31/12/24

Example: Reduce the eqn. $8x^2-4xy+5y^2-16x-14y+17=0$ into its standard form and find all its properties.

Solution: Comparing the given eqn with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get: $a=8$, $h=-2$, $b=5$, $g=-8$, $f=-7$, $c=17$ and also

$ab-h^2 = 8(5)-(-2)^2 = 36>0$ and

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$= abc+2fgh-af^2-bg^2-ch^2$

$= 8(5)17+2(-7)(-8)(-2)-8(-7)^2-5(-8)^2-17(-2)^2$

$= -324 \neq 0$

$\therefore$ The given equation represents an ellipse.

Co-ordinates of center:

$$x_1 = \frac{hf-bg}{ab-h^2} = \frac{(-2)(-7)-5(-8)}{8(5)-(-2)^2} = \frac{54}{36} = \frac{3}{2}$$

$$y_1 = \frac{gh-af}{ab-h^2} = \frac{(-8)(-2)-8(-7)}{8(5)-(-2)^2} = \frac{72}{36} = 2$$

center $(x_1, y_1) = \left(\frac{3}{2}, 2\right)$.

Now, shifting the origin at the center:

$8x^2 - 4xy + 5y^2 + c_1 = 0$, when, $c_1 = gx_1 + fy_1 + c$

$\Rightarrow c_1 = (-8)\left(\frac{3}{2}\right) + (-7)(2) + 17 = -9$

$\therefore 8x^2 - 4xy + 5y^2 = 9$

$\Rightarrow \frac{8}{9}x^2 - \frac{4}{9}xy + \frac{5}{9}y^2 = 1$

$\Rightarrow Ax^2 + 2Hxy + By^2 = 1$

where: $A = \frac{8}{9}$, $H = \frac{-2}{9}$, $B = \frac{5}{9}$

Now, for the eqn. of length of major and minor axis:

$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$

$\Rightarrow \frac{1}{r^4} - \frac{13}{9}\left(\frac{1}{r^2}\right) + \frac{40}{81} - \frac{4}{81} = 0$

$\Rightarrow \frac{1}{r^4} - \frac{13}{9}\left(\frac{1}{r^2}\right) + \frac{36}{81} = 0$

$\Rightarrow \frac{1}{r^4} - \frac{13}{9}\left(\frac{1}{r^2}\right) + \frac{4}{9} = 0$

$\Rightarrow \frac{9 - 13r^2 + 4r^4}{9r^4} = 0$

$\Rightarrow 4r^4 - 13r^2 + 9 = 0$ is a quadratic equation in r^2:

$r_1^2 = \frac{13 + \sqrt{(-13)^2 - 4(4)9}}{2(4)} = \frac{18}{8} = \frac{9}{4}$

$\Rightarrow r_1 = \frac{3}{2}$

Length major axis $= 2r_1 = 2\left(\frac{3}{2}\right) = 3$ units.

$\therefore r_2^2 = \frac{13-\sqrt{(-13)^2-4.(4)9}}{2(4)} = \frac{8}{8} = 1$

$\therefore r_2 = 1$

$\therefore$ Length of minor axis $= 2r_2 = 2(1) = 2$ units.

Now, the standard equation of ellipse is:-

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$$

$$\therefore \frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{(1)^2} = 1.$$

Now, the eqn of major axis is given by:

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$$

$$\Rightarrow \left(\frac{8}{9} - \frac{1}{\left(\frac{3}{2}\right)^2}\right)x - \left(\frac{-2}{9}\right)y = 0$$

$$\Rightarrow \frac{4}{9}x - \frac{2}{9}y = 0$$

$$\Rightarrow \frac{4x-2y}{9} = 0$$

$$\Rightarrow 4x - 2y = 0$$

$$\Rightarrow 2x - y = 0$$

$\therefore y = 2x$ with respect to center as origin.

Also, $2(x-x_1) - 2(y-y_1) = 0$

$$\Rightarrow 2\left(x - \frac{3}{2}\right) - (y-2) = 0$$

$\Rightarrow 2x - y - 1 = 0$ with respect to old origin.

CamScanner

Similarly, the eqn of minor axis is given by:

$$\left(A-\frac{1}{r^2}\right)x+Hy=0$$

$$\Rightarrow \left(\frac{8}{9}-\frac{1}{(1)^2}\right)x+\cancel{\left(\frac{-2}{9}\right)}\left(\frac{-2}{9}\right)y=0$$

$$\Rightarrow \frac{-x}{9}-\frac{24}{9}=0$$

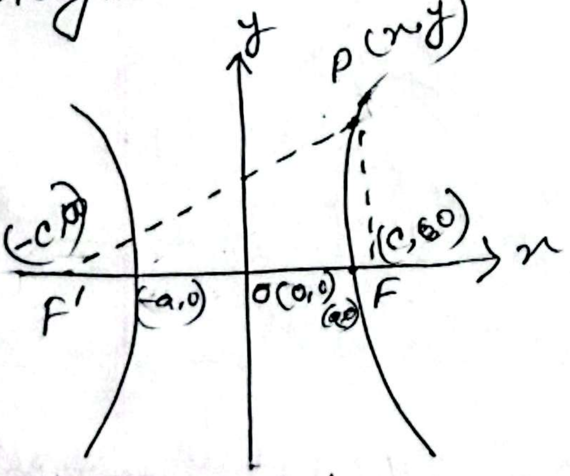
$$\Rightarrow x+2y=0$$

$\therefore y=-\frac{1}{2}x$ with respect to the center as the origin.

$$(x-x_1)+2(y-y_1)=0$$

$$\Rightarrow \left(x-\frac{3}{2}\right)+2(y-2)=0$$

$\therefore x+2y-\frac{11}{2}=0$ with respect to the old origin.



A hyperbola is a set of points whose difference of distances from 2 foci is constant value.

A hyperbola has two foci:

Foci: A hyperbola has two foci $F(c,0)$ & $F'(-c,0)$

center: Mid-point of the line joining the two foci is called the center of the hyperbola

15/01/25

Example: Reduce the eqn $32x^2+52xy+7y^2-64x-52y-148=0$ to the standard form.

Soln: Comparing the given eqn with the eqn. $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get:

$a=32,\ h=26,\ b=-7,\ g=-32,\ f=-26,\ c=-148$

Let, $f(x,y) \equiv 32x^2+52xy-7y^2-64x-52y-148=0$

$\frac{\partial f}{\partial x} = 64x+52y-0-64-0-0 \Rightarrow 16x+13y-16=0$ — (i)

$\frac{\partial f}{\partial y} = 0+52-14y-0-52-0 \Rightarrow 26x-7y-26=0$ — (ii)

$$\begin{bmatrix} 16 & 13 \\ 26 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \end{bmatrix}$$

Now, we solve (i) & (ii) for x & y

$\therefore x = \dfrac{D_x}{D} = \dfrac{-450}{-450} = 1$

$\therefore y = \dfrac{D_y}{D} = \dfrac{0}{-450} = 0 \rightarrow$ Center is at (x, y).

$= (1, 0)$

Now, new constant $c_1 = gx + fy + c = 32(1) - 26(0) - 148$

$= -180$

The equation of the conic becomes

$0 = 32x^2 + 52xy - 7y^2 + c_1 = 32x^2 + 52xy - 7y^2 - 180$

when the xy term is removed by rotation of axes, let the reduced eqn be:

$a_1x^2 + b_1y^2 - 180 = 0$

$a_1x^2 + b_1y^2 = 180$ —— (iii)

Then $a_1 + b_1 = 32 + (-7) = 25$ & $a_1b_1 = ab - h^2 = (32)7 - (26)^2$

$= -900$

Now, $a_1 + b_1 = 25$, $a_1b_1 = -900$

$a_1 = \dfrac{-900}{b_1} \rightarrow \dfrac{-900}{b_1} + b_1 = 25 \rightarrow \dfrac{-900 + b_1^2}{b_1} = 25$

$\rightarrow b_1^2 - 25b_1 - 900 = 0$

$\rightarrow b_1 = \dfrac{-(-25) \pm \sqrt{(25)^2 - 4(1)(900)}}{2(1)} = \dfrac{25 \pm 65}{2} = 45 - 20$

So $a_1 = 45$, $b_1 = 20$

(iii) becomes: $45x^2 - 20y^2 = 180 \rightarrow \frac{45x^2}{180} - \frac{20y^2}{180} = 1$

$\rightarrow \frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$ which represents a hyperbola.

The general equation of the first degree is x, y, z i.e. $ax+by+cz+d=0$ represents a plane.

General eqn of a given plane that passes through a given point.

The general eqn of a plan is $ax+by+cz+d=0$ —(i)

Since it passes through the point (x, y, z) we get: $ax_1+by_1+cz_1+d=0$ —(ii)

Substracting (ii) from (i) we get $a(x-x_1)+b(y-y_1)+c(z-z_1)$ is the general eqn of a given plane that passes through a given point.

General eqn. of a plane through the origin.

In the case the plane passes through the origin

general equation ~~will~~ b $ax+by+cz+d=0$ will

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Now, if P be the length of the perpendicular distance of

Vector Analysis:

Introduction: We define complex numbers by introducing an abstract symbol 'i' which satisfies the usual rules of algebra and additionally the rule:

$$i^2 = -1$$

$$(a+bi)(c+di) = ac + adi + bic + bdi^2 = ac + adi + bci + bdi$$
$$= (ac-bd) + (bc+ad)i$$

Similarly, the quartinious can be defined by introducing abstract symbols i, j, k which satisfy:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k, \; jk = i, \; ki = j, \; ji = -k$$
$$kj = -i, \; ik = -j.$$

That is unlike complex plane. 4-D set of quartinious are not commutative in multiplication:

$$(a+bi+cj+dk)(e+fi+gj+hk)$$
$$= (ae-bf-cg-dh) + (af+be+ch-dg)i + (ag-bh+ce+df)j$$
$$+ (ah+bg-cf+de)k$$

The imaginary part $bi+cj+dk$ of a quaternion behaves like a vector $\vec{v}=(b,c,d)$ in 3-D vector space and the real part 'a' behaves like a scalar in $\mathbb{R}$.

-geometrically, quaternions can be defined as a scalar plus a vector:

$$a+bi+cj+dk = a+\vec{v}$$
$$=[a+(0i+0j+0k)]+[0+(ei+fj+gk)]$$
$$=(a,\vec{0})+(0,\vec{v})$$

Modern dot product (of two vectors) is defined as:

$$\vec{v}\times\vec{\omega}=(bi+cj+dk).(fi+gj+dk)$$
$$=\begin{vmatrix} i & j & k \\ b & c & d \\ f & g & h \end{vmatrix}$$
$$=i(ch-dg)-j(bh-df)+k(bg-cf)$$
$$=(ch-dg)i+(df-bh)j+(bg+cf)k,$$

Thus, we get

$$\vec{v},\vec{\omega}=-(\vec{v}.\vec{\omega})+(\vec{v}\times\vec{\omega})$$
$$=(\vec{v}\times\vec{\omega})-(\vec{v}.\vec{\omega})$$

Vectors:

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- in 3-D a vector is an ordered triple (a, b, c), $a, b, c \in \mathbb{R}$;

Scaler $\longrightarrow$ আমরা 

→ (a, b, c) can be viewed as a co-ordinates of a point;

→ (a, b, c) can be also be viewed as an arrow from the origin to the point (a, b, c);

→ if viewed as an arrow, then it has direction from $(0, 0, 0)$ to (a, b, c) [by standing at the origin], and a magnatude [which is the length of the arrow or equivalently, the distance from $(0, 0, 0)$ to (a, b, c), which is $\sqrt{(a-0)^2+(b-0)^2+(c-0)^2}$

$$= \sqrt{a^2+b^2+c^2}.$$

→ if we think (a, b, c) as a vector, then a, b and c are called first, second and third component respectively;

Def^n: ordered triplets of real numbers subject to addition: $(a_1, a_2, a_3) + (b_1, b_2, b_3)$
$= (a_1 b_1, a_2 + b_2, a_3 + b_3)$

and multiplication by scalars ($\alpha \in \mathbb{R}$):
$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$ are called vectors;

Examples: Find the eqn of the straight line L, through $(1, -2, 4)$ and $(6, 2, -3)$ by using vectors.

Sol^n: Let (x, y, z) be any point on L. Then using point $(1, -2, 4)$ on L, we can write the arrow from $(1, -2, 4)$ to (x, y, z) which is along L:

$$(x-1)\hat{i} + (y+2)\hat{j} + (z-4)\hat{j} \quad \ldots (i)$$

Since, $(6, 2, -3)$ is also on L, we can write arrow from $(1, -2, 4)$ to $(6, 2, -3)$ as:

$$(6-1)\hat{i} + (2+2)\hat{j} + (-3-4)\hat{k} = 5\hat{i} + 4\hat{j} - 7\hat{k} \quad — (ii)$$

vector (i) & vector (ii) are parallel and hence are scalar multiples of each other ; and for some $t \in \mathbb{R}$:

$$(x-1)\hat{i}+(y+2)\hat{j}+(z-4)\hat{k} = t(5\hat{i}+4\hat{j}-7\hat{k}) = 5t\hat{i}+4t\hat{j}-7t\hat{k} \quad (*)$$

Thus, we get from (*):

$$x-1 = 5t \rightarrow x = 1+5t \quad —①$$

$$y+2 = 4t \rightarrow y = -2+4t \quad —②$$

$$z-4 = -7t \rightarrow z = 4-7t \quad —③$$

Equations ①, ② & ③ are called parametric equations in parameter t;

Now, from ① we get : $t = \frac{x-1}{5}$

from ② we get: $t = \frac{y+2}{4}$

from ③ we get: $t = \frac{z-4}{-7}$

Thus, $\frac{x-1}{5} = \frac{y+2}{4} = \frac{z-4}{-7}$ gives us an equivalent expression for L.

Ans:

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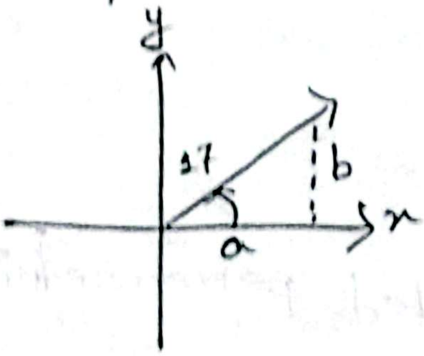
Example: Find a vector $\vec{F}$ of length 17 in the xy-Plane making an angle of 42° with positive x-axis.



From the figure the vector $\vec{F} = a\hat{i} + b\hat{j}$; $a = ?$, $b = ?$

Now, $\cos(42^\circ) = \frac{a}{17}$ and

$\sin(42^\circ) = \frac{b}{17} \rightarrow a = 17\cos 42^\circ$ &

$b = 17 \sin 42^\circ$.

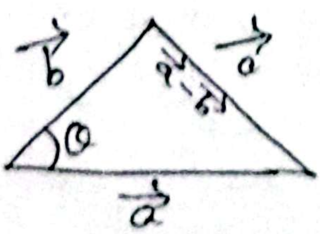
Thus, $\vec{F} = (17\cos 42^\circ)\hat{i} + (17 \sin 42^\circ)\hat{j}$.

Ans:

Dot Product: Suppose that $\vec{F} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ & $\vec{G} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. Then the dot product of $\vec{F}$ and $\vec{G}$ is the scalar:

$$\vec{F}\cdot\vec{G} = a_1a_2 + b_1b_2 + c_1c_2 .$$

Geometrical Interpretation of Dot Product:



From the figure,

$\vec{b} + \vec{c} = \vec{a} \rightarrow \vec{c} = \vec{a} - \vec{b}$

$\rightarrow$ the sides of this triangle have length $\|\vec{a}\|$, $\|\vec{b}\|$ and $\|\vec{a} - \vec{b}\|$

→ thus, from the law of cosines we get:

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\| \cdot \cos\theta$$

$$\Rightarrow 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a}-\vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$

$\rightarrow \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|.\|\vec{b}\|}$ for $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$

$\vec{a}\cdot\vec{b} = 0$, iff (if ~~only~~ and only if), or either $\vec{a}$ and/or $\vec{b}$ is zero vector.

<u>Example:</u> Find the angle between $\vec{a}$ and $\vec{b}$ if $\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-3\hat{k}$

<u>Soln:</u> We know, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$, θ is the angle between $\vec{a}$ and $\vec{b}$.

$$\Rightarrow \theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|.\|\vec{b}\|}\right)$$

Now, $\vec{a}\cdot\vec{b} = \{(2)1\} + \{1(2)\} + \{1\cdot(-3)\} = 1$

$\|\vec{a}\| = \sqrt{6}$, $\|\vec{b}\| = \sqrt{14}$

$$\theta = \arccos\left(\frac{1}{2\sqrt{4}}\right)$$

Unit: In 3-D space: $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$.

→ any vector can be expressed as a linear combination of unit co-ordinates vectors.

Example: Let $\vec{a} = (a_1, a_2, a_3)$, then

$$\vec{a} = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3)$$

$$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1).$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

→ unit vector $\vec{u}_{\vec{a}}$ of vector $\vec{a}$ in the same direction of $\vec{a}$ is the vector of unit length in the same direction of $\vec{a}$;

Derivation of Formula for finding Unit vector in the Direction of any vector $\vec{a} \neq \vec{0}$.

Let $\vec{u}_{\vec{a}}$ be the unit vector in the director of $\vec{a}$, then:

$\|\vec{u}_{\vec{a}}\| = 1$, and $\vec{u}_{\vec{a}} = \alpha\vec{a}$ where $\alpha \in \mathbb{R}$.

$\rightarrow \|\vec{u}_{\vec{a}}\| = \|\alpha \vec{a}\| \rightarrow 1 = \|\alpha \vec{a}\| \rightarrow 1 = |\alpha| \|\vec{a}\| \rightarrow$

$$\alpha = \frac{1}{\|\vec{a}\|} \rightarrow \vec{u}_{\vec{a}} = \alpha \vec{a} = \frac{1}{\|\vec{a}\|} \vec{a} \rightarrow$$

$$\vec{K}_{\vec{a}} = \frac{\vec{a}}{\|\vec{a}\|}$$

Ans:

→ thus, the unit vector in the opposite direction of $\vec{a}$ is:

$$\vec{u}_{\vec{a}} = -\frac{\vec{a}}{\|\vec{a}\|}$$

Ans:

2/2/25

<u>Example:</u> Let, $\vec{R}(t) = \cos t\hat{i} - 2t\hat{j} + e^t k$. Find the domain of $\vec{R}(t)$.

Sol^n: The general form: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$x(t) = \cos(t) \rightarrow \text{dom}[x(t)] = (-\infty, \infty)$

$y(t) = -2t \rightarrow \text{dom}[y(t)] = (-\infty, \infty)$

$z(t) = e^t \rightarrow \text{dom } z(t) = (-\infty, \infty)$

$\therefore \text{dom } \vec{R}(t) = (-\infty, \infty)$

Derivative: Let $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Then $\vec{R}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$ is the derivative of $\vec{R}(t)$ wherever $x'(t), y'(t), z'(t)$ exist.

$\rightarrow$ The derivative of $\vec{R}(t)$, evaluated at $t = t_0$ is $\vec{R}'(t_0)$ or $\frac{d}{dt}\vec{R}(t_0)$.

<u>Example:</u> Given $\vec{R}(t) = \cos t\hat{i} - 2t^2\hat{j} + e^t\hat{k}$. Find:

(a) $\vec{R}(0)$ and $\vec{R}'\left(\frac{\pi}{2}\right)$.

<u>Sol^n:</u> $\vec{R}'(t) = -\sin t\hat{i} - 4t\hat{j} + e^t\hat{k}$. Hence:

(a) $\vec{R}'(0) = 0 - 0 + e^0 k = k$

(b) $\vec{R}'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2}\hat{i} - 2\pi\hat{j} + e^{\frac{\pi}{2}}\hat{k}$

Length of $C = \int_a^b \|\vec{R}'(t)\|\, dt$

Example: Let $\vec{R}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t\hat{k}$. find the length of curve C swept out by $\vec{R}(t)$ as t varies from 0 to 2.

Solⁿ: We know, length of $C = \int_a^b \|\vec{R}'(t)\| dt$.

Here, $a=0$, $b=2$. $\rightarrow \vec{R}'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + \hat{k}$.

$\|\vec{R}'(t)\| = \left[(-2\sin t)^2 + (2\cos t)^2 + (1)^2\right]^{\frac{1}{2}}$

$= (4\sin^2 t + 4\cos^2 t + 1)^{\frac{1}{2}} = \sqrt{4(1)+1}$

$= \sqrt{5}$.

$\therefore$ Length of $C = \int_0^2 \sqrt{5}\, dt = \sqrt{5}[t]_0^2$

$= \sqrt{5}[2-0] = 2\sqrt{5}$ Units.

Ans:

Integration of vector valued functions:

$$\int_a^b \vec{f}(t)\,dt = \left[\int_a^b f_1(t)\,dt\right]\hat{i} + \left[\int_a^b f_2(t)\,dt\right]\hat{j} + \left[\int_a^b f_3(t)\,dt\right]\hat{k}$$

Example: Find $\int_0^1 \vec{f}(t)\,dt$, for $\vec{f}(t) = t\hat{i} + \sqrt{t+1}\,\hat{j} - e^{-t}\hat{k}$

Solⁿ: $\int_0^1 \vec{f}(t)\,dt = \left(\int_0^1 t\,dt\right)\hat{i} + \left(\int_0^1 \sqrt{t+1}\,dt\right)\hat{j} + \left(\int_0^1 (-e^{-t})\,dt\right)\hat{k}$

$$= \left[\frac{t^2}{2}\right]_0^1 \hat{i} + \left[\frac{2}{3}(t+1)^{\frac{3}{2}}\right]_0^1 \hat{j} + \left[e^{-t}\right]_0^1 \hat{k}$$

[Note: for $\int \sqrt{t+1}\, dt$, let $u = t+1$, then $du = dt \rightarrow \int (t+1)^{\frac{1}{2}} dt = \int u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + c$]

$$= \frac{1}{2}\hat{i} + \left(\frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}\right)\hat{j} + \left(\frac{1}{e} - 1\right)\hat{k}$$

$$= \frac{1}{2}\hat{i} + \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right)\hat{j} + \left(\frac{1}{e} - 1\right)\hat{k}.$$

Ans:

Facts: $\int_a^b [\vec{f}(t) \cdot \vec{g}(t)]\, dt$

$$= \vec{f}(t) \cdot \int_a^b \vec{g}(t)\, dt + \vec{g}(t) \int_a^b \vec{f}(t)\, dt.$$

$$\left\| \int_a^b \vec{f}(t)\, dt \right\| \leq \int_a^b \|\vec{f}(t)\|\, dt$$

$$(\vec{f} \cdot \vec{g})' = \vec{f}(t) \cdot \vec{g}\,'(t) + \vec{f}\,'(t) \cdot \vec{g}(t)$$

$$(\vec{f} \times \vec{g})'(t) = [\vec{f}(t) \times \vec{g}\,'(t)] + [\vec{f}\,'(t) \times \vec{g}(t)]$$

$$\frac{d}{dt} \vec{f}(u(t)) = \frac{d}{du} f(u(t)) \frac{d}{dt} u(t)$$

Arc Length:

Example: find the length of the curve $c: \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t^2 k$ from $t = 0$ to $t = 1$.

Solⁿ: We know, the length of the curve c is $l(c) = \int_a^b \| r'(t) \| dt$, here $a = 0$, $b = 1$ and

$$\vec{r}'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 2t\,\hat{k}.$$

$$\therefore \| \vec{r}'(t) \| = \left[(-2\sin t)^2 + (2\cos t)^2 + (2t)^2 \right]^{\frac{1}{2}}$$

$$= 2\sqrt{1+t^2}$$

$$\therefore \int_a^b \| \vec{r}'(t) \| dt = \int_0^1 \left(2\sqrt{1+t^2} \right) dt = 2\int_0^1 (1+t^2)^{\frac{1}{2}} dt$$

4/2/25 ~~30/1/25~~ 9/2/25

Some Properties of Gradient:

(1) The directional derivative of $\phi(x,y,z)$ at $P_0(x_0,y_0,z_0)$ in the direction given by a unit vector $\vec{u}$ is the dot product of the gradient of ϕ at P_0 with u:

$$\phi_{\vec{u}}(x_0,y_0,z_0) = \vec{\nabla}\phi(x_0,y_0,z_0)\cdot\vec{u}$$

If $\vec{u}$ is not a unit vector, then the directional derivative of $\phi(x,y,z)$ at $P_0(x_0,y_0,z_0)$ in the direction of $\vec{v}$ is the dot product of the gradient of $\phi(x,y,z)$ at the $P(x_0,y_0,z_0)$ with the unit vector in the direction of $\vec{v}$:

$$\phi_{\vec{u}_{\vec{v}}}(x_0,y_0,z_0) = \nabla\phi(x_0,y_0,z_0)\cdot\frac{\vec{v}}{\|\vec{v}\|}.$$

(2) $\vec{\nabla}\phi$ points in the direction in which ϕ increases at ~~ind~~ its greatest rate.

(3) $\vec{\nabla}\phi(P_0)$ has magnitude equal to the maximum rate of increase of ϕ per unit

distance at $P_0\,(x_0, y_0, z_0)$.

4. If $\vec{\nabla}\phi(x,y,z) \neq \vec{0}$ then $\vec{\nabla}\phi$ is normal (perpendicular) to the surface $\phi(x,y,z) = \text{constant}$.

Example: Prove that $\vec{\nabla}(r^n) = n r^{n-1} \vec{r}$.

Solution: Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{\nabla}(r^n) = \hat{i}\frac{\partial}{\partial x}(r^n) + \hat{j}\frac{\partial}{\partial y}(r^n) + \hat{k}\frac{\partial}{\partial z}(r^n)$$

$$= \hat{i} n r^{n-1}\frac{\partial r}{\partial x} + \hat{j} n r^{n-1}\frac{\partial r}{\partial y} + \hat{k} n r^{n-1}\frac{\partial r}{\partial z}$$

Now, $\frac{\partial r}{\partial x} = ?$, $\frac{\partial r}{\partial y} = ?$, $\frac{\partial r}{\partial z} = ?$

Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \|\vec{r}\| = r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\therefore r^2 = x^2+y^2+z^2$$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z.$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\Rightarrow \vec{\nabla} r^n = \hat{i} n r^{n-1}\left(\frac{x}{r}\right) + \hat{j} n r^{n-1}\left(\frac{y}{r}\right) + \hat{k} n r^{n-1}\left(\frac{z}{r}\right)$$

$$= \frac{n r^{n-1}}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n r^{n-2}\,\vec{r}$$

Ans.

Example: Find the directional derivative (D.D) of $\phi(x,y,z) = xy^2 + yz^3$ at point $P(2,-1,1)$ in the direction of PQ where Q is the point $(3,1,3)$.

Solⁿ: Given $\phi = xy^2 + yz^3 \rightarrow \vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$= \hat{i}(y^2) + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$

$\rightarrow \vec{\nabla}\phi(2,-1,1) = \hat{i} - 3\hat{j} - 3\hat{k}$.

Given, $\vec{a} = \vec{PQ} = \vec{OQ} - \vec{OP}$

$= (3\hat{i} + \hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= \hat{i} + 2\hat{j} + 2\hat{k}$.

Now, $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 2^2} = 3 \rightarrow \vec{a}$ is not a unit vector.

$\therefore \vec{u}_{\vec{a}} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

$\therefore$ The required directional derivative (D.D) is:

$\phi'(2,-1,1) = \vec{\nabla}\phi(2,-1,1) \cdot \vec{u}_{\vec{a}}$

$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot (\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k})$

$= 1(\frac{1}{3}) - 3(\frac{2}{3}) - 3(\frac{2}{3})$

$= -\frac{11}{3}$.

Ans:

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Example: What does it mean by an irrotational vector field? Find the constants a, b, c so that the vector $\vec{F} = (x+2y+az)\hat{i} + (bx+3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

Sol^n: $\vec{F}$ is irrotational means $\vec{\nabla}\times\vec{F} = \vec{0}$.

Ans:

Now, $$\vec{\nabla}\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx+3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$= \hat{i}(c+1) + \hat{j}(4-a) + \hat{k}(b-2)$$

$\therefore c+1=0 \rightarrow c=-1$, $4-a=0 \rightarrow a=4$, & $(b-2)=0 \rightarrow b=2$

Thus, $a=4$, $b=2$, $c=-1$.

Ans:

Example: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Sol^n: $$\nabla^2 f(r) = \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} f(r)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} f(r)\right]$$

$\frac{\partial}{\partial x} f(r)$

$= \frac{\partial}{\partial x} f\left((x^2+y^2+z^2)^{\frac{1}{2}}\right)$

$= f'(r)\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}}$

$= f'(r)\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x)$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{1}{2}}\right]$$
$$+ \frac{\partial}{\partial z}\left[f'(r)\frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{1}{2}}\right]$$
$$\left[\because r = \|\vec{r}\| = (x^2+y^2+z^2)^{\frac{1}{2}}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= f'(r)x\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r)(1)\frac{1}{r} + f''(r)\left(\frac{\partial r}{\partial x}\right)\frac{x}{r} +$$
$$+ f'(r)y\left[-\frac{1}{r^2}\frac{\partial r}{\partial y}\right] + f'(r)(1)\frac{1}{r} + f''(r)\frac{\partial r}{\partial x}\left(\frac{x}{r}\right)$$
$$+ f'(r)z\left[-\frac{1}{r^2}\frac{\partial r}{\partial z}\right]$$
$$+ f'(r)(1)\frac{1}{r} + f''(r)\frac{\partial r}{\partial z}\left(\frac{z}{r}\right)$$

$$= f'(r)x\left(-\frac{1}{r^2}\frac{x}{r}\right) + f'(r)\frac{1}{r} + f''(r)\frac{x}{r}x\frac{1}{r} + f'(r)y\left(-\frac{1}{r^2}\right)\frac{y}{r}$$
$$+ f'(r)\frac{1}{r} + f''(r)\frac{y}{r}y\left(\frac{1}{r}\right)$$
$$+ f'(r)z\left(-\frac{1}{r^2}\right)\frac{z}{r} + f'(r)\frac{1}{r} + f''(r)\frac{z}{r}z\frac{1}{r}$$

$$= f'(r)\left(-\frac{1}{r^3}\right)x^2 + f'(r)\frac{1}{r} + f''(r)\frac{1}{r^2}x^2 + f'(r)\left(-\frac{1}{r^3}\right)y^2 +$$
$$f'(r)\frac{1}{r} + f''(r)\frac{1}{r^2}y^2 + f'(r)\left(-\frac{1}{r^3}\right)z^2 + f'(r)\frac{1}{r}$$
$$+ f''(r)\frac{1}{r^2}z^2$$

$$= -f'(r)\left(\frac{1}{r^3}\right)(x^2+y^2+z^2) + \frac{3}{r}f'(r) + f''(r)\frac{1}{r^2}(x^2+y^2+z^2)$$

$$= -f'(r)\frac{1}{r^3}(r^2) + \frac{3}{r}f'(r) + f''(r)\frac{1}{r^2}r^2$$

$$= -f'(r)\frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

$$= \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

Ans.

Line Integral: Let $\vec{F}(x,y,z) = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ be continuous at least on the graph of a regular curve C which is given by : $x = x(t)$, $y = y(t)$, $z = z(t)$; $t: a \to b$; then —

the line integral of $\vec{F}$ over C is denoted $\int_C \vec{F}$ and is defined by:

$$\int_C \vec{F} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\, dt \quad \text{where}$$

$$\vec{F}(x(t), y(t), z(t)) = f_1(x(t), y(t), z(t))\hat{i} + f_2(x(t), y(t), z(t))\hat{j} + f_3(x(t), y(t), z(t))\hat{k} \text{ and}$$

$$\vec{R}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}.$$

Thus,

$$\int_C \vec{F} = \int_a^b \Big[f_1(x(t), y(t), z(t))\, x'(t) + f_2(x(t), y(t), z(t))\, y'(t) + f_3(x(t), y(t), z(t))\, z'(t)\Big]\, dt.$$

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Example: Let $\vec{F}(x,y,z) = x^2\hat{i} - 2xyz\hat{j} + z^2\hat{k}$ and let c is curve given by: $x(t) = t,\ y(t) = 2t,\ z(t) = -4t$; $t: 0 \to 3$. Find $\int_c \vec{F}$, if it exists.

Sol^n: Here, $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ for $t: 0 \to 3$.

Now, $\vec{F}(x(t), y(t), z(t)) = [x(t)]^2\hat{i} - 2[x(t)][y(t)][z(t)]\hat{j} + [z(t)]^2\hat{k}$.

$$\therefore \vec{F}(x(t), y(t), z(t)) = t^2\hat{i} - 2t(2t)(-4t)\hat{j} + (-4t)^2\hat{k}$$
$$= t^2\hat{i} + 16t^3\hat{j} + 16t^2\hat{k}$$

and, $\vec{R}(t) = t\hat{i} + 2t\hat{j} - 4t\hat{k} \to \vec{R}'(t) = \hat{i} + 2\hat{j} - 4\hat{k}$.

Now, we know $\int_c \vec{F} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\,dt$ exists if $\vec{F}(x(t), y(t), z(t))$ is continuous for $t: a \to b$ and c is a regular curve.

Here, $\vec{F}(x(t), y(t), z(t)) = f_1(x(t), y(t), z(t))\hat{i} + f_2(x(t), y(t), z(t))\hat{j} + f_3(x(t), y(t), z(t))\hat{k}$
$$= t^2\hat{i} + 16t^3\hat{j} + 16t^2\hat{k}.$$

Now, $f_1 = t^2$ is continuous for $t: 0 \to 3$
$f_2 = 16t^3$ " " " $t: 0 \to 3$
$f_3 = 16t^2$ " " " $t: 0 \to 3$ $\Big\} \to$

$\vec{F}(x, y, z)$ is continuous at least on the graph of C.

Now, $\vec{R}(t_1) = t_1\hat{i} + 2t_1\hat{j} - 4t_1\hat{k}$ and $R(t_2) = t_2\hat{i} + 2t_2\hat{j} - 4t_2\hat{k}$

For $x(t_i)$, $t_1 = t_2 \rightarrow t_1 = t_2$

for $y(t_i)$, $2t_1 = 2t_2 \rightarrow t_1 = t_2$

For $z(t_i)$, $-4t_1 = -4t_2 \rightarrow t_1 = t_2$

If only one of the three $x(t_i)$, $y(t_i)$ and $z(t_i)$ implies $t_1 = t_2$ then

$\vec{R}(t_1) = \vec{R}(t_2)$ implies $t_1 = t_2$

Hence, C is a simple curve.

Now, $x'(t) = 1$ is continuous for $0 < t < 3$

$y'(t) = 2$ " " " $0 < t < 3$ and

$z'(t) = -4$ " " " $0 < t < 3$

and, $(x'(t), y'(t), z'(t)) = (1, 2, -4)$ is never equal to $(0, 0, 0)$ for any $t \in (0, 3)$.

Hence, C is smooth implies at least piecewise.

That is, C is regular as because C is simple and piecewise smooth.

Hence, $\displaystyle\int_C \vec{F} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\, dt$

$$= \int_0^3 (t^2\hat{i} + 16t^3\hat{j} + 16t^2\hat{k}) \cdot (\hat{i} + 2\hat{j} - 4\hat{k})\, dt$$

$$= \int_0^3 (t^2 + 32t^3 - 64t^2)\, dt$$

$$= \int_0^3 (32t^3 - 63t^2)\, dt$$

$$= 32\left[\frac{t^4}{4}\right]_0^3 - 63\left[\frac{t^3}{3}\right]_0^3$$

$$= 8\left[t^4\right]_0^3 - 21\left[t^3\right]_0^3$$

$$= 8\left[3^4 - 0\right] - 21\left[3^3 - 0\right] = 8(81) - 21(27)$$

$$= 81$$

Ans:

Line Integral Using Notation $\int \vec{F} \cdot d\vec{R}$ instead of $\int_c \vec{F}$:

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

one often sees $\int_c \vec{F}$ written as $\int_c \vec{F} \cdot d\vec{R}$.

Pf we have to show that. $\int_c \vec{F} = \int_c \vec{F} \cdot d\vec{R}$

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Line Integral Using Notation $\int_C \vec{F}\cdot d\vec{R}$:

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

one often sees $\int_C \vec{F}$ written as $\int_C \vec{F}\cdot d\vec{R}$.

pf We have to show that : $\int_C \vec{F} = \int_C \vec{F}\cdot d\vec{R}$

We know, $\int_C \vec{F} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\,dt$

For $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$

continous on a regular curve C given by

$x = x(t),\ y = y(t),\ t = z(t);\ t: a \to b$.

Thus, we get, $\int_C \vec{F} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot (x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k})\,dt$;

$[\because \vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}]$

$$= \int_a^b [F_1(x(t), y(t), z(t))\hat{i} + F_2(x(t), y(t), z(t))\hat{j} + F_3(x(t), y(t), z(t))\hat{k}] \cdot [x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}]\,dt$$

$$= \int_a^b [F_1(x(t), y(t), z(t))x'(t) + F_2(x(t), y(t), z(t))y'(t) + F_3(x(t), y(t), z(t))z'(t)]\,dt$$

$$= \int_a^b \left[F_1 x'(t) + F_2 y'(t) + F_3 z'(t) \right] dt$$

$$= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

Now, for $\vec{F} \cdot d\vec{R}$, from $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

we get : $d\vec{R} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\left[\begin{array}{l} \because \vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \\ \longrightarrow \frac{d}{dt} R(t) = \hat{i}\frac{d}{dt} x(t) + \hat{j}\frac{d}{dt} y(t) + \hat{k}\frac{d}{dt} z(t) \\ \longrightarrow d \cdot \vec{R}(t) = \left[dx(t)\hat{i} + dy(t)\hat{j} + dz(t)\hat{k} \right] \frac{dt}{dt} \\ \longrightarrow d\vec{R} = dx\hat{i} + dy\hat{j} + dz\hat{k} \end{array} \right]$$

and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ gives :

$$\vec{F} \cdot d\vec{R} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= F_1 dx + F_2 dy + F_3 dz .$$

where,

$F_1 = F_1(x(t), y(t), z(t))$, $F_2 = F_2(x(t), y(t), z(t))$,

and $F_3 = F_3(x(t), y(t), z(t))$; and $x = x(t)$, $y = y(t)$

$z = z(t)$

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$$\therefore \int_a^b \vec{F} \cdot d\vec{R} = \int_a^b (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)\, dt = \int_C \vec{F}$$

Example:

Evaluate $\int_C (F_1 dx + F_2 dy + F_3 dz)$, where $\vec{F} = 2xy\,\hat{i} - y^2\hat{j} + e^x z\hat{k}$

and $C = \{x(t), y(t), z(t) : x(t) = -t, y(t) = \sqrt{t}, z(t) = 3t : t : 1 \to 4\}$

Sol^n:

On C, $\frac{dx}{dt} = \frac{d}{dt}(-t)$, $\frac{dy}{dt} = \frac{d}{dt}(t^{1/2})$, $\frac{dz}{dt} = \frac{d}{dt}(3t)$

$\therefore \frac{dx}{dt} = -1 \to dx = -dt$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}} \longrightarrow dy = \frac{1}{2\sqrt{t}} dt$

$\frac{dz}{dt}(3t) = 3, \longrightarrow dz = 3dt$

$F_1(x(t), y(t), z(t)) = 2x(t)y(t) = 2(-t)\sqrt{t} = -2t^{3/2}$.

$F_2(x(t), y(t), z(t)) = -[y(t)]^2 = -(\sqrt{t})^2 = -t$.

$F_3(x(t), y(t), z(t)) = e^x z = e^{-t}(3t) = 3te^{-t}$.

$$\therefore \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_1^4 (-2t^{3/2})(-dt) + \int_1^4 (-t)\frac{dt}{2\sqrt{t}} + \int_1^4 (t \cdot 3t\, e^{-t})\, 3\,dt.$$

$$= 2\int_1^4 t^{3/2}\,dt - \tfrac{1}{2}\int_1^4 t^{1/2}\,dt + 9\int_1^4 t\,e^{-t}\,dt$$

Now, $\int t^{3/2}\,dt = \frac{2}{5} t^{5/2}$, $\int t^{1/2}\,dt = \frac{2}{3} t^{3/2}$,

an for $\int t e^{-t}\,dt$, let $u = t$ $\quad dv = e^{-t}\,dt$

$du = dt \quad v = -e^{-t}$

$$\therefore \int t e^{-t}\,dt = \int u\,dv = uv - \int v\,du = -te^{-t} + \int e^{-t}\,dt$$

$$= -te^{-t} - e^{-t}$$

$$\therefore \int_C (F_1 dx + F_2 dy + F_3 dz) = \frac{2(2)}{5}\left[t^{5/2}\right]_1^4 - \tfrac{1}{2}\left(\tfrac{2}{3}\right)\left[t^{3/2}\right]_1^4 + 9\left[-te^{-t} - e^{-t}\right]_1^4$$

$$= \frac{4}{5}\left[4^{5/2} - 1^{5/2}\right] - \frac{1}{3}\left[4^{3/2} - 1^{3/2}\right] + 9\left[-\frac{4}{e^4} - \frac{1}{e^4} + \frac{1}{e} + \frac{1}{e}\right]$$

$$= \frac{4}{5}\left[4^{5/2} - 1\right] - \frac{1}{3}\left[4^{3/2} - 1\right] + 9\left[2/e - 5/e^4\right] \simeq 28.26$$