06-03-21

⇒ Axioms of Indifference curve :-

1. Completeness [Constant IC]
2. Transitivity [IC never intersect]
3. Continuity [IC smooth, twice differentiate, without jumps]
4. Non Satiation [↑ higher IC, ↑ higher satisfy, -ve IC]
5. Convexity [IC are convex] MRS → diminishing the MU

⇒ Contour Set :-

A Set which has same or constant value

Functions	Their Contour Set
① Utility	IC
② Production	IQ
③ Profit	Iso Profit

IF CONTOUR ~~FUNCTION~~ SET OF ANY FUNCTION IS CONVEX THEN ANY FUNCTION WOULD BE QUASI CONCAVE.

⇒ Utility Function :-

$U = f(\text{commodities})$ i.e $U(X_i)$

(i) $x' \sim x'' \rightarrow u(x') = u(x'')$

(ii) $x' > x'' \rightarrow u(x') > u(x'')$

(iii) $x' < x'' \rightarrow u(x') < u(x'')$

if $u(x') = a$, let $u(x'')$ call it b where $a = 10$ & $b = 5$ then $a > b$

⇒ Properties of utility function :-

① Increasing in X_i → (non satiation) → More commodity More utility

i.e $\frac{\partial U_i}{\partial X_i} > 0$

② Continuous in X_i → (continuity) → $\frac{\partial\left(\frac{\partial U}{\partial x}\right)}{\partial X_i} > 0$

③ Quasi concave → (MRS is diminishing) → Convexity

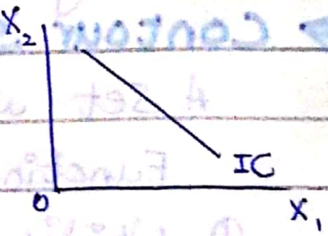
I = SOC of U maxi

MRS −ve = ایک بڑھنے سے دوسرا گھٹتا ہے
MRS +ve = Second derivative

$\bar{U} = (X_1, X_2)$

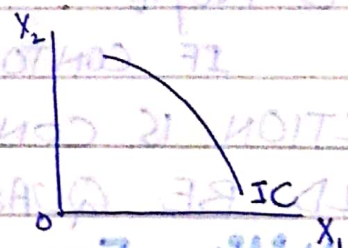
→ **Constant MRS**

X_1	X_2	$MRS_{12} = \Delta X_2 / \Delta X_1$
1	20	−
2	18	−2
3	16	−2
4	14	−2



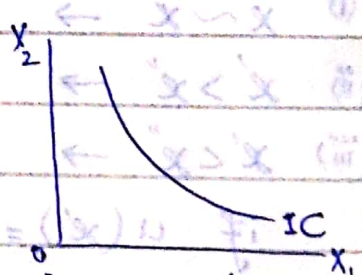
→ **Increasing MRS**

X_1	X_2	$MRS_{12} = \Delta X_2 / \Delta X_1$
1	20	−
2	15	−5
3	9	−6
4	2	−7



→ **Diminishing MRS**

X_1	X_2	$MRS_{12} = \Delta X_2 / \Delta X_1$
1	20	−
2	15	−5
3	11	−4
4	8	−3



MRS کو جب بھی تصویری میں ڈسکس کریں تو (−ve) سائن کو اگنور کر دیں۔

→ **Mathematical :-**

$$\bar{U} = (X_1, X_2)$$

$$d\bar{U} = \frac{\partial U}{\partial X_1} \cdot dX_1 + \frac{\partial U}{\partial X_2} \cdot dX_2 \qquad \because d\bar{U} = 0$$

$$0 = U_1 dX_1 + U_2 dX_2$$

$$-U_1 dX_1 = U_2 dX_2$$

By cross xing

$$\frac{dX_2}{dX_1} = -\frac{U_1}{U_2} \quad (MRS_{12})$$

$\Rightarrow$ $U_1 > 0$ & $U_2 > 0$ due to non satiation

Now taking 2nd derivative:

as $\bar{U} = (X_1, X_2)$

$$\rightarrow \frac{dX_2}{dX_1} = -\frac{U_1(X_1, X_2)}{U_2(X_1, X_2)}$$

quotient rule derivation

$$\rightarrow -\frac{1}{U_2^2}\left[U_2\left\{U_{11} + U_{12}\frac{dX_2}{dX_1}\right\} - U_1\left\{U_{21} + U_{22}\frac{dX_2}{dX_1}\right\}\right]$$

As $\frac{dX_2}{dX_1} = -\frac{U_1}{U_2}$

Therefore

$$\Rightarrow \frac{d^2X_2}{dX_1^2} = -\frac{1}{U_2^2}\left[U_2\left\{U_{11} - U_{12}\cdot\frac{U_1}{U_2}\right\} - U_1\left\{U_{21} - U_{22}\cdot\frac{U_1}{U_2}\right\}\right]$$

$$\Rightarrow \frac{d^2X_2}{dX_1^2} = -\frac{1}{U_2^2}\left[U_2U_{11} - U_{12}\cdot\frac{U_1}{\cancel{U_2}}\cdot\cancel{U_2} - U_1U_{12} + U_{22}\cdot\frac{U_1}{U_2}\cdot U_1\right]$$

we multiply & divide & expression with U_2,

$\Rightarrow$ For cancellation of denominator (U_2)

$$\frac{d^2X_2}{dX_1^2} = -\frac{1}{U_2^2}\cdot\frac{1}{U_2}\left[U_{11}U_2^2 - U_1U_2U_{12} - U_1U_2U_{12} + U_1^2U_{22}\right]$$

$$\frac{d^2X_2}{dX_1^2} = -\frac{1}{U_2^3}\left[U_{11}U_2^2 - 2U_1U_2U_{12} + U_1^2U_{22}\right] > 0$$

it is greater than zero bcoz MRS is diminishing.

Absolute diminishing

④

Thus $\frac{d^2 x_2}{dx_1^2} \gtreqless$ ·Convex
·Linear
·Concave

The overall sign of the $\frac{d^2x_2}{dx_1^2}$ is depends upon the sign of following $(U_1, U_2, U_{11}, U_{12}, U_{22})$ where $U_1 > 0$; $U_2 > 0$, $U_{11} \gtreqless 0$, $U_{22} \gtreqless 0$, $U_{12} \gtreqless 0$.

Normally, utility is diminishing but in some cases it can be constant or increasing.

→ $U_{12} > 0$ (complements) → combination
→ $U_{12} < 0$ (Substitutes) → no combination
→ $U_{12} = 0$ (Unrelated commodities)

$$\frac{d^2x_2}{dx_1^2} = -\frac{1}{U_2^3}\left[U_{11}U_2^2 - 2U_1U_2U_{12} + U_{22}U_1^2\right]$$

U_2^3: +, U_{11}: −0+, U_2^2: +, U_1: +, U_2: +, U_{12}: −0+, U_{22}: −0+, U_1^2: +

★ For convexity $(U_{11}U_2^2 - 2U_1U_2U_{12} + U_1^2U_{22}) < 0$

★ For concavity $(U_{11}U_2^2 - 2U_1U_2U_{12} + U_1^2U_{22}) > 0$

★ For Linearity $(U_{11}U_2^2 - 2U_1U_2U_{12} + U_1^2U_{22}) = 0$

⇒ Substitute (U_{12} -ve)

$$U_{11}U_2^2 + U_{22}U_1^2 - 2U_1U_2U_{12} \qquad \because U_{12}\ (-ve)$$

$$= U_{11}U_2^2 + U_{22}U_1^2 - 2U_1U_2(-ve)$$

$$= U_{11}U_2^2 + U_{22}U_1^2 + 2U_1U_2$$

(Substi)		$\because U_{11}U_2^2 + U_1^2U_{22} + 2U_1U_2$	Overall Sign
$U_{11}>0$	$U_{22}>0$	$+(+) + (+)(+) + 2$	$+$
$U_{11}<0$	$U_{22}>0$	$-(+) + (+)(+) + 2$	$+, -, 0$
$U_{11}=0$	$U_{22}>0$	$0(+) + (+)(+) + 2$	$+$
$U_{11}>0$	$U_{22}<0$	$+(+) + (+)(-) + 2$	$+, -, 0$
$U_{11}<0$	$U_{22}<0$	$-(+) + (+)(-) + 2$	$+, -, 0$
$U_{11}=0$	$U_{22}<0$	$0(+) + (+)(-) + 2$	$+, -, 0$
$U_{11}>0$	$U_{22}=0$	$+(+) + (+)(0) + 2$	$+$
$U_{11}<0$	$U_{22}=0$	$-(+) + (+)(0) + 2$	$+, -, 0$
$U_{11}=0$	$U_{22}=0$	$0(+) + (+)(0) + 2$	$+$

As there is no (-ve) Result so convexity is not guaranted in this case.

(-ve) ہونا ضروری ہے اندر والی ٹرم کا کیونکہ اندر کا
(-ve) جب باہر کے (-ve) کیساتھ آ جائیگا تو پوری ٹرم +ve ہو
جائیگی جو convexity بتاتی ہے۔

→ Complements (u_{12} +ve)

$u_{11}u_2^2 + u_{22}u_1^2 - 2u_1u_2u_{12}$

$= u_{11}u_2^2 + u_{22}u_1^2 - 2u_1u_2(+ve)$

$= u_{11}\oplus + u_{22}\oplus - 2\oplus\oplus$

Comp.	$u_{11}\overset{\oplus}{u_2^2} + u_{22}\overset{\oplus}{u_1^2} - 2\overset{\oplus}{u_1}\overset{\oplus}{u_2}$	Overall Sign
$u_{11}>0, u_{22}>0$	$+\oplus + \oplus - 2$	$+, -, 0$
$u_{11}>0, u_{22}<0$	$+\oplus + \ominus - 2$	$+, -, 0$
$u_{11}>0, u_{22}=0$	$+\oplus + 0 - 2$	$+, -, 0$
$u_{11}<0, u_{22}>0$	$\ominus + + \oplus - 2$	$+, -, 0$
$u_{11}<0, u_{22}<0$	$-\oplus + \ominus - 2$	$-$
$u_{11}<0, u_{22}=0$	$-\oplus + 0 - 2$	$-$
$u_{11}=0, u_{22}>0$	$0 + + \oplus - 2$	$+, -, 0$
$u_{11}=0, u_{22}<0$	$0\oplus + \ominus - 2$	$-$
$u_{11}=0, u_{22}=0$	$0\oplus \quad \oplus 0 - 2$	$-$

→ **Unrelated ($u_{12} = 0$)**

$$u_{11}u_2^2 + u_{22}u_1^2 - 2u_1u_2u_{12}$$
$$= u_{11}u_2^2 + u_{22}u_1^2 - 2u_1u_2(0)$$
$$= u_{11}u_2^2 + u_{22}u_1^2$$
$$= u_{11}(+) + u_{22}(+) \Rightarrow u_{11} + u_{22}$$

		u_{11} + u_{22}	Result
$u_{11} > 0$	$u_{22} > 0$	(+) + (+)	+ve
$u_{11} > 0$	$u_{22} < 0$	(+) + (−)	+ve −ve 0
$u_{11} > 0$	$u_{22} = 0$	(+) + (0)	+ve
$u_{11} = 0$	$u_{22} < 0$	0 + (−)	−ve
$u_{11} = 0$	$u_{22} > 0$	0 + (+)	+ve
$u_{11} = 0$	$u_{22} = 0$	0 + 0	zero
$u_{11} < 0$	$u_{22} > 0$	(−) + (+)	+ve −ve 0
$u_{11} < 0$	$u_{22} < 0$	(−) + (−)	−ve
$u_{11} < 0$	$u_{22} = 0$	(−) + 0	−ve

THEORY OF CONSUMER BEHAVIOUR

⇒ Concept of Consumption Sets:-

Cosumption set comprises of all consumption bundles which consumer can conceive wheather they are achieveable or not.

If we are consuming for our satisfaction so we are consumer & have ability to satisfy our desire.

⇒ Properties of Consumption Set:-

1. Each & every set is satisfiable.
2. Our consumption set are non identical.

⇒ All consumption bundles have ability to satisfy our desire.

⇒ consumption set is always non empty at least one consumption bundle

⇒ Consumption set is closed

* Boundries are defined & the boundries are the part of the set.

EX #

$$0 \leq x \leq 10 \rightarrow \text{closed Set}$$

while

$$\left.\begin{array}{l} 0 < x \leq 10 \\ 0 < x < 10 \end{array}\right\} \text{Bounded but not closed set}$$

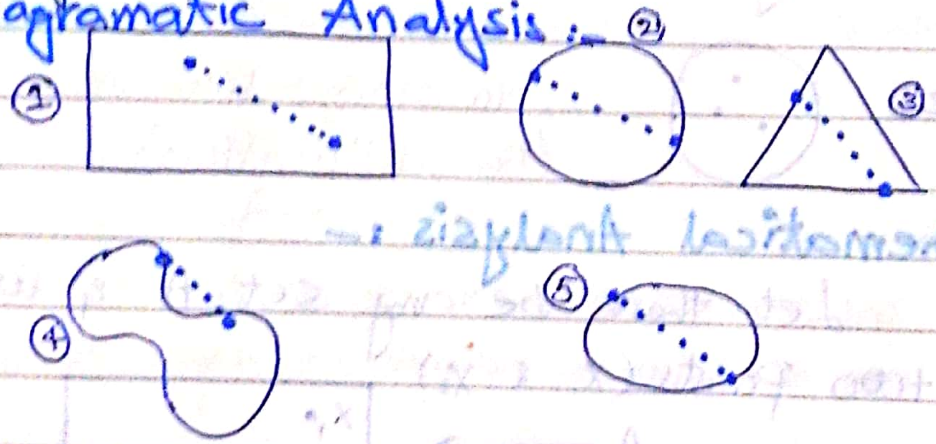
Every closed set is bounded set but every bounded set is not closed set

→ Consumption sets are convex set

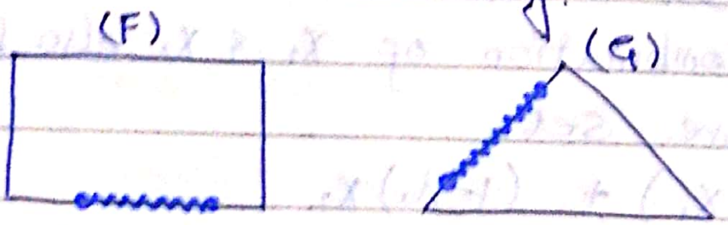
★ **Convex Set :-**

A set whose linear combination of any two consumption bundles also belong to the same set

⇒ **Diagramatic Analysis :-**



In (1), (2), (3) & (5) linear combinations are within the boundry.



In the above diagrams sets are along the boundries, these sets are called weakly convex set.

⇒ **Weakly Convex Set :-**

If any possibility along the boundry at least one straight line.

Ex # (F) & (G)

⇒ Strong convex Set :-

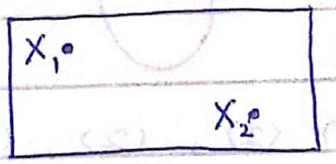
If there is no possibility along the boundry :

- No straight line
- strictly inside the boundries

i.e  → No straight line (For justification)

⇒ Mathematical Analysis :-

Let there be any set A & we took two Points (X_1 & X_2)

A ⟶ 

Then set A will be convex set. it's linear combination of X_1 & X_2 also belong to the same set.

$$A = \lambda(X_1) + (1-\lambda)X_2$$

where $0 \le \lambda \le 1$

Example

$N = 1, 2, 3 \ldots 10$

$X_1 = 3$ & $X_2 = 8$

$\lambda X_1 + X_2(1-\lambda) \Rightarrow 3\lambda + 8(1-\lambda)$

$\Rightarrow 3\lambda + 8 - 8\lambda \Rightarrow$ Let $\lambda = 0$

$\Rightarrow 0 + 8 \Rightarrow 8 \in N \rightarrow$ Convex set

Let $\lambda = 0.1$

$= 3(0.1) + 8 - 8(0.1)$

$= 7.5$

as $7.5 \notin N \rightarrow$ not convex set

If Set belongs to Rational numbers

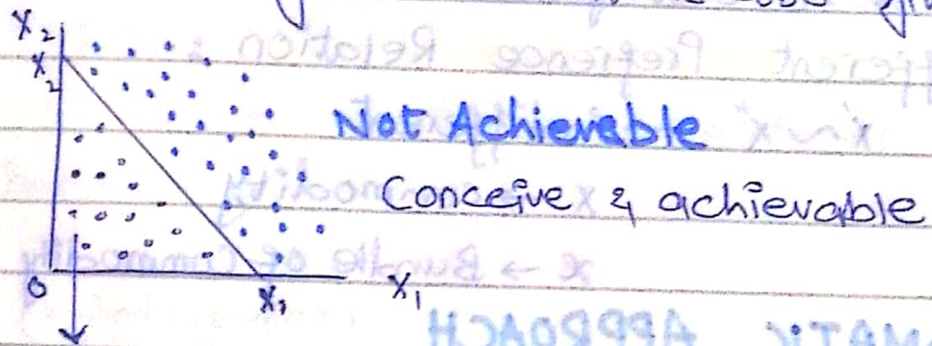
i.e $R = 1 \sim 10$

Here $7.5 \in R \rightarrow$ convex set

Because rational numbers also include the fractions.

FEASIBLE SET (Achieveable)

It's a subset of consumption set where budget constraint is also given



Feasible Set [We talk about budget line]

TWO APPROACHES OF CONSUMER BEHAVIOUR

1. Axiomatic Approach
2. Revealed Preference Theory

→ **Revealed Preference Approach** (Behaviour → Axiom)

* Something which consumer directly revealed. (Demand revealed)
* Something which is Observable.

* Something Opposite to axiom approach
Demand → equilibrium condition → Axioms

→ **Preference Relations :-**

1. Strong Preference relation :
One bundle is superior than Others
$x' > x''$ or $x' < x''$
x' is strongly Prefered.

2. Weak Prefrence Relation :
$x' \geq x''$ OR $x' \leq x''$
where x' is atleast good as x''.

3. Indifferent Prefrence Relation :-
$x' \sim x''$ → indifferent
X → commodity
x → Bundle of Commodity

AXIOMATIC APPROACH (Axiom → Behaviour)

It depend upon different axiom
Axioms (well maintained/established assumption)

① **Completeness :-**
Consumer has ability to rank different consumption bundle according to his/her Preference. For example x' & x'' are two consumption bundles & have certain bundles.

Then according to completeness Consumer Should be able to give any

One of the following Statement.

$x' > x''$ → One bundle strictly Prefer Over second

OR

$x' \geq x''$ or

$x'' > x'$ or

$x'' \geq x'$ or

$x' \sim x''$ So There no loop holes (سوراخ) in consumer Preference.

② Transitivity :- when 2 IC intersect

(Non Contradiction in Prefrences)

Rule of mathematics

Let there will be three different consumption bundle x', x'' & x'''

$x' > x''$ &

$x'' > x'''$

Compare & give ranking

Then according to transity

$x' > x'''$

We cannot compare them directly.

→ Consistence in Prefrence

→ There are not contradiction in consumer Preference.

→ Consumer can construct Indifference curve & IC dont intersect.

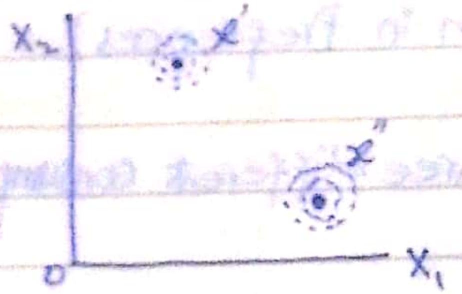
OR

IC construct through through completness never intersect

③ **Continuity**:-
There are no sudden jumps in consumer preferences

let:
$x' > x''$

Then continuity implies that all consumption bundles closer to x' & will also be prefered over x''.



→ consumer is more inclined towards second commodity (X_2)

→ No reversal due to small change

Sudden jump or Change:
If there is any change in consumption bundle that change does not reverse your Preferences.
OR
A Minor Change (Δ) in consumption bundle will not effect the Preference.

Minor Change : That does'nt Change taste & behaviour of Commodity.

④ **Non Satiation :-**

Non Satiation implies more is prefer over less.

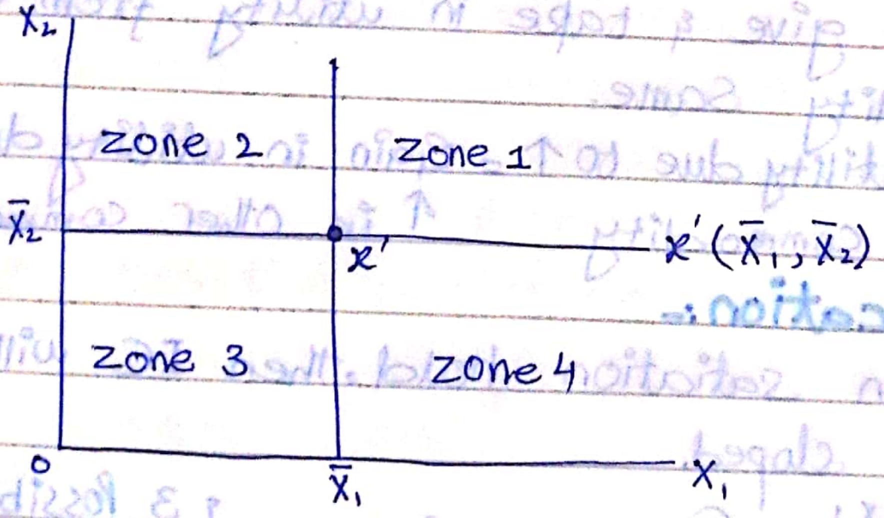
Q When more is prefer over less ?

when only & only MU is Positive.

All Commodities have Positive marginal utility.

Implication :

MU > zero



Zone 1 :- (including boundries)

* Superior
* Strictly Prefer to x'
* There is a point which is as good as x'

Zone 2 :- (excluding boundries)

Give & take

There can be few Consumption bundles

which are indifferent to x'. Only & only if give & take in utility from each commodity is same.

Loss in utility due to ↓ in one commodity = Gain in utility due to ↑ in other commodity

Zone 3:- (including boundries)

It is inferior as compare to x'

Zone 4:- (excluding boundries)

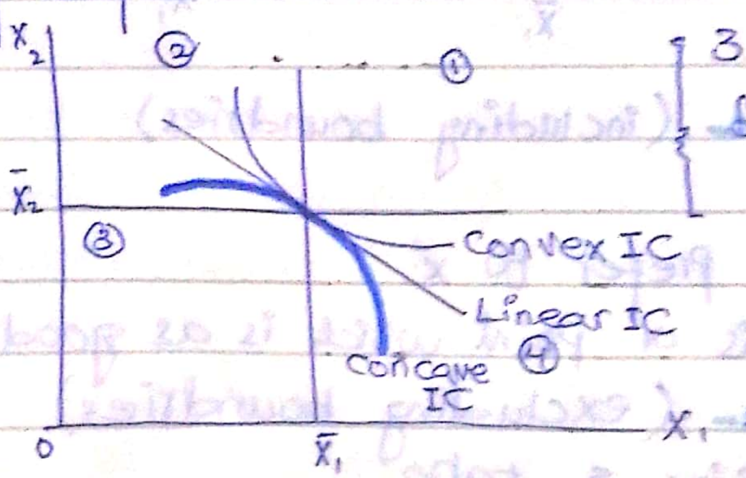
Give & take

There are few consumption bundles which are indifferent to x' only & only if give & take in utility from each commodity same.

Loss in utility due to ↑ in one commodity = Gain in utility due to ↑ in other commodity

Implication:-

If non satiation hold, then IC will be negatively sloped.

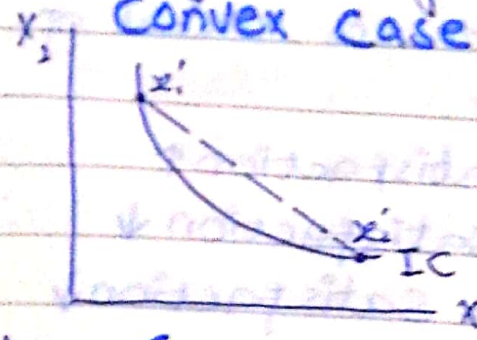


3 Possible Shapes of IC

5 ~~Continuity~~ Convexity

Convexity means That the linear combination of any two indifferent point / indifferent consumption bundles yeild higher utility

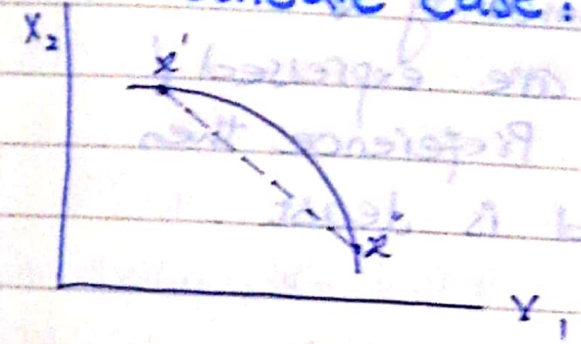
Convex case:



where $MRS_{12} \downarrow$

As the consumption bundles along the straight line are superior to x' & x''.

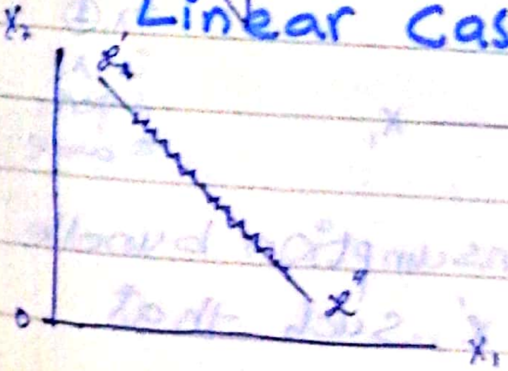
Now **Concave case:**



here $MRS_{12} \uparrow$

* Convexity does ~~dot~~ not hold
* All linear combinations below IC
* consumption bundles on straight line are inferior to x' & x''
* utility $\downarrow$ as compare to x' & x''

Linear Case:



Here $\overleftrightarrow{MRS_{12}}$

when consumption bundles along the straight line are indifferent to x' & x''.

THUS HIGHER UTILITY HOLD ON CONVEX IC

minor A B
minor A C

⇒ Concept of Lexocographic Ordering

→ continuity

* Commodities are written according to their Preferences.

* Consumer is always concern amount of 1st commodity.

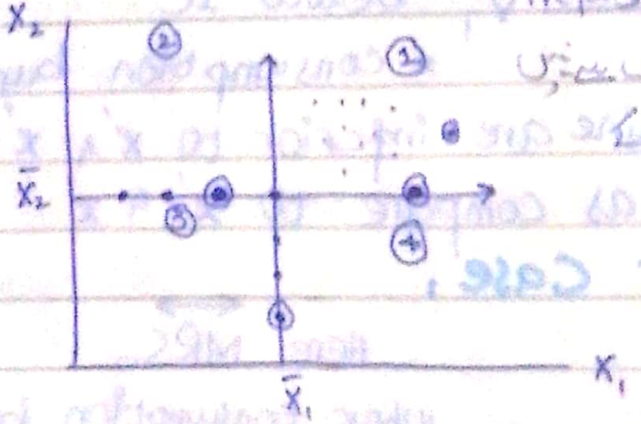
∵ 1st commodity ↑ → satisfaction ↑

1st commo. ↓ → satisfaction ↓

∵ All Other commodity ↑ → satisfaction ↓

→ 1st commodity ↓

When commodities are expressed / written according to Preference then 1st is most & second is least.



Prefer [illegible]

[illegible]

continuity [illegible]

[illegible]

↑ x_1 ① is x_1 superior to bundle

Let find one more consumption bundle which is indifferent to x', such that Lexocographic Ordering holds

bundle [illegible]

[illegible] change

IC lies & convex ← zone 2, 4
continuity does not hold 1, 3

Zone 1: Including boundries
amount of x' ↑ satisfaction ↑ & including boundries prefered over x'

Zone 3: Including boundries
Clearly inferior to x'
amount of x' ↓ → Satisfaction ↓ & x' is prefered over zone 3 including boundries

Zone 2: Inferior
Clearly inferior to x' wher x' is prefer over zone 2, amount of x_1 is less

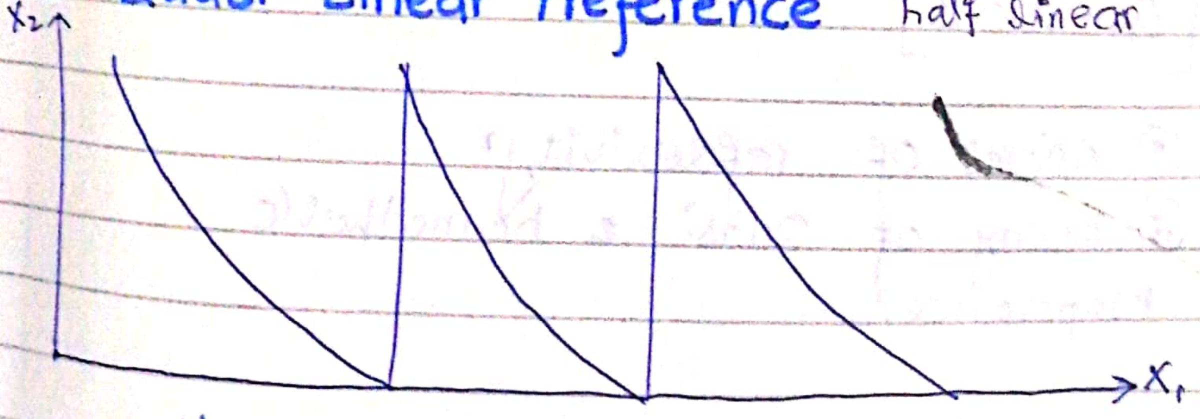
Zone 4: Superior
zone 4 is prefer over x'

There is no point which is indifferent in this? each & every consumption bundle has unique level of Satisfaction.

Application: We cannot construct IC because it ~~convexity~~ does not hold the continuity.

TWO CONCEPT :-

① Quasi Linear Preference half linear



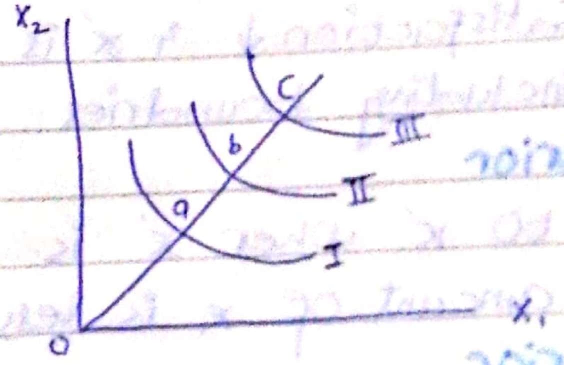
$U = X_1, X_2$

but $U = X_1, \sqrt{X_2}$ → Quasi linear

* All axioms do hold
* In every higher IC we have more of one commodity at all the points.

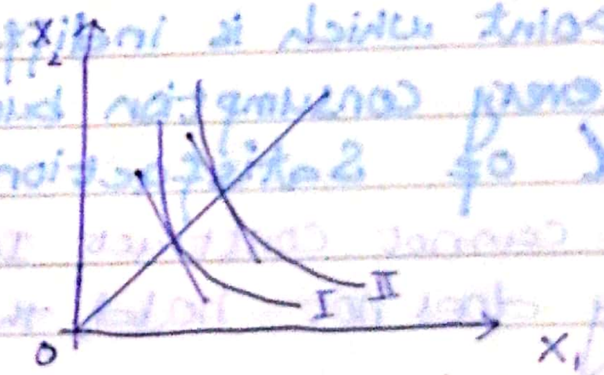
② **Homothetic Preferences**

IC are radical Projection of each other.



* Same Prices & Y ↑
* Same slope
* MRS is same at all intersecting Point

⇒ **Non Homothetic Preference**



* Different slope
* MRS is different on straight line

[Questions]

① Axioms of reflexivity?
② Theory of Quasi & homothetic Preference?

Utility Maximization

with constant prices & fixed budget where budget constraint is given

$$U(x_1, x_2, x_3 \dots x_n) \longrightarrow Max$$

s.t $\sum_{i=1}^{n} p_i x_i = M \qquad x_i \geq 0 \quad (i = 1, 2, 3 \dots n)$

$$\mathcal{L} = U(x_1, x_2, x_3 \dots x_n) + \lambda \left(M - \sum_{i=1}^{n} p_i x_i\right)$$

λ = Lagrange multiplier

FOC

$$\frac{\partial \mathcal{L}}{\partial x_i} = U_i - \lambda p_i = 0 \qquad (i = 1, 2, 3 \dots n)$$

$$\Rightarrow \lambda = \frac{U_i}{P_i} \qquad \text{with } n \rightarrow \text{equation}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - \sum_{i=1}^{n} p_i x_i = 0$$

$$\Rightarrow M - P_1 x_1 - p_2 x_2 - p_3 x_3 \dots p_n x_n = 0$$

nth equation

By solving simultaneously, we get UTILITY MAXIMUM DEMAND FUNCTION:

From FOC

$$\Rightarrow \lambda = \frac{U_1}{P_1} = \frac{U_2}{P_2} = \frac{U_3}{P_3} \dots \frac{U_n}{P_n}$$

$X_i(P_i, M) \Rightarrow$ ① Marshallian demand function
② Ordinary demand function
③ Utility maximization d fn
④ Money income held constant demand function

X_i = Commodities
P_i = Prices
M = Constraint/Income

Thus the solution of FOC will yield utility maximizing consumption bundle such that quantity demand will be the function of prices & given income.

⇒ Mathematical of Marshallian demand function $[X_i (P_i, M)]$ i.e $\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}$ & $\frac{\partial L}{\partial \lambda}$

GENERAL:-

$U = X_1 X_2 \quad s.t \quad M = P_1 X_1 + P_2 X_2$

$\mathcal{L} = X_1 X_2 + \lambda (M - P_1 X_1 - P_2 X_2)$

$\frac{\partial \mathcal{L}}{\partial X_1} = U_1 - \lambda P_1 = 0$

$\Rightarrow \lambda = U_1 / P_1$

$\frac{\partial \mathcal{L}}{\partial X_2} = U_2 - \lambda P_2 = 0$

$\Rightarrow \lambda = U_2 / P_2$

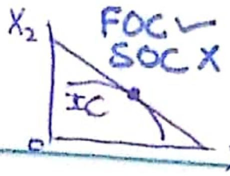
simultaneously FOC for 2 commodities

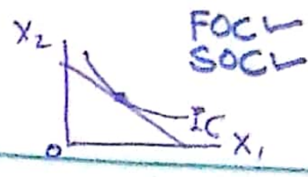
$\frac{\lambda}{\lambda} = \frac{U_1}{P_1} \times \frac{P_2}{U_2}$

$\frac{U_2}{P_2} = \frac{U_1}{P_1}$

OR $\frac{U_1}{U_2} = \frac{P_1}{P_2} \rightarrow$ Equilibrium level

Personal evaluation   Market evaluation





Market Value :- based upon demand & supply & is the price or amount that someone is willing to pay in the market.

Economic Value :- personal evaluation is the measurement of the benefit derived from a good or service to an individual or company.

SOC
$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > \text{zero}$$

$$= U_{11}\begin{vmatrix} U_{22} & -P_2 \\ -P_2 & 0 \end{vmatrix} - (U_{12})\begin{vmatrix} U_{21} & -P_2 \\ -P_1 & 0 \end{vmatrix} + (-P_1)\begin{vmatrix} U_{21} & U_{22} \\ -P_1 & -P_2 \end{vmatrix}$$

$$= U_{11}\left[(U_{22}(0) - (-P_2 . -P_2)\right] - U_{12}\left[U_{21}(0) - (-P_2 . -P_1)\right] - P_1\left[U_{21}(-P_2) - U_{22}(-P_1)\right]$$

$$= -U_{11}P_2^2 + U_{12}P_1P_2 + U_{21}P_1P_2 - U_{22}P_1^2 > 0$$

$$= -U_{11}P_2^2 - U_{22}P_1^2 + 2U_{12}P_1P_2 > 0$$

Quasi concave

As $\lambda = U_i/P_i$ so $P_i = U/\lambda$

Multiplying the whole term with $\lambda . U/\lambda$

$$\Rightarrow -U_{11}U_2^2 - U_{22}U_1^2 + 2U_{12}U_1U_2 > 0 \checkmark$$

OR

$$U_{11}U_2^2 + U_{22}U_1^2 - 2U_{12}U_1U_2 < 0 \checkmark$$

So IC Convex to Origin & Quasi Concave

This is condition for a utility function to be Quasi concave & alternatively for an IC to be convex. Therefore convexity of IC is SOC of utility maximization.

AN EXAMPLE OF MAXIMIZATION OF UTILITY FUNCTION

$U = X_1^{\alpha} X_2^{\beta}$ s.t $M = X_1P_1 + X_2P_2$

$\mathcal{L} = X_1^{\alpha} X_2^{\beta} + \lambda(M - X_1P_1 - X_2P_2)$

FOC

$$\frac{\partial \mathcal{L}}{\partial X_1} = \alpha X_1^{\alpha-1} X_2^{\beta} - \lambda P_1 = 0$$

$$\lambda P_1 = \alpha X_1^{\alpha-1} X_2^{\beta}$$

$$\lambda = \frac{\alpha X_1^{\alpha-1} X_2^{\beta}}{P_1} \quad \text{——— (i)}$$

$$\frac{\partial \mathcal{L}}{\partial X_2} = \beta X_1^{\alpha} X_2^{\beta-1} - \lambda P_2 = 0$$

$$\lambda = \frac{\beta X_1^{\alpha} X_2^{\beta-1}}{P_2} \quad \text{——— (ii)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - X_1P_1 - X_2P_2 = 0 \quad \text{——— (iii)}$$

Solving simultaneous (i) & (ii)

$$\frac{\lambda}{\lambda} = \frac{\alpha X_1^{\alpha-1} X_2^{\beta}}{P_1} \div \frac{\beta X_1^{\alpha} X_2^{\beta-1}}{P_2}$$

$$1 = \frac{\alpha X_1^{\alpha-1} X_2^{\beta}}{P_1} \times \frac{P_2}{\beta X_1^{\alpha} X_2^{\beta-1}}$$

$$1 = \frac{\alpha X_1^{\alpha-1} X_2^{\beta} . P_2}{\beta X_1^{\alpha} X_2^{\beta-1} . P_1}$$

$$1 = \frac{\alpha \cancel{X_1^{\alpha}} X_2^{\beta} . X_2 P_2}{\beta \cancel{X_1^{\alpha}} X_2^{\beta} . X_1 P_1}$$

$$1 = \frac{\alpha X_2 P_2}{\beta X_1 P_1}$$

$$\boxed{\frac{\beta X_1 P_1}{\alpha P_2} = X_2 \; ; \; X_1 = \frac{\alpha X_2 P_2}{\beta P_1}}$$

now for getting X_1 & X_2 we put above in (iii)

$$M = X_1 P_1 + X_2 P_2$$

$$M = \frac{\alpha X_2 P_2}{\beta \cancel{P_1}} . \cancel{P_1} + X_2 P_2$$

$$M = \frac{\alpha}{\beta} . X_2 P_2 + X_2 P_2$$

$$M = X_2 P_2 \left(\frac{\alpha}{\beta} + 1 \right)$$

$$M = X_2 P_2 \left(\frac{\alpha + \beta}{\beta} \right)$$

$$M\beta = X_2 P_2 (\alpha + \beta)$$

$$\boxed{X_2 = \frac{M}{P_2} \left(\frac{\beta}{\alpha + \beta} \right)}$$

Again for X_2

$$M = X_1P_1 + X_2P_2$$

$$M = X_1P_1 + \frac{\beta X_1 P_1}{\alpha \cancel{P_2}} \cdot \cancel{P_2}$$

$$M = X_1P_1\left(1 + \frac{\beta}{\alpha}\right)$$

$$\boxed{X_1 = \frac{M}{P_1}\left(\frac{\beta\alpha}{\alpha+\beta}\right)}$$

Here the Cross Price effect will be zero

SOC
$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > 0$$

$$= -U_{11}P_2^2 - U_{22}P_1^2 + 2U_{12}U_1U_2 > 0$$

$$\because \lambda = U_i/P_i$$
$$P_i = U_i/\lambda$$

x'ing by $\lambda \cdot U_i/\lambda$

$$= -U_{11}U_2^2 - U_{22}U_1^2 + 2U_{12}U_1U_2 > 0$$

$U_1 \Rightarrow \alpha X_1^{\alpha-1} X_2^{\beta}$

$U_2 \Rightarrow \beta X_1^{\alpha} X_2^{\beta-1}$

$U_{11} \Rightarrow \alpha(\alpha-1) X_1^{\alpha-2} X_2^{\beta}$

$U_{22} \Rightarrow \beta(\beta-1) X_1^{\alpha} X_2^{\beta-2}$

$U_{12} \Rightarrow \alpha\beta X_1^{\alpha-1} X_2^{\beta-1}$

$U_{21} \Rightarrow \alpha\beta X_1^{\alpha-1} X_2^{\beta-1}$

Thus

$$\Rightarrow -\left[(\beta X_1^{\alpha} X_2^{\beta-1})^2 \alpha(\alpha-1) X_1^{\alpha-2} X_2^{\beta}\right] - \left[(\alpha X_1^{\alpha-1} X_2^{\beta})^2 \beta(\beta-1) X_1^{\alpha} X_2^{\beta-2}\right]$$

$$+ 2\left[(\alpha X_1^{\alpha-1} X_2^{\beta})(\beta X_1^{\alpha} X_2^{\beta-1})(\alpha\beta X_1^{\alpha-1} X_2^{\beta-1})\right] > 0$$

⇒ WHAT IS LEMDA (λ) :-

It is the rate of change of optimal value of objective function due to a given change in constraint/income.

$$\lambda = \frac{dU^*}{dM}$$

Proof :- [Interpretation of Lagrangian Multiplier]

As $= U = U(x_1, x_2, x_3 \ldots x_n)$

$$\frac{dU}{dM} = \frac{\partial U}{\partial x_1} \cdot \frac{\partial x_1}{\partial M} + \frac{\partial U}{\partial x_2} \cdot \frac{\partial x_2}{\partial M} + \cdots + \frac{\partial U}{\partial x_n} \cdot \frac{\partial x_n}{\partial M}$$

$$\because \frac{\partial U}{\partial x_i} = U_i$$

$$\frac{dU}{dM} = U_1 \frac{\partial X_1}{\partial M} + U_2 \frac{\partial X_2}{\partial M} + \cdots + U_n \frac{\partial X_n}{\partial M}$$

From FOC $\lambda = U_i/p_i$ so $U_i = \lambda p_i$

$$\frac{dU}{dM} = \lambda p_1 \frac{\partial X_1}{\partial M} + \lambda p_2 \frac{\partial X_2}{\partial M} + \cdots + \lambda p_n \frac{\partial X_n}{\partial M}$$

After using FOC on U, it will become (U^*) Optimal.

$$\frac{dU}{dM} = \lambda \underbrace{\left[p_1 \frac{\partial x_1}{\partial M} + p_2 \frac{\partial x_2}{\partial M} + \cdots + p_n \frac{\partial x_n}{\partial M} \right]}_{=1}$$

$$M = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$$

Differentiating budget constraint

$$1 = P_1 \frac{\partial x_1}{\partial M} + P_2 \frac{\partial x_2}{\partial M} + \cdots + p_n \frac{\partial x_n}{\partial M}$$

$$\lambda = \frac{dU^*}{dM}$$

SOC :-

$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > 0$$

$$= 2U_{12}P_1P_2 - U_{11}P_2^2 - U_{22}P_1^2 > 0$$

Recall FOC $U_i/\lambda = P_i$

Thus

$$\Rightarrow U_1^2 U_{22} - 2U_1U_2U_{12} + U_2^2 U_{11} < 0$$

⇒ Properties of Marshallian demand funct.

(1) MDF is homogeneous of degree zero in all Prices & income

If all Prices & income of a consumer changes by same proportion it would not change MDF.

i.e 20% ↑ in Prices with 20% ↑ in income, so the MDF is homogeneous of degree zero.

$$x_i^M(tp_1, tp_2, tp_3, \ldots, tp_n, tM)$$

$$= t^0 x_i^M(p_1, p_2, p_3 \ldots p_n, M)$$

$$= x_i^M(p_1, p_2, p_3 \ldots p_n, M) \quad \because t>0$$

Thus 't' is factor out with power zero so it's homogeneous.

Application of Euler theorem:

~~$x_1P_1 + x_2P_2 + \cdots + x_nP_n$~~

$$\frac{\partial x_1^M}{\partial P_1} P_1 + \frac{\partial x_2^M}{\partial P_2} . P_2 + \cdots + \frac{\partial x_n^M}{\partial P_n} . P_n + \frac{\partial x_1^M}{\partial M} . M = 0$$

dividing by x_1^M

$$\frac{\partial x_1^M}{\partial P_1} . \frac{P_1}{x_1^M} + \frac{\partial x_2^M}{\partial P_2} . \frac{P_2}{x_2^M} + \cdots + \frac{\partial x_n^M}{\partial P_n} . \frac{P_n}{x_n^M} + \frac{\partial x_1^M}{\partial M} . \frac{M}{x_1^M} = 0$$

-ve (Inferior) ε_{11} +ve (superior) ε_{12} -ve (inferior) η_{14}

Own Price elasticities & Cross Price elasti. = 0

Thus: income price elasticity

$\varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{1n} + \eta_1 M = 0$

(2) **Balance Budgetness** : (MDF satisfy the adding up Property)

$$M = P_1\left[x_1^M(P_i, M)\right] + \dots + P_n\left[x_n^M(P_i, M)\right]$$

As $M = P_1X_1 + P_2X_2 + \dots + P_nX_n$

Now we substitute MDF into budget Constraint :

$$\frac{dM}{dM} = ? \quad \& \quad \frac{dM}{dP_i} = ?$$

(A) Engel Aggregation $[dM/dM]$

The engel Aggregation means that not all goods are luxuries $(\eta_i > 1)$, necessities $(\eta_i < 1)$ or inferior goods $(\eta_i < 0)$

→ **Implication**

As $M = P_1\left[x_1^M(P_i, M)\right] + P_2\left[x_2^M(P_i, M)\right] \dots + P_n\left[(x_n^M(P_i, M)\right]$

Now differentiating w.r.t M

$$1 = P_1\left[\frac{\partial x_1^M}{\partial M}\right] + P_2\left[\frac{\partial x_2^M}{\partial M}\right] + \dots + P_n\left[\frac{\partial x_n^M}{\partial M}\right]$$

Multiplying & dividing by M & x_i $\left[\frac{M x_i}{x_i M}\right]$

$$1 = \frac{\partial x_1^M}{\partial M} \cdot \frac{M}{x_1} \cdot \frac{P_1 x_1}{M} + \frac{\partial x_2^M}{\partial M} \cdot \frac{M}{x_2} \cdot \frac{P_2 x_2}{M} + \dots + \frac{\partial x_n^M}{\partial M} \cdot \frac{M}{x_n} \cdot \frac{P_n x_n}{M}$$

31 • Engel Aggregation → Differenciate budget constraint w.r.t income.

• Cournot Aggregation → Differenciate budget constraint w.r.t Price.

$$\Rightarrow 1 = \eta_{1M} \cdot \frac{P_1 X_1}{M} + \eta_{2M} \cdot \frac{P_2 X_2}{M} + \dots + \eta_{nM} \cdot \frac{P_n X_n}{M}$$

$$1 = \eta_{1M}(S_1) + \eta_{2M}(S_2) + \dots + \eta_{nM}(S_n)$$

$1 = \sum \eta_{iM}(S_i) \rightarrow$ Share weighted income elasticities sum to unity

All commodities cannot be inferior at same time.

$$\underset{-ve}{\eta_{1M}} S_1 + \underset{+ve}{\eta_{2M}} (S_2) = 1$$

(B) Cournot Aggregation [dM/dP_i]

Cournot aggregation predict the own & cross price elasticity.

As $M = P_1[X_1^M(P_i, M)] + P_2[X_2^M(P_i, M)] + P_3[X_3^M(P_i, M)]$
$+ \dots + P_n[X_n^M(P_i, M)]$

Differenciating w.r.t P_1, P_2

$$\frac{\partial M}{\partial P_1} = \left[X_1^M + P_1 \cdot \frac{\partial X_1^M}{\partial P_1}\right] + \left[P_2 \frac{\partial X_2^M}{\partial P_1}\right] + \left[P_3 \frac{\partial X_3^M}{\partial P_1}\right] + \dots + \left[P_n \frac{\partial X_n^M}{\partial P_1}\right] = 0$$

Multiplying & ÷ing by P_1 & X_i^M

$$\Rightarrow 0 = \left[X_1^M + \frac{\partial X_1^M}{\partial P_1} \cdot \frac{P_1}{X_1}(X_1)\right] + \left[\frac{\partial X_2^M}{\partial P_1} \cdot \frac{P_1}{X_2} \cdot \frac{P_2 X_2}{P_1}\right]$$
$$+ \left[\frac{\partial X_3^M}{\partial P_1} \cdot \frac{P_1}{X_3} \cdot \frac{P_3 X_3}{P_1}\right] + \dots + \left[\frac{\partial X_n^M}{\partial P_1} \cdot \frac{P_1}{X_n} \cdot \frac{P_n X_n}{P_1}\right]$$

IF WE TAKE DERIVATIVE OF IUF, WE WILL GET MDF (Roy's identity)

$$\Rightarrow 0 = \left[X_1^M + \varepsilon_{11}(X_1)\right] + \left[\varepsilon_{21}\,\frac{P_2 X_2}{P_1}\right] + \left[\varepsilon_{31}\cdot\frac{P_3 X_3}{P_1}\right] + \cdots + \left[\varepsilon_{n1}\cdot\frac{P_n X_n}{P_1}\right]$$

Multiplying by $P_1 \Rightarrow$

$$\Rightarrow 0 = \left[P_1 X_1^M + \varepsilon_{11}(X_1 P_1)\right] + \left[\varepsilon_{21}(P_2 X_2)\right] + \left[\varepsilon_{31}(P_3 X_3)\right] + \cdots + \left[\varepsilon_{n1}\, P_n X_n\right]$$

$$\Rightarrow 0 = P_1 X_1^M + \varepsilon_{11}(P_1 X_1) + \varepsilon_{21}(P_2 X_2) + \varepsilon_{31}(P_3 X_3) + \cdots + \varepsilon_{n1}(P_n X_n)$$

Dividing by M

$$\Rightarrow 0 = S_1 + \varepsilon_{11}(S_1) + \varepsilon_{21}(S_2) + \cdots + \varepsilon_{n1}(S_n)$$

$$\Rightarrow -S_1 = \sum_{i=1}^{n} \varepsilon_{i1}(S_i)$$

Thus share weighted own & cross price elasticity sum to negative of the expenditure share on ith commodity.

Roy's identity = IUF

Summary (Properties of MDF) :-

→ Homogenity of MDF tells that demand do not respond to an overall equi Proportionate change in all prices & income simultaneously.

→ Balance budgetness of MDF tells that demand always exhaust the consumer income.

INDIRECT UTILITY FUNCTION [IUF]

As we know earlier that the utility is the function of commodities:

$$U(x_1, x_2, x_3 \dots x_n)$$

& the commodities function of Prices & income

$$X_i = x(P_i, M)$$

thus there will be indirect relation & after this utility will be the function of price & income:

$$U = f(P_i, M) \longrightarrow \text{IUF}$$

utility fn → Maximization (1) → MDF (3) → IUF (2)

Note :-

Optimization

✓ IUF   ✗ MDF

IUF give maximum achieveable utility.

⇒ Properties of IUF

(1) Homogeneous of degree zero.

IUF is homogeneous of degree zero in Prices & income. Change in same Proportion so utility remain uneffected.

$$V(P_i, M) \qquad t>0$$
$$= V(tP_i, tM)$$
$$= t^0 V(P_i, M) \rightarrow t \text{ factor Out with Power zero}$$
$$= V(P_i, M)$$

(2) IUF is non increasing/decreasing in Prices

$$\frac{\partial V}{\partial P_i} < 0$$

$$U(x_1, x_2, x_3 \dots x_n)$$

Substituting MDF

$$V = [x_1(P_i, M), x_2(P_i, M), x_3(P_i, M) + \dots + x_n(P_i, M)]$$

$$\frac{\partial V}{\partial P_1} = \frac{\partial V}{\partial x_1} \cdot \frac{\partial x_1}{\partial P_1} + \frac{\partial V}{\partial x_2} \cdot \frac{\partial x_2}{\partial P_1} + \dots + \frac{\partial V}{\partial x_n} \cdot \frac{\partial x_n}{\partial P_1}$$

$$\mathcal{L} = u(x_i) + \lambda[M - P_i x_i]$$

Taking FOC

$$\frac{\partial \mathcal{L}}{\partial x_i} = u_i - \lambda p_i = 0$$

$$\rightarrow u_i = \lambda P_i$$

B. constraint used to find x_1 expression & put x_1 into $\frac{\partial v}{\partial P_1}$

$$\frac{\partial v}{\partial P_1} = \lambda\left[P_1\frac{\partial x_1}{\partial P_1} + P_2\frac{\partial x_2}{\partial P_1} + \dots + P_n\frac{\partial x_n}{\partial P_1}\right]$$

Let's consider budget constraint

$$M = P_1x_1 + P_2x_2 + P_3x_3 \dots P_nx_n$$

$$\frac{dM}{dP_1} = \left[P_1\frac{\partial x_1}{\partial P_1} + x_1\right] + P_2\frac{\partial x_2}{\partial P_1} + P_3\frac{\partial x_3}{\partial P_1} \dots P_n\frac{\partial x_n}{\partial P_1}$$

$$dM/dP_1 = 0$$

Thus

$$-x_1 = \frac{\partial x_1}{\partial P_1}.P_1 + P_2.\frac{\partial x_2}{\partial P_1} + \dots + P_n\frac{\partial x_n}{\partial P_1}$$

$$\therefore \frac{\partial v}{\partial P_1} = -\lambda x_1 \quad \text{so} \quad \frac{\partial v}{\partial P_1} < 0$$

Therefore IUF is decreasing in Prices

(3) IUF is increasing/non decreasing in income

$$v = [x_1(P_i,M), x_2(P_i,M), x_3(P_i,M) \dots x_n(P_i,M)]$$

$$\frac{\partial v}{\partial M} = \frac{\partial v}{\partial x_1}.\frac{\partial x_1}{\partial M} + \frac{\partial v}{\partial x_2}.\frac{\partial x_2}{\partial M} + \dots + \frac{\partial v}{\partial x_n}.\frac{\partial x_n}{\partial M}$$

From FOC $\frac{\partial v}{\partial M} = \lambda P_i$

$$\frac{\partial v}{\partial M} = \lambda\left[P_1\frac{\partial x_1}{\partial M} + P_2\frac{\partial x_2}{\partial M} + \dots P_n\frac{\partial x_n}{\partial M}\right]$$

Consider the budget constraint

$$M = x_1P_1 + x_2P_2 + \dots + x_nP_n$$

Differentiating w.r.t M

$$1 = P_1 \frac{\partial X_1}{\partial M} + P_2 \frac{\partial X_2}{\partial M} + \dots + P_n \frac{\partial X_n}{\partial M}$$

$\because \frac{\partial V}{\partial M} = \lambda$ as the marginal utility w.r.t income is "λ" that is positive.

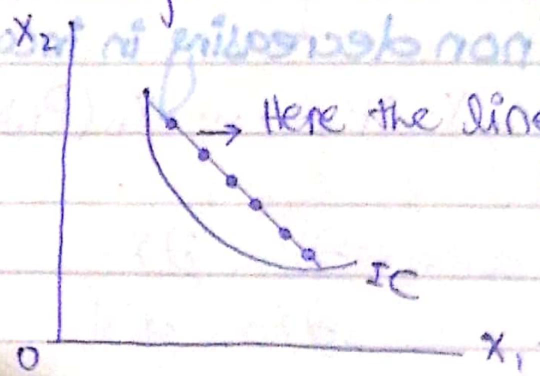
$\frac{\partial V}{\partial M} > 0$, thus IUF is increasing in income

(4) Roy's identity

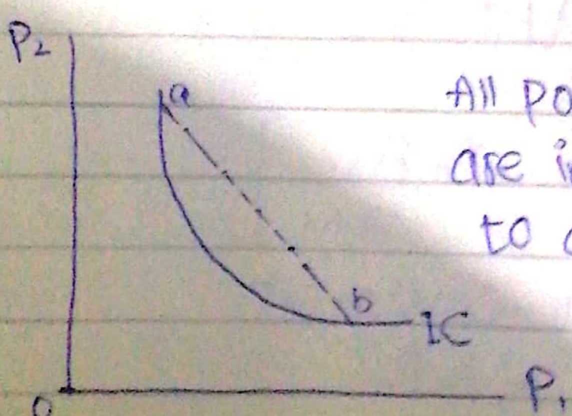
$$\frac{\partial V}{\partial P_i} \div \frac{\partial V}{\partial M} = -X_i \rightarrow \text{MDF}$$

(5) Shape of utility

IUF is Quasi Convex in Prices. The linear combinations of contour set are inferior.



Here the linear combinations are prefered on higher IC



All points on doted line are inferior as compare to a & b so higher the curve lower will be the satisfaction.

⇒ Derivation of IUF with help of utility fn

(A) $U = X_1 X_2$

s.t $M = X_1 P_1 + X_2 P_2$

$\mathcal{L} = X_1 X_2 + \lambda(M - X_1 P_1 - X_2 P_2)$

$\frac{\partial \mathcal{L}}{\partial X_1} = X_2 - \lambda P_1 = 0$ ——— ①

$\lambda = X_2 / P_1$

$\frac{\partial \mathcal{L}}{\partial X_2} = X_1 - \lambda P_2 = 0$ ——— ②

$\lambda = X_1 / P_2$

$\frac{\partial \mathcal{L}}{\partial \lambda} = M - X_1 P_1 - X_2 P_2 = 0$ ——— ③

solving simultaneously

$\frac{\lambda}{\lambda} = \frac{X_2}{P_1} \cdot \frac{P_2}{X_1}$

$X_1 P_1 = X_2 P_2$

$X_1 = \frac{X_2 P_2}{P_1}$ & $X_2 = \frac{X_1 P_1}{P_2}$

putting above in ——— ③ eq. for finding X_1 & X_2

$M - X_1 P_1 - X_2 P_2 = 0$

$M - \frac{X_2 P_2}{\cancel{P_1}} \cdot \cancel{P_1} - X_2 P_2 = 0$

$M = 2 X_2 P_2$

$$\boxed{X_2 = \frac{M}{2 P_2}}$$

similarly

$$M - X_1P_1 - X_2P_2 = 0$$

$$M - X_1P_1 - \cancel{P_2} \cdot \frac{X_1P_1}{\cancel{P_2}} = 0$$

$$M - X_1P_1 - X_1P_1 = 0$$

$$\boxed{X_1 = \frac{M}{2P_1}}$$

putting X_1 & X_2 in utility fn

$$U = X_1X_2$$

$$U = \frac{M}{2P_1} \cdot \frac{M}{2P_2}$$

$$U = \frac{M^2}{4P_1P_2} \longrightarrow \text{IUF}$$

Taking SOC

$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > 0$$

$$\Rightarrow U_{11}\begin{vmatrix} U_{22} & -P_2 \\ -P_2 & 0 \end{vmatrix} - U_{12}\begin{vmatrix} U_{21} & -P_2 \\ -P_1 & 0 \end{vmatrix} - P_1\begin{vmatrix} U_{21} & U_{22} \\ -P_1 & -P_2 \end{vmatrix} > 0$$

$$\Rightarrow -U_{11}P_2^2 + P_1P_2U_{12} + P_1P_2U_{21} - U_{22}P_1^2 > 0$$

$$= -U_{11}P_2^2 + 2P_1P_2U_{12} - U_{22}P_1^2 > 0$$

where $U_{11} = 0$, $U_{22} = 0$ & $U_{12} = 1$

Thus

$$2P_1P_2(1) > 0$$

(B) $U = x_1^2 x_2^2$ s.t $M = x_1P_1 + x_2P_2$

$\mathcal{L} = x_1^2 x_2^2 + \lambda(M - x_1P_1 - x_2P_2)$

$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1x_2^2 - \lambda P_1 = 0$ ———— ①

$\Rightarrow \lambda = 2x_1x_2^2 / P_1$

$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2x_1^2 - \lambda P_2 = 0$ ———— ②

$\Rightarrow \lambda = 2x_2x_1^2 / P_2$

$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1P_1 - x_2P_2 = 0$ ———— ③

Simultaneously solve λ

$\frac{\lambda}{\lambda} = \frac{2x_1x_2^2}{P_1} \times \frac{P_2}{2x_2x_1^2}$

$1 = \frac{\cancel{2}\cancel{x_1}x_2^2 P_2}{\cancel{2}x_1^2\cancel{x_2}P_1}$

$1 = \frac{x_2P_2}{x_1P_1}$

$x_1P_1 = x_2P_2$

$x_1 = \frac{x_2P_2}{P_1}$ & $x_2 = \frac{x_1P_1}{P_2}$

putting above in ③ eq for x_1 & x_2

$M - x_1P_1 - x_2P_2 = 0$

$M - \frac{x_2P_2}{\cancel{P_1}}\cancel{P_1} - x_2P_2 = 0$

$M - 2x_2P_2 = 0$

$\boxed{x_2 = \frac{M}{2P_2}}$

Similarly for X_1

$$M - X_1P_1 - \cancel{P_2} \cdot \frac{X_1P_1}{\cancel{P_2}} = 0$$

$$M - 2X_1P_1 = 0$$

$$\boxed{X_1 = \frac{M}{2P_1}}$$

putting X_1 & X_2 in U

$$U = X_1^2 X_2^2$$

$$U = \left(\frac{M}{2P_1}\right)^2 \left(\frac{M}{2P_2}\right)^2$$

$$U = \frac{(M^2)^2}{16 P_1^2 P_2^2} \longrightarrow \text{IUF}$$

SOC

$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > 0$$

$$= -U_{11}P_2^2 - U_{22}P_1^2 + 2U_{12}P_1P_2 > 0$$

where

$$U_{11} = 2X_2^2$$

$$U_{12} = 4X_1X_2$$

$$U_{22} = 2X_1^2$$

Thus

$$-2X_2^2P_2^2 - 2X_1^2P_1^2 + 8X_1X_2P_1P_2 > 0$$

(C) $U = X_1^{\alpha} X_2^{\beta}$ s.t $M = X_1P_1 + X_2P_2$

$\mathcal{L} = X_1^{\alpha} X_2^{\beta} + \lambda(M - X_1P_1 - X_2P_2)$

$\frac{\partial \mathcal{L}}{\partial X_1} = \alpha X_1^{\alpha-1} X_2^{\beta} - \lambda P_1 = 0$ ——— ①

$\Rightarrow \lambda = \frac{\alpha X_1^{\alpha-1} X_2^{\beta}}{P_1}$

$\frac{\partial \mathcal{L}}{\partial X_2} = \beta X_1^{\alpha} X_2^{\beta-1} - \lambda P_2 = 0$ ——— ②

$\Rightarrow \lambda = \frac{\beta X_1^{\alpha} X_2^{\beta-1}}{P_2}$

$\frac{\partial \mathcal{L}}{\partial \lambda} = M - X_1P_1 - X_2P_2 = 0$ ——— ③

Simultaneously solving ① & ②

$\frac{\lambda}{\lambda} = \frac{\alpha X_1^{\alpha-1} X_2^{\beta}}{P_1} \times \frac{P_2}{\beta X_1^{\alpha} X_2^{\beta-1}}$

$1 = \frac{\alpha \cancel{X_1^{\alpha}} \cancel{X_2^{\beta}} P_2 X_2}{P_1 \beta \cancel{X_1^{\alpha}} \cancel{X_2^{\beta}} X_1}$

$P_1 \beta X_1 = \alpha P_2 X_2$

$X_1 = \frac{\alpha P_2 X_2}{P_1 \beta}$ & $X_2 = \frac{P_1 \beta X_1}{\alpha P_2}$

Putting above in ——— ③ eq

$M = X_1P_1 + X_2P_2 = 0$

$M = \frac{\alpha X_2 P_2}{\beta \cancel{P_1}} \cdot \cancel{P_1} + X_2P_2$

$M = X_2P_2 \left[\frac{\alpha}{\beta} + 1\right]$

$$M = X_2 P_2 \cdot \frac{\alpha+\beta}{\beta}$$

$$\boxed{X_2 = \frac{M\beta}{P_2(\alpha+\beta)}}$$

Similarly

$$M = X_1 P_1 + X_2 P_2$$

$$M = X_1 P_1 + \frac{\beta X_1 P_1}{\alpha \cancel{P_2}} \cdot \cancel{P_2}$$

$$M = X_1 P_1 \left[1 + \frac{\beta}{\alpha}\right]$$

$$M = X_1 P_1 \cdot \frac{\alpha+\beta}{\alpha}$$

$$\boxed{X_1 = \frac{\alpha M}{P_1(\alpha+\beta)}}$$

Now $U = X_1^{\alpha} X_2^{\beta}$

$$U = \left[\frac{\alpha M}{P_1(\alpha+\beta)}\right]^{\alpha} \left[\frac{M\beta}{P_2(\alpha+\beta)}\right]^{\beta} \longrightarrow \text{IUF}$$

SOC

$$\begin{vmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_2 \\ -P_1 & -P_2 & 0 \end{vmatrix} > 0$$

$$= -U_{11} P_2^2 - U_{22} P_1^2 + 2U_{12} P_1 P_2 > 0$$

where

$$U_{11} = \alpha(\alpha-1) X_1^{\alpha-2} X_2^{\beta}$$

$$U_{12} = \alpha\beta X_1^{\alpha-1} X_2^{\beta-1}$$

$$U_{22} = \beta(\beta-1) X_2^{\beta-2} X_1^{\alpha}$$

Thus

$$\Rightarrow -\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}p_2^2 - \beta(\beta-1)x_2^{\beta-2}x_1^{\alpha}p_1^2$$
$$+ 2\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}p_1p_2 > 0$$

⇒ EXPENDITURE MINIMIZATION

$M = P_1X_1 + P_2X_2 + \ldots + P_nX_n$ → Objective function

$U = U(X_1, X_2, X_3 \ldots X_n)$ → Budget Constraint

$\mathcal{L} = \sum_{i=1}^{n} P_i X_i + \lambda' [\bar{U} - U(X_i)]$

∵ λ' → Expenditure Min

FOC

$$\frac{\partial \mathcal{L}}{\partial X_1} = P_1 - \lambda' U_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial X_2} = P_2 - \lambda' U_2 = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial \mathcal{L}}{\partial X_n} = P_n - \lambda' U_n = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda'} = \bar{U} - U(X_1, X_2 \cdots X_n) = 0 \quad \rightarrow n+1$$

Simultaneous Solution of these FOC will yeild expenditure minimization demand function / Hicksian demand function / compansated demand function.

$X_i(P_i, \bar{U})$ → utility Min demand function

$P\downarrow$ −ve compansation

$P\uparrow$ +ve compansation

⇒ Properties of Hicksian demand function

(1) Homogeneous = degree zero

It is homogeneous function of degree zero in all Prices

(2) Satisfy the utility Constraint

$U = (X_1, X_2 \dots X_n)$

$u[x_1(P_i, \bar{u}), x_2(P_i, \bar{u}), x_3(P_i, \bar{u}) \dots x_n(P_i, \bar{u})]$

Differentiating w.r.t P

$$= \frac{\partial U}{\partial X_1} \cdot \frac{\partial X_1}{\partial P_1} + \frac{\partial U}{\partial X_2} \cdot \frac{\partial X_2}{\partial P_1} + \frac{\partial U}{\partial X_3} \cdot \frac{\partial X_3}{\partial P_1} + \dots + \frac{\partial U}{\partial X_n} \cdot \frac{\partial X_n}{\partial P_1}$$

As $U(X_i) = \bar{U}$

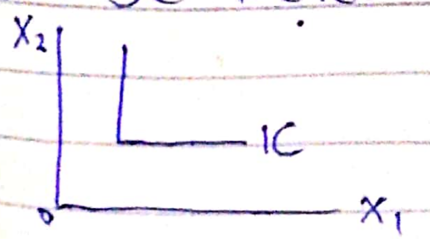
For all $U > 0$ & $U(P_i, X_i)$

$$\Rightarrow \sum_{i=1}^{n} p_i x_i$$

(3) -ve Sloped

Hicksian demand function always is -ve slope but only is case of complements it would be L-Shape.

Because hicksian demand function [HDF] based on Substitution effect [SE] & in case of complement goods there will be no SE so there demand function will be zero.



HDF (-ve)

MDF (+ve, -ve OR vertical)

EXPENDITURE FUNCTION

As IUF shows the maximum level of utility achieveable against given level of Prices & income.

Similarly, Expenditure function shows the minimum level of utility achievable against given level of Prices & income.

EF (Expenditure function) is Obtained by Substituting the Optimal values of HDF into minimizing expenditure equation.

$$e(P_i, U) = \min\left[\sum_{i=1}^{n} p_i x_i\right]$$

$$\text{s.t } \bar{U} = e(P_i, U) = \sum_{i=1}^{n} p_i \left[x_i^H(P_i, U)\right]$$

It is the minimum level of money expenditure the consumer must make facing a given set of Prices to achieve a given level of utility.

More specifically EF shows the minimum level of expenditure necessary to achieve a given level utility as a function of Prices & the required level of utility.

Hence expenditure function gives the minimum cost of achieving a fixed level of utility.

⇛ Properties of Expenditure function:-

- **1st Property :-**

EXPENDITURE FUNCTION IS HOMOGENOUS OF DEGREE "ONE" IN PRICES

If all Prices Change by Same Proportion, it will Change expenditure function by same proportion

$$e_i = e(tp_i, u) = te_i$$

ALL PRICES change by Same Proportion & utility is held constant.

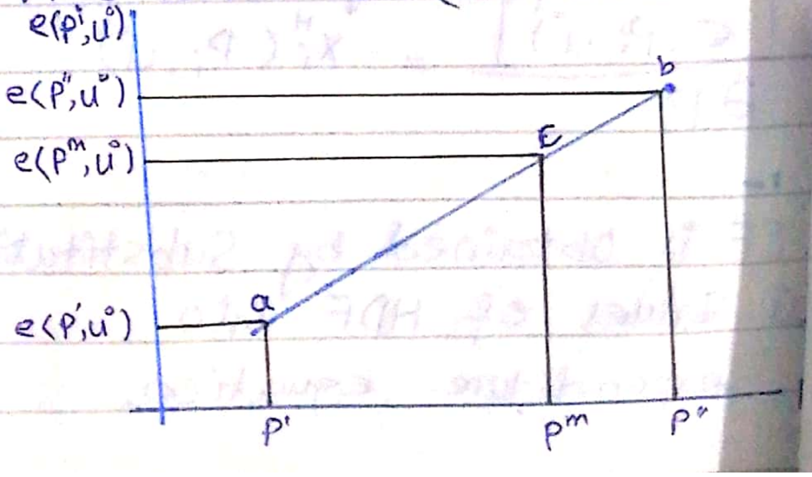
- **2nd Property :-** EF is concave in Prices

If Only one Price Change by ~~the same Proportion~~ & utility & other Prices held constant.

Proportionate ↑ in expenditure on all goods < Proportionate ↑ in Price of a good

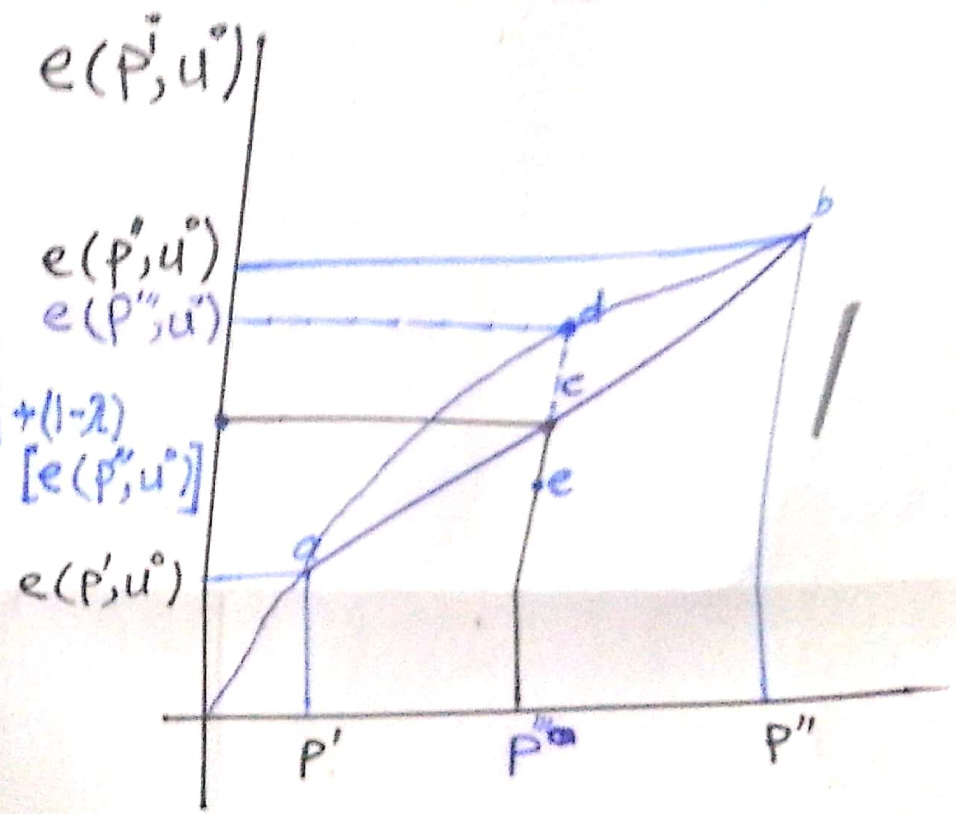
Proof :

Let there be two Price vectors (P' & P''). Suppose that when utility is held fixed at u° & Optimal expenditure minimizing bundles corresponding to these Price vectors are (x' & x'')



3 Possibilities :-

1. EF will be concave fn of Other two (d)
2. EF will be linear fn of Other two (c)
3. EF will be convex fn of Other two (e)



$p^i \rightarrow$ Price Vector

$\Rightarrow p^m = \lambda p' + (1-\lambda)p''$

① when $\lambda = 0 \rightarrow p^m = (1-\lambda)p''$ ② when $\lambda = 1$, $p^m = p'$

$p'x' = e(p', u^\circ)$
$p''x'' = e(p'', u^\circ)$
$p^m x^m = e(p^m, u^\circ)$
$p''x' \rightarrow$ X False

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∴ Then by definition of expenditure fn:

$p'x' = e(p', u^\circ)$ & $p''x'' = e(p'', u^\circ)$

→ Let p^m be the linear combination of p' & p'' such that $p^m = \lambda p' + (1-\lambda)p''$.

Further assume that the corresponding expenditure minimizing quantity is x^m.

BY DEFINITION OF EF :

$p^m x^m = e(p^m, u^\circ)$ m = 1/4

Concavity means that the cost of reaching utility level u° at price vector p^m must not be less than the weighted average of $e(p', u^\circ)$ & $e(p'', u^\circ)$

This concavity means :

$$e(p^m, u^\circ) \geq \lambda [e(p', u^\circ)] + (1-\lambda)[e(p'', u^\circ)]$$

- 3rd Property

SHEPHARD'S LEMMA

Derivative of EF w.r.t Price of nth good gives HDF of ith good.

$$\Rightarrow \frac{\partial [e(P_i, u^\circ)]}{\partial P_i} = x_i^H(P_i, u^\circ)$$

Proof :-

Since EF is obtained by substituting the optimal values of HDF into minimizing expenditure equation.

$\lambda \to$ always +ve

Slutsky $\to \Delta$ in dd of commodity due to Δ in P

$\frac{\partial EF}{\partial P}$ = HDF

$$\Rightarrow e(P_i, U) = \sum_{i=1}^{n} P_i [x_i^H (P_i, U)]$$

$$= P_1 x_1^H (P_1, U) + P_2 x_2^H (P_2, U) + \dots + P_n x_n^H (P_n, U)$$

differenciating w.r.t P_1

$$\Rightarrow \frac{\partial e}{\partial P_1} = \left[P_1 \frac{\partial x_1^H}{\partial P_1} + x_1^H (P_1, U) \right] + \left[P_2 \frac{\partial x_2^H}{\partial P_1} + \dots + P_n \frac{\partial x_n^H}{\partial P_1} \right]$$

$$\Rightarrow \frac{\partial e}{\partial P_1} = x_1^H (P_1, U) + \left[P_1 . \frac{\partial x_1^H}{\partial P_1} + P_2 . \frac{\partial x_2^H}{\partial P_1} + \dots + P_n . \frac{\partial x_n^H}{\partial P_1} \right]$$

$$\text{FOC} \Rightarrow P_i = \lambda' u_i$$

$$\Rightarrow \frac{\partial e}{\partial P_1} = x_1^H (P_1, U) + \lambda' \left[\frac{\partial x_1^H}{\partial P_1} . u_1 + \frac{\partial x_2^H}{\partial P_1} . u_2 + \dots + \frac{\partial x_n^H}{\partial P_1} . u_n \right]$$

$$\because \frac{\partial U}{\partial P_1} = u(x_1, x_2, \dots x_n) \to \frac{\partial U}{\partial P_1} = u_1 \frac{\partial x_1^H}{\partial P_1} + \dots + u_n \frac{\partial x_n^H}{\partial P_1} = 0$$

$$\Rightarrow \frac{\partial e}{\partial P_1} = x_1^H (P_1, U) \to \text{Shephard lemma}$$

- **4th Property :-**

EF IS NON DECREASING IN PRICES

This follows from Shephard lemma. Since at least one good must be brought, the EF is non decreasing in Price vector P & strictly ↑ing in atleast one Price.

Higher Price mean higher expenditure to reach a given utility.

- **5th Property :-**

EF IS INCREASIN IN UTILITY

Higher utility at a given Prices require higher expenditure.

Since value of the Lagrangian multiplier (λ') is the rate at which under optimal conditions the objective function ↑es as the result of ↑ in constraint parameters.

Here objective function is the expenditure function & constraint is utility.

$$\lambda' = \frac{\partial [e(P_i, U)]}{\partial U} \rightarrow$$ Rate at which expenditure changes as utility changes, under optimal conditions.

DUALITY IN CONSUMER THEORY

If we take the constraint in utility maximization Problem, the level of expenditure from the solution of expenditure minimization Problem then the solution to two Problems will be identical.

OR

If we take the constraint in the expenditure minimization Problem, the level of utility from the solution of utility maximization Problem then the solution to two Problems will be identical.

$$\Rightarrow MDF \cong HDF$$

HDF $\rightarrow X_{ih}(P_i, U)$ $\qquad X = x$

MDF $\rightarrow X_{im}(P_i, M)$

Substituting the expenditure fn in MDF

$$x_{ih} = (P_i, U) \cong x_{im}(P_i, M(P_i, U))$$

$$\frac{\partial x_{ih}}{\partial P_i} = \frac{\partial X_{im}}{\partial P_i} + \frac{\partial x_{im}}{\partial M} \cdot \frac{\partial M}{\partial P_i}$$

(Shephard lemma) : $\frac{\partial M}{\partial P_i} = x_{ih}$

$$\frac{\partial x_{ih}}{\partial P_i} - \frac{\partial x_{im}}{\partial M} x_{ih} = \frac{\partial x_{im}}{\partial P_i} \rightarrow \text{Slutsky eq.}$$

$\downarrow$ SE $\qquad \downarrow$ IE $\qquad \downarrow$ PE converting into elasticities

$$\frac{\partial X_{im}}{\partial P_i}\cdot\frac{P_i}{X_i} = \frac{\partial x_{ih}}{\partial P_i}\cdot\frac{P_i}{x_i} - \frac{\partial x_{im}}{\partial M}\cdot\frac{M}{x_i}\, x_{ih}\,\frac{P_i}{M}$$

$$\frac{\partial X_i}{\partial P_i}\cdot\frac{P_i}{X_i} = \frac{\partial x_{ih}}{\partial P_i}\cdot\frac{P_i}{x_i} - \frac{\partial x_{im}}{\partial M}\cdot\frac{M}{x_i}\, x_{ih}\,\frac{P_i}{M}$$

$$\frac{\partial X_i}{\partial P_i}\cdot\frac{P_i}{x_i} = \frac{\partial x_i}{\partial P_i}\cdot\frac{P_i}{x_i} - \frac{\partial x_i}{\partial M}\cdot\frac{M}{x_i}\left[\frac{P_i x_i}{M}\right]$$

$$\varepsilon_{ij} = \sigma_{ij} - \gamma_i\,[S_i]$$