CHAPTER # 04

MEAN:

Also called Expected value, of RV X.

$$\boxed{\mu = E(X) = \sum_x x f(x)}$$

mean — expected value

X	0	1	2	---	n
$P(X=x)$ $f(x)$	P_1	P_2	P_3	---	P_n

$$0(P_1) + 1(P_2) + 2(P_3) + \dots n(P_n) = \mu = E(X)$$

if continuous,

$$\boxed{\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx}$$

Example: A lot containing 7 components is sampled by a quality inspector; the lots contains 4 good components & 3 defective components. A sample of 3 is taken by the customer. Find the expected value of no. of good components in this sample.

X = no. of good comp. = 0, 1, 2, 3

X	0	1	2	3
$f(x)$	$1/35$	$12/35$	$18/35$	$4/35$
	$\frac{^3C_3}{^7C_3}$	$\frac{^3C_2\,^4C_1}{^7C_3}$	$\frac{^3C_1\,^4C_2}{^7C_3}$	$\frac{^4C_3}{^7C_3}$

$$\mu = E(X) = 0(1/35) + 1(12/35) + 2(18/35) + 3(4/35) = 1.7$$

→ it mean if a sample size of 3 is selected at random over & over again from a lot of 4 good & 3 def comp., it will on average contain 1.7 good components.

▷ Let X be a random variable with probability dist. $f(x)$, the expected value of random variable $g(x)$ is

$$\boxed{\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x)}$$

for continuous,

$$\boxed{\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx}$$

Example: Suppose the no. of cars X that pass through a car wash b/w 4 till 5 pm on any Sunny Friday has the following probability distribution,

x	4	5	6	7	8	9
$P(X=x)$	$1/12$	$1/12$	$1/4$	$1/4$	$1/6$	$1/6$

Let $g(X) = 2X-1$ represent the amount of money, in dollars paid to attendant by the manager. Find the attendants' expected earnings for this particular time period.

$$E[g(X)] = E(2X-1) = \sum_{x=4}^{9} (2x-1) f(x)$$

$$= (2(4)-1)(1/12) + (2(5)-1)(1/12) + (2(6)-1)(1/12) + (2(7)-1)(1/4) + (2(8)-1)(1/6) + (2(9)-1)(1/6)$$

$$E[g(X)] = 12.67$$

Example: $f(x) = \begin{cases} x^2/3 & , -1 < x < 2, \\ 0 & , \text{elsewhere} \end{cases}$ → Find expected value of $g(X) = 4X+3$

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)\, dx = 8$$

- Joint Probability :

Let X & Y be random variable with JPD $f(x,y)$. The mean or expected value, of random variable $g(X,Y)$ is

$$\boxed{\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)}$$

for continuous,

$$\boxed{\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy}$$

Example 4.6: $g(X,Y) = XY$

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xy\, f(x,y).$$

$$= (0)(0) f(0,0) + (0)(1) f(0,1) + (1)(0) f(1,0) + (1)(1) f(1,1) + (2)(0) f(2,0) +$$

$$= 3/14$$

Without ~~Story~~ dependent variable.

marginal distribution of X

$$E(X) = \begin{cases} \sum_x \sum_y x\, f(x,y) = \sum_x x\, g(x) & , \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x\, g(x)\, dx & , \text{continuous} \end{cases}$$

marginal distribution of Y

$$E(Y) = \begin{cases} \sum_y \sum_x y\, f(x,y) = \sum_y y\, h(y) & , \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} y\, h(y)\, dy & , \text{continuous} \end{cases}$$

Q4.4: A coin is biased such that a head is 3 times as likely to occur as a tail. Find the expected no of tails when this coin is tossed twice.

$w + 3w = 1 \Rightarrow w = \frac{1}{4}$

prob of tail $= \frac{1}{4}$

prob of head $= \frac{3}{4}$

no. of tails

X	0	1	2
f(x)	9/16	3/8	1/16
	HH	$\frac{3}{16}+\frac{3}{16}=\frac{3}{8}$ TH+HT	$\frac{1}{4}\times\frac{1}{4}$ TT

$E(X) = 0(9/16) + 1(3/8) + 2(1/16) = \frac{1}{2}$

Q4.10: Find u_x & u_y. (See table on pg # 117)

X	1	2	3
g(x)	0.17	0.50	0.33

$u_x = E(X) = \sum x\, g(x) = 1(0.17) + 2(0.50) + 3(0.33)$

$\boxed{u_x = 2.16}$

y	1	2	3
h(y)	0.23	0.50	0.27

$u_y = E(Y) = \sum_y y\, h(y) = 1(0.23) + 2(0.50) + 3(0.27) = 2.04$

Q4.11: $f(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$E(X) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)} dx = \int_0^1 \frac{4x}{\pi(1+x^2)} dx$

$= \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}$ → multiply & divide by 2

"ln" wala rule lagega → neeche function upar derivative

$= \frac{2}{\pi}\int_0^1 \frac{2x}{1+x^2} dx = \cancel{\frac{2}{\pi}\int_0^1 (1+x^2)^{-1} 2x\, dx}$

$= \frac{2}{\pi}\left(\ln(1+x^2)\right)\Big|_0^1 = \frac{2}{\pi}[\ln(2) - \ln(1)]$

$= \frac{2}{\pi}\ln(2/1) = \frac{1}{\pi}\ln 2^2$

$= \frac{1}{\pi}\ln(4)$

Q4.13: $f(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 \le x < 2 \\ 0 & , \text{elsewhere} \end{cases}$

$E(X) = \int_0^1 x \cdot x\, dx + \int_1^2 x(2-x)\, dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx$

$= \frac{x^3}{3}\Big|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2 = \frac{1}{3} + \left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right)$

$= 1$

So the avg no. of hrs/yr that families run their vacuum cleaner is $1 \times 100 = 100$ hrs.

"VARIANCE & COVARIANCE"

▷ **Variance:**

Let X be a random variable with probability distribution $f(x)$ & mean μ. The variance of X is,

$$\boxed{\begin{array}{ll} \sigma^2 = E[(X-\mu^2)] = \sum_x (x-\mu^2) f(x) & , \text{discrete} \\ \sigma^2 = E[(X-\mu^2)] = \int_{-\infty}^{\infty} (x-\mu^2) f(x)\, dx & , \text{continuous.} \end{array}}$$

→ The +ve sq root of variance, σ, is St. deviation of X.

Another Formula ⇒ $\boxed{\sigma^2 = E(X^2) - \mu^2}$

Q:

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Find σ^2

$\mu = 0(0.51) + 1(0.38) + 2(0.10) + 3(0.01) = 0.61$

$E(X^2) = 0^2(0.51) + 1^2(0.38) + 2^2(0.10) + 3^2(0.01) = 0.87$

$\sigma^2 = 0.87 - (0.61)^2$

$\boxed{\sigma^2 = 0.4979}$

Q: $f(x) = \begin{cases} 2(x-1) & , 1 < x < 2 \\ 0 & , \text{else} \end{cases}$

Find σ^2 of X.

$$\mu = \int_2^1 x \cdot 2(x-1)\, dx = \frac{5}{3}$$

$$E(X^2) = \int_2^1 x^2 \cdot 2(x-1)\, dx = \frac{17}{6}$$

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18} \text{ Ans.}$$

Δ Let X be a random variable with probability distn f(x). The variance of the random variable g(X) is,

$$\boxed{\sigma^2_{g(x)} = E\{[g(X) - \mu_{g(x)}]^2\} = \sum [g(x) - \mu_{g(x)}]^2 f(x)}$$

for continuous,

$$\sigma^2_{g(x)} = E\{[g(x) - \mu_g(x)]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_g(x)]^2 f(x)\, dx$$

Q: Calculate the variance of $g(X) = 2X + 3$, where X is RV with

x	0	1	2	3
$f(x)$	1/4	1/8	1/2	1/8

$$\mu_{g(x)} = (2(0)+3)\tfrac{1}{4} + (2(1)+3)\tfrac{1}{8} + (2(2)+3)\tfrac{1}{2} + (2(3)+3)\tfrac{1}{8}$$

$$\mu_{g(x)} = \tfrac{3}{4} + \tfrac{5}{8} + \tfrac{7}{2} + \tfrac{9}{8} = 6$$

$$\sigma^2 = E\{[(2X+3) - 6]^2\} = E\{(2X+3-6)^2\}$$

$$= E(4X^2 - 12X + 9) = \sum_{x=0}^{3} (4x^2 - 12x + 9) f(x)$$

$$= (4(0)^2 - 12(0) + 9)(\tfrac{1}{4}) + (4(1)^2 - 12(1) + 9)(\tfrac{1}{8}) + (4(2)^2 - (12)(2) + 9)(\tfrac{1}{2})$$
$$(4(3)^2 - 12(3) + 9)(\tfrac{1}{8})$$

$$= 4$$

Q: $g(X) = 4X + 3$. $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsew} \end{cases}$

$$\mu_{g(x)} = \int_{-1}^{2} (4x+3) \frac{x^2}{3}\, dx = 8$$

$$\sigma^2_{g(x)} = \int_{-1}^{2} (4x+3-8)^2 f(x)\, dx = \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3}\, dx$$

$$= \int_{-1}^{2} (16x^4 - 40x^3 + 25x^2)\, dx$$

$$\sigma^2_{g(x)} = \frac{51}{5}$$

For Joint Probability :

Δ Covariance:

Let X & Y be random variables with joint probability fn f(x,y). The covariance of X & Y is,

$$\boxed{\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_x)(y - \mu_y) f(x,y)}$$

for continuous,

$$\boxed{\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y) f(x,y)\, dx\, dy}$$

→ Alternative & proffer formulas

$$\boxed{\sigma_{xy} = E(XY) - \mu_x \mu_y}$$

→ The covariance b/w two RV is the a measure of association b/w two.

$(X-\mu_x)(Y-\mu_y)$ → if one is +ve then is +ve the relationship b/w
↳ if one -ve the neg relationship.

→ if X & Y are statistically independent then there Co-variance is zero but it's vice verso is not True.

Q: Example 4.13 pg # 124.
Find covariance of X & Y

$\mu_x = \sum_{x=0}^{2} x\,g(x) = 0(5/14) + 1(15/28) + 2(3/28) = 3/4$

$\mu_y = \sum_{y=0}^{2} y\,h(y) = 0(15/28) + 1(3/7) + 2(1/28) = \frac{1}{2}$

$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xy\,f(x,y) = (0)(0)f(0,0) + (0)(1)f(0,1) + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0)$

$= 3/14$

$\sigma_{xy} = 3/14 - (3/4)(1/2)$

$\boxed{\sigma_{xy} = -9/56}$

Q: $f(x,y) = \begin{cases} 8xy & , 0 \le y \le x \le 1 \\ 0 & , \text{elsewhere} \end{cases}$

Find covariance of X & Y

$g(x) = \int_0^x 8xy\,dy = 4xy^2\Big|_0^x = 4x^3$

$h(y) = \int_y^1 8xy\,dx = 4x^2y\Big|_y^1 = 4y(1-y^2)$

$\mu_x = \int_0^1 x \cdot g(x)\,dx = \int_0^1 4x^4\,dx = \frac{4}{5}$

$\mu_y = \int_0^1 y \cdot h(y)\,dy = \int_0^1 4y^2(1-y^2)\,dy = \frac{8}{15}$

$E(XY) = \int_0^1\int_y^1 (xy) f(x,y)\,dx\,dy = \int_0^1\int_y^1 8x^2y^2\,dx\,dy = \frac{4}{9}$

$\sigma_{xy} = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}$

$\frac{8}{3}x^3y^2\Big|_0^1 = \frac{8}{3}y^2$

$\frac{8}{9}$

"CORRELATION COEFFICENT"

▷ Let X & Y be random variable with covariance σ_{xy} and standard deviation σ_x & σ_y resp. The correlation coefficient of X & Y is

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

→ It is unit free
→ it tells the strength of relation b/w X & Y