2001-2002学年第一学期数理方程期末试题

注：考试时间两小时, 前七题中选做六题, 第八题必做. 试卷中$a>0$是常数.

一. (15分)解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + 2x, \ (t>0, -\infty < x < \infty), \\ u(t,x)|_{t=0} = 0, \ \frac{\partial u}{\partial t}|_{t=0} = 3x^2. \end{cases}$$

二. (15分)线性偏微分算子$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x \partial y} - 2\frac{\partial^2}{\partial y^2}$,

1. 求方程$L[u]=0$的通解;

2. 解定解问题

$$\begin{cases} L[u] = 0, \ (y>0, -\infty < x < +\infty), \\ u(x,y)|_{y=0} = \sin x, \ \frac{\partial}{\partial y}|_{y=0} = 0. \end{cases}$$

三. 解定解问题(15分)

1.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \ (t>0, 0<x<l), \\ u(t,x)|_{x=0} = \frac{\partial u}{\partial x}|_{x=l} = 0, \\ u(t,x)|_{t=0} = \phi(x), \ (\phi(0)=0). \end{cases}$$

2.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \ (t>0, 0<x<l), \\ u(t,x)|_{x=0} = u_0, \ \frac{\partial u}{\partial x}|_{x=l} = \frac{q_0}{k}, \\ u(t,x)|_{t=0} = u_0. \end{cases}$$

其中u_0, q_0, k为常数.

四. (15分)

1. 求解Laplace方程的边值问题

$$\begin{cases} \Delta_2 u = 0, \ (r = \sqrt{x^2+y^2} < 1), \\ \frac{\partial u}{\partial r}|_{r=1} = \cos^2\theta - \sin^2\theta. \end{cases}$$

2. 如果把边界条件改为$\frac{\partial}{\partial r}|_{r=1} = f(\theta)$, $f(\theta) = f(\theta+2\pi)$且有一阶连续导数及分段二阶连续导数，上述边值问题是否一定有解？为什么？

五. (15分)解定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},\ (t>0,x>0),\\ \left(u-\frac{\partial u}{\partial x}\right)|_{x=0}=0,\\ u(t,x)|_{t=0}=1,\ \frac{\partial u}{\partial t}|_{t=0}=0.\end{cases}$$

六. (15分)

1. 解定解问题

$$\begin{cases}\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=-\delta(x-\xi,y-\eta),\ (x>0,\xi<+\infty;\ y>0,\eta<+\infty),\\ G(x,y;\xi,\eta)|_{x=0}=G(x,y;\xi,\eta)|_{y=0}=0.\end{cases}$$

2. 利用1)中的$G(x,y;\xi,\eta)$写出定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ (x>0;\ y>0),\\ u(x,y)|_{x=0}=\phi(y),\ u(x,y)|_{y=0}=\psi(x).\ (\phi(0)=\psi(0))\end{cases}$$

解的积分公式.

七. (15分)求初值问题

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\Delta_2 u+b_1\frac{\partial u}{\partial x}+b_2\frac{\partial u}{\partial y}+cu+f(t,x,y),\ (t>0,-\infty<x,y<+\infty),\\ u(t,x,y)|_{t=0}=\phi(x,y).\end{cases}$$

的基本解，并利用基本解写出此定解问题解的积分公式 (b_1,b_2,c是常数).

八. (10分)用分离变量法求解边值问题

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+x\frac{\partial}{\partial x}(x\frac{\partial}{\partial x})=0,\ (1<x<e,0<y<1,0<z<+\infty),\\ u(x,y,z)|_{x=1}=u(x,y,z)|_{x=e}=0,\\ \frac{\partial u}{\partial y}|_{y=0}=\frac{\partial u}{\partial y}|_{y=1}=0,\\ (u-\frac{\partial}{\partial z})|_{z=0}=\psi(x,y),\ \text{且}z\to\infty\text{时},\ u(x,y,z)\text{有界}.\end{cases}$$

参考公式

$\int_0^{+\infty}e^{-a^2x^2}\cos bx dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}$; $L[\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}]=\frac{e^{-a\sqrt{p}}}{\sqrt{p}}$; $L[l^n]=\frac{n!}{p^{n+1}}$, $n=0,1,2,3,\cdots$;

$L[e^{\lambda t}f(t)]=\bar{f}(p-\lambda)$; $L[f(t-\tau)]=e^{-p\tau}\bar{f}(p)$, 其中$\bar{f}(p)=L[f(t)]$.

2001-2002学年第二学期数理方程期末试题

一. (20分)

1. 利用镜像法写出上半圆($x^2+y^2<a^2, y>0$)内场位方程第一边值问题的Green函数.

2. 利用达朗贝尔公式求出一维波动方程初值问题的基本解.

二. (45分)解下列定解问题

1.

$$\begin{cases} \Delta_2 u = 0,\ (r<1, 0<\phi<\pi/4), \\ u|_{\phi=0} = \frac{\partial u}{\partial \phi}|_{\phi=\pi/4} = 0, \\ u|_{r=1} = \sin 2\phi + \sin 6\phi. \end{cases}$$

2.

$$\begin{cases} \Delta_3 u = 0,\ (r \neq 1), \\ u|r=1 = f(\theta), \\ \lim\limits_{r\to\infty} u = 0. \end{cases}$$

3.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},\ (t>0, -\infty<x<\infty), \\ \frac{\partial u}{\partial x}|_{x=0} = q(t),\ u|_{t=0} = 0, \\ u_x(t,\infty) = u(t,\infty) = 0. \end{cases}$$

三. (20分)

1. 解定解问题($G = G(t,x;\xi)$)

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x-\xi),\ (0<t, 0<x<l, 0<\xi<l), \\ G|_{x=0} = G|_{x=l} = 0, \\ G|_{t=0} = 0,\ G_t|_{t=0} = 0. \end{cases}$$

2. 利用1)得到的$G(t,x;\xi)$, 写出定解问题

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x),\ (t>0, 0<x<l), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = 0,\ u_t|_{t=0} = 0 \end{cases}$$

的解.

四. (15分)(任选一题)

1. 设$G(x,y,z;\xi,\eta,\zeta)$为场位方程第三边值问题的Green函数，即定解问题

$$\begin{cases}\Delta_3 G=-\delta(x-\xi,y-\eta,z-\zeta),\ ((x,y,z)\in V,(\xi,\eta,\zeta)\in V),\\ (\alpha G+\beta\frac{\partial G}{\partial n})|_S=0,\ \alpha,\beta\text{是任意常数},\ S\text{是}V\text{的边界}\end{cases}$$

的解, 试利用第二Green公式, 推出定解问题

$$\begin{cases}\Delta_3 u=0,\ ((x,y,z)\in V),\\ (\alpha u+\beta\frac{\partial u}{\partial n})|_S=\phi(x,y,z),\ \alpha,\beta\text{是任意常数},\ S\text{是}V\text{的边界}\end{cases}$$

的解的积分表达式.

2. 利用积分变换求出三维波动方程初值问题的基本解.

附录

1. 设$u(x,y,z)$和$v(x,y,z)$在区域V及边界曲面S上有一阶连续偏导数, 在V内有二阶连续偏导数, 则有

$$\iiint_V (u\Delta v-v\Delta u)dV=\iint_S\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial v}\right)dS$$

2.

$$L[f(t-\tau)]=e^{-p\tau}L[f(t)],\ L\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}\right]=\frac{e^{-a\sqrt{p}}}{\sqrt{p}}$$

3.

$$\int_{-\infty}^{+\infty}e^{a\lambda-\beta^2\lambda^2}d\lambda=\frac{\sqrt{\pi}}{\beta}e^{\frac{\alpha^2}{4\beta^2}},\ \beta\neq 0$$

4.

$$\int_0^{+\infty}e^{-a^2x^2}\cos bx dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}$$

2002-2003学年第二学期数理方程期末试题

一. (20分)解定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},\ (0<x<l,t>0),\\ u|_{t=0}=0,\ \frac{\partial u}{\partial t}|_{t=0}=\sin\frac{\pi}{l}x+\sin\frac{2\pi}{l}x,\\ u|_{x=0}=0,\ u|_{x=l}=0.\end{cases}$$

二. (20分)解定解问题

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-u,\ (r=\sqrt{x^2+y^2}<1,t>0),\\ u|_{t=0}=x^2+y^2,\\ u|_{r=1}=e^{-t}.\end{cases}$$

三. (15分)用Laplace变换求解

$$\begin{cases}\frac{\partial^2 u}{\partial x\partial y}+c^2u=0,\ (x>0,y>0),\ c>0\text{为常数},\\ u|_{x=0}=y,\\ u|_{y=0}=0.\end{cases}$$

四. (10分)求边值问题

$$\begin{cases}\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\delta(x-\xi,y-\eta),\ (0<x,\xi<+\infty,0<y,\eta<+\infty),\\ G|_{x=0}=0,G|_{y=0}=0\end{cases}$$

的解$G(x,y;\xi,\eta)$.

五. (20分)现有初值问题

$$\begin{cases}\frac{\partial u}{\partial t}=9\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}-u+f(t,x,y),\ ((x,y)\in R^2,t>0),\\ u|_{t=0}=\phi(x,y),\end{cases}$$

1. 求此初值问题的基本解$U(t,x,y)$;

2. 利用此基本解写出上述初始问题解的积分表达式.

六. (15分)设$L[u]=x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}$, $xy\neq 0$, 试

1. 求出方程$L[u]=0$的特征曲线族$\phi(x,y)=c_1$, $\psi(x,y)=c_2$;

2. 在区域$x>0,y>0$内求方程$L[u]=0$的通解;

3. 求定解问题

$$\begin{cases} L[u]=0,\ (x>0,xy>1,y>x), \\ u|_{xy=1}=\frac{1}{x^2}, \\ u|_{y=x^2}=x^2. \end{cases}$$

参考公式

1. 在柱坐标(r,θ,z)下,

$$\Delta_3 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$$

2. 在球坐标(r,θ,ϕ)下,

$$\Delta_3 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

3. ν阶Bessel方程$x^2y''+xy'+(x^2-\nu^2)y=0$, 再$0<x<+\infty$上得基础解组为$J_\nu(x),N_\nu(x)$, 其中

$$J_\nu(x)=\sum_{l=0}^{+\infty}(-1)^k\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}.$$

2003-2004学年第一学期数理方程B期末试题

一. (20分)解定解问题

$$\begin{cases} u_{tt} - u_{xx} = \sin 2x, \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = 0, \ u_t|_{t=0} = 6x^2. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} \Delta_3 u = 0, \ (1 < r < 2, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi), \\ u|_{r=1} = 1 + \cos^2\theta, \\ u_r|_{r=2} = 0. \end{cases}$$

三. (20分)解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + u, \ (t > 0, 0 < x < 1), \\ u|_{x=0}\text{有界}, \ u|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

四. (20分)解定解问题

$$\begin{cases} u_t = a^2 u_{xx} + b u_x + cu + f(t, x), \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x). \end{cases}$$

其中a, b, c为常数.

五. (20分)求平面区域$D : x > 0, y > 0$的格林函数$G(x, y; \xi, \eta)$, 并求下列定解问题的解:

$$\begin{cases} \Delta_2 u = -f(M), \ M(x, y) \in D : x > 0, y > 0, \\ u|_l = \varphi(M), \ M(x, y) \in l : l\text{为}D\text{的边界}. \end{cases}$$

注: $\Delta_3 u = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u}{\partial\varphi^2}$.

2005-2006学年第一学期数理方程B期末试题

一. (30分)填空

1. 方程$u_{xy}+u_y=1$的通解是________________________.

2. 固有值问题: $y''+\lambda y=0, y(0), y'(\pi)=0$的固有值$\lambda_n=$ ________________________, 对应的固有函数$y_n(x)=$ ________________________.

3. 设$P_{2006}(x)$是2006阶勒让德多项式, 计算$\int_{-1}^{1} 2^{2005}P_{2006}(x)dx=$ ________________________.

4. 计算$\delta(x-a)$的傅里叶变换$F(\delta(x-a))=$ ________________________.

5. 试将函数$f(x)=x^3(-1<x<1)$按勒让德多项式展开: $f(x)=$ ________________________.

二. (15分)求解定解问题

$$\begin{cases} u_{tt}=u_{xx}+2x,\ (t>0, -\infty<x<+\infty), \\ u(0,x)=0,\ u_t(0,x)=0. \end{cases}$$

三. (15分)求解定解问题

$$\begin{cases} u_t=u_{xx},\ (t>0, 0<x<\pi), \\ u(t,0)=0,\ u(t,\pi)=100, \\ u(0,x)=\frac{100}{\pi}x+\delta(x-\frac{\pi}{x}). \end{cases}$$

四. (15分)求解定解问题

$$\begin{cases} \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial t^2},\ (0<x<l, t>0), \\ u(t,0)\text{有限},\ u(t,l)=0, \\ u|_{t=0}=f(x),\ u_t|_{t=0}=0. \end{cases}$$

五. (10分)

1. 求出区域$D=\{(x,y):x^2+y^2<1, y>0\}$上的格林函数$G(x,y;\xi,\eta), (\xi,\eta)\in D$, 即求解定解问题
$$\begin{cases} \Delta_2 G=-\delta(x-\xi, y-\eta),\ (x,y)\in D, \\ G(x,y)|_{x^2+y^2=1}=0,\ G(x,0)=0. \end{cases}$$

2. 写出定解问题
$$\begin{cases} \Delta_2 u=-f(x,y),\ (x,y)\in D, \\ u(x,y)|_{x^2+y^2=1}=0,\ u(x,0)=\phi(x) \end{cases}$$
的解的积分表达式.

六. (15分)

1. 求出方程$u_t = a^2 u_{xx} + bu$的柯西问题的基本解$U(t,x)$, 其中a和b是常数, 即求定解问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, \ (t > 0, -\infty < x < +\infty), \\ u(0,x) = \delta(x). \end{cases}$$

2. 求解柯西问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, \ (t > 0, -\infty < x < +\infty), \\ u(0,x) = 1 + x^2. \end{cases}$$

参考公式

1. 勒让德方程式$(1-x^2)y'' - 2xy' + n(n+1)y = 0$, $(n = 0, 1, 2, \cdots, -1 < x < 1)$; 勒让德多项式: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, 特别地, $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1), P_3(x) = \frac{1}{2}(5x^2-3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

2. 贝塞尔方程是$x^2y'' + xy' + (x^2 - \nu^2)y = 0$, $(\nu \geq 0, 0 < x < a)$, 贝塞尔函数具有微分关系式:
$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$$
和
$$\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\frac{J_{\nu+1}(x)}{x^\nu}.$$
贝塞尔函数在第一、二类边界条件下的模平方$N_\nu^2 = \int_0^n xJ_\nu^2(\omega x)dx$分别是
$$N_{\nu 1}^2 = \frac{a^2}{2}J_{\nu+1}^2(\omega a), \ N_{\nu 2}^2 = \frac{1}{2}\left[a^2 - \left(\frac{\nu}{\omega}\right)^2\right]J_\nu^2(\omega a).$$

3. 积分$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$. $f(x)$的傅里叶变换定义为$F(\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}dx$. $F(\lambda) = e^{-a|\lambda|}$的傅里叶反变换是 $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x}d\lambda = \frac{a}{\pi(x^2+a^2)}$, $F(\lambda) = e^{-\lambda^2 t}$的傅里叶反变换是$f(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$.

2006-2007学年第一学期数理方程B期末试题

一. (20分)求解定解问题

$$\begin{cases} u_{tt} - u_{xx} = x + t, \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \sin x, \ u_t|_{t=0} = 4x. \end{cases}$$

二. (20分)求解定解问题

$$\begin{cases} \Delta_3 u = 0, \ (1 < r < 2), \\ u|_{r=1} = 0, \ u|_{r=2} = 1 + \cos\theta, \end{cases}$$

其中(r, θ, φ)为球坐标.

三. (24分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1.
$$\begin{cases} Y''(x) + \lambda Y(x) = 0, \ (0 < x < 1), \\ Y'(0) = Y'(1) = 0. \end{cases}$$

2.
$$\begin{cases} x^2 Y'' + xY' + (\lambda x^2 - 1)Y = 0, \ (0 < x < b), \\ |Y(0)| < +\infty, \ Y(b) = 0. \end{cases}$$

3.
$$\begin{cases} \Delta_2 u + \lambda u = 0, \ (0 < x < 2, 0 < y < 3), \\ u|_{x=0} = u|_{x=2} = u|_{y=0} = u|_{y=3} = 0. \end{cases}$$

四. 设初值问题

$$(*) \begin{cases} u_t = 2u_x + f(t, x), \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x). \end{cases}$$

1. (10分)求上述初值问题的基本解$U(t, x)$.

2. (10分)求初值问题$(*)$的解.

五. 设平面区域$D = \{(x, y)|y > x\}$,

1. (10分)求D内格林函数G:

$$\begin{cases} \Delta_2 G = -\delta(x - \xi, y - \eta),\ ((x, y) \in D, (\xi, \eta) \in D), \\ G|_{y=x} = 0. \end{cases}$$

2. (6分)求边值问题

$$\begin{cases} \Delta_2 u = -f(x, y), ((x, y) \in D), \\ u|_{y=x} = \varphi(x) \end{cases}$$

   的解.

参考公式

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

2013-2014学年第二学期数理方程B期末试题

一. (16分)求下列偏微分方程的通解$u=u(x,y)$:

1. $\dfrac{\partial^2 u}{\partial x\partial y}=x^2y.$

2. $y\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial u}{\partial x}=xy.$

二. (10分)求下列固有值问题的解, 要求明确指出固有值及其所对应的固有函数:

$$\begin{cases} x^2y''+xy'+\lambda x^2y=0,\ (0<x<2),\\ |y(0)|<+\infty,\ y'(2)=0. \end{cases}$$

三. (12分)求第一象限$D=\{(x,y)\in\mathbb{R}^2|x>0,y>0\}$的第一边值问题的Green函数.

四. (12分)用积分变换法求解下列方程:

$$\begin{cases} u_t=a^2u_{xx}+u,\ (-\infty<x<+\infty,t>0),\\ u(0,x)=\varphi(x). \end{cases}$$

五. (15分)用分离变量法求解下列方程:

$$\begin{cases} \Delta_2 u=0,\ (r<2),\\ u|_{r=2}=\sin\theta+2\sin 5\theta-7\cos 4\theta. \end{cases}$$

六. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{tt}=4u_{xx},\ (0<x<1,t>0),\\ u(t,0)=0,\ u(t,1)=1,\\ u(0,x)=\varphi(x)+x,\ u_t(0,x)=\delta(x-\frac{1}{2}). \end{cases}$$

七. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{xx}+u_{yy}+u_{zz}=z,\ (x^2+y^2+z^2<1),\\ u|_{x^2+y^2+z^2=1}=0 \end{cases}$$

八. (5分)求解下列定解问题:

$$\begin{cases} 4u_{xx}=u_{tt}+2u_t+u,\ (-\infty<x<+\infty,t>0),\\ u(0,x)=2\cos x,\ u_t(0,x)=2x. \end{cases}$$

提示: 先对泛定方程进行变换成为一个较为简单的泛定方程, 再根据初始条件进行求解.

参考公式: 包括极坐标和球坐标下的Laplace算子表达式, Fourier级数及其系数的公式, Laplace和Fourier所有性质和变换公式及求解过程中用到的反变换公式, 勒让德方程的固有值和固有函数以及勒让德函数n=1-5时的表达式.

注: 本卷为考后回忆版本, 未给具体公式内容, 请同学自行参考其它卷子的相关公式.

2015-2016学年第二学期数理方程B期末试题

一. (12分)求以下固有值问题的固有值和固有函数:

$$\begin{cases} Y''(x) + \lambda Y(x) = 0,\ (0 < x < 16), \\ Y'(0) = 0,\ Y'(16) = 0. \end{cases}$$

二. (16分)利用分离变量法求解定解问题:

$$\begin{cases} u_t = 4u_x x,\ (t > 0, 0 < x < 5), \\ u(t,0) = u(t,5) = 0, \\ u(0,x) = \phi(x). \end{cases}$$

并求$\phi(x) = \delta(x-2)$时此定解问题的解.

三. (14分)考虑初值问题:

$$\begin{cases} u_{tt} = 4u_{xx} + f(t,x),\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2,\ u_t|_{t=0} = \sin 2x. \end{cases}$$

1. 如取$f(t,x) = 0$, 求此初值问题的解.

2. 如取$f(t,x) = t^2x^2$, 求此初值问题相应的解.

四. (14分)求解以下初值问题

$$\begin{cases} u_{tt} = 4u_{xx} + 5u,\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \phi(x). \end{cases}$$

并指出当$\phi(x) = e^{-x^2}$时此定解问题的解.

五. (16分)求解以下定解问题:

$$\begin{cases} u_t = u_{rr} + \frac{1}{r}u_r,\ (0 < r < 1), \\ |u(t,0)| < +\infty,\ u(t,1) = 0, \\ u|_{t=0} = \phi(r). \end{cases}$$

并算出$\phi(r) = J_0(ar) + 3J_0(br)$时的解, 其中$0 < a < b$, 且$J_0(a) = J_0(b) = 0$.

六. (14分)已知下半空间$V=\{(x,y,z)|x<0,-\infty<x,y<+\infty)\}$.

1. 求出V内泊松方程第一边值问题的Green函数.

2. 求解定解问题:

$$\begin{cases} 4u_{xx}+u_{yy}+u_{zz}=0,\ (z<0,-\infty<x,y<+\infty), \\ u|_{z=0}=\varphi(x,y). \end{cases}$$

七. (6分)对于三维波动方程

$$u_{tt}=a^2\Delta_3 u,\ (a>0,t>0,-\infty<x,y,z<+\infty)$$

它的形如$u=u(t,r)=T(t)R(r)$的解称为方程的可分离变量的径向对称解, 求方程满足$\lim\limits_{t\to+\infty}u=0$的可分离变量的径向对称解, 这里$r=\sqrt{x^2+y^2+z^2}$.

八. (8分)考虑固有值问题

$$\begin{cases} \frac{d}{dx}[(1-x^2)y']+\lambda y=0,\ (0<x<1), \\ y'(0)=0,\ |y(1)|<+\infty. \end{cases}$$

1. 求此固有值问题的固有值和固有函数.

2. 把$f(x)=2x+1$按此固有值问题所得到的固有函数系展开.

参考公式

1. 直角坐标系: $\Delta_3 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$, 柱坐标系: $\Delta_3 u=\frac{1}{r}r\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$,
   球坐标系: $r^2\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$.

2. 若ω是$J_\nu(\omega a)=0$的一个正根, 则有模平方$N_{\nu 1}^2=||J_\nu(\omega x)||_1^2=\frac{a^2}{2}J_{\nu+1}^2(\omega a)$.
   若ω是$J_\nu'(\omega a)=0$的一个正根, 则有模平方$N_{\nu 2}^2=||J_\nu(\omega x)||_2^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega^2}\right]J_\nu^2(\omega a)$.

3. 勒让德多项式: $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, $\int_{-1}^1 P_n^2(x)dx=\frac{2}{2n+1}, n=0,1,2,\cdots$,
   母函数: $(1-2xt+t^2)^{-\frac{1}{2}}=\sum\limits_{n=0}^{+\infty}P_n(x)t^n$, 递推公式: $P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$.

4. $\displaystyle\frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2 t}\right)$

5. 设$G(M;M_0)$是三维Poisson方程第一边值问题

$$\begin{cases} \Delta_3 u=-f(M),\ (M=(x,y,z)\in V), \\ u|S=\phi(M) \end{cases}$$

   对应的Green函数,则

$$u(M_0)=-\iint_S \phi(M)\frac{\partial G}{\partial n}(M;M_0)dS+\iiint_V f(M)G(M;M_0)dM,$$

   其中$M_0=(\xi,\eta,0)$.

2016-2017学年第二学期数理方程B期末试题

一. (10分)求方程$u_x + yu_{xy} = 0$的一般解.

二. (10分)求解一维半无界弦的自由振动问题:

$$\begin{cases} u_{tt} = 9u_{xx},\ (t > 0, 0 < x < +\infty), \\ u|_{x=0} = 0, \\ u|_{t=0} = x,\ u_t|_{t=0} = 2\sin x. \end{cases}$$

三. (20分)考察一维有界限振动问题:

$$\begin{cases} u_{tt} = u_{xx} + f(t,x),\ (t > 0, 0 < x < \pi), \\ u|_{x=0} = 0,\ u_x|_{x=\pi} = 0, \\ u|_{t=0} = \sin\frac{3}{2}x,\ u_t|_{t=0} = \sin\frac{x}{2}. \end{cases}$$

1. 当$f(t,x) = 0$时, 求出上述定解问题的解$u_1(x)$.

2. 当$f(t,x) = \sin\frac{x}{2}\sin\omega t,\ (\omega \neq k + \frac{1}{2}, k \in \mathbb{N})$时, 求出上述定解问题的解$u_2(t,x)$.

3. 指出定解问题中方程非齐次项$f(t,x)$, 边界条件和初始条件的物理意义.

四. (15分)求解定解问题:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + u,\ (t > 0, 0 < x < 1), \\ u|_{x=0}\text{有界},\ u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

五. (15分)求解如下泊松方程的边值问题:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = z,\ (x^2 + y^2 + z^2 < 1), \\ u|_{x^2+y^2+z^2=1} = 0. \end{cases}$$

六. (15分)设区域$\Omega = \{(x,y)|y \geq x\}$.

1. 求区域Ω上的Poisson方程Dirichlet边值问题的Green函数.

2. 求解如下Poisson方程的Dirichlet边值问题:

$$\begin{cases} \Delta_2 u = 0,\ ((x,y) \in \Omega), \\ u(x,x) = \phi(x). \end{cases}$$

七. (15分)考察定解问题:

$$\begin{cases} u_t = 4u_{xx} + 3u, \ (-\infty < x < +\infty, t > 0), \\ u(0,x) = \varphi(x). \end{cases}$$

1. 求出上述定解问题相应的基本解.

2. 当$\varphi(x) = x$时, 求解上述定解问题.

参考公式

1. 拉普拉斯算子Δ_3在各个坐标系下的表达形式
$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}.$
2. 二阶欧拉方程: $x^2y'' + pxy' + qy = f(x)$, 在作变量代换$x = e^t$下,可以约化为常系数线性微分方程: $\frac{d^2y}{dt^2} + (p-1)\frac{dy}{dt} + qy = f\left(e^t\right)$.
3. Legendre方程: $\left[(1-x^2)y'\right]' + \lambda y = 0$; n阶Legendre多项式:
$P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n-2k)!}{2^nk!(n-k)!(n-2k)!}x^{n-2k} = \frac{1}{2^nn!}\frac{d^n}{dx^n}(x^2-1)^n$
; Legendre多项式的母函数: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n, \ (|t|<1)$;
Legendre多项式的模平方: $||P_n(x)||^2 = \frac{2}{2n+1}$.
4. ν阶Bessel方程: $x^2y'' + xy' + (x^2-\nu^2)y = 0$; ν阶Bessel函数: $J_\nu(x) = \sum_{l=0}^{+\infty}(-1)^k\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$
Bessel函数的母函数: $e^{\frac{x}{2}}(\zeta - \zeta^{-1}) = \sum_{n=-\infty}^{+\infty} J_n(x)\zeta^n$; Bessel函数在三类边界条件下的模平方: $N_{\nu 1n}^2 = \frac{a^2}{2}J_{\nu+1}^2(\omega_{1n}a), N_{\nu 2n}^2 = \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_{2n}^2}\right]J_\nu^2(\omega_{2n}a), N_{\nu 3n}^2 = \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_{2n}^2} + \frac{a^2\alpha^2}{\beta^2\omega_{3n}^2}\right]J_\nu^2(\omega_{3n}a)$.
5. 傅里叶变换和逆变换: $\mathcal{F}[f](\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}dx$; $\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x}d\lambda$; $\mathcal{F}^{-1}[e^{-\lambda^2}] = \frac{1}{2\sqrt{\pi}}e^{-\frac{x^2}{4}}$.
6. 拉普拉斯变换: $L[f(t)] = \int_0^{+\infty}, p = \sigma + is$; $L[e^{\alpha t}] = \frac{1}{p-\alpha}$, $L[t^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$, $L[\sin t] = \frac{1}{p^2+1}$, $L[\cos t] = \frac{p}{p^2+1}$, $L\left[\frac{1}{\sqrt{\pi t}}e^{\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}$.
7. 拉普拉斯方程$\Delta_3 u = \delta(M)$的基本解:
二维, $U(x,y) = -\frac{1}{2\pi}\ln\frac{1}{r}, \ r = \sqrt{x^2+y^2}$;
三维, $U(x,y,z) = -\frac{1}{4\pi r}, \ r = \sqrt{x^2+y^2+z^2}$.

2019-2020学年第二学期数理方程B期末试题(毕业年级重修)

一. (18分)求解下列Cauchy问题:

1.

$$\begin{cases} u_{tt} = 4u_{xx},\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2,\ u_t|_{t=0} = \cos 2x. \end{cases}$$

2.

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 20, \\ u(0, y) = y^2,\ u(x, 0) = \sin x. \end{cases}$$

二. (18分)求以下固有值问题的固有值和固有函数:

1.

$$\begin{cases} Y''(x) + \lambda Y(x) = 0,\ (0 < x < \pi), \\ Y'(0) = 0,\ Y'(\pi) = 0. \end{cases}$$

2.

$$\begin{cases} x^2 Y''(x) + xY'(x) + \lambda Y(x) = 0,\ (1 < x < b), \\ Y(1) = 0,\ Y'(b) = 0. \end{cases}$$

三. (18分)

1. 求周期边界条件下

$$\begin{cases} u_{tt} = u_{xx},\ (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1),\ u_x(t, 0) = u_x(t, 1) \end{cases}$$

的分离变量解$u = T(t)X(x)$.

2. 求解

$$\begin{cases} u_{tt} = u_{xx},\ (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1),\ u_x(t, 0) = u_x(t, 1), \\ u(0, x) = \sin 2\pi x,\ u_t(0, x) = 2\pi \cos 2\pi x. \end{cases}$$

四. (14分)求解

$$\begin{cases} u_t = u_{xx} + u,\ (t > 0, \infty < x < +\infty), \\ u|_{t=0} = \delta(x + 1). \end{cases}$$

五. (18分)

1. P_n为n-阶勒让德函数, 写出$P_0(x), P_1(x), P_2(x)$, 并计算积分$\int_{-1}^{1}(20+x)P_2(x)dx$.

2. 求解以下定解问题, 其中(r,θ,ϕ)为球坐标:

$$\begin{cases}\Delta_3 u = 0,\ (r<2),\\ u|_{r=2} = 3\cos 2\theta.\end{cases}$$

六. (14分)已知平面区域$D=\{(x,y)|-\infty<x<+\infty, y<1\}$.

1. 写出D内泊松方程第一边值问题的Green函数所满足的定解问题, 并求出Green函数.

2. 求解定解问题:

$$\begin{cases}u_{xx}+a^2u_{yy}=0,\ (-\infty<x<+\infty, y<1),\\ u|_{y=1}=\varphi(x).\end{cases}$$

参考公式

1. 直角坐标系: $\Delta_3 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$,
   柱坐标系: $\Delta_3 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$,
   球坐标系: $\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$.

2. 若ω是$J_\nu(\omega a)=0$的一个正根, 则有模平方 $N_{\nu 1n}^2 = \frac{a^2}{2}J_{\nu+1}^2(\omega a)$.
   若ω是$J'_\nu(\omega a)=0$的一个正根, 则有模平方$N_{\nu 2n}^2 = \frac{1}{2}\left[a^2-\frac{\nu^2}{\omega^2}\right]J_\nu^2(\omega a)$.

3. 勒让德多项式: $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, $n=0,1,2,3,\cdots$, 母函数: $(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}P_n(x)t^n$, $(|t|<1)$, 递推公式: $P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x)$.

4. $\frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}e^{i\lambda x}d\lambda = \frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}}\exp(-\frac{x^2}{4a^2t})$

5. 二维泊松方程基本解为$u=\frac{1}{2\pi}\ln r$, 这里(r,θ)为极坐标.

6. 由平面区域D内Poisson方程第一边值问题的Green函数$G(M;M_0)$, 求得Poisson方程第一边值问题解$u(M)$的公式是:

$$u(M)=\int_S \varphi(M_0)\frac{\partial G}{\partial n}(M;M_0)dS+\iint_D f(M_0)G(M;M_0)dM_0,$$

   其中S是D的边界.

考察定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + f(t,x),\ (0 < x < \pi, t > 0), \\ u|_{x=0} = 0, u|_{x=\pi} = 0,\ (t > 0), \\ u|_{t=0} = \phi(x), u_t|_{t=0} = \psi(x),\ (0 < x < \pi). \end{cases}$$

1) 当$f(t,x) = 0$时, 求此定解问题的解u_1;

2) 当$f(t,x) = \sin 2x \sin \omega t$(其中$\omega \neq 4$), $\phi(x) = 0, \psi(x) = 0$时, 求此定解问题的解u_2.