数学物理方程 B　第六周作业　3 月 26 日 周四

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1.6 设$u = u(x,t)$,求下列方程的通解：

$$u_{tt} = a^2 u_{xx} + 3x^2$$

解：先找该方程的一个特解：可设 $u^{(*)} = kx^4$待定系数k

$$0 = 12a^2kx^2 + 3x^2 \quad \Longrightarrow \quad k = -\frac{1}{4a^2}$$

$$\Longrightarrow \quad u^{(*)} = -\frac{1}{4a^2}x^4$$

再求其齐次方程的通解$u^{(0)}$：即 $u_{tt} = a^2 u_{xx}$，$u = u(x,t)$

可作变量代换$\xi = x + at$，$\eta = x - at$,通过计算可将原方程化为$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$,再两边积分，可得

$$u^{(0)} = f(x+at) + g(x-at)$$

其中f, g为任意的二次可微函数。所以原方程的通解为：

$$u = u^{(0)} + u^{(*)} = f(x+at) + g(x-at) - \frac{1}{4a^2}x^4$$

1.9 求下列定解问题的解：

(1) $u_t = x^2$,$u(0,x) = x^2$；

(2) 球对称的三维波动方程的初始问题：

$$\begin{cases} u_{tt} = a^2 \Delta_3 u \ , \\ u|_{t=0} = \varphi(r) \, , \\ u_t|_{t=0} = \psi(r) \, ; \end{cases}$$

(3) $\begin{cases} \Delta_3 u = 0 \quad (x^2+y^2+z^2<1) \\ u|_{x^2+y^2+z^2=1} = (5+4y)^{-\frac{1}{2}} \end{cases}$

(4) 古尔萨(Goursat)问题：

$$\begin{cases} u_{tt} = u_{xx} \ , \\ u|_{t+x=0} = \varphi(x) \, , \\ u|_{t-x=0} = \psi(x) \, , \\ \varphi(0) = \psi(0) \, ; \end{cases}$$

解：(1) 对该方程直接两边积分：

$$u_t = \frac{\partial u}{\partial t} = x^2 \ \Longrightarrow \ \partial u = x^2 \partial t \ \ \Longrightarrow \ \ u = x^2 t + f(x)$$

带入边界条件 $u(0,x) = x^2$：

$$u(0,x) = x^2 * 0 + f(x) \equiv x^2 \ \ \Longrightarrow \ \ f(x) = x^2$$

因此，该方程的解为：$u = x^2 t + x^2 = x^2(t+1)$

(2) 在球坐标下，$u = u(t,r,\theta,\varphi)$，再写出$\Delta_3 u$在球坐标下的表达式：

$$\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

又由于该问题是球对称的，所以u对θ, φ的导数都为 0.因此原方程化为：

$$u_{tt} = a^2 \Delta_3 u = a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{a^2}{r^2}\left(2r\frac{\partial u}{\partial r} + r^2 \frac{\partial^2 u}{\partial r^2}\right) = a^2\left(\frac{2}{r}u_r + u_{rr}\right)$$

此时$u = u(t,r)$，再令$v = ru$，$v = v(t,r)$，将原方程化为弦振动方程：

$$v_t = ru_t \, ; \ \ v_{tt} = ru_{tt} \, ; \ \ v_r = u + ru_r \, ; \ \ v_{rr} = 2u_r + ru_{rr} \ \ \text{反解:}$$

$$\Longrightarrow \ u_t = \frac{v_t}{r} \, ; \quad u_{tt} = \frac{v_{tt}}{r} \, ; \quad u_r = \frac{1}{r}(v_r - u) = \frac{v_r}{r} - \frac{v}{r^2} \, ; \quad u_{rr} = \frac{1}{r}(v_{rr} - 2u_r) = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}$$

带入原方程，得到：

$$\frac{v_{tt}}{r} = a^2\left(\frac{2}{r}\left(\frac{v_r}{r} - \frac{v}{r^2}\right) + \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}\right) \quad \Longrightarrow \quad v_{tt} = a^2 v_{rr}$$

上一问已经解过，该方程的通解为：$v = f(r+at) + g(r-at)$

所以可以得到u的通解：$u = \frac{1}{r}\big(f(r+at) + g(r-at)\big)$

再将边界条件带入其中：

$$\begin{cases} u|_{t=0} = \varphi(r) \quad \Longrightarrow \quad f(r) + g(r) = r\varphi(r) \cdots\cdots\cdots\cdots ① \\ u_t|_{t=0} = \psi(r) \quad \Longrightarrow \quad a\big(f'(r) - g'(r)\big) = r\psi(r) \cdots\cdots ② \end{cases}$$

对第二式积分，有：

$$\int_0^r \big(f'(r) - g'(r)\big)\, dr = \frac{1}{a}\int_0^r r\psi(r)\, dr \quad \Rightarrow \quad f(r) - g(r) = \frac{1}{a}\int_0^r r\psi(r)\, dr + C \cdots\cdots ③$$

分别进行 ①+③，①−③，求出$f(r)$，$g(r)$：

$$\begin{cases} f(r) = \frac{1}{2}r\varphi(r) + \frac{1}{2a}\int_0^r r\psi(r)\, dr + \frac{C}{2} = \frac{1}{2}r\varphi(r) + \frac{1}{2a}\int_0^r \xi\psi(\xi)\, d\xi + \frac{C}{2} \\ g(r) = \frac{1}{2}r\varphi(r) - \frac{1}{2a}\int_0^r r\psi(r)\, dr - \frac{C}{2} = \frac{1}{2}r\varphi(r) - \frac{1}{2a}\int_0^r \xi\psi(\xi)\, d\xi - \frac{C}{2} \end{cases}$$

最后带入$u(t,r)$：

$$u = \frac{1}{r}\big(f(r+at) + g(r-at)\big)$$

$$= \frac{1}{r}\left(\frac{1}{2}(r+at)\varphi(r+at) + \frac{1}{2a}\int_0^{r+at} \xi\psi(\xi)\, d\xi + \frac{C}{2} + \frac{1}{2}(r-at)\varphi(r-at) - \frac{1}{2a}\int_0^{r-at} \xi\psi(\xi)\, d\xi - \frac{C}{2}\right)$$

$$= \frac{1}{2r}\left((r+at)\varphi(r+at) + (r-at)\varphi(r-at) + \frac{1}{a}\int_{r-at}^{r+at} \xi\psi(\xi)\, d\xi\right)$$

(3) 由提示给出，当$x_0^2+y_0^2+z_0^2>1$时，$u=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$ 满足该方程。

所以只要将x_0,y_0,z_0定出即可。当$x^2+y^2+z^2=1$时：

$$u=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=\frac{1}{\sqrt{5+4y}}$$

分别将$(x,y,z)=(1{,}0{,}0),(0{,}1{,}0),(0{,}0{,}1)$带入：（缩小可能的取值范围）

$$\begin{cases}(1-x_0)^2+y_0^2+z_0^2=5\\ x_0^2+(1-y_0)^2+z_0^2=9\\ x_0^2+y_0^2+(1-z_0)^2=5\end{cases}\Rightarrow\ \text{解:}\begin{cases}x_0=0\\ y_0=-2\\ z_0=0\end{cases}\ or\ \begin{cases}x_0=2\\ y_0=0\\ z_0=2\end{cases}$$

而第二组解将其展开，$(x-2)^2+(y-0)^2+(z-2)^2=x^2+y^2+z^2-4x-4z+8\neq 5+4y$

舍去这组解。同时第一组解：$x^2+(y+2)^2+z^2=x^2+y^2+z^2+4y+4=5+4y$

因此，原方程的解为：$u=\dfrac{1}{\sqrt{x^2+(y+2)^2+z^2}}$

(4) 此时$u=u(t,x)$. 满足$u_{tt}=u_{xx}$.这是弦振动问题，$a^2=1$.上一题已经解过，通解$u^{(0)}$为：

$$u^{(0)}=f(x+at)+g(x-at)=f(x+t)+g(x-t)$$

还需要将f,g用已知的φ,ψ来表示。将其边界条件带入。$t+x=0$时：

$$u^{(0)}(-x,x)=f(0)+g(2x)\equiv\varphi(x)$$

同样的，$t-x=0$时：

$$u^{(0)}(x,x)=f(2x)+g(0)\equiv\psi(x)$$

令$x=0$,可得：$f(0)+g(0)=\varphi(x)=\psi(x)$。再对上两式做变换：

$$\begin{cases}f(0)+g(x)=\varphi\left(\dfrac{x}{2}\right)\\ f(x)+g(0)=\psi\left(\dfrac{x}{2}\right)\end{cases}\Rightarrow\begin{cases}g(x)=\varphi\left(\dfrac{x}{2}\right)-f(0)\\ f(x)=\psi\left(\dfrac{x}{2}\right)-g(0)\end{cases}$$

$$\Rightarrow\ u^{(0)}=f(x+t)+g(x-t)=\psi\left(\frac{x+t}{2}\right)-f(0)+\varphi\left(\frac{x-t}{2}\right)-g(0)=\psi\left(\frac{x+t}{2}\right)+\varphi\left(\frac{x-t}{2}\right)-\varphi(0)$$

1.10 利用叠加原理与齐次化原理求解：

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = f(t,x) \ , \ (t > 0, x \in \mathbb{R}) \\ u(0,x) = \varphi(x) \ , \qquad \big(a \neq 0, \text{常数}\big) \end{cases}$$

解：先求其齐次方程的解：$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$，令 $\xi = x - at,\ \eta = t$.变换为$u = u(\xi,\eta)$：

$$\Longrightarrow \ \frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \ \ \text{带入，得：}$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -a\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} + a\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta} = 0$$

其齐次通解为 $u^{(0)} = g(\xi) = g(x - at)$.其中$f$需要通过边界条件确定：（$u(0,x) = \varphi(x)$）

令$t = 0\ , x = x$,有$u(0,x) = g(x) \equiv \varphi(x)$.所以 $u^{(0)} = \varphi(x - at)$

·再考虑非齐次方程：$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f(t,x) \\ u(0,x) = 0 \end{cases}$ 解记为$u^{(1)}$

由齐次化原理 2，可以先求解方程：$\begin{cases} \frac{\partial \omega}{\partial t} + a\frac{\partial \omega}{\partial x} = 0 \\ \omega|_{t=\tau} = f(\tau,x) \end{cases}$ 上面已经解过：$\omega = h(x - at)$

只需确定函数h，再利用齐次化原理 2 即可得到原方程的解：利用边界条件，$t = \tau$时，

$$\omega(\tau,x) = h(x - a\tau) = f(\tau,x)\ ; \text{令 } \alpha = x - at\text{，则}x = \alpha + at$$

$$\Longrightarrow \ h(\alpha) = f(\tau,x) = f(\tau,\alpha + a\tau) = f(\tau,x - at + a\tau)$$

再由齐次化原理 2，得：

$$u^{(1)} = \int_0^t f(\tau,x - at + a\tau)\,d\tau$$

所以，题中的方程最终解为：

$$u = u^{(0)} + u^{(1)} = \varphi(x - at) + \int_0^t f(\tau,x - at + a\tau)\,d\tau$$

附：半直线波动方程带第二类边界条件，即解：$\left(u = u(t,x)\right)$

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,x) = \varphi(x) \\ u_x(t,0) = 0 \\ u_t(0,x) = \psi(x) \end{cases} \quad x,t > 0$$

解：直接做辅助函数，进行偶延拓：

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x), & x < 0 \end{cases} \quad ; \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x), & x < 0 \end{cases}$$

利用达朗贝尔公式：

$$u(t,x) = \frac{1}{2}\left(\Phi(x-at) + \Phi(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)\ d\xi$$

$$= \begin{cases} \dfrac{1}{2}\left(\Phi(x-at) + \Phi(x+at)\right) + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \Psi(\xi)\ d\xi & ,t \leq \dfrac{x}{a} \\ \dfrac{1}{2}\left(\Phi(at-x) + \Phi(x+at)\right) + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \Psi(\xi)\ d\xi & ,t > \dfrac{x}{a} \end{cases}$$

检验这个解是否满足边界条件$u_x(t,0) = 0$：

$$u_x(t,0) = \frac{1}{2}\left(\Phi'(-at) + \Phi'(at)\right) = 0 \quad (\Phi\text{ 为偶函数，}\Phi'\text{为奇函数})$$

所以这个解符合题目给出的所有约束。