22K-4187 → SYED SAMROZE ALI ①

ASSIGNMENT #02

QUESTION #02

(a) $x^2y'' - 3xy' + 5y = 0$; $y_1 = x^2\cos(\ln x)$

$$y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0$$

$$y_2 = x^2\cos(\ln x)\int \frac{e^{-\int -\frac{3}{x}dx}}{x^4\cos^2(\ln x)}dx$$

$$= x^2\cos(\ln x)\int \frac{e^{\ln x^3}}{x^4\cos^2(\ln x)}dx$$

$$= x^2\cos(\ln x)\int \frac{\cancel{x^3}}{x^4\cos^2(\ln x)}dx$$

$$= x^2\cos(\ln x)\int \sec^2(\ln x)\cdot\frac{1}{x}dx$$

$$= x^2\cos(\ln x)\cdot\tan(\ln x)$$

$$y_2 = x^2\sin(\ln x)$$

$$\boxed{y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)}$$ ↳ Ans.

(b) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$

$y_1 = x+1$

$$y'' + \frac{2(1+x)}{(1-2x-x^2)}y' - \frac{2y}{(1-2x-x^2)} = 0$$

$$y'' + \frac{2(1+x)}{1-(2x+x^2)}y' - \frac{2y}{(1-2x-x^2)} = 0$$

$$y'' + \frac{2(1+x)}{1-[2x+x^2+1-1]}y' - \frac{2y}{(1-2x-x^2)} = 0$$

$$y'' + \frac{2(1+x)}{2-(1+x)^2}y' - \frac{2y}{(1-2x-x^2)} = 0$$

$$y_2 = (x+1)\int \frac{e^{-\int \frac{2(1+x)}{2-(1+x)^2}}}{(x+1)^2}$$

e_x

$$y_2 = (x+1)\int \frac{e^{\ln[2-(x+1)^2]}}{(x+1)^2}$$

$$= (x+1)\int \frac{2-(x+1)^2}{(x+1)^2}$$

$$= (x+1)\int \left(\frac{2}{(x+1)^2} - 1\right) dx$$

$$= (x+1)\left[\frac{-2}{(x+1)} - x\right]$$

$$= -(x+1)\left[\frac{2+x(x+1)}{(x+1)}\right]$$

$$y_2 = -2 - x^2 - x.$$

$$y_2 = 2 + x^2 + x$$

$$\boxed{y = c_1(x+1) + c_2(2+x^2+x)}$$

↳ Ans

22K-4187 ③

(QUESTION #03)

1. $y'' - y = \cosh x$

$y_c = c_1 \cosh x + c_2 \sinh x$

$y_p = Ax\cosh x + Bx\sinh x$

2. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

$y_c = e^{x}(c_1\cos x + c_2 \sin x)$

$y_p = Ae^{2x}\cos x + Be^{2x}\sin x$

3. $y'' + 2y' + y = \sin x + 3\cos 2x$

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_p = A\cos x + B\sin x + C\cos 2x + D\sin 2x$

4. $y''' + 8y = 2x - 5 + 8e^{-2x}$

$y_c = c_1 e^{-2x} + e^{x}(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x)$

$y_p = Ax + B + Cxe^{-2x}$

5. $y'' - 4y = (x^2 - 3)\sin 2x$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$y_p = (Ax^2 + Bx + C)\cos 2x + (Dx^2 + Ex + F)\sin 2x$

6. $y'' - 2y' + 5y = e^{x}\cos 2x$

$y_c = e^{x}(c_1\cos 2x + c_2\sin 2x)$

$y_p = Axe^{x}\cos 2x + Bxe^{x}\sin 2x$

④

22K-4187

7. $y''' - 3y'' + 3y' - y = x - 4e^x$

$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$y_p = Ax + B + Cx^3 e^x$

8. $y^{(4)} - y'' = 4x + 2xe^{-x}$

$y_c = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$

$y_p = Ax^3 + Bx^2 + (Cx^2 + Dx)e^{-x}$

9. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}$

$y_c = c_1 + c_2 e^x + c_3 x e^x$

$y_p = Ax + Bx^2 e^x + Ce^{5x}$

10. $y''' - 6y'' = 3 - 3\cos x$

$y_c = c_1 + c_2 x + c_3 e^{6x}$

$y_p = Ax^2 + B\cos x + C\sin x$

11. $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

$y_c = c_1 e^{-6x} + c_2 e^{4x}$

$y_p = A + (Bx^2 + Cx)e^{4x}$

QUESTION #02

(a) $y'' + y = 2xe^{2x} \rightarrow (1)$

$D^2y + y = 0$

$(D^2 + 1)y = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

For y_p:

$\because g(x) = 2xe^{2x}$

$y_p = (Ax + B)e^{2x}$

$y_p' = 2e^{2x}(Ax+B) + Ae^{2x}$

$y_p'' = 4e^{2x}(Ax+B) + 2Ae^{2x} + 2Ae^{2x}$

$= e^{2x}[4Ax + 4B + 4A]$

eq (1) ⇒

$e^{2x}[4Ax + 4B + 4A] + e^{2x}(Ax + B) = 2xe^{2x}$

$e^{2x}[5Ax + 5B + 4A] = 2xe^{2x}$

$5A = 2$

$\boxed{A = 2/5}$

$5B + 4A = 0$

$B = \frac{-8}{25}$

$y_p = \left(\frac{2}{5}x - \frac{8}{25}\right)e^{2x}$

$$\boxed{y = c_1 \cos x + c_2 \sin x + \left(\frac{2x}{5} - \frac{8}{25}\right)e^{2x}}$$

22K-4187

6

(b) $y'' - 3y' = 8e^{3x} + 4\sin x$] ①

$D^2y - 3Dy = 0$

$(D^2 - 3D)y = 0$

$m^2 - 3m = 0$

$m(m-3) = 0$

$m = 0 \quad , m = 3$

$y_c = c_1 + c_2 e^{3x}$

For y_p:

$\Rightarrow g(x) = 8e^{3x} + 4\sin x$

Let,

$y_p = Axe^{3x} + B\cos x + C\sin x$

$y_p' = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$

$y_p'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x} - B\cos - C\sin x$

$= e^{3x}[9Ax + 6A] - B\cos x - C\sin x$

eq ① ⇒

$e^{3x}[9Ax + 6A] - B\cos x - C\sin x$
$- 9Axe^{3x} - 9Ae^{3x} + 9B\sin x$
$- 9C\cos x = 8e^{3x} + 4\sin x$

$e^{3x}[\cancel{9Ax} + 6A - \cancel{9Ax} + 3A]$
$+ \cos x[-B - 3C]$
$+ \sin x[-C + 3B] =$
$8e^{3x} + 4\sin x$

$3A = 8$

$\boxed{A = \frac{8}{3}}$

$-C + 3B = 4 \rightarrow ②$

$-3C + B = 0 \rightarrow ③$

(x) eq ③ by '3' & add with ②

$-C + \cancel{3B} = 4$

$-9C - \cancel{3B} = 0$

$-10C = 4$

$\boxed{C = -2/5}$

$-\left(\frac{-2}{5}\right) + 3B = 4$

$2 + 15B = 20.$

$$\boxed{B = \frac{6}{5}}$$

$y_p = \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x + \frac{2}{5}\sin x$

$y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x$

$- \frac{2}{5}\sin x \rightarrow$ Ans

(c) $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$ ↳ (1)

$D^2 y - 6Dy + 9y = 0$

$(D^2 - 6D + 9)y = 0$

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3, 3$

$y_c = c_1 e^{3x} + c_2 x e^{3x}$

For y_p:

$\Rightarrow g(x) = 6x^2 + 2 - 12e^{3x}$

Let,

$y_p = Ax^2 + Bx + C + Dx^2 e^{3x}$

$y_p' = 2Ax + B + 2Dxe^{3x} + 3Dx^2 e^{3x}$

$y_p'' = 2A + 6Dxe^{3x} + 6Dxe^{3x} + 9Dx^2 e^{3x}$

$= 2A + e^{3x}[12Dx + 2D + 9Dx^2]$

eq(1) ⇒

$2A + e^{3x}[12Dx + 2D + 9Dx^2] - 12Ax - 6B - 12Dxe^{3x} - 18Dx^2 e^{3x} + 9Ax^2 + 9Bx + 9C + 9Dx^2 e^{3x} = 6x^2 + 2 - 12e^{3x}$

$e^{3x}[\cancel{12D}x + 2D + \cancel{9D}x^2 - \cancel{12D}x - \cancel{18D}x^2 + \cancel{9D}x^2] - 12Ax + 9Bx + 9Ax^2 + 2A - 6B + 9C = 6x^2 + 2 - 12e^{3x}$

Comparing Coefficents

$2D = -12$

$\boxed{D = -6}$

$9A = 6$

$\boxed{A = 2/3}$

$-12A + 9B = 0$

$-12(2/3) + 9B = 0$

$\boxed{B = \frac{8}{9}}$

$2A - 6B + 9C = 2$

$2(2/3) - 6(8/9) + 9C = 2$

$\boxed{C = 2/3}$

$$\boxed{y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} + 6x^2e^{3x}}$$

↳ Ans.

(d) $y''' + 12y'' + 36y' = 0$

$y(0)=0,\ y'(0)=1,\ y''(0)=-7$

$m^3 + 12m^2 + 36m = 0$

$m(m^2 + 12m + 36) = 0$

$m = 0 \quad m^2 + 12m + 36 = 0$

$(m+6)^2 = 0$

$m = -6, -6$

$y_c = c_1 + c_2e^{-6x} + c_3xe^{-6x}$

$\because y(0) = 0$

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$y' = -6c_2e^{-6x} - 6c_3xe^{-6x} + c_3e^{-6x}$

$1 = -6c_2 + c_3 \rightarrow (1)$

$y'' = 36c_2 + 36c_3xe^{-6x} - 6c_3e^{-6x} - 6c_3e^{-6x}$

$-7 = 36c_2 - 12c_3 \rightarrow (2)$

× eq ① by '6' & add with eq ②

$6 = -\cancel{36c_2} + 6c_3$

$-7 = \cancel{36c_2} - 12c_3$

$-1 = -6c_3$

$\boxed{c_3 = 1/6}$

$1 = -6c_2 + 1/6$

$\boxed{c_2 = \frac{-5}{36}}$

$\boxed{c_1 = \frac{5}{36}}$

$\boxed{y_c = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}}$

↳ Ans.

22K-4187

QUESTION # 04

b)

$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\omega t$; $x(0)=0$ → (1)

$x'(0)=0$

$D^2x + \omega^2 x = 0$

$(D^2+\omega^2)x = 0$

$m^2+\omega^2 = 0$

$m = \pm\omega i$

$x_c = c_1\cos\omega t + c_2\sin\omega t$

For y_p:-

$\Rightarrow g(x) = F_0\sin\omega t$

Let,

$x_p = A t\cos\omega t + B t\sin\omega t$

$x_p' = -A\omega t\sin\omega t + A\cos\omega t + B\omega t\cos\omega t + B\sin\omega t$

$x_p'' = -A\omega^2 t\cos\omega t - A\omega\sin\omega t - A\omega\sin\omega t - B\omega\cos\omega t - B\omega^2 t\sin\omega t + B\omega\cos\omega t$

eq (1) ⇒

$\cos\omega t[-A\omega^2 t + 2B\omega + At] + \sin\omega t[-B\omega^2 t - 2A\omega + Bt] = F_0\sin\omega t$

$\Rightarrow -2A\omega = F_0$

$\boxed{A = \frac{-F_0}{2\omega}}$

$2B = 0$

$\boxed{B=0}$

$x_p = -\frac{F_0}{2\omega} t\cos\omega t$

$x = c_1\cos\omega t + c_2\sin\omega t - \frac{F_0}{2\omega} t\cos\omega t$

$\because x(0)=0$

$0 = c_1 - \frac{F_0 t}{2\omega} \Rightarrow \boxed{c_1 = \frac{F_0 t}{2\omega}}$

$x' = -c_1\omega\sin\omega t + c_2\omega\cos\omega t + \frac{F_0 t\sin\omega t}{\omega} - \frac{F_0}{2\omega}\cos\omega t$

$\because x'(0)=0$

$0 = 0 + c_2\omega - \frac{F_0}{2\omega}$

$\boxed{c_2 = \frac{F_0}{2\omega^2}}$

$$x = \frac{F_0 t}{2\omega}\cos\omega t + \frac{F_0}{2\omega^2}\sin\omega t - \frac{F_0 t}{2\omega}\cos\omega t$$

↳ Ans.

c) $y'' - 2y' + 10y = g(x)$

$g(0) = 0 , \quad y'(0) = 0$

$$g(x) = \begin{cases} 20 & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

$D^2y - 2Dy + 10y = 0$

$m^2 - 2m + 10 = 0$

$m = 1 \pm 3i$

$y_c = e^x c_1 \cos 3x + e^x c_2 \sin 3x$

WHEN $g(x) = 20 \quad 0 \le x \le \pi$

$y_p = A , \; y_p' = 0 , \; y_p'' = 0$

$0 - 2(0) + 10A = 20$

$\boxed{A = 2}$

$y_p = 2$

$y = e^x c_1 \cos 3x + e^x c_2 \sin 3x + 2$

$\because y(0) = 0$

$0 = c_1 + 2$

$\boxed{c_1 = -2}$

$y' = e^x c_1 \cos x - 3e^x c_1 \sin x$
$\quad e^x c_2 \sin 3x + 3e^x c_2 \cos 3x$

$\because y'(0) = 0$

$0 = c_1 + 3c_2$

$\boxed{c_2 = 2/3}$

$y = -2e^x \cos 3x + 2/3 e^x \sin 3x + 2$

As the function is continuous

$\lim_{x \to \pi^-} (-2e^x \cos 3x + 2/3 e^x \sin 3x + 2)$
$= \lim_{x \to \pi^+} (e^x c_3 \cos 3x + e^x c_4 \sin 3x)$

Applying limit

$2e^\pi + 2 = -e^\pi c_3$

$\boxed{c_3 = -2e^{-\pi}(e^\pi + 1)}$

Now applying Continuity for y'

$\lim_{x \to \pi^-} (-2e^x \cos 3x + 6e^x \sin x +$
$2/3 e^x \sin 3x + 2e^x \cos 3x) =$
$\lim_{x \to \pi^+} (e^x c_3 \cos x - 3e^x c_3 \sin x$
$+ e^x c_4 \sin 3x + 3e^x c_4 \cos 3x)$

Applying limit,

$-2e^\pi - 2e^\pi = -e^\pi c_3 - 3e^\pi c_4$

$e^{\pi}C_3 = -3e^{\pi}C_4$

$C_4 = -\frac{2e^{-\pi}(e^{\pi}+1)}{3}$

$$y(x) = \begin{cases} e^{x}(-2\cos 3x + \frac{2}{3}\sin 3x) + 2 & 0 \le x \le \pi \\ (1+e^{\pi})e^{x-\pi}(-2\cos 3x + \frac{2}{3}\sin 3x) & x > \pi \end{cases}$$

↳ Ans.

(a) $y'' + y = 4x + 10\sin x,\ y(\pi) = 0,\ y'(\pi) = 2$

$D^2y + y = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = C_1\cos x + C_2\sin x$

For y_p:-

$\Rightarrow g(x) = 4x + 10\sin x$

Let,

$y_p = (Ax+B) + Cx\cos x + Dx\sin x$

$y_p' = A + C\cos x - Cx\sin x + D\sin x + Dx\cos x$

$y_p'' = -C\sin x - Cx\cos x - C\sin x + D\cos x + D\cos x - Dx\sin x$

$-2C\sin x - \cancel{Dx\sin x} + 2D\cos x - \cancel{Cx\cos x} + Ax + B + \cancel{Cx\cos x} + \cancel{Dx\sin x} = 4x + 10\sin x$

$\sin x[-2C] + \cos x[2D] + Ax + B = 4x + 10\sin x$

$\boxed{A = 4}$

$\boxed{B = 0}$

$-2C = 10$

$\boxed{C = -5}$

$2D = 0 \Rightarrow \boxed{D = 0}$

$y = C_1\cos x + C_2\sin x + 4x - 5x\cos x$

$y(\pi) = 0$

$0 = -C_1 + 4\pi + 5\pi$

$\boxed{C_1 = 9\pi}$

$y' = -C_1\sin x + C_2\cos x + 4 - 5\cos x + 5x\sin x$

$y'(\pi) = 2$

$2 = -C_2 + 4 + 5$

$\boxed{C_2 = 7}$

$\boxed{y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x}$

↳ Ans.

$y'' + 4y = g(x)$; →(1)
$y(0) = 1$, $y'(0) = 2$

$g(x) = \begin{cases} \sin x & 0 \le x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}$

$m^2 + 4 = 0$
$m^2 = -4$
$m = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

~~For y_p:-~~

WHEN $g(x) = \sin x$ $0 \le x \le \frac{\pi}{2}$

Let,
$y_p = A\cos x + B\sin x$
$y_p' = -A\sin x + B\cos x$
$y_p'' = -A\cos x - B\sin x$

eq(1) ⇒

$-A\cos x - B\sin x + 4A\cos x + 4B\sin x$
$= \sin x$

$3A\cos x + 3B\sin x = \sin x$

~~$(-A + 4B)\cos x + (-4A - B)\sin x$~~
~~$= \sin x$~~

$3A = 0$
$\boxed{A = 0}$
$3B = 1$
$\boxed{B = 1/3}$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x$

$y(0) = 1$
$\boxed{1 = c_1}$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x + \frac{1}{3}\cos x$

$y'(0) = 2$
$2 = 2c_2 + \frac{1}{3}$

$\boxed{c_2 = 5/6}$

$y = \cos 2x + \frac{5}{6}\sin 2x + \frac{1}{3}\sin x$

22K-4187

(14)

As the function is continuous,

$$\lim_{x\to\frac{\pi}{2}^-}\left(\cos 2x+\frac{5}{6}\sin 2x+\frac{1}{3}\sin x\right)=\lim_{x\to\frac{\pi}{2}^+}\left(c_3\cos 2x+c_4\sin 2x\right)$$

Applying limit,

$$\cancel{\frac{5}{6}}+\cancel{\frac{1}{3}}\ \cancel{=c_4}$$

$$-1=-c_3$$

$$\boxed{c_3=1}$$

Now for y',

$$\lim_{x\to\frac{\pi}{2}^-}\left(-2\sin 2x+\frac{5}{3}\cos 2x+\frac{1}{3}\cos x\right)=\lim_{x\to\frac{\pi}{2}^+}\left(-2c_3\sin 2x+2c_4\cos 2x\right)$$

$$-\frac{5}{3}=-2c_4$$

$$\boxed{c_4=5/6}$$

$$\boxed{y(x)=\begin{cases}\cos 2x+\frac{5}{6}\sin 2x+\frac{1}{3}\sin x & 0\le x\le \pi/2\\ \frac{2}{3}\cos 2x+\frac{5}{6}\sin 2x & x>\pi/2\end{cases}}$$

↳ Ans.