22K-4187 → SYED SAMROZE ALI ①

ASSIGNMENT #01

(QUESTION #01)

GIVEN:

⇒ $A(0) = 1$ gram

⇒ $A(3.3) = \frac{1}{2}$

⇒ When 90% is decayed

$A(t) = 0.1 \quad t = ?$

SOLUTION:-

$$\frac{dA}{dt} = RA$$

$$\frac{dA}{dt} - RA = 0$$

$P(x) = -R \quad , \quad Q(x) = 0$

$I.F = e^{\int -k\,dt}$

$I.F = e^{-Rt}$

$I.F \times A = \int I.F \times Q(t)\,dt$

$e^{-Rt}.A = \int e^{-Rt}.0\,dt$

$e^{-Rt}A = c$

$A = ce^{Rt} \rightarrow ①$

Using $A(0) = 1$

$1 = ce^{R(0)}$

$\boxed{c = 1}$ → put in eq ①

$A = e^{Rt}$

FOR R:

$A(3.3) = \frac{1}{2}$

$\frac{1}{2} = e^{R(3.33)}$

$\ln(\frac{1}{2}) = k(3.33)$

$k = \frac{1}{3.33}\ln(\frac{1}{2})$

$k = -0.208$

FOR $A(t) = 0.1 \quad t = ?$

$0.1 = \cancel{(6/06.200)}e^{(-0.208)t}$

$\ln(0.1) = (-0.208)t$

$\boxed{t = 11.07 \text{ hours}}$

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$T_1(0) = 100°$
$T_1(1) = 90°$
$T_1(2) = ?$
$T_2(0) = T_1(2)$
$T_2(t) = 99.9°$
$t = ?$
$T_2(1) = 91°$

(QUESTION #02)

GIVEN:-

$T_1(0) = 100°C$

$T_1(1) = 90°C$

$T_1(2) = ?$

$T_2(0) = T_1(2) = ?$

$T_2(t) = 99.99°C \rightarrow t = ?$

$T_2(1) = T_2(0) + 10$

SOLUTION:-

$\frac{dT}{dt} = k(T - T_s)$

CONSIDERING CONTAINER "A"

$\frac{dT_1}{dt} = k_1(T_1 - 0)$

$\frac{dT_1}{dt} = k_1 T_1$

$\frac{dT_1}{dt} - k_1 T_1 = 0$

$P(x) = -k_1 \quad Q(x) = 0$

$I.F = e^{-k_1 t}$

$e^{-k_1 t} \cdot T_1 = \int e^{-k_1 t} \cdot 0 \, dt$

$e^{-k_1 t} \cdot T = c$

$T_1(t) = c_1 e^{k_1 t} \rightarrow (1)$

$T_1(0) = 100°$

$100° = c_1 e^{k_1(0)}$

$\boxed{100 = c_1}$

$T_1(t) = 100 e^{k_1 t} \rightarrow (2)$

After 1 minute,

$T_1(1) = 90°$

$9\cancel{0} = 10\cancel{0} e^{k_1(1)}$

$\ln(0.9) = k_1$

$\boxed{k_1 = -0.105}$

eq (2) $\Rightarrow$

$T_1(2) = 100 e^{-0.105 \times 2}$

$\boxed{T_1(2) = 81°}$

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(3)

$\because T_1(2) = T_2(0)$

$T_2(0) = 81°C$

CONSIDERING CONTAINER "B"

Similarly,

$\frac{dT_2}{dt} = k(T_2 - 100)$

$\int \frac{dT_2}{T_2 - 100} = \int k\,dt$

$\ln(T_2 - 100) = kt + c$

$T_2 - 100 = C_2 e^{k_2 t}$

$T_2(t) = 100 + C_2 e^{k_2 t} \rightarrow (3)$

$T_2(0) = 81$

$81 = 100 + C_2 e^{k_2(0)}$

$\boxed{C_2 = -19}$

$T_2(t) = 100 - 19e^{k_2 t} \rightarrow (4)$

$T_2(1) = 91°C$

$91 = 100 - 19e^{k_2(1)}$

$k_2 = \ln \frac{9}{19}$

$\boxed{k_2 = -0.74}$

For $T_2(t) = 99.9 \rightarrow t = ?$

eq (4)

$99.9 = 100 - 19e^{-0.74 \times t}$

$\frac{0.1}{19} = e^{-0.74 \times t}$

$\ln\left(\frac{0.1}{19}\right) = -0.74 \times t$

$\boxed{t = 7.02}$ min

The total time until bar reaches to 99.9°C is $\boxed{9.02 \text{ minutes}}$

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(QUESTION #03)

(Q# 32)

GIVEN:

$E(t) = 200V$

$R = 1000$

$C = 5\times10^{-6} F$

$q(t) = ? \rightarrow i(0) = 0.4$

$q(0.005) = ?$

$i(0.005) = ?$

$q(t)\ t\rightarrow\infty$

SOLUTION:-

$\because R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

$\frac{dq}{dt} + \frac{1}{CR}q = \frac{E(t)}{R}$

$\frac{dq}{dt} + \frac{q}{5\times10^{-6}\times1000} = \frac{200}{1000}$

$\frac{dq}{dt} + 200q = 0.2 \rightarrow ①$

$P(x) = 200 \quad Q(x) = 0.2$

$I.F = e^{\int 200dt}$

$I.F = e^{200t}$

$e^{200t}.q = \int e^{200t}.0.2\,dt$

$e^{200t}.q = \frac{2}{10\times200}e^{200t} + C$

$q = \frac{1}{1000} + Ce^{-200t} \rightarrow ②$

eq ①⇒

$i + 200\left(\frac{1}{1000} + Ce^{-200t}\right) = \frac{2}{10}$

$i = -200Ce^{-200t} \rightarrow ③$

at $i(0) = 0.4$

eq ③⇒

$0.4 = -200Ce^{-200(0)}$

$C = -1/500$

FOR $q(0.005) = ?$

eq ②⇒

$q = \frac{1}{1000} + \left(\frac{-1}{500}\right)e^{-200(0.005)}$

$\boxed{q = 0.00026}$

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For $i(0.005) = ?$

eq (3)

$$i = \frac{200}{500} e^{-200(0.005)}$$

$$\boxed{i = 0.147 \text{ Amp}}$$

For $t \to \infty$

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left[\frac{1}{1000} + \left(\frac{-1}{500}\right)e^{-200t}\right]$$

$$\boxed{\lim_{t\to\infty} q(t) = \frac{1}{1000}}$$

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Q#33

GIVEN:-

$L = 20$ henries

$R = 2\Omega$

$$E(t) = \begin{cases} 120 & 0 \le t \le 20 \\ 0 & t > 20 \end{cases}$$

$i(0) = 0$

SOLUTION:-

$$\because L\frac{di}{dt} + Ri = E(t)$$

$$20\frac{di}{dt} + 2i = E(t)$$

$$\frac{di}{dt} + 0.1i = \frac{E(t)}{20}$$

$P(x) = 0.1$, $Q(x) = \frac{E(t)}{20}$

$$I.F = e^{\int 0.1\, dt}$$

$$I.F = e^{1/10\, t}$$

For $E(t) = \frac{120}{20}$, $0 \le t \le 20$

$$e^{1/10\, t} . i = \int e^{t/10} . \frac{120}{20} dt$$

$$e^{t/10} . i = 60e^{t/10} + C$$

$$i = 60 + Ce^{-t/10} \rightarrow (1)$$

$\because i(0) = 0$

$$0 = 60 + Ce^{-t/10}$$

$\boxed{C_1 = -60}$ → put in (1)

$$i = 60 - 60e^{-t/10} \rightarrow (2)$$

When $E(t) = 0$ $t > 20$

$$e^{1/10\, t} . i = \int e^{t/10} . 0\, dt$$

$$i = C_2 e^{-t/10}$$

As function is continuous

$$\lim_{t \to 20^-} \left(60 - 60e^{-t/10}\right) = \lim_{t \to 20^+} \left(C_2 e^{-t/10}\right)$$

$$60 - 60e^{-2} = C_2 e^{-2}$$

$$\boxed{C_2 = 60(e^2 - 1)}$$

$$i(t) = \begin{cases} 60(1 - e^{-t/10}) & 0 \le t \le 20 \\ 60(e^2 - 1)e^{-t/10} & t > 20 \end{cases}$$

Ans.

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Q # 34

GIVEN:-

$R = 0.2$ Ohms

$E(t) = 4$

$i(0) = 0$

$L(t) = \begin{cases} 1 - \frac{1}{10}t & 0 \le t < 10 \\ 0 & t > 10 \end{cases}$

SOLUTION:-

$L \frac{di}{dt} + Ri = E(t)$

$\left(1 - \frac{1}{10}t\right)\frac{di}{dt} + Ri = E(t)$

$\left(\frac{10-t}{10}\right)\frac{di}{dt} + 0.2i = 4$

$\frac{di}{dt} + \frac{2}{10-t} i = \frac{40}{10-t}$

$P(x) = \frac{2}{10-t}$, $Q(x) = \frac{40}{10-t}$

$\cancel{I.F = e^{\int \frac{2}{10-t} dt}}$

$\cancel{= e^{-2\ln(10-t)}}$

$\cancel{I.F = (10-t)^{-2}}$

$\cancel{(10-t)^{-2} \cdot i = \int (10-t)^{-2} \cdot \frac{40}{10-t} \cdot dt}$

$\cancel{= -40 \int (10-t)^{-3} dt}$

$\cancel{(10-t)^{-2} i = -40 \frac{(10-t)^{-2}}{-2} + C}$

$\frac{di}{dt} = \frac{40}{10-t} - \frac{2i}{10-t}$

$= \frac{40-2i}{10-t}$

$\frac{di}{40-2i} = \frac{1}{10-t} dt$

$-\frac{1}{2}\ln(40-2i) = -\ln(10-t) + \ln c$

$\ln(40-2i) = 2\ln(10-t) + \ln c$

$\ln(40-2i) = \ln(c(10-t)^2)$

Taking 'e'

$40 - 2i = c(10-t)^2$

$2i = 40 - c(10-t)^2$

$i = 20 - \frac{c(10-t)^2}{2} \rightarrow (1)$

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at $i(0)=0$

$0=20-\frac{1}{2}k(10-0)^2$

$\boxed{k=\frac{2}{5}}$

eq (1) ⇒

$i(t)=20-\frac{1}{5}(10-t)^2$

Now for $t>10$;

$L=0$

$0\frac{di}{dt}+0.2i=4$

$\frac{1}{5}i=4$

$\boxed{i=20}$

$i(t)=\begin{cases}20-\frac{1}{5}(10-t)^2 & 0\le t\le 10\\ 20 & t>10\end{cases}$

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EX: 7.7

LOGISTIC EQUATION

$$\frac{dp}{dt} = P(a - bP)$$

Where,

$P =$ size of population

$a \& b =$ constant

$\frac{dp}{dt} =$ Rate of change of population.

—(SOLUTION)—

$$\frac{dp}{dt} = P(a-bP)$$

$$\int \frac{dp}{P(a-Pb)} = \int dt \rightarrow ①$$

Consider,

$$\int \frac{1}{P(a-Pb)} = \frac{A}{P} + \frac{B}{a-bP} \rightarrow ②$$

$1 = A(a-bP) + BP$

at $P = 0$

$A = 1/a$

at $P = a/b$

$B = b/a$

put in ①

$$\cancel{\frac{1/a}{P} + \frac{b/a}{a-bP}}$$

$$\int \frac{1}{P(a-bP)} = \frac{1/a}{P} + \frac{b/a}{a-bP}$$

$$= \frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP|$$

using in ②

$$\frac{1}{a}\ln\left(\frac{P}{a-bP}\right) = t + c$$

$$\ln\left(\frac{P}{a-bP}\right) = at + c$$

$$\frac{P}{a-bP} = ce^{at}$$

$P = ace^{at} - bPce^{at}$

$P(1 + bce^{at}) = ace^{at}$

$$P(t) = \frac{ace^{at}}{1 + bce^{at}} \rightarrow ③$$

at $P(0) = P_0$

$$P_0 = \frac{ac}{1+bc}$$

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$P_0 + P_0 bc = ac$

$P_0 = ac - P_0 bc$

$P_0 = c(a - bP_0)$

$c = \dfrac{P_0}{a - bP_0}$ → put in ③

eq ③ ⇒

$$P(t) = \frac{a\left(\frac{P_0}{a-bP_0}\right)e^{at}}{1 + \frac{P_0}{(a-bP_0)}be^{at}}$$

$$= \frac{a\left(\frac{P_0}{\cancel{a-bP_0}}\right)e^{at}}{\frac{(a-bP_0)+P_0be^{at}}{\cancel{(a-bP_0)}}}$$

$$= \frac{ae^{at}P_0}{(a-bP_0)+P_0be^{at}}$$

$$\boxed{P(t) = \frac{aP_0}{(a-bP_0)e^{-at} + bP_0}}$$

Q1: Given:

$N(0) = 1$

$N(t) = ?$ → $t = 10$

SOLUTION:-

as we are given,

$$\frac{dN}{dt} = N(1 - 0.0005N)$$

$$\int \frac{dN}{N(1-0.0005N)} = \int dt \rightarrow ①$$

Consider,

$$\int \frac{dN}{N(1-0.0005N)} = \frac{A}{N} + \frac{B}{1-0.0005N}$$

$1 = A(1-0.0005N) + BN$

at $N = 0$

$\boxed{A = 1}$

at $N = 1/0.0005$

$1 = B \cdot \dfrac{1}{0.0005}$

$\boxed{B = 0.0005}$

$$\int \frac{dN}{N(1-0.0005N)} = \int \frac{1}{N}dN + \int \frac{0.0005\,dN}{1-0.0005N}$$

$= \ln|N| - \ln|1-0.0005N|$

put in ①

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(4)

$\ln|N| - \ln|1-0.0005N| = t + c$

$\ln\left(\dfrac{N}{1-0.0005N}\right) = t + c$

$\dfrac{N}{1-0.0005N} = ce^t$

$N = ce^t - 0.005Nce^t$

$N(1+0.005ce^t) = ce^t$

$N = \dfrac{ce^t}{1+0.0005ce^t}$

$1 = \dfrac{c}{1+0.0005c}$

$1 = c - 0.0005c$

$c = \dfrac{2000}{1999}$

$N = \dfrac{\frac{2000}{1999} \cdot e^t}{1 + \frac{0.0005 \cdot 2000e^t}{1999}}$

$= \dfrac{(2000/1999) \cdot e^t}{\frac{1999 + e^t}{1999}}$

$N(t) = \dfrac{2000e^t}{1999 + e^t}$

at $t = 10$

$N(10) = \dfrac{2000e^{10}}{1999 + e^{10}}$

$\boxed{N(10) = 1834}$

Q2: Given:-

- $N(0) = 500$
- $N(1) = 1000$
- $N(t) = ?$ $t \to \infty$

SOLUTION:-

> we have

$\dfrac{dN}{dt} = N(a - bN)$

we get,

$N(t) = \dfrac{aN_0}{(a-bN_0)e^{-at} + bN_0}$

$N = \dfrac{500a}{(a-500b)e^{-at} + 500b}$

$N(1) =$

$\overset{2}{1000} = \dfrac{500a}{(a-500b)e^{-a} + 500b}$

$2(a-500b)e^{-a} + 1000b = a \rightarrow (1)$

> $\lim\limits_{t\to\infty} N(t) = \dfrac{a}{b}$ $50000 = \dfrac{a}{b}$

$50000 = \dfrac{a}{b}$

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$a = 50{,}000b \rightarrow$ put in (1).

eq (i) $\Rightarrow$

$2(50{,}000b - 500b)e^{-50{,}000b} + 100b = 50{,}000b$

$99000be^{-50{,}000b} = 49{,}000b$

$e^{-50000b} = \frac{49}{99}$

$-50{,}000b = \ln(49/99)$

$\boxed{b = 0.000014}$

$a = 50{,}000 \times 0.000014$

$\boxed{a = 0.703}$

$\lim_{t\to\infty} N(t) = \frac{a}{b}$

$= \frac{0.703}{0.000014}$

$\boxed{\lim_{t\to\infty} N(t) = 50214}$

Q3: Given:

- $P(0) = 5000$
- $P(t) = ?$ $\Rightarrow t \rightarrow \infty$
- $P(t) = \frac{1}{2}$ limiting Population

$t = ?$

SOLUTION:-

$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P)$

we get,

$P(t) = \frac{10^{-1}P_0}{(10^{-1} - 10^{-7}P_0)e^{-at} + 10^{-7}P_0} \rightarrow (1)$

at $P(0) = 5000$

$5000 = \frac{10^{-1}P_0}{(10^{-1} - 10^{-7}P_0) + 10^{-7}P_0}$

$50{,}000(10^{-1} - 10^{-7}P_0) + \frac{1}{200}P_0 = P_0$

$5000 - \frac{1}{200}P_0 = \frac{199P_0}{200}$

$1000000 - P_0 = 199P_0$

$1000000 = 200P_0$

$P_0 = 5000 \rightarrow$ put in (1)

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eq (1)

$$P(t) = \frac{(5000)(10^{-1})}{(10^{-1} - 5000 \times 10^{-7}) e^{-at} + 10^{-7} \times 5000}$$

$$P(t) = \frac{500}{0.0995 e^{-0.1t} + 0.0005}$$

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \left(\frac{500}{0.0995 e^{-0.1t} + 0.005} \right)$$

$$= \lim_{t \to \infty} \left(\frac{500}{0.005} \right)$$

$$\lim_{t \to \infty} P(t) = 1000000$$

If $P(t) = \frac{1}{2} \lim_{t \to \infty} P(t)$

$$P(t) = 500000$$

$$500000 = \frac{500}{0.0995 e^{-0.1t} + 0.0005}$$

$$1000 = \frac{1}{0.0995 e^{-0.1t} + 0.0005}$$

$$99.5 e^{-0.1t} + 0.5 = 1$$

$$99.5 e^{-0.1t} = 0.5$$

$$e^{-0.1t} = \frac{1}{1999}$$

$$-0.1t = \ln(1/199)$$

$$\cancel{-}0.1t = \cancel{-}5.293$$

$$\boxed{t = 52.93} \text{ months}$$

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Q4: Given:

at 1790 → $t=0$

at 1850 → $t=60$ (1790-1850)

at 1910 → $t=120$ (1790-1910)

Solutions:-

$$\frac{dP}{dt} = P(a-bP)$$

we get

$$P(t) = \frac{aP_0}{(a-bP_0)e^{-at}+bP_0}$$

at $P(0) = 3.929$

∴ initial population, $P_0 = 3.929$

~~$3.929 = \frac{aP_0}{(a-bP_0)e^{0}+bP_0}$~~

$$P(t) = \frac{3.929a}{(a-3.929b)e^{-at}+3.929b}$$

$$= \frac{3.929b\,(a/b)}{3.929b\left[\left(\frac{a}{3.929b}-1\right)e^{-at}+1\right]}$$

~~at P(60)~~ =

Let, $c = a/b$ → (A)

$$P(t) = \frac{c}{\left(\frac{c}{3.929}-1\right)e^{-at}+1}$$

at $P(60) = 23.192$

$$23.192 = \frac{c}{1+\left(\frac{c}{3.929}-1\right)e^{-60a}}$$

$$e^{-60a} = \frac{c-23.192}{23.192\left(\frac{c}{3.929}-1\right)} \rightarrow (1)$$

at $P(120) = 91.972$

$$91.972 = \frac{c}{1+\left(\frac{c}{3.929}-1\right)\left(e^{-60a}\right)^2}$$

$$c = 91.972 + 91.97\left(\frac{c}{3.929}-1\right)\left(e^{-60a}\right)^2$$

→

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$$\frac{c-91.972}{91.972} = \left(\frac{c}{3.929}-1\right)\left(e^{60a}\right)^2$$

$$\frac{c-91.972}{91.972} = \left(\frac{c}{3.929}-1\right)\left[\frac{c-23.192}{23.192\left(c/3.929-1\right)}\right]^2$$

$$\frac{c-91.972}{91.972} = \left(\frac{\cancel{c-3.929}}{\cancel{3.929}}\right)\frac{\left(c-23.192\right)^2}{(23.192^2)\frac{\left(c-3.929\right)^2}{3.929^2}}$$

$$\frac{c-91.972}{91.972} = \frac{\left(c-23.192\right)^2(3.929)}{(23.192)^2(c-3.929)}$$

$$(c-91.972)(23.192)(c-3.929) = (91.972)(c-23.192)^2(3.929)$$

$$(c^2-95.901c+361.357)(23.192)^2 = (c^2-46.384+537.86)(361.357)$$

$$c^2-95.901c+361.357 = (c^2-46.384-537.86)(0.671)$$

$$'' \quad = 0.671c^2-31.162c+360.464$$

$$0.329c^2-64.739c+0.453=0$$

Using Quadratic Formula

$a = 0.329$, $b = -64.739$, $c = 0.453$

$$c = \left(-b \pm \sqrt{b^2-4ac}\right)/2a$$

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$$c = \frac{-(-64.739) \pm \sqrt{(-64.739)^2 - 4(0.329)(0.453)}}{2(0.329)}$$

$\boxed{c = 196.76}$ or $c = 7 \times 10^{-3}$

invalid

eq(I) $\Rightarrow$

$$e^{-60a} = \frac{196.76 - 23.192}{23.192\left(\frac{196.76}{3.929} - 1\right)}$$

$e^{-60a} = 0.152$

$-60a = \ln(0.152)$

$-60a = -1.88$

$\boxed{a = 0.031}$

eq(A)

$196.76 = 0.031/b$

$\boxed{b = 0.000159}$

Logistic Model:

$$\boxed{P(t) = \frac{0.122}{0.0306e^{-0.0313t} + 0.0062}}$$

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(b)

YEAR	POPULATION (Given) in millions	Population (Predicted) in million	Percentage error
1790	3.929	3.929	0%
1800	5.308	5.282	0.4 ~~0.49~~ %
1810	7.240	7.222	0.2 ~~1.47~~ %
1820	9.638	9.745	-1.1 ~~0.469~~ %
1830	12.866	13.088	-1.6 ~~0.257~~ %
1840	17.669	17.471	-2.3 ~~1.06~~ %
1850	23.192	23.137	0.23 ~~0~~ %
1860	31.433	30.330	3.5 ~~0.82~~ %
1870	38.558	39.253	-1.8 ~~2.59~~ %
1880	50.156	50.012	0.28 ~~10.49~~ %
1890	62.948	~~154.376~~ 62.550	0.63 ~~15.76~~ %
1900	75.996	76.590	-0.78 %
1910	91.972	91.630	0.37%
1920	105.711	106.994	-1.21 %
1930	122.775	121.945	0.65 ~~21.07~~ %
1940	131.669	~~261.300~~ 135.225	-3.1 ~~74.9~~ %
1950	150.697	~~751.694~~ 148.156	1.68 %