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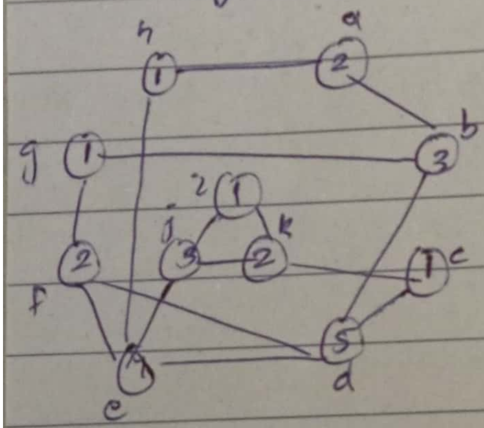
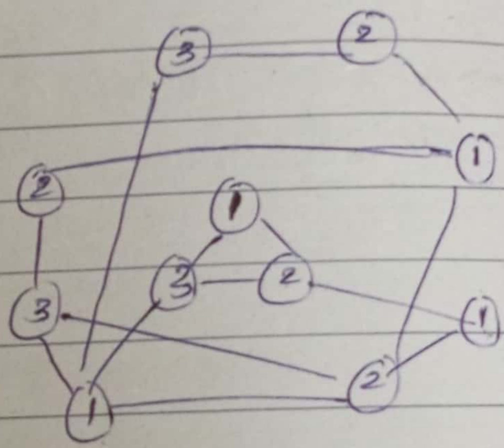
"GRAPH COLOURING"

→ A proper k-coloring of a graph G is an assignment of colors to the vertices of G so that no two adjacent vertices are given the same color & exactly 'k' colors are used.



G_1 proper 5-coloring



G_2 proper 3-coloring

→ Color classes: Given a proper k-coloring, the sets $S_1, S_2 \ldots S_k$ consist of all vertices of color i.
Mean color #1 ki jitni vertices hongi woh S_1 men 2 wale ki S_2 mein & so on....

In G_1 → $S_1 = \{c, g, h, i\}$, $S_2 = \{a, f, k\}$,
$S_3 = \{b, j\}$, $S_4 = \{e\}$, $S_5 = \{d\}$

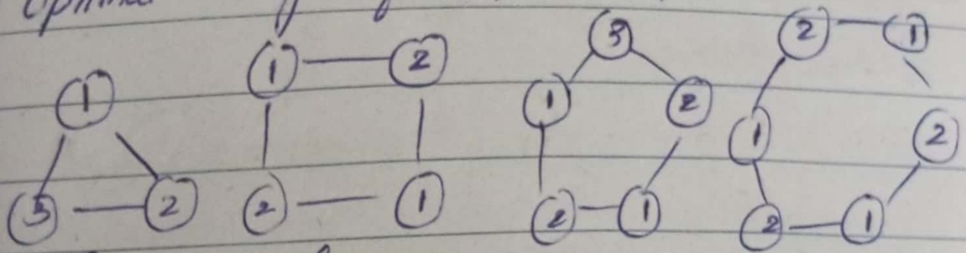
→ Independence Number: No. of vertices in the ~~indepe~~ maximum independence set (set of vertices in which no two vertices are adj to each other)
Denoted by $\alpha(G) = n$

Day

Vertex Colouring:

→ Chromatic Number: The smallest value k for which G has a proper k-coloring, denotion $\chi(G)$.

→ Optimal colorings of C_3, C_4, C_5, C_6

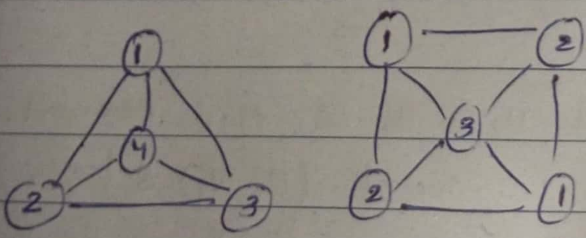


C_3 C_4 C_5 C_6

~~odd~~

↳ when n = odd → 3 colors
n = even → 2 colors } $n \geq 2$

→ Wheel Graphs:

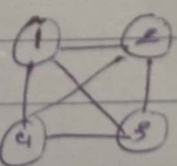


$n=4$ $n=5$

↳ when n = even → 4 colors
n = odd → 3 colors } $n \geq 3$

→ Complete Graphs:

$\chi(K_n) = n$



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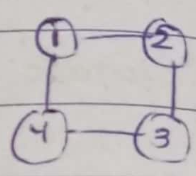
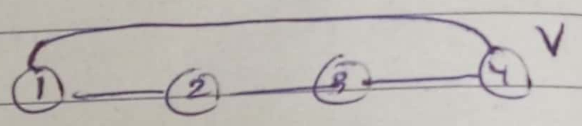
→ To get lower bound for the chromatic no. we check clique size of a graph.



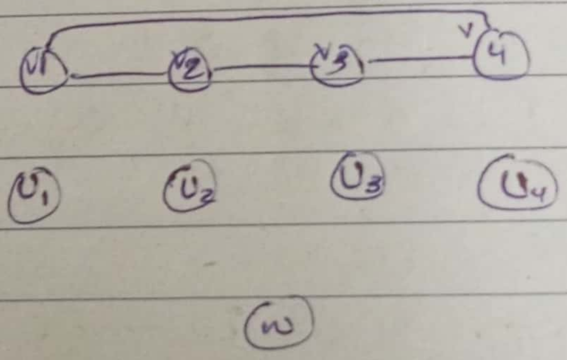
⊅ MYCIELSKI CONSTRUCTIONS

→ Sabse phele jo given graph G hai usko as it is draw krlenge.

G →

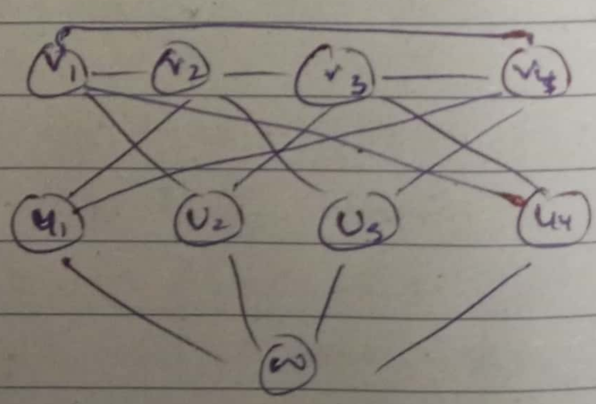


→ Usko bead har v_i k corresponding u_i vertice bnayenge. Aur ek w vertice bhi



→ No we connect all u_i with w and ~~connect~~ for each (v_i, v_j) we conect (u_i, v_j) & (u_j, v_i)

v_1, v_2 u_1, v_2 u_2, v_1



→ If original graph has chromatic number K than Mys— Graph will have K+1.

→ Coloring in My— Graph: Jo clor V_1 ko diya hai wohi U_1 ko dedenge. Means Color (V_i) = Color (U_i) and than ω will have a diff color.

▷ **BROOK'S THEOREM:**

Let G be a connected graph and Δ denote maximum degree among all vertices in G.

Then, $\chi(G) \leq \Delta$, but G != complete || odd cycle graph

If G is complete or odd cycle then

$\chi(G) = \Delta + 1$ → If all neighbours of a vertex 'x' have been given diff color, then one additional color is needed for x.

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

clique size — chromatic number — ↳ max degree

▷ **BASIC COLORING STRATEGIES:**

→ Begin with the highest degree vertices

→ Then look for locations where colors are forced (cliques, wheels, odd cycles) rather than choosen.

→ When these strategies have been exhausted, color the remaining vertices while trying to avoid using any additional colors.

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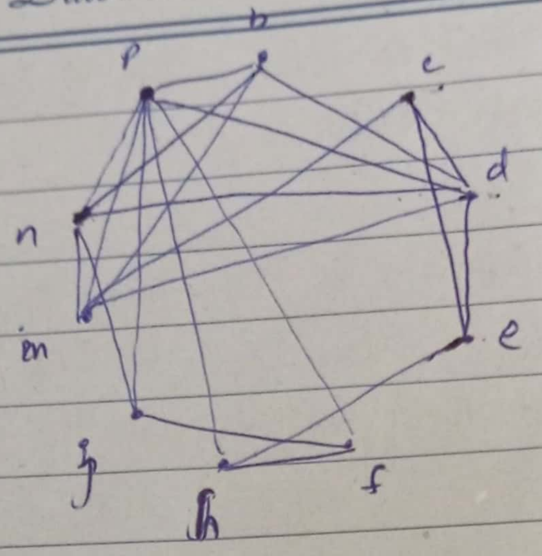
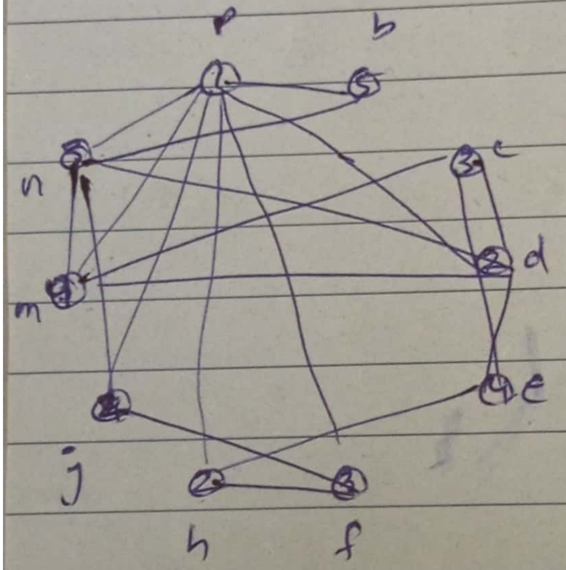


- choose 'p' (highest degree ver) give color 1.
- Now look for $N(p)$ with high degree
- $d \to 6$, m & $n \to 5$
- p, d, m & n form K_4 so they all will be assigned diff colors.



- Now the other vertice with high degree is b & it is also adjacent to p, d, m & n.
- So 5^{th} color is needed.
- The remaining vertices all have deg 3 and can be colored with any additional colors.

⇒ **Equitable Coloring:** It is a minimal proper coloring of G so that the no. of vertices of each color differs by at most one.

⇒ Let G be a graph then

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

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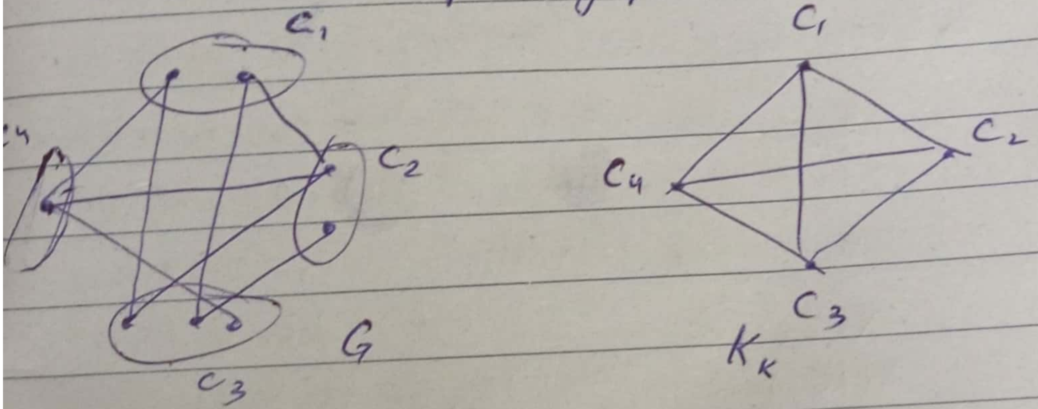
~~Let G be a graph~~

→ Assume $\chi(G)=k$. First note that there must be at least one edge b/w each color class because if two classes don't have at least one edge then they should be in same color class.

Now consider each color class as single vertex and representing ~~all the edges~~ any edges between color class as single edge we obtain a complete graph K_k.



Thus G must have at least as many edges as K_k. There no. of edges in

$$m \geq \frac{k(k-1)}{2} \rightarrow \text{total no of edges in } K_k$$

$$2m \geq k^2 - k$$

Completing the sq,

$$2m + \frac{1}{4} \geq k^2 - k - \frac{1}{4} = \left(k - \frac{1}{2}\right)^2$$

$$k \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

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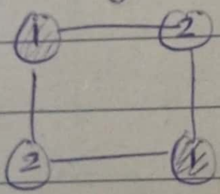
Let G be a graph and $l(G)$ be the length of of the longest path in G. Then $\chi(G) \leq 1 + l(G)$

Induced Subgraph:
Ismein saare vertices nhi hote, only subset of vertice hote hai. Lekn jitne bhi vertices honge unke darmiyan ki tamam edges include hongi.

Perfect: A graph G is perfect iff $\chi(H) = \omega(H)$ for all induced subgraphs H.
clique size

↳ Even cycles are perfect.

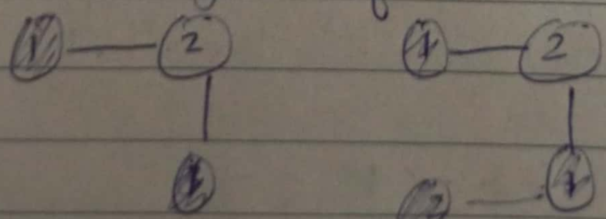


$\chi(C_4) = 2$
$\omega(C_4) = 2$

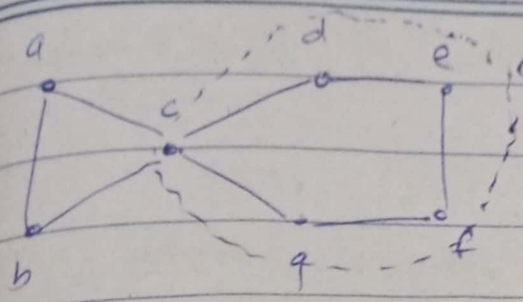
q k jitne bhi even cycles hote hai unka $\chi(C_n)$ 2 hi hota hai. Or jitn bhi $C_n \mid n \geq 4$ hote hai unka $\omega(C_n)$ bhi 2 hi hota hai. So they are perfect!

↳ ~~But odd cycles of length > 3~~ are not perfect.
↳ Q k inka "χ" hamesha 3 hota hai or clique size cycle graph ka where $n \geq 4$ 2 hi hota ha as discussed above.

Induced Subgraphs of C_n, $n \geq 4$



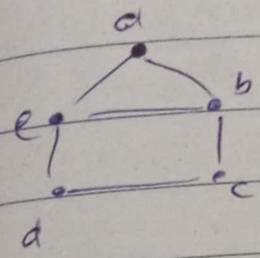
C_4 C_5



→ this is C_5 so it's $\omega(C_5)=2$ & $\chi(G)=3$

So not perfect.

G_2 → overall G_2 to perfect ki condition satisfy krsha leken subgraph satisfy nhi karhe.



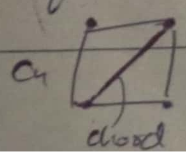
→ Here all subgraph satisfy the condition

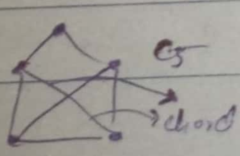
" A graph G is perfect iff $\bar{G}$ is perfect "

" A graph G is perfect iff no induced subgraph of G or $\bar{G}$ is an odd cycle of length at least 5. "

Following graphs are known to be perfect.
(1) Trees (2) Bipartite (3) Chordal
(4) Interval

CHORDAL GRAPH :
A graph G is chordal if any cycle of length 4 or larger has an edge (called chord) b/w two non-consecutive vertices of the cycle.





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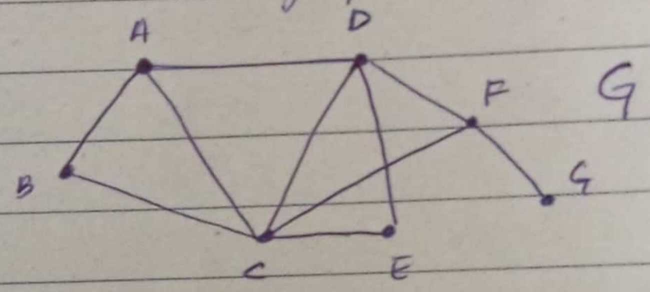
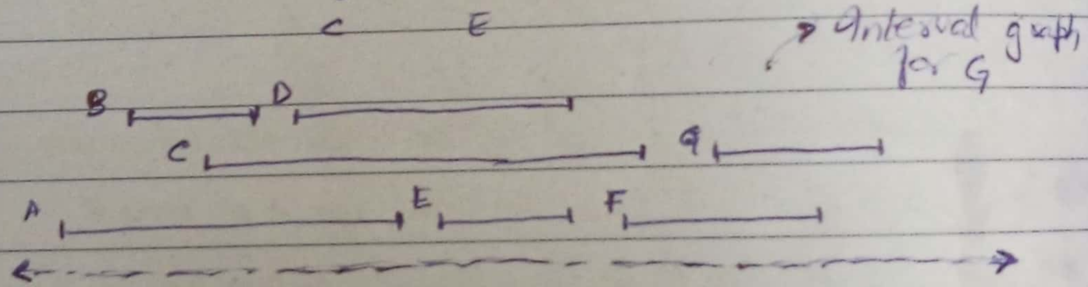
→ Complete graph superset of chordal graph.

$\chi(G) = \omega(G)$

INTERVAL GRAPH:

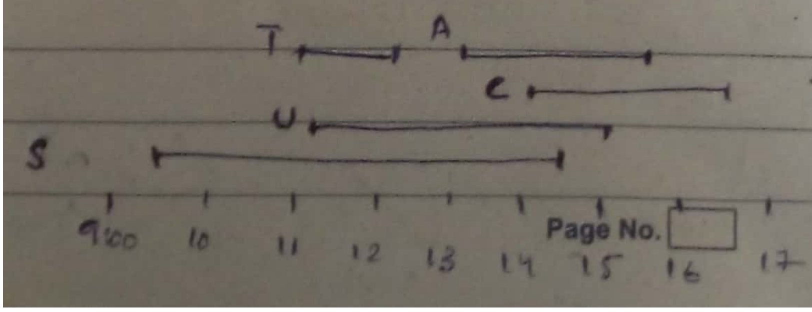
A graph that can be represented by a set of intervals on a line such that two intervals have an intersection if & only if the corresponding vertices in the graph are adjacent.

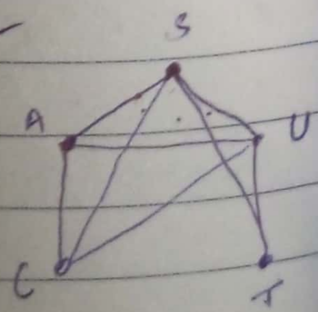




Example:

Students	Time
A	13 – 15:30
C	14 – 16:30
S	9:30 – 14:30
T	4:00 – 12:00
U	11:15 – 15:00





a b c d e f g
1 3 3 2 2 1 2 4

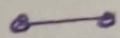
as there is K_4 {A, S, U, C} so it will get 4-colors.

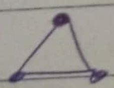
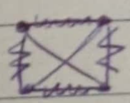
"EDGE COLORING"

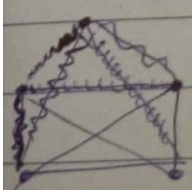
→ Assignment of colors to the edges so that if an edge share an endpoint they have different colors.

Minimum no. of colors needed for edge coloring is called chromatic index → $\chi'(G)$

Edge coloring of Complete graphs K_n n 1→6

 $\chi'(K_1) = 0$  $\chi'(K_2) = 1$

 $\chi'(K_3) = 3$  $\chi'(K_4) = 3$



$\chi'(K_n) = n - 1$ n = even

$\chi'(K_n) = n$ n = odd

ex: K_5 ki ~~minimum~~ max degree 4 hai to laazmi 4 se zyada hi colors use honge. Ek vertex se start kro or saari edges ko wahe color kode.

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↳ Therefore the chromatic index must be at least the maximum degree, $\Delta(G)$, of the graph.

↳ But for odd values of n, K_n req one ~~or~~ more ~~than~~ ~~one~~ color than the max degree.

↳ In fact any graph will either require $\Delta(G)$ or $\Delta(G)+1$ colors to its edges.

Vizing's Theorem: $\Delta(G) \leq X'(G) \leq \Delta(G)+1$

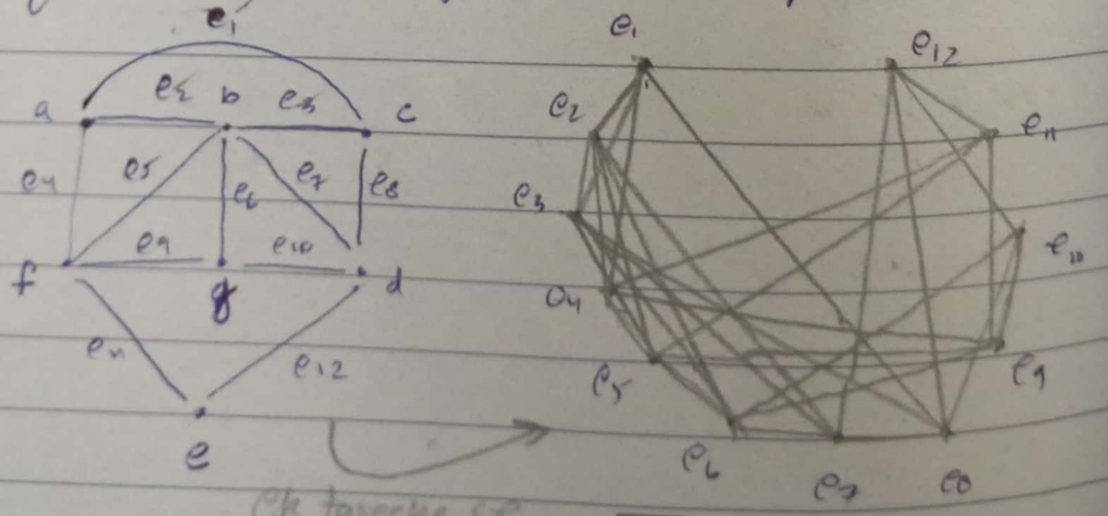
for all simple graphs G.

if $X'(G) = \Delta(G)$ → class 1 (Bipartite graphs)

$X'(G) = \Delta(G)+1$ → class 2 (regular graphs with odd vertices)

▷ Line Graph:

Line graph $L(G) = (V', E')$ is the graph formed from G where each vertex "x'" in $L(G)$ represents the edge "x" from G & "$x'y'$" is an edge of $L(G)$ if the edges x' & y' share an endpoint in G.



Ek taraf se edge ko vertex mein convert kardia.

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Given a line graph $L(G)$ then
$$\chi'(G) = \chi(L(G))$$

▷ **Ramsey Number:**

Given +ve integers m & n, the Ramsey Number $R(m,n)$ is the minimum no. of vertices "r" so that all simple graphs on 'r' vertices contain either clique of size 'm' or an independent set of size 'n'.

↳ Can be described as

"$R(3,2)$ must be interpreted as how many guests must be invited so that at least 3 people all know each other or at least 2 people donot know each other?"

"ONLINE - COLORING"

Consider a graph G with the vertices ordered as $x_1 < x_2 < \cdots < x_n$.

An on-line algorithm colors the vertices one at a time where the color for x_i depends on the induced subgraph $G[x_1, \ldots, x_i]$ which consist of the vertices upto & including x.

The max number of colors a specific algo A uses on any possible ordering of the vertices is denoted $\chi_A(G)$.

Bipartite can be colored with 2 colors

Hello

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• First Fit Coloring Algo:

→ Given G with vertices ordered as $x_1 < x_2 < \ldots < x_n$

① Assign x_1 color 1

② Assing x_2 color 1 if x_1 & x_2 not adjacent; otherwise assign x_2 color 2.

③ For all future vertices, assign x_i the least color available ~~ab to~~ x_i in $G[x_1, \ldots, x_i]$; that is give x_i the 1st color not used by any neighbour of x_i that has already been colored.

→ On-line coloring algo works well on interval graphs

→ First Fit give the uper bound for chromatic number.

• Weighted Coloring:

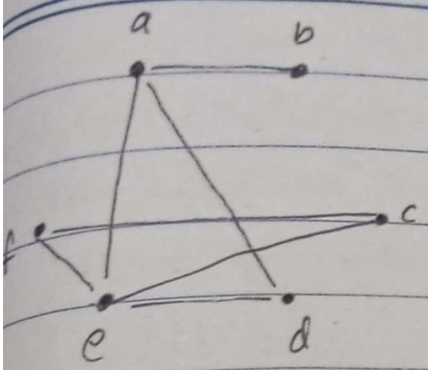
Given a weighted graph $G = (V, E, \omega)$, a proper weighted coloring of G assigns each vertex a set of colors so that

weight

i) the set consist of consective colors (or numbers)

ii) the no. of colors assigned to a vertex equals its weight; and

iii) if two vertices are adj, then their set of colors must be disjoint.



$w(a) = 2$ $w(e) = 2$

$w(b) = 1$ $w(f) = 4$

$w(c) = 4$

$w(d) = 2$

→ Suppose place "e" color range because it forms K_3 with aed & fec.

$e \to \{1,2\}$ now $f \to \{3,4,5,6\}$ then

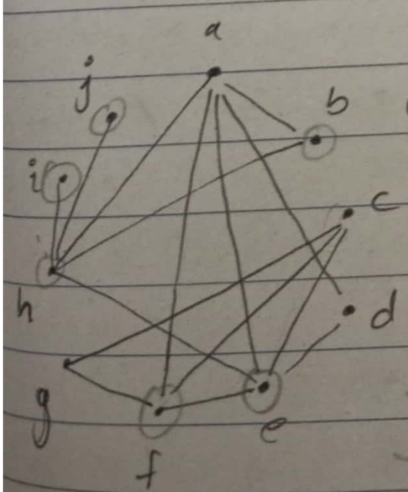
$c \to \{7,8,9,10\}$ → $a \to \{3,4\}$, $d \to \{5,6\}$ g

$b \to \{1\}$

→ Find a clique with largest weight.

→ Focus less on vertex degree & more on vertex weight

→ if a vertex has high weight, then it needs a larger range from which to pick set of colors.



$w(a)=2$ ~~$w(i)=4$~~

~~$w(b)=5$~~ ~~$w(j)=5$~~

~~$w(c)=2$~~ $b = 1,2,3,4,5$

~~$w(d)=1$~~ $j = 1,2,3,4,5$

~~$w(e)=4$~~ $e = 1,2,3,4$

~~$w(f)=3$~~ $i = 1,2,3,4$

~~$w(g)=1$~~ $f = 1,2,3$ $d = 5$

~~$w(h)=3$~~ $h = 6,7,8$ $g = 4$

$c = 5,6,\cancel{7}$

$a = 9$ ...

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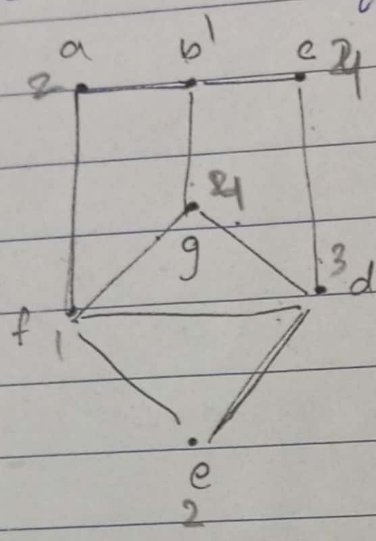
▷ List Coloring :

Here we are concerned with assigning colors to the vertices (or edges) with added restriction that the colors must come from some surprising results,



	List 1	List 2
a	{1,2}	{1}
b	{1,3,5}	{1,4,5}
c	{3,4}	{3,4}
d	{3,4}	{3,4}
e	{2,3}	{2,3}
f	{1,2,3}	{1,2,3}
g	{4,5}	{4,5}

~~har vertex ke color ka~~ number isi mein se choose karna hai.

no coloring exist for this list.

⊖ For List #1 : $f \to 1$, $d \to 3$, $e \to 2$, $a \to 2$, $b \to 1$

$c \to 4$, $g \to 4$

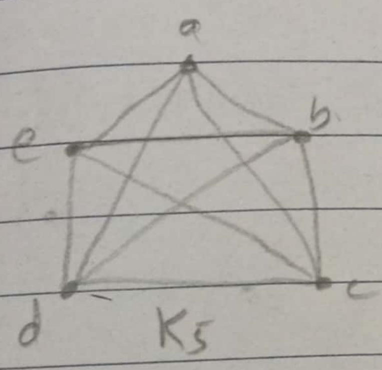
→ If for every collection of lists, each of size k, a proper list coloring exists then G is k-choosable. The min value for k for which G is k choosable is called choosablity of $G \to ch(G)$

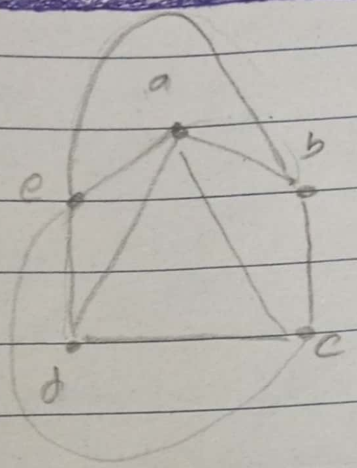
For any simple graph G, $ch(G) \geq \chi(G)$

" let G satisfy $\chi(G) = k$ & give each vertex of G the list $\{1, 2, \ldots k\}$. Then there is a proper coloring for G from these lists, namely the one exhibited by the fact that $\chi(G) = k$. However, if we remove the same one element from each of these lists, then G cannot be colored since otherwise $\chi(G) < k$.

→ for any simple graph G, $ch(G) \leq \Delta(G)+1$.

Planarity to pic



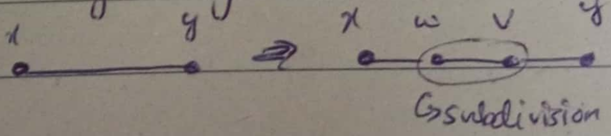


So K_5 is no planar!

→ Having either K_5 or $K_{3,3}$ as a subgraph will guarantee that a graph is non-planar

→ If a subgraph is non planar than whole graph is non-planar.

Subdivision : Ek edge "xy" k darmian vertices insert krna.



↳ Subdivision of a graph can be obtained by dividing one, two or even all of the edges.