

CSC 411: Lecture 02: Linear Regression

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(Most plots in this lecture are from Bishop's book)

Problems for Today

- What should I watch this Friday?

Find Movies, TV shows, Celebrities and more... All

IMDb

Movies, TV & Showtimes Celebs, Events & Photos News & Community Watchlist

The Martian (2015)
PG-13 | 144 min | Adventure, Comedy, Drama | 2 October 2015 (USA) 9
8.1 Your rating: ★★★★★★★★★★ /10
Ratings: 8.1/10 from 271,829 users Metascore: 80/100
Reviews: 750 user | 499 critic | 46 from Metacritic.com

During a manned mission to Mars, Astronaut Mark Watney is presumed dead after a fierce storm and left behind by his crew. But Watney has survived and finds himself stranded and alone on the hostile planet. With only meager supplies, he must draw upon his ingenuity, wit and spirit to subsist and find a way to signal to Earth that he is alive.

Director: Ridley Scott
Writers: Drew Goddard (screenplay), Andy Weir (book)
Stars: Matt Damon, Jessica Chastain, Kristen Wiig | See full cast and crew »

+ Watchlist Watch Trailer Share...

Problems for Today

- What should I watch this Friday?



The image shows a screenshot of the IMDb website for the movie "Point Break (2015)". The page features the classic yellow "IMDb" logo at the top left. A search bar is positioned above the main navigation menu, which includes links for "Movies, TV & Showtimes", "Celebs, Events & Photos", "News & Community", and "Watchlist". The main content area displays the movie's poster, which features a surfer riding a large wave with the text "FIND YOUR BREAKING POINT" overlaid. Below the poster, the movie's title "Point Break (2015)" is shown in large letters, followed by its rating of "PG-13 | 114 min | Action, Crime, Sport | 25 December 2015 (USA)". To the right of the title is a small red icon with the number "15". The movie's rating is displayed as "5.4" in a yellow star icon, with a full 10-star scale above it. Below the rating, it says "Ratings: 5.4/10 from 7,322 users | Metascore: 34/100 | Reviews: 60 user | 84 critic | 19 from Metacritic.com". A detailed plot summary follows, mentioning a young FBI agent infiltrating an extraordinary team of extreme sports athletes suspected of masterminding sophisticated corporate heists, noting the film is inspired by the 1991 hit. Below the plot, director Ericson Core, writer Kurt Wimmer (screenplay), and stars Édgar Ramírez, Luke Bracey, and Ray Winstone are listed, along with a link to the full cast and crew. At the bottom of the page are three buttons: "+ Watchlist" (with a dropdown arrow), "Watch Trailer", and "Share...".

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All

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Writers: Kurt Wimmer (screenplay), Rick King (story), 5 more credits »

Stars: Édgar Ramírez, Luke Bracey, Ray Winstone | See full cast and crew »

+ Watchlist ▾

Watch Trailer

Share...

Problems for Today

- **Goal:** Predict movie rating automatically!

The image shows a screenshot of the IMDb website for the movie "Point Break" (2015). The page includes the movie poster, basic info (PG-13, 114 min, 25 December 2015), and a call-to-action box: "Predict this automatically!". A red circle highlights the yellow star rating of 5.4.

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Point Break (2015)

PG-13 | 114 min | 25 December 2015

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FIND YOUR BREAKING POINT
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See More on IMDb Pro »

+ Watchlist Watch Trailer Share...

Problems for Today

- **Goal:** How many followers will I get?

Red Leather Jacket
Updated on Jan 09, 2016



< >

From This User

+1 282 VOTES

5 COMMENTS

67 FAVORITES

Like 0

Tweet

G+1 0

...

Pinit 2

Tags

Chic
Everyday
Winter

SHARE

Problems for Today

- **Goal:** Predict the price of the house

The screenshot shows the homepage of the Nationwide House Price Index. At the top, there is a navigation bar with links: 'Why choose Nationwide?', 'Have your say', 'Corporate information', 'Media, Policy & Legal', 'House Price Index' (which is highlighted in blue), and 'Investor relations'. Below the navigation bar is a large image of a row of houses. Overlaid on this image is a white rectangular box containing the text 'Nationwide' in red and 'House Price Index' in large blue letters. Below this box is a smaller white box containing the text 'House Price calculator'. At the bottom of the page, there is a navigation bar with links: 'Headlines', 'House Price calculator' (which is highlighted in red), 'Report archive', 'Download data', and 'Methodology'.

House Price Calculator

Instructions

- Property Value: Enter the price paid for, or a more recent valuation of your property. Please ensure the value is entered without commas, for example 150000, rather than 150,000.
- Valuation Date 1: The date when your property was purchased, or revalued.
- Valuation Date 2: Date for which you would like a new estimate of your property's value.
- Region: Select region which the property is situated in. If you are not sure which region the property is in, click on the link below to find your region.

Please note: The Nationwide House Price Calculator is intended to illustrate general movement in prices only.

The calculator is based on the Nationwide House Price Index. Results are based on movements in prices in the regions of the UK rather than in specific towns and cities. The data is based on movements in the price of a typical property in the region, and cannot take account of differences in quality of fixtures

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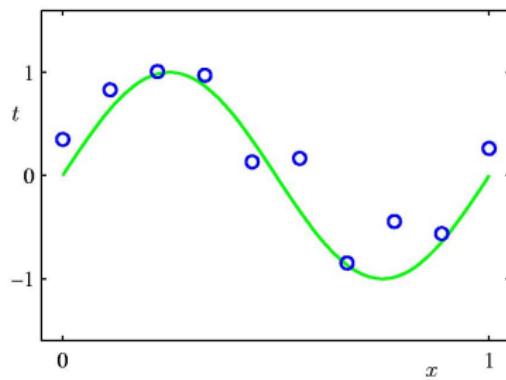
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 - ▶ A **loss** or a **cost** or an **objective** function, which tells us how well our model approximates the training examples
 - ▶ **Optimization**, a way of finding the parameters of our model that minimizes the loss function

Today: Linear Regression

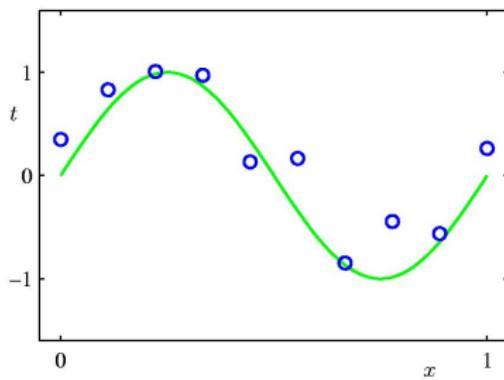
- Linear regression
 - ▶ continuous outputs
 - ▶ simple model (linear)
- Introduce key concepts:
 - ▶ loss functions
 - ▶ generalization
 - ▶ optimization
 - ▶ model complexity
 - ▶ regularization

Simple 1-D regression



- Circles are data points (i.e., training examples) that are given to us

Simple 1-D regression

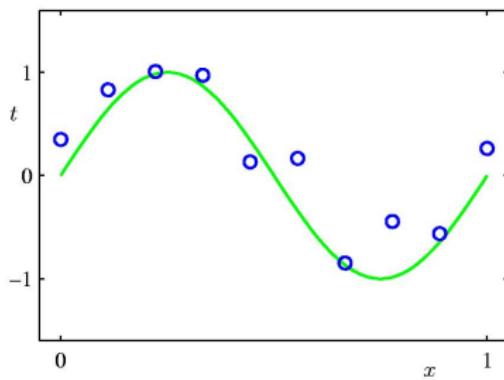


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$$t(x) = f(x) + \epsilon$$

with ϵ some noise

Simple 1-D regression



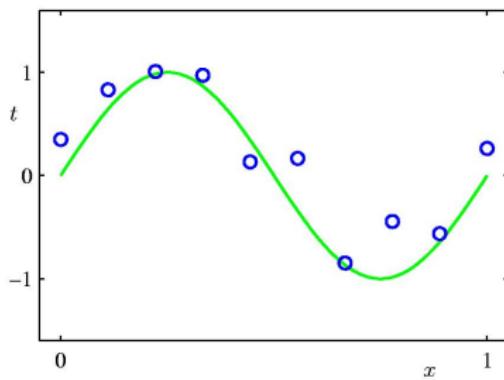
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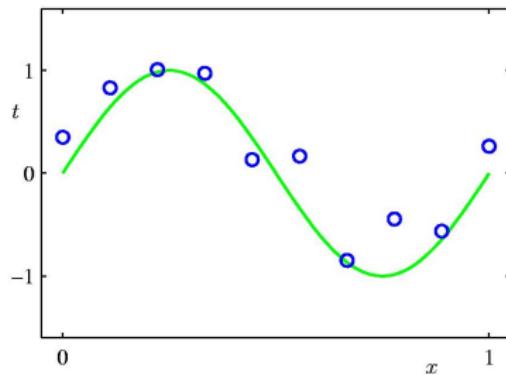
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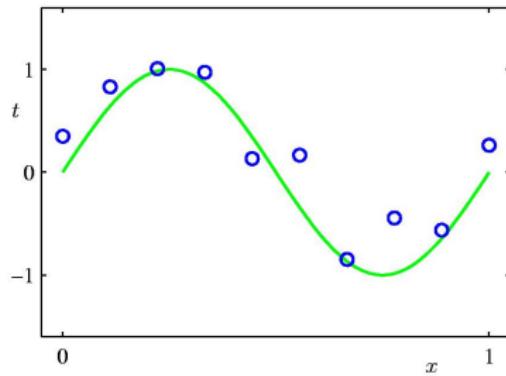
- In green is the "true" curve that we don't know
- Goal: We want to fit a curve to these points

Simple 1-D regression



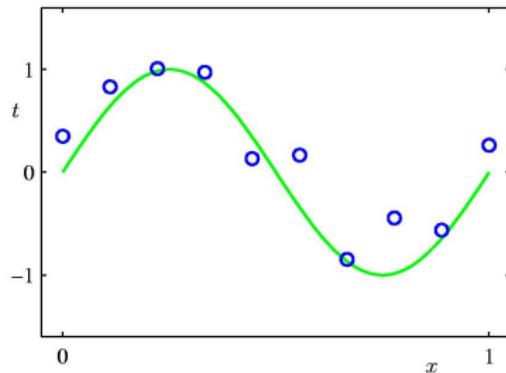
- Key Questions:

Simple 1-D regression



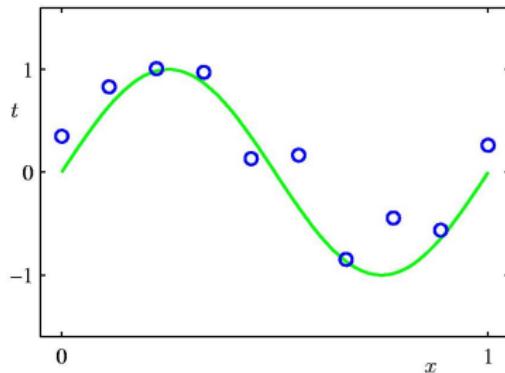
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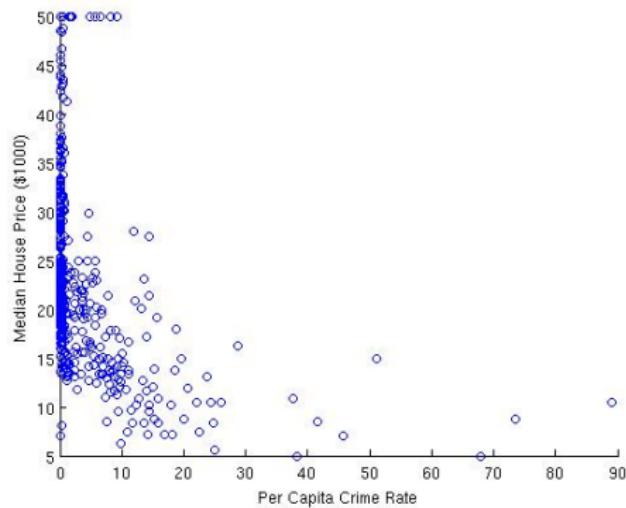
- Key Questions:
 - ▶ How do we parametrize the **model**?
 - ▶ What **loss (objective) function** should we use to judge the fit?
 - ▶ How do we optimize fit to unseen test data (**generalization**)?

Example: Boston Housing data

- Estimate median house price in a neighborhood based on neighborhood statistics

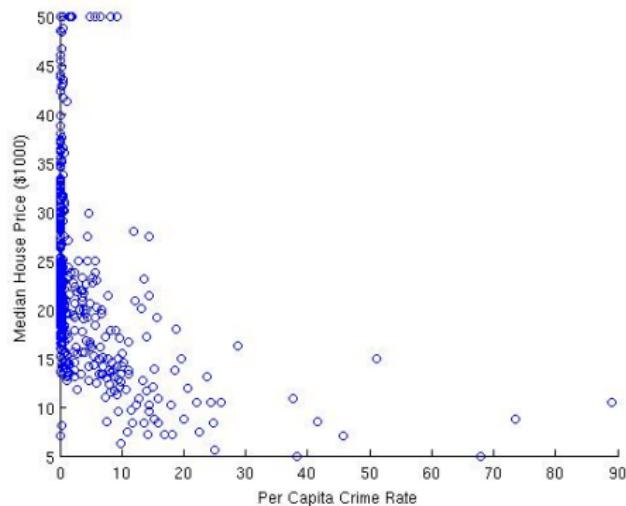
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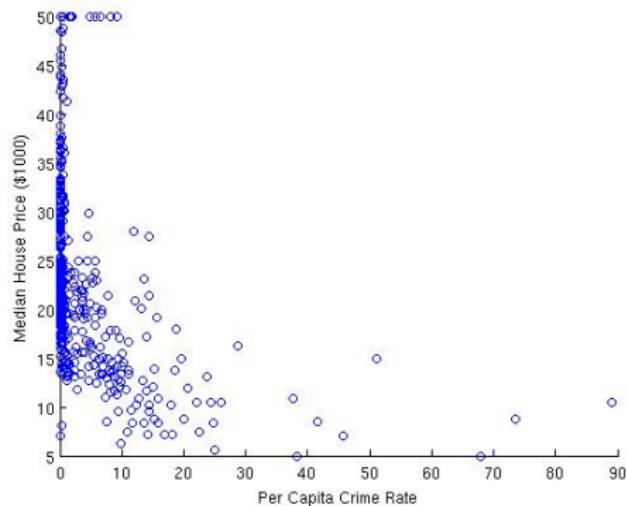
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Example: Boston Housing data

- Estimate median house price in a neighborhood based on neighborhood statistics
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- Use this to predict house prices in other neighborhoods
- Is this a good input (attribute) to predict house prices?

Represent the Data

- Data is described as pairs $\mathcal{D} = \{(x^{(1)}, t^{(1)}), \dots, (x^{(N)}, t^{(N)})\}$
 - ▶ $x \in \mathbb{R}$ is the **input feature** (per capita crime rate)
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- Divide the dataset into training and testing examples
 - ▶ Use the training examples to construct hypothesis, or function approximator, that maps x to predicted y
 - ▶ Evaluate hypothesis on test set

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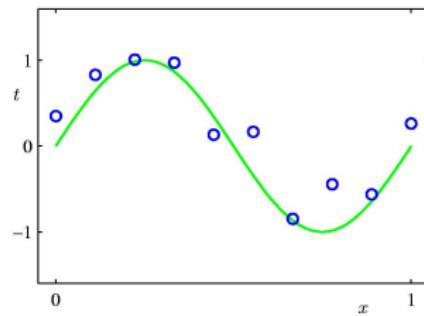
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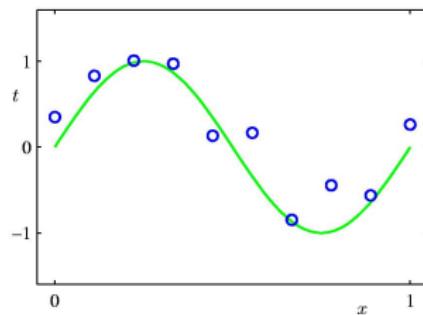
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 - ▶ **Model** may be **too simple** to account for data targets

Least-Squares Regression



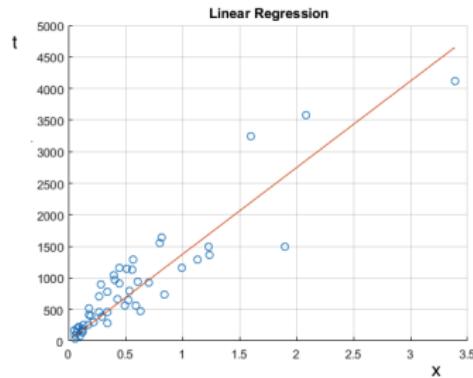
Least-Squares Regression



- Define a model

$$y(x) = \text{function}(x, \mathbf{w})$$

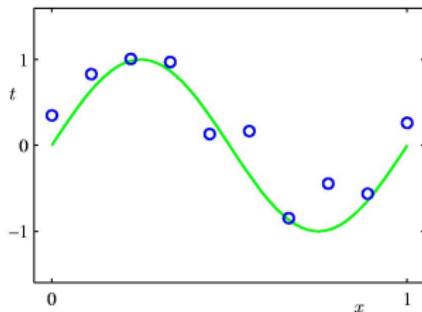
Least-Squares Regression



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Linear: $y(x) = w_0 + w_1 x$

Least-Squares Regression



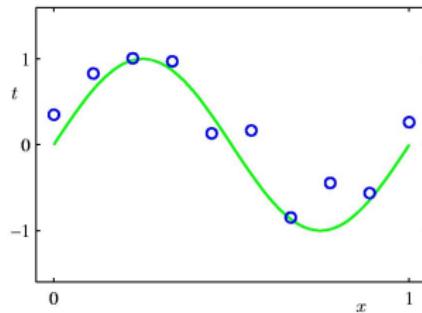
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Linear: $y(x) = w_0 + w_1 x$

- Standard loss/cost/objective function measures the squared error between y and the true value t

$$\ell(\mathbf{w}) = \sum_{n=1}^N [t^{(n)} - y(x^{(n)})]^2$$

Least-Squares Regression



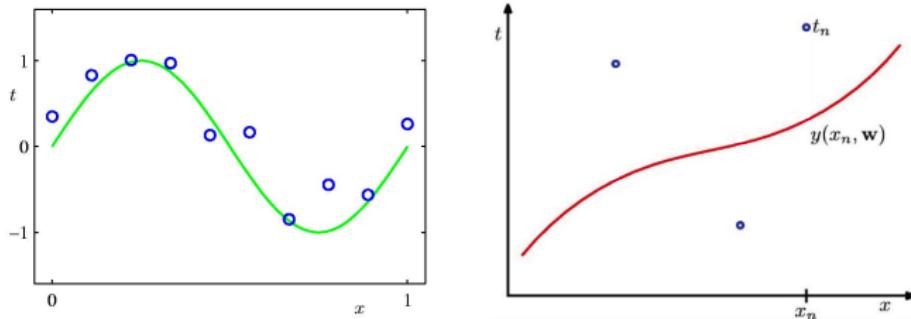
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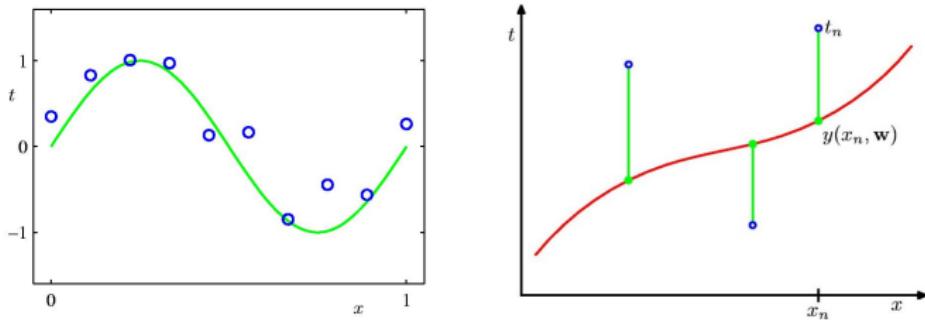
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- For a particular hypothesis ($y(x)$ defined by a choice of \mathbf{w} , drawn in red), what does the loss represent geometrically?

Least-Squares Regression



- Define a model

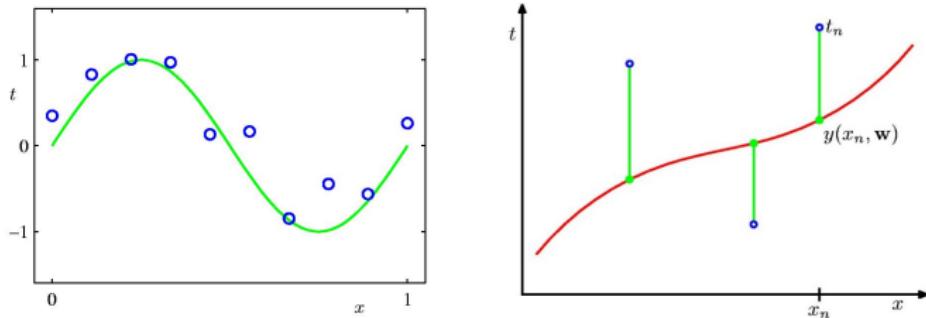
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- The loss for the red hypothesis is the **sum of the squared vertical errors** (squared lengths of green vertical lines)

Least-Squares Regression



- Define a model

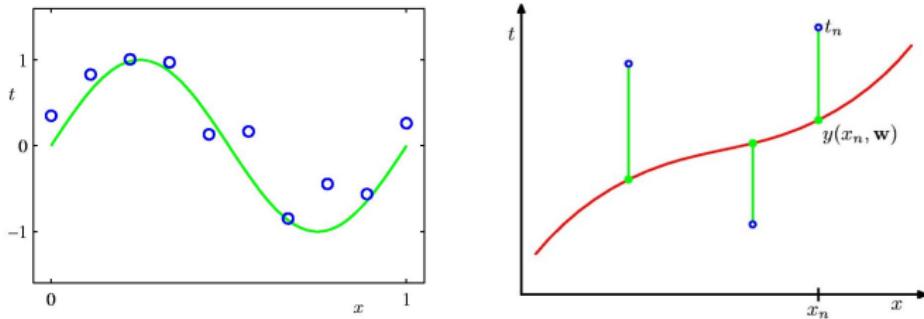
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- How do we obtain weights $\mathbf{w} = (w_0, w_1)$?

Least-Squares Regression



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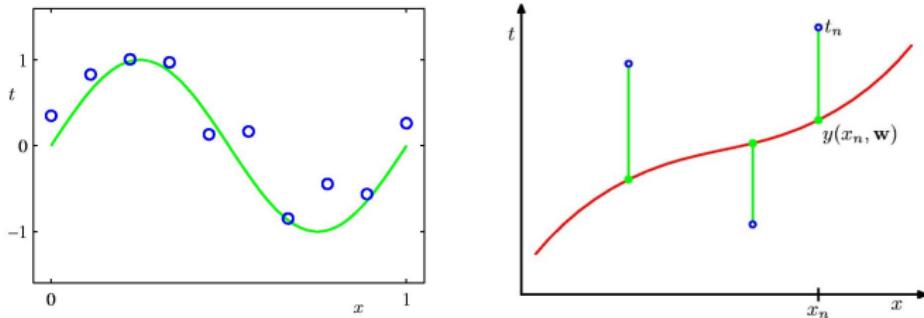
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- How do we obtain weights $\mathbf{w} = (w_0, w_1)$? Find \mathbf{w} that minimizes loss $\ell(\mathbf{w})$

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- How do we obtain weights $\mathbf{w} = (w_0, w_1)$?
- For the linear model, what kind of a function is $\ell(\mathbf{w})$?

Optimizing the Objective

- One straightforward method: [gradient descent](#)

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- One straightforward method: *gradient descent*
 - ▶ initialize \mathbf{w} (e.g., randomly)

Optimizing the Objective

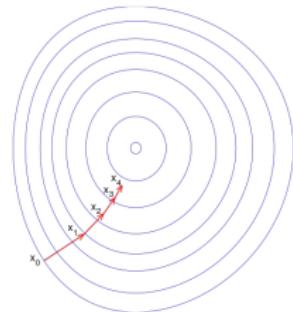
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 - ▶ initialize \mathbf{w} (e.g., randomly)
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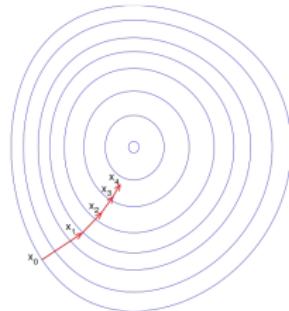


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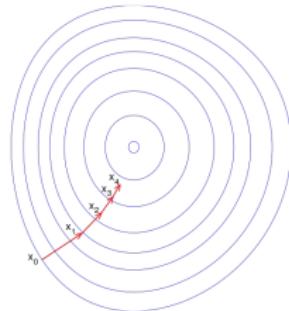
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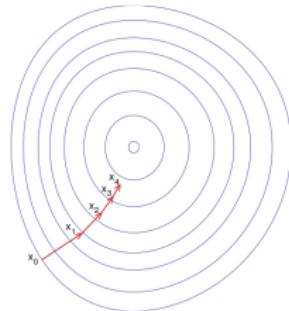
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- Note: As error approaches zero, so does the update (\mathbf{w} stops changing)

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Algorithm 1 Stochastic gradient descent

- 1: Randomly shuffle examples in the training set
- 2: **for** $i = 1$ to N **do**
- 3: Update:

$$\mathbf{w} \leftarrow \mathbf{w} + 2\lambda(t^{(i)} - y(x^{(i)}))x^{(i)} \quad (\text{update for a linear model})$$

- 4: **end for**
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 - ▶ Underlying assumption: sample is independent and identically distributed (i.i.d.)

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- Then:

$$\mathbf{w} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{t}$$

(work it out!)

Multi-dimensional Inputs

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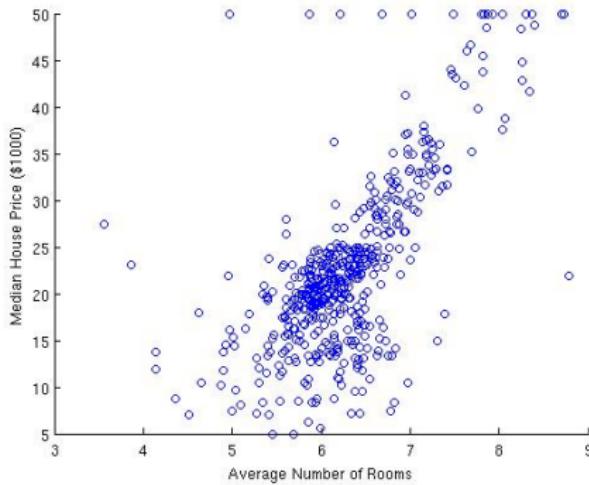
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- In the Boston housing example, we can look at the number of rooms



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- We can use gradient descent to solve for each coefficient, or compute \mathbf{w} analytically (how does the solution change?)

More Powerful Models?

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Fitting a Polynomial

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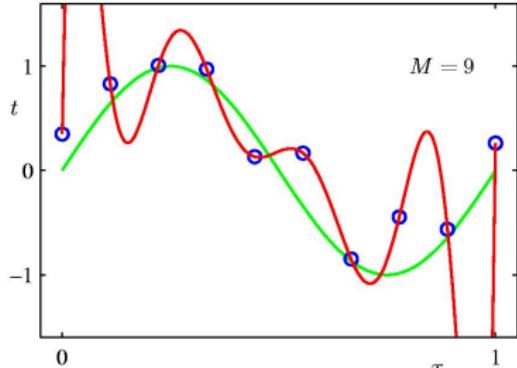
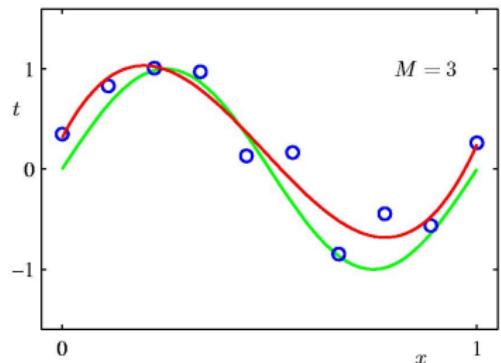
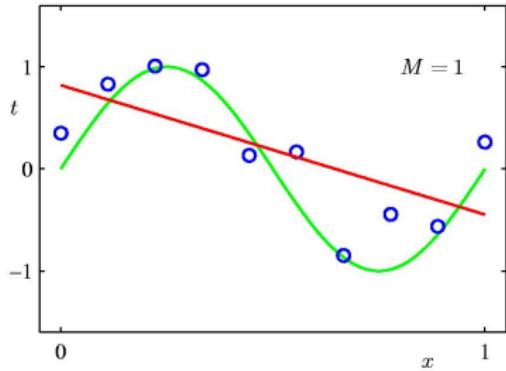
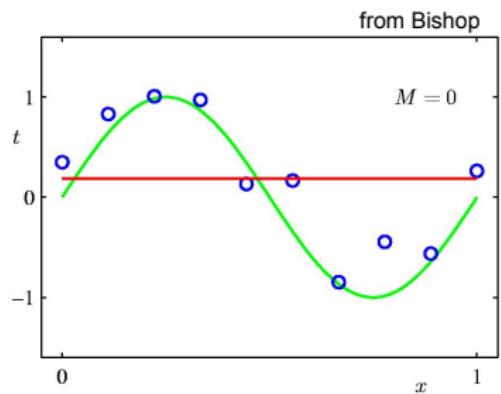
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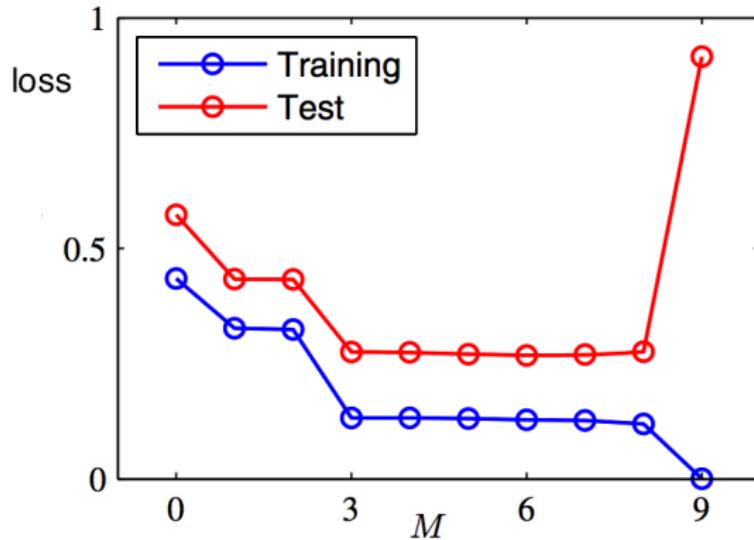
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- How do we do that?

Which Fit is Best?



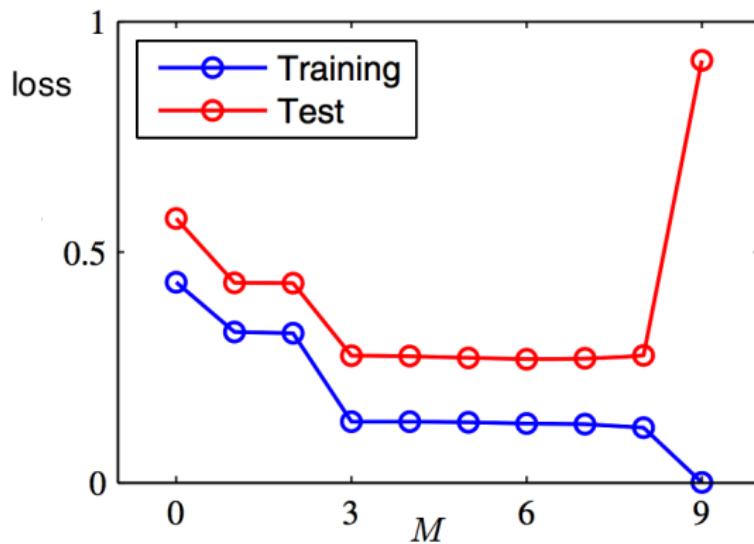
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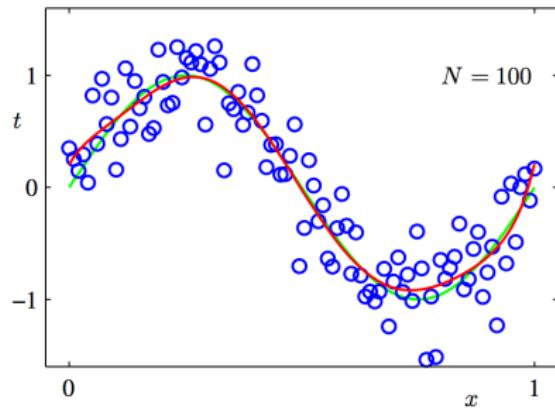
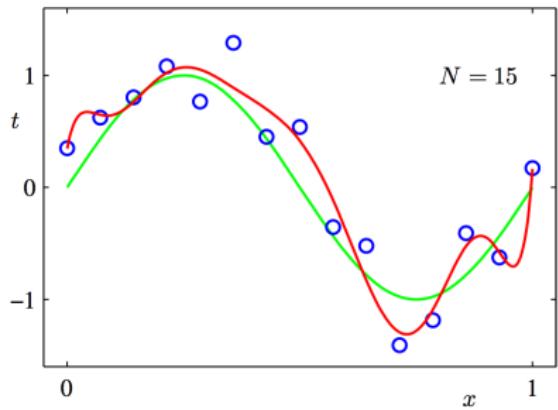
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	$M = 0$	$M = 1$	$M = 6$	$M = 9$
w_0^*	0.19	0.82	0.31	0.35
w_1^*		-1.27	7.99	232.37
w_2^*			-25.43	-5321.83
w_3^*			17.37	48568.31
w_4^*				-231639.30
w_5^*				640042.26
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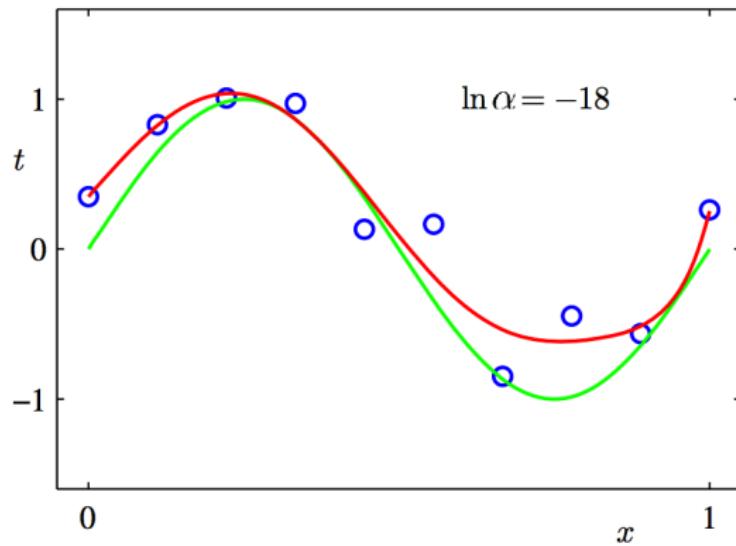
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- Also has an analytical solution: $\mathbf{w} = (\mathbf{X}^T \mathbf{X} + \alpha \mathbf{I})^{-1} \mathbf{X}^T \mathbf{t}$ (verify!)

Regularized least squares

- Better generalization
- Choose α carefully



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- [Optimization](#) is essential: stochastic and batch iterative approaches; analytic when available

So...

- Which movie will you watch?



Now Playing

REFINE YOUR SEARCH

Boys V. Girls
1h 32m | Comedy, Family
View Ratings and Warnings

BUY TICKETS **TRAILER**

Anamalai
1h 31m | Comedy, Animation, Fantasy
View Ratings and Warnings

BUY TICKETS **TRAILER**

Bajrangi Bhaijaan
2h 30m | Foreign Language, Drama, Romance, History
View Ratings and Warnings

BUY TICKETS

**IMAGE NOT AVAILABLE
PAID FOR IN ADVANCE**
1h 59m | Action, Foreign Language, Comedy
View Ratings and Warnings

BUY TICKETS

Brooklyn
1h 52m | Drama
View Ratings and Warnings

BUY TICKETS **TRAILER**