LU DECOMPOSITION

Every square matrix A can be expressed as the product of a lower triangular matrix & an upper triangular matrix.

Working Rules:- Consider a system of equations.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

in matrix form : $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{31} \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let, $A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Hence, $\underbrace{LU}_{A}X = B$

Let $UX = Y \longrightarrow LY = B$

where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, solve to find y_1, y_2, y_3

then from $UX = Y$

we can find X (i.e x, y, z req sol).

"Dolittle's METHOD"

Q: $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

agr L k diagonals ko 1 krdenge b
doolittle.

Let $A = LU = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}$

now we will ⊗ both

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}.U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} \end{bmatrix}$$

$\Rightarrow \boxed{U_{11} = 1}$, $\boxed{U_{12} = 1}$, $\boxed{U_{13} = 1}$

$l_{21}.U_{11} = 4 \Rightarrow \boxed{l_{21} = 4}$

$l_{31}.U_{11} = 3 \Rightarrow \boxed{l_{31} = 3}$

$l_{21}.U_{12} + U_{22} = 3 \Rightarrow \boxed{U_{22} = -1}$

$l_{31}.U_{12} + l_{32}.U_{22} = 5 \Rightarrow \boxed{l_{32} = -2}$

$l_{21}U_{13} + U_{23} = -1 \Rightarrow \boxed{U_{23} = -5}$

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 3 \Rightarrow \boxed{U_{33} = -10}$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

$LUX = B$

Let $UX = Y \rightarrow ① \Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

By forward subsitution we get,

$y_1 = 1$, $y_2 = 2$, $y_3 = 5$

$$\therefore Y = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

eq ① ⇒

$UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

→ By backward subsitution we get,

$$, y \boxed{\begin{matrix} x_3 = -1/2 \\ z \nearrow \end{matrix}} , \boxed{\begin{matrix} x_2 = 1/2 \\ y \nearrow \end{matrix}} \boxed{\begin{matrix} x_1 = 1 \\ x \nearrow \end{matrix}}$$

" CROUT'S METHOD "

$$\text{Let, } A = LU = \overset{L}{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}$$

↳ U k agr diagonal 1 krdein to Crout

Baaqi poora method same hi hai.

Cholesky's Method.

Let , 1. $A = LL^T$

$$= \underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}} \underset{L^T}{\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}}$$

~~Baagi poori~~

For Cholesky A should be symetric & positive difinite.
↓
submatrixes ka det is greater than 0. →

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} \rightarrow \det > 0 \\ \rightarrow \det > 0 \\ \rightarrow \det > 0 \end{matrix}$$

2. $LY = B$

3. $L^T X = Y$

Ex: 6.5

Q1 : Solve the following linear sys.

(b)

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$LUX = B$

$LY = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \leftarrow$$

$y_1 = -1/2$

$y_2 = \cancel{7/2}\ 5/2$

$y_3 = 7/2$

- Strictly Diagonally Dominant:

→ A matrix is diagonally dominant agr tamam diagonals ki value unke corresponding raos k sum s greater or equal ho.

$$A = \begin{bmatrix} \textcircled{7} & 2 & 0 \\ 3 & \textcircled{5} & -1 \\ 0 & 5 & \textcircled{-6} \end{bmatrix} \qquad \begin{aligned} |7| &\geq |2| + |0| \\ |5| &\geq |3| + |-1| \\ |-6| &\geq |0| + |5| \end{aligned}$$

if the values are greater than it's called strictly diagonlly dominant matrix.

→ A^t is not strictly diagonal dominant bcz
$[2 \ 5 \ 5] \rightarrow |5| < |2| + |5|$

- Positive Definite:

→ The determinants of all leading principal subm of A must be > 0.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\det A_1 = \det[2] = 2 > 0$

$\det A_2 = \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 > 0$

$\det A_3 = \det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 4 > 0$

- Symmetric Matrix:

A square matrix whose $A = A^t$ is called symmetric matrix.

- LDL^t Factorization:

L = Lower triangular with 1's in the diagonal.

D = Diagonal matrix

L^t = Transpose of lower triangular.

→ A matrix need to be +ve definite for LDL^t factorization.

Example: $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e & 0 & 0 \\ ae & f & 0 \\ be & cf & g \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix} = \begin{bmatrix} e & ea & eb \\ ae & a^2e+f & aeb+fc \\ be & bea+cf & b^2e+c^2f \end{bmatrix} + g$$

$\boxed{e=4}$ $ea=-1 \Rightarrow \boxed{a=-1/4}$, $eb=1 \Rightarrow \boxed{b=1/4}$

$a^2e+f=4.25 \Rightarrow \boxed{f=4}$

$aeb+fc=2.75 \Rightarrow \boxed{c=3/4}$

$b^2e+c^2f+g=3.5 \Rightarrow \boxed{g=1}$

$$A = CDC^t = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}$$

• Cholesky Decomposition:

Ex: 6.6

Q2 : Determine whether the (d) matrix is (i) ~~singula~~symmetric (ii) singular (iii) strictly diagonally dominat (iv) positive definite

$$A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ -2 & 4 & -1 & 5 \\ 3 & 7 & 1.5 & 1 \\ 6 & -9 & 3 & 7 \end{bmatrix}$$

(i) $A^t = \begin{bmatrix} 2 & -2 & 3 & 6 \\ 3 & 4 & 7 & -9 \\ 1 & -1 & 1.5 & 3 \\ 2 & 5 & 1 & 7 \end{bmatrix}$

Not symmetric

(ii) $\det(A) = 2\begin{bmatrix} 4 & -1 & 5 \\ 7 & 1.5 & 1 \\ -9 & 3 & 7 \end{bmatrix} - 3\begin{bmatrix} \\ 3 \\ 6 \end{bmatrix}$

$+1\begin{bmatrix} -2 & 4 & 5 \\ 3 & 7 & 1 \\ 6 & -9 & 7 \end{bmatrix} - 2\begin{bmatrix} -2 & 4 \\ 3 & 7 \\ 6 & -9 \end{bmatrix}$

$-3(0)$

$\det(A) = 2(260\cdot5)\cancel{-3(36)}$

$+1(-521) - 2(0)$

$\det(A) = \cancel{-90}\ 0$

~~Non~~ singular

(ii) $|2| < |3| + |1| + |2|$

Not strictly diagonal dominant

Q3: Use LDL^T Factorization (d) to form $A = LDL^T$ for the following matrices.

$$A = \begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & 1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} g & ag & bg & dg \\ ag & a^2g+h & abg+hc & adg+he \\ bg & abg+ch & b^2g+c^2h+i & bdg+che+fi \\ dg & adg+eh & bdg+ceh+fi & \downarrow \end{bmatrix}$$

$d^2g+e^2h+f^2i+j$

$\boxed{g=6}$, $ag = 2 \Rightarrow \boxed{a = 1/3}$

$bg = 1 \Rightarrow \boxed{b = 1/6}$

$dg = -1 \Rightarrow \boxed{d = -1/6}$

$a^2g + h = 4 \Rightarrow \boxed{h = 10/3}$

$abg + hc = 1 \Rightarrow \boxed{c = 1/5}$

$adg + he = 0 \quad \boxed{e = 1/10}$

$b^2g + c^2h + i = 4 \Rightarrow \boxed{i = 37/10}$

$bdg + che + fi = -1 \Rightarrow \boxed{f = -9/37}$

$d^2g + e^2h + f^2i + j = 3$

$\hookrightarrow \boxed{j = 191/74}$

$A = LDL^T$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{bmatrix} \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & j \end{bmatrix} \begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} g & 0 & 0 & 0 \\ ag & h & 0 & 0 \\ bg & ch & i & 0 \\ dg & eh & fi & j \end{bmatrix} \begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q6 : Use Cholesky Algo to find
(b) factorizon of form $A = LL^T$.

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A = LL^T$$

$$= \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ec \\ ad & bd+ec & d^2+e^2+f^2 \end{bmatrix}$$

$a^2 = 4 \Rightarrow \boxed{a = 2}$, $ab = 2 \rightarrow \boxed{b = 1}$

$ad = 2 \rightarrow \boxed{d = 1}$, $b^2 + c^2 = 6 \Rightarrow \boxed{c = \sqrt{5}}$

$bd + ec = 2 \rightarrow \boxed{e = 1/\sqrt{5}}$, $d^2 + e^2 + f^2 = 5$, $\boxed{f = \sqrt{95}/5}$

$$A = LL^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & 1/\sqrt{5} & \sqrt{95}/5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{5} & 1/\sqrt{5} \\ 0 & 0 & \sqrt{95}/5 \end{bmatrix}$$

✓

"GAUSS JACOBI METHOD"

→ Gauss Jacobi is applied on a strictly dominant linear eqn of system.

Q: Solve the following system of eqns by Gauss Jacobi n

$$6x + 2y - z = 4$$
$$x + 5y + z = 3$$
$$2x + y + 4z = 27$$

→ Har ek eqn se x, y, z ki values nikaal lenge.

$$\left.\begin{aligned} x &= \frac{1}{6}(4 - 2y + z) \\ y &= \frac{1}{5}(3 - \overset{x}{\cancel{5y}} - z) \\ z &= \frac{1}{4}(27 - 2x - y) \end{aligned}\right\} \longrightarrow ①$$

Iteration # 01: $x = x_0 = 0$, $y = y_0 = 0$, $z = z_0 = 0$ in ①

$$x_1 = \frac{1}{6}(4 - 2y_0 + z_0) = 0.6667$$

$$y_1 = \frac{1}{5}(3 - x_0 - z_0) = 0.6$$

$$z_1 = \frac{1}{4}(27 - 2x_0 - y_0) = 6.75$$

Iteration # 02:

$$x_2 = \frac{1}{6}(4 - 2y_1 + z_1) = 1.5917$$
$$y_2 = \frac{1}{5}(3 - x_0 - z_0) = -0.8833$$

Iteration #03,

$x_3 = 1/6(4 - 2y_2 + z_2) = 2.0055$

$y_3 = 1/5(3 - x_2 - z_2) = -0.9717$

$z_3 = 1/4(27 - 2x_2 - y_2) = 6.1750$

we will continue till we get the same values for all x, y, z.

Ex: 7.3

Q1: Find 1st two iterations (c) of the Jacobi method for following linear system. $x^{(0)} = 0$

$10x_1 + 5x_2 = 6$

$5x_1 + 10x_2 - 4x_3 = 25$

$-4x_2 + 8x_3 - x_4 = -11$

$-x_3 + 5x_4 = -11$

$$x_1 = \frac{1}{10}(6 - 5x_2)$$

$$x_2 = \frac{1}{10}(25 - 5x_1 + 4x_3)$$

$$x_3 = \frac{1}{8}(-11 + 4x_2 + x_4)$$

$$x_4 = \frac{1}{5}(-11 + x_3)$$

$x_1 = 1/10(6 - 5x_2^{(0)}) = 0.6$

$x_2 = 1/10(25 - 5x_1^{(0)} + 4x_3^{(0)}) = 2.5$

$x_3 = 1/8(-11 + 4x_2^{(0)} + x_4^{(0)}) = -1.375$

$x_4 = 1/5(-11 + x_3^{(0)}) = -2.2$

$x_1 = 1/10(6 - 5x_2^{(1)}) = -0.65$

$x_2 = 1/10(25 - 5x_1^{(1)} + 4x_3^{(1)}) = 1.65$

$x_3 = 1/8(-11 + 4x_2^{(1)} + x_4^{(1)}) = -0.4$

$x_4 = 1/5(-11 + x_3^{(1)}) = -2.475$

"GAUSS - SEIDEL"

→ Applied only on strictly diagonally dominant linear system of eqns.

→ Same procedure like Jacobi only difference is that har iteration ~~only~~ mein $x_1 \ldots x_n$ ki recent values ko lenge.

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

$x = 1/27(85 - 6y + z)$

$y = 1/15(72 - 6x - 2z)$

$z = 1/54(110 - x - y)$

Iteration #01:

$x_1 = 1/27(85 - 6y_0 + z_0) = 3.1481$

$y_1 = 1/15(72 - 6x_1 - 2z_0) = 3.5408$

$z_1 = 1/54(110 - x_1 - y_1) = 1.9132$

Iteration #02:

$x_2 = 1/27(85 - 6y_1 + z_1) = 2.4322$

$y_2 = 1/27(85 - 6x_2 - 2z_1) = 3.5720$

$z_2 = 1/54(110 - x_2 - y_2) = 1.9258$

POWER METHOD

(Ex : 9.3)

→ It is an iterative method used to find dominant eigenvalue of a matrix – that is, the eigen value with the largest magnitude.

Q1: Find the first 3 iterations (d) Obtained by the Power Method applied to the folloing matrix.

$$\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} ; \quad \text{use } x^{(0)} = (1,-2,0,3)^t$$

$$x_1 = Ax_0 = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 4-2+3 \\ 1-6+3 \\ 1+2 \\ 1-2+6 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \\ 5 \end{bmatrix}$$

$$x_1 = \underset{\substack{\downarrow \\ \text{eigen value.}}}{(5)} \underbrace{\begin{bmatrix} 1 \\ -2/5 \\ 3/5 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$$

$$x_2 = Ax_1 = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2/5 \\ 3/5 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 4-2/5+3/5+1 \\ 1-6/5-3/5+1 \\ 1+2/5+6/5 \\ 1-2/5+2 \end{bmatrix} = \begin{bmatrix} 26/5 \\ 1/5 \\ 13/5 \\ 13/5 \end{bmatrix}$$

$$x_2 = \frac{26}{5} \begin{bmatrix} 1 \\ 1/26 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$x_3 = Ax_2 = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/26 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 4+1/26+1/2+1/2 \\ 1+3/26-1/2+1/2 \\ 1-1/26+1 \\ 1+1/26+1 \end{bmatrix} = \begin{bmatrix} 131/26 \\ 29/26 \\ 51/26 \\ 53/26 \end{bmatrix}$$

$$= \frac{131}{26} \begin{bmatrix} 1 \\ 29/131 \\ 51/131 \quad 53/ \end{bmatrix}$$

$\mu^{(3)} = 5.038462$ largest eigen value.

$x_3^{t} = (1, 0.2213741, 0.3893130, 0.4045802)^{t}$