数学物理方程 B　第十五周作业　4 月 23 日 周四

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5.9 在用基本解方法求解下列柯西问题：

(1) $\begin{cases} u_t + a u_x = f(t,x) \\ u|_{t=0} = \varphi(x) \end{cases}$ $t>0,\ x\in\mathbb{R}$

(2) $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial t} - 2u \\ u|_{t=0} = 0,\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$ $a,t>0,\ \ x\in\mathbb{R}$

解：(1) 这是一个$u_t = Lu$型方程柯西问题，先求它的基本解，即求 $U=U(t,x)$，满足：

$$\begin{cases} U_t + aU_x = 0 \\ U|_{t=0} = \delta(x) \end{cases} \quad t>0,\quad x\in\mathbb{R}$$

解这个问题，利用傅里叶变换求解：对x作傅里叶变换：

$$\overline{U}(t,\lambda) = F[U(t,x)] = \int_{-\infty}^{\infty} U(t,\xi)e^{i\lambda\xi}\,\mathrm{d}\xi$$

$$F[U_x] = -i\lambda F[U] = -i\lambda\overline{U}\ ;\quad F[U_t] = \frac{\mathrm{d}\overline{U}}{\mathrm{dt}}\ ;\quad F[\delta(x)] = 1$$

原问题变换为问题：

$$\begin{cases} \dfrac{\mathrm{d}\overline{U}}{\mathrm{dt}} - i\lambda a\overline{U} = 0 \\ \overline{U}|_{t=0} = 1 \end{cases} \quad\Longrightarrow\quad \text{解出：} \overline{U} = e^{ia\lambda t}$$

再对得到的解作傅里叶逆变换，立即可得：

$$U = U(t,x) = F^{-1}\left[\,\overline{U}\,\right] = F^{-1}\left[\,e^{ia\lambda t}\,\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ia\lambda t}\cdot e^{-i\lambda x}\,\mathrm{d}\lambda = \delta(x-at)$$

至此解出原问题的基本解为：

$$U = U(t,x) = \delta(x-at)$$

再由教材 331 页结论，原问题的解为：

$$\begin{aligned} u = u(t,x) &= U(t,x) * \varphi(x) + \int_0^t U(t-\tau,x) * f(\tau,x)\,\mathrm{d}\tau \\ &= \delta(x-at) * \varphi(x) + \int_0^t \delta\big(x - a(t-\tau)\big) * f(\tau,x)\,\mathrm{d}\tau \\ &= \varphi(x-at) + \int_0^t f\big(\tau, x - a(t-\tau)\big)\,\mathrm{d}\tau \end{aligned}$$

(2) 这是一个$u_{tt} = Lu$型方程柯西问题，求它的基本解即为求 $U = U(t,x)$，满足：

$$\begin{cases} U_{tt} = a^2U_{xx} - 2U_t - 2U \\ U|_{t=0} = 0,\ U_t|_{t=0} = \delta(x) \end{cases} \quad t>0,\quad x\in\mathbb{R}$$

解这个问题，利用傅里叶变换求解：对x作傅里叶变换：

$$\overline{U}(t,\lambda) = F[U(t,x)] = \int_{-\infty}^{\infty} U(t,\xi)e^{i\lambda\xi}\,\mathrm{d}\xi$$

$$F[U_{tt}] = \overline{U}_{tt}\,,\ \ F[U_t] = \overline{U}_t\,,\ \ F[U_{xx}] = (-i\lambda)^2F[U] = -\lambda^2\overline{U}\,,\quad F[\delta(x)] = 1$$

原问题变换为问题：

$$\begin{cases} \overline{U}_{tt} = -a^2\lambda^2\overline{U} - 2\overline{U}_t - 2\overline{U} \\ \overline{U}|_{t=0} = 0,\ \overline{U}_t|_{t=0} = 1 \end{cases}$$

二阶常系数齐次线性微分方程用特征方程求解：

$$r^2 + 2r + a^2\lambda^2 + 2 = 0\,,\text{显然 }\Delta < 0\text{，有一对共轭复根}$$

$$r = \alpha + i\beta,\qquad r_1 = -1 + i\sqrt{a^2\lambda^2+1}\,,\qquad r_2 = -1 - i\sqrt{a^2\lambda^2+1}$$

则这种情况的通解公式为：

$$\overline{U} = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t) = e^{-t}(C_1\cos\sqrt{a^2\lambda^2+1}\,t + C_2\sin\sqrt{a^2\lambda^2+1}\,t)$$

再带入两个边界条件确定系数：

$$\overline{U}|_{t=0} = C_1 = 0 \quad\Longrightarrow\quad C_1 = 0$$

$$U_t|_{t=0} = C_2\sqrt{a^2\lambda^2+1} = 1 \quad\Longrightarrow\quad C_2 = \frac{1}{\sqrt{a^2\lambda^2+1}}$$

$$\Longrightarrow\quad \overline{U} = \frac{1}{\sqrt{a^2\lambda^2+1}}e^{-t}\sin\sqrt{a^2\lambda^2+1}\,t$$

因此原问题的基本解为：

$$U = U(t,x) = F^{-1}\left[\frac{1}{\sqrt{a^2\lambda^2+1}}e^{-t}\sin\sqrt{a^2\lambda^2+1}\,t\right]$$

根据提示：

$$F^{-1}\left[\frac{\sin\sqrt{\lambda^2+b}\,a}{\sqrt{\lambda^2+b}}\right] = \frac{1}{2}J_0\left(b\sqrt{a^2-x^2}\right)h(a-|x|)$$

在本例中凑出这样的形式：

$$\begin{aligned} U(t,x) = F^{-1}\left[\frac{e^{-t}\sin\sqrt{a^2\lambda^2+1}\,t}{\sqrt{a^2\lambda^2+1}}\right] &= e^{-t}F^{-1}\left[\frac{1}{a}\frac{\sin\sqrt{\lambda^2+\dfrac{1}{a^2}}\,at}{\sqrt{\lambda^2+\dfrac{1}{a^2}}}\right] \\ &= \frac{e^{-t}}{2a}J_0\left(\frac{1}{a^2}\sqrt{a^2t^2-x^2}\right)h(at-|x|) \end{aligned}$$

这就是原问题的基本解。再由教材 335 页定理，此时原问题的解可以表示为：

$$\begin{aligned} u(t,x) = 0 + U(t,x) * \psi(x) + 0 &= \int_{-\infty}^{\infty}\frac{e^{-t}}{2a}J_0\left(\frac{1}{a^2}\sqrt{a^2t^2-\xi^2}\right)h(at-|\xi|)\,\psi(x-\xi)\,\mathrm{d}\xi \\ &= \frac{e^{-t}}{2a}\int_{-at}^{at}J_0\left(\frac{1}{a^2}\sqrt{a^2t^2-\xi^2}\right)\,\psi(x-\xi)\,\mathrm{d}\xi \end{aligned}$$

5.10. 试写出定解问题的解的积分表达式：　　【教材 340 页】

$$\begin{cases} u_{tt} = a^2\left(u_{xx} + u_{yy}\right) + f(t,x,y) \\ u(0,x,y) = 0, \quad u_t(0,x,y) = 0 \end{cases}$$

解：这是一个$u_{tt} = Lu$型方程柯西问题，求它的基本解即为求 $U = U(t,x,y)$，满足：

$$\begin{cases} U_{tt} = a^2\left(U_{xx} + U_{yy}\right) \\ U(0,x,y) = 0, \quad U_t(0,x,y) = \delta(x,y) \end{cases}$$

直接由 341 页的结论，可知：

$$U(t,x,y) = \frac{1}{2\pi a} \iint_{D_{at}} \frac{\delta(x,y)}{\sqrt{a^2t^2 - (\xi - x)^2 - (\eta - y)^2}} \, \mathrm{d}\xi \mathrm{d}\eta = \frac{1}{2\pi a} \frac{1}{\sqrt{a^2t^2 - x^2 - y^2}}$$

再由 335 页定理，可知：

$$u(t,x,y) = 0 + 0 + \int_0^t U(t-\tau,x,y) * f(\tau,x,y) \, \mathrm{d}\tau$$

$$= \frac{1}{2\pi a} \int_0^t \left(\iint_{D_{at}} \frac{f(\tau,\xi,\eta)}{\sqrt{a^2(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2}} \mathrm{d}\xi \mathrm{d}\eta \right) \mathrm{d}\tau$$

其中，$(x-\xi)^2 + (y-\eta)^2 \leq a^2(t-\tau)^2$

5.12. 根据已知公式直接求下列问题的解：

(1) $\begin{cases} u_t = a^2 u_{xx} \\ u(0,x) = e^{-x^2} \end{cases}$

(2) $\begin{cases} u_{tt} = a^2 \Delta_2 u \\ u(0,x,y) = x^2(x+y), \quad u_t(0,x,y) = 0 \end{cases}$

(3) $\begin{cases} u_{tt} = a^2 \Delta_2 u + x + y \\ u|_{t=0} = 0 \\ u_t|_{t=0} = x + y \end{cases}$

(4) $\begin{cases} u_t = a^2 \Delta_3 u + x + y + z \\ u|_{t=0} = x + y + z \\ u_t|_{t=0} = x + y + z \end{cases}$

解：(1) 这是一维热传导方程的柯西问题，也是$u_t = Lu$型方程。先求其基本解，即是求$U = U(t,x)$满足：

$$\begin{cases} U_t = a^2 U_{xx} \\ u(0,x) = \delta(x) \end{cases} t > 0, x \in \mathbb{R}$$

由教材 333 页已知公式，可得这个方程的解为：

$$U = U(t,x) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

再由教材 331 页的定理：

$$\begin{aligned}
u(t,x) &= U(t,x) * e^{-x^2} + \int_0^t U(t-\tau) * f(\tau,x)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} U(t,x-\xi) e^{-\xi^2}\,\mathrm{d}\xi + 0 \quad = \quad \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} e^{-\xi^2}\,\mathrm{d}\xi \\
&= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-\xi)^2}{4a^2 t} + \xi^2\right)}\,\mathrm{d}\xi \quad = \quad \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left(\frac{(4a^2 t+1)\xi^2 - 2x\xi + x^2}{4a^2 t}\right)}\,\mathrm{d}\xi \\
&= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left((4a^2t+1)\frac{\xi^2 - \frac{2x}{4a^2t+1}\xi + \frac{x^2}{4a^2t+1}}{4a^2 t}\right)}\,\mathrm{d}\xi \\
&= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left((4a^2t+1)\frac{\left(\xi - \frac{x}{4a^2t+1}\right)^2 + \frac{x^2}{4a^2t+1} - \left(\frac{x}{4a^2t+1}\right)^2}{4a^2 t}\right)}\,\mathrm{d}\xi \\
&= \frac{e^{-\frac{\frac{x^2}{4a^2t+1} - \left(\frac{x}{4a^2t+1}\right)^2}{4a^2t}(4a^2t+1)}}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\left(\xi - \frac{x}{4a^2t+1}\right)^2 (4a^2t+1)}{4a^2 t}} \frac{2a\sqrt{t}}{\sqrt{4a^2t+1}}\,\mathrm{d}\left(\frac{\left(\xi - \frac{x}{4a^2t+1}\right)\sqrt{4a^2t+1}}{2a\sqrt{t}}\right) \\
&= \frac{e^{-\frac{\frac{x^2}{4a^2t+1} - \left(\frac{x}{4a^2t+1}\right)^2}{4a^2t}(4a^2t+1)}}{2a\sqrt{\pi t}} \cdot \frac{2a\sqrt{t}}{\sqrt{4a^2t+1}} \cdot \sqrt{\pi} \quad = \quad \frac{1}{\sqrt{4a^2t+1}} e^{-\frac{x^2}{4a^2t+1}}
\end{aligned}$$

(2) 这是 $u_{tt} = Lu$ 型方程，为二维波动方程的柯西问题。由教材 341 页的已知公式：

这个问题的解的公式为： ($\varphi(x,y) = x^2(x+y)$； $\psi(x,y) = 0$)

$$u = u(t,x,y) = \frac{1}{2\pi a} \frac{\partial}{\partial t} \iint_{D_{at}} \frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2 - (\xi - x)^2 - (\eta - y)^2}}\,\mathrm{d}\xi\mathrm{d}\eta$$

其中积分区域为 $(\xi - x)^2 + (\eta - y)^2 \le a^2 t^2$. 换元，令$\xi - x = r\cos\theta$，$\eta - y = r\sin\theta$

$$u = u(t,x,y) = \frac{1}{2\pi a} \frac{\partial}{\partial t} \int_0^{2\pi} \int_0^{at} \frac{(x + r\cos\theta)^2 (x + y + r\cos\theta + r\sin\theta)}{\sqrt{a^2t^2 - r^2}}\, r\,\mathrm{d}r\mathrm{d}\theta$$

考察积分：

$$I = \int_0^{2\pi} \int_0^{at} \frac{(x + r\cos\theta)^2 (x + y + r\cos\theta + r\sin\theta)}{\sqrt{a^2t^2 - r^2}}\, r\,\mathrm{d}r\mathrm{d}\theta$$

先对θ积分，可以化简掉很多项：

$$\int_0^{2\pi} \sin\theta\,\mathrm{d}\theta = \int_0^{2\pi} \cos\theta\,\mathrm{d}\theta = \int_0^{2\pi} \sin\theta\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi} \cos^3\theta\,\mathrm{d}\theta = \int_0^{2\pi} \cos^2\theta\sin\theta\,\mathrm{d}\theta = 0$$

$$\int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = \pi$$

化简得：

$$\begin{aligned}
I &= \int_0^{at} \frac{r}{\sqrt{a^2t^2 - r^2}} \cdot \left(2\pi x^2(x+y) + \pi r^2(3x+y)\right)\mathrm{d}r \\
&= 2\pi x^2(x+y) \int_0^{at} \frac{r}{\sqrt{a^2t^2 - r^2}}\,\mathrm{d}r + \pi(3x+y) \int_0^{at} \frac{r^3}{\sqrt{a^2t^2 - r^2}}\,\mathrm{d}r \\
&= 2\pi x^2(x+y) \cdot \left(-\sqrt{a^2t^2 - r^2}\Big|_{r=0}^{at}\right) + \pi(3x+y) \int_0^{(at)^2} \frac{k}{2\sqrt{a^2t^2 - k}}\,\mathrm{d}k \\
&= 2\pi a t x^2(x+y) + \frac{\pi(3x+y)}{2}\left(\frac{2}{3}(-k - 2a^2t^2)\sqrt{-k + a^2t^2}\Big|_{k=0}^{a^2t^2}\right) \\
&= 2\pi a t x^2(x+y) + \frac{2\pi(3x+y)a^3t^3}{3}
\end{aligned}$$

再将这个结果代回$u = u(t,x,y)$的表达式：

$$u = u(t,x,y) = \frac{1}{2\pi a} \frac{\partial}{\partial t}\left(2\pi a t x^2(x+y) + \frac{2\pi(3x+y)a^3t^3}{3}\right) = x^2(x+y) + a^2t^2(3x+y)$$

这就是原问题的解。

(3) 这是 $u_{tt}=Lu$ 型方程的柯西问题，利用基本解方法求解：

求它的基本解即为求 $U=U(t,x,y)$，满足：

$$\begin{cases} U_{tt}=a^2\left(U_{xx}+U_{yy}\right) \\ U(0,x,y)=0,\quad U_t(0,x,y)=\delta(x,y) \end{cases}$$

直接由 341 页的结论，可知：

$$U(t,x,y)=\frac{1}{2\pi a}\iint_{D_{at}}\frac{\delta(x,y)}{\sqrt{a^2t^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\mathrm{d}\eta=\frac{1}{2\pi a}\frac{1}{\sqrt{a^2t^2-x^2-y^2}}\quad,\quad x^2+y^2<a^2t^2$$

再由 335 页定理，可知原问题的解为：

$$\begin{aligned} u=u(t,x,y)&=U(t,x,y)*(x+y)+\int_0^t U(t-\tau,x,y)*(x+y)\mathrm{d}\tau \\ &=U(t,x,y)*(x+y)+\int_0^t U(\tau,x,y)*(x+y)\mathrm{d}\tau \end{aligned}$$

为此需要计算卷积：(其中积分区域为 $\xi^2+\eta^2\le a^2t^2$)

$$I=U(t,x,y)*(x+y)=\frac{1}{2\pi a}\iint_{D_{at}}\frac{x+y-\xi-\eta}{\sqrt{a^2t^2-\xi^2-\eta^2}}\,\mathrm{d}\xi\mathrm{d}\eta$$

换元，令 $\xi=r\cos\theta$，$\eta=r\sin\theta$：为简单先对θ进行积分，利用关系

$$\int_0^{2\pi}\sin\theta\ \mathrm{d}\theta=\int_0^{2\pi}\cos\theta\ \mathrm{d}\theta=0$$

化简后仅有第一项积分：

$$I=\frac{1}{2\pi a}\int_0^{at}r\,\mathrm{d}r\int_0^{2\pi}\frac{x+y-r(\sin\theta+\cos\theta)}{\sqrt{a^2t^2-r^2}}\,\mathrm{d}\theta=\frac{1}{a}\int_0^{at}\frac{x+y}{\sqrt{a^2t^2-r^2}}\,r\,\mathrm{d}r=(x+y)t$$

将这个结果代回原问题的解中，立即可知：

$$u=u(t,x,y)=I+\int_0^t(x+y)\tau\,\mathrm{d}\tau=(x+y)\left(t+\frac{t^2}{2}\right)$$

(4) 这是 $u_{tt}=Lu$ 型方程的柯西问题，利用基本解方法求解：

求它的基本解即为求 $U=U(t,x,y,z)$，满足：

$$\begin{cases} U_{tt}=a^2\left(U_{xx}+U_{yy}+U_{zz}\right) \\ U(0,x,y,z)=0,U_t(0,x,y,z)=\delta(x,y) \end{cases}$$

由教材 338 页的已知公式，可知这个问题的解为：

$$U=U(t,x,y,z)=\frac{1}{4\pi a\sqrt{x^2+y^2+z^2}}\delta\left(\sqrt{x^2+y^2+z^2}-at\right)$$

再由教材 335 页定理，原问题的解为：

$$u=u(t,x,y,z)=\frac{\partial I}{\partial t}+I+\int_0^t I(t=\tau)\,\mathrm{d}\tau$$

其中还需要计算卷积积分I：

$$I=U(t,x,y,z)*(x+y+z)=\iiint U(t,\xi,\eta,\zeta)(x+y+z-\xi-\eta-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

作球面换元，令 $\xi=r\sin\theta\cos\varphi$，$\eta=r\sin\theta\sin\varphi$，$\zeta=r\cos\theta$

积分顺序为：$\theta\to\varphi\to r$

$$\begin{aligned} I&=\int_{-\infty}^{\infty}\frac{\delta(r-at)}{4\pi ar}\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}(x+y+z-r(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta))r^2\sin\theta\,\mathrm{d}\theta \\ &=\int_{-\infty}^{\infty}\frac{\delta(r-at)}{4\pi ar}\,\mathrm{d}r\cdot4\pi r^2(x+y+z)\quad=\quad\int_{-\infty}^{\infty}\frac{\delta(r-at)r(x+y+z)}{a}\,\mathrm{d}r \\ &=t(x+y+z) \end{aligned}$$

再将这个积分结果代回解的表达式中，立即可得：

$$u=u(t,x,y,z)=\frac{\partial I}{\partial t}+I+\int_0^t\tau(x+y+z)\,\mathrm{d}\tau=(1+t+\frac{t^2}{2})(x+y+z)$$