_{-\pi/8} & \text{if } y = 1 \end{cases}$$

He then measures B and sends the result to the referee.

CHSH game strategy

Alice and Bob's strategy

Alice and Bob share an e-bit (A, B).

Alice's actions

Alice applies an operation to A:

$$\begin{cases} U_0 & \text{if } x = 0 \\ U_{\pi/4} & \text{if } x = 1 \end{cases}$$

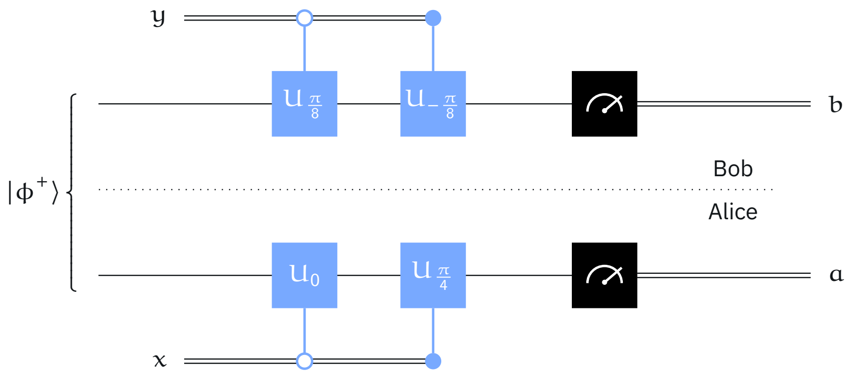
She then measures A and sends the result to the referee.

Bob's actions

Bob applies an operation to B:

$$\begin{cases} U_{\pi/8} & \text{if } y = 0 \\ U_{-\pi/8} & \text{if } y = 1 \end{cases}$$

He then measures B and sends the result to the referee.



Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 1: $(x, y) = (0, 0)$

Alice performs U_0 and Bob performs $U_{\frac{\pi}{8}}$.

$$\begin{aligned}(U_0 \otimes U_{\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_0 \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_0 \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &\quad + |10\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &= \frac{\cos(-\frac{\pi}{8})|00\rangle + \cos(-\frac{5\pi}{8})|01\rangle + \cos(\frac{3\pi}{8})|10\rangle + \cos(-\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

(a, b)	Probability	Simplified
$(0, 0)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(0, 1)$	$\frac{1}{2}\cos^2(-\frac{5\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 0)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

$$\Pr(a \neq b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 2: $(x, y) = (0, 1)$

Alice performs U_0 and Bob performs $U_{-\frac{\pi}{8}}$.

$$\begin{aligned}(U_0 \otimes U_{-\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_0 \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_0 \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\ &\quad + |10\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\ &= \frac{\cos(\frac{\pi}{8})|00\rangle + \cos(-\frac{3\pi}{8})|01\rangle + \cos(\frac{5\pi}{8})|10\rangle + \cos(\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

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$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 3: $(x, y) = (1, 0)$

Alice performs $U_{\frac{\pi}{4}}$ and Bob performs $U_{\frac{\pi}{8}}$.

$$\begin{aligned}(U_{\frac{\pi}{4}} \otimes U_{\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &\quad + |10\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &= \frac{\cos(\frac{\pi}{8})|00\rangle + \cos(-\frac{3\pi}{8})|01\rangle + \cos(\frac{5\pi}{8})|10\rangle + \cos(\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

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$(1, 0)$	$\frac{1}{2}\cos^2(\frac{5\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

$$\Pr(a \neq b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha\otimes\psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 4: $(x, y) = (1, 1)$

Alice performs $U_{\frac{\pi}{4}}$ and Bob performs $U_{-\frac{\pi}{8}}$.

$$\begin{aligned}(U_{\frac{\pi}{4}}\otimes U_{-\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_{\frac{\pi}{4}}\otimes\psi_{-\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_{\frac{\pi}{4}}\otimes\psi_{\frac{3\pi}{8}}|\phi^+\rangle\\ &\quad + |10\rangle\langle\psi_{\frac{3\pi}{4}}\otimes\psi_{-\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{3\pi}{4}}\otimes\psi_{\frac{3\pi}{8}}|\phi^+\rangle\\ &= \frac{\cos(\frac{3\pi}{8})|00\rangle + \cos(-\frac{\pi}{8})|01\rangle + \cos(\frac{7\pi}{8})|10\rangle + \cos(\frac{3\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

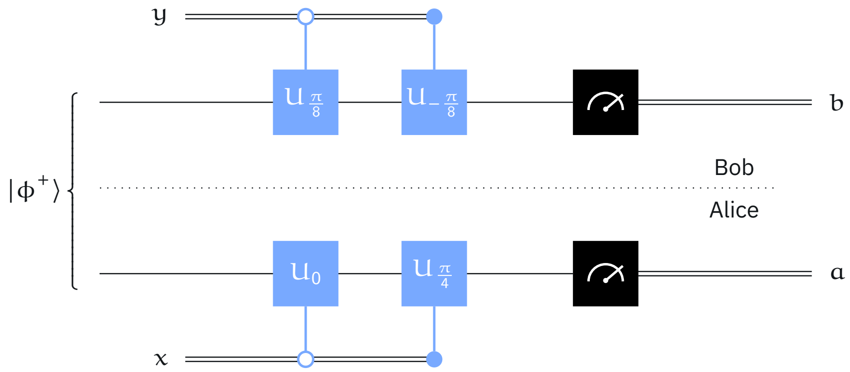
(a, b)	Probability	Simplified
$(0, 0)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(0, 1)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(1, 0)$	$\frac{1}{2}\cos^2(\frac{7\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

$$\Pr(a \neq b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy



The strategy wins with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$ (in all four cases, and therefore overall).

Analysis of the strategy

We can also think about the strategy geometrically.

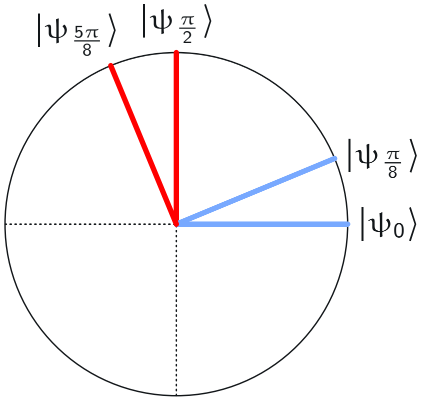
Using the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle$$

the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (0, 0)$

(a, b)	Probability
$(0,0)$	$\frac{1}{2}\lvert\langle \psi_0 \vert \psi_{\frac{\pi}{8}} \rangle\rvert^2$
$(0,1)$	$\frac{1}{2}\lvert\langle \psi_0 \vert \psi_{\frac{5\pi}{8}} \rangle\rvert^2$
$(1,0)$	$\frac{1}{2}\lvert\langle \psi_{\frac{\pi}{2}} \vert \psi_{\frac{\pi}{8}} \rangle\rvert^2$
$(1,1)$	$\frac{1}{2}\lvert\langle \psi_{\frac{\pi}{2}} \vert \psi_{\frac{5\pi}{8}} \rangle\rvert^2$



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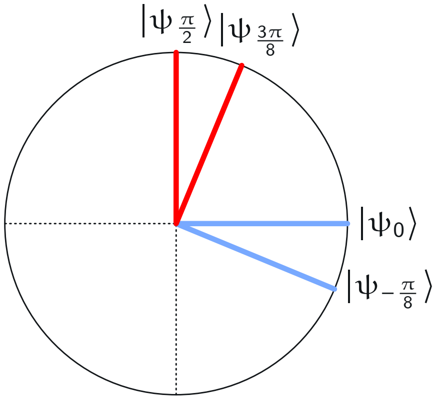
the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (0, 1)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_0 \vert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_0 \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \vert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$



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Using the formula

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the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (1, 0)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{\frac{\pi}{8}} \rangle \rvert^2$
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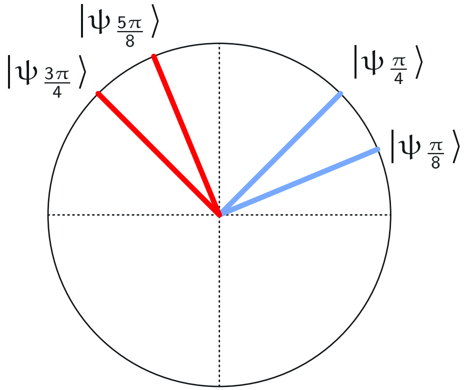

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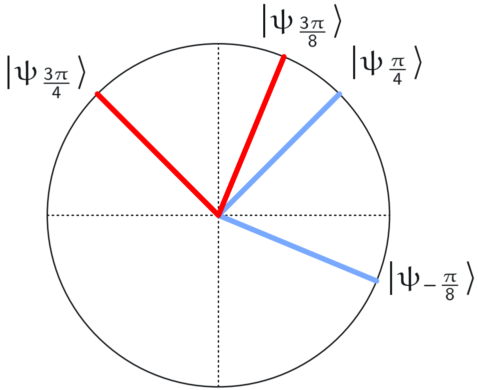
the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (1, 1)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{-\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$



Remarks on the CHSH game

- The CHSH game is not always described as a game — it's often described as an experiment, or an example of a *Bell test.*
- The CHSH game offers a way to *experimentally test* the theory of quantum information.

  (The 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger for experiments that do this with entangled photons.)
- The study of nonlocal games more generally is a fascinating and active area of research that still holds many mysteries.