ITEC- 322 Discrete Structures

Functions

Functions(Introduction)

- In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first).
- For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F}.
- And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure in book.

Functions(Introduction)

- This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science.

- Let A and B be nonempty sets. A function *f* from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function *f* to the element a of A. If f is a function from A to B, we write $f : A \rightarrow B$.

- Functions are sometimes also called mappings or transformations.

Functions (Cont…)

- Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in Figure 1.
- Often we give a formula, such as $f(x) = x + 1$, to define a function.
- Other times we use a computer program to specify a function.

Functions (Cont…)

- A function $f : A \to B$ can also be defined in terms of a relation from A to B.
- Recall from Section of sets that a relation from A to B is just a subset of $A \times B$.
- A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B.
- This function is defined by the assignment $f(a) = b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

Functions (Cont…)

- If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.
- If f (a) = b, we say that b is the image of a and a is a preimage of b.
- The range, or image, of f is the set of all images of elements of A.
- Also, if f is a function from A to B, we say that f maps A to B.

Functions (Cont…)

- EXAMPLE 1 :

  What are the domain, codomain, and range of the function that assigns grades to students described in the first paragraph of the introduction of this section?
- Solution: Let G be the function that assigns a grade to a student in our discrete mathematics class.
- Note that G(Adams) = A, for instance. The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set {A, B, C, D, F}.
- The range of G is the set {A, B, C, F}, because each grade except D is assigned to some student.

Functions (Cont…)

- Let f : Z → Z assign the square of an integer to this integer.
- Then, f (x) = x power 2, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9,...}.

- EXAMPLE 5 The domain and codomain of functions are often specified in programming languages.
- For instance, the Java statement int floor(float real){...} and the C++ function statement int function (float x){...} both tell us that the domain of the floor function is the set of real numbers (represented by floating point numbers) and its codomain is the set of integers.

Functions (Cont…)

- DEFINITION

  Let $f1$ and $f2$ be functions from A to R. Then $f1 + f2$ and $f1f2$ are also functions from A to R defined for all $x \in A$ by

  $$(f1 + f2)(x) = f1(x) + f2(x),$$

  $$(f1f2)(x) = f1(x)f2(x).$$

Functions (Cont…)

- EXAMPLE

  Let f1 and f2 be functions from R to R such that

  f1(x) = x square 2 and f2(x) = (x − x square 2). What are the functions f1 + f2 and f1f2?

  Solution: From the definition of the sum and product of functions, it follows that (f1 + f2)(x) = f1(x) + f2(x) = x square 2 + (x − x2) = x and (f1f2)(x) = x2(x − x2) = x3 − x4.

Types of Functions

- A function f is said to be one-to-one, or an injunction, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

- Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with f (a) = 4, f (b) = 5, f (c) = 1, and f (d) = 3 is one-to-one.

  Solution: The function f is one-to-one because f takes on different values at the four elements of its domain.

Types of Functions

- Determine whether the function f (x) = x power 2 from the set of integers to the set of integers is one-to-one.

- The function f (x) = x2 is not one-to-one because, for instance, f (1) = f (−1) = 1, but 1 is not equal to −1.

- Note that the function f (x) = x power 2 with its domain restricted to Z+ is one-to-one. .

Types of Functions

- Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

- Solution: The function $f(x) = x + 1$ is a one-to-one function.

Types of Functions

- A function f from A to B is called onto, or a surjection, if and only if for every element b ∈ B there is an element a ∈ A with f (a) = b. A function f is called surjective if it is onto.

- Is the function f (x) = x power 2 from the set of integers to the set of integers onto?

- The function f is not onto because there is no integer x with x square 2 = −1, for instance.

Types of Functions

- The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

- Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f (a) = 4, f (b) = 2, f (c) = 1, and f (d) = 3. Is f a bijection?

- The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value.
- It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Types of Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

- If a function f is not a one-to-one correspondence, we cannot define an inverse function of f.

- When f is not a one-to-one correspondence, either it is not one-to-one or it is not onto.

Types of Functions

- If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain.
- If f is not onto, for some element b in the codomain, no element a in the domain exists for which f (a) = b.

- Consequently, if f is not a one-to-one correspondence, we cannot assign to each element b in the codomain a unique element a in the domain such that f (a) = b.

- A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Types of Functions

- Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

- Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Types of Functions

- Let f be the function from R to R with f (x) = x square 2. Is f invertible?

- Because f (−2) = f (2) = 4, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

Inverse Functions and Compositions of Functions

- Consider a one-to-one correspondence f from the set A to the set B. Because f is an onto function, every element of B is the image of some element in A. Furthermore, because f is also a one-to-one function, every element of B is the image of a unique element of A. Consequently, we can define a new function from B to A that reverses the correspondence given by f.

Inverse Functions and Compositions of Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

- A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist. See figure on page 145.

Inverse Functions and Compositions of Functions

- Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

  Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Inverse Functions and Compositions of Functions

- Let f be the function from R to R with f (x) = x2. Is f invertible?

Solution: Because f (−2) = f (2) = 4, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.)

Inverse Functions and Compositions of Functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to g(a). That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain g(a) and then we apply the function f to the result g(a) to obtain $(f \circ g)(a) = f(g(a))$.

Inverse Functions and Compositions of Functions

- Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined.

Moreover, $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$