Ch#07

Orthogonal Matrices

$$A^T \cdot A = A \cdot A^T = I$$
$$A^T = A^{-1}$$

Q.1) Determine whether the matrix as orthogonal and of so find ~~that~~ its inverse.

(A) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^T \cdot A = I$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix is orthogonal

$$A^T = A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Q.) show that the matrix is orthogonal in these ways:

① $A^T \cdot A = I$

② By rows

③ By columns

$$A = \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$$

$$\begin{bmatrix} 4/5 & -9/25 & 12/25 \\ 0 & 4/5 & 3/5 \\ -3/5 & -12/25 & 16/25 \end{bmatrix} \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$$

$$A \cdot A^T = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By Rows

$$\left.\begin{array}{l} r_1 \cdot r_2 = 0 \\ r_2 \cdot r_3 = 0 \\ r_1 \cdot r_3 = 0 \end{array}\right\} \text{so orthogonal}$$

$$\left.\begin{array}{l} \|r_1\| = 1 \\ \|r_2\| = 1 \\ \|r_3\| = 1 \end{array}\right\} \text{so orthonormal}$$

By col

$$\left.\begin{aligned} C_1 \cdot C_2 &= 0 \\ C_1 \cdot C_3 &= 0 \\ C_2 \cdot C_3 &= 0 \end{aligned}\right\}$$ By orthogonal

$$||C_1|| = 1$$
$$||C_2|| = 1$$
$$||C_3|| = 1$$

CH# 06

Date

" INNER PRODUCT SPACES "

$(U, V) = (V, U)$

$(U, V) = U_1V_1 + U_2V_2 \dots U_nV_n$

$|V| = \sqrt{V \cdot V}$

$d(U, V) = |U - V| = \sqrt{(U-V)\cdot(U-V)}$

$(U, V) = AU \cdot AV$

A is any matrix

<u>Standard Inner product on M_{22}:</u>

$(U, V) = tr(U^T V)$

Ex: 6.1

Q1: Let R^2 have the weighted Euclidean inner product

$(U, V) = 2U_1V_1 + 3U_2V_2$

$$\begin{array}{ccc} U_1 \; U_2 & V_1 \; V_2 & W_1 \; W_2 \\ U = (1,1), & V = (3,2), & W = (0,-1) \end{array}$$

$K = 3$

(a) (U, V)

$(U, V) = 2U_1V_1 + 3U_2V_2$

$= 2(1)(3) + 3(1)(2)$

$= 12$

(b) (KV, W)

$= 2KV_1W_1 + 3KV_2W_2$

$= 2(3)(1)(0) + 3(3)(2)(-1)$

$= -18$

(c) $(U+V, W)$

$= 2(U_1 + V_1)W_1 + 3(U_2+V_2)$

$= -9$

(d) $|V| = \sqrt{V \cdot V} = \sqrt{2V_1V_1 + 3V_2V_2}$

$= \sqrt{30}$

(e) $d(U, V)$

$= |U - V|$

$= \sqrt{(U-V)\cdot(U-V)}$

$(U - V) = (1-3, 1-2) = (-2, -1)$

$= \sqrt{(-2,-1)\cdot(-2,-1)}$

$= \sqrt{2(-2)(-2) + 3(-1)(-1)}$

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$= \sqrt{11}$

b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$A^T \cdot A \neq I$

The matrix is not orthogonal

c) $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{6} + \frac{1}{3} & 0 \frac{-2}{6} + \frac{1}{3} & \frac{-1}{2} + \frac{1}{6} + \frac{1}{3} \\ 0 \frac{-2}{6} + \frac{1}{3} & 0 \frac{+4}{6} + \frac{1}{3} & 0 \frac{-2}{6} + \frac{1}{3} \\ -\frac{1}{2} + \frac{1}{6} + \frac{1}{3} & 0 \frac{-2}{6} + \frac{1}{3} & \frac{1}{2} + \frac{1}{6} + \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Orthogonal}}$

$\begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}3 & 5\end{bmatrix}$ 2×1 $1\times2 = 4 \times 1$ 2×1

(v, v)

Date

① $\|U - kV\|$

$= \sqrt{(U-kV)\cdot(U-kV)}$

$= \sqrt{203}$

Q2: Compute ⓐ–ⓕ of Q.1. $A = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$

$(U, V) = AU \cdot AV$

$= \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \cdot \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}$

$= \begin{bmatrix}2+1\\1+1\end{bmatrix} \begin{bmatrix}6+2\\3+2\end{bmatrix}$

$= \begin{bmatrix}3\\2\end{bmatrix} \cdot \begin{bmatrix}8\\5\end{bmatrix}$

$\because (U, V) = U_1V_1 + U_2V_2.$

$= (3)(8) + (2)(5)$

$= 34$

Q3: Find a matrix that generate the stated weighted inner product on R^2.

ⓐ $2U_1V_1 + 3U_2V_2$

$\begin{bmatrix}\sqrt{2} & 0\\0 & \sqrt{3}\end{bmatrix}$ Ans

ⓑ $\frac{1}{2}U_1V_1 + 5U_2V_2$

$\begin{bmatrix}\sqrt{\frac{1}{2}} & 0\\0 & \sqrt{5}\end{bmatrix}$ Ans

Q 4: Compute the standard inner product on M_{22} of the given Matrix.

$U = \begin{bmatrix}3 & -2\\4 & 8\end{bmatrix}, V = \begin{bmatrix}-1 & 3\\1 & 1\end{bmatrix}$

(U, V) 1st inner product M_{22}

$(U, V) = Tr\left[U^T V\right]$

$= Tr\left[\begin{bmatrix}3 & 4\\-2 & 8\end{bmatrix} \cdot \begin{bmatrix}-1 & 3\\1 & 1\end{bmatrix}\right]$

$= Tr\begin{bmatrix}1 & 13\\10 & 2\end{bmatrix}$

$= 1 + 2$

$(U, V) = 3$

(V, V)

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Q 4: Compute the standard inner product on M_{22} of the given Matrix.

$$U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

(U, V) std inner product M_{22}

$$(U, V) = Tr\left[U^T V\right]$$

$$= Tr\left[\begin{bmatrix} 3 & 4 \\ -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\right]$$

$$= Tr\begin{bmatrix} 1 & 13 \\ 10 & 2 \end{bmatrix}$$

$$= 1 + 2$$

$$(U, V) = 3$$

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Q: Find the St. inner Product of Polynomial

$\rightarrow p = -2 + x + 3x^2$

$q = 4 - 7x^2$

(p, q) St. Inner Product

$(p, q) = (-2)(4) + (1)(0) + 3(-7)$

$= -8 + 0 - 21$

$\boxed{(p, q) = -29}$ Ans

$(p, q) = ?$

$p = x + x^3, \ q = 1 + x^2$

$x_0 = -2, \ x_1 = -1, \ x_2 = 0, \ x_3 = 1$

$(p, q) = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$

$\boxed{(p, q) = -50}$

Q1: a

Euclia

$(U, V) =$

$U_1 \ U_2$

$U = (1, 1),$

$k = 3$

a) (U, V)

Ex: 6.2

Angle b/w U & V:

$$\cos\theta = \frac{(U, V)}{|U||V|}$$

$=$

Orthogonal b/w U & V:

$(U, V) = 0$

Q1: Find the cosine angle b/w the vector.

$U = (-1, 5, 2), \ V = (2, 4, -9)$

$$\cos\theta = \frac{(U, V)}{|U||V|}$$

$$= \frac{(-1)(2) + (5)(4) + 2(-9)}{\sqrt{(-1)^2 + 5^2 + 2^2} \cdot \sqrt{(2)^2 + (4)^2 + (-9)^2}}$$

$= 0$

$\theta = \cos^{-1}(0)$

$\theta = \pi/2$

Q3: Find angle b/w ~~A&~~ the vector w.r.t inner product P_2.

$p = -1 + 5x + 2x^2$

$q = 2 + 4x - 9x^2$

$\cos\theta = \dfrac{(p, q)}{|p| \cdot |q|}$

$= \dfrac{(-1)(2) + 5(4) + 2(-5)}{\sqrt{(-1)^2 + 5^2 + 2^2} \cdot \sqrt{2^2 + 4^2 + (-9)^2}}$

$\cos\theta = 0$

$\theta = \dfrac{\pi}{2}$

Q3: Find angle b/w A & B matrice M_{22} w.r.t inner product.

$A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

$\cos\theta = \dfrac{(A, B)}{|A| \cdot |B|} = \dfrac{(2)(3) + 6(2) + (1)(1) + (-3)(0)}{\sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \cdot \sqrt{3^2 + 2^2 + 1^2 + 0^2}}$

$= \dfrac{19}{10\sqrt{7}}$

$\theta =$

Q4: Determine whether the vectors are orthogonal w.r.t inner product.

$U = (-1, 3, 2), V = (4, 2, -1)$

$(U, V) = (-1)(4) + (3)(2) + (2)(-1)$

$= 0$

Hence U & V are orthogonal

Q5: Show that vectors are orthognal w.r.t inner product P_2.

$p = -1 - x + 2x^2, q = 2x + x^2$

$(p, q) = (-1)(0) + (-1)(2) + (2)(1)$

$= 0$ Hence proved

Q6: Show that matrices are orthogonal on M_{22}

$U = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, V = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$(U, V) = (2)(-3) + (1)(0) + (-1)(0) + (3)(2)$

$= 0$

Hence proved

IAW → 017

Q7: Show that the vector

$U = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, V \begin{bmatrix} 5 \\ -8 \end{bmatrix}$

are orthognal,

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$C(U, V) = AU \cdot AV$

$= 0$

Q8: Let P_2 have the evaluation inner product at the points.

$x_0 = -2, x_1 = 0, x_2 = 2$

Show that vector $p = x$ & $q = x^2$ are orthogonal w.r.t inner product.

$= 0$

CHP # 04

Let V be the non empty set of object for which two operation are defined addition & multiplication.

Let $V = \{1, 2, 3 \ldots 10\}$

If we add any two no.'s in set V and the result is also in V then result is said to be closed.

$1 + 4 = 5 \rightarrow$ in V matlab closed

$10 + 1 = 11 \rightarrow$ not closed

① If U & V are objects in V, then $U + V$ is in V

② $U + V = V + U$

③ $U + (V + W) = (U + V) + W$

④ Zero Vector,

$U + 0 = 0 + U = U$

⑤ Negative U

$U + (-U) = 0$

⑥ If K is any scalar & U in any obj in V, KU is in V.

⑦ $K(U + V) = KU + KV$

⑧ $(K + m)U = KU + mU$

⑨ $K(mU) = (km)U$

⑩ $1U = U$

Q1: The set of all real no with the standard operation of add & multiplication or Write down all properties of ...

Ismein saari properties likhni hongi.

Q2: The set of all pair of real no. of the form $(x, 0)$ with the statement operation

R^2. $V = R^2$

$U = (U_1, U_2)$

$V = (V_1, V_2)$

which condition not satisfy on $x \geq 0$.

Date

(4) $(x,0)+(y,0)=(x+y,0)$, in R^2, x, y are real

(5) $(x,0)+(y,0)=(y,0)+(x,0)$, in R^2, x, y are real

(3) $(x,0)+[(y,0)+(z,0)]=[(x,0)+(y,0)]+(z,0)$, in R^2, x, y, z are real no.

(4) $(x,0)+0=0+(x,0)=(x,0)$, ", "

(5) $(x,0)+[-(x,0)]=0$, ", "

(6) If k is any scalar vector & $(x,0)$ is real object in R^2, then $k(x,0)=k(x,0)$ in R^2, ", "

(7) $k[(x,0)+(y,0)]=k(x,0)+k(y,0)=[(kx+ky,0)=k[(x+y),0]$

(8) $(k+m)(x,0)=k(x,0)+m(x,0)$

(9) :

(10) /

Q3: The set of all pairs of real no. of the form $(x,4)$, where $x \geq 0$, with the stand. oper. on R^2

The 5th & 6th proper. will fail.

Ex: 4.1

eg #07

Q9, 7

Ex. 4.1

Date

⇒ LINEAR COMBINATION:

$k_1 v_1 + k_2 v_2 + k_3 v_3 + \dots k_n v_n = \omega$

where, k = scalar
v = vector space
ω = sub space vectors

Q#1: Which are the following are linear Combination of $U = (0, -2, 2)$, $V = (1, 3, -1)$?

(a) $(2, 2, 2) = \omega$
b) $(0, 4, 5) = \omega$
c) $(0, 0, 0) = \omega$

Sol: L.C:
(a) $k_1 U + k_2 V = \omega$

$k_1 (0, -2, 2) + k_2 (1, 3, -1) = (2, 2, 2)$

$0k_1 + k_2 = 2$
$-2k_1 + 3k_2 = 2$
$2k_1 - k_2 = 2$

Augmented

$$\begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

Reduced

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$k_1 = 2$
$k_2 = 2$

The system is consi... U & V are linear... $(2, 2, 2)$.

inconsistent ...

Q2: Which of the following are L.C.

$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$?

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} \omega$

$k_1 A + k_2 B + k_3 C = \omega$

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$K_1\begin{bmatrix}4 & 6\\ -7 & -2\end{bmatrix} + K_2\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} + K_3\begin{bmatrix}0 & 2\\ 1 & 4\end{bmatrix} = \begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix}$

Augmented

$$\begin{array}{cccc} A & B & C & W \\ \end{array}$$

$$\left[\begin{array}{cccc} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -7 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -6 \end{array}\right]$$

Reduce,

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 = 1$

$K_2 = 2$

$K_3 = -3$

3: In each part Express the vectors in a L·C of

$p_1 = 2 + x + 4x^2$

$p_2 = 1 - x + 3x^2$

$p_3 = 3 + 2x + 5x^2$

a) $-9 - 7x - 15x^2 = w$

$K_1(2 + x + 4x^2) + K_2(1 - x + 3x^2) + K_3(3 + 2x + 5x^2) = 9 - 7x - 15x^2$

$$\left[\begin{array}{cccc} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & +15 \end{array}\right]$$

Reduce,

Q4: In each part, determine whether the vectors span R^3

we order mazeed subspace bo span karte hai.

$v_1 = (2, 2, 2)$

$v_2 = (0, 0, 3)$

$v_3 = (0, 1, 1)$

$K_1(2,2,2) + K_2(0,0,3) + K_3(0,1,1) = (w_1, w_2, w_3)$

Augmented

$$\left[\begin{array}{cccc} 2 & 0 & 0 & w_1 \\ 2 & 0 & 1 & w_2 \\ 2 & 3 & 1 & w_3 \end{array}\right]$$

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$\omega_1, \omega_2, \omega_3$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

Now taking det(A).
if det(A) = 0, not span
otherwise span.

$\det(A) = -6 \neq 0$
so span.

⇒ determinant isliye nikala h
q k $\omega_1, \omega_2, \omega_3$ maaloom nhi
hain agr unki value maaloom
hoti to echelon hi lagate

Q5: Suppose that,
$v_1 = (2, 1, 0, 3)$
$v_2 = (3, -1, 5, 2)$
$v_3 = (-1, 0, 2, 1)$

which of the following vectors
are in span $[v_1, v_2, v_3]$
(a) $(2, 3, -7, 3)$

By LC

$K_1 V_1 + K_2 V_2 + K_3 V_3 = \ldots$

$K_1(2, 1, 0, 3) + K_2(3, \ldots$
$K_3(-1, 0, 2, 1) = (2, \ldots$

Augment

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

$K_1 = 2,\ K_2 = -1,\ K_3 \ldots$
The system is consistent
hence span $[v_1, v_2, v_3]$
of $(2, 3, -7, 3)$

Ex: 4.4

Date

LINEAR INDEPENDENCE

$K_1V_1 + K_2V_2 + K_3V_3 \dots + K_nV_n = 0$

If $K_1 = K_2 = \dots K_n = 0$, then system is linearly independence otherwise linearly dependent

Q:1. Determine whether the vectors are l. inde- or l. dep?

(3,8,7,-3), (1,5,3,-1), (2,-1,2,6), (4,2,6,4)

By linear indep, —

$K_1V_1 + K_2V_2 + K_3V_3 = 0$

$K_1(3,8,7,-3) + K_2(1,5,3,-1)$

$+K_3(2,-1,2,6) + K_4(4,2,6,4) =$

$(0,0,0,0)$.

Augmented Matrix,

$$\begin{bmatrix} 3 & 1 & 2 & 4 & 0 \\ 8 & 5 & -1 & 2 & 0 \\ 7 & 3 & 2 & 6 & 0 \\ -3 & -1 & 6 & 4 & 0 \end{bmatrix}$$

Reduce Echelon,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$K_4 = t$

$K_1 = -t$

$K_2 = t$

$K_3 = -t$

This system is consistent

This system is l. depen

Q2: P_1 $2 - 4 + 4x^2$, P_2 $3 + 6x + 2x^2$, P_3 $2 + 10x - 4x^2$

By l. indep —

$K_1P_1 + K_2P_2 + K_3P_3 = 0$

$k_1(2-4x-4x^2)+k_2(3+6x+2x^2)$
$+k_3(2+10x-4x^2)=0$

Augmented Matrix,

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -4 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{bmatrix}$$

~~Echelon se krenge~~

$$A=\begin{bmatrix} 2 & 3 & 2 & 0 \\ -4 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{bmatrix}$$

$\det(A)=\cancel{-18}$

Q3: $\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

$k=?$

Matrix are L. Independence

i.e $\det(A)=0$

By L.Ind. P

$k_1A+k_2B+k_3C=0$

$k_1\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}+k_2\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}+$

$k_3\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Augmented Matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & k & 1 & 0 \\ k & 1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & k & 1 & 0 \\ k & 1 & 3 & 0 \end{bmatrix}$$ 2nd ~~[illegible]~~ hta di

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$$

$\det(A)=(k-1)(k+2)$

$0=(k-1)(k+2)$

$\boxed{k=1}, \boxed{k=-2}$

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4. In each part, let $T_A: R^2 \to R^2$ be Multiplication by A and let $U_1 = (1, 2)$ & $U_2 = (-1, 1)$. Determine whether the set $\{T_A(U_1), T_A(U_2)\}$ are l. indep in R^2.

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

$T_A(U_1) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$T_A(U_2) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

By l. indepen.

$K_1 T_A(U_1) + K_2 T_A(U_2) = 0$

$K_1(-1, 4) + K_2(-2, 2) = (0, 0)$

Augmented Matrix

$\begin{bmatrix} -1 & -2 & 0 \\ 4 & 2 & 0 \end{bmatrix}$

By reducing to echelon

$K_1 = 0$, $K_2 = 0$

System is l. independent.

WRONSKIAN DETERMINANT

$W(x) = \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ f_1^{(n)} & f_2^{(n)} & \dots & f_n^{(n)} \end{bmatrix}$

If $W(x) \neq 0$, l. independ

otherwise, l. dependent

Q.1 $f_1(x) = x$, $f_2(x) = \cos x$

By Wronskian set

$W(x) = \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}$

$W = \begin{bmatrix} x & \cos x \\ 1 & -\sin x \end{bmatrix}$

$= -x\sin x - \cos x$

$\neq 0$

so $f_1(x)$ & $f_2(x)$ are linearly independent.

Ex: 4.5

Date

Basis:

If $S = \{V_1, V_2, \dots V_n\}$ is a set of vector in a finite dimensional vectors space V, then S is called a basis for V if:

(a) S spans V.

(b) linearly independent

if $det(A) \neq 0$, Basis
otherwise non-basis

Q1: Show that the following polynomial form a basis for P_2.

$x^2+1, x^2-1, 2x-1$

By linear indepent.

$K_1P_1 + K_2P_2 + K_3P_3 = 0 \rightarrow (1)$

By linear Combination.

$K_1P_1 + K_2P_2 + K_3P_3 = \omega \rightarrow (2)$

eq(1) $K_1(x^2+1) + K_2(x^2-1) + K_3(2x-1) = 0$

Augmented Matrix

eq (1)

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

eq (2)

$$\begin{bmatrix} 1 & -1 & -1 & \omega_1 \\ 0 & 0 & 2 & \omega_2 \\ 1 & 1 & 0 & \omega_3 \end{bmatrix}$$

→ dono left side same hai to ek krdia

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

expanding from row 2

$det(A) = -4 \neq 0$ so basis of vector

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Q: ~~First~~

→ Linear combination means ~~taking~~ vector finding value of K_1, K_2, K_3.

→ Co-ordinate Vector are also K_1, K_2, K_3

M_{22} means Matrix 2×2

Date

By linear Indep

$$K_1\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+K_2\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}+K_3\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$$

$$K_4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=0$$

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 0\end{bmatrix}\to ①$$

By linear combination

$$K_1A_1+K_2A_2+K_3A_3+K_4A_4=W$$

$$K_1\begin{bmatrix} & \end{bmatrix}+K_2\begin{bmatrix} & \end{bmatrix}+K_3\begin{bmatrix} & \end{bmatrix}+K_4\begin{bmatrix} & \end{bmatrix}$$

$$=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

eq ②

$$\begin{bmatrix}1 & 0 & 0 & 0 & a\\ 1 & 1 & 0 & 0 & b\\ 1 & 1 & 1 & 0 & c\\ 1 & 1 & 1 & 1 & d\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}$$

$\det(A)=1\neq 0$ a basis of vector space.

By Linear Combination

$K_1A_1+K_2A_2+\dots K_4A_4=A$

eq ③

$$\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 0\end{bmatrix}$$

$K_1=1$

$K_2=-K_1=-1$

$K_3=-1-K_1-K_2=1$

$K_4=1$

This l.combination of $\{A_1, A_2, A_3, A_4\}$.

These co-ordinate vectors

$(K_1, K_2, K_3, K_4)=(1,-1,1,-1)$

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Q3: $T_A : R^3 \to R^3$ be multiplication by A, and let $v = (1,-2,-1)$. Find the cordinate vector of $T_A(v)$ relative to basis $S = \{(1,1,0), (0,1,1), (1,1,1)\}$ for R^3.

(a) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

$$T_A(v) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

By Linear Combination

$K_1 V_1 + K_2 V_2 + K_3 V_3 = T_A(u)$

$K_1(1,1,0) + K_2(0,1,1) + K_3(1,1,1)$
$= (4,-2,0)$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Reduced Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$K_1 = -2$, $K_2 = -6$, $K_3 = 6$

Cordinate vector $(-2,-6,6)$

Q4: The first four hermite polynomial. $1, 2t, -2+4t^2, -12t + 8t^3$

(a) Show that H.P form a basis for P_3.

(b) Find the cordinate vector of the polynomial $p(t) = -1 - 4t + 8t^2 + 8t^3$ relative basis.

By Linear Ind.

$K_1 P_1 + K_2 P_2 + K_3 P_3 + K_4 P_4 = 0 \to (1)$

By Linear Combi

$" \quad " = w \to (2)$

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Ex 4.9

Date

RANK & NULLITY

DIMENSION THEOREM FOR MATRICES

$Rank(A) + Nullity(A) = n$

if A is a matrix with n col.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ → the no. of rows having leading 1 is called rank

↓ the no. of ~~rows~~ rows with all zeros.

Q1: Find the rank & Nullity of the matrix A by reading or it to row echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$$

Nullity(A) = 3 , Rank(A) = 1

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → rank,] NULLITY

Q2: The matrix R is the reduced row echelon form of the matrix A.

(a) Find rank and Nullity of A.
(b) Conform that rank & nullity
(c) Find the no. of leading variables and no. of parameters in the general sol $Ax = 0$

(a) $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol: (a) Rank = 3
nullity = 0

(b) rank + nullity = n
3 + 0 = 3
3 = 3

(c) No. of leading variable.
parametric. = 0

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(b) $A = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{bmatrix}$, $R = \begin{bmatrix} 1 & -1/2 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank = 1

nullity = 2

$x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_3$

let, $x_3 = t$

$x_2 = s$

$x_1 = \frac{1}{2}s + \frac{3}{2}t$

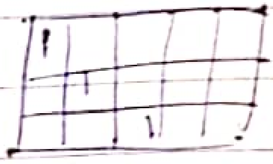
are parametric eqn.

(b) 5×3

Rank(A) = 3

nullity (A) = 2

(c) 3×5

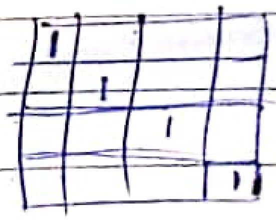


rank = 3

nullity = 0

Q3: Find the largest possible value for the rank of A and the smallest possible value for the nullity of A

(a) A is 4×4



rank = 4

nullity = 0

Q4: Verify that rank(A) = rank

$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$

By reducing to reduced Row.

Rank(A) = 2

nullity = 1

&

rank$(A^T) = 2$

Ex: 5.2

Date

Diagonalization

Similar Matrix :

$|A| = |B|$

$P^{-1}A.P$

Q1: Show that Similar matric

$A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

$\det(A) = -1$

$\det(B) = -2$

$\Rightarrow \det(A) \neq \det(B)$

so not similar

Q2: Find a matrix P that diagonalizes A and check your work by computing $P^{-1}A.P$.

$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

By Characteristic Eqn.

$\det(\lambda I - A) = 0$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det\left[\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\right]$

$\det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\right] =$

$\det\left[\begin{bmatrix} \lambda-1 & 0 \\ -6 & \lambda+1 \end{bmatrix}\right] = 0$

$(\lambda-1)(\lambda+1) - 0 = 0$

$\cancel{\lambda^2 - 1 + 0} = 0$

$\lambda = 1 \;,\; \lambda = -1$ → ei

at $\lambda = 1$

$\begin{bmatrix} \lambda-1 & 0 \\ -6 & \lambda+1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}$

$\begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix}$

By Reduce ~~REE~~ Ech

$R_1 \div -6$

$\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$

$x_2 = t$

$x_1 = 1/3\, t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} t \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ is eigen space at $\lambda = 1$

$\hookrightarrow P_1$

fraction khatam kidhar 3 se ⊗ kiya

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \to P_1$

at $\lambda = -1$

$$\begin{bmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}$$

$-3R_1 + R_2 \to R_2$

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$R_1 \div -2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$

$x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigen space at $\lambda = -1$

$\hookrightarrow P_2$

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} A P = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$\Rightarrow \lambda_1 = -1$

$\lambda_2 = 1$

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3. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

Find the eigen value of A
For each eigen value λ,
Find the $\lambda I - A$.
Is A diagnolizationable?
Justify your conclusion.

Q: The characteristic eqn of matrix A is given. Find the size of the matrix & dimension of eigen-space.

a) $(\lambda-1)(\lambda+3)(\lambda-5)=0$

Size $= 3\times 3$

Dim $= 1$

b) $A^2(\lambda-1)(\lambda-2)^3 = 0$

Size $= 6\times 6$

Dim $= 1$

Q: Use the method to compute A^{10} Matrix.

$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

By Characteristic Eqn

$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}\right) = 0$

$\det\begin{bmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{bmatrix} = 0$

$\lambda(\lambda+1) - (6) = 0$

$\lambda^2 + \lambda - 6 = 0$

$\lambda^2 + 3\lambda - 2\lambda - 6 = 0$

$\lambda = 2, -3$

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$$D^{10} = \begin{bmatrix} (2)^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix}$$

$A^{10} = P \cdot D^{10} \cdot P^{-1}$

$$A^{10} = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & 59049 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1536 & -59049 \\ 1024 & 59049 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

Again calculating P^{-1}:

$$P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$= 3/2 + 1$

$= 5/2$

$$P^{-1} = \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

Now again,

$$A^{10} = \begin{bmatrix} 1536 & -59049 \\ 1024 & 59049 \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 24234 & -34015 \\ -23210 & 35839 \end{bmatrix} \text{Ans}$$

Q: det

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

Compute A^{11}.

Sol:

$A^{11} = P D^{11} \cdot P^{-1}$

$D = P^{-1} A P$

$$A^{11} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

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$$\begin{bmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{bmatrix} x = 0$$

For $\lambda = 2$

$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} = 0$$

$$2x_1 - 3x_2 = 0$$

Let $x_2 = t$

$$2x_1 = 3t$$

$$\boxed{x_1 = \tfrac{3}{2} t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} t \\ t \end{bmatrix}$$

$$P_1 = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

For $\lambda = -3$

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

Let $x_2 = t$

$$x_1 + t = 0$$

$$\boxed{x_1 = -t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$P^{-1} A . P$

$$P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$P^{-1} A . P = \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3/2 & \\ 1 & \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = D$$

P^{-1} was wrong

Date $\vec{P}$

eq (i) $\Rightarrow$

$$K_1[1] + K_2[2t] + K_3[-2+4t^2] + K_4[-12t+8t^3] = 0$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & -12 & -4 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 8 & 8 \end{bmatrix}$$

$K_1 = 3$

$K_2 = 4$

$K_3 = 2$

$K_4 = 1$

eq (ii) $\Rightarrow$

$$\omega = \omega$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & \omega_1 \\ 0 & 2 & 0 & -12 & \omega_2 \\ 0 & 0 & 4 & 0 & \omega_3 \\ 0 & 0 & 0 & 8 & \omega_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\det(A) = 64 \neq 0$$

(b) $K_1P_1 + K_2P_2 + K_3P_3 + \lambda_4 P_4 = P$

→ Every positive definite is a positive semidefinite but not vice versa

→ Same for neg definite

[illegible] diagonal [illegible] [illegible] non-diagonal [illegible] [illegible] diagonal [illegible] eigen values are same as diagonal elements

---

Q is −ve semidefinite if:

$Q \leq 0,\ x \neq 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

eigenvalues non-positive

Q: Express as $Q = X^T A X$ in the matrix form

a) $3x_1^2 + 7x_2^2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $4x_1^2 - 9x_2^2 - 6x_1x_2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 + x_1x_3 + x_2x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -1 & \frac{1}{2} & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q2) $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$Q = 2x^2 + 5y^2 - 6xy$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} -2 & 7/2 & 1 \\ 7/2 & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$Q = -2x_1^2 + 3x_3^2 + 7x_1x_2 + 2x_1x_3 + 12x_2x_3$

Q: Find an orthogonal change variable & express Q.

$Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$

$y = px$ (change of variable)

$X^TAX = Y^TDY$

$= \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$Q = X^TAX = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$A \cdot x^{\circ} = 3$ Date

① $a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^T \cdot A \cdot X$$

② $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= X^T A \cdot X$

$$\boxed{Q = X^T \cdot A \cdot X}$$

A quadratic form of $X^T A X$ is said to be:

① Positive definite if $Q > 0$; $x \neq 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

if all eigen values of A are +ve

② Negative definite if $Q < 0$; $x \neq 0$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

if all eigen are -ve

③ Indefinite if Q has both +ve & -ve values

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

if atleast one +ve & one -ve

④ Q is +ve semidefinite if $Q \geq 0$; $x \neq 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

if all eigen are non negative

<u>Principal Axes Theorem:</u>

$$X^T \cdot A \cdot X = \underbrace{y^T D \cdot y}_{D = P^{-1}AP} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

change of variable $= X = Py$

Date

Ang GSP:

$v_1 = P_1$

$v_2 = P_2 - \frac{\langle P_2, P_1 \rangle \cdot v_1}{\|v_1\|^2}$

$= (1,1) - \frac{(1)(1) + (1)(-1) \cdot (1,-1)}{(1^2 + 1^2)}$

$= (1,1) - \frac{1 + (-1) \cdot (1,-1)}{\sqrt{2}}$

$v_2 = (1,1)$

$q_1 = \frac{v_1}{|v_1|} = \frac{(1,-1)}{\sqrt{2}}$

$= \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$P^t = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$P^t = P^{-1}$

$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$P^{-1} \cdot A \cdot P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ Ans

$\frac{1}{2} + \frac{1}{2}$

$\begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$[R3+(v)]$

Spectral Decomposition:

$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T \dots \lambda_n q_n q_n^T$

Date

Q3 Find a matrix P that orthogonally diagonalizes A ($P^T = P^{-1}$) and determine $P^{-1}AP$ and also find spectral dec— of matrix A:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$\det(\lambda I - A) = 0$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{bmatrix} = 0$$

$(\lambda - 3)(\lambda - 3) - (1) = 0$

$\lambda^2 - 3\lambda - 3\lambda + 9 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 4, 2$

$\lambda = 4$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

reducing row echelon

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

$x_2 = 5$

$x_1 = -5$

$(x_1, x_2) = (5, -5)$

$= 5(1, -1)$

$\therefore P_1 = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2$

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$-R_1 + R_2 \to R_1$, $R_1 \times -1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

$x_2 = 5$

$(x_1, x_2) = (5, 5)$

$P_2 = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Date

Ex: 7.2

orthogonal

ORTHOGONAL DIAGONALIZATION

$\|\cdot\|=1, \|x_j\|=1$

Q: Find the characteristic eqn of the given system of matrix and then inspect the dimension of eigen spaces.

$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & 2 & -2 \end{bmatrix}$$

$\det(\lambda I - A) = 0$

$=1$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -\cdots & \\ & & \\ & & \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & -2 & \lambda+2 \end{bmatrix}\right)$$

$\lambda^3 - 27\lambda - 54 = (\lambda-6)(\lambda+3)^2$

$\lambda = 6, -3$

$\because (\lambda-6)$ ki power is 1 so $\lambda = 6$ is dimension is 1.

$\because (\lambda+3)^2$ ki power is 2 so $\lambda = -3$ has dimension 2.

Date

$\det(\lambda I - A) = 0$

$$\det\begin{bmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{bmatrix} = 0$$

$(\lambda-2)(\lambda-2) - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 3$

$u_1 = (-1, 1), u_2 = (1, 1)$

By GSP :-

$v_1 = u_1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

$q_1 = $ —

$q_2 = $ —

$$P = [q_1 \; q_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$P^{-1} = P^T$

change of variables:

$y = Px$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$P^{-1}AP = [\ \ ]$

New variable.

$Q = x^T A x = y^T D y$

$$= [y_1 \; y_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$= y_1^2 + 3y_2^2$ Ans

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