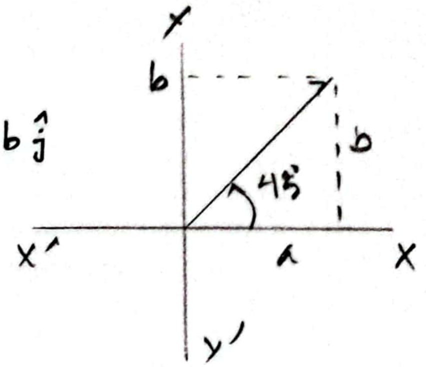
Question - 01

[a] (i) Find a vector $\vec{F}$ of length 13 in the xy-plane making an angle of 45° with the positive x-axis.



Sol^n:

From the figure the vector, $\vec{F} = a\hat{i} + b\hat{j}$

Now,

$$\cos(45°) = \frac{a}{13} \quad \text{and}$$

$$\sin(45°) = \frac{b}{13}$$

$\therefore a = 13\cos(45°) \qquad b = 13\sin 45°$

Thus, $\vec{F} = [13\cos(45°)]\hat{i} + [13\sin(45°)]\hat{j}$

(ii) Find the angle between $\vec{a}$ and $\vec{b}$ if $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$

Sol^n: Let the angle between $\vec{a}$ and $\vec{b}$ is α.

then: $$\alpha = \text{arc}\cos\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

Now, $$\vec{a}\cdot\vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k})\cdot(2\hat{i} + \hat{j} + \hat{k})$$
$$= 2 + 2 - 3 = 1$$

$\|\vec{a}\| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}$ and

$\|\vec{b}\| = \sqrt{(2)^2 + (1)^2 + (1)^2} = \sqrt{6}$

$$\therefore \alpha = \text{arc}\cos\left(\frac{1}{\sqrt{14}\sqrt{6}}\right) \cong 1.46 \text{ radians}$$

[b] Given $\vec{F} = -2\hat{i}+\hat{j}+4\hat{k}$ and $\vec{G} = \hat{i}+2\hat{j}-3\hat{k}$.

(i) Find $\vec{F}\times\vec{G}$ (ii) Find $\vec{G}\times\vec{F}$

(i) Ans: $\vec{F}\times\vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 4 \\ 1 & 2 & -3 \end{vmatrix} = \hat{i}(-3-8) - \hat{j}(6-4) + \hat{k}(-4-1)$

$= -11\hat{i} - 2\hat{j} - 5\hat{k}$

(ii) Ans: $\vec{G}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ -2 & 1 & 4 \end{vmatrix} = \hat{i}(8+3) - \hat{j}(4-6) + \hat{k}(1+4)$

$= 11\hat{i} + 2\hat{j} + 5\hat{k}$

$= -(\vec{F}\times\vec{G})$

Question-02

[a] Let $\vec{R}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + 2t\hat{k}$. Find the length of curve C swept out by $\vec{R}(t)$ as t varies from 0 to 3.

Soln: We know,

$$\text{length of } C = \int_a^b \|\vec{R}'(t)\|\,dt$$

Here, $a=0$, $b=3$, $\vec{R}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j} + 2\hat{k}$

$\therefore \|\vec{R}'(t)\| = \left[[(-\sin t)]^2 + [\cos(t)]^2 + (2)^2\right]^{1/2}$

$= \sqrt{\sin^2 t + \cos^2 t + 4}$ $\quad [\because \sin^2 t + \cos^2 t = 1]$

$= \sqrt{5}$

$\therefore \text{Length of } C = \int_0^3 \sqrt{5}\,dt = \sqrt{5}\int_0^3 dt = \sqrt{5}[t]_0^3 = 3\sqrt{5}$ units

[b] (i) Given $\vec{f}(t) = \sqrt{(t+1)}\,\hat{i} + t\hat{j} - e^t\hat{k}$. Find dom $\vec{f}(t)$.

(ii) Find $||\vec{f}(t)||$ (iii) Find $\vec{f}(t).\vec{f}'(t)$, & (iv) Find

$\int_0^1 \vec{f}(t)\,dt$.

(i) Ans: $f_1(t) = \sqrt{(t+1)}$ → for $\sqrt{t+1} \geq 0$

$\Rightarrow (t+1) \geq 0$

$\Rightarrow t \geq -1$

$\therefore$ dom $(f_1(t)) = [-1, \infty)$;

$f_2(t) = t$, $\Rightarrow$ dom $(f_2(t)) = (-\infty, \infty)$

$f_3(t) = -e^t$ $\Rightarrow$ dom $[f_3(t)] = (-\infty, \infty)$

$\therefore$ dom $(\vec{f}(t)) = [-1, \infty)$

(ii) Ans: $||\vec{f}(t)|| = \sqrt{(\sqrt{t+1})^2 + t^2 + (e^t)^2} = \sqrt{t^2+t+1+e^{2t}}$

(iii) Ans: $\vec{f}(t).\vec{f}'(t) = (\sqrt{t+1})\left(\frac{1}{2\sqrt{t+1}}\right) + t.(1) + \{(-e^t)(-e^t)\}$

$= \frac{1}{2} + t + e^{2t}$

(iv) Ans: $\int_0^1 \vec{f}(t)\,dt = \left(\int_0^1 \sqrt{t+1}\,dt\right)\hat{i} + \left[\int_0^1 t\,dt\right]\hat{j} + \left(\int_0^1 (-e^t)\,dt\right)\hat{k}$

$= \left[\frac{2}{3}(t+1)^{3/2}\right]_0^1 \hat{i} + \left[\frac{t^2}{2}\right]_0^1 \hat{j} + \left[-e^t\right]_0^1 \hat{k}$

$= \left[\frac{2}{3}(2)^{3/2} - \frac{2}{3}\right]\hat{i} + \frac{1}{2}\hat{j} - (e^1 - e^0)\hat{k}$

$= \frac{2}{3}(2^{3/2} - 1)\hat{i} + \frac{1}{2}\hat{j} - (e-1)\hat{k}$

Question - 03

[a] Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = \|\vec{r}\|$. Find

(i) $\vec{\nabla} r$ (ii) $\vec{\nabla}\left(\frac{1}{r}\right)$ (iii) $\vec{\nabla}(\ln r)$

(i) **Ans:** Gradient of r:

The magnitude of $\vec{r}$ is: $r = \sqrt{x^2+y^2+z^2}$

$$\therefore \vec{\nabla} r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

Using chain rule:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Therefore, $\vec{\nabla} r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$

(ii) $\Rightarrow$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$$

Using chain rule:

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{i}, \quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{j}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{k}$$

$$\therefore \vec{\nabla}\left(\frac{1}{r}\right) = +\frac{-x}{r^3}\hat{i} + \frac{-y}{r^3}\hat{j} + \frac{-z}{r^3}\hat{k}$$

$$= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3} = -\frac{\vec{r}}{r^3}$$

(iii) Ans: Since, $\ln r = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$

we differentiate using chain rule:

$$\frac{\partial}{\partial x}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right] = \frac{1}{2}\,\frac{2x}{x^2+y^2+z^2} = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial y}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right] = \frac{1}{2}\,\frac{2y}{x^2+y^2+z^2} = \frac{y}{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right] = \frac{z}{x^2+y^2+z^2}$$

$$\therefore \vec{\nabla}(\ln r) = \frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^2} = \frac{\vec{r}}{r^2}$$

[b] Given $\phi = 4xz^2 + x^2yz$ (i) Find $\|\vec{\nabla}\phi\|$ at $(1,-2,1)$ (ii) Find the directional derivative of ϕ at $(1,-2,-1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$ & (iii) Find the unit normal vector $\left(\hat{n} = \frac{\vec{\nabla}\phi}{\|\vec{\nabla}\phi\|}\right)$ to the surface $x^2+y^2=z$ at the point $(1,-2,5)$.

(i) Ans: Here, $\phi = 4xz^2 + x^2yz$

Compute partial derivatives:

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(4xz^2+x^2yz) = 4z^2 + 2xyz$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(4xz^2+x^2yz) = x^2z$$

$$\frac{\delta\phi}{\partial z} = \frac{\delta}{\partial z}\left(4xz^2 + x^2yz\right) = 8xz + x^2y$$

$$\therefore \vec{\nabla}\phi(x,y,z) = \frac{\delta\phi}{\partial x}\hat{i} + \frac{\delta\phi}{\partial y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$$

$$= (4z^2 + 2xyz)\hat{i} + (x^2z)\hat{j} + (8xz + x^2y)\hat{k}$$

$$\therefore \vec{\nabla}\phi(1,-2,-1) = \{4(-1)^2 + 2.1(-2).1\}\hat{i} + \{(1)^2.1\}\hat{j}$$

$$+ \{8.1.1 + (1)^2.(-2)\}\hat{k}$$

$$= 0\hat{i} + \hat{j} + 6\hat{k} = \hat{j} + 6\hat{k}$$

$$\therefore \|\vec{\nabla}\phi\| = \sqrt{(1)^2 + (6)^2} = \sqrt{37}$$

(11) Ans: let, $P_0 = (1,-2,-1)$

$$\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$$

Here, $\|\vec{a}\| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = 3 \neq 1$

Here, $\vec{a}$ is not a unit vector and the unit vector in the direction of $\vec{a}$ is:

$$\vec{u}_{\vec{a}} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{3}\left(2\hat{i} - \hat{j} - 2\hat{k}\right)$$

Now, from (1) no. math we get:

$$\vec{\nabla}\phi = (4z^2 + 2xyz)\hat{i} + (x^2z)\hat{j} + (8xz + x^2y)\hat{k}$$

Now,

$$\vec{\nabla}\phi(1,-2,-1) = \{4(-1)^2 + 2\cdot 1(-2)(-1)\}\hat{i} + \{1^2\cdot(-1)\}\hat{j} + \{8\cdot 1(-1) + 1^2(-2)\}\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k} = \vec{\nabla}\phi(P_0)$$

$$\therefore \phi'_{\vec{a}}(P_0) = \vec{\nabla}\phi(P_0)\cdot\vec{u}_{\vec{a}}$$

$$= (8\hat{i} - \hat{j} - 10\hat{k})\cdot\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3}$$

iii) Ans: Let, $\phi = x^2 + y^2 - z$

compute partial derivatives:

$$\vec{\nabla}(x^2+y^2-z) = 2x\hat{i} + 2y\hat{j} - \hat{k} = \text{Normal vector}$$

Normal vector at $(1,-2,5)$ $= 2\hat{i} - 4\hat{j} - \hat{k}$

$\therefore$ Unit Normal vector, $\hat{n} = \dfrac{\vec{\nabla}\phi}{\|\vec{\nabla}\phi\|}$

$$= \frac{2\hat{i} - 4\hat{j} - \hat{k}}{\sqrt{2^2 + (-4)^2 + (-1)^2}} = \frac{1}{\sqrt{21}}(2\hat{i} - 4\hat{j} - \hat{k})$$

Question - 4 (a) If $\vec{\nabla}\phi = 2xyz\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ find ϕ if $\phi(-1,2,1) = 4$

(b) (i) If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A}\times\vec{B}$ is solenoidal (ii) Find $\vec{\nabla}^2(\ln r)$

(a) Ans: we integrate each component:

$$\frac{\delta\phi}{\delta x} = 2xyz \quad \therefore \phi = x^2yz + f_1(y,z) \quad (1)$$

$$\frac{\delta\phi}{\delta y} = x^2z^3 \quad \therefore \phi = x^2z^3y + f_2(x,z) \cdot (2)$$

$$\frac{\delta\phi}{\delta z} = 3x^2yz^2 \quad \therefore \phi = x^2yz^3 + f_3(y,x) \quad (3)$$

combining (1), (2) & (3), we get:

$$\phi = x^2yz + x^2yz^3$$

$$\phi(-1,2,1) = (-1)^2 \cdot 2 \cdot 1 + (-1)^2 \cdot 2 \cdot (1)^3$$
$$= 4$$

(b) $\Rightarrow$ (i) Ans! A vector field is irrotational if its curl is zero:

$$\vec{\nabla}\times\vec{A} = 0$$
$$\vec{\nabla}\times\vec{B} = 0$$

We need to show that the divergence of $\vec{A}\times\vec{B}$ is zero

$$\vec{\nabla}\cdot(\vec{A}'\times\vec{B}') = 0$$

using the identity:

$$\vec{\nabla}\cdot(\vec{A}'\times\vec{B}') = \vec{B}'\cdot(\vec{\nabla}\times\vec{A}') - \vec{A}'\cdot(\vec{\nabla}\times\vec{B}')$$

since both $\vec{A}'$ and $\vec{B}'$ are irrotational,

$\vec{\nabla}\times\vec{A}' = 0$ and $\vec{\nabla}\times\vec{B}' = 0$ so;

$$\vec{\nabla}\cdot(\vec{A}'\times\vec{B}') = 0$$

Hence, $\vec{A}'\times\vec{B}'$ ~~are~~ is solenoidal.

11 Ans: We know: $\vec{\nabla}(\ln r) = \dfrac{\vec{r}}{r^2}$

Taking divergence: $\vec{\nabla}^2(\ln r) = \vec{\nabla}\cdot\left(\dfrac{\vec{r}}{r^2}\right) = \dfrac{2}{r^2}$

Question: 5 (a) If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$

Ans: Parameterize the curve:

$$x = t, \quad y = 2t^2, \quad 0 \le t \le 1$$

compute:

$$dx = dt, \quad dy = 4t\,dt$$

$\therefore \vec{F}\cdot d\vec{r} = (3xy\,dx - y^2\,dy)$

substitute $x = t$, $y = 2t^2$: $\quad 3t(2t)^2 dt - (2t^2)^2(4t\,dt)$

$$= (6t^3 - 16t^5)\,dt$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt$$

$$= \left[\frac{6t^4}{4} - \frac{16t^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}$$

(b) **Question:** Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$

Ans! Parameterize the path:

Let, $x = t^2$, $y = t$, $0 \leq t \leq 1$

compute: $dx = 2t\,dt$, $dy = dt$

Evaluate $\vec{F}.\vec{dr} = \left[(t^4 - t^2 + t)\,2t\,dt - (2t^3 + t)\,dt\right]$

$$= (2t^5 - 2t^3 + 2t^2 - 2t^3 - t)\,dt$$

$$= \int_0^1 (2t^5 - 4t^3 + 2t^2 - t)\,dt$$

$$= \left[\frac{2t^6}{6} - \frac{4t^3}{4} + \frac{2t^3}{3} - \frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{3} - 1 + \frac{2}{3} - \frac{1}{2}$$

$$= \frac{-1}{2}$$

Question: 6 Show that $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\hat{n}\, ds = \frac{3}{8}$

where S is the surface of the sphere $x^2+y^2+z^2=1$ in the first octant.

Answer! We use the Divergence Theorem, which states:

$$\iiint_V (\vec{\nabla}\cdot\vec{F})\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

where V is the volume enclosed by the surfaces.

compute the divergence of $\vec{F} = (yz, zx, xy)$:

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy)$$

$$= 0+0+0 = 0$$

Since the divergence is zero, the flux integral is just the integral over a fraction of the total surface.

compute the surface integral for the first octant:

The total surface integral over the full sphere is zero (because divergence is zero). The first octant contributes $\frac{1}{8}$ of the total flux. Given the flux over the whole sphere is $\frac{3}{1} = 3$, we take: $\frac{3}{8}$

Hence, $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{8}$

Question : 07 If $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate $\iiint_V \vec{\nabla}\times\vec{F}\, dv$ where v is the volume of the region bounded by $x=0, y=0, z=0$ and $2x+2y+z=4$

Ans: $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x^2-3z) & -2xy & -4x \end{vmatrix}$

$= \hat{i}\left\{\frac{\partial}{\partial y}(4x) - \frac{\partial}{\partial z}(2xy)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2-3z)\right\}$
$+ \hat{k}\left\{\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right\}$

$= (0-0)\hat{i} - (-4+3)\hat{j} + (-2y-0)\hat{k}$

$= \hat{j} - 2y\hat{k}$

By the divergence theorem of curl:

$$\iiint_V (\nabla\times\vec{F})\cdot d\vec{v} = \iint_S \vec{F}\cdot d\vec{s}$$

Since the divergence of a curl is always zero:

$$\nabla\cdot(\nabla\times\vec{F}) = 0$$

By the volume of integral property:

$$\iiint_V (\nabla\times\vec{F})\cdot d\vec{V} = 0$$

Question-8 Verify Green's theorem in the xy-plane for $\oint_C [(3x^2-8y^2)\,dx + (4y-6xy)\,dy]$ where C is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$

Answer: Green's theorem states that for a vector field

$$F = P\hat{i} + Q\hat{j}$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dA$$

From given integral:

$$P = 3x^2 - 8y^2$$
$$Q = 4y - 6xy$$

compute partial derivatives:

$$\frac{\delta Q}{\delta x} = \frac{\delta}{\partial x}(4y - 6xy) = -6y$$

$$\frac{\delta P}{\partial y} = \frac{\delta}{\partial y}(3x^2 - 8y^2) = -16y$$

Thus,

$$\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} = -6y + 16y = 10y$$

The given region is a right triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. The limits are:

$$\int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

Evaluating the inner integral: $\int_0^{1-x} 10y\,dy = [5y^2]_0^{1-x} = 5(1-x)^2$

Now, integrate over x: $\int_0^1 5(1-x)^2\,dx$

$$= \int_0^1 5(1-x)^2 dx$$

$$= \int_0^1 5(1-2x+x^2) dx$$

$$= 5\left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$= 5\left(1 - 1 + \frac{1}{3}\right) = \frac{5}{3}$$

Thus, the left-hand line integral equals the right hand double integral, verifying Green's theorem.

Question: 09 Verify stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ in a rectangular region in the xy plane bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$.

Answer: Stoke's theorem states:

$$\oint_C F.dr = \iint_S (\nabla \times \vec{F}).ds$$

Now, $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2y+2y) = 4y\hat{k}$$

Since the surface lies in the xy-plane, the normal vector is $\hat{k}$, so: $ds = \hat{k}\, dx\, dy$

$$\iint_S (\nabla \times F) \cdot ds = \iint_S 4y\,dA$$

The limits of integration: $\int_0^a \int_0^b 4y\,dy\,dx$

$$= \int_0^a [2y^2]_0^b\,dx$$

$$= \int_0^a 2b^2\,dx = 2ab^2$$

Thus, the circulation integral equals the surface integral, verifying stoke's theorem.

Question: 10 Verify Gauss's Divergence Theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Answer: Gauss's theorem state that;

$$\iiint_V (\nabla \cdot F)\,dv = \oiint_S \vec{F} \cdot ds$$

Now, $\nabla \cdot F = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$

$$= 4z - 2y + y = 4z - y$$

compute volume integral: $\iiint_V (4z - y)\,dv$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\,dx\,dy\,dz$$

Computing Each term:

First term: $\int_0^1 4z\,dz = 4 \times \frac{1}{2} = 2$

Second term: $\int_0^1 (-y)\,dy = \frac{-1}{2}$

Since the integration over x is straightforward ($\int_0^1 dx = 1$) we get:

$$\iiint_V (4z - y)\,dV = 2 - \frac{1}{2} = \frac{3}{2}$$

We compute: $\oiint_S \vec{F} \cdot d\vec{s}$ for each of the six faces of the cube.

Face $x=0$: $(-\hat{i})$ $\therefore F \cdot (-\hat{i}) = -4xz$ at $x = 0 \Rightarrow 0$

Face $x=1$: $(+\hat{i})$ $\therefore F \cdot \hat{i} = 4xz$ at $x = 1$

Integral: $\int_0^1 \int_0^1 4z\,dy\,dz = 4 \times \frac{1}{2} = 2$

Face $y=0$ $(-\hat{j})$: $F \cdot (-\hat{j}) = y^2$ at $y = 0 \Rightarrow 0$

Face $y=1$ $(+\hat{j})$: $F \cdot \hat{j} = -y^2$ at $y = 1 \Rightarrow -1$

Integral: $\int_0^1 \int_0^1 (-1)\,dx\,dz = -1$

Face $z=0$ $(-\hat{k})$: $\vec{F} \cdot (-\hat{k}) = -yz$ at $z = 0$

Face $z=1$ $(\hat{k})$: $\vec{F}(\hat{k}) = yz$ at $z = 1 \Rightarrow y$

Integral: $\int_0^1 \int_0^1 y\,dx\,dy = \frac{1}{2}$

After summing we get, $2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}$

Since both the volume and surface integral are equal so, Gauss's theorem is verified.