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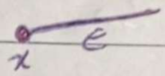
GRAPH THEORY

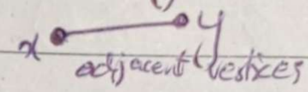
- A graph G consist of two sets: $V(G)$ called the vertex set & $E(G)$ called the edge set.
  An edge denoted by xy is an unordered pair of vertices. Short hand notation $G = (V, E)$

- No. of vertices in $G = |V(G)|$ or $|G|$
  No. of edges in $G = |E(G)|$ or $\|G\|$

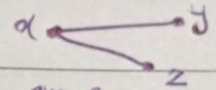
- If xy is an edge than x & y are the endpoints for that edge. "x is incident to edge e if x is an endpoint of e"



- If two vertices incident to same edge they are called adjacent, denoted $x \sim y$

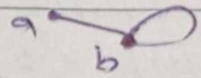


- x & y are neighbors
- $N(x) \xrightarrow{} $ set of all neighbors of x

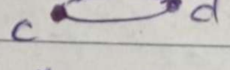


xy & xz are adjacent edges

- If both end points of an edge on same vertex than called loop



- More than one edge with same endpoints than called multi-edges



- If no multi-edges & loop than graph is called simple.

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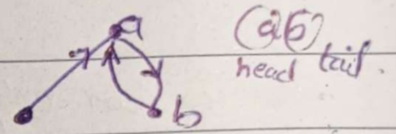
- A loop will count as 2 to the degree of a vertex.
- K-Regular Graph: If all vertices in graph G have the same degree K.
  Ex: If $K=3$ the graph is cubic.

- Directed Graph: or a digraph is a graph $G=(V,A)$ that consist of vertex set $V(G)$ and an arc set $A(G)$. An arc is an ordered pair of vertices.
  ↳ In-degree: No. of arcs for which x is tail. $deg^-(x)$
  ↳ out-degree: No. of arcs for which x is head. $deg^+(x)$

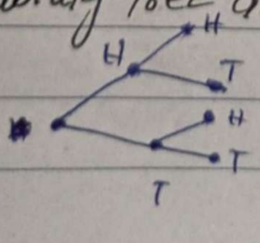


$a \to b$

- Underlying Graph: for a digraph is the graph $G' = (V, E)$ which is formed by removing direction for each arc to form an edge.

- Weighted Graph: A weighted graph $G=(V,E,\omega)$ is a graph in which each edge has a weight $\omega(xy)$ → weigh for edge xy.

- Probability Tree Graph:



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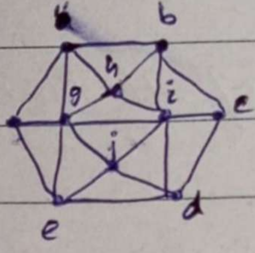
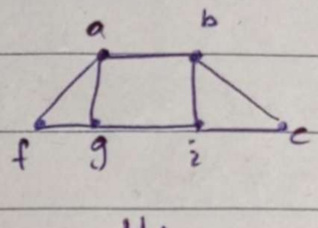
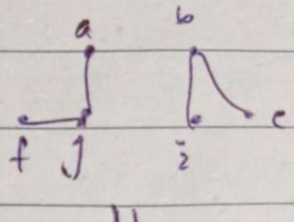
- **Subgraphs:** A subgraph H of a graph G is a graph where H contains some of the edges & vertices of G; i.e $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$







Both H_1 & H_2 are sub graphs. and have same set of V. But H_1 mein tamam edges included hai whereas H_2 mein kuch edges missing hai..

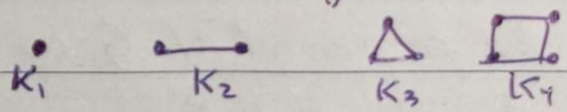
! Induced Subgraph: Given graph $G = (V, E)$, an Induced subgraph $G[V']$, wher $V' \subseteq V$ and every available edge from G b/w the V' include. Like H_1.

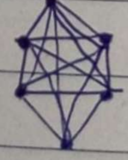
! Spanning Subgraph: If it contains all the vertices but not necessarily all edges of G, that is $V(H) = V(G)$ and $E(H) \subseteq E(G)$

- **Complete Graphs:**

Har vertex ki har vertex k saath edge ho.

Denoted by K_n.





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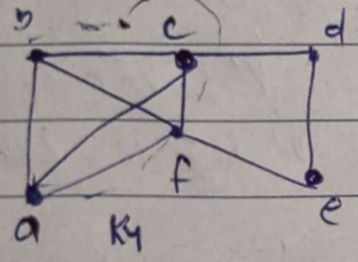
- Properties of K_n :
  - ↳ Each vertex has degree $n-1$
  - ↳ K_n has $\dfrac{n(n-1)}{2}$ edges
  - ↳ K_n contains the most edges out of all simple graphs on n vertices

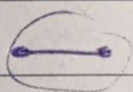
- Clique-Size : The clique size of a graph $\omega(G)$ is the largest integer 'n' such that K_n.

Humein agr ek graph mein koi maximum ~~complete~~ possible complete mil jate hai to wo clique khelaiga.



↓ Is mein K_3 bhi hai (b,c,f) or (a,c,f) or (a,b,c) but we need to take the maximum K_n. or max possible K_4 hai.

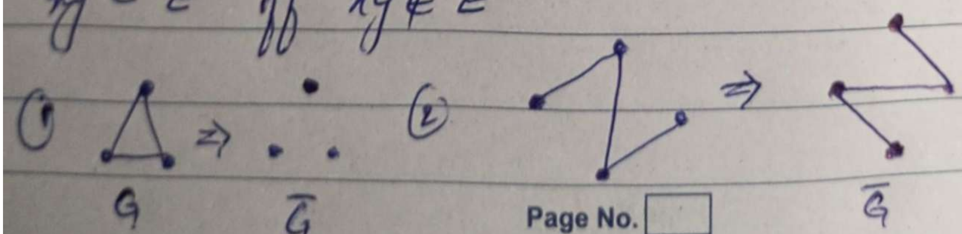
so clique-size = 4

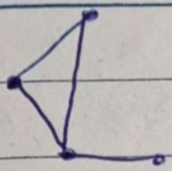
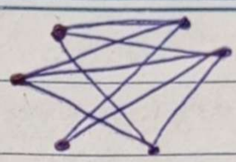
$K_2 \rightarrow$ 

ye to hr us mein hi hota hai.

$K_3 \rightarrow \Delta \Rightarrow$ isko bhi identify krna easy hai.

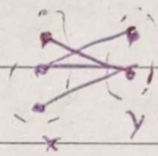
- Graph Complements : Given a simple graph $G = (V, E)$, define the complement of G as $\overline{G} = (V, \overline{E})$, where $xy \in \overline{E}$ iff $xy \notin E$



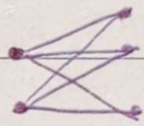
(5)  $\Rightarrow$ 

G $\overline{G}$

- Bipartite Graph: A graph G is bipartite if the vertices can be partitioned into two sets X & Y so that every edge has one endpoint in X and other in Y.

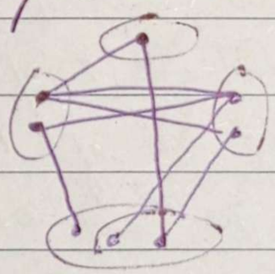


- Complete Bipartite Graph: $K_{m,n}$, where $|X| = m$ and $|Y| = n$. Every vertex in X is adjacent to every vertex in Y.

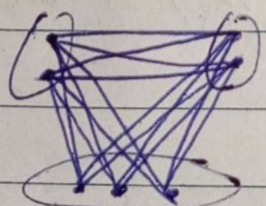


$K_{1,3}$

- K-Partite Graph: Vertices partitioned into K-sets $X_1, X_2 \ldots X_K$.



4-partite



$K_{2,2,3}$

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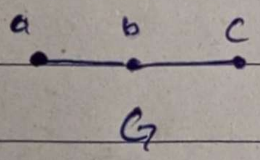
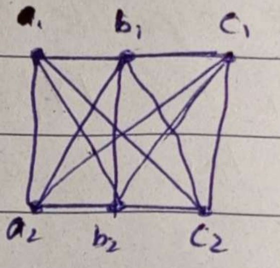
- Union of Graph: $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ & edge set $E(G) \cup E(H)$
  ↳ if vertex set are disjoint ie $V(G) \cap V(H) = \phi$ then we called the disjoint union the sum, denoted $G+H$
  ↳ sum ke andar bas dono graphs ko alag alag draw kardenge

- Join of Graph: Denoted by $G \vee H$, is the sum $G+H$ together with all edges of the form xy where $x \in V(G)$ & $y \in V(H)$.



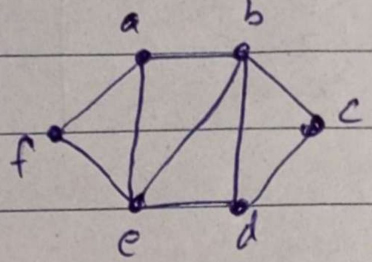


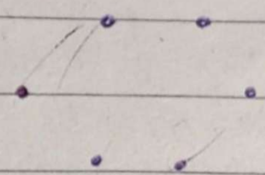
- Theorem 1·33:
  Let $|A|$ be no. of arcs in digraph G.
  Then the sum of in-degrees of the vertices or out-degrees equals the no. of arcs.
  $V = \{v_1, v_2 \ldots v_n\}$
  $$\deg^-(v_1) + \cdots + \deg^-(v_n) = |A|$$
  $$= \deg^+(v_1) + \cdots + \deg^+(v_n)$$

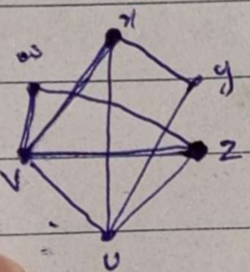
- Isomorphism of Graphs :

Two graphs G_1 & G_2 are isomorphic, denoted by $G_1 \cong G_2$, if there exist a bijection.

$f: V(G_1) \rightarrow V(G_2)$ so that $xy \in E(G_1)$ iff $f(x)f(y) \in E(G_2)$







$V(G_1) \rightarrow V(G_2)$	Edges
$a \rightarrow x$	$ab \rightarrow xv$
$b \rightarrow v$	$ae \rightarrow xu$
$c \rightarrow w$	$af \rightarrow xy$
$d \rightarrow z$	$bc \rightarrow vw$
$e \rightarrow u$	$bd \rightarrow vz$
$f \rightarrow y$	$be \rightarrow vu$
	$cd \rightarrow wz$
	$de \rightarrow zu$
	$ef \rightarrow uy$

→ Properties to Check
↳ is connected
↳ has n vertices
↳ has m edges
↳ has m vertices of deg K
↳ has a cycle of length K
↳ has an eulerian circuit
↳ has a hamiltonian cycle

then so too must G_2

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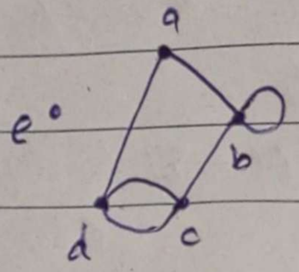
- Matrix Representation :



$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & & & & & \\
b & & & & & \\
c & & & & & \\
d & & & & & \\
e & & & & &
\end{array}
\quad
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

loop will be counted as 1

d & c have 2 edges b/w them

" PROOF TECHNIQUES "

- Direct Proofs:

" The sum of 2 odd integer is even "

Let x & y be two odd integers, then

$x = 2n+1$ & $y = 2m+1$

$$
\begin{aligned}
x+y &= 2n+1+2m+1 \\
&= 2n+2m+2 \\
&= 2(n+m+1) \\
&= 2(k) \rightarrow \text{EVEN}
\end{aligned}
$$

- Handshaking Lemma:

$$\sum_{i=1}^{n} \deg(v_i) = \deg(v_1) + \dots \deg(v_n) = 2|E|$$

no. of edges

PROOF: Let $G = (V, E)$ be a graph with $V = \{v_1 \dots v_n\}$

Any edge $e = v_i v_j$ we be counted once in the

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total $|E|$. Since each edge is defined by it's 2 endpoints, this edge will add one to the count of both $deg(v_i)$ & $deg(v_j)$. Thus every edge of G will add 2 to the count of sum of degrees.

Thus $deg(V_1) + deg(V_2) + \dots + deg(V_n) = 2|E|$

" Every graph has an even no. of vertices of odd degree " → means every graph has an even no. of ~~vertices that have an odd degree.~~

Proof: By Handshaking Lemma

$$\sum_{i=1}^{n} deg(V_i) = 2m \quad , \quad m = |E|$$

$$\sum_{V \in X} deg(V) + \sum_{V \in Y} deg(V) = 2m$$

Here, $X = \{ v \in V \mid deg(v) \text{ is even})$

$Y = \{ \text{set of odd deg vertices})$

$$2k + \sum_{V \in Y} deg(V) = 2m$$

$$\sum_{V \in Y} deg(V) = 2(m-k)$$

$$\sum_{V \in Y} deg(V) = 2n$$ → This can only satisfy if we add even no. of odd vertices degree.

$\underbrace{1+3+5+7}_{4} = 16$

$\underbrace{1+3+5}_{3} = 9$

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- Indirect Proofs :

"For any integer n, if n^2 is odd then n is odd

▷ Proof by contradiction:

n^2 is odd but n is even

Let, $n = 2k$

$n^2 = 4k^2$

But here n^2 is even & it's a contradiction.

The given statement is true.

▷ Proof by contraposition:

Suppose n is ~~odd~~ not odd. Then n is even.

$n = 2k$.

$n^2 = 4k^2$

so n^2 is even.

Thus if n^2 is odd then n must also be odd

Proposition 1.24:

"For every simple graph G on at least 2 vertices, there exist two vertices of same deg"

Let there be a graph $G = (V, E)$ with n vertices. The possible ~~options~~ values for a deg in a simple graph can be $n-1$. And as there are total n vertices so by pigeon hole principle any two vertice must have same degree value.

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• Mathematical Induction :

From discrete notes.

• Degree Sequence :

→ listing of degree of the vertices in descending order
→ If a sequence is a degree sequence of a simple graph then we call it graphical.

• HAVEL HAKIMI THEOREM :

Used to determine whether a given degree seq is graphical or not.

1st no will be cut. Then in next step we will minus one from the next 4 vertex. 4 why? bcz 1st no. is 4.

	4	3	3	2	1	1
		2	2	1	0	1
reorder if req		~~2~~	2	1	1	0
			1	0	1	0
reorder			~~1~~	1	0	0
			0	0	0	0

✓ graphical as it is a valid sequence

→ agr kisi step mein koi neg term agaye ya subtract krne k liye numbers kam par jaye to wo graphical nhi hai.

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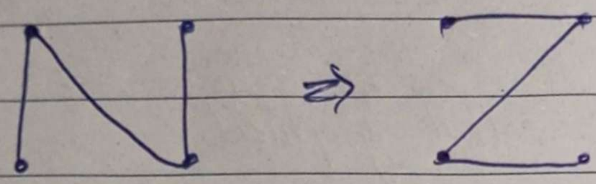
- Self Complementary Graph:
  A graph G is said to be self-complementary if $\bar{G} \cong G$.
  A graph which is isomorphic to it's complement



G $\qquad$ $\bar{G}$

"GRAPH ROUTES"

▷ **Walk:** Simple bas graph ko traverse karenge. Edge or vertex alone repeat ho sakte hai.
↳ closed walk: agr 'b' se start karoge to 'b' par end

▷ **Trail:** It is a walk but EDGES not repeated. Vertex repeat hosakti hai.
↳ closed trail: jaha se start wahi khatam. also called circuit

▷ **Path:** Neither edge nor vertex can be repeated.
↳ closed path / cycle:

- Path is more restrictive version of trail
- Trail " " " " " walk
- Any path can also be view as a trail & as a walk
- A walk might not be a trail or a path (if it repeats E or V)

- A cycle is a circuit or a closed walk.

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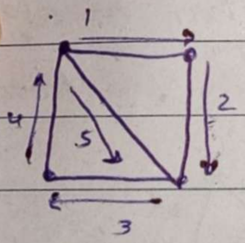
- Connected Graph:

The graph G is connected if every pair of distinct vertices is connected.

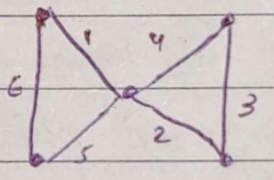
- Theorem 2.3: Every x-y walk contain x-y path
- Theorem 2.5: If every vertex of a graph has degree at least 2 then G contains a cycle

"EULERIAN GRAPHS" (Edge cannot repeat)

- An eulerian circuit (or trial) is one that contains every EDGE and every VERTEX of G.
- If G contains eulerian circuit it's called eulerian
- If G contains a trial but not circuit then it's called semi-eulerian



euler path



euler path & circuit.

Characteristics:

1. G is connected
2. Every vertex has an even degree (eulerian)
   Exactly two vertices have odd deg (semi-eulerian)

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Properties:

① Let G be connected multigraph, & u & v be the only 2 odd vertices in G. Then the no. of u-v path in G is odd.

② If G is connected eulerian multigraph, then any edge in G is contained in an odd no. of cycles.

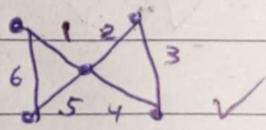
▸ Fleury's Algo: (if no odd vertice than any starting point, if two odd Vertice then starting ~~must be any one of them~~)
Youtube Video.

▸ Hierholzer's Algo: (lecture 8 & 9)
Youtube Video.

"HAMILTONIAN GRAPHS"

Hamiltonian graph k liye hamil cycle honi chahiye

Saare k saare vertices cover krne hai or edges bhi repeat nhi krne. Or saare edges ko traverse krna bhi zaroori nhi.



Properties:

① No vertex of G can have deg less than 2

② G cannot contain cut-vertex, i.e a vertex whose removal disconnects the graph.

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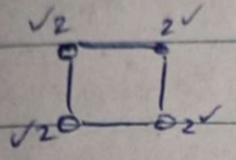
- **Dirac's Theorem:**

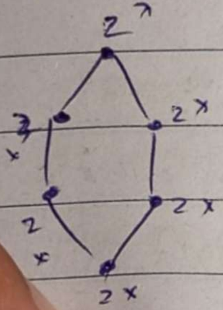
If G is connected & $\forall_v \deg(v) \geq n/2$ then G is a hamiltonian graph. ↓



$n = 4$, $\frac{n}{2} = 2$

Sabki deg check karenge or wo $n/2$ se badi honi chahiye.
n = no. of vertices

Agr ye condition satisfy horhi hai to laazmi hai ke G → hamiltonian.
Lekin agr ye condition satisfy nhi horhi then we donot know ke ~~wt~~ wether it's hamiltonian or not



$n = 6 = \frac{6}{2} = 3$ (∴ Dirac's theorm fails)
but still this is a hamiltonian graph.

this is a unidirectional condition.

- **Ore's Theorem:**

If G is connected & $\forall_{v,u} \deg(v) + \deg(u) \geq n$ & u & v are non-adjacent. then G → hamiltonian.
This is also unidirection condition.

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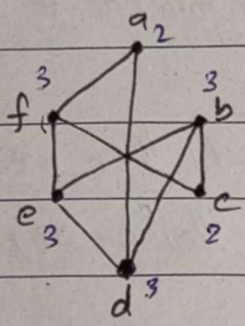
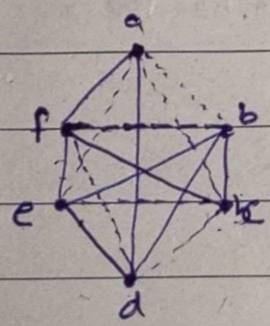
- Hamiltonian Closure :

The hamiltonian closure of graph G is obtained by, jitni bhi non-adjacent vertices are unke degree ka sum agr $\geq$ no. of ~~edo~~ vertices hoa to unhe graph G mein add krlenge. Ye jab tak krte rhenge jab tak non-adjacent vertices k pair khatam na hojayein.



ab $\rightarrow 2+3 \geq 6$ ×
ae $\rightarrow 2+3 \geq 6$ ×
ac $\rightarrow 2+2 \geq 6$ ×
bf $\rightarrow 3+3 \geq 6$ ✓ (we add b to f)
ce
cd
df

$n=6$



ce $\rightarrow 2+3 \geq 6$ ×
cd $\rightarrow 2+3 \geq 6$ ×
df $\rightarrow 3+3 \geq 6$ ✓ (add edge)

As new edges are added we repeat again.

ab $\rightarrow 2+4 \geq 6$ ✓
ae $\rightarrow 3+3 \geq 6$ ✓
ac $\rightarrow 4+2 \geq 6$ ✓
ce $\rightarrow 3+4 \geq 6$ ✓
cd $\rightarrow$ ~~3~~ $+4 \geq 6$ ✓

$\therefore cl(G) = K_6$

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Theorem 2.15: The closure of G is well defined

Theorem 2.16: A graph G is hamiltonian iff it's closure cl(G) is hamiltonian

Lemma 2.17: If G is a graph with at least 3 vertices such that its closure cl(G) is complete, then G is hamiltonian.

- BruteForce Algo:

Is mein hum ek graph k andar saare hamiltonian nikaalenge or sabse least sum of weights vala choose krenge.

Nearest Neighbour:

Is mein hum starting vertex se us vertex par jayenge jiska kam weight hoga. phir next vertex se bhi wi vertex ko choose krenge jiska kam weight hai.

- Repetitive Nearest Neighbour Algo:

Is mein har ek vertex ko as starting vertex choose krke nearest neighbour lagayenge or total weight jis ka kam hoga wo choose krenge.

|Lecture #10|

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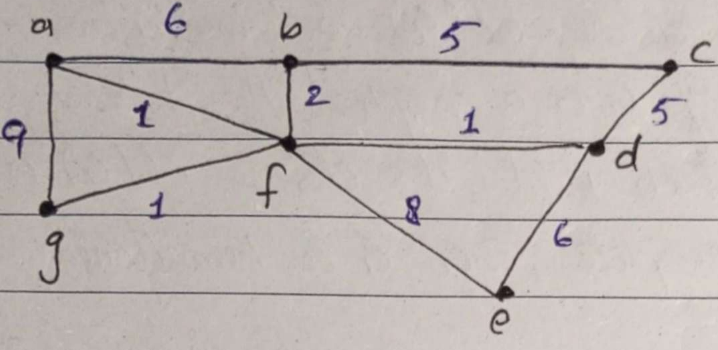
"DIJKSTRA ALGORITHM"



	a	b	c	d	e	f	g
g	∞	∞	∞	∞	∞	∞	0
g	9,g	∞	∞	∞	∞	1,g	
gf	2,f	3,f	∞	2,f	9,f		
gfa		3,f	∞	2,f	9,f		
gfad		3,f	7,d		8,d		
gfadb			7,d		8,d		
gfadbc					8,d		
gfadbce							

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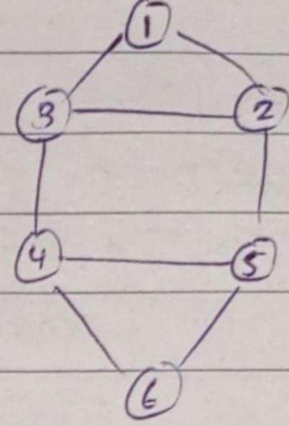
"DISTANCE, RADIUS, DIAMETER & ECCENTRICITY"

• **Distance :** The distance b/w two vertices in a graph is no. of edges in a shortest path.



$d(1,2)=1$	$d(2,3)=1$	$d(3,5)=2$
$d(1,3)=1$	$d(2,4)=2$	$d(3,6)=2$
$d(1,4)=2$	$d(2,5)=1$	$d(4,5)=1$
$d(1,5)=2$	$d(2,6)=2$	$d(4,6)=1$
$d(1,6)=3$	$d(3,4)=1$	$d(5,6)=1$

• **Eccentricity :** Ek vertex 'x' ka maximum distance of a vertex

$$E(x) = \max_{y \in V(G)} d(x,y)$$

∴ $E(1)=3$, $E(2)=2$, $E(3)=2$, $E(4)=2$, $E(5)=2$
$E(6)=3$

• **Diameter of Graph :** Maximum of ecentricity of the vertices. (maximum distance b/w two vertices)

$$\text{diam}(G) = \max_{y \in V(G)} d(x,y) \text{ or } E(y)$$

$$\text{diam}(G) = 3$$

• **Radius of Graph :** Minimum of the all eccentricities

$$\text{rad}(G) = \min_{x \in V(G)} E(x)$$

$\text{rad}(G)=2$

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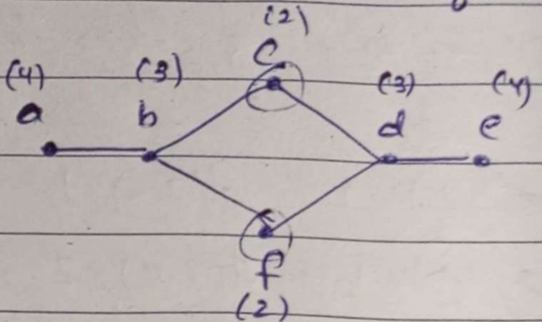
→ A vertex w in G is called a central vertex if $e(w) = rad(G)$ and the center of G, denoted $C(G)$, is the set of all centeral vertices



$diam(G) = 4$

$Rad(G) = 2$

$C(G) = \{c, f\}$

↳ centeral vertices, $e(c) = e(f) = rad(G) = 2$

Theorem 2.26: If G is disconnected then $\bar{G}$ is connected & $diam(\bar{G}) \leq 2$

Theorem 2.27: For a simple graph G if $rad(G) \geq 3$ then $rad(\bar{G}) \leq 2$.

Theorem 2.28: For any simple graph, ~~rad~~ $rad(G) \leq diam(G) \leq 2\,rad(G)$

Girth: Minimum length of a cycle in G, denoted $g(G)$

Circumference: Max length of a cycle.

↳ ~~max~~ the circ—nce of a graph can be atmost n.

↳ if G doesnot have cycle (tree graph) then $g(G) = \infty$ and circum $= 0$

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Theorem 2.31: If G is a graph with at least one cycle the $g(G) \leqslant 2\,diam(G) + 1$

Theorem 2.32: Let G be a graph with n vertices, radius at most k, and max degree at most d, with $d \geqslant 3$, then

$$n < \frac{d}{d-2}(d-1)^k$$