Q1. The "enclosed" region is unbound,
so the area would be infinite

Q2.

$$x^3 - 3x = x$$

$$x^3 - 4x = 0 \qquad x(x^2-4) = 0$$

$$x(x-2)(x+2) = 0$$

$$x = -2 \quad x = 0 \quad x = 2$$

$$\int_{-2}^{0} (x^3 - 3x - x)\,dx + \int_{0}^{2} (x - (x^3 - 3x))\,dx$$

$$\int_{-2}^{0} x^3 - 4x \;dx + \int_{0}^{2} -x^3 + 4x \;dx$$

$$\frac{x^4}{4} - 2x^2 \Big|_{-2}^{0} + 2x^2 - \frac{x^4}{4} \Big|_{0}^{2}$$

$$(0-0) - \left(\frac{16}{4} - 8\right) + \left(8 - \frac{16}{4}\right) - (0-0)$$

$$-(-4) + 4 = \underline{8}$$

Q3.

$$\int_1^e \ln x - \frac{1}{x} \, dx$$

$$\int \ln x = x \ln x - \int x \cdot \frac{1}{x} = x \ln x - x$$

$$\int \frac{1}{x} = \ln x$$

$$x \ln x - x \Big|_1^e - \ln x \Big|_1^e$$

$$\left[(e \ln e - e) - (1 \ln 1 - 1)\right] - (\ln e - \ln 1)$$

$$(e - e) - (0 - 1) - (1 - 0)$$

$$0 + 1 - 1 = 0$$

Q4. $\sqrt{x} = \frac{x}{4}$

$x = \frac{x^2}{16}$ $\quad 16x = x^2 \quad x^2 - 16x = 0$

$x(x-16) = 0 \quad \therefore x = 0 \quad x = 16$

$\pi \int_0^{16} (\sqrt{x})^2 - \left(\frac{x}{4}\right)^2 dx = \pi \int_0^{16} x - \frac{x^2}{16} \, dx$

~~$\pi \left[\left(16 - \frac{16^2}{16}\right) - 0 - 0\right] dx$~~

$\pi \left[\frac{x^2}{2} - \frac{x^3}{48}\right]_0^{16} \cancel{dx}$

$\pi \left(\frac{16^2}{2} - \frac{16^3}{48}\right) \quad \pi \left(128 - \frac{256}{3}\right)$

$= \frac{128}{3}\pi$

Qs. $x = \sqrt{9-x^2}$

$x^2 = 9 - x^2$

$2x^2 = 9$

$y = x = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

$V = 2\pi \int_a^b x \cdot h(x)\, dx$

$V = 2\pi \int_0^{\frac{3\sqrt{2}}{2}} x(\sqrt{9-x^2} - x)\, dx$

$= 2\pi \int_0^{\frac{3\sqrt{2}}{2}} x\sqrt{9-x^2} - x^2\, dx$

$u = 9 - x^2 \cdot \quad du = -2x\, dx \quad x = \frac{-1}{2} du$

$\int_9^{9/2} \sqrt{u}\left(-\frac{1}{2}\right) du = \frac{-1}{2}\left[\frac{2}{3} u^{3/2}\right]_9^{9/2} = \frac{-1}{3}\left[\left(\frac{9}{2}\right)^{3/2} - 9^{3/2}\right]$

$= \frac{-1}{3}\left[\frac{27}{2\sqrt{2}} - 27\right] = 9 - \frac{9}{2\sqrt{2}} = 9 - \frac{9\sqrt{2}}{4}$

$\int_0^{\frac{3\sqrt{2}}{2}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{\frac{3\sqrt{2}}{2}} = \frac{1}{3}\left(\frac{3\sqrt{2}}{2}\right)^3 = \frac{1}{3} \frac{27 \cdot 2\sqrt{2}}{8}$

$= \frac{9\sqrt{2}}{4} \qquad V = 2\pi\left(9 - \frac{9\sqrt{2}}{4} - \frac{9\sqrt{2}}{4}\right) = 2\pi\left(9 - \frac{9\sqrt{2}}{2}\right)$

$V = 18\pi - 9\pi\sqrt{2} = 9\pi(2 - \sqrt{2})$

Q6. a) $\int x^2 \ln x \, dx$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$

$(\ln x)\frac{x^3}{3} - \int \frac{x^2}{3} \, dx$

$\frac{x^3}{3} \ln x - \frac{x^3}{9} = \frac{x^3}{3}\left(\ln - \frac{1}{3}\right) + c$

b) $\int \frac{x^3}{x^4+1} \, dx$

$\frac{1}{4}\int \frac{4x^3}{x^4+1} \, dx \Rightarrow \frac{1}{4} \ln(x^2+1) + c$

c) $\int_0^{\pi/2} x \sin x \, dx$

$-x\cos x - \int -\cos x \, dx$

$= -x\cos x + \sin x \Big|_0^{\pi/2}$

$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (0 + 0)$

$0 + 1 = \underline{1}$

d) $\int \frac{3x+1}{(x^2+x+1)^2} dx$

$3x+1 = 3/2\,(2x+1) - 1/2$

$\frac{3}{2}\int \frac{2x+1}{(x^2+x+1)^2} - \frac{1}{2}\int \frac{1}{(x^2+x+1)^2}$

$du = 2x+1\,dx \qquad u = x^2+x+1$

$\frac{3}{2}\int \frac{1}{u^2}\,du = \frac{3}{2}\left(-\frac{1}{u}\right) = -\frac{3}{2(x^2+x+1)}$

$u = x+\frac{1}{2} \qquad du = dx \qquad a = \frac{\sqrt{3}}{2}$

$\int \frac{1}{((x+\frac{1}{2})^2+3/4)^2} dx = \frac{1}{(u^2+a^2)^2}$

$u = a\tan\theta \qquad du = a\sec^2\theta$

$\int \frac{a\sec^2\theta}{(a^2\tan^2\theta+a^2)^2} d\theta = \frac{a\sec^2\theta}{(a^2\sec^2\theta)^2} = \frac{1}{a^3}\int \frac{1}{\sec^2\theta} d\theta$

$= \frac{1}{2a^3}(1+\cos 2\theta) = \frac{1}{2a^3}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$

$\frac{1}{2}\left(\frac{\cancel{2}}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2x+1}{2(x^2+x+1)}\right) + c$

$-\frac{3x+7}{2(x^2+x+1)} - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

) $\displaystyle\int_1^{\infty} \frac{1}{x^2\sqrt{x}}\,dx$

$$\int_1^{\infty} x^{-5/2}\,dx = -\frac{2}{3}x^{-3/2}\Big|_1^{\infty}$$

$$-\frac{2}{3}\left(\frac{1}{\infty^{-3/2}} - \frac{1}{1}\right) = \frac{-2}{3}(0-1)$$

$$\frac{2}{3}$$

f) $\displaystyle\int \frac{\sqrt{1+4x^2}}{x}\,dx$

$2x = \tan\theta \qquad x = \frac{1}{2}\tan\theta \qquad dx = \frac{1}{2}\sec^2\theta\,d\theta$

$$\sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta|$$

$$\int \frac{\sec\theta}{1/2\tan\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta = \int \frac{\sec^3\theta}{\tan\theta}\,d\theta$$

$$\int \sec^2\theta\csc\theta\,d\theta$$

$2x = \sinh u \quad x = \frac{1}{2}\sinh u \quad dx = \frac{1}{2}\cosh u \, du$

$\cosh^2 u = 1 + \sinh^2 u$

$\int \frac{\sqrt{1+\sinh^2 u}}{1/2 \sinh u} \left(\frac{1}{2}\cosh u\right) du$

$\int \frac{\cosh u}{\sinh u} \cosh u \, du = \int \frac{\cosh^2 u}{\sinh u} du$

$\int \frac{1+\sinh^2 u}{\sinh u} = \int \operatorname{csch} u + \sinh u \, du$

$= \ln\left|\tanh\left(\frac{u}{2}\right)\right| + \cosh u + C$

$\sqrt{1+4x^2} - \frac{1}{2} \ln \left| \frac{\sqrt{1+4x^2}+1}{\sqrt{1+4x^2}-1} \right| + C$

g) $\int \frac{x^3+8}{(x^2+1)(x-2)} dx$

$\frac{x^3+8}{(x^2+1)(x-2)} = \frac{x^3+8}{x^3-2x^2-x+2} = 1 + \frac{2x^2+x+6}{x^3-2x^2-x+2}$

$2x^2+x+6 = A(x+1)(x-2) + B(x-1)(x-2) + C(x+1)(x-1)$

$A = -9/2 \qquad B = 7/6 \qquad C = 16/3$

$$\int \frac{-9}{2(x-1)} + \frac{7}{6(x+1)} + \frac{16}{3(x-2)}\,dx$$

$$= -\frac{9}{2}\ln(x-1) + \frac{7}{6}\ln(x+1) + \frac{16}{3}\ln(x-2) + c$$

2) $$\int \frac{\sec^2}{\tan^3\theta - \tan^2\theta}\,d\theta$$

$u = \tan\theta \qquad du = \sec^2\theta\,d\theta$

$$\int \frac{du}{u^3 - u^2} = \int \frac{du}{u^2(u-1)}$$

$$1 = A(u)(u-1) + B(u-1) + C(u^2)$$

$A = -1 \qquad B = -1 \qquad C = 1$

$$\int \frac{-1}{u} - \frac{1}{u^2} + \frac{1}{u-1}\,du = -\ln|u| + \frac{1}{u} + \ln|u-1| + c$$

$$\ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + \cot\theta + c$$