(12.4)

Q) $a^2 U_{xx} = U_{tt}$

$a^2 X''T = XT''$

$\frac{X''}{X} = \frac{T''}{a^2 T}$

$X'' = \lambda X$, $T'' = a^2 T\lambda$

$m^2 = \pm\sqrt{\lambda}$ $\quad m = \pm a\sqrt{\lambda}$

For $\lambda = 0$

$m = 0$

$X = c_1 + c_2 x$ $\quad T = c_3 + c_4 x$

For $\lambda > 0$

$m = \pm p$

For $\lambda = \cancel{\;}\ -p^2 < 0$

$m = \pm pi$ $\quad m = \pm api$

$U = (c_1 \cos px + c_2 \sin px)$
$(c_3 \cos apt + c_3 \sin apt)$

$\hookrightarrow$ ~~eqn~~ (A)

$U(0,t) = 0$, $U(L,t) =$
$U(x,0) = \frac{1}{4} x(L-x)$,

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

putting $U(0,t) = 0$ in (A)

$U(0,t) = c_1(c_3 \cos apt + \ldots$

$0 = c_1(c_3 \cos apt + \sin \ldots$

$c_1 = 0$ $\hookrightarrow$ put in (A)

$U(x,t) = c_2 \sin px(c_3 \cos \ldots$
$+ c_3 \sin apt)$

$\hookrightarrow$ (B)

Now $U(L,t) = 0$

$0 = c_2 \sin pL(c_3 c\ldots + \ldots)$

$\sin pL = 0$

$p = \frac{n\pi}{L}$

$\because g(x) = 0$

$B_n = 0$

$$U(x,t) = \sin\frac{n\pi}{L}x\left(A\cos\left(\frac{n\pi}{L}\right)at + B\sin\left(\frac{n\pi}{L}\right)at\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\right)x\left[A_n\cos\left(\frac{n\pi}{L}\right)at + B_n\sin\left(\frac{n\pi}{L}\right)at\right]$$

$\hookrightarrow$ (C)

Now $U(x,0) = f(x)$

$$U(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx}$$

$\hookrightarrow$ (D)

Now,

$$\left.\frac{\partial u}{\partial t}(x,0)\right|_{t=0} = g(x)$$

Diff (C)

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n\frac{an\pi}{L}\sin\left(\frac{n\pi}{L}\right)at + B_n\frac{an\pi}{L}\cos\left(\frac{n\pi}{L}\right)at\right]\sin\left(\frac{n\pi}{L}\right)x$$

$$g(x) = \sum_{n=1}^{\infty} B_n\frac{an\pi}{L}\sin\left(\frac{n\pi}{L}\right)x$$

$$B_n\frac{an\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$\because f(x) = \frac{1}{4}x(L-x)$

put in (D)

$$A_n = \frac{1}{2L}\int_0^L (xL - x^2)\sin\frac{n\pi}{L}x\,dx$$

$xL - x^2$	$\sin\left(\frac{n\pi}{L}\right)$
$L - 2x$	$-\cos\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)$
-2	$-\sin\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)^2$
0	$\cos\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)^3$

$$A_n = \frac{1}{2L}\left[-\frac{(xL - x^2)\cos\frac{n\pi}{L}}{\left(\frac{n\pi}{L}\right)} + \frac{(L-2x)\sin}{\left(\frac{n\pi}{L}\right)^2} - \frac{2\cos\left(\frac{n\pi}{L}\right)}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L$$

$$A_n = \frac{2L}{(n\pi)^3}\left[1 - (-1)^n\right]$$

Q2) $U(0,t)=0$

$U(L,t)=0$

$U(x,0)=0$

$\frac{\partial u}{\partial t}\Big|_{t=0} = x(L-x)$

$U(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos apt + C_4 \sin apt) \to (A)$

using $U(0,t)=0$

$0 = C_1(C_3 \cos apt + C_4 \sin apt)$

$C_1 = 0$, $C_3 \cos \text{—} \neq 0$

$\to$ put in (A)

$U(x,t) = C_2 \sin px (C_3 \cos apt + C_3 \sin apt)$

Using $U(L,t)=0$ $\to B$

$0 = C_2 \sin pL (C_3 \text{———})$

$C_2 \neq 0$, $\sin pL = 0$

$p = \frac{n\pi}{L}$

put in (B)

$U(x,t) = C_2 \sin \frac{p\pi}{L} (\text{———})$

$U(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi}{L} at + B_n \sin \ldots) \sin \frac{n\pi}{L} x \to C$

Now using $U(x,0) = f(x)$

$f(x) = \sum_{n=1}^{\infty} (A_n \sin \frac{n\pi}{L} x)$

$\therefore A_n = \frac{2}{L}\int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \, dx$

$\because f(x) = 0$

$\boxed{A_n = 0}$

Now $\partial U/\partial t\big|_{t=0} = g(x)$

as

$\partial U/\partial t = \sum (A_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} at + B_n \frac{n\pi a}{L} \cos \frac{n\pi}{L} at) \sin \frac{n\pi}{L} x$

using eq

$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x$

$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$

$$B_n = \frac{2}{n\pi a}\int_0^l (xL - x^2)\sin\frac{n\pi}{l}x\,dx$$

$$B_n = \frac{4L^3}{n^4\pi^4}\left[1-(-1)^n\right]$$

$$U(x,t) = \sum_{n=1}^{\infty}\left[\frac{4L^3}{n^4\pi^4}\left[1-(-1)^n\right]\sin\frac{n\pi}{l}at\right]\sin\frac{n\pi}{l}x$$

(3) $U(0,t)=0$, $U(\pi,t)=0$

$U(x,0)=0$, $\left.\frac{\partial U}{\partial t}\right|_{t=0} = \sin x$

By using 1st condition :-

$$U(x,t) = c_2\sin px\,(c_3\cos apt + c_3\sin apt) \to (B)$$

Now $U(\pi,t)=0$

$0 = c_2\sin p\pi\,(c_3\cos apt + \dots)$

$c_2 \neq 0$

$\sin p\pi = 0$

$p\pi = n\pi$

$p = n$

eq (B)

$$U(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos ant + B_n\sin ant\right]\sin nx \to C$$

Now using $U(x,0)=0$

$f(x) = \sum A_n \sin nx$

$f(x)=0$

$\boxed{A_n = 0}$

$$\frac{\partial U}{\partial t} = \left[-A_n an\sin ant + B\,an\cos ant\right]\sin nx$$

using condition,

$g(x) = \sum B_n an \sin nx$

$$B_n an = \frac{2}{L}\int_0^L g(x)\sin nx\,dx$$

$$B_n = \frac{2}{anL}\int_0^L \sin x\sin nx\,dx$$

$$B_n = \frac{2}{anL}\cdot\frac{1}{2}\int_0^L \cos(1-n)x - \cos(1+n)x\,dx$$

$\therefore L = \pi$

$$B_n = \frac{1}{a n\pi}\left[\frac{1}{1-n}\sin(1-n)x - \frac{1}{1+n}\sin(1+n)x\right]_0^{\pi}$$

$B_n = 0$ for $n = 1, 2, 3 \ldots$

For B_0;

$$B_n = \frac{2}{a\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{2}{a n\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{2}{a\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{2}{a\pi}\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= \frac{2}{a\pi}\left[\frac{\pi}{2} - 0\right]$$

$$B_1 = \frac{1}{a}$$

$$U(x,t) = B_1 \sin at \sin x = \frac{1}{a}\sin at \sin x$$

(4) $U(0,t) = 0$

$U(\pi,t) = 0$

$U(x,0) = \frac{1}{6}x(\pi^2 - x^2)$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = 0$$

By using 1st 2 conditions

$$U(x,t) = \sum\left(A_n \cos ant + B_n \sin ant\right)\sin nx$$

$\hookrightarrow C$

Now Using,

$U(x,0) = f(x)$

$f(x) = \sum A_n \sin nx$

$$A_n = \frac{2}{L}\int_0^{L} f(x) \cdot \sin nx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{6}(x\pi^2 - x^3)\sin nx$$

$x\pi^2 - x^3 \quad + \quad \sin nx$

$\pi^2 - 3x^2 \quad - \quad -\cos nx/n$

$-6x \quad + \quad -\sin nx/n$

$-6 \quad - \quad \cos nx/n$

$0 \quad \sin nx/n$

$$= \frac{1}{3\pi}\left[-\frac{(x\pi^2 - x^3)}{n}\cos nx + \frac{(\pi^2 - 3x^2)}{n^2}\sin nx \right.$$

$$\left. \frac{(-6x)\cos nx}{n^3} - \frac{6\sin nx}{n^4} \right]_0^{\pi}$$

$$= \frac{1}{3\pi}\left[\frac{(-6\pi)(-1)^n}{n^3} \right]$$

$$A_n = \frac{2}{n^3}\left[(-1)^{n+1}\right]$$

$g(x) = 0$

$B_n = 0$

$$U(x,t) = \sum_{n=1}^{\infty} \left(\frac{2[(-1)^{n+1}]}{n^3} \cos ant \right) \sin nx$$

(5) $U(0,t) = 0, \; U(1,t) = 0$

$U(x,0) = x(1-x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = x(1-x)$$

using 1st cond.

$U(x,t) = C_2 \sin px \, (C_3 \cos apt + C_4 \sin apt)$

using $U(1,t) = 0$

$U(1,t) = C_2 \sin p (\text{——})$

$\sin p = 0$

$p = n\pi$

$U(x,t) = \sum A_n \cos an\pi t +$

$B_n \sin a n\pi t)$

$\sin n\pi x$

$\hookrightarrow C$

Using $U(x,0) = x(1-x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin n\pi x \, dx$$

$$= 2\int_0^1 (x - x^2) \sin n\pi x \, dx$$

$x - x^2$	$+$	$\sin n\pi x$
$1 - 2x$	$-$	$-\cos n\pi x / n\pi$
-2	$+$	$-\sin n\pi x / n^2\pi^2$
0		$\cos n\pi x / n^3\pi^3$

$$= 2\left[\underbrace{-\frac{(x-x^2)\cos n\pi x}{n\pi}}_{0} + \underbrace{\frac{(1-2x)\sin n\pi x}{n^2\pi^2}}_{0} - \frac{2\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2\left[-\frac{2(-1)^n}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right]$$

$$A_n = \frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]$$

$$\frac{\partial U}{\partial t} = \sum \left[A_n a n\pi \sin a n\pi t + B_n a n\pi \cos a n\pi t\right] \sin n\pi x$$

Using

$$g(x) = \sum B_n a n\pi \sin n\pi x$$

$$B_n a n\pi = \frac{2}{L}\int_0^L g(x) \sin n\pi x \, dx$$

$$B_n = \frac{2}{a n\pi}\int_0^1 x(1-x) \sin n\pi x \, dx$$

$$= \frac{1}{a n\pi}\left[\frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]\right]$$

$$= \frac{4}{a n^4\pi^4}\left[1 + (-1)^{n+1}\right]$$

$$U(x,t) = \sum \left(\frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right] \cos a n\pi t + \frac{4}{a n^4\pi^4}\left[1 + (-1)^{n+1}\right] \sin a n\pi t\right) \sin n\pi x$$

1

(6) $U(0,t)=0$

$U(1,t)=0$

$U(x,0) = 0.01 \sin 3\pi x$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = 0$$

Using 1st two condn

$$U(x,t) = \sum (A_n \cos a n\pi t + B_n \sin a n\pi t) \sin n\pi x$$

$\hookrightarrow C$

Now $U(x,0) = f(x)$

$$f(x) = \sum A_n \sin n\pi x$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin n\pi x \, dx$$

$$A_n = 2\int_0^{\pi} 0.01 \sin 3\pi x \sin n\pi x$$

$$= 0.02 \int_0^{\pi} \sin 3\pi x \sin n\pi x$$

$$\int_0^{\pi} \sin nx \cdot \sin mx = 0 \quad \forall\, m \neq 0$$

So, $\int_0^1 \sin 3\pi x \sin n\pi x \, dx = 0$

for $n = 1, 2, 4, \ldots$

For $n = 3$, New

$$A_3 = 0.02 \int_0^1 \sin^2 3\pi x \, dx$$

$$= 0.02 \int_0^1 \left(\frac{1-\cos 6\pi x}{2}\right) dx$$

$$= 0.02 \left[\frac{x}{2} - \frac{\sin 6\pi x}{12}\right]_0^1$$

$$= A_3 = 0.01$$

$g(x) = 0$

$B_n = 0$

$$U(x,t) = \sum A_n \cos a n\pi t \sin n\pi x$$

For $n = 3$

$$= A_3 \cos 3a\pi t \sin 3\pi x$$

$$= 0.01 C \text{———}$$