解 15. 记$Y_1 = X_1 - 2X_2$，$Y_2 = 3X_3 - 4X_4$，则Y_1, Y_2仍为$\mu = 0$的正态分布.
由于X_1, X_2, X_3, X_4独立同分布，故$Var(Y_1) = Var(X_1) + 2^2 Var(X_2)$，$Var(Y_2) = 3^2 Var(X_3) + 4^2 Var(X_4)$
即$Y_1 \sim N(0, 20)$，$Y_2 \sim N(0, 100)$
若要$T = aY_1^2 + bY_2^2 \sim \chi^2$，则必有$\sqrt{a}Y_1, \sqrt{b}Y_2 \sim N(0,1)$.
解得$a = \dfrac{1}{20}, b = \dfrac{1}{100}$

解 16. 由于$X_i (1 \le i \le 9)$为独立同分布的正态分布. 设$X_i \sim N(\mu, \sigma^2)$.
因为$Y_1 = \dfrac{1}{6}(X_1 + X_2 + \cdots + X_6)$，$Y_2 = \dfrac{1}{3}(X_7 + X_8 + X_9)$，故$Y_1 \sim N\left(\mu, \dfrac{1}{6}\sigma^2\right)$，$Y_2 \sim N\left(\mu, \dfrac{1}{3}\sigma^2\right)$.
同时又因为Y_1与Y_2相互独立，故$Y' = (Y_1 - Y_2) \sim N\left(0, \dfrac{1}{2}\sigma^2\right)$，$\dfrac{\sqrt{2}Y'}{\sigma} \sim N(0,1)$.
对于$S^2 = \dfrac{1}{2}\sum\limits_{i=7}^{9}(X_i - Y_2)^2$，有$\dfrac{2S^2}{\sigma^2} \sim \chi_2^2$，故$Z = \dfrac{\sqrt{2}(Y_1 - Y_2)}{S} = \dfrac{\sqrt{2}Y'/\sigma}{\sqrt{(2S^2/\sigma^2)/2}} \sim \dfrac{N(0,1)}{\sqrt{\chi_2^2/2}} = t_2$.
故$Z = \dfrac{\sqrt{2}(Y_1 - Y_2)}{S} \sim t_2$.

解 19. 记$Y = X_1 + X_2 + \cdots + X_n$，则$\bar{X} = \dfrac{1}{n}Y$.
由于$X_1, X_2, \ldots, X_n$之间互相独立，且$X_i \sim B(1,p)$，故$Y \sim B(n,p)$.
由$E(Y) = np$，$Var(Y) = np(1-p)$，可得：$E(\bar{X}) = p$，$Var(\bar{X}) = \dfrac{1}{n}p(1-p)$
故$E(X_i - \bar{X})^2 = EX_i^2 - 2EX_i\bar{X} + E\bar{X}^2 = Var(X_i) + (EX_i)^2 - 2\dfrac{1}{n}EX_i(X_1 + \cdots + X_i + \cdots + X_n) + Var(\bar{X}) + (E\bar{X})^2$
由于对$1 \le i \ne j \le n$，X_i, X_j相互独立，故$EX_iX_j = EX_iEX_j = p^2$，代入上式有：
$E(X_i - \bar{X})^2 = p(1-p) + p^2 - \dfrac{2}{n}\left(p + (n-1)p^2\right) + \dfrac{p(1-p)}{n} + p^2 = \dfrac{(n-1)p(1-p)}{n}$
故$ES_n^2 = E\left(\dfrac{1}{n}\sum\limits_{i=1}^{n}(X_i - \bar{X})^2\right) = \dfrac{1}{n}\sum\limits_{i=1}^{n}E(X_i - \bar{X})^2 = \dfrac{1}{n}n\dfrac{(n-1)p(1-p)}{n} = \dfrac{(n-1)p(1-p)}{n}$

解 2. 总体均值为$EX = p_1 + 2p_2 + 3(1 - p_1 - p_2) = -2p_1 - p_2 + 3$
总体二阶原点矩为$EX^2 = p_1 + 4p_2 + 9(1 - p_1 - p_2) = -8p_1 - 5p_2 + 9$
样本均值为$a_1 = \dfrac{n_1}{n} + 2\dfrac{n_2}{n} + 3\dfrac{n_3}{n}$，样本二阶原点矩为$a_2 = \dfrac{n_1}{n} + 4\dfrac{n_2}{n} + 9\dfrac{n_3}{n}$
根据矩估计近似有：$EX = a_1$，$EX^2 = a_2$，解得：$\widehat{p}_1 = \dfrac{n_1}{n}$，$\widehat{p}_2 = \dfrac{n_2}{n}$

解 4.(2). 若总体X的概率密度函数为

$$f(x;\theta) = \begin{cases} (\theta+1)x^\theta & 0 < x < 1, \theta > 0 \\ 0 & \text{其他} \end{cases}$$

则总体均值为$EX = \displaystyle\int_{-\infty}^{+\infty} xf(x;\theta)dx = \int_0^1 x(\theta+1)x^\theta dx = \frac{\theta+1}{\theta+2}$
根据矩估计，样本均值$\bar{X} = \dfrac{1}{n}(X_1 + X_2 + \cdots + X_n) = EX$，解得：$\widehat{\theta} = \dfrac{2\bar{X} - 1}{1 - \bar{X}}$

解 4.(5). 若总体X的概率密度函数为

$$p(x;\theta) = \begin{cases} \dfrac{6x}{\theta^3}(\theta - x) & 0 < x < \theta, \theta > 0 \\ 0 & \text{其他} \end{cases}$$

则总体均值为$EX = \displaystyle\int_{-\infty}^{+\infty} xp(x;\theta)dx = \int_0^\theta \frac{6x^2}{\theta^3}(\theta - x)dx = \frac{\theta}{2}$
根据矩估计，样本均值$\bar{X} = \dfrac{1}{n}(X_1 + X_2 + \cdots + X_n) = EX$，解得：$\widehat{\theta} = 2\bar{X}$

解 5. *(1).* 若总体X的概率密度函数为

$$f(x)=\begin{cases}\dfrac{4x^2}{\theta^3\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}} & x\geq 0,\theta>0\\ 0 & \text{其他}\end{cases}$$

则总体均值为$EX=\displaystyle\int_{-\infty}^{+\infty}xf(x)dx=\int_0^{+\infty}\frac{4x^3}{\theta^3\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}}dx=\frac{2\theta}{\sqrt{\pi}}$

根据矩估计，样本均值$\bar{X}=\dfrac{1}{n}(X_1+X_2+\cdots+X_n)=EX$，解得：$\widehat{\theta}=\dfrac{\sqrt{\pi}\bar{X}}{2}$

(2). $EX^2=\displaystyle\int_{-\infty}^{+\infty}x^2f(x)dx=\int_0^{+\infty}\frac{4x^3}{\theta^3\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}}dx=\frac{3\theta^2}{2}$

$Var(X)=EX^2-(EX)^2=\dfrac{3\theta^2}{2}-\dfrac{4\theta^2}{\pi}$

则$Var(\widehat{\theta})=\dfrac{\pi}{2}\dfrac{Var(X)}{n}=\dfrac{(3\pi-8)\theta^2}{8n}$

解 10.(3). 由于$p(x;\theta)=(x-1)\theta^2(1-\theta)^{x-2}$，$x=2,3,\cdots$，$0<\theta<1$，则

$$L=\prod_{i=1}^{n}\left[(X_i-1)\theta^2(1-\theta)^{X_i-2}\right]$$

$$\ln L=\sum_{i=1}^{n}\left[\ln(X_i-1)+2\ln\theta+(X_i-2)\ln(1-\theta)\right]=\sum_{i=1}^{n}\ln(X_i-1)+\ln\theta^{2n}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n\right)}$$

故

$$\frac{\partial\ln L}{\partial\theta}=\frac{1}{\theta^{2n}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n\right)}}\left(2n\theta^{2n-1}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n\right)}-\left(\sum_{i=1}^{n}X_i-2n\right)\theta^{2n}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n-1\right)}\right)$$

令$\dfrac{\partial\ln L}{\partial\theta}=0$，则有：

$$2n\theta^{2n-1}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n\right)}-\left(\sum_{i=1}^{n}X_i-2n\right)\theta^{2n}(1-\theta)^{\left(\sum\limits_{i=1}^{n}X_i-2n-1\right)}=0$$

解得：$\widehat{\theta}=\dfrac{2}{\bar{X}}$，其中$\bar{X}=\dfrac{1}{n}\sum\limits_{i=1}^{n}X_i$

解 11.(1). 由于

$$f(x;\theta)=\begin{cases}\dfrac{2}{\theta^2}(\theta-x) & 0<x<\theta\\ 0 & \text{其他}\end{cases}$$

则

$$L=\prod_{i=1}^{n}\left[\frac{2}{\theta^2}(\theta-X_i)\right]$$

$$\ln L=\sum_{i=1}^{n}\left[\ln 2+\ln\frac{(\theta-X_i)}{\theta^2}\right]=\ln 2^n+\sum_{i=1}^{n}\ln\frac{(\theta-X_i)}{\theta^2}$$

故

$$\frac{\partial\ln L}{\partial\theta}=\sum_{i=1}^{n}\frac{2X_i-\theta}{\theta(\theta-X_i)}$$

令$\dfrac{\partial\ln L}{\partial\theta}=0$，即$\widehat{\theta}$为方程$\sum\limits_{i=1}^{n}\dfrac{2X_i-\theta}{\theta-X_i}=0$的根.

解 11.(2). 由于

$$f(x;\theta)=\begin{cases}(\theta+1)x^\theta & 0<x<1,\theta>0\\ 0 & \text{其他}\end{cases}$$

则

$$L=\prod_{i=1}^{n}\left[(\theta+1)X_i^\theta\right]$$

$$\ln L=\sum_{i=1}^{n}\left[\ln(\theta+1)+\theta\ln X_i\right]=n\ln(\theta+1)+\theta\ln\left(\prod_{i=1}^{n}X_i\right)$$

则

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta+1}+\ln\left(\prod_{i=1}^{n}X_i\right)$$

令$\dfrac{\partial\ln L}{\partial\theta}=0$, 解得：$\widehat{\theta}=-\dfrac{n}{\ln\left(\prod_{i=1}^{n}X_i\right)}-1$

解 15. 由于$X_i\sim N(\mu,\sigma^2)$, $1\le i\le n$, 则似然函数为：

$$L=\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(X_i-\mu)^2}{2\sigma^2}}\right]$$

$$\ln L=\sum_{i=1}^{n}\left[\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right)-\frac{(X_i-\mu)^2}{2\sigma^2}\right]=n\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

则

$$\frac{\partial\ln L}{\partial\mu}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)$$

$$\frac{\partial\ln L}{\partial\sigma}=-\frac{n}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)^2$$

分别令$\dfrac{\partial\ln L}{\partial\mu}=0$, $\dfrac{\partial\ln L}{\partial\sigma}=0$, 分别得到$\mu$和$\sigma$的估计值为：

$$\widehat{\mu}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

$$\widehat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\widehat{\mu})^2$$

则$X_i\sim N(\widehat{\mu},\widehat{\sigma}^2)$

$$\theta=P(X\ge 2)=1-P(X<2)=1-P\left(\frac{X-\widehat{\mu}}{\widehat{\sigma}}<\frac{2-\widehat{\mu}}{\widehat{\sigma}}\right)=1-\Phi\left(\frac{2-\widehat{\mu}}{\widehat{\sigma}}\right)$$

解 17. *(1).* 总体的概率密度函数为：

$$f(x;\theta)=(F(x;\theta))'=\begin{cases}\dfrac{2x}{\theta}e^{-\frac{x^2}{\theta}} & x\ge 0\\ 0 & \text{其他}\end{cases}$$

故

$$\begin{aligned}EX &= \int_{-\infty}^{+\infty} xf(x;\theta)dx = \int_0^{+\infty} \frac{2x^2}{\theta}e^{-\frac{x^2}{\theta}}dx = -\int_0^{+\infty} xd\left(e^{-\frac{x^2}{\theta}}\right)\\
&= -xe^{-\frac{x^2}{\theta}}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\frac{x^2}{\theta}}dx = \int_0^{+\infty} e^{-\frac{x^2}{\theta}}dx\\
&= \sqrt{\theta}\int_0^{+\infty} e^{-\frac{x^2}{\theta}}d\left(\frac{x}{\sqrt{\theta}}\right) = \frac{\sqrt{\theta\pi}}{2}\end{aligned}$$

$$\begin{aligned}EX^2 &= \int_{-\infty}^{+\infty} x^2 f(x;\theta)dx = \int_0^{+\infty} \frac{2x^3}{\theta}e^{-\frac{x^2}{\theta}}dx = -\int_0^{+\infty} x^2 d\left(e^{-\frac{x^2}{\theta}}\right)\\
&= -x^2e^{-\frac{x^2}{\theta}}\Big|_0^{+\infty} + \int_0^{+\infty} 2xe^{-\frac{x^2}{\theta}}dx = -\theta\int_0^{+\infty} d\left(e^{-\frac{x^2}{\theta}}\right)\\
&= -\theta e^{-\frac{x^2}{\theta}}\Big|_0^{+\infty} = \theta\end{aligned}$$

(2). 似然函数为

$$L = \prod_{i=1}^{n}\left[\frac{2X_i}{\theta}e^{-\frac{X_i^2}{\theta}}\right]$$

$$\ln L = \sum_{i=1}^{n}\left[\ln\left(\frac{2X_i}{\theta}\right) - \frac{X_i^2}{\theta}\right]$$

则

$$\frac{\partial \ln L}{\partial\theta} = \sum_{i=1}^{n}\left[-\frac{1}{\theta} + \frac{X_i^2}{\theta^2}\right] = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} X_i^2$$

令$\frac{\partial \ln L}{\partial\theta} = 0$, 即可得$\theta$的估计值为$\widehat{\theta} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i^2$

解 33. *(1)*. 矩估计：记$t = \lambda x^\alpha$, 则有$\lambda\alpha x^{\alpha-1}dx = dt$

$$EX = \int_{-\infty}^{+\infty} xf(x,\lambda) = \int_0^{+\infty} x\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}dx = \lambda^{-\frac{1}{\alpha}}\int_0^{+\infty} e^{-t}t^{\frac{1}{\alpha}}dt = \lambda^{-\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}+1\right)$$

样本均值为$\bar{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$. 由矩估计, 令$EX = \bar{X}$, 解得$\lambda$的估计值：$\widehat{\lambda} = \left(\Gamma\left(\frac{1}{\alpha}+1\right)/\bar{X}\right)^\alpha$

(2). 极大似然估计：

似然函数为：

$$L = \prod_{i=1}^{n}\left[\lambda\alpha X_i^{\alpha-1}e^{-\lambda X_i^\alpha}\right]$$

$$\ln L = \sum_{i=1}^{n}\left[\ln\lambda\alpha + (\alpha-1)\ln X_i - \lambda X_i^\alpha\right] = n\ln\lambda\alpha + (\alpha-1)\sum_{i=1}^{n}\ln X_i - \lambda\sum_{i=1}^{n} X_i^\alpha$$

则

$$\frac{\partial \ln L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i^\alpha$$

令$\frac{\partial \ln L}{\partial\lambda} = 0$, 可解得$\lambda$的估计值为：$\widehat{\lambda} = \frac{n}{\sum\limits_{i=1}^{n} X_i^\alpha}$