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CHP # 12

PARTIAL DE :

$U \rightarrow$ dep on x & y

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = x+y$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$\frac{\partial^2 U}{\partial x} + \frac{\partial^2 U}{\partial y} + \frac{\partial^2 U}{\partial x \partial z} = xyz$$

ORDER AND DEGREE OF PDE

- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x+yz$ order = 1, d = 1, linear
- $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial^3 z}{\partial y^3}\right) = 2x\left(\frac{\partial z}{\partial x}\right)$, O = 3, d = 1, non-linear
- $z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x$ O = 1, d = 1, non-linear
- $\frac{\partial^2 z}{\partial x^2} = \left(1+\frac{\partial z}{\partial y}\right)^{1/2}$

Sq b/s

$$\left(\frac{\partial^2 z}{\partial x^2}\right)^2 = 1+\frac{\partial z}{\partial y}$$ non-linear, O = 2, d = 2

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \sin(x+y) = u$ linear

If dependent variable or the power of its derivative is $\neq 1$ than rules wohi same hi hain jese ODE k the.

$$\frac{\partial U}{\partial x} + \sin U + \frac{\partial U}{\partial y} = 1$$ non-linear

$$\frac{\partial U}{\partial x} = U_x = p$$

$$\frac{\partial U}{\partial y} = U_y = q$$

$$\frac{\partial^2 U}{\partial x^2} = U_{xx} = r$$

$$\frac{\partial^2 U}{\partial x \partial y} = U_{yx}$$ → pdde w.r.t y then w.r.t x

$$\frac{\partial^2 U}{\partial y \partial x} = U_{xy}$$

U_{yx}, U_{xy} → s

$$\frac{\partial^2 U}{\partial y^2} = U_{yy} = t$$

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2nd ORDER PDE :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F.U = f(x,y)$$

A or C apas mein change hosakte hai lekin B hamesh $\frac{\partial^2 u}{\partial x \partial y}$ k saath hi ayega.
Baaqi D or E bhi fixed hai.

$f(x,y)=0 \longrightarrow$ Homogenous
$f(x,y)\neq 0 \longrightarrow$ Non-homogenous

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ homo

$\frac{\partial^2 u}{\partial x^2} + (x+y)^2 = \frac{\partial u}{\partial y} \rightarrow$ non-homo

$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \rightarrow$ homo as there is no x or y function

CLASSIFICATION OF 2nd ORDER PDE

1. $B^2 - 4AC = 0 \rightarrow$ parabolic eq.
2. $B^2 - 4AC < 0 \rightarrow$ elliptic eqn
3. $B^2 - 4AC > 0 \rightarrow$ Hyperbolic eq.

$\rightarrow 3\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} = 6$

$A=3, \; B=-5, \; C=2$

$B^2 - 4AC = (-5)^2 - 4(3)(2)$
$= 1 > 0$

Hyperbolic

$\rightarrow a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

$A = a^2, \; B = 0, \; C = -1$

$(0)^2 - 4(a^2)(-1)$

$4a^2 \rightarrow a=0 \rightarrow$ parabolic

$\left.\begin{matrix} a>0 \\ a<0 \end{matrix}\right\} \rightarrow$ Hyperbolic

(1) $U_t = c^2 U_{xx}$ → Heat Eqn

$$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$$

$A = c^2$, $B = 0$, $C = 0$

$B^2 - 4AC = 0$

$0^2 - 4(c^2)(0) = 0$

$0 = 0$ → parabolic

(2) $U_{tt} = c^2 U_{xx}$ → Wave eqn

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

$A = c^2$, $B = 0$, $C = 1$

$B^2 - 4AC$

$0 - 4(c^2)$

$-4c^2$ always >0 so hyperbolic

(3) Laplace Eqn

$U_{xx} + U_{yy} = 0$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$A = 1$, $B = 0$, $C = 1$

$B^2 - 4AC$

$-4 < 0$ so elliptical

FORMATION OF PDE by Eliminating Arbitary Constant:

(1) $z = ax + by + a^2 + b^2$ → (1)

> if no. of independ variab = no. of arbitary const than we will take derivative w.r.t x than w.r.t y.

Taking w.r.t x

$$\frac{dz}{dx} = a$$

Taking w.r.t y

$$\frac{dz}{\partial y} = b$$

put in eq (1)

$$z = \left(\frac{\partial z}{\partial x}\right)x + \left(\frac{\partial z}{\partial y}\right)y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2$$

(2) $z = ax^3 + by^3$ → (1)

2 independ & 2 constants (a & b)

$$\frac{\partial z}{\partial x} = 3ax^2 \Rightarrow a = \frac{1}{3x^2}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = 3by^2 \Rightarrow b = \frac{1}{3y^2}\frac{\partial z}{\partial y}$$

put values in eq (1)

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$$z = \frac{1}{3}\left(\frac{\partial z}{\partial x}\right)x + \frac{1}{3}\left(\frac{\partial z}{\partial y}\right)y$$

3) $4z = \left[ax + \frac{y}{a} + b\right]^2 \rightarrow (1)$

W.r.t x

$$4\frac{\partial z}{\partial x} = 2\left(ax + \frac{y}{a} + b\right)a \rightarrow (2)$$

w.r.t y

$$4\frac{\partial z}{\partial y} = 2\left(ax + y/a + b\right)\cdot\frac{1}{a} \rightarrow (3)$$

eq (2)

$$a = 2\left(\frac{\partial z}{\partial x}\right)\cdot\frac{1}{(ax + y/a + b)}$$

put in (3)

$$4\left(\frac{\partial z}{\partial y}\right) = \frac{(ax + y/a + b)}{2\left(\frac{\partial z}{\partial x}\right)}\cdot\frac{1}{\frac{1}{(ax + y/a + b)}}$$

$$4\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) = (ax + y/a + b)^2$$

$$4\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) = 4z$$

4) $z = ax^2 + bxy + cy^2$

If no. of arbitrary constant > no. of independent variable than we can take more than one derivative.

w.r.t x

$$\frac{\partial z}{\partial x} = 2ax + by \rightarrow (2)$$

w.r.t y

$$\frac{\partial z}{\partial y} = bx + 2cy \rightarrow (3)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a$$

$$\frac{\partial^2 z}{\partial y^2} = 2c$$

Now we can take derivative of eq (2) w.r.t y or derivative of eq (3) w.r.t x. (own choice)

eq (2) $\frac{\partial^2 z}{\partial y x} = b$

ab agr abc ki values nhi milti to 3rd lena parega.

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put values in eq (1)

$$z = \frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2}\right)x^2 + \left(\frac{\partial^2 z}{\partial y \partial x}\right)xy + \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)y^2$$

* $z = ax + by \rightarrow (1)$

If no. of arbitary constant < no. of independent variable than there can be more than one ans.

w.r.t x

$$\frac{\partial z}{\partial x} = a \rightarrow \text{Now put in (1)}$$

$$z = \left(\frac{\partial z}{\partial x}\right)x + y$$

w.r.t y

$$\frac{\partial z}{\partial y} = 1$$

as 'a' is eliminated

* $z = Ae^{pt} \cdot \sin px$

If no. of indep = no. of arbitary constant and If two functions are multiplied and both have same arbitary constant them we will go till 2nd derivati

$$\frac{\partial z}{\partial x} = pAe^{pt}\cos px \rightarrow (2)$$

$$\frac{\partial^2 z}{\partial x^2} = -p^2Ae^{pt}\sin px \rightarrow (3)$$

$$\frac{\partial z}{\partial t} = pAe^{pt}\sin px \rightarrow (4)$$

$$\frac{\partial^2 z}{\partial t^2} = p^2Ae^{pt}\sin px \rightarrow (5)$$

Adding eq (3) & (5)

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0}$$

$* \quad z = ae^{-b^2x} \cdot \cos by$

$$\frac{\partial z}{\partial x} = -ab^2 e^{-b^2x} \cos by \rightarrow (1)$$

$$\frac{\partial^2 z}{\partial x^2} = ab^4 e^{-b^2x} \cos by$$

$$\frac{\partial z}{\partial y} = -bae^{-b^2x} \sin by$$

$$\frac{\partial^2 z}{\partial y^2} = -b^2 ae^{-b^2x} \cos by \rightarrow (2)$$

Subtracting (1) & (2)

$$\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2} = 0$$

Solution Of PDE by Seperation of Variable

$U(x,y) \rightarrow U$ depend on x & y

Let,

$U(x,y) = X(x)Y(y)$

$X(x) \rightarrow X$ depends on x
$Y(y) \rightarrow Y$ " " y

$* \quad \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} \longrightarrow (1)$

Let,

$U = XY \longrightarrow (2)$

w.r.t x jab derivative lenge X ka x k saath to simple aata hai X lenge & Y sirf x par depend kr raha hai

Taking derivative w.r.t x

$$\frac{\partial U}{\partial x} = \frac{dX}{dx} \cdot Y \rightarrow (2)$$

Taking derivative w.r.t y

$$\frac{\partial U}{\partial y} = X\frac{dY}{dy} \rightarrow (3)$$

Y ka derivative w.r.t y lenge to ordinary derivative hi lenge

eq (3) & (4) put (1)

$$\frac{dX}{dx} Y = X \frac{dY}{dy}$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy}$$

first order eqn ... $\lambda = 0$ check wrong
2nd " " " λ +ve or λ = -ve
will check wrong

Let,

$$\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy} = \lambda$$

then,

$$\frac{1}{X}\frac{dX}{dx} = \lambda \quad , \quad \frac{1}{Y}\frac{dY}{dy} = \lambda$$

$$\frac{1}{X} dX = \lambda dx \quad , \quad \frac{1}{Y} dY = \lambda dy$$

$$\int \frac{1}{X} dX = \lambda \int dx \quad , \quad \int \frac{1}{Y} dY = \lambda \int dy$$

$$\ln X = \lambda x + c_1 \quad , \quad \ln Y = \lambda y + c_2$$

$$\ln X = \lambda x + \ln c_1 \quad , \quad \ln Y = \lambda y + \ln c_2$$

$$\ln X - \ln c_1 = \lambda x \quad , \quad \ln Y - \ln c_2 = \lambda y$$

$$\ln \frac{X}{c_1} = \lambda x \qquad \ln \frac{Y}{c_2} = \lambda y$$

$$\frac{X}{c_1} = e^{\lambda x} \quad , \quad \frac{Y}{c_2} = e^{\lambda y}$$

$$X = c_1 e^{\lambda x} \quad , \quad Y = c_2 e^{\lambda y}$$

$$U(x,y) = X(x)\,Y(y)$$
$$= c_1 e^{\lambda x} \cdot c_2 e^{\lambda y}$$

$$U(x,y) = c_3 e^{(x-y)\lambda}$$

→ λ is any constant. it can be +ve, -ve or 0

If $\lambda = 0$

$$U(x,y) = c_3$$

(*) $U_x = U_y + U \rightarrow$ (A)

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} + U$$

Let, $U(x,y) = X(x)\,Y(y)$

Taking derivat. w.r.t x

$$\frac{\partial U}{\partial x} = \frac{dX}{dx} \cdot Y \rightarrow \text{(I)}$$

T—— w.r.t y

$$\frac{\partial U}{\partial y} = X\frac{dY}{dy}$$

eq (A) ⇒

$$\frac{dX}{dx} Y = X \cdot \frac{dY}{dy} + XY$$

$$\frac{dX}{dx} Y = X\left(\frac{dY}{dy} + Y\right)$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\left(\frac{dY}{dy} + Y\right)$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy} + 1$$

Let,

$$\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy} + 1 = \lambda$$

$$\frac{1}{X}\frac{dX}{dx} = \lambda \quad , \quad \int \frac{1}{Y} dY = \int (\lambda - 1) dy$$

$$\int \frac{1}{X} dX = \int \lambda dx$$

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$\ln X = \lambda x + \ln c_1$, $\ln Y = (\lambda-1)y + \ln c_2$

$\frac{X}{c_1} = e^{\lambda x}$, $\frac{Y}{c_2} = e^{(\lambda-1)y}$

$X = c_1 e^{\lambda x}$, $Y = c_2 e^{(\lambda-1)y}$

$U(x,y) = c_1 e^{\lambda x} \cdot c_2 e^{(\lambda-y)y}$

$= c_3 e^{\lambda x + (\lambda-1)y}$

For $\lambda = 0$

$U(x,y) = c_3 e^{-y}$

⑥ $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \rightarrow (A)$

Let, $U(x,y) = X(x)\cdot Y(y)$

Taking deriv w.r.t x

$\frac{\partial U}{\partial x} = \frac{dX}{dx}\cdot Y \rightarrow (2)$

T——— w.r.t y

$\frac{\partial U}{\partial y} = X\cdot\frac{dY}{dy}$

eq (A)

$y\cdot\frac{dX}{dx}\cdot Y + x\cdot X\frac{dY}{dy} = 0$

$y\cdot\frac{dX}{dx} Y = -x\cdot X\frac{dY}{dy}$

$\frac{1}{x\cdot X}\frac{dX}{dx} = \frac{-1}{y\cdot Y}\frac{dY}{dy} = \lambda$

$\frac{1}{x\cdot X}\frac{dX}{dx} = \lambda$, $\frac{1}{y\cdot Y}\frac{dY}{dy} = -\lambda$

$\frac{1}{X}dX = \lambda x\,dx$, $\frac{1}{Y}dy = -\lambda y\,dy$

$\ln X = \frac{\lambda x^2}{2} + \ln c_1$, $\ln Y = \frac{-\lambda y^2}{2} + \ln c_2$

$X = c_1 e^{\lambda x^2/2}$, $Y = c_2 e^{-\lambda y^2/2}$

$U(x,y) = c_3 e^{\frac{\lambda}{2}(x^2-y^2)}$

For $\lambda = 0$

$U(x,y) = c_3$

⑦ $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x\partial y} + \frac{\partial^2 U}{\partial y^2} = 0 \rightarrow (A)$

Let,

$U = XY$

$U_{xx} = X''Y$ $\left[X'' = \frac{d^2X}{dx^2}\right]$

$U_{yy} = XY''$ $\left[Y'' = \frac{d^2Y}{dy^2}\right]$

$U_{yx} = X'Y'$ $X' = \frac{dX}{dx}$ $Y' = \frac{dY}{dy}$

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eq(A) $\Rightarrow X''Y + X'Y' - XY' = 0$

Not possible! as variables can't be seperated.

(B) $y\dfrac{\partial^2 U}{\partial x \partial y} + U = 0 \rightarrow (A)$

$U = XY$

$U_{yx} = X'Y' \rightarrow$ put (A)

$yX'Y' + XY = 0$

$yX'Y' = -XY$

$y \cdot \dfrac{dX}{dx} \cdot \dfrac{dY}{dy} = -XY$

$\dfrac{1}{X}\dfrac{dX}{dx} = -\dfrac{Y}{y}\dfrac{1}{\frac{dY}{dy}} = \lambda$

$\dfrac{1}{X}\dfrac{dX}{dx} = \lambda \quad , \quad -\dfrac{Y}{y} \cdot \dfrac{1}{\frac{dY}{dy}} = \lambda$

$, \quad -\dfrac{Y}{y} = \lambda \dfrac{dY}{dy}$

$-\dfrac{1}{y} dy = \dfrac{\lambda}{Y} dY$

$= -\dfrac{1}{\lambda y} dy = \dfrac{1}{Y} dY$

$-\dfrac{1}{\lambda} \ln y + \ln c_2 = \ln Y$

$\ln y^{-1/\lambda} = \ln\left(\dfrac{Y}{c_2}\right)$

$c_2 y^{-1/\lambda} = Y$

$U = c_3 e^{\lambda x} \cdot y^{-1/\lambda}$

$U = \dfrac{c_3 e^{\lambda x}}{y^{1/\lambda}} \quad \lambda \neq 0$

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this is 2 dimensional eqn becz 2 variable x & y.

Q3 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$

this is laplace eqn and laplace is an elliptical eqn.

$U = XY \rightarrow (1)$

$\frac{\partial^2 u}{\partial x^2} = X''Y$ $\quad X'' = \frac{d^2X}{dx^2}$

$\frac{\partial^2 u}{\partial y^2} = XY''$ $\quad Y'' = \frac{d^2Y}{dy^2}$

$X''Y + XY'' = 0$

$X''Y = -XY''$

$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

$\frac{X''}{X} = \lambda$, $\frac{-Y''}{Y} = \lambda$

$X'' = \lambda X$, $Y'' = -\lambda Y$

$X'' - \lambda X = 0$, $Y'' - \lambda Y = 0$

$\frac{d^2X}{dx^2} - \lambda X = 0$, $\frac{d^2Y}{dy^2} + \lambda Y = 0$

ab ye 2nd order eqn ban gayi hai to auxialary waale treeke se solve krenge.

$D^2X - \lambda X = 0$, $D^2Y + \lambda Y = 0$

$(D^2 - \lambda)X = 0$ $\quad (D^2 + \lambda)Y = 0$

$m^2 - \lambda = 0$ $\quad m^2 = -\lambda$

$m^2 = \lambda$ $\quad m = \pm\sqrt{-\lambda}$

$m = \pm\sqrt{\lambda}$

For $\lambda = 0$

$m = 0$, $m = 0$

$X = e^{0x}(C_1 + C_2x)$

$X = C_1 + C_2x$ $\quad Y = C_3 + C_4y$

$U = (C_1 + C_2x)(C_3 + C_4y)$

For $\lambda > 0$ let $\lambda = p^2$ qk hum +ve daalein ya -ve ans +ve hi milega.

$m = \pm\sqrt{p^2}$, $m = \pm\sqrt{-p^2}$

$m = \pm p$, $m = \pm pi$

$X = C_1e^{px} + C_2e^{-px}$, $Y = C_3\cos py + C_4\sin py$

$U = (C_1e^{px} + C_2e^{-px})(C_3\cos py + C_4\sin py)$

For $\lambda < 0$: let $\lambda = -p^2$

$m = \pm\sqrt{-p^2}$, $m = \pm\sqrt{p^2}$

$m = \pm pi$ $\quad m = \pm p$

$X = C_1\cos px + C_2\sin px$ $\quad Y = C_3e^{py} + C_4e^{-py}$

$U = (\quad)(\quad)$

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One dimension bcz direction is only changing in 'x'. 't' is the time
no change in 'y'.

(9) $k\frac{\partial^2 U}{\partial x^2} - U = \frac{\partial U}{\partial t}$ $k>0$

$U(x,t) = XT$

$\frac{\partial^2 U}{\partial x^2} = X''T$, $X'' = \frac{d^2X}{dx^2}$

$\frac{\partial U}{\partial T} = XT'$

$kX''T - XT = XT'$

$(kX'' - X)T = XT'$

$\frac{(kX'' - X)}{X} = \frac{T'}{T} = \lambda$

$kX'' - X = \lambda X$, $T' = \lambda T$

$kX'' - X - \lambda X = 0$

$kX'' - (1+\lambda)X = 0$

$km^2 - (1+\lambda) = 0$

$m^2 = \frac{1+\lambda}{k}$

$m = \sqrt{\frac{1+\lambda}{k}}$

$dT/dt = \lambda dt$ agr 2nd order ho or 'm' ki value root

$\int \frac{1}{T} dT = \lambda \int dt$ mein aaye to hum poori root waali eqn ko >0 & <0 let krenge. Jese $m = \sqrt{\frac{1+\lambda}{k}}$ hai $\frac{1+\lambda}{k} > 0$ & $\frac{1+\lambda}{k} < 0$.

$\ln T = \lambda t + \ln c_3$

$\ln\left(\frac{T}{c_3}\right) = \lambda t$

$T = c_3 e^{\lambda t}$

~~ab $\lambda > 0$~~ ab let $\frac{1+\lambda}{k} = p^2$, $\frac{1+\lambda}{k} > 0$
$\hookrightarrow$ (1)

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$T = c_3 e^{\lambda t}$, $\lambda = kp^2 - 1$ from eq (1)

$T = c_3 e^{(kp^2-1)t}$

$U = (\quad)(\quad)$

Let, $\frac{1+\lambda}{k} = -p^2$ $\frac{1+\lambda}{k} < 0$

at $\lambda = 0$

$m = \sqrt{1/k}$, $T = c_3$

$X = c_1 e^{\frac{1}{\sqrt{k}}x} + c_2 e^{-\frac{1}{\sqrt{k}}x}$

$U = \left(c_1 e^{\frac{1}{\sqrt{k}}x} + c_2 e^{-\frac{1}{\sqrt{k}}x}\right) c_3$

$U = Ae^{\frac{1}{\sqrt{k}}x} + Be^{-\frac{1}{\sqrt{k}}x}$ c_3 jab c_1 se (x) hoa to A
c_3 jab c_2 se hoa to B.

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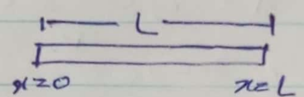
HEAT EQN: → 1 dimensio
depends on x & t
→ parabolic.

i) $K\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ → (A)



Boundary Condition:

$U(0,t)=0$ $t \geq 0$

$U(L,t)=0$ $t \geq 0$

I-C: → represents initial temp

$U(x,0) = f(x)$

$0<x<L$

~~U = XT~~ → (A)

ii) $U(x,t) = XT$

$\frac{\partial^2 U}{\partial x} = X''T$

$\frac{\partial U}{\partial t} = XT'$

put in (A)

$K X''T = XT'$

$\frac{X''}{X} = \frac{T'}{KT} = \lambda$

$X'' = X\lambda$, $T' = \lambda KT$

$m^2 - \lambda = 0$ $\frac{dT}{dt} = \lambda KT$

$m = \pm\sqrt{\lambda}$ $\frac{dT}{T} = K\lambda dt$

$\ln T = \lambda Kt + \ln c$

$\ln\left(\frac{T}{c_3}\right) = \lambda KT$

$T = C_3 e^{\lambda KT}$

For $\lambda=0$

$m=0$

$X = e(c_1 + c_2 x)$

$X = c_1 + c_2 x$

$T = C_3$

$U = (c_1 + c_2 x)(c_3)$

For $\lambda = p^2 > 0$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$T = c_3 e^{p^2 kt}$

$U = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{p^2 kt}$

$U = (Ae^{px} + Be^{-px}) e^{p^2 kt}$

For $\lambda = -p^2 < 0$

$m = \pm pi$

$X = c_1 \cos px + c_2 \sin px$

$T = c_3 e^{-p^2 kt}$

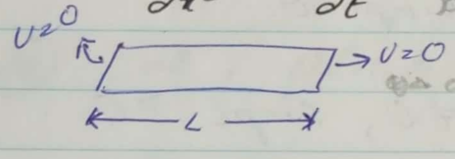
$U = (A\cos px + B\sin px) e^{-p^2 kt}$

We only consider the last case / solution where $\lambda < 0$ for HEAT EQN,
bcz in 1st one has no 't' in it,
2nd one give ∞ ans when $t \to \infty$

3rd $$\boxed{U(x,t) = (A\cos px + B\sin px)e^{-p^2kt}}$$

↳ 3rd eqn isliye select kia ⓐ ke jese jese time barhta hai wese wese heat kam hoti hai to e^{-p^2kt} mein jab $t \to \infty$ hoga to value 0 hojayegi.
jab ke 1st wali mein 't' hi nhi hai or 2nd mein at $t \to \infty$ 'U' also increases.

Q. $$k\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \to Ⓐ$$



B.C (Bondary condition);
$U(0,t) = 0$ phele $x=0$ par
$U(l,t) = 0$ phir $x=l$ par
dono par U zero h.

IC (Initial Condition):
$U(x,0) = f(x)$
↳ initial temperature is function of x.

⇒ For $U(x,0) = f(x)$ put Ⓓ.
$$U(x,0) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \left(e^{-\frac{n^2\pi^2}{L^2}k(0)}\right) = 1$$
$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \to \text{Half Range Sine series}$$
bcz $U(x,0) = f(x)$, $n=0$
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x$$
$$\boxed{b_n = \frac{2}{L}\int_0^b f(x)\cdot \sin\left(\frac{n\pi}{l}\right)x \, dx}$$

(SOLUTION OF HEAT EQN)

When $\lambda < 0$, $\lambda = -p^2$

eqn $U(x,t) = (A\cos px + B\sin px)e^{-p^2kt} \to Ⓐ$

$U(0,t) = 0$, $U(l,t) = 0$

$U(0,t) = 0$,
$0 = (A\cos(0) + B\sin(0))e^{-p^2kt}$
$0 = Ae^{-p^2kt}$
$A = 0$ but $e^{-p^2kt} \neq 0$
isko zero nhi kr sakte warna time = 0 hojayega. To agr $0=0$ krna h to $A=0$ kr sakte h.

put in Ⓐ

eq Ⓐ ⇒
$U(x,t) = (B\sin px)e^{-p^2kt} \to Ⓒ$

$U(l,t) = 0$
$0 = (B\sin plx)e^{-p^2kt}$
$B \neq 0$, $\sin pl = 0$, $e^{-p^2kt} \neq 0$

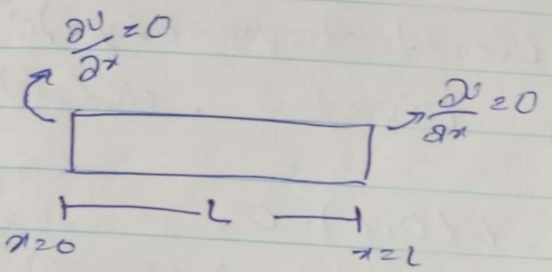
$\sin pl = 0$
$pl = \sin^{-1}(0)$
$pl = n\pi$, $n = 0, 1, 2, 3 \cdots$
$$p = \frac{n\pi}{l}$$

$$U(x,t) = B\sin\left(\frac{n\pi}{L}\right)x\, e^{-\frac{n^2\pi^2}{l^2}kt}$$
$$\boxed{U(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x\, e^{-\frac{n^2\pi^2}{L^2}kt}} \to Ⓓ \quad \because B = b_n$$

Now calculating value of b_n,

Q3: Find temp $U(x,t)$ in a rod of length l. If the initial temp is $f(x)$ throughout and if the ends $x=0$ and $x=l$ are insulated

Sol:



I.C:

$U(x,0) = f(x)$ → represent initial temp.

B.C:

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \quad \text{bcz insulated}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=l} = 0$$

$$K\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

Sol of Heat eqn $\lambda < 0, \lambda = -p^2$

$$U(x,t) = (A\cos px + B\sin px)\cdot e^{-p^2kt} \rightarrow (A)$$

$$\frac{\partial U}{\partial x} = (-Ap\sin px + Bp\cos px)\cdot e^{-p^2kt} \rightarrow (B)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \rightarrow 1^{st} \text{ putting } 1^{st} \text{ condition}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = Bp\cdot e^{-p^2kt}$$

$$Bp e^{-p^2kt} = 0$$

$B=0$ put → (B)

$$\Rightarrow \frac{\partial U}{\partial x} = -Ap\sin px \cdot e^{-p^2kt}$$

Now using 2nd condition

$$\left.\frac{\partial U}{\partial x}\right|_{x=l} = 0$$

$$0 = -Ap\sin pl\cdot e^{-p^2kt}$$

$A \neq 0$, $p \neq 0$, $\sin pl = 0$

$pl = \sin^{-1}(0)$

$pl = n\pi$, $n = 0, 1, 2, 3 \ldots$

$p = \frac{n\pi}{l}$ → put (1)

put $B=0$, $p = \frac{n\pi}{l}$, put (A)

$$U(x,t) = A\cos\left(\frac{n\pi}{l}\right)x\, e^{-\frac{n^2\pi^2 kt}{l^2}}$$

$$U(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x\, e^{-\frac{n^2\pi^2 kt}{l}}$$

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B.C: $U(0,t) = 0$
$U(L,t) = 0$
I.C: $U(x,0) = f(x)$, $\left.\frac{\partial U}{\partial t}\right|_{t=0} = g(x)$

Ex: 12-4

(WAVE EQUATION)

$U(x,0) = f(x)$

$U(x,0) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$

$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x \rightarrow$ Half range

$a_0 = \frac{2}{L}\int_0^L f(x)\cdot dx$

$a_n = 2/L \int_0^L f(x)\cdot \cos\left(\frac{n\pi}{L}\right)x\, dx$

$a^2 U_{xx} = U_{tt} \rightarrow$ (A)

$U = XT$

$U_{xx} = X''T$, $U_{tt} = XT''$

$a^2 X''T = XT''$

$\frac{X''}{X} = \frac{T''}{a^2 T} = \lambda$

(4) $L = 2$, $U(x,0) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

$f(x) = \begin{cases} x , & 0 < x < 1 \\ 0 , & 1 < x < 1 \end{cases}$

$X'' = \lambda X$, $T'' = a^2\lambda T$

$m^2 - \lambda = 0$, $T'' - a^2\lambda T = 0$

$m = \pm\sqrt{\lambda}$, $m^2 = \pm a\sqrt{\lambda}$

For $\lambda = 0$

$m = 0$, $m = 0$

$X = C_1 + C_2 x$, $T = C_1 + C_3 t$

$U(x,t) = (C_1 + C_2 x)(C_3 + C_4 t)$

For $\lambda = p^2 > 0$

$m = \pm p$, $m = \pm ap$

$X = C_1 e^{px} + C_2 e^{-px}$, $T = C_3 e^{apt} + C_4 e^{-apt}$

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For $\lambda \xi = -p^2 < 0$

$m = \pm p i$, $m = \pm a p i$

$X = c_1 \cos px + c_2 \sin px$, $T = c_3 \cos apt + c_4 \sin apt$

$U(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt)$ ⟶ (D)

We will use only (D) eqn as it contains sin cos function (bcz of wave).

(D) ⟹ $U(x,t) = (c_1 \cos px + c_2 \sin px) \cdot (c_3 \cos apt + c_4 \sin apt)$ ⟶ (i)

Use Boundary Condition

$U(0,t) = 0$ put (i)

$U(0,t) = (c_1 \cos(0) + c_2 \sin(0))(c_3 \cos apt + c_4 \sin apt)$

$0 = c_1 (c_3 \cos apt + c_4 \sin apt)$, $U(0,t) = 0$

$c_1 = 0$, $c_3 \cos pt + c_4 \sin apt \neq 0$

put $c_1 = 0$ in → (i)

eq (i)

$U(x,t) = c_2 \sin px (c_3 \cos apt + c_4 \sin apt)$

$U(x,t) = \sin px (A \cos apt + B \sin apt)$

$\because c_2 c_3 = A$, $c_2 c_4 = B$ ⟶ (ii)

Use, $U(L,t) = 0$ put in (ii)

eq (ii)

$U(L,t) = \sin pL \cdot (A \cos apt + B \sin apt)$

$0 = \sin pL (A \cos$ —— $)$

$\sin pL = 0$, $A \cos apt + B \sin apt \neq 0$

$pL = \sin^{-1}(0)$

$pL = n\pi$

$p = \frac{n\pi}{L}$ → put in (ii)

eq (ii)

$U(x,t) = \sin\left(\frac{n\pi}{L}\right)x \left[A \cos\left(\frac{n\pi}{L}\right)at + B \sin\left(\frac{n\pi}{L}\right)at\right]$

use Σ,

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{L}\right)at + B_n \sin\left(\frac{n\pi}{L}\right)at\right] \sin \frac{n\pi}{L} x}$$ ⟶ (iii) solution of wave eqn

$U(x,0) = f(x)$ ⟶ put in (iii)

$\therefore U(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$

$f(x) = \sum_{n=1}^{\infty} \left[A_n \cos(0) + B_n \sin(0)\right] \sin\left(\frac{n\pi}{L}\right)x$

$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$, $A_n = b_n$ ⟶ half range sine serie

⟶ (ii) $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$ here $A_n = b_n$

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Where,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x} \rightarrow (i)$$

Now, $\left.\frac{\partial U}{\partial t}(x,0)\right|_{t=0} = g(x)$

Now taking derivative w.r.t t of (iii)

$$\frac{\partial U}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\right)x\left[-A\frac{an\pi}{L}\sin\left(\frac{n\pi}{L}\right)at + B\frac{an\pi}{L}\cos\left(\frac{n\pi}{L}\right)at\right]$$

at $t=0$

$$g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\right)x \cdot B\frac{an\pi}{L}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{Ban\pi}{L}\sin\left(\frac{n\pi}{L}\right)x \rightarrow \text{Half range sine series}$$

$\hookrightarrow$ constant

$$B_n\frac{an\pi}{L} = \frac{2}{L}\int_0^L g(x)\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$\boxed{B_n = \frac{2}{an\pi}\int_0^L g(x)\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx}$$

CamScanner

$\lambda > 0$

(17) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} - 2k\dfrac{\partial u}{\partial t}$

$\dfrac{\partial^2 u}{\partial x^2} = X''T$

$\dfrac{\partial u}{\partial t} = XT'$

$\dfrac{\partial^2 u}{\partial t^2} = XT''$

$a^2 X''T = XT'' + 2kXT'$

$a^2 X''T = X(T'' + 2kT')$

$\dfrac{a^2 X''}{X} = \dfrac{1}{T}(T'' + 2kT') = \lambda$

$a^2 X'' = X\lambda \qquad , T'' + 2kT' = \lambda T$

$a^2 m^2 = \lambda$

$m = \pm \sqrt{\lambda}/a$

$\boxed{\text{at } \lambda = 0}$

$m = 0 \qquad , T'' + 2kT' = (0)T$

$X = c_1 + c_2 x \qquad T'' + 2kT' = 0$

$m^2 + 2km = 0$

$m(m + 2k) = 0$

$m = 0, m = -2k$

$T = c_3 + c_4 e^{-2kt}$

$u = (c_1 + c_2 x)(c_3 + c_4 e^{-2kt})$

at $\lambda > 0$ let $\lambda = p^2$

$m = \pm p/a$

$X = c_1 e^{p/a\, x} + c_2 e^{-p/a\, x}$

$T'' + 2kT' = p^2 T$

$T'' + 2kT' - p^2 T = 0$

$m^2 + 2km - p^2$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \dfrac{-2k \pm \sqrt{4k^2 + 4p^2}}{2}$

$= \dfrac{2[-k \pm \sqrt{k^2 + p^2}]}{2}$

$m = -k \pm \sqrt{k^2 + p^2}$

$T = c_3 e^{(-k + \sqrt{k^2+p^2})} + c_4 e^{(-k - \sqrt{k^2+p^2})}$

$u = (\qquad)(\qquad)$

at $\lambda < 0 \qquad \lambda = -p^2$

$m = \pm \dfrac{p}{a} i$

$X = c_1 \cos\dfrac{p}{a}x + c_2 \sin\dfrac{p}{a}x$

$T'' + 2kT' + p^2 T = 0$

$m^2 + 2km + p^2 = 0$

$m =$

(13) $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$

$X''Y + XY'' = 0$

$X''Y = -XY''$

$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$

$X'' = X\lambda$, $Y'' = -Y\lambda$

$m^2 = \lambda$, $m^2 = -\lambda$

$m = \pm\sqrt{\lambda}$, $m = \pm\sqrt{-\lambda}$

at $\lambda = 0$

$m = 0$, $m = 0$

$X = c_1 + c_2 x$, $Y = c_3 + c_4 y$

$U = (\quad)(\quad)$

at $\lambda > 0 = p^2$

$m = \pm p$, $m = \pm p i$

$X = c_1 e^{px} + c_2 e^{-px}$, $Y = c_3 \cos py + c_4 \sin py$

$U = (\quad)(\quad)$

at $\lambda = -p^2 < 0$

(14) $x^2 \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$

$x^2 X''Y + XY'' = 0$

$x^2 X''Y = -XY''$

$\frac{x^2 X''}{X} = \frac{-Y''}{Y} = \lambda$

$x^2 X'' = X\lambda$, $Y'' = -\lambda Y$

$m^2 = -\lambda$

$m = \pm\sqrt{-\lambda}$

$\boxed{\text{at } \lambda = 0}$

$m = 0$

$x^2 X'' = 0$, $Y = c_3 + c_4 y$

$x^2 m^2 = 0$

$m = 0$

$X = c_1 + c_2 x$

$U = (c_1 + c_2)(c_3 + c_4 y)$

at $\lambda = p^2 > 0$

$x^2 X'' = p^2 X$, $m = \pm$

$x^2 X'' - p^2 X = 0$, $m = c_3 \cos$

Cauchy Euler

$m(m-1) - p^2 = 0$

$m^2 - m - p^2 = 0$

$m = \frac{1 \pm \sqrt{1 + 4p^2}}{2}$

$\sqrt{-3}$

$\sqrt{?}$

$X = c_1 x^{\left(\frac{1+\sqrt{1-4p^2}}{2}\right)} + c_2 x^{\left(\frac{1-\sqrt{1-4p^2}}{2}\right)}$

$U = (\quad)(\quad)$

at $\lambda = -p^2 < 0$

$x^2 X'' = -p^2 X$

$x^2 X'' + p^2 X = 0$

$m(m-1) + p^2 = 0$

$m^2 - m + p^2 = 0$

$, m = \pm p$

$Y = c_3 e^{py} + c_4 e^{-py}$

$m = \dfrac{1 \pm \sqrt{1-4p^2}}{2}$

If $1 - 4p^2 = 0$

$m = \dfrac{1}{2}$.

$X = 6x^{1/2}(c_1 + c_2 \ln x)$

$U = (\quad)(\quad)$

If $1 - 4p^2 > 0$

$m = \dfrac{1 \pm \sqrt{1-4p^2}}{2}$

$X = c_1 x^{\left(\frac{1+\sqrt{\quad}}{2}\right)} + c_2 x^{\left(\frac{1-\sqrt{\quad}}{2}\right)}$

$U = (\quad)(\quad)$

If $1 - 4p^2 < 0$

$m = \dfrac{1}{2} \pm \dfrac{\sqrt{-(1-4p^2)}}{2}$

$= \dfrac{1}{2} \pm \dfrac{\sqrt{4p^2-1}}{2}\, i$

$X = x^{1/2}\left(c_1 \cos\left(\dfrac{\sqrt{4p^2-1}}{2}\right)\ln x + c_2 \sin \dfrac{\sqrt{4p^2-1}\ln x}{2}\right)$

$U = (x)(y)$

Q1: $K\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$

$U = XT$

$\frac{\partial^2 U}{\partial x^2} = X''T$

$\frac{\partial U}{\partial t} = XT'$

$KX''T = XT'$

$\frac{X''}{X} = \frac{T'}{KT}$

$\frac{X''}{X} = \lambda \quad , \quad \frac{T'}{KT} = \lambda$

$X'' = \lambda X \qquad T' = \lambda KT$

$m^2 = \lambda \qquad \frac{dT}{dt} = \lambda KT$

$m = \pm\sqrt{\lambda}$

$\int \frac{1}{T} dT = \lambda K \int dt$

$\ln T = \lambda Kt - \ln c$

$T = c_2 e^{\lambda Kt}$

at $\lambda = 0$

$m = 0$

$X = c_1 + c_2 x$

$U = (c_1 + c_2 x) c_3 e^{\lambda Kt}$

at $\lambda = p^2 > 0$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$U = (\qquad) c_3 e^{p^2Kt}$

at $\lambda = -p^2 < 0$

$m = \pm pi$

$U = (A\cos px + B\sin)e^{-p^2Kt}$ → (A) → Ans

→ $U(0,t) = 0$ → using in A

$U(L,t) = 0$

$U(x,0) = \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$

$U(0,t) = Ae^{-p^2Kt}$

$0 = Ae^{-p^2Kt}$

$A = 0 \qquad e^{-p^2Kt} \neq 0$

put in (A)

$U(x,t) = (B\sin px)e^{-p^2Kt}$ → B

Now $U(l,t) = 0$

$0 = (B\sin pl)e^{-p^2Kt}$

$B \neq 0$, — $\sin pl = 0$

$xL-x^2 \xrightarrow{+} -\cos n\pi/n$
$L-2x \Rightarrow -\sin$
$-2 \xrightarrow{+} \cos$

$pl = \sin^{-1}(0)$
$= n\pi$

$p = \frac{n\pi}{l}$
put in B

$U(x,t) = \left[B\sin\left(\frac{n\pi}{L}\right)x\right]e^{-p^2 k t}$

$U(x,t) = \sum_{n=1}^{\infty}\left(B_n \sin\left(\frac{n\pi}{L}\right)x\right)e^{-\frac{n^2\pi^2}{l^2}kt}$ ↳ C

at $\{U(x,0) =$

$U(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}\right)x$

$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}\right)x$

$b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx$

$\because f(x) = \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$

$b_n = \frac{2}{l}\int_0^{L/2} 1.\sin\left(\frac{n\pi}{l}\right)x\,dx$

$b_n = \frac{2}{L}\left[\frac{L}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)\right]_0^{L/2}$

$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right].$

(2) Boundary Condition Same so

$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)e$ ↳ (A)

for b_n:

$b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx$

$\Rightarrow U(x,0) = \begin{cases} x(L-x) \\ 0<x<L \end{cases}$

$= \frac{2}{L}\int_0^L (xL-x^2)\cdot\sin\frac{n\pi}{L}x\,dx$

$= \frac{2}{L}\left[\cancel{-\frac{(xL-x^2)\cos\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)}}\right.$ $\quad 0$

$+\cancel{\frac{(L-2x)\sin}{\frac{n^2\pi^2}{L^2}}} - \left.\frac{2\cos}{\frac{n^3\pi^3}{L^3}}\right]_0^L$

$= \frac{2}{L}\left[-\frac{2\cos n\pi}{\frac{n^3\pi^3}{L^3}} + \frac{2}{\frac{n^3\pi^3}{L^3}}\right]$

$= \frac{4}{L\,\frac{n^3\pi^3}{L^3}}\left[-(-1)^n + 1\right]$

$= \frac{4L^2}{n^3\pi^3}\left[1-(-1)^n\right]$

Q4:

$U(x,t) = (A\cos px + B\sin p)e^{-p^2kt}$

$U'(x,t) = (-Ap\sin px + Bp\cos px)e^{-p^2kt}$ → (B)

$U'(0,t) = Bp\cos e^{-p^2kt}$ → (B)

$0 = Bpe^{-p^2kt}$

$B = 0 \quad p \neq e^{-p^2kt} \neq 0$

~~put in (A)~~ put (A)

~~$U(x,t) = (A\cos px)e^{-p^2kt}$~~

$U'(x,t) = (-Ap\sin px)e^{-p^2kt}$

$U(x,L) = -Ap\sin pL\, e^{-}$

$A \neq 0, \; p \neq 0, \; 0^{-}$

$\sin pL = 0$

$pL = n\pi$

$p = \frac{n\pi}{L}$

put

$p = \frac{n\pi}{L}$, $B = 0$ in eq (A)

$U(x,t) = \sum_{n}^{\infty}\left[A_n\cos\frac{n\pi}{2}x\right]e^{-\frac{n^2\pi^2}{4}kt}$

$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{2}\, e^{-\frac{n^2\pi^2}{4}kt}$

For A_n:

$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}dx$

$L = 2$

~~$a_n = \int_0^2 f(x)\cdot\cos\frac{n\pi}{2}x\,dx$~~

a_n $f(x) = \begin{cases} x & 0<x<1 \\ 0 & 1<x<2 \end{cases}$

For a_0:

$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$= \int_0^2 f(x)\,dx$

$= \int_0^1 x\,dx$

$= x^2/2 \Big|_0^1$

$\boxed{a_0 = 1/2}$

For a_n:-

$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$

$= \int_0^2 f(x)\cos\frac{n\pi}{2}x\,dx$

$x \searrow \cos n\pi/2$

$1 \searrow \sin n\pi/2$

$0 \searrow -\cos n\pi/2$

$$a_n = \int_0^1 x \cos \frac{n\pi}{2} x \, dx$$

$$= \left[\frac{x \sin \frac{n\pi}{2}x}{\frac{n\pi}{2}} + \frac{\cos \frac{n\pi}{2}x}{\frac{n^2\pi^2}{4}} \right]_0^1$$

$$= \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \right]$$

$$= \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \left[\cos \frac{n\pi}{2} - 1 \right]$$

Q5: $K \frac{\partial^2 U}{\partial x^2} - hU = \frac{\partial U}{\partial t}$

$U = XT$

$KX''T - hXT = XT'$

$\cancel{T(KX'' - hX) = XT'}$

$\cancel{KX'' - hX = \frac{T'}{T}}$

$\cancel{KX'' - hX = \lambda X}$

$\cancel{KX'' = \lambda X - hX}$

$KX''T = XT' + hXT$

$KX''T = X(T' + hT)$

$$\frac{X''}{\lambda} = \frac{(T' + hT)}{KT}$$

$X'' = X\lambda$, $T' + hT = KT\lambda$

$m^2 = \lambda$, $T' = T(K\lambda - h)$

$m = \pm\lambda$, $\frac{dT}{dt} = T(-)$

$\frac{1}{T} dT = (-) dt$

$\ln T = (-)t + \ln$

$T = c\, e^{(-)t}$

$\lambda = 0$

$m = 0$

$X = c_1 + c_2 x$, $T = c_3 e^{-ht}$

$U = (\quad)(\quad)$

$\lambda = -p^2 < 0$

$X = \pm pi$, $T = c_3 e^{(-kp^2 - h)t}$

$U = (A\cos px + B\sin px) e^{(-kp^2 - h)t}$ → (A)

$\cancel{U(t, 0, x) = 0}$

$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0$

$\left.\frac{\partial U}{\partial x}\right|_{x=L} = 0$ → (B)

$U' = (-Ap\sin px + Bp\cos px)$

applying 1st

$U(0,0) = 0$

$U(L,0) = 0$

$0 = (Bp\cos px)e^{(-kp^2-h)t}$

$B = 0 \rightarrow$ put in (B)

$0 = (Ap\sin pL)e^{-kp^2-ht}$

$A \neq 0,\ p \neq 0,\ \sin pL = 0$

$p = \frac{n\pi}{L}$

put $p = \frac{n\pi}{L}$, $B = 0$ in eq (A)

$U(x,t) = \left[A_n\cos\frac{n\pi}{L}x\right]e^{\left(-k\frac{n^2\pi^2}{L^2}-h\right)t}$

$= \frac{a_0}{2}e^{-ht} + e^{-ht}\sum_{n=1}\left(A_n\cos\frac{n\pi}{P}x\right)e^{\left(-k\frac{n^2\pi^2}{L^2}\right)}$

For a_0:

$a_0 = \frac{2}{L}\int_0^L f(x)\,\cancel{\cos\frac{n\pi}{P}x}\,dx$

$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{P}x\,dx$

06:

$U = (A\cos px + B\sin px)e^{(-kp^2-h)t}$ ↳ (A)

$U(0,t) = 0$

$U(L,t) = 0$

using 1st cond

$U(0,t) = A\,\cancel{\cos p0}\,e^{-kp^2-ht}$

$A = 0 \rightarrow$ put in (A)

$U(x,t) = (B\sin px)e^{(-)t}$

$U(L,t) = 0$ ↳ (B)

$0 = B\sin pL\,e^{(-)}$

$B \neq 0,\ e^{(-)} \neq 0$

$\sin pL = 0$

$p = \frac{n\pi}{L}$

put in (B)

$U(x,t) = B_n\sin\frac{n\pi}{L}x\,e^{(-)}$

$U(x,t) = \sum B_n\sin\frac{n\pi}{L}x\,e^{(-)}$

$\frac{\partial u}{\partial x}$

$B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi}{L}x\, dx$

Q7: $K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

→ $u(-L,t) = u(L,t)$

→ $\left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}$

→ $u(x,0) = f(x)$

$u(x,t) = (A\cos px + B\sin px)e^{-kp^2t}$

$(A\cos pL - B\sin pLx)e^{(-)} = (A\cos pL + B\sin pL)e^{-(-)}$

$0 = 2B\sin pL\, e^{-p^2kt}$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u(x,t) = (A\cos px + B\sin px)e^{-}$

$\cancel{u_x(0,t) = Ae^{-p^2t}}$

$\cancel{u_x(0,t) = u_x}$

$\cancel{(-Ap\sin px + Bp\cos px)e^{-p^2t} = Ae^{-p^2t}}$

$u_x = (-Ap\sin p + Bp\cos px)e^{-}$

$u_x(0,t) = Bp\, e^{-p^2t}$

$(-Ap\sin px + Bp\cos px)e^{-p^2t} = Bp\, e^{-p^2t}$

(12.4)

Q) $a^2 U_{xx} = U_{tt}$

$a^2 X'' T = X T''$

$\frac{X''}{X} = \frac{T''}{a^2 T}$

$X'' = \lambda X$, $T'' = a^2 T \lambda$

$m^2 = \pm\sqrt{\lambda}$ $\quad m = \pm a\sqrt{\lambda}$

For $\lambda = 0$

$m = 0$

$X = c_1 + c_2 x$ $\quad T = c_3 + c_4 x$

For $\lambda > 0$

$m = \pm p$

For $\lambda = -p^2 < 0$

$m = \pm pi$ $\quad m = \pm api$

$U = (c_1 \cos px + c_2 \sin px)$
$(c_3 \cos apt + c_3 \sin apt)$

$\hookrightarrow$ ~~eqn~~ (A)

$U(0,t) = 0$, $U(L,t) =$

$U(x,0) = \frac{1}{4} x(L-x)$,

$\left.\frac{\partial v}{\partial t}\right|_{t=0} = 0$

putting $U(0,t) = 0$ in (A)

$U(0,t) = c_1 (c_3 \cos apt + \ldots$

$0 = c_1 (c_3 \cos apt + \sin\ldots$

$c_1 = 0$ $\hookrightarrow$ put in (A)

$U(x,t) = c_2 \sin px (c_3 \cos apt$
$+ c_3 \sin apt)$

$\hookrightarrow$ (B)

Now $U(L,t) = 0$

$0 = c_2 \sin pL (c_3 c\ldots + \ldots$

$\sin pL = 0$

$p = \frac{n\pi}{L}$

$$U(x,t) = \sin\frac{n\pi}{L}x\left(A\cos\left(\frac{n\pi}{L}\right)at + B\sin\left(\frac{n\pi}{L}\right)at\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\right)x\left[A_n\cos\left(\frac{n\pi}{L}\right)at + B_n\sin\left(\frac{n\pi}{L}\right)at\right]$$

$\hookrightarrow$ (C)

Now $U(x,0) = f(x)$

$$U(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx}$$

$\hookrightarrow$ (D)

Now,

$$\left.\frac{\partial U}{\partial t}(x,0)\right|_{t=0} = g(x)$$

n.o.(C)

$$\frac{\partial U}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n\frac{an\pi}{L}\sin\left(\frac{n\pi}{L}\right)at + B_n\frac{an\pi}{L}\cos\left(\frac{n\pi}{L}\right)at\right]\sin\left(\frac{n\pi}{L}\right)x$$

$$g(x) = \sum_{n=1}^{\infty} B_n\frac{an\pi}{L}\sin\left(\frac{n\pi}{L}\right)x$$

$$g(x) = 0$$

$$B_n = 0$$

$$B_n\frac{an\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$\because f(x) = \frac{1}{4}x(L-x)$

put in (D)

$$A_n = \frac{1}{2L}\int_0^L (xL - x^2)\sin\frac{n\pi}{L}x\,dx$$

$xL - x^2$ $\sin\left(\frac{n\pi}{L}\right)$

$L - 2x$ $+$ $-\cos\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)$

-2 $-$ $-\sin\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)^2$

0 $+$ $\cos\frac{n\pi}{L}/\left(\frac{n\pi}{L}\right)^3$

$$A_n = \frac{1}{2L}\left[-\frac{(xL - x^2)\cos\frac{n\pi}{L}}{\left(\frac{n\pi}{L}\right)} + \frac{(L-2x)\sin}{\left(\frac{n\pi}{L}\right)^2} - \frac{2\cos\left(\frac{n\pi}{L}\right)}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L$$

$$A_n = \frac{2L}{(n\pi)^3}\left[1 - (-1)^n\right]$$

Q2) $U(0,t)=0$

$U(L,t)=0$

$U(x,0)=0$

$\frac{\partial u}{\partial t}\Big|_{t=0} = x(L-x)$

$U(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt)$ ↳ (A)

Using $U(0,t)=0$

$0 = c_1 (c_3 \cos apt + c_4 \sin apt)$

$c_1 = 0$, $c_3 \cos \text{—} \neq 0$

↳ put in (A)

$U(x,t) = c_2 \sin px (c_3 \cos apt + c_3 \sin apt)$ ↳ B

Using $U(L,t)=0$

$0 = c_2 \sin pL (c_3 \text{———})$

$c_2 \neq 0$, $\sin pL = 0$

$p = \frac{n\pi}{L}$

put in (B)

$U(x,t) = c_2 \sin \frac{n\pi}{L} (\text{———})$

$U(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi}{L} at + B_n \sin \text{—}) \sin \frac{n\pi}{L} x$ ↳ C

Now using $U(x,0) = f(x)$

$f(x) = \sum_{n=1}^{\infty} (A_n \sin \frac{n\pi}{L} x)$

$\therefore A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \, dx$

$\because f(x) = 0$

$\boxed{A_n = 0}$

Now $\partial u / \partial t \big|_{t=0} = g(x)$

as

$\partial u / \partial t = \sum (A_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} at + B_n \frac{n\pi a}{L} \cos \frac{n\pi}{L} at) \sin \frac{n\pi}{L}$

using ce—

$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x$

$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$

$$B_n = \frac{2}{n\pi a}\int_0^l x(L - x^2)\sin\frac{n\pi}{l}x\,dx$$

ⓧ

$$B_n = \frac{4L^3}{n^4\pi^4}\left[1 - (-1)^n\right]$$

$$U(x,t) = \sum_{n=1}^{\infty}\left[\frac{4L^3}{n^4\pi^4}\left[1-(-1)^n\right]\sin\frac{n\pi}{l}at\right]\sin\frac{n\pi}{l}x$$

(3) $U(0,t)=0$, $U(\pi,t)=0$

$U(x,0)=0$, $\left.\frac{\partial U}{\partial t}\right|_{t=0} = \sin x$

By using 1^{st} condⁿ :-

$$U(x,t) = c_2\sin px\,(c_3\cos apt + c_3\sin apt)$$

↳ (B)

Now $U(\pi,t)=0$

$$0 = c_2\sin p\pi\,(c_3\cos apt + \ldots)$$

$c_2 \neq 0$

$\sin p\pi = 0$

$p\pi = n\pi$

$p = n$

eq (B)

$$U(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos ant + B_n\sin ant\right]\sin nx$$

↳ C

Now using $U(x,0)=0$

$f(x) = \sum A_n \sin nx$

$\rightarrow f(x)=0$

$\boxed{A_n = 0}$

$$\frac{\partial U}{\partial t} = \left[-A_n an\sin ant + B\,an\cos ant\right]\sin nx$$

using condition,

$$g(x) = \sum B_n an\sin nx$$

$$B_n an = \frac{2}{L}\int_0^L g(x)\sin nx\,dx$$

$$B_n = \frac{2}{anL}\int_0^L \sin x\sin nx\,dx$$

$$B_n = \frac{\cancel{2}}{anL}\cdot\frac{1}{\cancel{2}}\int_0^L \cos(1-n)x - \cos(1+n)x\,dx$$

$\therefore L = \pi$

$$B_n = \frac{1}{an\pi}\left[\frac{1}{1-n}\sin(1-n)x - \frac{1}{1+n}\sin(1+n)x\right]_0^{\pi}$$

$B_n = 0$ for $n = 1, 2, 3 \ldots$

For B_0;

$$B_n = \frac{2}{a\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{2}{an\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{2}{a\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{2}{a\pi}\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= \frac{2}{a\pi}\left[\frac{\pi}{2} - 0\right]$$

$$B_1 = \frac{1}{a}$$

$$U(x,t) = B_1 \sin at \sin x = \frac{1}{a}\sin at \sin x$$

(4) $U(0,t) = 0$

$U(\pi,t) = 0$

$U(x,0) = \frac{1}{6}x(\pi^2 - x^2)$

$\left.\frac{\partial U}{\partial t}\right|_{t=0} = 0$

By using 1st 2 conditions

$U(x,t) = \Sigma(A_n \cos ant + B_n \sin ant)\sin nx$

$\hookrightarrow C$

Now Using,

$U(x,0) = f(x)$

$f(x) = \Sigma A_n \sin nx$

$$A_n = \frac{2}{L}\int_0^{L} f(x) \cdot \sin nx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{6}(x\pi^2 - x^3)\sin nx$$

$x\pi^2 - x^3 \quad + \quad \sin nx$

$\pi^2 - 3x^2 \quad - \quad -\cos nx/n$

$-6x \quad + \quad -\sin nx/n$

$-6 \quad - \quad \cos nx/n$

$0 \quad \sin nx/n$

$$= \frac{1}{3\pi}\left[-\frac{(x\pi^2 - x^3)}{n}\cos nx + \frac{(\pi^2 - 3x^2)}{n^2}\sin nx \right.$$

$$\left. \frac{(-6x)}{n^3}\cos nx - \frac{6\sin nx}{n^4} \right]_0^{\pi}$$

$$= \frac{1}{3\pi}\left[\frac{-6\pi(-1)^n}{n^3} \right]$$

$$A_n = \frac{2}{n^3}\left[(-1)^{n+1}\right]$$

$g(x) = 0$

$B_n = 0$

$$U(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n^3}\left[(-1)^{n+1}\right] \cos ant \right) \sin nx$$

(5) $U(0,t) = 0$, $U(1,t) = 0$

$U(x,0) = x(1-x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = x(1-x)$$

Using 1st condn

$U(x,t) = C_2 \sin px\,(C_3 \cos apt + C_4 \sin apt)$

Using $U(1,t) = 0$

$U(1,t) = C_2 \sin p(\text{——})$

$\sin p = 0$

$p = n\pi$

$U(x,t) = \sum (A_n \cos an\pi t + B_n \sin a n\pi t)\sin n\pi x$

$\hookrightarrow C$

Using $U(x,0) = x(1-x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin n\pi x \, dx$$

$$= 2\int_0^1 (x - x^2)\sin n\pi x \, dx$$

$x - x^2$	+	$\sin n\pi x$
$1 - 2x$	−	$-\cos n\pi x / n\pi$
-2	+	$-\sin n\pi x / n^2\pi^2$
0		$\cos n\pi x / n^3\pi^3$

$$= 2\left[\frac{-(x-x^2)\cos n\pi x}{n\pi} + \frac{(1-2x)\sin n\pi x}{n^2\pi^2} - \frac{2\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2\left[\frac{-2(-1)^n}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right]$$

$$A_n = \frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]$$

1

$$\frac{\partial U}{\partial t} = \sum\left[-A_n a n\pi \sin a n\pi t + B_n a n\pi \cos a n\pi t\right]\sin n\pi x$$

Using ...

$$g(x) = \sum B_n a n\pi \sin n\pi x$$

$$B_n a n\pi = \frac{2}{L}\int_0^L g(x)\sin n\pi x \, dx$$

$$B_n = \frac{2}{a n\pi}\int_0^1 x(1-x)\sin n\pi x \, dx$$

$$= \frac{1}{a n\pi}\left[\frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]\right]$$

$$= \frac{4}{a n^4\pi^4}\left[1 + (-1)^{n+1}\right]$$

$$U(x,t) = \sum \frac{4}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]\cos a n\pi t + \frac{4}{a n^4\pi^4}\left[1 + (-1)^{n+1}\right]\sin a n\pi t \; \sin n\pi x$$

(6) $U(0,t)=0$

$U(1,t)=0$

$U(x,0)=0.01\sin 3\pi x$

$\left.\frac{\partial U}{\partial t}\right|_{t=0}=0$

Using 1st two condn

$U(x,t)=\sum(A_n\cos a n\pi t+B_n\sin a n\pi t)\sin n\pi x$ ↳ C

Now $U(x,0)=f(x)$

$f(x)=\sum A_n \sin n\pi x$

$A_n=\frac{2}{L}\int_0^L f(x)\sin n\pi x\,dx$

$A_n=\frac{2}{\pi}\int_0^{\pi} 0.01\sin 3\pi x\sin n\pi x$

$=\frac{0.02}{\pi}\int_0^{\pi}\sin 3\pi x\sin n\pi x$

$\int_0^{a}\sin nx\cdot\sin mx=0 \;\forall\, m\neq 0$

So, $\int_0^1 \sin 3\pi x\sin n\pi x\,dx=0$

for $n=1,2,4,$ —

For $n=3$, N—

$A_3=0.02\int_0^1\sin^2 3\pi x\,dx$

$=0.02\int_0^1\left(\frac{1-\cos 6\pi x}{2}\right)dx$

$=0.02\left[\frac{x}{2}-\frac{\sin 6\pi x}{12}\right]_0^1 dx$

$= A_3=0.01$

$\because g(x)=0$

$B_n=0$

$U(x,t)=\sum A_n\cos a n\pi t\sin n\pi x$

For $n=3$

$=A_3\cos 3a\pi t\sin 3\pi x$

$=0.01C$ —— ✓

Ex: 11.2

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}\right)x + b_n \sin\left(\frac{n\pi}{p}\right)x$$

in $(-p, p)$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}\right)x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}\right)x\,dx$$

(3) $f(x) = \begin{cases} 1 & -1 \le x < 0 \\ x & 0 \le x < 1 \end{cases}$ (−1 → min, 1 → max)

$(-1, 1)$ (-1 ↓ min value, 1 → max value) $\quad p = 1$

or $p = (\text{max} - \text{min})/2$

$\boxed{\text{period} = \text{max} - \text{min}}$

$$f(x) = \frac{a_0}{2} + \sum(a_n \cos n\pi x + b_n \sin n\pi x) \rightarrow (1)$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

piece wise hai is waja se do mastab ese likha hai

$$a_0 = \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx$$

$$a_0 = x\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$$

$$\boxed{a_0 = 3/2}$$

For a_n:

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx$$

$$a_n = \int_{-1}^{0} 1\cdot\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x$$

↳ by parts

$$a_n = \frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0} + \left[x\cdot\frac{\sin n\pi x}{n\pi} + (1)\cdot\frac{\cos n\pi x}{n^2\pi^2}\right]_{0}^{1}$$

$$a_n = \left(\frac{\sin 0}{n\pi} - \frac{\sin n\pi(-1)}{n\pi}\right) + \left[\left(\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 + \frac{\cos 0}{n^2\pi^2}\right)\right]$$

$\therefore \cos n\pi = (-1)^n$

$$a_n = \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}$$

$$\boxed{a_n = \frac{(-1)^n - 1}{n^2\pi^2}}$$

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$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\ \sin n\pi x\,dx$$

$$b_n = \int_{-1}^{0} \sin n\pi x + \int_0^1 x \sin n\pi x\, dx$$

↳ by parts

$$b_n = \frac{-\cos n\pi}{n\pi} + \left[\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$b_n = -\frac{\cos 0}{n\pi} + \frac{\cos n\pi}{n\pi} + \left[-\frac{\cos n\pi}{n\pi} - 0\right]$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi}$$

$$b_n = \frac{-1}{n\pi}$$

put in (1)

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)x + \frac{1}{n\pi}\sin(n\pi)x\right)$$

→ If function is continous at all points then fourier series is convergent else it is divergent.

$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = 0$$

$\therefore \lim_{x\to 0^-} \neq \lim_{x\to 0^+}$ ∴ not continuous

If the function is not continous then we will find out the convergence.

agr function or uska derivative continous hai jese 1 ka derivative '0' which is conti. & x ka derivative 1 or ye bhi continous. ~~To~~ convergence nikaalne ka formula.

$$f(x) = \frac{\overset{\nearrow \text{Right limit}}{f(x^+)} + \overset{\nearrow \text{left limit}}{f(x^-)}}{2}$$

$$f(0) = \frac{f(0^+) + f(0^-)}{2}$$

$$= 0+1/2 =$$

$$\boxed{f(0) = \frac{1}{2}}$$

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$f(x) = x+\pi \quad , \quad -\pi < x < \pi$

$(-p,p) = (-\pi,\pi) \Rightarrow p = \pi$

$\because f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}\right)x + b_n \sin\left(\frac{n\pi}{p}\right)x$

putting value of $p=\pi$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow (1)$

$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\,dx$

$= \frac{1}{\pi}\left(\frac{x^2}{2} + \pi x\right)_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \pi^2\right) - \left(\frac{\pi^2}{2} - \pi^2\right)\right]$

$\boxed{a_0 = 2\pi}$

$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\cos n\pi\,dx$

↳ using by parts

$a_n = \frac{1}{\pi}\left[\cancel{\frac{(x+\pi)\sin nx}{n}} + \frac{(1)\cos nx}{n^2}\right]_{-\pi}^{\pi}$

as $\sin\pi$ & $\sin(-\pi) = 0$

$a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(-\pi)}{n^2}\right]$ → even function hai is waja se ye negative nhi hoga

$a_n = \frac{1}{\pi}\left[\cancel{\frac{(-1)^n}{n^2}} - \cancel{\frac{(-1)^n}{n^2}}\right]$

$\boxed{a_n = 0}$

$\because b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}\right)x\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)\sin nx\,dx$

$= \frac{1}{\pi}\left[\frac{-(x+\pi)\cos nx}{n} + \frac{(1)\sin nx}{n^2}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{-(x+\pi)\cos nx}{n} + \cancel{\frac{(1)\sin nx}{n^2}}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{-(\pi+\pi)\cos n\pi}{n} + \frac{\cancel{(-\pi+\pi)}\cos n(-\pi)}{n}\right]$

$b_n = \frac{1}{\pi}\left[\frac{-2\pi(-1)^n}{n}\right]$

$b_n = \frac{-2(-1)^n}{n}$

$b_n = \frac{2\cdot(-1)(-1)^n}{n}$

$\boxed{b_n = \frac{2(-1)^{n+1}}{n}}$

putting values in eq (1)

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as function is continous at $-\pi$ to π therefore we donot need to find convergence

(21) Use the result of Q#7 to show that,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

proof,

ye 7 ka ans hai

$$f(x) = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

agr fourier series mein cos aaye to 0 or π agr sin hai to $\frac{\pi}{2}$ x ki jaga put krenge.

$\because f(x) = x + \pi$, $x = \pi/2$ bcz sin is in fourier

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \pi$$

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2}$$

Now puting values of n, ie 1, 2, 3, 4...

$$\frac{3\pi}{2} = \pi + 2\Big[\underbrace{\sin\frac{\pi}{2}}_{\text{when } n=1} \underbrace{- \frac{1}{2}\cancel{\sin\pi}}_{n=2} + \underbrace{\frac{1}{3}\sin\frac{3\pi}{2}}_{n=3} \underbrace{- \frac{1}{4}\sin 2\pi}_{n=4} + \cdots\Big]$$

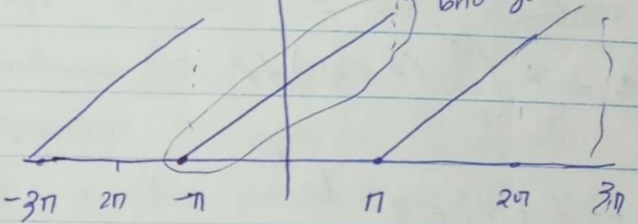
agr sin cos dono ho or agr series odd numbers ki ho to $\frac{\pi}{2}$ or agr continous 1, 2, 3, 4, 5, 6... esi counting ho to 0 ya π.

$$\frac{3\pi - \pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right]$$

$$\frac{\pi}{2} = 2[\;\_\_\_\_\;]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

ye wali question ko dekh kr bnaayi hai.



$(-\pi, \pi)$

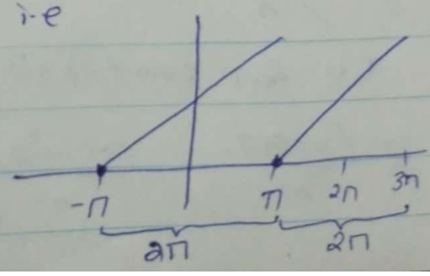
~~$(-2\pi, 2\pi)$~~

humein graph bnaana hai -2π se 2π tak $(-2\pi, 2\pi)$ so period

$=$ max $-$ min $= \cancel{(-\pi - \pi)}$ $\rightarrow$ $\cancel{2\pi}$

humein swaal mein jo interval diya hai usmein max $= \pi$ and min is $-\pi$ so

$\pi - (-\pi) = 2\pi$ phele hum apne swaal k according graph bnayenge i.e



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5 $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 \le x \le \pi \end{cases}$

$p = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx$$

→ phele waali term 0 hojayegi is wajah se nhi likhi

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\cancel{\frac{x^2 \sin x}{n}} + \frac{2x\cos nx}{n^2} + \cancel{\frac{2\sin x}{n^3}}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right]$$

$$a_n = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[\frac{-x^2\cos n}{n} + \cancel{\frac{2x\sin nx}{n^2}} - \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\left(\frac{-\pi^2\cos n\pi}{n} - \frac{2\cos n\pi}{n^3}\right) - \left(\frac{0 - 2\cos n(0)}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2(-1)^n}{n} - \frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi^2(-1)^{n+1}}{n} + \frac{2}{n^3}\left(1 - (-1)^n\right)\right]$$

$$b_n = \frac{\pi(-1)^{n+1}}{n} + \frac{2}{n^3\pi}\left(1 - (-1)^n\right) \rightarrow \text{put (1)}$$

eq (1) ⇒

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\left(1 - (-1)^n\right)\right)\sin nx\right]$$

(π) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$f(\pi) = ?$

ye q k undefined hai to hum average nikalenge.

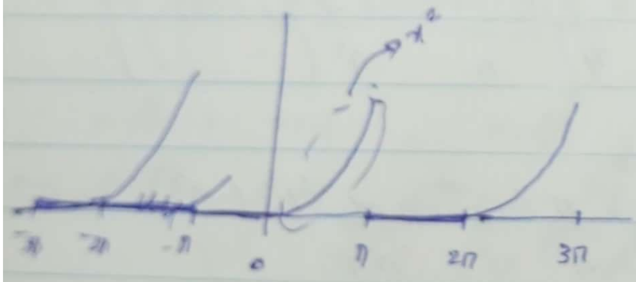
ye graph breaks dekhenge

$f(x) = \frac{f(x^+) + f(x^-)}{2}$

$f(\pi) = \frac{0 + \pi^2}{2} = \frac{\pi^2}{2}$

$f(0) = (0)^2 = 0$

$f(\pi/2) = x^2 = \frac{\pi^2}{4}$



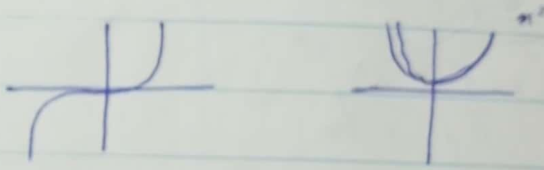
Ex: 11-3

Even / Odd Function

If $f(x)$ is a function x replaced by $-x$ then if

$f(-x) = f(x) \rightarrow$ even

$f(-x) = -f(x) \rightarrow$ odd



odd　　　　even

$E \pm E = E$

$O \pm O = O$

$E \pm O = O \pm E = N.E\ N.O$

$E \times E = E$

$O \times O = E$

$E \times O = O \times E = \text{Odd}$

$\frac{E}{E} = \frac{O}{O} = E$

$\frac{E}{O} = \frac{O}{E} = \text{Odd}$

$f(x) = x^2 + \sin x$

$E + O = \text{Neither}$

$f(x) = x^2 \sin x$

$E \times O = \text{Odd}$

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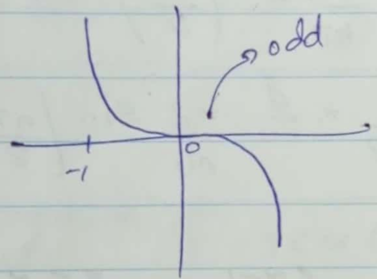
by graph

$$f(x) = \begin{cases} x^2 & , -1 < x < 0 \\ -x^2 & 0 \le x < 1 \end{cases}$$

$$f(-x) = \begin{cases} (-x)^2 & , -1 < -x < 0 \\ -(-x)^2 & , 0 \le -x < 2 \end{cases}$$

$$f(-x) = \begin{cases} x^2 & 1 > x > 0 \\ -x^2 & 0 \ge x > -1 \end{cases}$$

$f(-x) = \{ x^2$



Cosine Series $(p, -p)$

If $f(x)$ is even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}\right)x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

ismein b_n odd banjayega v k $f(x)$ even or sin odd to $E \times O = O$ & odd $= 0$ when limit $= p, -p$

$$a_n = \frac{2}{p}\int_0^p \underbrace{f(x)}_{\text{even}} \times \underbrace{\cos\left(\frac{n\pi}{p}\right)x}_{\text{even}} = \text{even}$$

is wajase bahir 2 se $\times$ kodia

$$\int_{-1}^{1} x^2 = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\therefore 2\int_0^1 x^2 = 2\left[\frac{x^3}{3}\right]_0^1 = 2/3$$

jab even hota hai to bahir 2 likh dete hai

odd par jab limit $(p, -p)$ ho to ans $= 0$ hota hai

$$\int_{-1}^{1} x = \left.\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Sine Series:

If $f(x)$ is odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}\right)x$$

$a_0 = 0$, $a_n = 0$ bcz odd $\times$ even $= 0$

& odd function ans $= 0$ if limit $= p, -p$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}\right) * dx$$

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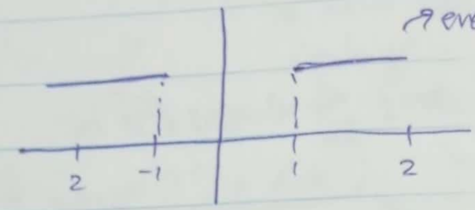
(12) $f(x) = \begin{cases} 1 , & -2 < x < -1 \\ 0 , & -1 \leq x < 1 \\ 1 , & 1 < x < 2 \end{cases}$ $\nearrow p = 2$



Cosine Series

$p = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right)x \rightarrow (1)$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$= \cancel{\tfrac{2}{2}}\int_0^2 f(x)\,dx$$

$=$

0 se 2 par f(x) given nhi hai to humne ~~???~~ integeration ~~mein~~ ko break kardenge. ~~So~~ humein 0 se 2 par f(x) given hai to humne f(x) ko $\int_0^1$ or $\int_1^2$ mein break kardia, as it will bcm 'c'

$$a_0 = \cancel{\int_0^1 0\,dx}^{\,0} + \int_1^2 1\,dx$$

defined on $-1 < x < 1$; $1 < x < 2$

$$= [x]_1^2$$

$$\boxed{a_0 = 1}$$

$$a_n = \cancel{\frac{2}{2}}\int_0^2 f(x)\cos\left(\frac{n\pi}{2}\right)x$$

$$= \cancel{\int_0^1 0 \cdot \cos\left(\frac{n\pi}{2}\right)x\,dx}^{\,0} + \int_1^2 1 \cdot \cos\left(\frac{n\pi}{2}\right)x\,dx$$

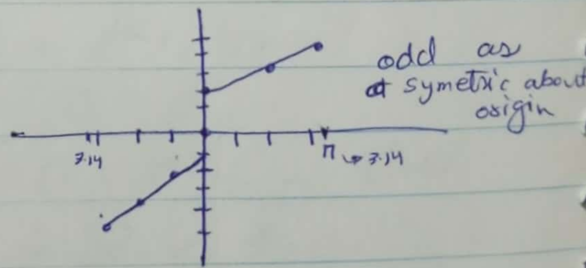
$$a_n = \left[\frac{\sin\left(\frac{n\pi}{2}\right)x}{\frac{n\pi}{2}}\right]_1^2$$

$$a_n = \frac{2}{n\pi}\left[\cancel{\sin n\pi}^{\,0} - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$a_n = \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

(19) $f(x) = \begin{cases} x-1 & -\pi < x < 0 \\ x+1 & 0 \leq x < \pi \end{cases}$



odd as symetric about origin

To ab ye swaal sine series se krenge as f(x) is odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}\right)x$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

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$b_n = \frac{2}{\pi}\int_0^{\pi} (x+1)\sin nx \, dx$

$\therefore b_n = \frac{2}{2}\int_0^{p} f(x)\cdot \sin\left(\frac{n\pi}{p}\right)x \, dx$

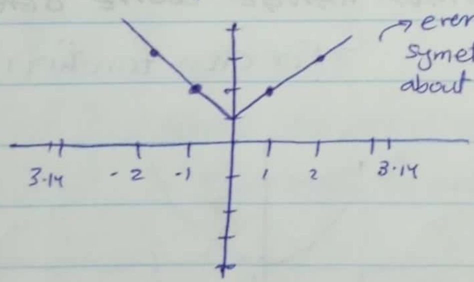
$= \frac{2}{\pi}\left[-\frac{(x+1)\cos nx}{n} + \cancel{\frac{(1)\sin nx}{n^2}}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-\frac{(x+1)\cos n\pi}{n} + \frac{(0+1)\cos 0}{n^2}\right]$

$b_n = \frac{2}{\pi}\left[-\frac{(\pi+1)(-1)^n}{n} + \frac{1}{n}\right] \rightarrow ①$

$f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

graph is necessary



even as symetric about y-axis

$x = 0, \ \frac{\pi^2}{8} = ?$

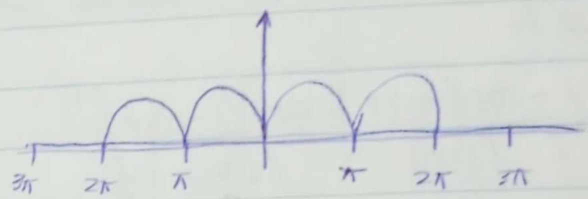
↳ as $f(x)$ is not defined at $x = 0$

so we take average.

Q: Find the cosine & sine series of the function

(23) $f(x) = |\sin x|, \quad -\pi < x < \pi$

$f(x) = \begin{cases} -\sin x & -\pi < x < 0 \\ \sin x & 0 \leq x < \pi \end{cases}$



For Cosine:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}\right)x$

$p = \pi$

$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$

$a_0 = 2/p \int_0^{p} f(x)\cdot dx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx$

$= 2/\pi\left[-\cos x\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-\cos\pi + \cos(0)\right]$

$\boxed{a_0 = 4/\pi}$

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$$a_n = \frac{2}{P}\int_0^P f(x)\cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cdot \cos(nx)dx$$

$$\because \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$a_n = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^\pi \sin(1+n)x + \sin(1-n)x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\right]_0^\pi$$

For $n = 2,3,4 \ldots$ — $(n \neq 1)$

$\cos(1+n)\pi \to n=2 \to -1$
$n=3 \to 1$
$n=4 \to -1$
$n=5 \to 1$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{(1-n)} + \frac{\cos(0)}{(1+n)} + \frac{\cos(0)}{(1-n)}\right]$$

yaha phelese mojood hai — iswaja se yaha nhi lagaya.

$\therefore \cos(1+n)\pi = -(-1)^n$ take if

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{(1+n)} + \frac{(-1)^n}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

n = even then phir bahir wala

$$= \frac{1}{\pi}\left[\frac{(-1)^n(1-n) + (-1)^n(1+n) + 1-n+1+n}{(1+n)(1-n)}\right]$$

minus bg jayeg

$$= \frac{1}{\pi}\left(\frac{2(-1)^n+2}{(1-n^2)}\right)$$

or if n = odd phir andar or bahir dono cancel hojayenge

agr yaha 1 put kra to ∞ hojayega so we cannot put n=1. Uske liye alag se a_n ki value nikalni hogi.

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n+1}{(1-n^2)}\right]$$

For a_1:

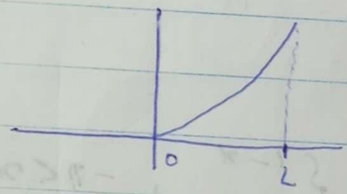
$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cdot \cos 1x\, dx \quad (n=1)$$

$$a_1 = 2/\pi \int_0^\pi \sin x \cdot \cos x\, dx$$

$$= \frac{2}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^\pi$$

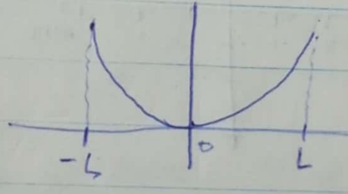
$$\boxed{a_1 = 0}$$

★ Half Range : $(0, L)$



Half Range Cosine Serie

↳ For even function



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$$

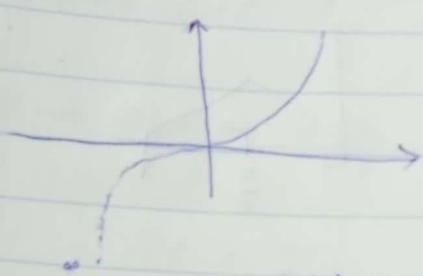
$$a_0 = \frac{2}{L}\int_0^L f(x)\cdot dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\, dx$$

28 — 34

* Half Range Sine Series

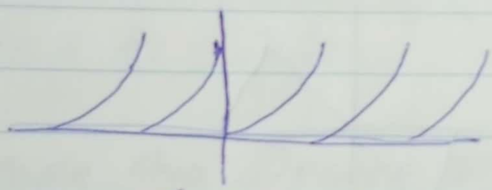
↳ for odd functions



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\left(\frac{n\pi}{L}\right)dx$$

Half Range Fourier Series



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + b_n \sin\left(\frac{n\pi}{L}\right)x$$

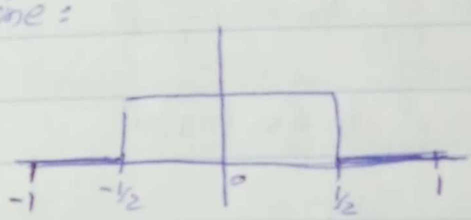
$a_0 =$ 〃 Same as previous
$a_n =$ 〃 half range formulas.
$b_n =$ 〃

Find the Half range sine & cosine

$$f(x) = \begin{cases} 1 & , \ 0 < x < 1/2 \\ 0 & , \ 1/2 \le x < 1 \end{cases}$$

$L = 1$

For cosine:



$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$$

$$= a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{1}\right)x \rightarrow (1)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\cdot dx \Rightarrow a_0 = 2\int_0^1 f(x)\,dx$$

$$a_0 = 2\int_0^{1/2} 1\,dx + 2\int_{1/2}^1 0\cdot dx$$

$$a_0 = 2\left[x\right]_0^{1/2} \Rightarrow a_0 = 1$$

$$a_n = 2/L \int_0^1 f(x)\cdot \cos(n\pi x)\,dx$$

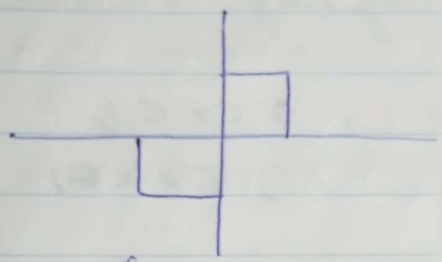
$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx$$

$$a_n = 2\int_0^{1/2} \cos n\pi x\,dx + 2\int_{1/2}^1 \cancel{0\cdot \cos n\pi x}$$

$$a_n = \frac{2}{n\pi}\cdot\left[\sin n\pi x\right]_0^{1/2}$$

$$\boxed{a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}}$$

FOR SINE SERIES :



$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$

$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi)x$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx$

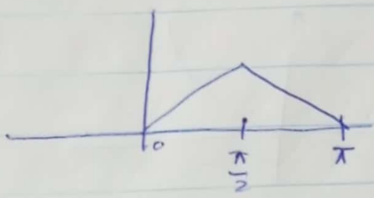
$b_n = 2\int_0^{1/2} \sin n\pi x\,dx + 2\int_{1/2}^1 \cancel{0\cdot\sin n\pi x\,dx}$ 0

$= 2\left[-\frac{\cos(n\pi)x}{n\pi}\right]_0^{1/2}$

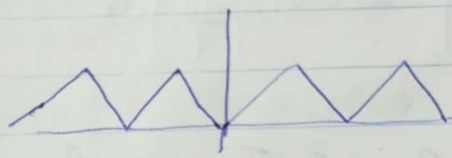
$= 2\left[\frac{-\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{\cos(0)}{n\pi}\right]$

$= \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + 1\right]$

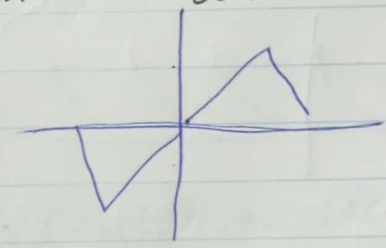
Q: $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$



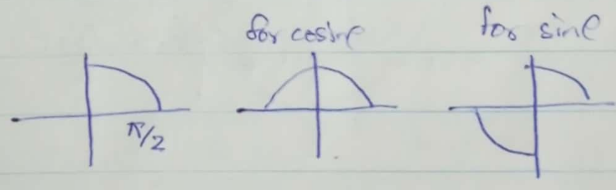
FOR COSINE SERIES:



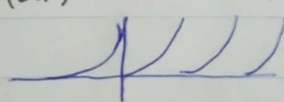
FOR SINE SERIES:



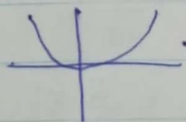
$f(x) = \cos x \qquad 0 \le x < \pi/2$



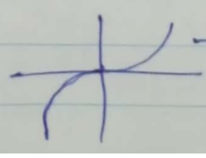
$f(x) = x^2$ → neither even nor odd



→ fourier series



→ cosine serie



→ sine series

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~~original~~ ordinary derivative mein ek dependent variable ek hi independent par depend krega.

$y \rightarrow$ dep x

$\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$

$y^{(4)} \rightarrow$ iska matlab 4th derivative.

$\ddot{y} \rightarrow$ iska matlab bhi 2nd derivative

Partial derivative mein one depending variable depends on more than one variable.

$U \rightarrow$ dep x and y

$\frac{\partial u}{\partial x} = U_x$, $\frac{\partial^2 u}{\partial x^2} = U_{xx}$

$\frac{\partial u}{\partial y} = U_y$

$\frac{\partial^2 u}{\partial x \partial y} = U_{yx}$

iska matlab U ka double derivative liya hai phele w.r.t y phir w.r.t x.

$U = xy + x^2y^2 + x + y$

iska jab hum derivative w.r.t x leng gaani $\frac{\partial U}{\partial x}$ lenge to y ko as coefficent ya constant treat krenge.

$\frac{\partial u}{\partial x} = y + 2xy^2 + 1$

$\frac{\partial}{\partial y} \cdot \left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(y + 2xy^2 + 1\right)$

$\frac{\partial^2 U}{\partial y \partial x} = 1 + 4xy$

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ordinary eq.

hum yaha kisi bhi ek ko dependent consider krenge

ORDER AND DEGREE of ODE:

(1) $$\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^3 = \sin x$$

order = 2 , degree = 1

- highest derivative is called order
- The highest power of highest derivative is called degree.

(2) $$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial^3 u}{\partial x \partial y^2}\right)^{2} = 3$$

degree

order = 3 degree = 2

(3) degree & order is not in fraction

(3) $$3\frac{dy}{dx} + \frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^5 = \sin(x+y)$$

order = 3 degree = 1

(4) $(y''+1) = \sqrt{(y'')^2 + y'}$

without simplification hum order nhi dekhenge.

Sq. on b/s

$(y''+1)^2 = (y'')^2 + y'$

$\cancel{(y'')^2} + 2y'' + 1 = \cancel{(y'')^2} + y'$

$2y'' + 1 - y' =$

Order = 2 degree = 1

(5) $$\frac{dy}{dx} = x + \frac{dx}{dy}$$

$$\frac{dy}{dx} = x + \frac{1}{\frac{dy}{dx}}$$

$$y' = x + \frac{1}{y'}$$

$$y' = \frac{xy'+1}{y'}$$

$$(y')^2 = xy' + 1$$

order = 1 degree 2

✓ dependent variable ya uske derivative ki power 1 ho to linear agr 1 se greater hai to non-linear.

- $y^2 \to y \cdot y$
- $\left(\frac{dy}{dx}\right)^2 \to \left(\frac{dy}{dx}\right)\cdot\left(\frac{dy}{dx}\right)$
- $y \cdot \frac{dy}{dx}$, $\frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$ agr is tarha product form mein ho tab bhi non-linear

$\sin y, \cos y, \tan^{-1} y, e^y, \ln y$ (function k saath bhi y ho to bhi non-linear)

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General solution → • involves arbitary constant c $y = 2x + c$ ← arbitary constant.

Particular Solution → that contains no arbitary constant

SOLUTION TO DE

dependend variable bhi agr product ki form mein aajye to non-linear.

$y^2 \to y \cdot y$, $y \cdot \frac{dy}{dx} =$

$\frac{d^2y}{dx} \cdot \frac{dy}{dx} \to$ non linear.

Ex # 1-1

① $(1-x)y'' + xy' + y = 0$

linear, order = 2, degree = 1

angle mein dependent variable

② $\frac{d^2u}{dr^2} + r \cdot \frac{du}{dr} = \cos(u+r)$

non-linear, order = 2

degree = 1

③ $xy'' + y'y = x \sin x$

non-linear

bcz y is multiplied by $\frac{dy}{dx}$

if you write dependent variable in terms of independent variable is called explicit funct.

Ex : 1-1

explicit function — if dep = f(ind), $y = f(x)$

(*) $2y' + y = 0$, $y = e^{-x/2}$

doesn't contain an arbitary constant hence it is a particular solu

first finding y';

$y' = \frac{d}{dx} e^{-x/2}$

$y' = \frac{-1}{2} e^{-x/2}$ → putting in eqn.

$2\left(-\frac{1}{2} e^{-x/2}\right) + e^{-x/2} = 0$

$-e^{-x/2} + e^{-x/2} = 0$

$0 = 0$

hence it is a solution!

implicit func.

(*) $\frac{dx}{dt} = (x-1)(2x-1)$, $\ln\left(\frac{2x-1}{x-1}\right) = t$

phele L.H.S phir R.H.S Solve karenge L.H.S k liye dx/dt nikalna hoga or R.H.S k liye 'x' ki value.

as answer contain 'x' dependent.

$\ln\left(\frac{2x-1}{x-1}\right) = t$

$\frac{2x-1}{x-1} = e^t$

$2x - 1 = xe^t - e^t$

$x = \frac{xe^t - e^t}{2 - e^t}$

taking der on b/s

→

Ex : 2.3

Ist Order Linear D.E

→ where x & y cannot be seperated

step #01 ↓

① $\dfrac{dy}{dx} + p(x)y = Q(x)$ → linear D.E

↳ depends on x — should only be x term

② $\dfrac{dy}{dt} + p(t)y = Q(t)$

↳ depends on t

Step #02 ↓

I.F = Integerating Factor

$I.F = e^{\int p(x).dx}$ ⇒ it was derived

step #03 ↓

Solution →

$I.F \times y = \int (I.F \times Q(x)) \cdot dx$

Integeratingfactor × dependent variable $= \int (I.F \times Q(x))dx$

Q) $x\dfrac{dy}{dx} + y = x^2$

divide iski kaaya take ye hamare formule se match ho jaye.

x & y terms cannot be seperated

÷ by x on b/s :-

$\dfrac{dy}{dx} + \dfrac{y}{x} = x$ → ①

$\dfrac{dy}{dx} + p(x)y = Q(x)$

1. $p(x) = 1/x$ $Q(x) = x$

2. Integerating factor $= I.F = e^{\int p(x)\,dx}$

$= e^{\int 1/x \cdot dx} = e^{\ln(x)} = x$

$I.F = x$

3. $I.F \cdot y = \int (I.F \times Q(x))\, dx$

$x \cdot y = \int x \times x \, dx$

$= \int x^2 \, dx$

$xy = \dfrac{x^3}{3} + C$

$y = \dfrac{x^3}{3x} + \dfrac{C}{x}$

OR

directly :-

$(y')(x)^{-1} + (x'/y) = x^2$

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Ex: 2.3

$\frac{d}{dx}(y \cdot x) = x^2$

$d(y \cdot x) = \int x^2 dx$

$yx = \int x^2 dx$

$\frac{dy}{dx} + \frac{1}{x} y = x$

$\frac{dy}{dx} x + y = x^2$

so proved that integrating factor when multiplied by equation gives us product rule formula.

Q) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

$\Rightarrow \frac{dy}{dx} + p(x)y = Q(x) \rightarrow (1)$

$\Rightarrow \frac{dr}{d\theta} + p(\theta)r = Q(\theta)$

$\Rightarrow p(\theta) = \sec\theta, \quad Q(\theta) = \cos\theta$

$\Rightarrow I.F = e^{\int p(\theta) d\theta} = e^{\int \sec\theta \, d\theta}$

$= e^{\ln(\sec\theta + \tan\theta)}$

$I.F = \sec\theta + \tan\theta$

$I.F \cdot r = \int I.F \cdot Q(\theta) \, d\theta$

$= \int (\sec\theta + \tan\theta) \cos\theta \, d\theta$

$(\sec\theta + \tan\theta) \cdot r = \int (\sec\theta \cdot \cos\theta + \tan\theta \cdot \cos\theta) \, d\theta$

$= \int \left(\sec\theta \cdot \frac{1}{\sec\theta} + \frac{\sin\theta}{\cos\theta} \cdot \cos\theta \right) d\theta$

$= \int (1 + \sin\theta) \, d\theta$

$(\sec\theta + \tan\theta) r = \theta - \cos\theta + c$

$$\boxed{r = \frac{\theta}{\sec\theta - \tan\theta} - \frac{\cos\theta}{\sec\theta - \tan\theta} + \frac{c}{\sec\theta - \tan\theta}}$$

Q) $x(x+1)\frac{dy}{dx} + xy = 1 \; ; \; y(e) = 1$

$\div$ by $x(x+1)$ on b/s

$\frac{dy}{dx} + \frac{y}{x+1} = \frac{1}{x(x+1)}, \quad x \neq 0 \; \& \; x \neq -1$

$P(x) = \frac{1}{x+1}, \quad Q(x) = \frac{1}{x(x+1)}$

$I.F = e^{\int \frac{1}{x+1} dx}$

$= e^{\ln(x+1)}, \quad x > 0$

$I.F = (x+1)$

$I.F \times y = \int I.F \cdot Q(x) \, dx$

$(x+1) y = \int (x+1) \cdot \frac{1}{x(x+1)} \, dx$

$= \int \frac{1}{x} \, dx$

$(x+1) \cdot y = \ln x + c \rightarrow (A)$

put $x = e, \; y = 1$

$(e+1) \cdot 1 = \ln e + 1 \Rightarrow c = e$

$(x+1) y = \ln x + e \rightarrow$ Ans.

$$\boxed{y = \frac{\ln x}{x+1} + \frac{c}{x+1}}$$

Domain of sol: $(0, \infty)$

Q) $y\,dx - 4(x + y^6)\,dy = 0$

y ka form main linear nhi hosakti q k 'y' k power 6 hai.

$$y - 4(x+y^6)\frac{dy}{dx} = 0$$

$$y - 4x\frac{dy}{dx} - \left(y^6\frac{dy}{dx}\right) \rightarrow \text{non-linear}$$

w.r.t x ye linear hosakti hai.

$$y\,dx = 4(x+y^6)\,dy$$

$$\frac{dx}{dy} = \frac{4(x+y^6)}{y}$$

$$\frac{dx}{dy} = \frac{4x}{y} + \frac{4y^6}{y}$$

$$\frac{dx}{dy} - \frac{4x}{y} = 4y^5 \quad , y \neq 0$$

$$P(y) = -\frac{4}{y} \quad , Q(y) = 4y^5$$

$$I.F = e^{\int -4/y\,dy} = e^{-4\ln y} \quad , y > 0$$

bcz ln(0) ≠ ∞

$$= e^{\ln y^{-4}}$$

→ -4 ko power islye bnaya hai taake e or ln cancel hosake

$$\boxed{I.F = y^{-4}}$$

$$I.F.x = \int I.F.\,Q(y)\,dy$$

$$\frac{1}{y^4}\cdot x = \int \frac{1}{y^4}\cdot 4y^5\,dy$$

$$\frac{x}{y^4} = \frac{4y^2}{2} + C$$

$$\frac{x}{y^4} = 2y^2 + C$$

$$\boxed{x = 2y^6 + Cy^4 \quad (0, \infty)}$$

$$\because x(1) = 2$$

(c) $\frac{dy}{dx} + \overset{P(x)}{2x}y = \overset{Q(x)}{f(x)}$, $y(0)=2$

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

$I.F = e^{\int 2x\,dx} = e^{2x^2/2}$

$\boxed{I.F = e^{x^2}}$

• For $f(x) = x = Q(x)$, $0 \le x < 1$

$e^{x^2} \cdot y = \int e^{x^2} \cdot x\,dx$

$e^{x^2} \cdot y = \frac{1}{2}\int e^{x^2} \cdot (2x)\,dx$

$e^{x^2} \cdot y = \frac{e^{x^2}}{2} + C$

$y = \frac{1}{2} + C_1 e^{-x^2}$, $0 \le x < 1$

For $f(x) = 0 = Q(x)$ $\quad x \ge 1$

$e^{x^2} \cdot y = \int e^{x^2} \cdot (0)\,dx$

$e^{x^2} \cdot y = C_2$ → agr koi integeration nhi hoti to constant bachta hai.

$y = C_2 e^{-x^2}$

$$y = \begin{cases} \frac{1}{2} + C_1 e^{-x^2}, & 0 \le x < 1 \to (1) \\ C_2 e^{-x^2}, & x \ge 1 \to (2) \end{cases}$$

∵ $y(0) = 2$

$x = 0$, $y = 2$

using eq (1)

$y = \frac{1}{2} + C_1 e^{-x^2/2}$

$2 = \frac{1}{2} + C_1 e^0$

$2 - \frac{1}{2} = C_1$

$\boxed{C_1 = \frac{3}{2}}$

$y = \frac{1}{2} + \frac{3}{2} e^{-x^2} \to (3) \quad 0 \le x < 1$

$y = C_2 e^{-x^2} \to (2) \quad x \ge 1$

humara function continuous hai q k sawal mein given hai to L.H limit or right hand limit brabr hogi $x=1$ par.

$\lim_{x \to 1^-} \left(\frac{1}{2} + \frac{3}{2} e^{-x^2} \right) = \lim_{x \to 1^+} \left(C_2 e^{-x^2} \right)$

$\frac{1}{2} + \frac{3}{2} e^{-1} = C_2 e^{-1}$

$\frac{1}{2e^{-1}} + \frac{3}{2} = C_2$

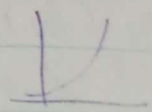
$\boxed{C_2 = \frac{e}{2} + \frac{3}{2}}$

Ex: 3.1

* EXPONENTIAL GROWTH MODEL

change in population is $\propto$ to population

$\rightarrow$ population



$$\frac{dp}{dt} \propto p$$

$\rightarrow$ proportionality constant

$$\frac{dp}{dt} = kP \quad , \quad P(0) = P_0$$

at $t=0$, $P = P_0$

$$\frac{dP}{dt} - kP = 0 \qquad = \frac{dy}{dt} + p(x)y = Q(x)$$

$p(x) = -k$, $Q(x) = 0$

$$I.F = e^{\int -k\,dt}$$

$$I.F = e^{-kt}$$

$$I.F \times P = \int \cancel{I.F \times Q(t)}^{\,0}\,dt$$

~~I.F~~ =

$$e^{-kt} \cdot P = c$$

$$\boxed{P = ce^{kt}} \rightarrow (1)$$

$t=0$, $P = P_0$ put in (1)

$$P_0 = ce^0 \Rightarrow \boxed{c = P_0}$$

$$\boxed{P(t) = P_0 e^{kt}}$$

(C) $\dfrac{dT}{dt} = k(T - T_s)$; $T(0) = T_0$

$$\int \frac{1}{T - t_s}\,dt = k\,dt$$

$$\ln(T - T_s) = kt + c$$

$$\ln(T - T_s) = kt + \ln c$$

$$\ln(T - T_s) - \ln c = kt$$

$$\ln\left(\frac{T - T_s}{c}\right) = kt$$

$$\frac{T - T_s}{c} = e^{kt}$$

$$T - T_s = ce^{kt}$$

$$T = ce^{kt} + T_s \rightarrow (1)$$

$t = 0$, $T = T_0$ put in (1)

$$T_0 = ce^0 + T_s$$

$$T_0 - T_s = c \rightarrow \text{put } (1)$$

$$T = (T_0 - T_s)e^{kt} + T_s$$

$$T(t) = (T_0 - T_s)e^{kt} + T_s$$

T_0 $\rightarrow$ initial temp

T_s $\rightarrow$ surrounding temp

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Ex: 3.1

Q#13: GIVEN:-

- $T(0) = T_0 = 70°F$

$T_s = 10°F$

$T(\frac{1}{2}) = 50°F$

$T(1) = ?$

$T(t) = 15°F$

SOLUTION:-

Peeche waali derivation hoke dikhani hai.

For k:

$T(t) = (T_0 - T_s)e^{kt} + T_s$

$T(\frac{1}{2}) = (70-10)e^{k(\frac{1}{2})} + 10$

$50 = 60e^{k/2} + 10$

$\frac{40}{60} = e^{k/2}$

$\frac{3}{2} = e^{k/2}$

$\ln(3/2) = k/2$

$k = 2\ln(2/3)$

For $T(1)$:

$T(1) = 60e^{(1)[2\ln(\frac{2}{3})]}$

$T(1) = 36.7 F$

- For $T(t) = 15°F$

$15 = (70-10)e^{kt} + 10$

$5 = 60e^{kt}$

$\frac{1}{12} = e^{kt}$

$\ln(1/12) = kt$

$\ln(1/12) = 2\ln(2/3)\, t$

$t = \frac{\ln(1/12)}{2\ln(2/3)}$ Ans → in decimal

$t = 3.06$ sec

Q#15: GIVEN:

$T_0 = T(0) = 20°C$

$T_s = 100°C$ → bcz boiling water.

$T(t) = 90°C$ — (i)

$T(t+1) = 92°C$ — (ii)

$T(t) = 98°C$

SOLUTION:-

$T(t) = (T_0 - T_s)e^{kt} + T_s$

$90 = (20-100)e^{kt} + 100$

$-10 = -80e^{kt}$

$\frac{1}{8} = e^{kt}$ → (1)

Now using (ii) condition

$92° = -80°e^{k(t+1)} + 100$

$\frac{1}{10} = e^{k(t+1)}$ → (2)

(2) ÷ (1)

$\frac{1/10}{1/8} = \frac{\cancel{e^{kt}} \cdot e^k}{\cancel{e^{kt}}}$

$\frac{8}{10} = e^k$

$\ln(8/10) = k \rightarrow$ put ①

$\frac{1}{8} = e^{\ln(8/10)t}$ → 't' power mei jayega

$\frac{1}{8} = \left(\frac{8}{10}\right)^{t}$

$\ln\left(\frac{1}{8}\right) = \ln\left(\frac{8}{10}\right)^{t}$

$\ln\left(\frac{1}{8}\right) = t \ln\left(\frac{8}{10}\right)$

$\frac{\ln\left(\frac{1}{8}\right)}{\ln(8/10)} = t$

$\boxed{t = 9.3 \text{ sec}}$

Q#17: GIVEN:

$T_0 = T(0) = 70°F$

$T(\frac{1}{2}) = 110°F$

$T(1) = 145F$

$T_s = ?$

$\because T(t) = (T_0 - T_s)e^{kt} + T_s$

$110° = (70 - T_s)e^{k\frac{1}{2}} + T_s \rightarrow ①$

$145 = (70 - T_s)e^{k} + T_s \rightarrow ②$

~~Subtracting~~

eq① $\frac{110 - T_s}{70 - T_s} = e^{k/2} \rightarrow ①$

eq② $\frac{145 - T_s}{70 - T_s} = e^{k} \rightarrow ②$

eq② $\frac{145 - T_s}{70 - T_s} = (e^{k})^{1/2}$

$\frac{145 - T_s}{70 - T_s} = \left(\frac{110 - T_s}{70 - T_s}\right)^{1/2}$

$\left(\frac{145 - T_s}{70 - T_s}\right)^2 = \frac{110 - T_s}{70 - T_s}$

$\frac{(110)^2 - 2(110)(T_s) + T_s^2}{(70 - T_s)^2} = \frac{110 - T_s}{\cancel{70 - T_s}}$

$12100 - 220T_s + T_s^2 = (110 - T_s)(70 - T_s)$

$12100 - 220T_s + \cancel{T_s^2} \quad 10150 - 215T_s - \cancel{T_s^2}$

$\boxed{T_s = 390}$

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Q#3: GIVEN:-

$P_0 = P(0) = 500$ $\quad 500 \times 15\% = 75$, $75+500$

$P(10) = 575$, $t = 10$

$P(30) = ?$

$\frac{dP}{dt} \rightarrow t = 30$

For k: pehle derive karna hai!

$P(10) = 575$

$P(t) = P_0 e^{kt}$

$P(10) = 500e^{k(10)}$

$\frac{575}{500} = e^{k(10)}$

$\ln\left(\frac{575}{500}\right) = 10k$

$\boxed{k = 0.014}$

$P(30) = 500e^{(30)(0.014)}$

$\boxed{P(30) = 760.\cancel{93}}$

$\Rightarrow \frac{dP}{dt} = kP(30)$

$\frac{dP}{dt} = (0.014)(\cancel{760})$ 760

$\boxed{\frac{dP}{dt} = 10.64\cancel{8}}$ persons/yr

Q#4: GIVEN:

$P(3) = 400$

$P(10) = 2000$

$P_0 = ?$

SOLUTION:-

$\Rightarrow P(t) = P_0 e^{kt}$

$P(3) = P_0 e^{3t} \Leftrightarrow \cancel{0}$

$400 = P_0 e^{3k} \rightarrow ①$

$2000 = P_0 e^{10k} \rightarrow ②$

$① \div ②$

$\frac{\cancel{2000}^{5}}{\cancel{400}} = \frac{\cancel{P_0} e^{10k}}{\cancel{P_0} e^{3k}}$

$5 = e^{7k}$

$\ln(5) = 7k$

$\boxed{k = 0.23}$

eq(1)

$400 = P_0 e^{3(0.23)}$

$P_0 = \frac{400}{e^{(3)(0.23)}}$

$\boxed{P_0 = 200.6}$

Ex : 2.5

"Bernulli's Equation" → Convert into linear:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$
$n \neq 1$ & $n \neq 0$

when $n = 0$

$$\frac{dy}{dx} + P(x)y = Q(x)y^0$$

$$\frac{dy}{dx} + P(x)y = Q(x) \rightarrow \text{linear eqn}$$

when $n = 1$

$$\frac{dy}{dx} + P(x)y = Q(x)y^1$$

$$\frac{dy}{dx} + [Q(x) - P(x)]y$$

- Bernulli eqn is a nonlinear eqn because as we cannot put $n=0$ or $n=1$ so if we put $n=2$ then it will be 'y^2' which is nonlinear.

① ÷ by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)y}{y^n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)(y^{1-n}) = Q(x) \rightarrow ②$$

Let $z = y^{1-n}$

Taking differentiation w.r.t 'x'

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{1}{1-n} \cdot \frac{dz}{dx} = y^{-n}\frac{dy}{dx} \rightarrow (B)$$

put in ②

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)z = Q(x)$$

Now it is linear eqn!

FAST
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Agr R.H.S par 'y' term aajye to ye bernoli eqn kehlaati h.

$\frac{dy}{dx} - y = y^2 e^x \longrightarrow (1)$

$\div$ b/s by 'y^2'

$y^{-2}\frac{dy}{dx} - y^{-1} = e^x \longrightarrow (2)$

Let, $z = y^{-1}$

Differentiating w.r.t 'x'

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} = y^{-2}\frac{dy}{dx} \rightarrow$ put in (2)

$-\frac{dz}{dx} - \cancel{y}z = e^x$

$\frac{dz}{dx} + z = -e^x \rightarrow$ linear eqn

Let, $P(x) = 1$, $Q(x) = -e^x$

$I.F = e^{\int 1\, dx}$

$I.F = e^x$

$e^x . z = \int e^x . -e^x \, dx$

$= -\int e^{2x} dx$

$e^x . z = -\frac{1}{2}e^{2x} + c$

$\boxed{z = \frac{-1}{2}e^x + ce^{-x}}$

$y^{-1} = \frac{-1}{2}e^x + ce^{-x}$

$\boxed{y = \frac{1}{\frac{-1}{2}e^x + ce^{-x}}}$

(15) $x\frac{dy}{dx} + y = \frac{1}{y^2}$ (x bhi hatana hoga.)

$\div$ b/s by y^{-2} & x

$y^2\frac{dy}{dx} + \frac{y^3}{x} = \frac{1}{x} \rightarrow (2)$

Let $z = y^3$

$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$

$\frac{1}{3}\frac{dz}{dx} = y^2\frac{dy}{dx} \rightarrow$ put in (2)

$\times$ by 3

$\frac{1}{3}\frac{dz}{dx} + \frac{3z}{x} = \frac{3}{x}$

$P(x) = 3/x$, $Q(x) = 3/x$

$I.F = e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3}$

$I.F = x^3$

$x^3 \cdot z = \int x^3 \cdot \frac{3}{x} \, dx$

$x^3 \cdot z = 3\int x^2$

$x^3 \cdot z = \frac{3x^3}{3} + C$

$x^3 \cdot z = x^3 + C$

$z = 1 + Cx^{-3}$

$y^3 = 1 + Cx^{-3}$

$\boxed{y = \sqrt[3]{1 + Cx^{-3}}}$

$\Rightarrow y(1) = 2$

$2 = \sqrt[3]{1 + C(1)}$

$8 = 1 + C$

$\boxed{C = 7}$

$\boxed{y = \sqrt[3]{1 + 7x^{-3}}}$

(22) $y^{1/2} \frac{dy}{dx} + y^{3/2} = 1 \rightarrow (1)$

$y(0) = 4$

let, $z = y^{3/2}$

$\frac{dz}{dx} = \frac{3}{2} y^{1/2} \frac{dy}{dx}$

$\frac{2dz}{3dx} = y^{1/2} \frac{dy}{dx}$ put in (1)

eq (1)

$\frac{2}{3} \frac{dz}{dx} + z = 1$

$\frac{dz}{dx} + \frac{3z}{2} = \frac{3}{2}$

$P(x) = 3/2 \qquad Q(x) = 3/2$

$I.F = e^{\int 3/2 \, dx} = e^{\frac{3}{2}x}$

$e^{\frac{3}{2}x} \cdot z = \int e^{\frac{3}{2}x} \cdot \frac{3}{2} \, dx$

$e^{3/2 x} \cdot z = e^{3/2 x} + C$

$z = 1 + Ce^{-3/2 x}$

$y^{3/2} = 1 + Ce^{-3/2 x}$

$\Rightarrow y(0) = 4$

$4^{3/2} = 1 + Ce^{-3/2 (0)}$

$8 = 1 + C$

$\boxed{C = 7}$

$\boxed{y = \left(1 + 7e^{-3/2 x}\right)^{2/3}}$

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(2) $3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$

$3(1+t^2)\frac{dy}{dt} = 2ty^4 - 2ty$

$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$

$\div$ b/s by $3(1+t^2)$ & y^4

$y^{-4}\frac{dy}{dt} + \frac{2t}{3(1+t^2)} y^{-3} = \frac{2t}{3(1+t^2)} \rightarrow (2)$

$z = y^{-3}$

$\frac{dz}{dt} = -3y^{-4}\frac{dy}{dt}$

$\frac{-1}{3}\frac{dz}{dt} = y^{-4}\frac{dy}{dt}$ put (2)

$\frac{-1}{3}\frac{dz}{dt} + \frac{2t}{3(1+t^2)} z = \frac{2t}{3(1+t^2)}$

$\frac{dz}{dt} - \frac{2t}{1+t^2} z = \frac{-2t}{1+t^2} \rightarrow (A)$

$P(t) = \frac{-2t}{1+t^2} = Q(t)$

$I.F = e^{-\int \frac{2t}{1+t^2} dt}$

$I.F = e^{-\ln(1+t^2)}$

$= e^{\ln(1+t^2)^{-1}}$

$I.F = (1+t^2)^{-1}$

$(1+t^2)^{-1} \cdot z = \int (1+t^2)^{-1} \cdot \frac{-2t}{1+t^2}$

$= \int -2t \cdot (1+t^2)^{-2} dt$

$(1+t^2)^{-1} \cdot z = (1+t^2)^{-1} + C$

$z = 1 + C(1+t^2)$

$y^{-3} = 1 + C(1+t^2)$

$\frac{1}{y^3}$

$y = \frac{1}{[1 + C(1+t^2)]^3}$

Q: $\frac{dy}{dx} + y\sec x = y^2 \sin x \cos x$

$\div$ b/s by 'y^2'

$y^{-2}\frac{dy}{dx} + y^{-1}\sec x = \sin x \cos x \rightarrow (1)$

Let $z = y^{-1}$

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} = y^{-2}\frac{dy}{dx} \rightarrow$ put in (1)

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$$-\frac{dz}{dx} + z\sec x = \sin x\cos x$$

$$\frac{dz}{dx} - z\sec x = -\sin x\cos x$$

$Q(x) = -\sin x\cos x \quad , \quad P(x) = -\sec x$

$$I.F = e^{-\int \sec x} = e^{-\ln(\sec x+\tan x)}$$

$$I.F = (\sec x+\tan x)^{-1}$$

$$(\sec x+\tan x)^{-1}\cdot z = \int(\sec x+\tan x)^{-1} \cdot -\sin x\cos x$$

$$= -\int \frac{\sin x\cos x}{\sec x+\tan x}$$

$$= -\int \frac{\sin x\cos x}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}$$

$$= -\int \frac{\sin x\cos x}{\frac{1+\sin x}{\cos x}}$$

$$= -\int \frac{\sin x\cos^2 x}{1+\sin x}$$

$$= -\int \frac{\sin x(1-\sin^2 x)}{1+\sin x}$$

$$= -\int \frac{\sin x(1-\sin x)\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}}$$

$$= -\int \sin x(1-\sin x)$$

$$-\int \sin x\,dx + \int \sin^2 x\,dx$$

$$-\int \sin x + \int \frac{1-\cos 2x}{2}\,dx$$

$$-\int \sin x + \frac{1}{2}\int(1-\cos 2x)\,dx$$

$$-\int \sin x + \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$$\left(\frac{\cos x}{1+\sin x}\right)\cdot z = \ldots$$

$$z = (1+\sin x) + \frac{(1+\sin x)}{2\cos x}x - \frac{1}{4}\sin 2x\left(\frac{1+\sin x}{\cos x}\right) + \left(\frac{1+\sin x}{\cos x}\right)c$$

(x) whole eqn by 4 and $\cos x$

$$z = \frac{\cos x\,4(1+\sin x) + 2x(1+\sin x) - \sin 2x(1+\sin x) + (1+\sin x)4c}{4\cos x}$$

$$y = \frac{4\cos x}{(1+\sin x)[4\cos x + 2x - \sin 2x + 4c]}$$

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Q: $\cot x \dfrac{dy}{dx} + y = y^2$

Dividing b/s by $\cot x$ & y^2

$y^{-2}\dfrac{dy}{dx} + y^{-1}\tan x = \tan x$ ↳ (1)

Let $z = y^{-1}$

$\dfrac{dz}{dx} = -y^{-2}\dfrac{dy}{dx}$

$-\dfrac{dz}{dx} = y^{-2}\dfrac{dy}{dx} \rightarrow$ put in (1)

$-\dfrac{dz}{dx} + z\tan x = \tan x$

$\dfrac{dz}{dx} - z\tan x = -\tan x$

$Q(x) = -\tan x$, $P(x) = -\tan x$

$I.F = e^{-\int \tan x\, dx}$

$= e^{-\ln \sec x}$

$I.F = (\sec x)^{-1}$

~~$(\sec x)^{-1}\cdot z = \int (\sec x)^{-1}\cdot -\tan x\, dx$~~

~~$= -\int \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}\, dx$~~

~~$= -\int (\cos x)^{-2}\sin x\, dx$~~

~~$= \int (\cos x)^{-2}\cdot(-\sin x)\, dx$~~

~~$(\sec x)^{-1}\cdot z = -(\cos x)^{-1} + C$~~

~~$(\sec x)^{-1} z = -\sec x + C$~~

~~$z = -\sec^2 x + C\sec x$~~

~~$y^{-1} = -\sec^2 x + C\sec x$~~

~~$y = (-\sec^2 x + C\sec x)^{-1}$~~

$(\sec)^{-1}\cdot z = \int (\sec x)^{-1}\cdot -\tan x\, dx$

$\cos x\cdot z = \int \cos x\cdot -\tan x\, dx$

$= \int \cos x\cdot -\dfrac{\sin x}{\cos x}$

$\cos x\cdot z = -\int \sin x\, dx$

$\cos x\cdot z = \cos x + C$

$z = 1 + C\sec x$

$\dfrac{1}{y} = 1 + C\sec x$

$y = (1 + C\sec x)^{-1}$

Ex: 2.5

Homogenous 1st Order DE

$\frac{dy}{dx} = f(x,y) = \frac{h(x,y)}{g(x,y)}$

degree of $h(x,y)$ = degree $g(x,y)$

$\frac{y^{+1}}{x^{+1}} \rightarrow$ degree 0

dono powers minus hojayengi

$\sqrt{xy} \rightarrow 1 \quad (x^1y^1)^{\frac{1}{2}} = ((xy)^2)^{\frac{1}{2}} = 1$

In homogenous eqn we Subsitute, $\rightarrow v$ dep on x

$\boxed{y = vx}$

$\frac{d}{dx}(y) = \frac{d}{dx}(vx) \quad \hookrightarrow u \cdot v$

$\boxed{\frac{dy}{dx} = x\frac{dv}{dx} + v}$

Q:2) $(x+y)dx + x\,dy = 0$

$x\,dy = -(x+y)dx$

$x\frac{dy}{dx} = -x-y \Rightarrow \frac{dy}{dx} = \frac{-x-y}{x} \rightarrow deg = 1$ deg=1

$x\frac{dy}{dx} + y = -x \rightarrow$ ye ab linear ban sakti hai hum isko linear se bhi solve karsakte hai.

Let, $y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$\frac{dv}{dx}x + v = \frac{-x - vx}{x}$

$\frac{dv}{dx}x + v = \frac{x(-1-v)}{x}$

$\frac{dv}{dx}x + v = -1-v$

$\frac{dv}{dx}x = -1-v-v$

$\frac{dv}{dx}x = -1-2v$

$dv \cdot x = (-1-2v)dx$

$\frac{1}{1+2v}dv = \frac{-1}{x}dx$

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$$\int \frac{1}{(1+2v)}\, dv = -\int \frac{1}{x}\, dx$$

$$\frac{1}{2}\ln(1+2v) = -\ln x + \ln c$$

$$\ln(1+2v)^{1/2} = \ln\left(\frac{c}{x}\right)$$

taking 'e' on b/s

$$(1+2v)^{1/2} = \frac{c}{x}$$

Sq. on b/s

$$1+2v = \frac{c^2}{x^2}$$

c^2 or c dono same hai

$\Rightarrow y = vx$

$$1 + \frac{2y}{x} = \frac{c}{x^2}$$

$$x^2 + 2yx = c$$

$$2yx = c - x^2$$

$$y = \frac{c - x^2}{2x}$$

(8) $(y^2 + yx)dx - x^2 dy = 0$

$$(y^2 + yx)dx = x^2 dy$$

$$\frac{dy}{dx} = \frac{y^2 + yx}{x^2} \quad \begin{matrix}\to deg = 2 \\ \to deg = 2\end{matrix} \quad \to (1)$$

Let, $y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v \quad \Big\} \text{ put in (1)}$$

$$\frac{dv}{dx}x + v = \frac{v^2x^2 + vx^2}{x^2}$$

$$= \frac{\cancel{x^2}(v^2+v)}{\cancel{x^2}}$$

$$\frac{dv}{dx}x + \cancel{v} = v^2 + \cancel{v}$$

$$\frac{dv}{dx}x = v^2$$

$$v^{-2}dv = \frac{1}{x}dx$$

$$\frac{-1}{v} = \ln x + \ln(c)$$

$$\frac{-1}{y/x} = \ln(xc)$$

$$\frac{-x}{y} = y\ln(xc)$$

$$\boxed{y = \frac{-x}{\ln(xc)}}$$

V

$-V$

$\int \frac{1}{x}\, dx$

$= \int \frac{1}{x}\, dx$

$dv = \int \frac{1}{x}\, dx$

$dv = \int \frac{1}{x}\, dx$

$) = \ln(xc)$

$)] = \ln(xc)$

$\ln(xc)$

yegi a property

$$\boxed{c = e^{-1}}$$

* By Subsitution :-

(23) $\frac{dy}{dx} = (\{x + y\} - 1)^2$

let,

$U = x + y \rightarrow (1)$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{du}{dx} - 1 = \frac{dy}{dx} \rightarrow (2)$ put (2)

$\frac{du}{dx} - 1 = (U-1)^2$

$\frac{du}{dx} = (U-1)^2 + 1$

$\frac{1}{(U-1)^2+1}\, du = dx$

$\tan^{-1}(U-1) = x$

$\tan^{-1}(x+y-1) = x + C$

$x + y - 1 = \tan(x+c)$

$y = \tan(x+c) - x - 1$

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ishki whole power 1 nhi hai yk x ki power 1 y ki power 1 or 1 ki power 0 hai.

24) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

$\frac{dy}{dx} = \frac{1-(x+y)}{x+y}$

$U = x+y$

$\frac{du}{dx} - 1 = \frac{1-U}{U}$

$\frac{du}{dx} = \frac{1-U}{U} + 1$

$\frac{du}{dx} = \frac{1}{U}$

$\int U\,du = \int dx$

$\frac{U^2}{2} = x$

$(x+y)^2 = 2x + C$

29) $\frac{dy}{dx} = \cos(x+y)$

let, $U = x+y$

$\frac{dU}{dx} = 1 + \frac{dy}{dx}$

$\frac{dU}{dx} - 1 = \cos U$

$\frac{du}{dx} = \cos U + 1$

$\frac{1}{\cos U + 1}\,du = dx$

$\times$ & $\div$ by $1-\cos U$

$\frac{1-\cos U}{1-\cos^2 U}\,du = dx$

$\frac{1-\cos U}{\sin^2 U}\,du = dx$

$\frac{1}{\sin^2 U}\,du - \frac{\cos U}{\sin^2 U}\,du = dx$

$\int \csc^2 U\,du - \int \csc U \cot U\,du = \int dx$

$-\cot(x+y) + \csc(x+y) = x + C$ →(1)

$y(0) = \pi/4$

$-\cot(0+\frac{\pi}{4}) + \csc(0+\frac{\pi}{4}) = C$

$\sqrt{2} - 1 = C$

eq(1)

$\csc(x+y) - \cot(x+y) = x + \sqrt{2} - 1$

$\frac{dP}{dt} = P(a-bP)$

Ex: 3.2 —(2.4)—

① $\frac{dN}{dt} = N(1-0.0005N)$

(b) $\int \frac{1}{N(1-0.0005N)} dN = \int dt$

↳ By partial fraction.

(4) 1790 → $t=0$ consider karenge
1850 → $t=60$ lenge $1790-1850 = 60$
1910 → $t=120$ $1790-1910 = 120$

in teen se a, b, N_0 nikaal lenge

Exact D.E :

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ → Exact D.E

Step → $\int M\,dx + \int N\,dy = c$

y-const (for $\int M\,dx$); only consider y-term (for $\int N\,dy$)

① $(x-y)dx - x\,dy = 0$

$M = x-y$, $N = -x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

∴ both are equal ∴ it is exact DE

$\int_{y\text{-const}} (x-y)\,dx + \int (0)\,dy = c$

as there is no y-term so we write ZERO

$\int x\,dx - y\int dx = c$ ($\because y$ constant)

$\frac{x^2}{2} - yx = c$

$\left(1+\ln x+\frac{y}{x}\right)dx = (1-\ln x)dy$

$\left(1+\ln x+\frac{y}{x}\right)dx - (1-\ln x)dy = 0$

$M = 1+\ln x+\frac{y}{x}$, $N = -1+\ln x$

$\frac{\partial M}{\partial y} = \frac{1}{x}$, $\frac{\partial N}{\partial x} = \frac{1}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{1}{x}$ Exact D.E

$= \int M dx + \int N dy = C$

$\int M dx$: $y =$ const; $\int N dy$: term does not consist of 'x' term (only 'y' terms)

ismein 'y' ko as constant treat hoga.

$\int \left(1+\ln x+ y/x\right)dx + \int (-1)\,dy = C$

$\int dx + \int \ln x\,dx + y\int \frac{1}{x} - \int dy = C$

$x + \left[\ln x\cdot x - \int \frac{1}{x}\cdot x\,dx\right] + y\ln x - y = C$

$x + [x\ln x - x] + y\ln x - y = C$

Ans.

(22) $(e^x+y)dx + (2+x+ye^y)dy = 0$

$y(0) = 1$

$M = e^x+y$, $N = 2+x+ye^y$

$\frac{\partial M}{\partial y} = 0+1 = 1$ $\quad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} = 1$

$\int (e^x+y)dx + \int (2+ye^y)dy = C$

$\int e^x dx + y\int dx + 2\int dy + \int ye^y = C$

by parts

$e^x + xy + 2y + [ye^y - e^y] = C$

$y(0) = 1$

C will be 3

$e^x + xy + 2y + [ye^y - e^y] = 3$

(25) $\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}\right)dt +$

$\left(ye^y + \frac{t}{t^2+y^2}\right)dy = 0$

$M = \frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}$

$\frac{\partial M}{\partial y} = -\left[\frac{(y)'(t^2+y^2) - (t^2+y^2)'(y)}{(t^2+y^2)^2}\right]$

$= -\frac{[t^2+y^2 - 2y \cdot y]}{(t^2+y^2)^2}$

$= -\frac{[t^2-y^2]}{(t^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{y^2-t^2}{(t^2+y^2)^2}$

→ isko w.r.t 't' derivate krenge

$N = ye^y + \frac{t}{t^2+y^2}$

$N = \frac{[(t)'(t^2+y^2) - (t^2+y^2)'(t)]}{(t^2+y^2)^2}$

$= \frac{t^2+y^2-2t^2}{(t^2+y^2)^2}$

$\frac{\partial N}{\partial t} = \frac{y^2-t^2}{(t^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ → Exact D.E

$\frac{x+y}{1+y^2}$ is ~~ka~~ denominator alag krenge to

$\frac{x}{1+y^2} + \frac{y}{1+y^2}$ → ye without 'x' hai to isko 'N' mein likh dete hai.

$\int\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}\right)dt + \int ye^y\,dy = c$

← By parts

$\ln t - \frac{1}{t} - y\int\frac{1}{t^2+y^2}\,dt + [ye^y - e^y] = c$

$\int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(x/a)$

$\ln t - \frac{1}{t} - y\left[\frac{1}{y}\tan^{-1}\left(\frac{t}{y}\right)\right] + \left[ye^y - e^y\right] = c$

$\ln t - \frac{1}{t} - \tan^{-1}\left(\frac{t}{y}\right) + ye^y - e^y = c$

(26) $\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+y\sin x)$

$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy = y(y+y\sin x)\,dx$

$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy - y(y+y\sin x)dx = 0$

$M = -y^2 - y\sin x$, $N = \frac{1}{1+y^2} + \cos x - 2xy$

$\frac{\partial M}{\partial y} = -2y - \sin x$, $\frac{\partial N}{\partial x} = -\sin x - 2y$

Sol:

$\int(-y^2 - y\sin x)\,dx + \int\left(\frac{1}{1+y^2}\right)dy = c$

$-y^2\int dx - y\int\sin x\,dx + \tan^{-1}(y) = c$

Non Exact D.E

$$M\,dx + N\,dy = 0 \rightarrow (A)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Case #01:

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x) \text{ or can be any constant}$$

only x term

Case #02:

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = p(y)$$

For case #01 →

$$I \cdot F = e^{\int p(x)dx} \text{ or } e^{\int p(y)dy}$$

For case #02:

$$I \cdot F \times (A) \rightarrow Q \rightarrow \text{Exact D.E}$$

$\tan^{-1}(y) = c$

$= 0$

$(y) = 1 + \frac{\pi}{4}$

D.E is Exact;

$x - (3xy^2 + 20x^2y^3)dy = 0$

$\cancel{3}y^2 + 40xy^3$

Coef xy^3

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(31) $(2y^2+3x)dx + 2xy\,dy = 0$

$M = 2y^2 + 3x \qquad N = 2xy$

$\frac{\partial M}{\partial y} = 4y \qquad \frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ Non-Exact

CASE # 01 :

$\frac{1}{2xy}\left(4y - 2y\right) = P(x)$

$p(x) = \frac{1}{x} \rightarrow$ only 'x' term

$\therefore$ case # 01 approved

$I.F = e^{\int \frac{1}{x} dx} = e^{\ln x}$

$I.F = x$

Now multiplying eq (A) by x

$(2y^2x + 3x^2)dx + 2x^2y\,dy = 0$

⋮

aage bhi same hoga

(38) $(x^2+y^2-5)dx = (y+xy)dy,\ y(0)=1$

$(x^2+y^2-5)dx - (y+xy)dy$

$M = x^2+y^2-5 \quad , \quad N = -y-xy$

$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial y} = -y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y x}$

CASE # 01 :

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$

$\frac{1}{-y-xy}(2y+y) = p(x)$

$\frac{1}{-y(1+x)}(3y) = p(x)$

$p(x) = \frac{-3}{1+x}$

$I.F = e^{\int p(x)dx} = e^{\int \frac{-3}{1+x}} = e^{-3\ln(1+x)}$

$= (1+x)^{-3}$

eq (A) $\times$ by $(1+x)^{-3}$

$$\frac{(x^2+y^2-5)}{(1+x)^3}\,dx - \frac{(y+xy)}{(1+x)^3}\,dy = 0$$

$$\int\left[\frac{x^2+y^2-5}{(1+x)^3}\right]dx + \int (0)\,dy = C$$

$$\int \frac{x^2}{(1+x)^3}\,dx + y^2\int\frac{1}{(1+x)^3}\,dx - 5\int\frac{1}{(1+x)^3} = C \quad \hookrightarrow (B)$$

Consider,

$$\int \frac{x^2}{(1+x)^3}$$

U	V
x^2	$(1+x)^{-3}$
$2x$ (+)	$1/(1+x)^{-2}(-2)$
2 (−)	$1/(1+x)(-2)(-1)$
0 (+)	$\ln(1+x)/2$

$$\frac{x^2}{-2(1+x)^2} - \frac{2x}{(-2)(-1)(1+x)} + \cancel{2}\frac{\ln(1+x)}{\cancel{2}}$$

$$\frac{-x^2}{-2(1+x)^2} - \frac{x}{(1+x)} + \ln(1+x) \rightarrow \text{put in } (B)$$

$$\frac{-x^2}{-2(1+x)^2} - \frac{x}{(1+x)} + \ln(1+x) - \frac{y^2}{2(1+x)^2} - \frac{5}{2(1+x)^2}$$

(35) $(10-6y+e^{-3x})dx - 2dy = 0 \quad \hookrightarrow (A)$

$M = 10-6y+e^{-3x}$, $N = -2$

$\frac{\partial M}{\partial y} = -6$, $\frac{\partial N}{\partial x} = 0$

CASE #01:

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$$

$$\frac{1}{-2}(-6-0) = p(x)$$

$$3 = p(x)$$

$$I.F = e^{\int p(x)\,dx} = e^{\int 3\,dx} = e^{3x}$$

Now $\otimes$ (A) by e^{3x}

$$(10e^{3x} - 6ye^{3x} + 1)dx - 2e^{3x}\,dy$$

$$\int (10e^{3x} - 6ye^{3x}+1)\,dx + \int (0)\,dy = C$$

$$10\int e^{3x}dx - 6y\int e^{3x}dx + \int 1\,dx = C$$

$$\frac{10e^{3x}}{3} - \frac{6ye^{3x}}{3} + x = C$$

$$\frac{10e^{3x}}{3} - 2ye^{3x} + x = C$$

Ex: 2.6

(32) $y(x+y+1)dx + (x+2y)dy = 0$

$M = xy + y^2 + y$, $N = x + 2y$

$\frac{\partial M}{\partial y} = x + 2y + 1$, $\frac{\partial N}{\partial x} = 1$

CASE # 01

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$

1. Variable Seperable
2. Bernoli Linear
~~3. Exact, non-exact~~
3. Bernoli
4. Subsitution
5. Exact, non-Exact

(Numeric Method)

Euler Method

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$f(x,y) = \frac{dy}{dx}$ if $\frac{dy}{dx} = 2xy$

then

$f(x,y) = 2xy$ or $f(x_n, y_n) = 2x_n y_n$

$y_{n+1} = y_n + 0.5(2x_n y_n)$

$y_{n+1} = y_n + x_n y_n$

we have been givenby $y(0) = 1$

$\left.\begin{array}{l} x_0 = 0 \\ y_0 = 1 \end{array}\right\} \rightarrow$ intial condition

ye given hoga

$n = 0$, $h = 0.5$

$y_{0+1} = y_0 + x_0 y_0$

$y_1 = 1 \rightarrow y(0.5) = 1$

$\hookrightarrow x_1 = 0.5$

x ki initial 0 value thi ab hum x mein h ko add krdiya &
x 0.5 bangaya
phir dobara add krdia

$n = 1$

$y_2 = y_1 + x_1 y_1$

$= 1 + 0.5(1) = 1.5$

$y_2(1) = 1.5$

hum jab tak krte rhenge jab tak 'x' given R equal na hojaye

$y_3 = 1.5 + (1)(1.5) = 3$

$y_3(1.5) = 3$

(8) $\frac{dy}{dx} = e^{x^2}$, $y(0) = 1$

$y(0.003) = ?$

$h = 0.001$

n	x_n	y_n	$y_{n+1} = y_n + h\underbrace{f(x_n, y_n)}_{e^{x^2}}$
0	0	1	1.001
1	0.001	1.001	1.002
2	0.002	1.002	1.003
3	0.003	1.003	

Q6: $y' = x^2 + y^2$, $y(0) = 1$

$y(0.5) = ?$

Use Euler's Method,

$h = 0.1$ & $h = 0.05$

For $h = 0.1$:

$y_{n+1} = y_n + h(x_n, y_n)$

$y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$

x_n	y_n	y_{n+1}	not necessary
0	1	1.1	$y_{0+1} = y_0 + 0.1(x_0^2 + y_0^2)$
0.1	1.1	1.222	$y_{1+1} = y_1 + 0.1(x_1^2 + y_1^2)$
0.2	1.222	1.3753	$y_{2+1} = y_2 + 0.1(x_2^2 + y_2^2)$
0.3	1.3753	1.573	$y_{3+1} = y_3 + 0.1(x_3^2 + y_3^2)$
0.4	1.573	1.837	⋮
0.5	1.837	2.1995	

For $h = 0.05$

x_n	y_n	y_{n+1}
0	1	1.05
0.05	1.05	⋮
0.1		
0.15		
0.2		
0.25		
0.3		
⋮		
0.5		

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$y = e^{x^2-1}$ se actual value nikal sakte h.

Q#4: Use Euler Method, also Find explicit solution and Find absolute error and % Relative error.

$y' = 2xy \quad y(1) = 1 \quad y(1.5) = ?$

$h = 0.1$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\, dy = \int 2x\, dx$$

$\ln y = x^2 + C \rightarrow (A)$

$y(1) = 1$

put $x = 1$, $y = 1$

$\ln(1) = 1^2 + C$

$\boxed{C = -1}$

$\ln y = x^2 - 1$

$\boxed{y = e^{x^2-1}}$

For $h = 0.1$ Euler Method,

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_{n+1} = y_n + 0.1 (2 x_n y_n)$

$y_{n+1} = y_n + 0.2 (x_n y_n)$

x_n	y_n	y_{n+1}	Actual Value	Abs error (Act. Value − Approx.)	% error
1	1	1.2	1	0.2	20%
1.1	1.2	1.464	1.2337	0.2303	18.67%
1.2	1.464	1.8154	1.5527	0.2627	16.91%
1.3	1.8154	2.874	1.9937	0.8803	44.2%
1.4	2.874	2.9278	2.6117	0.3161	12 %
1.5	2.9278	3.8062	3.4903	0.3169	9.05%

- Absolute Error $\rightarrow$ |Actual − Approx. Value|
- % Relative $\rightarrow \left(\frac{|\text{Actual} - \text{Approx}|}{\text{Actual}} \right) \times 100$

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EHD # 4

8, 29, 30, 31...

* **HIGHER D.E :**

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{(n-1)} y}{dx^{(n-1)}} + \dots + a(x)y = f(x)$$

$$\rightarrow 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 3x$$

$$\rightarrow \frac{d^3y}{dx^3} + 4x\frac{dy}{dx} + 4 = 0$$

* If $a_n(x), a_{n-1}(x) \dots$ and $f(x)$ are continuous in Interval I. and $a_n(x) \neq 0$ at $x = x_0$, then solution is unique or else it can be infinite many or no sol.

HOMOGENEOUS D.E	NON-HOMOGENEOUS D.E
$\Rightarrow f(x) = 0$	$\Rightarrow f(x) \neq 0$
$y'' + y' + 3y = 0$	$y'' + y' + 3y = 3x$
$xy'' + 3y' = 0$	$xy'' + 3y' + \sin x \neq 0$
no x terms	x term
ye 'y' ke saath hai.	
akela 'x' hona chahiye	$y'' + 2y' + 3 = 0$ ↳ $3x^0$

(Ex : 4.1)

Q: Find member of family of the sol of IVP.

$$y = c_1 e^{x} + c_2 e^{-x} \quad (-\infty, \infty)$$

$D.E \rightarrow y'' - y = 0, \; y(0) = 0$
$y'(0) = 1$

SOLUTION:-

$y = c_1 e^{x} + c_2 e^{-x} \rightarrow (A)$

$y(0) = 0 \quad , x = 0, y = 0$

$0 = c_1 + c_2$

$c_1 = -c_2 \rightarrow (1)$

$y'(0) = 1$

$x = 0 \;, \; y' = 1$

$y' = c_1 e^{x} - c_2 e^{-x}$

$1 = c_1 - c_2 \rightarrow (2)$

eq ? $c_1 = -c_2$

eq (2) $\Rightarrow$

$1 = -c_2 - c_2$

$2c_2 = -1$

$$\boxed{c_2 = -\tfrac{1}{2}}$$

$C_1 = -(-\tfrac{1}{2})$

$\boxed{C_1 = \tfrac{1}{2}}$

★ Existence & Uniqueness (IVP)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0(x)y = g(x)$$

initial value

$IV = y(x_0) = y_0,\ y'(x_0) = y_0 \dots$

If $a_n(x), a_{n-1}(x), \dots, a_0(x)$ and $g(x)$ are continuous on interval I and $a_n(x) \neq 0$ at $x = x_0$, then solution of $y(x)$ of IVP exist and unique.

⑤ $y = c_1 + c_2x^2$ is a solution of $xy'' - y' = 0$ on the intervals $(-\infty, +\infty)$ show that constant c_1, and c_2 but be found at $y(0) = 0,\ y'(0) = 1$

$y = c_1 + c_2x^2 \rightarrow (A)$

$y = 0,\ x = 0$

$\boxed{c_1 = 0}$

$y'(0) = 1$

$y' = 2c_2x$

$1 = 2c_2(0)$

$1 \neq 0$

No solution.

Q6: but $y(0) = 0,\ y'(0) = 0$

$c_1 = 0,$

$y' = 2c_2x$

$0 = 2c_2(0)$

$0 = 0$

c_2 mein hum koi bhi value daalien to zero hi ans thega to iske too many solutions honge.

$C_2 = 3, 4, 5, 6 \dots$

$c_1 = 0,\quad C_2 \in (-\infty, +\infty)$

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① coefficient continuous in given interval
② $a_n(x) \neq 0$, x_0 is initial condition

Q: In problem 9 and 10, find intervals centered about $x=0$ with the given IVP has a unique.

Solution:-

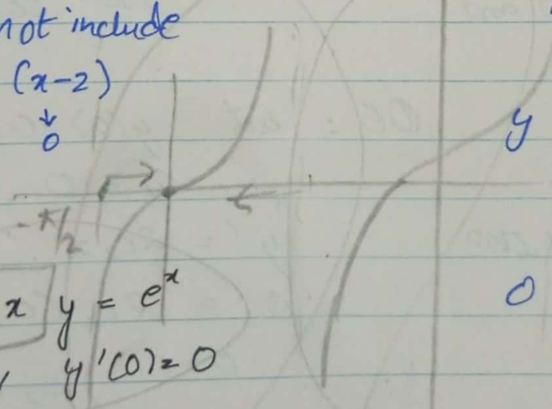
⑨ $(x-2)y'' + 3y = x$

$y(0)=0$, $y'(0)=1$

conti conti 1st condition

within Interval

$(-\infty, 2) \cup (2, \infty)$
↓
$(-\infty, 2)$
↓
2 must not include
$(x-2)$
↓
0



⑩ $y'' + \boxed{\tan x}\, y = e^x$
$y(0)=1$, $y'(0)=0$

$\tan x = \sin x / \cos x$

is me zero mid pe aarha hai

$\Rightarrow (-3\pi/2, 3\pi/2)$
$\hookrightarrow \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Interval,
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ Ans.
$(0, \pi/2)$

Boundary Value PROBLEM

$y(a)=y_1$, $y(b)=y_2$
$y(a)=y_1$, $y'(b)=y_2$

$\boxed{c_2} \sin \pi x = 0$

Q13: $y = c_1 e^x \cos x + c_2 e^x \sin x$
$y'' - 2y' + 2y = 0$

(a) $y(0)=1$, $y'(\pi)=0$

$y = 1$, $x=0$
$\boxed{1 = c_1}$

$y' = c_1[e^x - \sin x + \cos x\, e^x] + c_2[e^x \cos x + \sin x\, e^x]$

$0 = c_1[e^\pi(-\sin(\pi)) + \cos(\pi) e^\pi] + c_2[e^\pi \cos(\pi) + \sin(\pi) e^\pi]$

$0 = c_1(-e^\pi) + c_2(-e^\pi)$
$0 = (c_1 + c_2)e^{-\pi}$
$\boxed{c_2 = -1}$

IVP me x same hota BVP mein x different hota hai.

c_1 and c_2 ka abh hi answer hai to unique solution hai ye.

$$\boxed{y = e^x \cos x - e^x \sin x}$$

(*) "HOMOGENOUS DE"

$x \to$ is independent

↳ we say function of x

agar x waali term '0' hai to woe homogenous hai, agr x wali term 0 nhi hai to non-homogenous.

(*) $y'' + 2y' + y = 0$ } homogenous

(*) $3xy'' + 3y' - xy = 0$ } homogenous

(*) $3xy'' + 3y' - x = 0$

$f(x)$ = non-homogenous

(*) 2^{nd} Order D.E + Homogenous

eg: $y_1 = x$, $y_2 = x^2$

$$\boxed{y = c_1 x + c_2 x^2}$$

* doo solution hu agar combine karey tu linear combination

↳ kisi bhi D.E ke solution ko add karde.

Q $y_1 = x^2$, $y_2 = x^2 \ln x$

↳ $x^3 y''' - 2xy' + 4y = 0$

$x^3(0) - 2x(2x) + 4(x^2) = 0$

$-4x^2 + 4x^2 = 0$

$0 = 0$

$$\boxed{y = c_1 x^2 + c_2 x^2 \ln x}$$

↳ Linear Combinatio

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$y = x \quad y_2 = x^2$

$\boxed{y_g = c_1 x + c_2 x^2}$

general eqn

$c_1 = 0,\ c_2 = 1,\ y_2 = x^2$

$c_1 = 1,\ c_2 = 0,\ y = x$

apnimarzi se li hai

$y = x + x^2$

$y = 0,\ c_1 = c_2 = 0$

* Akh solution of all homogenous eqn is always zero.

agr c_1 & c_2 dono zero hogai, to $y = 0$ → always solution of Homogenous eqn

Linear Dependent Function

$c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots + c_n f_n = 0$

$c_1 = c_2 = c_3 = \dots = c_n = 0$

→ linear independent.

agr c_1, c_2, c_3 k ans 0 ho yaani zero put krne par eqn satisfy hoti hai to linearly independent warna agr koi doosri values zero k ilawa bhi put hoti ho to lineary depend. ZERO to laazmi hi hoga.

$c_1 y_1 + c_2 y_2 + c_3 y = 0$

eg:

$c_1 \cos^2 x + c_2 \sin^2 x + c_3 (2) = 0$

ab ye kya 0,0,0 put krenge to to satisfy hogi hi lekin agr is k ilawa jese $(2, 2, -1)$ put krein to bhi ans ZERO ayega to ye linear dependent hai.

eg: $f_1 = 0,\ f_2 = x,\ f_3 = e^x$

$c_1(0) + c_2 x + c_3 e^x = 0$ to ye bhi lineary dependent

$c_1 = 1,\ c_2 = 0,\ c_3 = 0$

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$y_1 = e^{3x}, \; y_2 = e^{-3x}$

$c_1 e^{3x} + c_2 e^{-3x} = 0$

$(c_1, c_2) = (0, 0)$

linearly independent is k ilawa koi values nhi daal sakte.

$f_1 = x, \; f_2 = x^2, \; f_3 = 4x - 3x^2$

$c_1 x + c_2 x^2 + c_3 (4x - 3x^2) = 0$

$(0, 0, 0), \; (-4, 3, 1)$

linear dependent

agr koi ek function bhi zero ho to wo hamesha linearly ~~inde~~ dependent hoga.

eg:

$f_1 = 0, \quad f_2 = x^2, \; f_3 = 4x - 3x^2$

$c_1(0) + c_2 x^2 + c_3 (4x - 3x^2)$

$(1, \; 0, \; 0)$

(16) $f_1 = 2 + x, \quad f_2 = 2 + |x|$

$2 + x \quad x \geq 0 \qquad 2 - x \quad x < 0$

$c_1(2+x) + c_2(2+x) = 0$ linearly dep

$c_1(2+x) + c_2(2-x) = 0 \rightarrow$ linearly indep

• For Linearly Independent:

WRONSKIEN:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

agr $=0$ hai to inconclusive we cannot tell

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$$

Ex: 4.2

y'' ko coefficient must be 1.

* 2nd ORDER HOMOGENOUS D.E

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

$a_2(x) \neq 0$

q k agr $a_2(x)=0$ hogaya to y'' khtam hojayega.

General Solution → $y = c_1y_1 + c_2y_2$

→ REDUCTION OF ORDER:

Given → y_1 (or y_2) → function of x

Find → y_2 or y_1

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$y_1 = y_2 \int \frac{e^{-\int p(x)\,dx}}{y_2^2}\,dx$$

$y = c_1y_1 + c_2y_2$

$y = c_1\sin 3x + c_2(c\cos 3x)$

$y = c_1\sin 3x + c_3\cos 3x$

⇒ we dont write minus sign or constants in G.S.

(1) $y'' - 4y' + 4y = 0 \qquad y_1 = e^{2x}$

Find 2nd sol → y_2

$p(x) = -4, \quad Q(x) = 4 \quad y_1 = e^{2x}$

$$\Rightarrow y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

$$= e^{2x}\int \frac{e^{-\int -4\,dx}}{(e^{2x})^2}\,dx$$

$$= e^{2x}\int dx$$

$$\boxed{y_2 = xe^{2x}}$$

G.S ⇒ $y = c_1y_1 + c_2y_2$

$y = c_1e^{2x} + c_2xe^{2x}$

(4) $y'' + 9y = 0, \quad y_1 = \sin 3x$

$P(x) = 0, \quad Q(x) = 9, \quad y_1 = \sin 3x$

$$y_2 = \sin 3x \int \frac{e^{-\int 0\,dx}}{\sin^2 3x}$$

$$= \sin 3x \int \operatorname{cosec}^2 3x\, e^{-c_1}\,dx$$

$$= e^{-c_1}\sin 3x \int \operatorname{cosec}^2 3x\,dx$$

$$= -\frac{1}{3}\sin 3x \cot 3x \cdot e^{-c_1}$$

C → all are constant

$$= -\frac{1}{3}\sin 3x \cdot \frac{\cos 3x}{\sin 3x} \Rightarrow -\frac{e^{-c_1}}{3}\cos 3x$$

Ex: 4.3

- $y_1 = x \cancel{\tan x} \sin(\ln x)$

$x^2 y'' - x y' + 2y = 0$

$y'' - \frac{1}{x} y' + \frac{2}{x^2} y = 0$

$P(x) = \frac{-1}{x}$, $Q(x) = \frac{2}{x^2}$, $y_1 = x\sin(\ln x)$

$y_2 = x\sin(\ln x) \int \frac{e^{-\int \frac{1}{x} dx}}{x^2 \sin^2(\ln x)}$

$= x\sin(\ln x) \int \frac{e^{\ln x}}{x^2 \sin^2(\ln x)}$

$= x\sin(\ln x) \int \frac{x}{x^3 \sin^2(\ln x)}$

$= x\sin(\ln x) \int \operatorname{cosec}^2(\ln x) \cdot \frac{1}{x} dx$

$= -x\sin(\ln x) \cdot \cot(\ln x)$

$= -x \cancel{\sin(\ln x)} \cdot \frac{\cos(\ln x)}{\cancel{\sin(\ln x)}}$

$y_2 = -x\cos(\ln x)$

$y = c_1 x \sin(\ln x) + c_2 x \cos(\ln x)$

2nd Order Homogenous Eqn With Constant Coefficent

$ay'' + by' + cy = 0$

$a, b, c \in \mathbb{R}$, $a \neq 0$

Let Sol is $y = e^{mx}$

$a(m^2 e^{mx}) + bme^{mx} + ce^{mx}$

$e^{mx}(am^2 + bm + c) = 0$

$e^{mx} \neq 0$ as it is equal to y

$\boxed{am^2 + bm + c = 0}$ ↳ auxilary eqn

(1) Roots Are Equal And Real

$y = e^{mx}(c_1 + c_2 x)$

(2) If Roots Are Unequal And Real m_1, m_2

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(3) IF ROOTS ARE UNEQUAL AND COMPLEX:

$m = \alpha + \beta i$

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

• → $D = \frac{d}{dx}, \frac{d}{dt}$ new representation of derivative

→ $D^2 = \frac{d^2}{dx^2}$

(1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$D^2y - 3Dy + 2y = 0$

$(D^2 - 3D + 2)y = 0$

Auxilary eqn — replacing D with m.

$m^2 - 3m + 2 = 0$

$m = 1, 2$

$y = c_1 e^{1x} + c_2 e^{2x}$

(2) $(D^2 + 2D + 1)y = 0$

$m^2 + 2m + 1 = 0$ → by using calculator

$m = -1$

$y = e^{-1x}(c_1 + c_2 x)$

(3) $y'' + 9y = 0$

$D^2y + 9y = 0$

$(m^2 + 9)y = 0$

$m^2 + 9 = 0$

$m = \pm\sqrt{-3}$

$m = \pm 3i$ → roots are complex

$m = \alpha \pm \beta i, \quad \alpha = 0, \beta = 3$

→ $y = e^{\alpha x}(c_1 \cos\beta + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$y = c_1 \cos 3x + c_2 \sin 3x$

(*) $m = 3, 5$

$y = c_1 e^{3x} + c_2 e^{5x}$

(*) $m = -3 \pm i$

$y = e^{-3x}(c_1 \cos x + c_2 \sin x)$

$m = 5$

$y = e^{5x}(c_1 + c_2 x)$

(28) $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2}$

, $D = \frac{d}{ds}$

$2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$

$m^2(2m^3 - 7m^2 + 12m + 8) = 0$

$m^2 = 0$ $\quad$ $2m^3 - 7m^2 + 12m + 8 = 0$

$m = 0, 0$ $\quad$ $m = \frac{-1}{2}, \; 2 \pm 2i$

$x = e^{0s}(c_1 + c_2 x) + c_3 e^{-\frac{1}{2}s} +$

$e^{2s}(c_4 \cos 2s + c_5 \sin 2s)$

(27) $m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5$

$m^4(m+5) - 2m^2(m+5) + 1(m+5) = 0$

$(m+5)(m^4 - 2m^2 + 1) = 0$

$m = -5$ $\quad$ $m^4 - 2m^2 + 1 = 0$

$(m^2 - 1)^2 = 0$

$m = \pm 1$

$m = +1, +1, -1, -1$

(34) $y'' + 10y' + 25y = 0$

$y(0) = 1 \quad , \quad y(1) = 0$

$m^2 + 10m + 25 = 0$

$m^2 + 5m + 5m + 25 = 0$

$m(m+5) + 5(m+5) = 0$

$(m+5)^2 = 0$

$m = 5, 5$

$y = e^{5x}(c_1 + c_2 x)$

$\boxed{1 = c_1}$

$0 = c_1 + c_2$

$\boxed{c_2 = -1}$

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Q# 49 — 58

(36) $y''' + 2y'' - 5y' - 6y = 0$

$y(0) = y'(0) = 0 \quad , y''(0) = 1$

$m^3 + 2m^2 - 5m - 6 = 0$

$m = 2, -1, -3$

$y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-3x}$ — (1)

$y' = 2c_1 e^{2x} - c_2 e^{-x} - 3c_3 e^{-3x} \rightarrow$ (2)

$y'' = 4c_1 e^{2x} + c_2 e^{-x} + 9c_3 e^{-3x} \rightarrow$ (3)

eq (1) $\Rightarrow$

$0 = c_1 + c_2 + c_3$

$0 = 2c_1 - c_2 - 3c_3$

$1 = 4c_1 + c_2 + 9c_3$

$c_1 = \frac{1}{15} \quad , c_2 = \frac{-1}{6}$

$c_3 = \frac{1}{10}$ put in (A)

$m^2 + 2m + 1 - (-1) = 0$

$m^2 + 2m + 2 = 0$

$(D^2 + 2D + 2)y = 0$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

(49) $y = c_1 e^{x} + c_2 e^{5x}$

agr sirf constant term involve hongi to isko matlab homogenous hai agr

$y = c_1 e^{x} + c_2 e^{5x} + x$

this is not homogeneous

$y = c_1 e^{(1)x} + c_2 e^{5x}$

$m = 1 \quad , m = 5$

$(m - r_1)(m - r_2) = 0 \rightarrow$ factor hamesh esi hi bnte hai

$(m-1)(m-5) = 0$

$m^2 - 5m - m + 5 = 0$

$m^2 - 6m + 5 = 0$ reverse kaam karna hai.

$(D^2 - 6D + 5)y = 0$

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$

(55) $y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

$y = e^{-x}(c_1\cos x + c_2 \sin x)$

$\rightarrow y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

Comparing both eqn

$\alpha = -1 \quad , \beta = 1$

$m = -1 + i \quad , m = -1 - i$

$(m - (-1+i))(m - (-1-i)) = 0$

$(\underbrace{m+1}_{a} - \underbrace{i}_{b})(\underbrace{m+1}_{a} + \underbrace{i}_{b}) = 0$

$a^2 - b^2 = (a+b)(a-b)$

$(m+1)^2 - i^2 = 0$

Ex: 4.4

⇒ Non-Homogenous D.E with Constant Coeff:

$ay'' + by' + cy = g(x)$
$a \neq 0$

G.S → $y = y_c + y_p$
general solution

y_c = Complementry Funct.
y_p = Particular Function.

For y_c: put $g(x)=0$

$ay'' + by' + cy = 0$
↳ Homogenous D.E

Now we will solve by previous method of auxilary eqn.

For y_p:

① $g(x)$ = Constant (2, 3, 4, etc)
let, $y_p = A$
But if constant appear in y_c: let $y_p = Ax$

② $g(x)$ = poly (degree 1)
x, $x+1$, $2x-1$ etc.

let, $y_p = Ax+B$ → ①
agr y_c mein $Ax+B$ ki form aa rahi hai to eq① ko 'x' se ⊗ kerdenge.
Jese $y_c = c_1 + c_2 x$ ye or $Ax+B$ same hi hai to ab hum eq① ko $Ax^2 + Bx$ likhenge.

③ $g(x)$ = poly (degree 2)
x^2+5x-1, x^2, $2x^2+1$ etc.

let, $y_p = Ax^2 + Bx + C$

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Q#1: $y'' + 3y' + 2y = \underbrace{6}_{g(x)} \quad \to (1)$

$\therefore y = y_c + y_p \to (A)$

For y_c:

$y'' + 3y' + 2y = 0$

$D^2y + 3Dy + 2y = 0$

$y(D^2 + 3D + 2) = 0$

$m^2 + 3m + 2 = 0$

$m = -1, \quad m = -2$

$y_c = c_1e^{-x} + c_2e^{-2x}$

For y_p:

$g(x) = 6 \to$ constant

Let, $y_p = A$

$y_p' = 0, \quad y_p'' = 0$

using values in eq (1)

$0 + 3(0) + 2A = 6$

$A = 3$

$\therefore y_p = 3$

y_p ka ans $g(x)$ ki form mein ayega hamesha agr $g(x)$ constant to y_p bhi constant agr $g(x)$ $\sin()$ to y_p bhi $\sin()$ ki form mein ayega

Now eq (A)

eq (A) $\Rightarrow$

$y = c_1e^{-x} + c_2e^{-2x} + 3$

agr G.S mein c_1 or c_2 k ilawa kuch or term ho to iska matlab non-homo ka G.S hai.

Q: $y'' + 2y' = 5$

$y = y_c + y_p \to (A)$

For y_c:

$y'' - 2y' = 0$

$m^2 - 2m = 0$

$m = 0, \quad m = 2$

$y_c = c_1e^{0x} + c_2e^{2x}$

$y_c = c_1 + c_2e^{2x}$

For y_p:

$g(x) = 5 \to$ constant

Let, $y_p = Ax$

b/c constant term appears in y_c.

$\therefore y'' - 2y' = 0$

$0 - 2A = 5$

$A = \frac{-5}{2}$ $\qquad y_p = \frac{-5}{2}x$

$y = c_1 + c_2 e^{2x} - \frac{5}{2}x$

(37) $y'' + y = x^2 + 1$ → y_p ka ans bhi x^2 ki form mein ayega.

$y(0) = 5, \quad y(1) = 0$

For y_c:

$m^2 + 1 = 0$

$m^2 = -1$

$m = i, \quad m = -i$

$y_c = e^0(c_1 \cos x + c_2 \sin x)$

$y_c = c_1 \cos x + c_2 \sin x$

For y_p:

$g(x) = x^2 + 1$

let, $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$\cancel{2A + A = x^2 + 1}$

$\cancel{3A = x^2 + 1}$

$\cancel{A = \frac{x^2}{3} + \frac{1}{3}}$

$2A + Ax^2 + Bx + C = x^2 + 1$

$Ax^2 + Bx + 2A + C = x^2 + 1$

Comparing coeff

$\boxed{A = 1}$

$\boxed{B = 0}$

$2A + C = 1$

$2(1) + C = 1$

$\boxed{C = -1}$

$y_p = (1)x^2 + (0)x - 1$

$y_p = x^2 - 1$

$y = c_1 \cos x + c_2 \sin x + x^2 - 1 \rightarrow (*)$

$5 = c_1 + \cancel{0^2} - 1$

$\boxed{c_1 = 6}$

$0 = c_1 \cos 1 + c_2 \sin(1) + \cancel{1} - \cancel{1}$

$0 = 6 \cos 1 + c_2 \sin(1)$

$-6 \cos(1) = c_2 \sin(1)$

$\frac{-6 \cos(1)}{\sin(1)} = c_2$

$c_2 = -6 \cot(1)$

(*) ⟹

$y = 6 \cos x + 6 \cot(1) \sin x + x^2 - 1$

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(5) $y^{(4)} + 2y'' + y = (x-1)^2$

$\Rightarrow y = y_c + y_p$

For y_c:

$m^4 + 2m^2 + 1 = 0$

$(m^2+1)^2 = 0 \Rightarrow (m^2+1)(m^2+1)=0$

$m^2+1 = 0$

$m = \pm i, \pm i$

q k power 4 hai,

$y_c = c_1 \cos x + c_2 \sin x + c_3 \cos x + c_4 \sin x$

~~$y_c = \cos x (c_1 + c_3) + \sin x (c_2 + c_4)$~~

m ki dono pairs same aarahe hai to ek pair k saath 'x' likh denge.

$y = c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)$

For y_p:

$g(x) = (x-1)^2$

Let, $y_p = Ax^2 + Bx + C$

$, y^{(4)} + 2y'' + y = (x-1)^2$

$0 + 2(2A) + Ax^2 + Bx + C = x^2 - 2x + 1$

$4A +$

$Ax^2 + Bx + 4A + C = x^2 - 2x + 1$

$\boxed{A = 1}$

$\boxed{B = -2}$

$4A + C = 1$

$4(1) + C = 1$

$\boxed{C = -3}$

$y_p = x^2 - 2x - 3$

$y = x^2 - 2x - 3 + c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)$.

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① If $g(x) = e^{ax}$
Let $\boxed{y_p = Ae^{ax}}$

If e^{ax} in y_c then

$y_p = \underline{Axe^{ax}}$

② If $g(x) = \sin ax$ or $\cos ax$
Let,
$\boxed{y_p = A\cos ax + B\sin ax}$

If, $\cos ax$ or $\sin ax$ in y_p
then,

$y_p = \underline{x(A\cos ax + B\sin ax)}$

⑩ $y'' + 2y' = 2x + 5 - e^{-2x}$

For y_c:

$y'' + 2y' = 0$

$m^2 + 2m = 0$

$m = 0, \quad m = -2$

$y_c = c_1 + c_2 e^{-2x}$

For y_p:

$g(x) = 2x + 5 - e^{-2x}$

$y_p = Ax^2 + Bx$

↳ ye jab likhte jab sirf $2x+5$ (ek constant phele se mojood hai is waja se x se (x) krdiya)

or doosra

$y_p = Ae^{ax}$

lekin q k phele ye hamare paas y_c mein mojood hai to humne 'x' se (x) krdiya to $y_p = Axe^{ax}$

so final y_p will be,

A phele liya wa tha to C le liya

$$y_p = \underbrace{Ax^2 + Bx}_{\text{ye } 2x+5 \text{ k liye}} + \underbrace{Cxe^{-2x}}_{-e^{-2x}}$$

$y'' + 2y' = 2x + 5 - e^{-2x} \rightarrow (A)$

$y_p' = 2Ax + B + C(e^{-2x} - 2xe^{-2x})$

$y_p'' = 2A + C(-2e^{-2x} - 2(e^{-2x} - 2xe^{-2x}))$

$y_p'' = 2A - 4Ce^{-2x} + 4Cxe^{-2x}$

eq(A) ⇒

$2A - 4Ce^{-2x} + \cancel{4Cxe^{-2x}} + 4Ax + 2B + 2Ce^{-2x} - \cancel{4Cxe^{-2x}} = 2x + 5 - e^{-2x}$

Comparing Coefficient

⇒ $4A = 2 \Rightarrow \boxed{A = 1/2}$

⇒ $2A + 2B = 5$

$1 + 2B = 5 \Rightarrow \boxed{B = 2}$

⇒ $+2C = +1 \Rightarrow C = 1/2$

$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

$y = y_c + y_p$

(1) $4y'' - 4y' - 3y = \cos 2x$

For y_c:

$4m^2 - 4m - 3 = 0$

$m = -\frac{1}{2}, \frac{3}{2}$

$y_c = c_1 e^{-\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}$

For y_p:

$g(x) = \cos 2x$

Let,

$y_p = A\cos 2x + B\sin 2x$

$\Rightarrow 4y'' - 4y' - 3y = \cos 2x$

$4(-4A\cos 2x - 4B\sin 2x) - 4(-2A\sin 2x + 2B\cos 2x)$
$-3(A\cos 2x + B\sin 2x) = \cos 2x$

$-16A\cos 2x - 16B\sin 2x + 8A\sin 2x - 8B\cos 2x -$
$-3A\cos 2x - 3B\sin 2x = \cos 2x$

$(-19A - 8B)\cos 2x + (-19B + 8A)\sin 2x = \cos 2x$

• Comparing coeff.

$-19A - 8B = 1 \rightarrow (1)$

$8A - 19B = 0 \rightarrow (2)$

$A = \frac{19B}{8} \rightarrow$ put in (1)

eq (1) $\Rightarrow$ $-19\left(\frac{19B}{8}\right) - 8B = 1$

$-361B - 64B = 8$

$-425B = 8$

$\boxed{B = \frac{-8}{425}}$

$A = \frac{19}{8} \cdot \left(\frac{-8}{425}\right)$

$\boxed{A = \frac{19}{425}}$

• can use calculator to solve this eqns.

(32) $y'' - y = \cosh x$

$y(0) = 3, \quad y'(0) = 1$

For y_c:

$y'' - y = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$y_c = c_1 e^{x} + c_2 e^{-x}$

For y_p:

$g(x) = \cosh x = \frac{e^{x} + e^{-x}}{2}$

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$y_p = Ae^{x} + Be^{-x}$

ye dono y_c mein mojood hai to $\times$ by 'x'

$y_p = Axe^{x} + Bxe^{-x}$

$\Rightarrow y'' - y' = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$

$y_p' = \vdots$

⊗ $y'' + 4y = 3\sin 2x$

$y_c = C_1\cos 2x + C_2 \sin 2x$

$g(x) = 3\sin 2x$

$y_p = Ax\sin 2x + Bx\cos 2x$

'x' se multiply karaya hai q k y_c mein mojood hai.

⊗ $g(x) = e^{2x} - 5\sin x$

$y_p = Ae^{2x} + B\sin x + C\cos x$

⊗ $g(x) = x^2 + 5\cos 3x$

$y_p = Ax^2 + Bx + C + D\cos 3x + E\sin 3x$

⊗ $g(x) = \sin x + \cos 2x$

$= A\sin x + B\cos x + C\cos 2x + D\sin 2x$

$g(x)$	y_p
xe^{ax}	$(Ax+B)e^{ax}$
$(x^2+1)e^{ax}$	$(Ax^2+Bx+C)e^{ax}$

agr kabhi bhi koi polynomial, exponential k saath $\times$ hoga usko let wali cheez ko exponential ki saath likhdenge.

$g(x) = xe^{3x}$

$y_c = e^{3x}(C_1 + Cx)$

Let,

$y_p = e^{3x}(Ax+B)$

to ye ab same same hogai to 'x' se $\times$ hojayega

$y_p = e^{3x}(Ax^2+Bx)$

$\sin ax \cdot e^{ax} \rightarrow (A\sin ax + B\cos ax)e^{ax}$

$\cos ax \cdot e^{ax} \nearrow$

agr koi trigonometric ajaye to bhi usko ese hi 'e' k saath likhdenge or agr repeat horha ho to 'x' se multiply kardenge.

$y'' + 3y = -10x^2e^{3x}$

$y = y_c + y_p$

For y_c:

$y'' + 3y = 0$

$D^2y + 3y = 0$

$(D^2+3)y = 0$

$m^2 + 3 = 0$

$m = \pm\sqrt{3}\,i$

$y_c = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x$

For y_p:

$g(x) = -10x^2e^{3x}$

Let,

$y_p = (Ax^2+Bx+C)e^{3x}$

$\Rightarrow y'' + 3y = -10x^2e^{3x} \rightarrow (i)$

$y_p' = 3(Ax^2+Bx+C)e^{3x} + (2Ax+B)e^{3x}$

$y_p' = e^{3x}(3Ax^2+3Bx+3C+2Ax+B)$

$y_p'' = e^{3x}(6Ax+3B+2A) + 3e^{3x}(3Ax^2+3Bx+3C+2Ax+B)$

eq (i) — 3 se multiply krke x common le liya or add krdiya

$e^{3x}(12Ax + 6B + 9Ax^2 + 9Bx + 9C + 2A) + 3(Ax^2+Bx+C)e^{3x} = -10x^2e^{3x}$

$e^{3x}[12Ax + 6B + 12Ax^2 + 12Bx + 12C + 2A] = -10x^2e^{3x}$

Comparing Coeffi —

$12A = -10 \Rightarrow \boxed{A = -5/6}$

$12B + 12A = 0 \Rightarrow 12B + 12\left(-\frac{5}{6}\right) = 0$

$\boxed{B = \frac{5}{6}}$

$2A + 6B + 12C = 0$

$2\left(-\frac{5}{6}\right) + 6\left(\frac{5}{6}\right) + 12C = 0$

$-\frac{5}{3} + 5 + 12C = 0$

$-5 + 15 + 36C = 0$

$36C = -10$

$C = \frac{7}{12}$

$y_p =$

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$y_c = e^x(c_1 + c_2 x)$

$y_c = c_1 e^x + c_2 x e^x$

$g(x) = e^x - 3e^{3x}$

$y_p = Ae^x + Be^{-x}$ ⊗ by 'x' as 1st term is repeat.

$y_p = Axe^x + Be^{-x}$ again ⊗ by 'x'

$y_p = Ax^2 e^x + Be^{-x}$ ✓

2. $y_c = c_1 \cos 2x + c_2 \sin 2x$

$g(x) = 3\cos 2x$

$y_p = A\cos 2x + B\sin 2x$ ⊗ by 'x'

$y_p = Ax\cos 2x + Bx\sin 2x$ ✓

3. $y_c = c_1 \cos 2x + c_2 \sin 2x$

$g(x) = x + \sin x$

$y_p = (Ax+B) + C\cos x + D\sin x$ ✓

4. $y_c = c_1 e^{3x} + c_2 e^{5x}$

$g(x) = 100x - 26xe^x$

$y_p = Ax + B + (Cx + D)e^x$

5. $y_c = c_1 e^x + c_2 e^{0.23x} + c_3 e^{-1.69x}$

$g(x) = 5 - e^x + e^{2x}$

$y_p = A + Bxe^x + Ce^{2x}$ ✓

6. $y_c = c_1 + c_2 e^{-2x}$

$g(x) = 2x - 5 - e^{-2x}$

$y_p = Ax + B + Ce^{-2x}$ ⊗ by 'x'

$y_p = (Ax+B)x + Cxe^{-2x}$ ✓

7. $y_c = e^x(c_1 \cos x + c_2 \sin x)$

$g(x) = e^{2x}(\cos x - 3\sin x)$

↳ dono k angle same hai to ek hi mastaba let karenge

$y_p = e^{2x}(A\cos x + B\sin x)$ ✓

y_c mein $A\cos x + B\sin x$ e^x se ⊗ hosta hai or neeche e^{2x} se to is waja se ⊗ nhi hoga.

8. $y_c = c_1 e^{-x} + c_2 x e^{-x}$

$g(x) = \sin x + 3\cos 2x$

dono k angle differ. hai to alag let kiya h.

$y_p = A\sin x + B\cos x + C\cos 2x + D\sin 2x$ ✓

$x \cdot \sin bx \rightarrow (Ax+B)\cos bx + (Cx+D)\sin bx$

$(x+1)\sin bx$ ↗

$(x^2 + 2x)\sin 3x \rightarrow (Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x$

$(x^2 - 3)\sin ax$ ↗

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Ex: 4.6

(x) $y'' - 2y'$ $y_c = e^x(c_1\cos 2x + c_2\sin 2x)$

$y_p = ...$

$g(x) = e^x\cos 2x$

$y_p = e^x(A\cos 2x + B\sin 2x)$ (x) by 'x'

$y_p = e^x x(A\cos 2x + B\sin 2x)$

(x) $y'' + a^2 y = b\sin\omega x \rightarrow ①$

For y_c:

$m^2 + a^2 = 0$

$m = \pm ai$

$y_c = c_1\cos ax + c_2\sin ax$

For y_p:

$g(x) = b\sin\omega x$

let $y_p = A\cos\omega x + B\sin\omega x$

$y_p' = -\omega A\sin\omega x + B\omega\cos\omega x$

$y_p'' = -A\omega^2\cos\omega x - B\omega^2\sin\omega x$

eq①

$-A\omega^2\cos\omega x - B\omega^2\sin\omega x + Aa^2\cos\omega x + Ba^2\sin\omega x = b\sin\omega x$

$-B\omega^2 + Ba^2 = b$ $\qquad B = \dfrac{b}{a^2-\omega^2}$

$-B\omega^2 + Aa^2 = 0$

$\boxed{A = 0}$

Solution of Non-Homogenous D.E by Variation of Parameters

↳ ye doosra method hai non-homogenous

$ay'' + by' + cy = f(x)$ to solve karne ke liye.

$y = y_c + y_p$

$y_c = c_1 y_1 + c_2 y_2$

$y_p = u_1 y_1 + u_2 y_2$

We have,

$u_1 = \int \frac{W_1}{W}dx$, $u_2 = \int \frac{W_2}{W}dx$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

$Q2 = y'' + y = \tan x$ coefficent must be '1'

$y = y_c + y_p$

For y_c:

$y'' + y = 0$

$D^2 y + y = 0$

$(D^2 + 1) y = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = \underbrace{c_1 \cos x}_{y_1} + \underbrace{c_2 \sin x}_{y_2}$

For y_p:

$\to y_p = u_1 y_1 + u_2 y_2$

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$y_1' = -\sin x \quad , \quad y_2' \cos x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}$

$= -\tan x \sin x$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}$

$W_2 = \cos x \tan x = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$

$u_1 = \int \frac{-\tan x \sin x}{1}$

$= -\int \frac{\sin^2 x}{\cos x}$

$= -\int \frac{(1-\cos^2 x)}{\cos x}$

$= -\int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$

$= -\int \sec x \, dx + \int \cos x \, dx$

$= -\ln(\sec x + \tan x) + \sin x$

$u_2 = \int \frac{\sin x}{1}$

$u_2 = -\cos x$

$y_p = [-\ln(\sec x + \tan x) + \sin x] \cos x - \cos x \sin x$

$\boxed{y = c_1 \cos x + c_2 \sin x + [-\ln(\sec x + \tan x) + \sin x] \cos x - \cos x \sin x.}$

$y'' + 9y = \sec 3x$

For y_c:

$y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \pm 3i$

$y_c = c_1 \cos 3x + c_2 \sin 3x$

$y_1 = \cos 3x \quad , \quad y_2 = \sin 3x$

$y_1' = -3\sin 3x \quad , \quad y_2' = 3\cos 3x$

by variation of parameter wale method se sare sawal solve harsakte hai lekin undertmined coefficent wala method sirf polynomial, exponential, constants, sinx, cosx par valid hai.

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$W = 3\cos^2 3x + 3\sin^2 3x$

$W = 3$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \sec 3x & 3\cos 3x \end{vmatrix}$$

$= \sec 3x \cdot \sin 3x$

$W_1 = \tan 3x$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \sec 3x \end{vmatrix}$$

$= \cos 3x \cdot \sec 3x$

$W_2 = 1$

$$U_1 = \int \frac{\tan 3x}{3}$$

$U_1 = \frac{1}{9} \ln(\sec 3x)$

$$U_2 = \int \frac{1}{3} dx$$

$U_2 = \frac{1}{3} x$

$y_p = \frac{1}{9} \ln(\sec 3x) \cos 3x + \frac{1}{3} x \sin 3x$

(8) $y'' - y = \sinh 2x$

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$y_c = C_1 e^x + C_2 e^{-x}$

$y_1 = e^x \qquad y_2 = e^{-x}$

$y_1' = e^x \qquad y_2' = -e^{-x}$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$= -e^0 - e^0 = -2$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \sinh 2x & -e^{-x} \end{vmatrix}$$

$= -\sinh 2x \cdot e^{-x}$

$= -e^{-x}\left[\frac{e^{2x} - e^{-2x}}{2}\right]$

$W_1 = \frac{-e^{x} + e^{-3x}}{2}$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \sinh 2x \end{vmatrix}$$

$= e^x \sinh 2x$

$= e^x\left[\frac{e^{2x} - e^{-2x}}{2}\right] = \frac{e^{3x} - e^{-x}}{2}$

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$$U_1 = \int \frac{\frac{-e^x + e^{-3x}}{2}}{+2}$$

$$= \int \frac{-e^x + e^{-3x}}{-4}$$

$$= -\frac{1}{4}\left[\int -e^{-x} + \int e^{-3x}\right] dx$$

$$\boxed{U_1 = \frac{-1}{4}\left[-e^x - \frac{1}{3}e^{-3x}\right]}$$

$$U_2 = \int \frac{\frac{e^{3x} - e^{-x}}{2}}{-2}$$

$$= -\frac{1}{4}\left[\int e^{3x} - \int e^{-x}\right] dx$$

$$\boxed{U_2 = -\frac{1}{4}\left[\frac{1}{3}e^{3x} + e^{-x}\right]}$$

(12) $y'' - 2y' + y = \frac{e^x}{1+x^2}$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y_c = e^x(c_1 + c_2 x)$

$y_c = c_1 e^x + c_2 x e^x$

$y_1 = e^x \qquad y_2 = xe^x$

$y_1' = e^x \qquad y_2' = e^x + xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$$

$$W = e^{2x} + \cancel{2xe^{2x}}$$

$=$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & e^x + xe^x \end{vmatrix}$$

$$= -xe^x\left[\frac{e^x}{1+x^2}\right]$$

$$W_1 = \frac{-xe^{2x}}{1+x^2}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix}$$

$$W_2 = \frac{e^{2x}}{1+x^2}$$

$$U_1 = \int \frac{\frac{-xe^{2x}}{1+x^2}}{e^{2x}}$$

$$= -\int \frac{x}{1+x^2}\, dx$$

$$U_1 = -\frac{1}{2}\ln(1+x^2)$$

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23 to 26
integeratio not possib!

$$U_2 = \int \frac{\frac{e^{x}}{1+x^2}}{e^{x}}$$

$$= \int \frac{1}{1+x^2}$$

$$U_2 = \tan^{-1}(x)$$

$$y''' + 4y' = \sec 2x$$

For y_c:

$$y''' + 4y' = 0$$

$$m^3 + 4m = 0$$

$$m = 0, \quad m = \pm 2i$$

$$y_c = C_1 + C_2 \cos 2x + C_3 \sin 2x$$

For y_p:

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

$y_1 = 1$ $\quad y_2 = \cos 2x$ $\quad y_3 = \sin 2x$

$y_p' = 0$ $\quad y_2' = -2\sin 2x$ $\quad y_3' = 2\cos 2x$

$y_p'' = 0$ $\quad y_2'' = -4\cos 2x$ $\quad y_3'' = -4\sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

Exp

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}$$

Ex: 4.7

Cauchy Euler D.E

$$ax^2y'' + bxy' + cy = f(x)$$

y'' k saath x^2 hona chahiye y' k saath x. To hi we Cauchy eqn kehlaigi.

Let, $y = x^m$

For y_c:

$$ax^2y'' + bxy' + cy = 0$$

Case #01 → Roots unequal / Real

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

Case #02 → Roots equal / Real

$$y_c = x^m (c_1 + c_2 \ln x)$$

Case #03 → Complex Roots

$$y_c = x^{\alpha}(c_1 \cos\beta \ln x + c_2 \sin\beta \ln x)$$

y_p → will be found by variation of parameter.

- The coefficent of y'' must be 1 to calculate y_p.

(6) $x^2y'' + 5xy' + 3y = 0 \rightarrow (1)$

ye cauchy euler D.E hai bcz

x^2 with y'' & x with y'.

This is homogenous so no y_p $(y_p = 0)$

$y = y_c + y_p$

$y = y_c + 0$

$y = y_c$

Let, $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

put in eq (1)

$x^2[m(m-1)x^{m-2}] + 5x[mx^{m-1}] + 3x^m = 0$

$m(m-1)x^m + 5mx^m + 3x^m = 0$

$(m(m-1) + 5m + 3)\, x^m = 0$

$m(m-1) + 5m + 3 = 0$ $\quad x^m \neq 0$ as it is $= to\ y$

$m^2 - m + 5m + 3 = 0$

$m^2 + 4m + 3 = 0$

$m = -1, \quad m = -3$

$y_c = c_1 x^{-1} + c_2 x^{-3}$

$y_c = \dfrac{c_1}{x} + \dfrac{c_2}{x^3}$

$y = y_c + y_p$

$y = \dfrac{c_1}{x} + \dfrac{c_2}{x^3}$

$\Rightarrow x^2y'' = m(m-1)$

$\Rightarrow xy' = m$

$\Rightarrow x^3y''' = m(m-1)(m-2)$

direct put krenge

• $m = 0, 1, 2$

$y = c_1 x^0 + C_2 x^1 + C_3 x^2$

• $m = 2, 3, 3$

$y = x^3(c_1 + c_2 \ln x) + C_3 x^2$

• $m = 1 \pm 2i, 5$

$y_c = c_1 x^5 + x^1(C_2 \cos 2\ln x + C_3 \sin 2\ln x)$

• $m = 2, 2, 4, 4$

$y_c = x^2(c_1 + c_2 \ln x) + x^4(c_3 + c_4 \ln x)$

(19) $xy'' - 4y' = x^4$

(x) b/s by x $\rightarrow$ take x^2 miljaye

$x^2y'' - 4xy' = x^5$

$y = y_c + y_p$

For y_c:

$x^2y'' + 4xy' = 0$

$m(m-1) - 4m = 0$

$m^2 - m - 4m = 0$

$m^2 - 5m = 0$

$m = 0, \quad m = 5$

$y_c = c_1 x^0 + c_2 x^5 = c_1 + c_2 x^5$

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For y_p:

$x^2y'' - 4xy' = x^5$

$\div$ b/s by x^2

$y'' - \frac{4}{x}y' = x^3 \longrightarrow f(x)$

$y_p = U_1y_1 + U_2y_2$

$y_1 = 1 \qquad y_2 = x^5$

$y_1' = 0 \qquad y_2' = 5x^4$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$W = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix}$

$W = 5x^4$

$W_1 = \begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}$

$W_1 = -x^8$

$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}$

$W_2 = x^3$

$U_1 = \int \frac{W_1}{W}dx \quad , \quad U_2 = \int \frac{W_2}{W}dx$

$U_1 = \int \frac{-x^8}{5x^4}dx \quad , \quad U_2 = \int \frac{x^3}{5x^4}dx$

$U_1 = \frac{-1}{5}\int x^4dx \qquad U_2 = \frac{1}{5}\int \frac{1}{x}dx$

$U_1 = \frac{-1}{25}x^5 \qquad U_2 = \frac{1}{5}\ln x$

$y = y_c + y_p$

(x) $x^2y'' - 2xy' + 2y = x^4e^x$

$m(m-1) - 2m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m = 1, \; m = 2$

$y_c = c_1x + c_2x^2$

For y_p: $\div$ by x^2

$y'' + \frac{2}{x}y' + \frac{3}{x^2}y = x^2e^x$

$y_1 = x \qquad y_2 = x^2$

$y_1' = 1 \qquad y_2' = 2x$

$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$$= \begin{vmatrix} 0 & x^2 \\ x^3 e^x & 2x \end{vmatrix}$$

$$= -x^4 e^x$$

$$= |$$

Ex # 13.5

(Chain Rule)

If $z = x^2 y$, $x = t^2$ & $y = t^3$ then find $\frac{dz}{dt}$

$$\begin{tikzcd} & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\ t & & t \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy)(2t) + (x^2)(3t^2)$$

$$\frac{dz}{dt} = 4xyt + 3x^2t^2$$

$$\frac{dz}{dt} = 4(t^2)(t^3)(t) + 3(t^2)^2 t$$

$$\frac{dz}{dt} = 4t^6 + 3t^6 = 7t^6$$

$W = \sqrt{x^2+y^2+z^2}$, $x = \cos\theta$

$y = \sin\theta$, $z = \tan\theta$

Find $\frac{dw}{d\theta}$

$$\begin{array}{ccc} & w & \\ \swarrow & \downarrow & \searrow \\ x & y & z \\ \downarrow & \downarrow & \downarrow \\ \theta & \theta & \theta \end{array}$$

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$=$

(23) $T = x^2y - xy^3 + 2$

$x = r\cos\theta$ + $y = r\sin\theta$

Find $\frac{\partial T}{\partial r}$ & $\frac{\partial T}{\partial \theta}$

$$\begin{array}{ccccccc} & & & T & & & \\ & & \swarrow^{\frac{\partial T}{\partial x}} & & \searrow^{\frac{\partial T}{\partial y}} & & \\ & x & & + & & y & \\ \swarrow^{\frac{\partial x}{\partial r}} & & \searrow^{\frac{\partial x}{\partial \theta}} & & \swarrow^{\frac{\partial y}{\partial r}} & & \searrow^{\frac{\partial y}{\partial \theta}} \\ r & & \theta & & r & & \theta \end{array}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}$$

humein yaha $\frac{\partial T}{\partial r}$ chahiye that is waja se $\frac{\partial x}{\partial r}$ wale ko choose kia hai, is waja se $\frac{\partial x}{\partial \theta}$ nhi likha

$$\frac{\partial T}{\partial r} = (2xy - y^3)(\cos\theta) + (x^2 - 3xy^2)(\sin\theta)$$

$= [2(r\cos\theta)(r\sin\theta) - r^3\sin^3\theta][\cos\theta] +$
$[r^2\cos^2\theta - 3(r\cos\theta)(r^2\sin^2\theta)][\sin\theta]$

$= 2r^2\cos^2\theta\sin\theta - r^3\sin^3\theta\cos\theta +$
$r^2\cos^2\theta\sin\theta - 3r^3\cos\theta\sin^3\theta$

$= 3r^2\cos^2\sin\theta - 4r^3\sin^3\theta\cos\theta$

$= \cancel{r^2[3\cos\theta}$

$= r^2\cos\theta\sin\theta[3\cos\theta - 4r\sin^2\theta]$

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta}$$

$= (2xy - y^3)(-r\sin\theta) + (x^2 - 3xy^2)(r\cos\theta)$

Find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$ if $z = \frac{x}{y}$

$x = 2\cos u$, $y = 3\sin v$

$$\begin{array}{ccc} & z & \\ \swarrow^{\frac{\partial z}{\partial x}} & & \searrow^{\frac{\partial z}{\partial y}} \\ x & & y \\ \frac{dx}{du}\downarrow & & \downarrow\frac{dy}{dv} \\ u & & v \end{array}$$

iskonhi lenge ∂v k isme saath v rakhna hai

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \cancel{\frac{\partial z}{\partial y}}(\quad)$$

ye req tha q k humein $\frac{\partial z}{\partial u}$ chahiye

→ yaha $\frac{\partial y}{\partial r}$ is liye nhi liya q k humein yaha $\frac{\partial z}{\partial v}$ req nhi hai.

Ex: 13-8

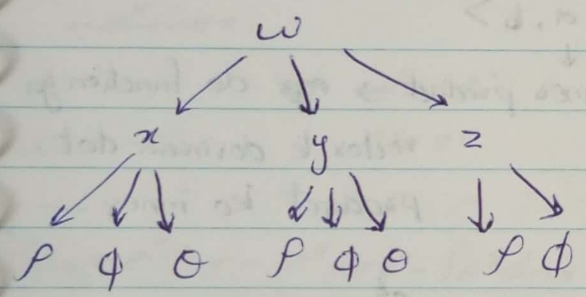
$$\frac{\partial z}{\partial v} = \cancel{\frac{\partial z}{\partial x}} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$w = 4x^2 + 4y^2 + z^2$

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \cos\theta$

$z = \rho \cos\phi$

Find $\frac{\partial w}{\partial \rho}$, $\frac{\partial w}{\partial \phi}$, $\frac{\partial w}{\partial \theta}$



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \phi} = \text{"} \frac{\partial x}{\partial \phi} + \text{" "} \frac{}{\partial \phi} + \text{"} \frac{\text{"}}{\partial \phi}$$

$$\frac{\partial w}{\partial \theta} = \text{"} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \cancel{\frac{\partial w}{\partial z}}$$

→ Max & Min value of Two variable Function.

$F(x,y) =$ → ①

$f_x = \frac{\partial f}{\partial x} = 0$

$f_y = \frac{\partial f}{\partial y} = 0$

inko solve karne se 2 eqn milenge un dono ko solve kar ke x or y nikaal lenge

Value of x & y will be as critical point. (x_0, y_0)

double partial derivative of f_x

f_{xx} → put (x_0, y_0)

f_{yy} → put (x_0, y_0)

f_{xy} → put (x_0, y_0)

first w.r.t x that w.r.t y (partial)

$D = f_{xx} f_{yy} - (f_{xy})^2$

① IF $D > 0$ & $f_{xx} > 0$ so, f is min at (x_0, y_0)

② IF $D > 0$ & $f_{xx} < 0$, so f is max at (x_0, y_0)

③ If $D < 0$ then f has saddle point

④ If $D = 0$, the no conclusion can be drawn.

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$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$ if $a>0, b>0$

$D>0$, $f_{xx}>0 \rightarrow$ min

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$, $a>0, b<0$

$D>0$, $f_{xx}<0$, maxima

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$, $a<0, b>0$

$f_{xx}<0$, $D>0$, maxima

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$ $a<0, b<0$

$D>0$, $f_{xx}>0$ max

—— x —— x —— x ——

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\sin\alpha\cancel{\cos}\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

[CHP # ...

Orthogonal

$\vec{a} = (1,0)$

$\vec{b} = (0,1)$

$a \cdot b = i + j = 0$

$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$

if ~~dot product~~

$\langle a, b \rangle$

inner product → ~~eg~~ ...

vector k do... product k...

$\langle f_1, f_2 \rangle = \int_a^b f_1 \cdot f_2 \, dx$

if ans of $\langle f_1, f_2 \rangle = 0$

f_1 & f_2 are orthogonal

(2) $f_1(x) = x^3$, $f_2(x) = x$...

$\langle f_1, f_2 \rangle = \int_a^b f_1 \cdot f_2 \, dx$

$\langle f_1, f_2 \rangle = \int_{-1}^{1} x^3 \cdot (x^2+1) \, d...$

$= \int_{-1}^{1} x^5 dx + \int_{-1}^{1} x^3 ...$

$= \frac{x^6}{6}\Big|_{-1}^{1} + \frac{x^4}{4}\Big|_{-1}^{1}$

$= \left(\frac{1}{6} - \frac{1}{6}\right) + \left(\frac{1}{4} ...\right.$

$= 0$

f_1 & f_2 are orthogonal