1-5 COMPLEMENTS

Complements are used in digital computers for simplifying the subtraction operation and for logical manipulations. There are two types of complements for each base-r system: (1) the r's complement and (2) the $(r - 1)$'s complement. When the value of the base is substituted, the two types receive the names 2's and 1's complement for binary numbers, or 10's and 9's complement for decimal numbers.

The r's Complement

Given a positive number N in base r with an integer part of n digits, the r's complement of N is defined as $r^n - N$ for $N \neq 0$ and 0 for $N = 0$. The following numerical example will help clarify the definition.

The 10's complement of $(52520)_{10}$ is $10^5 - 52520 = 47480$.

The number of digits in the number is $n = 5$.

The 10's complement of $(0.3267)_{10}$ is $1 - 0.3267 = 0.6733$. $10^0 - 0.3267$

No integer part, so $10^n = 10^0 = 1$.

The 10's complement of $(25.639)_{10}$ is $10^2 - 25.639 = 74.361$.

The 2's complement of $(101100)_2$ is $(2^6)_{10} - (101100)_2 = (1000000 - 101100)_2 = 010100$.

The 2's complement of $(0.0110)_2$ is $(1 - 0.0110)_2 = 0.1010$.

1101.10
1001.01
0100.01

1000000
−101100
010100

From the definition and the examples, it is clear that the 10's complement of a decimal number can be formed by leaving all least significant zeros unchanged, subtracting the first nonzero least significant digit from 10, and then subtracting all other higher significant digits from 9. The 2's complement can be formed by leaving all least significant zeros and the first nonzero digit unchanged, and then replacing 1's by 0's and 0's by 1's in all other higher significant digits. A third, simpler method for obtaining the r's complement is given after the definition of the $(r - 1)$'s complement.

The r's complement of a number exists for any base r (r greater than but not equal to 1) and may be obtained from the definition given above. The examples listed here use numbers with $r = 10$ (decimal) and $r = 2$ (binary) because these are the two bases of most interest to us. The name of the complement is related to the base of the number used. For example, the $(r - 1)$'s complement of a number in base 11 is named the 10's complement, since $r - 1 = 10$ for $r = 11$.

1 0 0 0 0 0 0
64 32 16 8 4 2 1

The $(r - 1)$'s Complement

Given a positive number N in base r with an integer part of n digits and a fraction part of m digits, the $(r - 1)$'s complement of N is defined as $r^n - r^{-m} - N$. Some numerical examples follow:

The 9's complement of $(52520)_{10}$ is $(10^5 - 1 - 52520) = 99999 - 52520 = 47479$.

No fraction part, so $10^{-m} = 10^0 = 1$.

The 9's complement of $(0.3267)_{10}$ is $(1 - 10^{-4} - 0.3267) = 0.9999 - 0.3267 = 0.6732$.

No integer part, so $10^n = 10^0 = 1$.

The 9's complement of $(25.639)_{10}$ is $(10^2 - 10^{-3} - 25.639) = 99.999 - 25.639 = 74.360$.

The 1's complement of $(101100)_2$ is $(2^6 - 1) - (101100) = (111111 - 101100)_2 = 010011$.

The 1's complement of $(0.0110)_2$ is $(1 - 2^{-4})_{10} - (0.0110)_2 = (0.1111 - 0.0110)_2 = 0.1001$.

dec → bin

From the examples, we see that the 9's complement of a decimal number is formed simply by subtracting every digit from 9. The 1's complement of a binary number is even simpler to form: the 1's are changed to 0's and the 0's to 1's. Since the $(r - 1)$'s complement is very easily obtained, it is sometimes convenient to use it when the r's complement is desired. From the definitions and from a comparison of the results obtained in the examples, it follows that the r's complement can be obtained from the $(r - 1)$'s complement after the addition of r^{-m} to the least significant digit. For example, the 2's complement of 10110100 is obtained from the 1's complement 01001011 by adding 1 to give 01001100.

It is worth mentioning that the complement of the complement restores the number to its original value. The r's complement of N is $r^n - N$ and the complement of $(r^n - N)$ is $r^n - (r^n - N) = N$; and similarly for the 1's complement.

Subtraction with r's Complements

The direct method of subtraction taught in elementary schools uses the borrow concept. In this method, we borrow a 1 from a higher significant position when the minuend digit is smaller than the corresponding subtrahend digit. This seems to be easiest when people perform subtraction with paper and pencil. When subtraction is implemented by means of digital components, this method is found to be less efficient than the method that uses complements and addition as stated below.

The subtraction of two positive numbers $(M - N)$, both of base r, may be done as follows:

1. Add the minuend M to the r's complement of the subtrahend N.
2. Inspect the result obtained in step 1 for an end carry:
   (a) If an end carry occurs, discard it.
   (b) If an end carry does not occur, take the r's complement of the number obtained in step 1 and place a negative sign in front.

The following examples illustrate the procedure:

EXAMPLE 1-5: Using 10's complement, subtract 72532 − 3250.

$$M = 72532 \qquad\qquad 72532$$
$$N = 03250$$

$$\text{10's complement of } N = 96750 \qquad + \;\; 96750$$

$$\text{end carry} \rightarrow 1 \quad 69282$$

answer: 69282

EXAMPLE 1-6: Subtract: $(3250 - 72532)_{10}$.

$$M = 03250 \qquad\qquad 03250$$
$$N = 72532$$

$$\text{10's complement of } N = 27468 \qquad + \;\; 27468$$

$$\text{no carry} \quad 30718$$

answer: −69282 = −(10's complement of 30718)

$r^n - N$

EXAMPLE 1-7 Use 2's complement to perform $M - N$ with the given binary numbers.

(a) $M = 1010100$
$N = 1000100$

$$\begin{array}{r} 1010100 \\ +\ 0111100 \\ \hline \text{end carry} \rightarrow 1\ \ 0010000 \end{array}$$

2's complement of $N = 0111100$

10000000
−1000100
0111100

answer: 10000

(b) $M = 1000100$
$N = 1010100$

$$\begin{array}{r} 1000100 \\ +\ 0101100 \\ \hline \text{no carry}\ \ 1110000 \end{array}$$

2's complement of $N = 0101100$

answer: $-10000 = -$ (2's complement of 1110000)

The proof of the procedure is: The addition of M to the r's complement of N gives $(M + r^n - N)$. For numbers having an integer part of n digits, r^n is equal to a 1 in the $(n + 1)$th position (what has been called the "end carry"). Since both M and N are assumed to be positive, then:

$$(a)\quad (M + r^n - N) \geq r^n \quad \text{if } M \geq N, \text{ or}$$
$$(b)\quad (M + r^n - N) < r^n \quad \text{if } M < N$$

In case (a) the answer is positive and equal to $M - N$, which is directly obtained by discarding the end carry r^n. In case (b) the answer is negative and equal to $-(N - M)$. This case is detected from the absence of an end carry. The answer is obtained by taking a second complement and adding a negative sign:

$$-[r^n - (M + r^n - N)] = -(N - M).$$

Subtraction with $(r - 1)$'s Complement

The procedure for subtraction with the $(r - 1)$'s complement is exactly the same as the one used with the r's complement except for one variation, called "end-around carry," as shown below. The subtraction of $M - N$, both positive numbers in base r, may be calculated in the following manner:

1. Add the minuend M to the $(r - 1)$'s complement of the subtrahend N.
2. Inspect the result obtained in step 1 for an end carry.
   (a) If an end carry occurs, add 1 to the least significant digit (end-around carry).

(b) If an end carry does not occur, take the $(r-1)$'s complement of the number obtained in step 1 and place a negative sign in front.

The proof of this procedure is very similar to the one given for the r's complement case and is left as an exercise. The following examples illustrate the procedure.

EXAMPLE 1-8: Repeat Examples 1-5 and 1-6 using 9's complements.

(a) $M = 72532$
$N = 03250$
9's complement of $N = 96749$

```
                          72532
                       +  96749
                      ---------
               ┌──── 1   69281
end-around carry│              +
               └────────→    1
                         -------
                          69282
```

answer: 69282

(b) $M = 03250$
$N = 72532$
9's complement of $N = 27467$.

```
                          03250
                       +  27467
                         -------
no carry                  30717
```

answer: $-69282 = -$ (9's complement of 30717)

EXAMPLE 1-9: Repeat Example 1-7 using 1's complement.

(a) $M = 1010100$
$N = 1000100$
1's complement of $N = 0111011$

```
                          1010100
                       +  0111011
                         ---------
end-around carry ┌──── 1   0001111
                 │                 +
                 └───────→       1
                           -------
                           0010000
```

answer: 10000