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SYED SHAHROZE ALI (BSC1-7J)

ASSIGNMENT #03

QUESTION #01

(i) $a_n = 1 + \frac{(-1)^n}{n}$

$= 1 + \lim_{n \to +\infty} (-1)^n \cdot \lim_{n \to +\infty} \frac{1}{n}$

$= 1 + \lim_{n \to \infty} (-1)^n \times 0$

$= 1$

The ~~term~~ sequence converges to 1.

(ii) $a_n = \frac{1-2^n}{2^n}$

$= \frac{1}{2^n} - 1$

$= \lim_{n \to +\infty} \frac{1}{2^n} - 1$

$= 0 - 1$

$= -1$

the ~~term~~ sequence converges to -1.

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(iii) $a_n = \frac{\ln(2n+1)}{n}$

∴ By applying limit we get $\frac{\infty}{\infty}$ ∴ applying L'Hopital

$$= \lim_{n\to\infty} \frac{2}{2n+1 \cdot (1)}$$

$$= \lim_{n\to\infty} \frac{2}{2n+1}$$

Applying limit

$\boxed{a_n = 0}$ | the sequence converges to 0

(iv) $a_n = \left(\frac{n+3}{n+1}\right)^n$

Let,

$$y = \lim_{n\to\infty} \left(\frac{n+3}{n+1}\right)^n$$

Applying L'Hopital

$$y = \lim_{n\to\infty} n \ln\left(\frac{n+3}{n+1}\right)$$

$$y = \lim_{n\to\infty} \frac{\ln\left(\frac{n+3}{n+1}\right)}{1/n}$$

Applying L'Hopital

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Replacing 'n' with 'x',

$$y = \lim_{x\to\infty} \frac{\frac{-2}{(x+1)^2}}{\frac{(x+3)}{(x+1)}} \cdot \frac{1}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2x^2}{(x+3)(x+1)}$$

$$= \lim_{x\to\infty} \frac{2x^2}{x^2+4x+3}$$

Applying L'Hopital

$$= \lim_{x\to+\infty} \frac{4x}{2x+4}$$

$$= \lim_{x\to+\infty} \frac{\cancel{4}\,2}{\cancel{2}}$$

$$= \lim_{n\to\infty} 2$$

$\ln y = 2$

$\boxed{y = e^2}$

$\boxed{\text{The sequence converges to } e^2.}$

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QUESTION #02

(1) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Applying integral Test. because terms are positive, decreasing and continuous.

$$\sum_{n=1}^{\infty} \frac{\ln}{n^3} = \int_{a=1}^{+\infty} n^{-3} \ln n \, dx$$

$$= \lim_{l \to +\infty} \int_1^l x^{-3} \ln x \, dx$$

$U = \ln x \qquad dV = x^{-3}$

$\frac{du}{dx} = \frac{1}{x} dx \qquad V = \frac{-1}{2x^2}$

$\int U dV = UV - \int V du$

$$\int x^{-3} \ln x = \ln x \cdot \left(\frac{-1}{2x^2}\right) + \int \frac{1}{2x^2} \cdot \frac{1}{x} dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2} \int \frac{1}{x^3} dx$$

$$= \frac{-\ln x}{2x^2} + \frac{1}{2} \left[\frac{-1}{2x^2}\right]$$

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$$= \left[\frac{-\ln x}{2x^2} - \frac{1}{4x^2} \right]_1^{+\infty}$$

$$= \lim_{l \to \infty} \left[\frac{-\ln x}{2x^2} - \frac{1}{4x^2} \right]_1^l$$

$$= \frac{-1}{2} \left[\lim_{l \to \infty} \left(\frac{\ln l}{l^2} - \frac{1}{2l^2} \right) - \left(\frac{\ln(1)}{1} + \frac{1}{2} \right) \right]$$

$$= -\frac{1}{2} \left[\lim_{l \to \infty} \frac{\ln l}{l^2} - \lim_{l \to \infty} \frac{1}{2l^2} - \frac{1}{2} \right]$$

$$= \frac{-1}{2} \left[\lim_{l \to \infty} \frac{1/l}{2l} - 0 - \frac{1}{2} \right]$$

$$= \frac{-1}{2} \left[\lim_{l \to \infty} \frac{1}{2l^2} \cdot \frac{1}{2} \right]$$

$$= \frac{1}{4}$$

Series is convergent.

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(ii) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

Using Ratio Test

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} = \lim_{n\to+\infty} \frac{(-3)^{n+1}}{(n+1)!} \Big/ \frac{(-3)^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(-3)^n}\cdot(-3)}{(n+1)\cancel{(n)!}} \cdot \frac{\cancel{n!}}{\cancel{(-3)^n}}$$

$$\rho = \lim_{n\to\infty} \frac{-3}{n+1}$$

Applying limit

$$\rho = 0$$

$\Rightarrow \rho < 1$ $\boxed{\therefore \text{ series converges.}}$

(iii) $\sum_{n=1}^{\infty} \frac{2}{n(2n+1)}$

As series is decreasing, positive and continuous therefore applying Integral Test.

$$\sum_{n=1}^{\infty} \frac{2}{n(2n+1)} = \int_{n=1}^{\infty} \frac{2}{n(2n+1)}\,dx$$

$$= \left\{ \lim_{l \to \infty} \int_1^l \frac{2}{x(2x+1)} \, dx \right.$$

Consider,

$$\frac{2}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

$$2 = A(2x+1) + Bx$$

at $x = 0$

$2 = A$

$\boxed{A = 2}$

at $x = -\frac{1}{2}$

$2 = \frac{-1}{2} B$

$\boxed{B = -4}$

$$\lim_{l \to \infty} \int_1^l \left[\frac{2}{x} + \frac{(-4)}{2x+1} \right] dx$$

$$= \lim_{l \to \infty} \left[2\ln x - 2\ln(2x+1) \right]_1^l$$

$$= 2 \lim_{l \to \infty} \left[\ln x - \ln(2x+1) \right]_1^l$$

$$= 2 \lim_{l \to +\infty} \left[\ln\left(\frac{x}{2x+1} \right) \right]_1^l$$

$$= 2 \left[\lim_{l \to \infty} \ln\left(\frac{l}{2l+1} \right) - \ln\left(\frac{1}{3} \right) \right]$$

$(2l+1) - 2$

$\frac{2l+1}{\frac{1}{2l+1}}$ 22K-4187 $\frac{l}{\frac{1}{2l+1}}$

$$2 \lim_{l \to \infty} \ln\left(\frac{1}{2l+1}\right) - 2\ln\left(\frac{1}{3}\right)$$

$$2 \lim_{l \to +\infty} \ln\left(\frac{1}{l(2+1/l)}\right) - 2\ln\left(\frac{1}{3}\right)$$

$$2 \lim_{l \to +\infty} \ln\left(\frac{1}{2+1/l}\right) - 2\ln\left(\frac{1}{3}\right)$$

$$2 \ln\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{3}\right)$$

$$= 0.810$$

$\boxed{\therefore \text{ series converges.}}$

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(iv) $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$

Applying comparision test.

$$\frac{2^n}{3^n+1} < \frac{2^n}{3^n}$$

Check the convergence of 2^{nd} series

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n}$$

it is geometric series

$\therefore a = \frac{2}{3}$, $r = \frac{2}{3}$

$S_n = \frac{a}{1-r}$ $\qquad |r| < 1$

$\therefore$ converge

$$= \frac{2/3}{1-2/3}$$

$$= 2$$

$\boxed{\text{as the bigger series converges then } \frac{2^n}{3^n+1} \text{ also converges.}}$

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(vi) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Applying Ratio Test

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)!}{2(n+1)!} \Big/ \frac{n!}{(2n)!}$$

$$= \lim_{n\to+\infty} \frac{(n+1)\cancel{n!}}{(2n+2)(2n+1)\cancel{2n!}} \cdot \frac{\cancel{2n!}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}}{2(2n+1)\cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \frac{1}{2(2n+1)}$$

$$= \cancel{\lim_{n\to\infty}}$$

$$\rho = 0$$

$\because \rho < 1$ $\boxed{\therefore \text{ series convergent.}}$

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(iii) $\sum_{n\geq 1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\rho = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n\times\cancel{n}}}$$

$$= \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^{n}$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^n}}{\cancel{n^n}}\left(\frac{1}{1+1/n}\right)^{n}$$

$$= \lim_{n\to+\infty} \frac{1}{(1+1/n)^n}$$

$$= \lim_{n\to+\infty} (1+1/n)^{-n}$$

$$= \lim_{n\to+\infty} -n\ln(1+1/n)$$

$$= \lim_{n\to+\infty} \frac{-\ln(1+1/n)}{1/n}$$

Applying L'Hopital

$$= \lim_{x\to+\infty} -\frac{\cancel{(-1/x^2)}}{(1+1/x)} \cdot \frac{1}{\cancel{-1/x^2}}$$

$$= \lim_{x\to+\infty} -\frac{1}{1+1/n}$$

$\ln y = -1$

$\boxed{y = e^{-1}}$ $\quad\because e^{-1} < 1 \quad \boxed{\therefore \text{ converges}}$

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(vii) $\sum_{n=1}^{\infty} n^2 e^{-n}$

Ratio Test

$$\sum_{n=1}^{n} \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \Big/ \frac{n^2}{e^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{e^n}.e} \cdot \frac{\cancel{e^n}}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{en^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{e \quad n^2}$$

$$= \lim_{n\to+\infty} \frac{1}{e}\left[\frac{\cancel{n^2}}{\cancel{n^2}} + \frac{2\cancel{n}}{n^{\cancel{2}}} + \frac{1}{n^2}\right]$$

$$= \lim_{n\to+\infty} \frac{1}{e}\left[1 + \frac{2}{n} + \frac{1}{n^2}\right]$$

$$= \frac{1}{e}$$

$\rho = e^{-1} < 1$ $\boxed{\therefore \text{converges.}}$

(viii) $1/2 + 2/3 + 3/4 + \ldots$

$\sum_{n=1}^{\infty} n/n+1$

Divergence test

$U_K = \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)}$ ~~$\Rightarrow U_K = \lim_{n}$~~

$$U_K = \lim_{n\to\infty} \frac{1}{1+1/n}$$

$$U_K = 1$$

$$\left(\frac{3}{2}\right)^{n-1}$$

$U_K \neq 0$ diverges

$$\sqrt[n]{\left(\frac{3}{2}\right)^n \cdot \left(\frac{3}{2}\right)^{-1}}$$

(1K) $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \dots$

$$\cancel{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n-1}}{2^{n-1}}}$$

$$\cancel{U_K = \frac{2}{3} \cdot \frac{3^n}{2^n}}$$

~~it is a gq~~

It is a geometric series

$a = 1 \quad r = \frac{-3}{2}$

$|r| \geq 1$ $\boxed{\text{series diverges}}$

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QUESTION #03

(i) $$\sum_{n=0}^{\infty} \frac{(n-1)(2n+1)^{n}}{(2n+1)2^{n}}$$

$$a_0 = \frac{(-1)(+1)^{0}}{(1)(1)} = -1$$

$$a_1 = \frac{(1-1)(2(1)+1)^{1}}{(2(1)+1)2^{1}} = 0$$

$$a_2 = \frac{(2-1)(2(2)+1)^{2}}{(2(2)+1)2^{2}} = \frac{5}{4}$$

$$a_3 = \frac{(3-1)(2(3)+1)^{3}}{(2(3)+1)2^{3}} = \frac{49}{4}$$

$$\boxed{-1 + 0 + \frac{5}{4} + \frac{49}{4} + \cdots}$$

(ii) $$\sum_{n=1}^{\infty} \frac{(n-1)^{2n-2}}{(2n-1)!}$$

$$a_1 = \frac{(1-1)^{2(1)-2}}{(2(1)-1)!} = 0$$

$$a_2 = \frac{(2-1)^{2(2)-2}}{(2(2)-1)!} = \frac{1}{6}$$

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$a_3 = \frac{(3-1)^{2(3)-2}}{(2(3)-1)!} = \frac{16}{120} = \frac{2}{15}$

$$\boxed{0 + \frac{1}{6} + \frac{2}{15} + \cdots}$$

$(1-x)^{-1}$ -1

$-\frac{1}{(1-x)^2}$ $\frac{1}{x}$

x^{-1}

x^{-2}

$-\frac{1}{x^2}$

QUESTION #04

(1) $f(x) = 1/_{1-x}$ $x = 2$

Let $x = x_0 = 2$

$-2(1-x)^{-3} \cdot \frac{d}{dx}(1-x)$

$f'(x) = \frac{1}{(1-x)^2}$

$f'(2) = 1$

$f''(x) = \frac{2}{(1-x)^3}$

$f''(2) = -2$

$f'''(x) = \frac{6}{(1-x)^4}$

$f'''(2) = 6$

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$$f(x) = f(x_0) + f'(x_0)(x-x_0)^1 + \frac{f''(x_0)(x-x_0)^2}{2!}$$

$$+ \frac{f'''(x_0)(x-x_0)^3}{3!}$$

$$= -1 + 1(x-2) + \frac{(-\cancel{2})(x-2)^2}{\cancel{2}\times 1} + \frac{\cancel{6}(x-2)^3}{\cancel{3\times 2}\times 1}$$

$$\boxed{= -1 + (x-2) - (x-2)^2 + (x-2)^3}$$

$$\cancel{f(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n}$$

(ii) $f(x) = \sqrt{3+x^2}$ $\qquad x = -1$

det $x = x_0 = -1$

$$f'(x) = x(x^2+3)^{-1/2}$$

$$f'(-1) = -\frac{1}{2}$$

$$f''(x) = 3(x^2+3)^{-3/2}$$

$$f''(-1) = \frac{3}{8}$$

$$f'''(x) = -9x(x^2+3)^{-5/2}$$

$$f'''(-1) = \frac{9}{32}$$

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$$f(x) \approx \sqrt{2} + \frac{-\frac{1}{\cancel{\sqrt{2}}}}{2} \times (x-(-1)) + \frac{\frac{3\cancel{\sqrt{2}}}{8}(x-(-1))^2}{2!}$$

$$+ \frac{\frac{9\cancel{\sqrt{2}}}{32}(x-(-1))^3}{3!} + \cdots$$

$$\cancel{f(x) \approx \sqrt{2} - \frac{\sqrt{2}(x+1)}{2} + \frac{3\sqrt{2}(x+1)^2}{8} + \frac{3\sqrt{2}(x+1)^3}{16}}$$

$$\boxed{\approx 2 - \frac{(x+1)}{2} + \frac{3(x+1)^2}{16} + \frac{3(x+1)^3}{64}}$$

QUESTION #05

(1) $\cos\sqrt{5x}$

$$f'(x) = -\frac{\sqrt{5}\sin\sqrt{5x}}{2\sqrt{x}}$$

$f'(0) =$ ~~does not e~~ undefined

$\boxed{\text{Maclaurin series not possible.}}$

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(ii) e^{-x^2}

$f'(x) = -2xe^{-x^2}$
$f'(0) = 0$

$f''(x) = (4x^2-2)e^{-x^2}$
$f''(0) = -2$

$f'''(x) = (-8x^3+12x)e^{-x^2}$
$f'''(0) = 0$

$f''''(x) = 16x^4e^{-x^2} + 48x^2e^{-x^2} + 12e^{-x^2}$
$f''''(0) = 12$

$\therefore$ Series will be

$$= 1 + 0 - \frac{2x^2}{2!} + 0 + \frac{12x^4}{4!} + \cdots$$

$$\boxed{= 1 - x^2 + \frac{x^4}{2} + \cdots}$$