ordinary ~~original~~ derivative mein ek dependent variable ek hi independent par depend krega.

$y \rightarrow$ dep x

$\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$

$y^{(4)} \rightarrow$ iska matlab 4th derivative.

$\ddot{y} \rightarrow$ iska matlab bhi 2nd derivative

Partial derivative mein one depending variable depends on more than one variable.

$U \rightarrow$ dep x and y

$\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 u}{\partial x^2} = u_{xx}$

$\frac{\partial u}{\partial y} = u_y$

$\frac{\partial^2 u}{\partial x \partial y} = u_{yx}$

iska matlab u ka double derivative liya hai phele w.r.t y phir w.r.t x.

$u = xy + x^2y^2 + x + y$

iska jab hum derivative w.r.t x lenge yaani $\frac{\partial U}{\partial x}$ lenge to y ko as coefficent ya constant treat krenge.

$\frac{\partial u}{\partial x} = y + 2xy^2 + 1$

$\frac{\partial}{\partial y} \cdot \left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(y + 2xy^2 + 1\right)$

$\frac{\partial^2 U}{\partial y \partial x} = 1 + 4xy$

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$\nearrow$ ordinary eq.

$\nearrow$ hum yaha kisi bhi ek ko dependent consider karenge

ORDER AND DEGREE of ODE:

① $\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^3 = \sin x$

order = 2 , degree = 1

- highest derivative is called order
- The highest power of highest derivative is called degree.

② $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial^3 u}{\partial x \partial y^2}\right)^{(2)} = 3$ → degree

order = 3   degree = 2

③ degree & order is not in fraction

⑤ $3\frac{dy}{dx} + \frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^5 = \sin(x+y)$

order = 3   degree = 1

④ $(y''+1) = \sqrt{(y'')^2 + y'}$

without simplification hum order nhi dekhonge.

Sq. on b/s

$(y''+1)^2 = (y'')^2 + y'$

$\cancel{(y'')^2} + 2y'' + 1 = \cancel{(y'')^2} + y'$

$2y'' + 1 - y' =$

Order = 2   degree = 1

⑥ $\frac{dy}{dx} = x + \frac{dx}{dy}$

$\frac{dy}{dx} = x + \frac{1}{\frac{dy}{dx}}$

$y' = x + \frac{1}{y'}$

$y' = \frac{xy' + 1}{y'}$

$(y')^2 = xy' + 1$

order = 1   degree 2

✓ dependent variable ya uske derivative ki power 1 ho to linear agr 1 se greater hai to non-linear.

- $y^2 \to y \cdot y$
- $\left(\frac{dy}{dx}\right)^2 \to \left(\frac{dy}{dx}\right) \cdot \left(\frac{dy}{dx}\right)$
- $y \cdot \frac{dy}{dx}$ = agr is tarha product form mein ho tab bhi non-linear
  $\frac{d^2y}{dx^2} \cdot \frac{dy}{dx}$

$\sin y, \cos y, \tan y, e^y, \ln y$ (function k saath bhi y ho to bhi non-linear)

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General solution → • involves arbitary constant C $y = 2x + C$ arbitary constant.

Particular Solution → that contains no arbitary constant

SOLUTION TO DE

dependend variable bhi agr product ki form mein aajye to non-linear.

$y^2 \to y \cdot y$, $y \cdot \frac{dy}{dx} =$

$\frac{d^2y}{dx} \cdot \frac{dy}{dx} \to$ non linear.

Ex # 1.1

① $(1-x)y'' + xy' + y = 0$

linear , order = 2 , degree = 1

② $\frac{d^2u}{dr^2} + r \cdot \frac{du}{dr} = \cos(u+r)$ angle mein dependent variable

non-linear , order = 2

degree = 1

③ $xy'' + y'y = x \sin x$

non-linear

bcz y is multiplied by $\frac{dy}{dx}$

if you write dependent variable in terms of independent variable is called explicit funct.

Ex : 1.1

explicit function

if dep = f(ind)

$y = f(x)$

Ⓐ $2y' + y = 0$, $y = e^{-x/2}$

doesn't contain an arbitary constant hence it is a particular solu

first finding y';

$y' = \frac{d}{dx} e^{-x/2}$

$y' = \frac{-1}{2} e^{-x/2}$ → putting in eqn.

$2(-\frac{1}{2} e^{-x/2}) + e^{-x/2} = 0$

$-e^{-x/2} + e^{-x/2} = 0$

$0 = 0$

hence it is a solution!

implicit func.

Ⓐ $\frac{dx}{dt} = (x-1)(2x-1)$, $\ln\left(\frac{2x-1}{x-1}\right) = t$

phele L.H.S phir R.H.S Solve krenge L.H.S k liye dx/dt nikalna hoga or R.H.S k liye 'x' ki value.

as answer contain 'x' dependent.

$\ln\left(\frac{2x-1}{x-1}\right) = t$

$\frac{2x-1}{x-1} = e^t$

$2x - 1 = xe^t - e^t$

$x = \frac{xe^t - e^t}{2 - e^t}$

taking der on b/s

→

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Ex: 2.3

1st ORDER LINEAR D.E

→ where x & y cannot be seperated

step #01 ↓

① $\frac{dy}{dx} + p(x)y = Q(x)$ → linear D.E

↳ depends on x — should only be x term

② $\frac{dy}{dt} + p(t)y = Q(t)$

↳ depends on t

Step #02 ↓

I.F = Integerating Factor

$I.F = e^{\int p(x).dx}$ ⇒ it was derived

step #03 ↓

Solution →

$I.F \times y = \int (I.F \times Q(x)) \cdot dx$

Integeratingfactor × dependent variable $= \int (I.F \times Q(x))dx$

Q) $x\frac{dy}{dx} + y = x^2$

divide isliye karaya take ye hamare formula se match kar jaye.

x & y terms cannot be seperated

÷ by x on b/s :-

$\frac{dy}{dx} + \frac{y}{x} = x$ → ①

$\frac{dy}{dx} + p(x)y = Q(x)$

1. $p(x) = 1/x$ $Q(x) = x$

2. Integerating factor $= I.F = e^{\int p(x)\,dx}$

$= e^{\int 1/x \cdot dx} = e^{\ln(x)} = x$

$I.F = x$

3. $I.F \cdot y = \int (I.F \times Q(x))\, dx$

$x \cdot y = \int x \times x\, dx$

$= \int x^2\, dx$

$xy = \frac{x^3}{3} + C$

$y = \frac{x^3}{3x} + \frac{C}{x}$

OR

directly :-

$(y')(x)^{-1} + (x')(y) = x^2$

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Ex: 2.3

$\frac{d}{dx}(y\cdot x) = x^2$

$\boxed{d(y\cdot x) = \int x^2 dx}$

$yx = \int x^2 dx$

$\frac{dy}{dx} + \frac{1}{x}y = x$

$\frac{dy}{dx}x + y = x^2$

So proved that integrating factor when multiplied by equation gives us product rule formula.

Q) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

$\Rightarrow \frac{dy}{dx} + p(x)y = Q(x) \rightarrow (1)$

$\Rightarrow \frac{dr}{d\theta} + p(\theta)r = Q(\theta)$

$\Rightarrow p(\theta) = \sec\theta, \quad Q(\theta) = \cos\theta$

$\Rightarrow I\cdot F = e^{\int p(\theta)d\theta} = e^{\int \sec\theta\, d\theta}$

$= e^{\ln(\sec\theta+\tan\theta)}$

$I\cdot F = \sec\theta + \tan\theta$

$I\cdot F\cdot r = \int I\cdot F\,.\,Q(\theta)\,d\theta$

$= \int(\sec\theta+\tan\theta)\cos\theta\, d\theta$

$(\sec\theta+\tan\theta)\cdot r = \int(\sec\theta\cdot\cos\theta + \tan\theta\cdot\cos\theta)\,d\theta$

$= \int\left(\cancel{\cos\theta}\cdot\frac{1}{\cancel{\cos\theta}} + \frac{\sin\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\right)d\theta$

$= \int(1+\sin\theta)\,d\theta$

$(\sec\theta - \tan\theta)r = \theta - \cos\theta + c$

$$\boxed{r = \frac{\theta}{\sec\theta-\tan\theta} - \frac{\cos\theta}{\sec\theta-\tan\theta} + \frac{C}{\sec\theta-\tan\theta}}$$

Q) $x(x+1)\frac{dy}{dx} + xy = 1 \quad ; y(e) = 1$

$\div$ by $x(x+1)$ on b/s

$\frac{dy}{dx} + \frac{y}{x+1} = \frac{1}{x(x+1)}, \quad x \neq 0 \;\&\; x \neq -1$

$P(x) = \frac{1}{x+1}, \quad Q(x) = \frac{1}{x(x+1)}$

$I\cdot F = e^{\int\frac{1}{x+1}dx}$

$= e^{\ln(x+1)}, \quad x > 0$

$I\cdot F = (x+1)$

$I\cdot F\cdot y = \int I\cdot F\,.\,Q(x)\,dx$

$(x+1)y = \int \cancel{(x+1)}\cdot\frac{1}{x\cancel{(x+1)}}\,dx$

$= \int \frac{1}{x}\,dx$

$(x+1)\cdot y = \ln x + C \rightarrow (A)$

put $x = e, \; y = 1$

$(e+1)\cdot 1 = \ln e + 1 \Rightarrow C = e$

$(x+1)y = \ln x + e \rightarrow$ Ans.

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$$\boxed{y = \frac{\ln x}{x+1} + \frac{c}{x+1}}$$

Domain of sol: $(0, \infty)$

Q) $y\,dx - 4(x+y^6)\,dy = 0$

y ka form main linear nhi hosakti q k 'y' k power 6 hai.

$$y - 4(x+y^6)\frac{dy}{dx} = 0$$

$$y - 4x\frac{dy}{dx} - y^6\frac{dy}{dx} \rightarrow \text{non-linear}$$

w.r.t x ye linear hosakti hai.

$$y\,dx = 4(x+y^6)\,dy$$

$$\frac{dx}{dy} = \frac{4(x+y^6)}{y}$$

$$\frac{dx}{dy} = \frac{4x}{y} + \frac{4y^6}{y}$$

$$\frac{dx}{dy} - \frac{4x}{y} = 4y^5 \quad , y \neq 0$$

$$P(y) = -\frac{4}{y} \quad , Q(y) = 4y^5$$

$$I.F = e^{\int -4/y\,dy} = e^{-4\ln y} \quad , y > 0$$

bcz ln(0) ≠ ∞

$$= e^{\ln y^{-4}}$$

→ −4 ko power islye bnaya hai taake e or ln cancel hosake

$$\boxed{I.F = y^{-4}}$$

$$I.F.x = \int I.F.\,Q(y)\,dy$$

$$\frac{1}{y^4}\cdot x = \int \frac{1}{y^4}\cdot 4y^5\,dy$$

$$\frac{x}{y^4} = \frac{4y^2}{2} + c$$

$$\frac{x}{y^4} = 2y^2 + c$$

$$\boxed{x = 2y^6 + cy^4 \quad (0,\infty)}$$

$$\because x(1) = 2$$

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(c) $\dfrac{dy}{dx} + \underbrace{2x}_{P(x)}y = \underbrace{f(x)}_{Q(x)}$, $y(0)=2$

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

$I.F = e^{\int 2x\,dx} = e^{2\frac{x^2}{2}}$

$\boxed{I.F = e^{x^2}}$

• For $f(x) = x = Q(x)$, $0 \le x < 1$

$e^{x^2}\cdot y = \int e^{x^2}\cdot x\,dx$

$e^{x^2}\cdot y = \dfrac{1}{2}\int e^{x^2}\cdot(2x)\,dx$

$e^{x^2}+y = \dfrac{e^{x^2}}{2} + C$

$y = \dfrac{1}{2} + C_1 e^{-x^2}$, $0 \le x < 1$

For $f(x) = 0 = Q(x)$ $x \ge 1$

$e^{x^2}\cdot y = \int e^{x^2}\cdot(0)\,dx$

$e^{x^2}\cdot y = C_2$ → agr koi integeration nhi hoti to constant bachta hai.

$y = C_2 e^{-x^2}$

$$y = \begin{cases} \dfrac{1}{2} + C_1 e^{-x^2}, & 0 \le x < 1 \to ① \\ C_2 e^{-x^2}, & x \ge 1 \to ② \end{cases}$$

∵ $y(0) = 2$

$x = 0$, $y = 2$

using eq ①

$y = \dfrac{1}{2} + C_1 e^{-x^2/2}$

$2 = \dfrac{1}{2} + C_1 e^{0}$

$2 - \dfrac{1}{2} = C_1$

$\boxed{C_1 = \dfrac{3}{2}}$

$y = \dfrac{1}{2} + \dfrac{3}{2}e^{-x^2} \to ③$ $0 \le x < 1$

$y = C_2 e^{-x^2} \to ②$ $x \ge 1$

humara function continous hai q k sawal mein given hai to L.H limit or Right hand limit brabr hogi $x=1$ par.

$$\lim_{x\to 1^-}\left(\frac{1}{2} + \frac{3}{2}e^{-x^2}\right) = \lim_{x\to 1^+}\left(C_2 e^{-x^2}\right)$$

$\dfrac{1}{2} + \dfrac{3}{2}e^{-1} = C_2 e^{-1}$

$\dfrac{1}{2e^{-1}} + \dfrac{3}{2} = C_2$

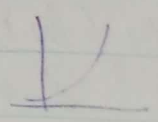
$\boxed{C_2 = \dfrac{e}{2} + \dfrac{3}{2}}$

Ex: 3.1

* Exponential Growth Model

change in population is $\propto$ to population → population

$$\frac{dp}{dt} \propto P$$



→ proportionality constant

$$\frac{dp}{dt} = kP \quad , \; P(0) = P_0$$

at $t=0$, $P = P_0$

$$\frac{dP}{dt} - kP = 0 \qquad = \frac{dy}{dt} + p(x)y = Q(x)$$

$P(x) = -k \quad , \; Q(x) = 0$

$$I.F = e^{\int -k\,dt}$$

$$I.F = e^{-kt}$$

$$I.F \times P = \int \cancel{I.F \times Q(t)}\,dt \quad 0$$

$$e^{-kt} \cdot P = C$$

$$\boxed{P = Ce^{kt}} \rightarrow (1)$$

$t=0$, $P = P_0$ put in (1)

$$P_0 = Ce^0 \Rightarrow \boxed{C = P_0}$$

$$\boxed{P(t) = P_0 e^{kt}}$$

(C) $\frac{dT}{dt} = k(T - T_s)$; $T(0) = T_0$

$$\int \frac{1}{T - T_s}\,dt = k\,dt$$

$$\ln(T - T_s) = kt + c$$

$$\ln(T - T_s) = kt + \ln c$$

$$\ln(T - T_s) - \ln c = kt$$

$$\ln\left(\frac{T - T_s}{c}\right) = kt$$

$$\frac{T - T_s}{c} = e^{kt}$$

$$T - T_s = Ce^{kt}$$

$$T = Ce^{kt} + T_s \rightarrow (1)$$

$t = 0$, $T = T_0$ put in (1)

$$T_0 = Ce^0 + T_s$$

$$T_0 - T_s = C \rightarrow \text{put (1)}$$

$$T = (T_0 - T_s)e^{kt} + T_s$$

$$T(t) = (T_0 - T_s)e^{kt} + T_s$$

T_0 ↳ initial temp ; T_s ↳ surrounding temp

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Ex: 3.1

Q#13: GIVEN:-

- $T(0) = T_0 = 70°F$
  $T_s = 10°F$
  $T(\frac{1}{2}) = 50°F$
  $T(1) = ?$
  $T(t) = 15°F$

SOLUTION:-

Peeche waali derivation Roke dikhani hai.

For k:

$\because T(t) = (T_0 - T_s)e^{kt} + T_s$

$T(\frac{1}{2}) = (70-10)e^{k(\frac{1}{2})} + 10$

$50 = 60e^{k/2} + 10$

$\frac{40}{60} = e^{k/2}$

$\frac{3}{2} = e^{k/2}$

$\ln(\frac{3}{2}) = \frac{k}{2}$

$k = 2\ln(\frac{2}{3})$

For $T(1)$:

$T(1) = 60e^{(1)[2\ln(\frac{2}{3})]}$

$T(1) = 86.7 F$

- For $T(t) = 15°F$

$15 = (70-10)e^{kt} + 10$

$5 = 60e^{kt}$

$\frac{1}{12} = e^{kt}$

$\ln(\frac{1}{12}) = kt$

$\ln(\frac{1}{12}) = 2\ln(\frac{2}{3})\,t$

$t = \frac{\ln(\frac{1}{12})}{2\ln(2/3)}$ Do $\rightarrow$ in decimal

$t = 3.06$ sec

Q#15: GIVEN:

$T_0 = T(0) = 20°C$

$T_s = 100°C$ $\rightarrow$ bcz boiling water.

$T(t) = 90°C$ — (i)

$T(t+1) = 92°C$ — (ii)

$T(t) = 98°C$

SOLUTION:-

$\because T(t) = (T_0 - T_s)e^{kt} + T_s$

$90 = (20-100)e^{kt} + 100$

$-10 = -80e^{kt}$

$\frac{1}{8} = e^{kt} \rightarrow (1)$

Now Using (ii) condition

$92° = -80°e^{k(t+1)} + 100$

$\frac{1}{10} = e^{k(t+1)} \rightarrow (2)$

(2) ÷ (1)

$\frac{1/10}{1/8} = \frac{\cancel{e^{kt}} \cdot e^{k}}{\cancel{e^{kt}}}$

$\frac{8}{10} = e^{k}$

$\ln(8/10) = k \rightarrow$ put (1)

$\frac{1}{8} = e^{\ln(8/10)t}$ → 't' power mei jayega

$\frac{1}{8} = \left(\frac{8}{10}\right)^{t}$

$\ln\left(\frac{1}{8}\right) = \ln\left(\frac{8}{10}\right)^{t}$

$\ln\left(\frac{1}{8}\right) = t\ln\left(\frac{8}{10}\right)$

$\frac{\ln\left(\frac{1}{8}\right)}{\ln(8/10)} = t$

$\boxed{t = 9.3 \text{ sec}}$

Q#17: GIVEN:

$T_0 = T(0) = 70°F$

$T(\tfrac{1}{2}) = 110°F$

$T(1) = 145°F$

$T_s = ?$

$\because T(t) = (T_0 - T_s)e^{kt} + T_s$

$110° = (70 - T_s)e^{k\frac{1}{2}} + T_s \rightarrow (1)$

$145 = (70 - T_s)e^{k} + T_s \rightarrow (2)$

eq(1)⇒ $\frac{110 - T_s}{70 - T_s} = e^{k/2} \rightarrow (1)$

eq(2) $\frac{145 - T_s}{70 - T_s} = e^{k} \rightarrow (2)$

eq(2)⇒ $\frac{145 - T_s}{70 - T_s} = (e^{k})^{1/2}$

$\frac{145 - T_s}{70 - T_s} = \left(\frac{110 - T_s}{70 - T_s}\right)^{1/2}$

$\left(\frac{145 - T_s}{70 - T_s}\right)^{2} = \frac{110 - T_s}{70 - T_s}$

$\frac{(110)^2 - 2(110)(T_s) + T_s^2}{(70 - T_s)^2} = \frac{110 - T_s}{\cancel{70 - T_s}}$

$12100 - 220T_s + T_s^2 = (110 - T_s)(70 - T_s)$

$12100 - 220T_s + \cancel{T_s^2}$ $10150 - 215T_s - \cancel{T_s^2}$

$\boxed{T_s = 390}$

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Q#3: GIVEN:-

$P_0 = P(0) = 500$ $\quad 500 \times 15\% = 75$ $\quad 75 + 500$

$P(10) = 575$, $t = 10$

$P(30) = ?$

$\frac{dP}{dt} \to t = 30$

For k: pehle derive krna hai.!

$P(10) = 575$

$P(t) = P_0 e^{kt}$

$P(10) = 500 e^{k(10)}$

$\frac{575}{500} = e^{k(100)}$

$\ln\left(\frac{575}{500}\right) = 10k$

$\boxed{k = 0.014}$

$P(30) = 500 e^{(30)(0.014)}$

$\boxed{P(30) = 760.\cancel{48}}$

$\Rightarrow \frac{dP}{dt} = kP(30)$

$\frac{dP}{dt} = (0.014)(\cancel{...})$ 760

$\boxed{\frac{dP}{dt} = 10.64\cancel{8}}$ persons/yr

Q#4: GIVEN:

$P(3) = 400$

$P(10) = 2000$

$P_0 = ?$

SOLUTION:-

$\Rightarrow P(t) = P_0 e^{kt}$

$P(3) = P_0 e^{3t}$

$400 = P_0 e^{3k} \to (1)$

$2000 = P_0 e^{10k} \to (2)$

$(1) \div (2)$

$\frac{\cancel{2000}^{5}}{\cancel{400}_{1}} = \frac{\cancel{P_0} e^{10k}}{\cancel{P_0} e^{3k}}$

$5 = e^{7k}$

$\ln(5) = 7k$

$\boxed{k = 0.23}$

eq (1)

$400 = P_0 e^{3(0.23)}$

$P_0 = \frac{400}{e^{(3)(0.23)}}$

$\boxed{P_0 = 200.6}$

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Ex : 2.5

"BERNULLI'S EQUATION" → Convert into linear:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad n \neq 1 \;\&\; n \neq 0$$

when $n = 0$

$$\frac{dy}{dx} + P(x)y = Q(x)y^0$$

$$\frac{dy}{dx} + P(x)y = Q(x) \rightarrow \text{linear eqn}$$

when $n = 1$

$$\frac{dy}{dx} + P(x)y = Q(x)y^1$$

$$\frac{dy}{dx} + [Q(x) - P(x)]y$$

- Bernulli eqn is a nonlinear eqn because as we cannot put $n=0$ or $n=1$ so if we put $n=2$ then it will be 'y^2' which is nonlinear.

① ÷ by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)y}{y^n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)\cdot(y^{1-n}) = Q(x) \rightarrow ②$$

Let $z = y^{1-n}$

Taking differentiation w.r.t 'x'

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\cdot\frac{dz}{dx} = y^{-n}\frac{dy}{dx} \rightarrow Ⓑ$$

put in ②

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)z = Q(x)$$

Now it is linear eqn!

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Agr R.H.S par 'y' term aa ge to ye bernoli eqn kehlaati h.

$\frac{dy}{dx} - y = y^2 e^x \longrightarrow (1)$

$\div$ b/s by 'y^2'

$y^{-2}\frac{dy}{dx} - y^{-1} = e^x \longrightarrow (2)$

Let, $z = y^{-1}$

Differentiating w.r.t 'x'

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dz}$

$-\frac{dz}{dx} = y^{-2}\frac{dy}{dz} \rightarrow$ put in (2)

$-\frac{dz}{dx} - \cancel{y}z = e^x$

$\frac{dz}{dx} + z = -e^x \rightarrow$ linear eqn

Let, $P(x) = 1$, $Q(x) = -e^x$

$I.F = e^{\int 1\, dx}$

$I.F = e^x$

$e^x . z = \int e^x . -e^x \, dx$

$= -\int e^{2x} dx$

$e^x . z = -\frac{1}{2}e^{2x} + c$

$z = -\frac{1}{2}e^x + ce^{-x}$

$y^{-1} = -\frac{1}{2}e^x + ce^{-x}$

$$\boxed{y = \frac{1}{-\frac{1}{2}e^x + ce^{-x}}}$$

x bhi hatana hoga.

(15) $x\frac{dy}{dx} + y = \frac{1}{y^2}$

$\div$ b/s by y^{-2} & x

$y^2\frac{dy}{dx} + \frac{y^3}{x} = \frac{1}{x} \rightarrow (2)$

Let $z = y^3$

$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$

$\frac{1}{3}\frac{dz}{dx} = y^2\frac{dy}{dx} \rightarrow$ put in (2)

$\times$ by 3

$\frac{1}{3}\frac{dz}{dx} + \frac{3z}{x} = \frac{3}{x}$

$P(x) = 3/x$, $Q(x) = 3/x$

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$I.F = e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3}$

$I.F = x^3$

$x^3 \cdot z = \int x^3 \cdot \frac{3}{x} dx$

$x^3 \cdot z = 3\int x^2$

$x^3 \cdot z = \frac{3x^3}{3} + C$

$x^3 \cdot z = x^3 + C$

$z = 1 + Cx^{-3}$

$y^3 = 1 + Cx^{-3}$

$\boxed{y = \sqrt[3]{1 + Cx^{-3}}}$

$\Rightarrow y(1) = 2$

$2 = \sqrt[3]{1 + C(1)}$

$8 = 1 + C$

$\boxed{C = 7}$

$\boxed{y = \sqrt[3]{1 + 7x^{-3}}}$

(22) $y^{1/2} \frac{dy}{dx} + y^{3/2} = 1 \rightarrow (1)$

$y(0) = 4$

let, $z = y^{3/2}$

$\frac{dz}{dx} = \frac{3}{2} y^{1/2} \frac{dy}{dx}$

$\frac{2dz}{3dx} = y^{1/2} \frac{dy}{dx}$ put in (1)

eq (1)

$\frac{2}{3}\frac{dz}{dx} + z = 1$

$\frac{dz}{dx} + \frac{3z}{2} = \frac{3}{2}$

$P(x) = 3/2 \qquad Q(x) = 3/2$

$I.F = e^{\int 3/2 \, dx} = e^{\frac{3}{2}x}$

$e^{\frac{3}{2}x} \cdot z = \int e^{\frac{3}{2} \cdot x} \cdot \frac{3}{2} dx$

$e^{3/2 x} \cdot z = e^{3/2 x} + C$

$z = 1 + Ce^{-3/2 x}$

$y^{3/2} = 1 + Ce^{-3/2 x}$

$\Rightarrow y(0) = 4$

$4^{3/2} = 1 + Ce^{-3/2 (0)}$

$8 = 1 + C$

$\boxed{C = 7}$

$\boxed{y = (1 + 7e^{-3/2 x})^{2/3}}$

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(2) $3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$

$3(1+t^2)\frac{dy}{dt} = 2ty^4 - 2ty$

$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$

$\div$ b/s by $3(1+t^2)$ & y^4

$y^{-4}\frac{dy}{dt} + \frac{2t}{3(1+t^2)} y^{-3} = \frac{2t}{3(1+t^2)} \rightarrow (2)$

$z = y^{-3}$

$\frac{dz}{dt} = -3y^{-4}\frac{dy}{dt}$

$\frac{-1}{3}\frac{dz}{dt} = y^{-4}\frac{dy}{dt}$ put (2)

$\frac{-1}{3}\frac{dz}{dt} + \frac{2t}{3(1+t^2)} z = \frac{2t}{3(1+t^2)}$

$\frac{dz}{dt} - \frac{2t}{1+t^2} z = \frac{-2t}{1+t^2} \rightarrow (A)$

$P(t) = \frac{-2t}{1+t^2} = Q(t)$

$I.F = e^{-\int \frac{2t}{1+t^2} dt}$

$I.F = e^{-\ln(1+t^2)}$

$= e^{\ln(1+t^2)^{-1}}$

$I.F = (1+t^2)^{-1}$

$(1+t^2)^{-1}. z = \int (1+t^2)^{-1} . \frac{-2t}{1+t^2}$

$= \int -2t.(1+t^2)^{-2} dt$

$(1+t^2)^{-1}. z = (1+t^2)^{-1} + C$

$z = 1 + C(1+t^2)$

$y^{-3} = 1 + C(1+t^2)$

$\frac{1}{y^3}$

$y = \frac{1}{[1+C(1+t^2)]^3}$

Q: $\frac{dy}{dx} + y\sec x = y^2 \sin x \cos x$

$\div$ b/s by 'y^2'

$y^{-2}\frac{dy}{dx} + y^{-1}\sec x = \sin x \cos x \rightarrow (1)$

Let $z = y^{-1}$

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} = y^{-2}\frac{dy}{dx} \rightarrow$ put in (1)

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$$-\frac{dz}{dx} + z\sec x = \sin x\cos x$$

$$\frac{dz}{dx} - z\sec x = -\sin x\cos x$$

$Q(x) = -\sin x\cos x$, $P(x) = -\sec x$

$$I.F = e^{-\int \sec x} = e^{-\ln(\sec x+\tan x)}$$

$$I.F = (\sec x+\tan x)^{-1}$$

$$(\sec x+\tan x)^{-1}\cdot z = \int (\sec x+\tan x)^{-1} \cdot -\sin x\cos x$$

$$= -\int \frac{\sin x\cos x}{\sec x+\tan x}$$

$$= -\int \frac{\sin x\cos x}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}$$

$$= -\int \frac{\sin x\cos x}{\frac{1+\sin x}{\cos x}}$$

$$= -\int \frac{\sin x\cos^2 x}{1+\sin x}$$

$$= -\int \frac{\sin x(1-\sin^2 x)}{1+\sin x}$$

$$= -\int \frac{\sin x(1-\sin x)\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}}$$

$$= -\int \sin x(1-\sin x)$$

$$-\int \sin x\,dx + \int \sin^2 x\,dx$$

$$-\int \sin x + \int \frac{1-\cos 2x}{2}\,dx$$

$$-\int \sin x + \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$-\int \sin x + \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$$\left(\frac{\cos x}{1+\sin x}\right)\cdot z = \text{" " " "}$$

$$z = (1+\sin x) + \frac{(1+\sin x)x}{2\cos x} - \frac{1}{4}\sin 2x\left(\frac{1+\sin x}{\cos x}\right) + \left(\frac{1+\sin x}{\cos x}\right)c$$

(x) whole eqn by 4 and $\cos x$

$$z = \frac{\cos x\,4(1+\sin x) + 2x(1+\sin x) - \sin 2x(1+\sin x) + (1+\sin x)4c}{4\cos x}$$

$$y = \frac{4\cos x}{(1+\sin x)[4\cos x + 2x - \sin 2x + 4c]}$$

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Q: $\cot x \frac{dy}{dx} + y = y^2$

Dividing b/s by $\cot x$ & y^2

$y^{-2}\frac{dy}{dx} + y^{-1}\tan x = \tan x$ ↳ (1)

let $z = y^{-1}$

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} = y^{-2}\frac{dy}{dx} \rightarrow$ put in (1)

$-\frac{dz}{dx} + z\tan x = \tan x$

$\frac{dz}{dx} - z\tan x = -\tan x$

$Q(x) = -\tan x$, $P(x) = -\tan x$

$I.F = e^{-\int \tan x\, dx}$

$= e^{-\ln \sec x}$

$I.F = (\sec x)^{-1}$

$(\sec x)^{-1} \cdot z = \int (\sec x)^{-1} \cdot -\tan x \, dx$

$= -\int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} dx$

$= -\int (\cos x)^{-2} \sin x \, dx$

$= \int (\cos x)^{-2} \cdot (-\sin x) dx$

$(\sec x)^{-1} \cdot z = -(\cos x)^{-1} + C$

$(\sec x)^{-1} z = -\sec x + C$

$z = -\sec^2 x + C \sec x$

$y^{-1} = -\sec^2 x + C \sec x$

$y = (-\sec^2 x + C\sec x)^{-1}$

$(\sec)^{-1} \cdot z = \int (\sec x)^{-1} \cdot -\tan x \, dx$

$\cos x \cdot z = \int \cos x \cdot -\tan x \, dx$

$= \int \cos x \cdot -\frac{\sin x}{\cos x}$

$\cos x \cdot z = -\int \sin x \, dx$

$\cos x \cdot z = \cos x + C$

$z = 1 + C \sec x$

$\frac{1}{y} = 1 + C \sec x$

$y = (1 + C\sec x)^{-1}$

Ex: 2.5

Homogenous 1^{st} Order DE

$$\frac{dy}{dx} = f(x,y) = \frac{h(x,y)}{g(x,y)}$$

degree of $h(x,y)$ = degree $g(x,y)$

$\frac{y^{+1}}{x^{+1}} \rightarrow$ degree 0

dono powers minus hojayengi

$\sqrt{xy} \rightarrow 1 \quad (x^1y^1)^{\frac{1}{2}} = ((xy)^2)^{\frac{1}{2}} = 1$

In homogenous eqn we Subsitute, $\rightarrow$ v dep on x

$$\boxed{y = vx}$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(vx) \quad \hookrightarrow u \cdot v$$

$$\boxed{\frac{dy}{dx} = \frac{x\,dv}{dx} + v}$$

Q:2) $(x+y)dx + x\,dy = 0$

$$x\,dy = -(x+y)dx$$

$$x\frac{dy}{dx} = -x - y \Rightarrow \frac{dy}{dx} = \frac{-x-y}{x} \quad \begin{matrix} \rightarrow deg=1 \\ \rightarrow deg=1 \end{matrix}$$

$$x\frac{dy}{dx} + y = -x$$

$\rightarrow$ ye ab linear ban sakti hai hum isko linear se bhi solve karsakte hai.

Let, $y = vx$

$$\frac{dy}{dx} = \frac{x\,dv}{dx} + v$$

$$\frac{dv}{dx}x + v = \frac{-x - vx}{x}$$

$$\frac{dv}{dx}x + v = \frac{x(-1-v)}{x}$$

$$\frac{dv}{dx}x + v = -1 - v$$

$$\frac{dv}{dx}x = -1 - v - v$$

$$\frac{dv}{dx}x = -1 - 2v$$

$$dv \cdot x = (-1 - 2v)\,dx$$

$$\frac{1}{1+2v}dv = \frac{-1}{x}dx$$

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Ex: 2.5

$$\int \frac{1}{(1+2v)} dv = -\int \frac{1}{x} dx$$

$$\frac{1}{2}\ln(1+2v) = -\ln x + \ln c$$

$$\ln(1+2v)^{1/2} = \ln(c/x)$$

taking 'e' on b/s

$$(1+2v)^{1/2} = \frac{c}{x}$$

Sq. on b/s

$$1+2v = \frac{c^2}{x^2}$$

c^2 or c done same hai

$\Rightarrow y = vx$

$$1 + \frac{2y}{x} = \frac{c}{x^2}$$

$$x^2 + 2yx = c$$

$$2yx = c - x^2$$

$$y = \frac{c - x^2}{2x}$$

(5) $(y^2 + yx)dx - x^2 dy = 0$

$$(y^2 + yx)dx = x^2 dy$$

$$\frac{dy}{dx} = \frac{y^2 + yx}{x^2} \quad \begin{matrix}\rightarrow deg=2 \\ \rightarrow deg=2\end{matrix} \quad \rightarrow (1)$$

Let, $y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v \quad \text{put in (1)}$$

$$\frac{dv}{dx}x + v = \frac{v^2x^2 + vx^2}{x^2}$$

$$= \frac{\cancel{x^2}(v^2+v)}{\cancel{x^2}}$$

$$\frac{dv}{dx}x + \cancel{v} = v^2 + \cancel{v}$$

$$\frac{dv}{dx}x = v^2$$

$$v^{-2}dv = \frac{1}{x}dx$$

$$\frac{-1}{v} = \ln x + \ln(c)$$

$$\frac{-1}{y/x} = \ln(xc)$$

$$\frac{-x}{y} = \ln(xc)$$

$$\boxed{y = \frac{-x}{\ln(xc)}}$$

V

$-V$

$\int \frac{1}{x}\, dx$

$= \int \frac{1}{x}\, dx$

$dv = \int \frac{1}{x}\, dx$

$dv = \int \frac{1}{x}\, dx$

$) = \ln(xc)$

$)] = \ln(xc)$

$\ln(xc)$

yegi ∝ property

$\boxed{c = e^{-1}}$

* **By Substitution:**

(23) $\frac{dy}{dx} = ((x+y)-1)^2$

Let,

$u = x + y \rightarrow (1)$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{du}{dx} - 1 = \frac{dy}{dx} \rightarrow (2)$ put (2)

$\frac{du}{dx} - 1 = (u-1)^2$

$\frac{du}{dx} = (u-1)^2 + 1$

$\frac{1}{(u-1)^2+1}\, du = dx$

$\tan^{-1}(u-1) = x$

$\tan^{-1}(x+y-1) = x + c$

$x + y - 1 = \tan(x+c)$

$y = \tan(x+c) - x - 1$

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jiski whole power 1 nhi hai yk x ki power 1 or y ki power 1 hai.

(24) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

$\frac{dy}{dx} = \frac{1-(x+y)}{x+y}$

$U = x+y$

$\frac{du}{dx} - 1 = \frac{1-U}{U}$

$\frac{du}{dx} = \frac{1-U}{U} + 1$

$\frac{du}{dx} = \frac{1}{U}$

$\int U\,du = \int dx$

$\frac{U^2}{2} = x$

$(x+y)^2 = 2x + C$

(29) $\frac{dy}{dx} = \cos(x+y)$

let, $U = x+y$

$\frac{dU}{dx} = 1 + \frac{dy}{dx}$

$\frac{dU}{dx} - 1 = \cos U$

$\frac{du}{dx} = \cos U + 1$

$\frac{1}{\cos U + 1}\,du = dx$

$\times$ & $\div$ by $1-\cos U$

$\frac{1-\cos U}{1-\cos^2 U}\,du = dx$

$\frac{1-\cos U}{\sin^2 U}\,du = dx$

$\frac{1}{\sin^2 U}\,du - \frac{\cos U}{\sin^2 U}\,du = dx$

$\int \csc^2 U\,du - \int \csc U \cot U\,du = \int dx$

$-\cot(x+y) + \csc(x+y) = x + C$ → (1)

$y(0) = \pi/4$

$-\cot(0+\frac{\pi}{4}) + \csc(0+\frac{\pi}{4}) = C$

$\sqrt{2} - 1 = C$

eq (1)

$\csc(x+y) - \cot(x+y) = x + \sqrt{2} - 1$

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$\frac{dP}{dt} = P(a-bP)$

~~Ex~~: 3.2 —(2.4)—

(1) $\frac{dN}{dt} = N(1-0.0005N)$

(b) $\int \frac{1}{N(1-0.0005N)} dN = \int dt$

↳ By partial fraction.

(4) 1790 → $t=0$ consider kesonge
1850 → $t=60$ lenge $1790-1850 = 60$
1910 → $t=120$ $1790-1910=120$

in teen se a, b, N_0 nikaal lenge

Exact D.E :

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ → Exact DE

Step → $\underbrace{\int M\,dx}_{y\text{-const}} + \underbrace{\int N\,dy}_{\text{only consider } y\text{-term}} = c$

(1) $(x-y)dx - x\,dy = 0$

$M = x-y$, $N = -x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

→ both are equal so it is Exact DE

$\underbrace{\int (x-y)\,dx}_{y\text{-const}} + \int (0)\,dy = c$

as there is no y-term so we write ZERO

y constant

$\int x\,dx - y\int dx = c$

$\frac{x^2}{2} - yx = c$

$$\left(1+\ln x+\frac{y}{x}\right)dx = (1-\ln x)dy$$

$$\left(1+\ln x+\frac{y}{x}\right)dx-(1-\ln x)dy=0$$

$$M = 1+\ln x+\frac{y}{x}, \quad N = -1+\ln x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}, \quad \frac{\partial N}{\partial x} = \frac{1}{x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{1}{x} \quad \text{Exact D.E}$$

$$= \int M\,dx + \int N\,dy = C$$

$y =$ const (under $\int M\,dx$); term does not consist of 'x' term (only 'y' terms) (under $\int N\,dy$)

ismein 'y' ko as constant treat hoga.

$$\int \left(1+\ln x+\frac{y}{x}\right)dx + \int(-1)\,dy = C$$

$$\int dx + \int \ln x\,dx + y\int\frac{1}{x} - \int dy = C$$

$$x + \left[\ln x\cdot x - \int \tfrac{1}{x}\cdot x\,dx\right] + y\ln x - y = C$$

$$x + [x\ln x - x] + y\ln x - y = C$$

Ans.

(22) $(e^x+y)dx + (2+x+ye^y)dy = 0$

$y(0)=1$

$$M = e^x+y, \quad N = 2+x+ye^y$$

$$\frac{\partial M}{\partial y} = 0+1 = 1, \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} = 1$$

$$\int (e^x+y)dx + \int(2+ye^y)dy = C$$

$$\int e^x dx + y\int dx + 2\int dy + \int ye^y = C$$

by parts

$$e^x + xy + 2y + [ye^y - e^y] = C$$

$y(0)=1$

C will be 3

$$e^x + xy + 2y + [ye^y - e^y] = 3$$

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$\frac{x+y}{1+y^2}$ is ~~ka~~ denominator alag krenge to $\frac{x}{1+y^2} + \frac{y}{1+y^2} \to$ ye without 'x' hai to isko 'N' mein likh(te) hai.

(25) $\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}\right)dt + \left(ye^y + \frac{t}{t^2+y^2}\right)dy = 0$

$M = \frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}$

$\frac{\partial M}{\partial y} = -\left[\frac{(y)'(t^2+y^2) - (t^2+y^2)'(y)}{(t^2+y^2)^2}\right]$

$= -\frac{[t^2+y^2 - 2y \cdot y]}{(t^2+y^2)^2}$

$= -\frac{[t^2-y^2]}{(t^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{y^2-t^2}{(t^2+y^2)^2}$

't' derivate krenge
→ isko w.r.t

$N = ye^y + \frac{t}{t^2+y^2}$

$N = \frac{[(t)'(t^2+y^2) - (t^2+y^2)'(t)]}{(t^2+y^2)^2}$

$= \frac{t^2+y^2-2t^2}{(t^2+y^2)^2}$

$\frac{\partial N}{\partial t} = \frac{y^2-t^2}{(t^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \Rightarrow$ Exact D.E

$\int\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2+y^2}\right)dt + \int ye^y\, dy = c$

↖ By parts

$\ln t - \frac{1}{t} - y\int \frac{1}{t^2+y^2}\, dt + [ye^y - e^y] = c$

$\Rightarrow \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\ln t - \frac{1}{t} - y\left[\frac{1}{y}\tan^{-1}\left(\frac{t}{y}\right)\right] + \left[ye^y - e^y\right] = c$

$\ln t - \frac{1}{t} - \tan^{-1}\left(\frac{t}{y}\right) + ye^y - e^y = c$

(26) $\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y + y\sin x)$

$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy = y(y + y\sin x)\,dx$

$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy - y(y+y\sin x)\,dx = 0$

$M = -y^2 - y\sin x \quad , \quad N = \frac{1}{1+y^2} + \cos x - 2xy$

$\frac{\partial M}{\partial y} = -2y - \sin x \quad , \quad \frac{\partial N}{\partial x} = -\sin x - 2y$

Sol:

$\int(-y^2 - y\sin x)\,dx + \int\left(\frac{1}{1+y^2}\right)dy = c$

$-y^2\int dx - y\int \sin x\, dx + \tan^{-1}(y) = c$

$\ldots an^{-1}(y) = c$

$\ldots = 0$

$\ldots (y) = 1 + \frac{\pi}{4}$

$\ldots$ D.E is Exact;

$\ldots x - (3xy^2 + 20x^2y^3)dy = 0$

$\cancel{3}y^2 + 40xy^3$

Coef xy^3

Non Exact D.E

$$Mdx + Ndy = 0 \rightarrow (A)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

CASE #01:

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x) \text{ or can be any constant}$$

only x term

CASE #02:

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = p(y)$$

For case #01 →

$$I \cdot F = e^{\int p(x)dx} \text{ or } e^{\int p(y)dy}$$

For case #02:

$$I \cdot F \times (A) \rightarrow Q \rightarrow \text{Exact D.E}$$

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(31) $(2y^2+3x)dx + 2xy\,dy = 0$

$M = 2y^2+3x \quad N = 2xy$

$\frac{\partial M}{\partial y} = 4y \quad \frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ Non-Exact

CASE # 01 :

$\frac{1}{2xy}\left(4y - 2y\right) = P(x)$

$p(x) = \frac{1}{x} \rightarrow$ only 'x' term

$\therefore$ case#01 approved

$I.F = e^{\int \frac{1}{x} dx} = e^{\ln x}$

$I.F = x$

Now multiplying eq (A) by x

$(2y^2x + 3x^2)dx + 2x^2y\,dy = 0$

aage bhi same hoga

(38) $(x^2+y^2-5)dx = (y+xy)dy, \; y(0)=1$

$(x^2+y^2-5)dx - (y+xy)dy$

$M = x^2+y^2-5 \quad , \quad N = -y-xy$

$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial y} = -y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial gx}$

CASE # 01 :

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$

$\frac{1}{-y-xy}(2y+y) = p(x)$

$\frac{1}{-y(1+x)}(3y) = p(x)$

$p(x) = \frac{-3}{1+x}$

$I.F = e^{\int p(x)dx} = e^{\int \frac{-3}{1+x}} = e^{-3\ln(1+x)}$

$= (1+x)^{-3}$

eq (A) $\times$ by $(1+x)^{-3}$

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$$\frac{(x^2+y^2-5)}{(1+x)^3}dx - \frac{(y+xy)}{(1+x)^3}dy = 0$$

$$\int\left[\frac{x^2+y^2-5}{(1+x)^3}\right]dx + \int(0)dy = C$$

$$\int\frac{x^2}{(1+x)^3}dx + y^2\int\frac{1}{(1+x)^3}dx - 5\int\frac{1}{(1+x)^3} = C$$

$\hookrightarrow$ (B)

Consider,

$$\int\frac{x^2}{(1+x)^3}$$

U	V
x^2 (+)	$(1+x)^{-3}$
$2x$ (−)	$1/(1+x)^{-2}(-2)$
2 (+)	$1/(1+x)(-2)(-1)$
0	$\ln(1+x)/2$

$$\frac{x^2}{-2(1+x)^2} - \frac{2x}{(-2)(-1)(1+x)} + \cancel{2}\frac{\ln(1+x)}{\cancel{2}}$$

$$\frac{-x^2}{-2(1+x)^2} - \frac{x}{(1+x)} + \ln(1+x) \rightarrow \text{put in (B)}$$

$$\frac{-x^2}{-2(1+x)^2} - \frac{x}{(1+x)} + \ln(1+x) - \frac{y^2}{2(1+x)^2} - \frac{5}{2(1+x)^2}$$

(35) $(10-6y+e^{-3x})dx - 2dy = 0$ $\hookrightarrow$ (A)

$M = 10-6y+e^{-3x}$, $N = -2$

$\frac{\partial M}{\partial y} = -6$, $\frac{\partial N}{\partial x} = 0$

CASE # 01 :

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$$

$$\frac{1}{-2}(-6-0) = p(x)$$

$$3 = p(x)$$

$$I.F = e^{\int p(x)\,dx}$$
$$= e^{\int 3dx} = e^{3x}$$

Now (x) (A) by e^{3x}

$$(10e^{3x} - 6ye^{3x} + 1)dx - 2e^{3x}\,dy$$

$$\int(10e^{3x} - 6ye^{3x} + 1)dx + \int(0)dy = c$$

$$10\int e^{3x}dx - 6y\int e^{3x}dx + \int 1dx = c$$

$$\frac{10e^{3x}}{3} - \frac{6ye^{3x}}{3} + x = c$$

$$\frac{10e^{3x}}{3} - 2ye^{3x} + x = c$$

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Ex: 2.6

(32) $y(x+y+1)dx + (x+2y)dy = 0$

$M = xy + y^2 + y$, $N = x + 2y$

$\frac{\partial M}{\partial y} = x + 2y + 1$, $\frac{\partial N}{\partial x} = 1$

CASE # 01

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$

1. Variable Seperable
2. ~~Bernoli~~ Linear
~~3. Exact, non-exact~~
3. Bernoli
4. Subsitution
5. Exact, non-Exact

(NUMERIC METHOD)

(7) EULER METHOD

$y_{n+1} = y_n + h f(x_n, y_n)$

$f(x,y) = \frac{dy}{dx}$ if $\frac{dy}{dx} = 2xy$

then

$f(x,y) = 2xy$ or $f(x_n, y_n) = 2x_n y_n$

$y_{n+1} = y_n + 0.5(2x_n y_n)$

$y_{n+1} = y_n + x_n y_n$

we have been givenby $y(0) = 1$

$\left.\begin{matrix} x_0 = 0 \\ y_0 = 1 \end{matrix}\right\} \rightarrow$ intial condition

$n = 0$, $h = 0.5$ → ye given hoga

$y_{0+1} = y_0 + x_0 y_0$

$y_1 = 1 \rightarrow y(0.5) = 1$

$\hookrightarrow x_1 = 0.5$

x ki initial 0 value thi ab hum x mein h ko add krdiya a x 0.5 ban gaya phir dobara add krdia

$n = 1$

$y_2 = y_1 + x_1 y_1$

$= 1 + 0.5(1) = 1.5$

$y_2(1) = 1.5$

hum jab tak krte rhenge jab tak 'x' given R equal na hojaye

$y_3 = 1.5 + (1)(1.5) = 3$

$y_3(1.5) = 3$

⑧ $\frac{dy}{dx} = e^{x^2}$, $y(0) = 1$

$y(0.003) = ?$

$h = 0.001$

n	x_n	y_n	$y_{n+1} = y_n + h\underbrace{f(x_n, y_n)}_{e^{x^2}}$
0	0	1	1.001
1	0.001	1.001	1.002
2	0.002	1.002	1.003
3	0.003	1.003	

Q#6: $y' = x^2 + y^2$, $y(0) = 1$

$y(0.5) = ?$

Use Euler's Method,

$h = 0.1$ & $h = 0.05$

For $h = 0.1$:

$y_{n+1} = y_n + h(x_n y_n)$

$y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$

x_n	y_n	y_{n+1}	
			not necessary
0	1	1.1	$y_{0+1} = y_0 + 0.1(x_0^2 + y_0^2)$
0.1	1.1	1.222	$y_{1+1} = y_1 + 0.1(x_1^2 + y_1^2)$
0.2	1.222	1.3753	$y_{2+1} = y_2 + 0.1(x_2^2 + y_2^2)$
0.3	1.3753	1.573	$y_{3+1} = y_3 + 0.1(x_3^2 + y_3^2)$
0.4	1.573	1.837	⋮
0.5	1.837	2.1995	

For $h = 0.05$

x_n	y_n	y_{n+1}
0	1	1.05
0.05	1.05	⋮
0.1		
0.15		
0.2		
0.25		
0.3		
⋮		
0.5		

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$y = e^{x^2-1}$ se actual value nikal sakte h.

Q#4: Use Euler Method, also Find explicit solution and Find absolute error and % Relative error.

$y' = 2xy \quad y(1) = 1 \quad y(1.5) = ?$

$h = 0.1$

x_n	y_n	y_{n+1}	Actual Value	Abs error |Act. Value − Approx.|	% error
1	1	1.2	1	0.2	20%
1.1	1.2 +1	1.464	1.2337	0.2303	18.67%
1.2	1.464 +2	1.8154	1.5527	0.2627	16.91%
1.3	1.8154 +3	2.874	1.9937	0.8803	44.2%
1.4	2.874 +4	2.9278	2.6117	0.3161	12 %
1.5	2.9278 +5	3.8062	3.4903	0.3169	9.05%

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\, dy = \int 2x\, dx$$

$\ln y = x^2 + C \rightarrow (A)$

$y(1) = 1$

put $x = 1$, $y = 1$

$\ln(1) = 1^2 + C$

$\boxed{C = -1}$

$\ln y = x^2 - 1$

$\boxed{y = e^{x^2-1}}$

For $h = 0.1$ Euler Method,

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_{n+1} = y_n + 0.1(2x_n y_n)$

$y_{n+1} = y_n + 0.2(x_n y_n)$

- Absolute Error → |Actual − Approx. Value|
- % Relative → $\left(\frac{|\text{Actual} - \text{Approx}|}{\text{Actual}}\right) \times 100$

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EHD # 4

8, 29, 30, 31...

* HIGHER D.E :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \dots + a(x)y = f(x)$$

$$\rightarrow \frac{2d^2y}{dx^2} + \frac{dy}{dx} + 2y = 3x$$

$$\rightarrow \frac{d^3y}{dx^3} + 4x\frac{dy}{dx} + 4 = 0$$

* If $a(x), a_{n-1}(x) \dots$ and $f(x)$ are continuous in interval I. and $a_n(x) \neq 0$ at $x = x_0$, then solution is unique or else it can be infinite many or no sol.

HOMOGENEOUS D.E	NON-HOMOGENEOUS D.E
$\curvearrowleft f(x) = 0$	$\curvearrowleft f(x) \neq 0$
$y'' + y' + 3y = 0$	$y'' + y' + 3y = 3x$
$xy'' + 3y' = 0$ no x terms	$xy'' + 3y' + \sin x \neq 0$ x term
ye 'y' ke saath hai. akela 'x' hona chahiye	$y'' + 2y' + 3 = 0$ $\hookrightarrow 3x^0$

(Ex : 4.1)

Q: Find member of family of the sol of IVP.

$$y = c_1 e^{x} + c_2 e^{-x} \quad (-\infty, \infty)$$

$$D.E \rightarrow y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

SOLUTION:-

$$y = c_1 e^{x} + c_2 e^{-x} \rightarrow (A)$$

$$y(0) = 0, \quad x = 0, y = 0$$

$$0 = c_1 + c_2$$

$$c_1 = -c_2 \rightarrow (1)$$

$$y'(0) = 1$$

$$x = 0, \quad y' = 1$$

$$y' = c_1 e^{x} - c_2 e^{-x}$$

$$1 = c_1 - c_2 \rightarrow (2)$$

eq ? $c_1 = -c_2$

eq (2) $\Rightarrow$

$$1 = -c_2 - c_2$$

$$2c_2 = -1$$

$$\boxed{c_2 = -\tfrac{1}{2}}$$

$c_1 = -(-\frac{1}{2})$

$\boxed{c_1 = \frac{1}{2}}$

* EXISTENCE & UNIQUENESS (IVP)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0(x)y = g(x)$$

initial value

$IV = y(x_0) = y_0,\ y'(x_0) = y_0 \dots$

If $a_n(x), a_{n-1}(x), \dots, a_0(x)$ and $g(x)$ are continuous on interval I and $a_n(x) \neq 0$ at $x = x_0$, then solution of $y(x)$ of IVP exist and unique.

(5) $y = c_1 + c_2x^2$ is a solution of $xy'' - y' = 0$ on the intervals $(-\infty, +\infty)$ show that constant c_1, and c_2 but be found at $y(0) = 0$, $y'(0) = 1$

$y = c_1 + c_2x^2 \rightarrow (A)$

$y = 0,\ x = 0$

$\boxed{c_1 = 0}$

$y'(0) = 1$

$y' = 2c_2x$

$1 = 2c_2(0)$

$1 \neq 0$

No solution.

Q6: but $y(0) = 0$, $y'(0) = 0$

$c_1 = 0$,

$y' = 2c_2x$

$0 = 2c_2(0)$

$0 = 0$

c_2 mein hum koi bhi value daalien to zero hi ans hoga to iske too many solutions honge.

$c_2 = 3, 4, 5, 6 \dots$

$c_1 = 0,\quad c_2 \in (-\infty, +\infty)$

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① coefficient continuous in given interval
② $a_n(x) \neq 0$, x_0 is initial condition

Q: In problem 9 and 10, find intervals centered about $x=0$ with the given IVP has a unique.

SOLUTION:-

⑨ $(x-2)y'' + 3y = x$

$y(0)=0$, $y'(0)=1$

conti conti 1st condition

within Interval

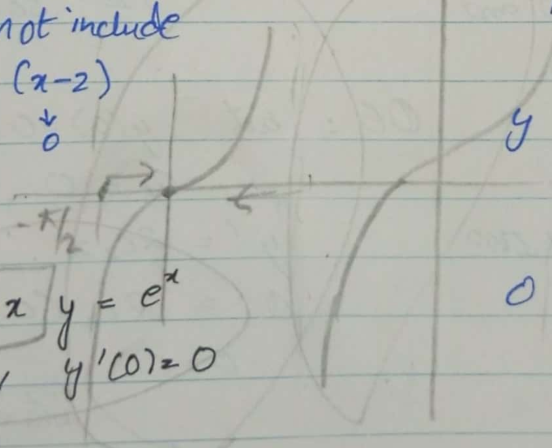
$(-\infty, 2) \cup (2, \infty)$

$\downarrow$

$(-\infty, 2)$

$\downarrow$

2 must not include $(x-2)$
$\downarrow$
0



⑩ $y'' + \boxed{\tan x}\, y = e^x$
$y(0)=1$, $y'(0)=0$

$\tan x = \sin x / \cos x$

is me zero mid pe aata hai

$\Rightarrow (-3\pi/2, 3\pi/2)$

$\hookrightarrow \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Interval,

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ Ans.

$(0, \pi/2)$

* Boundary Value PROBLEM

$y(a)=y_1$, $y(b)=y_2$
$y(a)=y_1$, $y'(b)=y_2$

$c_2 \sin \pi = 0$

Q13: $y = c_1 e^x \cos x + c_2 e^x \sin x$
$y'' - 2y' + 2y = 0$

(a) $y(0)=1$, $y'(\pi)=0$

$y = 1$, $x = 0$
$\boxed{1 = c_1}$

$y' = c_1 [e^x \cdot -\sin x + \cos x\, e^x] + c_2 [e^x \cos x + \sin x\, e^x]$

$0 = c_1 [e^\pi (-\sin(\pi)) + \cos(\pi) e^\pi] + c_2 [e^\pi \cos(\pi) + \sin(\pi) e^\pi]$

$0 = c_1(-e^\pi) + c_2(-e^\pi)$
$0 = (c_1 + c_2) e^{-\pi}$
$\boxed{c_2 = -1}$

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IVP me x same hota BVP mein x different hota hai.

c_1 and c_2 ka abh hi answer hai to unique solution hai ye.

$$\boxed{y = e^x \cos x - e^x \sin x}$$

* "HOMOGENOUS DE"

$x \rightarrow$ is independent

↳ we say y function of x

agar x waali term '0' hai to woh homogenous hai, agr x wali term 0 nhi hai to non-homogenous.

* $y'' + 2y' + y = 0$
* $3xy'' + 3y' - xy = 0$

homogenous

* $3xy'' + 3y' - x = 0$

$f(x)$ = non-homogenous

* 2nd Order D.E

Homogenous

eg: $y_1 = x$, $y_2 = x^2$

$$\boxed{y = c_1 x + c_2 x^2}$$

* doo solution hu agar combine karey tu linear combination

kisi bhi D.E ke solution ko add karde.

Q: $y_1 = x^2$, $y_2 = x^2 \ln x$

↳ $x^3 y''' - 2xy' + 4y = 0$

$$x^3(0) - 2x(2x) + 4(x^2) = 0$$

$$-4x^2 + 4x^2 = 0$$

$$0 = 0$$

$$\boxed{y = c_1 x^2 + c_2 x^2 \ln x}$$

↳ Linear Combinatio

$y = x \quad y_2 = x^2$

$\boxed{y_t = c_1 x + c_2 x^2}$

general eqn

$c_1 = 0,\ c_2 = 1,\ y_2 = x^2$

$c_1 = 1,\ c_2 = 0,\ y = x$

apnimarzi se li hai

$y = x + x^2$

$y = 0,\ c_1 = c_2 = 0$

- Ath solution of all homogenous eqn is always zero.

agr c_1 & c_2 dono zero hogai, to $y = 0$ → always solution of Homogenous eqn

- LINEAR DEPENDENT FUNCTION)

$c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots + c_n f_n = 0$

$c_1 = c_2 = c_3 = \dots = c_n = 0$

→ linear independent.

agr c_1, c_2, c_3 k ans 0 ho yaani zero put krne par eqn satisfy hothi hai to linearly independent warn agr koi doosri values zero k ilawa bhi put hothi ho to lineary depend. ZERO to laazmi hi hoga.

$c_1 y_1 + c_2 y_2 + c_3 y = 0$

eg:

$c_1 \cos^2 x + c_2 \sin^2 x + c_3 (2) = 0$

ab yhya 0,0,0 put krenge to to satisfy hogi hi lekn agr is k ilawa jese $(2, 2, -1)$ put krein to bhi ans ZERO ayega to ye linear dependent hai.

eg: $f_1 = 0,\ f_2 = x,\ f_3 = e^x$

$c_1(0) + c_2 x + c_3 e^x = 0$ to ye bhi lineary dependent

$c_1 = 1,\ c_2 = 0,\ c_3 = 0$

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(*) $y_1 = e^{3x}, \ y_2 = e^{-3x}$

$c_1 e^{3x} + c_2 e^{-3x} = 0$

$(c_1, c_2) = (0, 0)$
linearly independent is k
ilawa koi values nhi daal
sakte.

(*) $f_1 = x, \ f_2 = x^2, \ f_3 = 4x - 3x^2$

$c_1 x + c_2 x^2 + c_3 (4x - 3x^2) = 0$

$(0, 0, 0), \ (-4, 3, 1)$
linear dependent

agr koi ek function bhi zero
ho to wo hamesha linearly
~~inde~~ dependent hoga.

eg:
$f_1 = 0, \quad f_2 = x^2, \ f_3 = 4x - 3x^2$
$c_1(0) + c_2 x^2 + c_3(4x - 3x^2)$
$(1, \ 0, \ 0)$

(16) $f_1 = 2 + x, \quad f_2 = 2 + |x|$

$2 + x \quad x \geq 0 \qquad 2 - x \quad x < 0$

$c_1(2+x) + c_2(2+x) = 0$ linearly dep

$c_1(2+x) + c_2(2-x) = 0 \rightarrow$ linearly indep

• FOR LINEARLY INDEPENDENT :

WRONSKIEN :

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

agr $=0$ hai to inconclusive we cannot tell

$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$

Ex : 4.2

y'' ko coefficient must be 1.

* 2nd ORDER HOMOGENOUS D.E

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

$a_2(x) \neq 0$

q k agr $a_2(x)=0$ hogaya to y'' khtam hojayega.

General Solution → $y = c_1y_1 + c_2y_2$

→ REDUCTION OF ORDER :

Given → y_1 or y_2 → function of x

Find → y_2 or y_1

$y'' + P(x)y' + Q(x)y = 0$

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$y_1 = y_2 \int \frac{e^{-\int p(x)\,dx}}{y_2^2}\,dx$$

$y = c_1y_1 + c_2y_2$

$y = c_1\sin 3x + c_2(c\cos 3x)$

$y = c_1\sin 3x + c_3\cos 3x$

⇒ we dont write minus sign or constants in G.S.

① $y'' - 4y' + 4y = 0$ $y_1 = e^{2x}$

Find 2nd sol → y_2

$p(x) = -4$, $Q(x) = 4$ $y_1 = e^{2x}$

$$\Rightarrow y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

$$= e^{2x}\int \frac{e^{-\int -4\,dx}}{(e^{2x})^2}\,dx$$

$$= e^{2x}\int dx$$

$$\boxed{y_2 = xe^{2x}}$$

G.S ⇒ $y = c_1y_1 + c_2y_2$

$y = c_1e^{2x} + c_2xe^{2x}$

④ $y'' + 9y = 0$, $y_1 = \sin 3x$

$P(x) = 0$, $Q(x) = 9$, $y_1 = \sin 3x$

$$y_2 = \sin 3x \int \frac{e^{-\int 0\,dx}}{\sin^2 3x}$$

$$= \sin 3x \int \operatorname{cosec}^2 3x\, e^{-c_1}\,dx$$

$$= e^{-c_1}\frac{1}{3}\sin 3x \int \operatorname{cosec}^2 3x\,dx$$

$$= -\frac{1}{3}\sin 3x\cot 3x \cdot e^{-c_1}$$

C → all are constants

$$= -\frac{1}{3}\sin 3x \cdot \frac{\cos 3x}{\sin 3x} \Rightarrow \frac{-e^{-c_1}}{3}\cos 3x$$

12, 14, 13, 16

Ex: 4.3

- $y_1 = x \cancel{\ln x} \sin(\ln x)$

$x^2y'' - xy' + 2y = 0$

$y'' - \frac{1}{x}y' + \frac{2}{x^2}y = 0$

$P(x) = -\frac{1}{x}$, $Q(x) = \frac{2}{x^2}$, $y_1 = x\sin(\ln x)$

$y_2 = x\sin(\ln x)\int \frac{e^{-\int -\frac{1}{x}dx}}{x^2\sin^2(\ln x)}$

$= x\sin(\ln x)\int \frac{e^{\ln x}}{x^2\sin^2(\ln x)}$

$= x\sin(\ln x)\int \frac{\cancel{x}}{x^{\cancel{2}}\sin^2(\ln x)}$

$= x\sin(\ln x)\int \operatorname{cosec}^2(\ln x)\cdot\frac{1}{x}\,dx$

$= -x\sin(\ln x)\cdot\cot(\ln x)$

$= -x\cancel{\sin(\ln x)}\cdot\frac{\cos(\ln x)}{\cancel{\sin(\ln x)}}$

$y_2 = -x\cos(\ln x)$

$y = c_1 x\sin(\ln x) + c_2 x\cos(\ln x)$

2nd Order Homogenous Eqn With Constant Coefficent

$ay'' + by' + cy = 0$

$a, b, c \in R$, $a \neq 0$

Let Sol is $y = e^{mx}$

$a(m^2e^{mx}) + bme^{mx} + ce^{mx}$

$e^{mx}(am^2 + bm + c) = 0$

$e^{mx} \neq 0$ as it is equal to y

$\boxed{am^2 + bm + c = 0}$ ↳ auxilary eqn

(1) Roots Are Equal And Real

$y = e^{mx}(c_1 + c_2x)$

(2) If Roots Are Unequal And Real m_1, m_2

$y = c_1e^{m_1x} + c_2e^{m_2x}$

(3) IF ROOTS ARE UNEQUAL AND COMPLEX:

$m = \alpha + \beta i$

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

• → $D = \frac{d}{dx}, \frac{d}{dt}$ new representation of derivative

→ $D^2 = \frac{d^2}{dx^2}$

(1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$D^2y - 3Dy + 2y = 0$

$(D^2 - 3D + 2)y = 0$ } replacing D with m.

Auxilary eqn

$m^2 - 3m + 2 = 0$

$m = 1, 2$

$y = c_1 e^{1x} + c_2 e^{2x}$

(2) $(D^2 + 2D + 1)y = 0$

$m^2 + 2m + 1 = 0$ → by using calculator

$m = -1$

$y = e^{-1x}(c_1 + c_2 x)$

(3) $y'' + 9y = 0$

$D^2y + 9y = 0$

$(m^2 + 9)y = 0$

$m^2 + 9 = 0$

$m = \pm\sqrt{-3}$

$m = \pm 3i$ → roots are complex

$m = \alpha \pm \beta i, \quad \alpha = 0, \beta = 3$

$\Rightarrow y = e^{\alpha x}(c_1 \cos\beta + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$y = c_1 \cos 3x + c_2 \sin 3x$

$m = 3, 5$

$y = c_1 e^{3x} + c_2 e^{5x}$

$m = -3 \pm i$

$y = e^{-3x}(c_1 \cos x + c_2 \sin x)$

$m = 5$

$y = e^{5x}(c_1 + c_2 x)$

28) $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2}$

$, \; D = \frac{d}{ds}$

$2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$

$m^2(2m^3 - 7m^2 + 12m + 8) = 0$

$m^2 = 0 \qquad 2m^3 - 7m^2 + 12m + 8 = 0$

$m = 0, 0 \qquad m = \frac{-1}{2}, \; 2 \pm 2i$

$x = e^{0s}(c_1 + c_2 x) + c_3 e^{-\frac{1}{2}s} +$

$e^{2s}(c_4 \cos 2s + c_5 \sin 2s)$

27) $m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5$

$m^4(m+5) - 2m^2(m+5) + 1(m+5) = 0$

$(m+5)(m^4 - 2m^2 + 1) = 0$

$m = -5 \qquad m^4 - 2m^2 + 1 = 0$

$(m^2 - 1)^2 = 0$

$m = \pm 1$

$m = +1, +1, -1, -1$

34) $y'' + 10y' + 25y = 0$

$y(0) = 1 \;, \; y(1) = 0$

$m^2 + 10m + 25 = 0$

$m^2 + 5m + 5m + 25 = 0$

$m(m+5) + 5(m+5) = 0$

$(m+5)^2 = 0$

$m = 5, 5$

$y = e^{5x}(c_1 + c_2 x)$

$\boxed{1 = c_1}$

$0 = c_1 + c_2$

$\boxed{c_2 = -1}$

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Q# 49 — 58

(36) $y''' + 2y'' - 5y' - 6y = 0$

$y(0) = y'(0) = 0 \quad , y''(0) = 1$

$m^3 + 2m^2 - 5m - 6 = 0$

$m = 2, -1, -3$

$y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-3x}$ — (1)

$y' = 2c_1 e^{2x} - c_2 e^{-x} - 3c_3 e^{-3x} \to$ (2)

$y'' = 4c_1 e^{2x} + c_2 e^{-x} + 9c_3 e^{-3x} \to$ (3)

eq (1) $\Rightarrow$

$0 = c_1 + c_2 + c_3$

$0 = 2c_1 - c_2 - 3c_3$

$1 = 4c_1 + c_2 + 9c_3$

$c_1 = \frac{1}{15} \quad , c_2 = \frac{-1}{6}$

$c_3 = \frac{1}{10}$ put in (A)

(49) $y = c_1 e^{x} + c_2 e^{5x}$

↳ agr sirf constant term involve horgi to isko matlab homogenous hai agr

$y = c_1 e^{x} + c_2 e^{5x} + x$

this is not homogeus

$y = c_1 e^{(1)x} + c_2 e^{5x}$

$m = 1 \quad , m = 5$

$(m - r_1)(m - r_2) = 0 \to$ factor hamesh esi hi bnte hai

$(m-1)(m-5) = 0$

$m^2 - 5m - m + 5 = 0$

$m^2 - 6m + 5 = 0$ reverse kaam karna hai.

$(D^2 - 6D + 5)y = 0$

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$

(55) $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

$y = e^{-x}(c_1 \cos x + c_2 \sin x)$

$y = e^{\alpha x}(c_1 \cos\beta + c_2 \sin\beta x)$

Comparing both eqn

$\alpha = -1 \quad , \beta = 1$

$m = -1 + i \quad , m = -1 - i$

$(m - (-1+i))(m - (-1-i)) = 0$

$(\underbrace{m+1}_{a} - \underbrace{i}_{b})(\underbrace{m+1}_{a} + \underbrace{i}_{b}) = 0$

$a^2 - b^2 = (a+b)(a-b)$

$(m+1)^2 - i^2 = 0$

$m^2 + 2m + 1 - (-1) = 0$

$m^2 + 2m + 2 = 0$

$(D^2 + 2D + 2)y = 0$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

Ex: 4.4

⇒ Non-Homogenous D.E with Constant Coeff:

$ay'' + by' + cy = g(x)$
$a \neq 0$

G.S → $y = y_c + y_p$
general solution

y_c = Complementry Funct.
y_p = Particular Function.

For y_c:

put $g(x)=0$

$ay'' + by' + cy = 0$
↳ Homogenous D.E

Now we will solve by previous method of auxilary eqn.

For y_p:

① $g(x)$ = Constant (2, 3, 4, etc)
Let, $y_p = A$
But if constant appear in y_c: let $y_p = Ax$

② $g(x)$ = poly (degree 1)
$x,\ x+1,\ 2x-1$ etc.

Let, $y_p = Ax+B \rightarrow ①$
agr y_c mein $Ax+B$ ki form aarhi hai to eq① ko 'x' se ⊗ kardenge.
Jese $y_c = c_1 + c_2x$ ye or $Ax+B$ same hi hai to ab hum eq① ko Ax^2+Bx likhenge.

③ $g(x)$ = poly (degree 2)
$x^2+5x-1,\ x^2,\ 2x^2+1$ etc.

Let, $y_p = Ax^2+Bx+C$

Q#1: $y'' + 3y' + 2y = \underbrace{6}_{g(x)} \rightarrow ①$

$\therefore y = y_c + y_p \rightarrow Ⓐ$

For y_c:

$y'' + 3y' + 2y = 0$

$D^2y + 3Dy + 2y = 0$

$y(D^2 + 3D + 2) = 0$

$m^2 + 3m + 2 = 0$

$m = -1, \quad m = -2$

$y_c = c_1e^{-x} + c_2e^{-2x}$

For y_p:

$g(x) = 6 \rightarrow$ constant

Let, $y_p = A$

$y_p' = 0, \quad y_p'' = 0$

using values in eq ①

$0 + 3(0) + 2A = 6$

$A = 3$

$\therefore y_p = 3$

y_p ka ans $g(x)$ ki form mein ayega hamesha agr $g(x)$ constant to y_p bhi constant agr $g(x)$ sin() to y_p bhi sin() ki form mein ayega

Now eq Ⓐ

eq Ⓐ ⇒

$y = c_1e^{-x} + c_2e^{-2x} + 3$

agr G.S mein c_1 or c_2 k ilawa khuch or term ho to iska matlab non-homo ka G.S hai.

Q: $y'' + 2y' = 5$

$y = y_c + y_p \rightarrow Ⓐ$

For y_c:

$y'' - 2y' = 0$

$m^2 - 2m = 0$

$m = 0, \quad m = 2$

$y_c = c_1e^{0x} + c_2e^{2x}$

$y_c = c_1 + c_2e^{2x}$

For y_p:

$g(x) = 5 \rightarrow$ constant

Let, $y_p = Ax$

b/c constant term appears in y_c.

$\therefore y'' - 2y' = 0$

$0 - 2A = 5$

$A = \frac{-5}{2}$ $\quad y_p = \frac{-5}{2}x$

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$y = c_1 + c_2 e^{2x} - \frac{5}{2}x$

(37) $y'' + y = x^2 + 1$ → y_p ka ans bhi x^2 ki form mein ayega.

$y(0) = 5$, $y(1) = 0$

For y_c:

$m^2 + 1 = 0$

$m^2 = -1$

$m = i$, $m = -i$

$y_c = e^0(c_1 \cos x + c_2 \sin x)$

$y_c = c_1 \cos x + c_2 \sin x$

For y_p:

$g(x) = x^2 + 1$

Let, $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$\cancel{2A + A = x^2 + 1}$

$\cancel{3A = x^2 + 1}$

$\cancel{A = \frac{x^2}{3} + \frac{1}{3}}$

$2A + Ax^2 + Bx + C = x^2 + 1$

$Ax^2 + Bx + 2A + C = x^2 + 1$

Comparing coeff

$\boxed{A = 1}$

$\boxed{B = 0}$

$2A + C = 1$

$2(1) + C = 1$

$\boxed{C = -1}$

$y_p = (1)x^2 + (0)x - 1$

$y_p = x^2 - 1$

$y = c_1 \cos x + c_2 \sin x + x^2 - 1$ → (A)

$5 = c_1 + \cancel{x}^{(0)2} - 1$

$\boxed{c_1 = 6}$

$0 = c_1 \cos 1 + c_2 \sin(1) + \cancel{1} - \cancel{1}$

$0 = 6 \cos 1 + c_2 \sin(1)$

$-6\cos(1) = c_2 \sin(1)$

$\frac{-6\cos(1)}{\sin(1)} = c_2$

$c_2 = -6\cot(1)$

(A) ⟹

$y = 6\cos x + 6\cot(1)\sin x + x^2 - 1$

(5) $y^{(4)} + 2y'' + y = (x-1)^2$

$\Rightarrow y = y_c + y_p$

For y_c :

$m^4 + 2m^2 + 1 = 0$

$(m^2+1)^2 = 0 \Rightarrow (m^2+1)(m^2+1)=0$

$m^2+1 = 0$

$m = \pm i \, , \pm i$

q k power 4 hai,

$y_c = c_1 \cos x + c_2 \sin x + c_3 \cos x + c_4 \sin x$

$y_c = \cos x(c_1 + c_3) + \sin x(c_2 + c_4)$

m ki dono pairs same aarhe hai to ek pair k sath 'x' likh denge.

$y = c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)$

For y_p:

$g(x) = (x-1)^2$

Let, $y_p = Ax^2 + Bx + C$

$, \; y^{(4)} + 2y'' + y = (x-1)^2$

$0 + 2(2A) + Ax^2 + Bx + C = x^2 - 2x + 1$

$~~4A~~ +$

$Ax^2 + Bx + 4A + C = x^2 - 2x + 1$

$\boxed{A=1}$

$\boxed{B=-2}$

$4A + C = 1$

$4(1) + C = 1$

$\boxed{C=-3}$

$y_p = x^2 + 2x - 3$

$y = x^2 - 2x - 3 + c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)$.

① If $g(x) = e^{ax}$

Let $\boxed{y_p = Ae^{ax}}$

If e^{ax} in y_c then

$y_p = Axe^{ax}$

② If $g(x) = \sin ax$ or $\cos ax$

Let,

$\boxed{y_p = A\cos ax + B\sin ax}$

If, $\cos ax$ or $\sin ax$ in y_p

then,

$y_p = x(A\cos ax + B\sin ax)$

⑩ $y'' + 2y' = 2x + 5 - e^{-2x}$

For y_c:

$y'' + 2y' = 0$

$m^2 + 2m = 0$

$m = 0 \quad , \quad m = -2$

$y_c = c_1 + c_2 e^{-2x}$

For y_p:

$g(x) = 2x + 5 - e^{-2x}$

$y_p = Ax^2 + Bx$

↳ ye jab likhte jab sirf $2x+5$ (ek constant phele se mojood hai is waja se x se ⓧ krdiya)

or doosra

$y_p = Ae^{ax}$

lekin q k phele ye hamare paas y_c mein mojood hai to humne 'x' se ⓧ krdiya to $y_p = Axe^{ax}$

so final y_p will be,

$$y_p = \underbrace{Ax^2 + Bx}_{\text{ye } 2x+5 \text{ k liye}} + \underbrace{Cxe^{-2x}}_{-e^{-2x}}$$

A phele liya wa tha to C le liya

$y'' + 2y' = 2x + 5 - e^{-2x} \rightarrow$ Ⓐ

$y_p' = 2Ax + B + C(e^{-2x} - 2xe^{-2x})$

$y_p'' = 2A + C(-2e^{-2x} - 2(e^{-2x} - 2xe^{-2x}))$

$y_p'' = 2A - 4Ce^{-2x} + 4Cxe^{-2x}$

eq Ⓐ ⇒

$2A - 4Ce^{-2x} + \cancel{4Cxe^{-2x}} + 4Ax + 2B + 2Ce^{-2x} - \cancel{4Cxe^{-2x}} = 2x + 5 - e^{-2x}$

Comparing Coefficient

$\Rightarrow 4A = 2 \Rightarrow \boxed{A = 1/2}$

$\Rightarrow 2A + 2B = 5$

$1 + 2B = 5 \Rightarrow \boxed{B = 2}$

$\Rightarrow +2C = +1 \Rightarrow C = 1/2$

$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

$y = y_c + y_p$

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(1) $4y'' - 4y' - 3y = \cos 2x$

For y_c:

$4m^2 - 4m - 3 = 0$

$m = -\frac{1}{2}, \frac{3}{2}$

$y_c = c_1 e^{-\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}$

For y_p:

$g(x) = \cos 2x$

Let,

$y_p = A\cos 2x + B\sin 2x$

$\Rightarrow 4y'' - 4y' - 3y = \cos 2x$

$4(-4A\cos 2x - 4B\sin 2x) - 4(-2A\sin 2x + 2B\cos 2x)$
$-3(A\cos 2x + B\sin 2x) = \cos 2x$

$-16A\cos 2x - 16B\sin 2x + 8A\sin 2x - 8B\cos 2x -$
$- 3A\cos 2x - 3B\sin 2x = \cos 2x$

$(-19A - 8B)\cos 2x + (-19B + 8A)\sin 2x = \cos 2x$

- Comparing coeff.

$-19A - 8B = 1 \rightarrow (1)$

$8A - 19B = 0 \rightarrow (2)$

$A = \frac{19B}{8} \rightarrow$ put in (1)

eq (1) $\Rightarrow -19\left(\frac{19B}{8}\right) - 8B = 1$

$-361B - 64B = 8$

$-425B = 8$

$\boxed{B = \frac{-8}{425}}$

$A = \frac{19}{8} \cdot \left(\frac{-8}{425}\right)$

$\boxed{A = \frac{19}{425}}$

- can use calculator to solve this eqns.

(32) $y'' - y = \cosh x$

$y(0) = 3, \; y'(0) = 1$

For y_c:

$y'' - y = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$y_c = c_1 e^{x} + c_2 e^{-x}$

For y_p:

$g(x) = \cosh x = \frac{e^{x} + e^{-x}}{2}$

$y_p = Ae^x + Be^{-x}$

ye dono y_c mein mojood hai to ⊗ by 'x'

$y_p = Axe^x + Bxe^{-x}$

$\Rightarrow y'' - y' = \frac{e^x}{2} + \frac{e^{-x}}{2}$

$y_p' = \cdot$

⊗ $y'' + 4y = 3\sin 2x$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$g(x) = 3\sin 2x$

$y_p = Ax\sin 2x + Bx\cos 2x$

'x' se multiply karaya hai q k y_c mein mojood hai.

⊗ $g(x) = e^{2x} - 5\sin x$

$y_p = Ae^{2x} + B\sin x + C\cos x$

⊗ $g(x) = x^2 + 5\cos 3x$

$y_p = Ax^2 + Bx + C + D\cos 3x + E\sin 3x$

⊗ $g(x) = \sin x + \cos 2x$

$= A\sin x + B\cos x + C\cos 2x + D\sin 2x$

$g(x)$	y_p
xe^{ax}	$(Ax+B)e^{ax}$
$(x^2+1)e^{ax}$	$(Ax^2+Bx+C)e^{ax}$

agr kabhi bhi koi polynomial, exponential k saath ⊗ hoga usko let wali cheez ko exponential ki saath likhdenge.

$g(x) = xe^{3x}$

$y_c = e^{3x}(c_1 + c x)$

dot,

$y_p = e^{3x}(Ax+B)$

to ye ab same same hogai to 'x' se ⊗ hojayega

$y_p = e^{3x}(Ax^2 + Bx)$

$\sin ax \cdot e^{ax} \rightarrow (A\sin ax + B\cos ax)e^{ax}$

$\cos ax \cdot e^{ax}$

agr koi trigrometric ajaye to bhi usko ese hi 'e' k saath likhdenge or agr repeat horha ho to 'x' se multiply kardenge.

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$y'' + 3y = -10x^2e^{3x}$

$y = y_c + y_p$

For y_c:

$y'' + 3y = 0$

$D^2y + 3y = 0$

$(D^2+3)y = 0$

$m^2 + 3 = 0$

$m = \pm\sqrt{3}i$

$y_c = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x$

For y_p:

$g(x) = -10x^2e^{3x}$

Let,

$y_p = (Ax^2+Bx+C)e^{3x}$

$\Rightarrow y'' + 3y = -10x^2e^{3x} \rightarrow (1)$

$y_p' = 3(Ax^2+Bx+C)e^{3x} + (2Ax+B)e^{3x}$

$y_p' = e^{3x}(3Ax^2+3Bx+3C+2Ax+B)$

$y_p'' = e^{3x}(6Ax+3B+2A) + 3e^{3x}(3Ax^2+3Bx+3C+2Ax+B)$

eq (1) — 3 se multiply karke e^{3x} common le liya or add kardiya

$e^{3x}(12Ax + 6B + 9Ax^2 + 9Bx + 9C + 2A) + 3(Ax^2+Bx+C)e^{3x} = -10x^2e^{3x}$

$e^{3x}[12Ax + 6B + 12Ax^2 + 12Bx + 12C + 2A] = -10x^2e^{3x}$

Comparing Coeffi —

$12A = -10 \Rightarrow \boxed{A = -5/6}$

$12B + 12A = 0 \Rightarrow 12B + 12\left(\frac{-5}{6}\right) = 0$

$\boxed{B = \frac{5}{6}}$

$2A + 6B + 12C = 0$

$2\left(\frac{-5}{6}\right) + 6\left(\frac{5}{6}\right) + 12C = 0$

$\frac{-5}{3} + 5 + 12C = 0$

$-5 + 15 + 36C = 0$

$36C = -10$

$C = \frac{7}{12}$

$y_p =$

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$y_c = e^x(c_1 + c_2 x)$

$y_c = c_1 e^x + c_2 x e^x$

$g(x) = e^x - 3e^{3x}$

$y_p = Ae^x + Be^{-x}$ ⊗ by 'x' as 1st term is repeat.

$y_p = Axe^x + Be^{-x}$ again ⊗ by 'x'

$y_p = Ax^2 e^x + Be^{-x}$ ✓

2. $y_c = c_1 \cos 2x + c_2 \sin 2x$

$g(x) = 3\cos 2x$

$y_p = A\cos 2x + B\sin 2x$ ⊗ by 'x'

$y_p = Ax\cos 2x + Bx\sin 2x$ ✓

3. $y_c = c_1 \cos 2x + c_2 \sin 2x$

$g(x) = x + \sin x$

$y_p = (Ax+B) + C\cos x + D\sin x$ ✓

4. $y_c = c_1 e^{3x} + c_2 e^{5x}$

$g(x) = 100x - 26xe^x$

$y_p = Ax + B + (Cx + D)e^x$

5. $y_c = c_1 e^x + c_2 e^{0.23x} + c_3 e^{-1.69x}$

$g(x) = 5 - e^x + e^{2x}$

$y_p = A + Bxe^x + Ce^{2x}$ ✓

6. $y_c = c_1 + c_2 e^{-2x}$

$g(x) = 2x - 5 - e^{-2x}$

$y_p = Ax + B + Ce^{-2x}$ ⊗ by 'x'

$y_p = (Ax+B)x + Cxe^{-2x}$ ✓

7. $y_c = e^x(c_1 \cos x + c_2 \sin x)$

$g(x) = e^{2x}(\cos x - 3\sin x)$

↳ dono k angle same hai to ek hi mastaba let karenge

$y_p = e^{2x}(A\cos x + B\sin x)$ ✓

y_c mein $A\cos x + B\sin x$ 'x' se ⊗ hosta hai or neeche e^{2x} se to is waja se ⊗ nhi hoga.

8. $y_c = c_1 e^{-x} + c_2 x e^{-x}$

$g(x) = \sin x + 3\cos 2x$

dono k angle differ. hai to alag let kiya h.

$y_p = A\sin x + B\cos x + C\cos 2x + D\sin 2x$ ✓

$x \cdot \sin bx \rightarrow (Ax+B)\cos bx + (Cx+D)\sin bx$

$(x+1)\sin bx \nearrow$

$(x^2+2x)\sin 3x \rightarrow (Ax^2+Bx+C)\cos 3x + (Dx^2+Ex+F)\sin 3x$

$(x^2-3)\sin 3x \nearrow$

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Ex: 4.6

⊗ $y'' + 2y'$ $y_c = e^{x}(c_1\cos 2x + c_2\sin 2x)$

$y_p = Ae^{2x}$

$g(x) = e^{x}\cos 2x$

$y_p = e^{x}(A\cos 2x + B\sin 2x)$ ⊗ by 'x'

$y_p = e^{x}x(A\cos 2x + B\sin 2x)$

⊗ $y'' + a^2y = b\sin\omega x \rightarrow ①$

For y_c:

$m^2 + a^2 = 0$

$m = \pm ai$

$y_c = c_1\cos ax + c_2\sin ax$

For y_p:

$g(x) = b\sin\omega x$

let $y_p = A\cos\omega x + B\sin\omega x$

$y_p' = -\omega A\sin\omega x + B\omega\cos\omega x$

$y_p'' = -A\omega^2\cos\omega x - B\omega^2\sin\omega x$

eq①

$-A\omega^2\cos\omega x - B\omega^2\sin\omega x + Aa^2\cos\omega x + Ba^2\sin\omega x = b\sin\omega x$

$-B\omega^2 + Ba^2 = b$

$-B\omega^2 + Aa^2 = 0$

$B = \dfrac{b}{a^2-\omega^2}$

$\boxed{A = 0}$

Solution of Non-Homogenous D.E by Variation of Parameters

↳ ye doosra method hai non-homogenous Eq to solve karne ke liye.

$$ay'' + by' + cy = f(x)$$

$$y = y_c + y_p$$

$$y_c = c_1y_1 + c_2y_2$$

$$y_p = u_1y_1 + u_2y_2$$

We have,

$$u_1 = \int \frac{W_1}{W}\,dx, \quad u_2 = \int \frac{W_2}{W}\,dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Q2: $y'' + y = \tan x$ coefficent must be '1'

$y = y_c + y_p$

For y_c:

$y'' + y = 0$

$D^2 y + y = 0$

$(D^2 + 1) y = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = \underbrace{c_1 \cos x}_{y_1} + \underbrace{c_2 \sin x}_{y_2}$

For y_p:

$y_p = u_1 y_1 + u_2 y_2$

$y_1 = \cos x$, $y_2 = \sin x$

$y_1' = -\sin x$, $y_2 \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x = 1$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}$$

$= -\tan x \sin x$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}$$

$W_2 = \cos x \tan x = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$

$u_1 = \int \frac{-\tan x \sin x}{1}$

$= -\int \frac{\sin^2 x}{\cos x}$

$= -\int \frac{(1 - \cos^2 x)}{\cos x}$

$= -\int \frac{1}{\cos x} - \frac{\cos^2 x}{\cancel{\cos x}}$

$= -\int \sec x \, dx + \int \cos x \, dx$

$= -\ln(\sec x + \tan x) + \sin x$

$u_2 = \int \frac{\sin x}{1}$

$u_2 = -\cos x$

$y_p = [-\ln(\sec x + \tan x) + \sin x] \cos x -$

$\cos x \sin x$

$$\boxed{y = c_1 \cos x + c_2 \sin x \; [-\ln(\sec x + \tan x) + \sin x] \cos x - \cos x \sin x.}$$

$y'' + 9y = \sec 3x$

For y_c:

$y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \pm 3i$

$y_c = c_1 \cos 3x + c_2 \sin 3x$

$y_1 = \cos 3x$, $y_2 = \sin 3x$

$y_1' = -3 \sin 3x$, $y_2' = 3 \cos 3x$

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by variation of parameter wale method se sare sawal solve hosakte hai lekin undetermined coefficient wala method sirf polynomial, exponential, constants, sinx, cosx par valid hai.

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$W = 3\cos^2 3x + 3\sin^2 3x$

$W = 3$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \sec 3x & 3\cos 3x \end{vmatrix}$$

$= \sec 3x \cdot \sin 3x$

$W_1 = \tan 3x$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \sec 3x \end{vmatrix}$$

$= \cos 3x \cdot \sec 3x$

$W_2 = 1$

$$U_1 = \int \frac{\tan 3x}{3}$$

$U_1 = \frac{1}{9} \ln(\sec 3x)$

$$U_2 = \int \frac{1}{3}\, dx$$

$U_2 = \frac{1}{3} x$

$y_p = \frac{1}{9} \ln(\sec 3x) \cos 3x + \frac{1}{3} x \sin 3x$

(8) $y'' - y = \sinh 2x$

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$y_c = C_1 e^{x} + C_2 e^{-x}$

$y_1 = e^{x} \quad y_2 = e^{-x}$

$y_1' = e^{x} \quad y_2' = -e^{-x}$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}$$

$= -e^{0} - e^{0} = -2$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \sinh 2x & -e^{-x} \end{vmatrix}$$

$= -\sinh 2x \cdot e^{-x}$

$= -e^{-x}\left[\frac{e^{2x} - e^{-2x}}{2}\right]$

$W_1 = \frac{-e^{x} + e^{-3x}}{2}$

$$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \sinh 2x \end{vmatrix}$$

$= e^{x} \sinh 2x$

$= e^{x}\left[\frac{e^{2x} - e^{-2x}}{2}\right] = \frac{e^{3x} - e^{-x}}{2}$

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$$U_1 = \int \frac{\frac{-e^{x}+e^{-3x}}{2}}{+2}$$

$$= \int \frac{-e^{x}+e^{-3x}}{-4}$$

$$= -\frac{1}{4}\left[\int -e^{-x} + \int e^{-3x}\right] dx$$

$$\boxed{U_1 = \frac{-1}{4}\left[-e^{x} - \frac{1}{3}e^{-3x}\right]}$$

$$U_2 = \int \frac{\frac{e^{3x}-e^{-x}}{2}}{-2}$$

$$= -\frac{1}{4}\left[\int e^{3x} - \int e^{-x}\right] dx$$

$$\boxed{U_2 = -\frac{1}{4}\left[\frac{1}{3}e^{3x} + e^{-x}\right]}$$

(12) $y'' - 2y' + y = \frac{e^{x}}{1+x^2}$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y_c = e^{x}(c_1 + c_2 x)$

$y_c = c_1 e^{x} + c_2 x e^{x}$

$y_1 = e^{x}$ $\quad y_2 = xe^{x}$

$y_1' = e^{x}$ $\quad y_2' = e^{x} + xe^{x}$

$$W = \begin{vmatrix} e^{x} & xe^{x} \\ e^{x} & e^{x}+xe^{x} \end{vmatrix}$$

$$= e^{2x} + xe^{2x} - xe^{2x}$$

$W = e^{2x} + \cancel{2xe^{2x}}$

$\cancel{W_1} =$

$$W_1 = \begin{vmatrix} 0 & xe^{x} \\ \frac{e^{x}}{1+x^2} & e^{x}+xe^{x} \end{vmatrix}$$

$$= -xe^{x}\left[\frac{e^{x}}{1+x^2}\right]$$

$$W_1 = \frac{-xe^{2x}}{1+x^2}$$

$$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{e^{x}}{1+x^2} \end{vmatrix}$$

$$W_2 = \frac{e^{2x}}{1+x^2}$$

$$U_1 = \int \frac{\frac{-xe^{2x}}{1+x^2}}{e^{2x}}$$

$$= -\int \frac{x}{1+x^2}\, dx$$

$$U_1 = -\frac{1}{2}\ln(1+x^2)$$

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23 to 26
integeratio not possib!

$$U_2 = \int \frac{\frac{e^{x}}{1+x^2}}{e^{x}}$$

$$= \int \frac{1}{1+x^2}$$

$$U_2 = \tan^{-1}(x)$$

$$y''' + 4y' = \sec 2x$$

For y_c:

$$y''' + 4y' = 0$$

$$m^3 + 4m = 0$$

$$m = 0, \quad m = \pm 2i$$

$$y_c = C_1 + C_2 \cos 2x + C_3 \sin 2x$$

For y_p:

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

$y_1 = 1$ $\quad y_2 = \cos 2x$ $\quad y_3 = \sin 2x$

$y_p' = 0$ $\quad y_2' = -2\sin 2x$ $\quad y_3' = 2\cos 2x$

$y_p'' = 0$ $\quad y_2'' = -4\cos 2x$ $\quad y_3'' = -4\sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

Exp

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}$$

Ex: 4.7

Cauchy Euler D.E

$$ax^2y'' + bxy' + cy = f(x)$$

y'' k saath x^2 hona chahiye y' k saath x. To hi we Cauchy eqn kehlaigi.

Let, $y = x^m$

For y_c:

$$ax^2y'' + bxy' + cy = 0$$

CASE #01 → Roots unequal / Real

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

CASE #02 → Roots equal / Real

$$y_c = x^m(c_1 + c_2 \ln x)$$

CASE #03 → Complex Roots

$$y_c = x^{\alpha}(c_1 \cos\beta \ln x + c_2 \sin\beta \ln x)$$

y_p → will be found by variation of parameter.

- The coefficent of y'' must be 1 to calculate y_p.

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⑥ $x^2y'' + 5xy' + 3y = 0 \rightarrow ①$

ye cauchy euler D.E hai bcz

x^2 with y'' & x with y'.

This is homogenous so no y_p ($y_p = 0$)

$y = y_c + y_p$

$y = y_c + 0$

$y = y_c$

Let, $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

put in eq ①

$x^2[m(m-1)x^{m-2}] + 5x[mx^{m-1}] + 3x^m = 0$

$m(m-1)x^m + 5mx^m + 3x^m = 0$

$(m(m-1) + 5m + 3)x^m = 0$

$m(m-1) + 5m + 3 = 0$ $\quad x^m \neq 0$ as it is = to y

$m^2 - m + 5m + 3 = 0$

$m^2 + 4m + 3 = 0$

$m = -1, \quad m = -3$

$y_c = c_1x^{-1} + c_2x^{-3}$

$y_c = \frac{c_1}{x} + \frac{c_2}{x^3}$

$y = y_c + y_p$

$y = \frac{c_1}{x} + \frac{c_2}{x^3}$

$\Rightarrow x^2y'' = m(m-1)$

$\Rightarrow xy' = m$

$\Rightarrow x^3y''' = m(m-1)(m-2)$

direct put krenge

• $m = 0, 1, 2$

$y = c_1x^0 + c_2x^1 + c_3x^2$

• $m = 2, 3, 3$

$y = x^3(c_1 + c_2\ln x) + c_3x^2$

• $m = 1 \pm 2i, 5$

$y_c = c_1x^5 + x^1(c_2\cos 2\ln x + c_3\sin 2\ln x)$

• $m = 2, 2, 4, 4$

$y_c = x^2(c_1 + c_2\ln x) + x^4(c_3 + c_4\ln x)$

⑲ $xy'' - 4y' = x^4$

(x) b/s by x → take x^2 miljaye

$x^2y'' - 4xy' = x^5$

$y = y_c + y_p$

For y_c:

$x^2y'' + 4xy' = 0$

$m(m-1) - 4m = 0$

$m^2 - m - 4m = 0$

$m^2 - 5m = 0$

$m = 0, \quad m = 5$

$y_c = c_1x^0 + c_2x^5 = c_1 + c_2x^5$

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For y_p:

$x^2y'' - 4xy' = x^5$

$\div$ b/s by x^2

$y'' - \frac{4}{x}y' = x^3 \longrightarrow f(x)$

$y_p = U_1y_1 + U_2y_2$

$y_1 = 1 \qquad y_2 = x^5$

$y_1' = 0 \qquad y_2' = 5x^4$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$W = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix}$

$W = 5x^4$

$W_1 = \begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}$

$W_1 = -x^8$

$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}$

$W_2 = x^3$

$U_1 = \int \frac{W_1}{W} dx \quad , \quad U_2 = \int \frac{W_2}{W} dx$

$U_1 = \int \frac{-x^8}{5x^4} dx \quad , \quad U_2 = \int \frac{x^3}{5x^4} dx$

$U_1 = \frac{-1}{5}\int x^4 dx \qquad U_2 = \frac{1}{5}\int \frac{1}{x} dx$

$U_1 = \frac{-1}{25}x^5 \qquad U_2 = \frac{1}{5}\ln x$

$y = y_c + y_p$

(x) $x^2y'' - 2xy' + 2y = x^4e^x$

$m(m-1) - 2m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m = 1, \ m = 2$

$y_c = c_1x + c_2x^2$

For y_p: $\div$ by x^2

$y'' + \frac{2}{x}y' + \frac{3}{x^2}y = x^2e^x$

$y_1 = x \qquad y_2 = x^2$

$y_1' = 1 \qquad y_2' = 2x$

$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$= \begin{vmatrix} 0 & x^2 \\ x^3e^x & 2x \end{vmatrix}$

$= -x^4e^x$

$=$

Ex # 13.5

(CHAIN RULE)

If $z = x^2y$, $x = t^2$ & $y = t^3$ then find $\frac{dz}{dt}$

$$\begin{tikzcd} & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\ t & & t \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy)(2t) + (x^2)(3t^2)$$

$$\frac{dz}{dt} = 4xyt + 3x^2t^2$$

$$\frac{dz}{dt} = 4(t^2)(t^3)(t) + 3(t^2)^2t$$

$$\frac{dz}{dt} = 4t^6 + 3t^6 = 7t^6$$

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$W = \sqrt{x^2+y^2+z^2}$, $x = \cos\theta$

$y = \sin\theta$, $z = \tan\theta$

Find $\frac{dw}{d\theta}$

$$\begin{tikzcd} & w \arrow[dl] \arrow[d] \arrow[dr] & \\ x \arrow[d] & y \arrow[d] & z \arrow[d] \\ \theta & \theta & \theta \end{tikzcd}$$

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$=$

(23) $T = x^2y - xy^3 + 2$

$x = r\cos\theta$ + $y = r\sin\theta$

Find $\frac{\partial T}{\partial r}$ & $\frac{\partial T}{\partial \theta}$

$$\begin{tikzcd} & & T \arrow[dl, "\frac{\partial T}{\partial x}"'] \arrow[dr, "\frac{\partial T}{\partial y}"] & & \\ & x \arrow[dl, "\frac{\partial x}{\partial r}"'] \arrow[d, "\frac{\partial x}{\partial \theta}"] & + & y \arrow[d, "\frac{\partial y}{\partial r}"'] \arrow[dr, "\frac{\partial y}{\partial \theta}"] & \\ r & \theta & & r & \theta \end{tikzcd}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial r}$$

humein yaha $\frac{\partial T}{\partial r}$ chahiye that is wajah se $\frac{\partial x}{\partial r}$ wale ko choose kia hai, is wajah se $\frac{\partial x}{\partial \theta}$ nhi likha

$$\frac{\partial T}{\partial r} = (2xy - y^3)(\cos\theta) + (x^2 - 3xy^2)(\sin\theta)$$

$= [2(r\cos\theta)(r\sin\theta) - r^3\sin^3\theta][\cos\theta] + [r^2\cos^2\theta - 3(r\cos\theta)(r^2\sin^2\theta)][\sin\theta]$

$= 2r^2\cos^2\theta\sin\theta - r^3\sin^3\theta\cos\theta + r^2\cos^2\theta\sin\theta - 3r^3\cos\theta\sin^3\theta$

$= 3r^2\cos^2\sin\theta - 4r^3\sin^3\theta\cos\theta$

$= \cancel{r\,3\cos\theta}$

$= r^2\cos\theta\sin\theta[3\cos\theta - 4r\sin^2\theta]$

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta}$$

$= (2xy - y^3)(-r\sin\theta) + (x^2 - 3xy^2)(r\cos\theta)$

Find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$ if $z = \frac{x}{y}$

$x = 2\cos u$, $y = 3\sin v$

$$\begin{tikzcd} & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x \arrow[d, "\frac{dx}{du}"'] & & y \arrow[d, "\frac{dy}{dv}"] \\ u & & v \end{tikzcd}$$

iskohi lenge ∂v k isse saath vi manchhiki

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \cancel{\frac{\partial z}{\partial y}}\left(\quad\right)$$

ye req tha q k humein $\frac{\partial z}{\partial u}$ chahiye

→ yaha $\frac{\partial y}{\partial v}$ is liye nhi liya q k humein yaha $\frac{\partial z}{\partial v}$ req nhi hai.

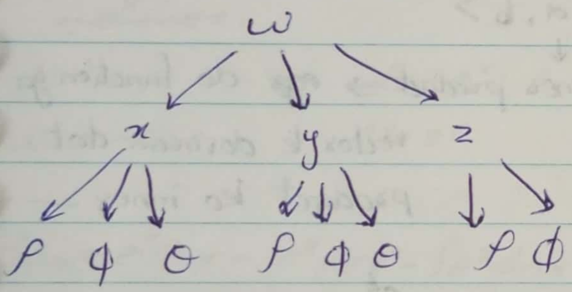
$$\frac{\partial z}{\partial v} = \cancel{\frac{\partial z}{\partial x}} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$w = 4x^2 + 4y^2 + z^2$

$x = \rho\sin\phi\cos\theta \quad , y = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$

Find $\frac{\partial w}{\partial \rho}$, $\frac{\partial w}{\partial \phi}$, $\frac{\partial w}{\partial \theta}$



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \phi} = " \frac{\partial x}{\partial \phi} + " " \frac{}{\partial \phi} + " \frac{"}{\partial \phi}$$

$$\frac{\partial w}{\partial \theta} = " \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \cancel{\frac{\partial w}{\partial z}}$$

13, 1b, 20

Ex: 13-8

→ Max & Min Value of Two variable Function.

$F(x,y) =$ → ①

$f_x = \frac{\partial f}{\partial x} = 0$

$f_y = \frac{\partial f}{\partial y} = 0$

inko solve karne se 2 eqn milenge un dono ko solve karke x or y nikaal lenge

value of x & y will be as critical point. (x_0, y_0)

double partial derivative of f_x

f_{xx} → put (x_0, y_0)

f_{yy} → put (x_0, y_0)

f_{xy} → put (x_0, y_0)

→ first w.r.t x that w.r.t y (partial)

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

① IF $D > 0$ & $f_{xx} > 0$ so, f is min at (x_0, y_0)

② IF $D > 0$ & $f_{xx} < 0$, so f is max at (x_0, y_0)

③ If $D < 0$ then f has saddle point

④ If $D = 0$, the no conclusion can be drawn.

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$ if $a>0,\ b>0$

$D>0$, $f_{xx}>0 \rightarrow$ min

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$, $a>0,\ b<0$

$D>0$, $f_{xx}<0$, maxima

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$, $a<0,\ b>0$

$f_{xx}<0$, $D>0$, maxima

$\Rightarrow f_{xx} = \frac{2b^3}{a^3}$ $a<0,\ b<0$

$D>0$, $f_{xx}>0$ max

—— x —— x —— x ——

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

[CHP

Orthogonal

$\vec{a} = (1,0)$
$\vec{b} = (0,1)$

$a\cdot b = i + j = 0$
$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$

If ~~dot product~~

$\langle a, b\rangle$
↓
inner product → ~~eg~~ vector k dot product k

$\langle f_1, f_2\rangle = \int_a^b f_1 \cdot f_2\, dx$

if ans of $\langle f_1, f_2\rangle = 0$
f_1 & f_2 are orthogonal

(2) $f_1(x) = x^3$, $f_2(x) = x$

$\langle f_1, f_2\rangle = \int_a^b f_1\cdot f_2\, dx$

$\langle f_1, f_2\rangle = \int_{-1}^{1} x^3\cdot(x^2+1)\,dx$

$= \int_{-1}^{1} x^5 dx + \int_{-1}^{1} x^3$

$= \frac{x^6}{6}\Big|_{-1}^{1} + \frac{x^4}{4}\Big|_{-1}^{1}$

$= \left(\frac{1}{6}-\frac{1}{6}\right) + \left(\frac{1}{4}\right.$

$= 0$

f_1 & f_2 are orthogonal