Introduction to Algorithms

Advanced Data Structures: II

Xiang-Yang Li and Haisheng Tan

School of Computer Science and Technology
University of Science and Technology of China (USTC)

Fall Semester 2021

Outline of Topics

Mergeable Heap (min-heap by default)

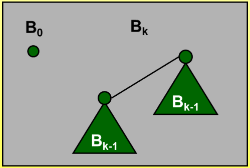
- A data structure supports the following operations:
  1. MAKE-HEAP(): Create and return a new heap containing no elements
  2. INSERT(H,x): Insert element x
  3. MINIMUM(H): Return min element
  4. EXTRACT-MIN(H): Return and delete minimum element
  5. UNION(H_1,H_2): Create and return a new heap that contains all the elements of heaps H_1 and H_2.
- Some other operations: Decrease key of element x to k; Delete an element.
- Applications: Dijkstra's shortest path algorithm, Prim's MST algorithm, Event-driven simulation, Huffman encoding, Heapsort. . .

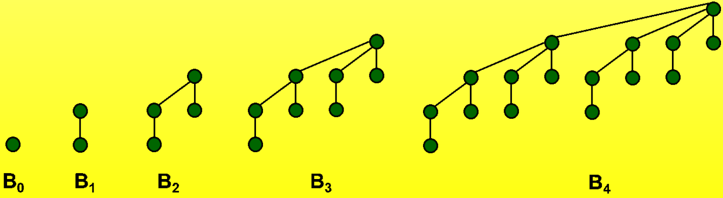
Mergeable Heap

		Heaps			
Operation	**Linked List**	**Binary**	**Binomial**	**Fibonacci**	**Relaxed**
make-heap	1	1	1	1	1
insert	1	log N	log N	1	1
find-min	N	1	log N	1	1
delete-min	N	log N	log N	log N	log N
union	1	N	log N	1	1
decrease-key	1	log N	log N	1	1
delete	N	log N	log N	log N	log N
is-empty	1	1	1	1	1

Binomial Tree

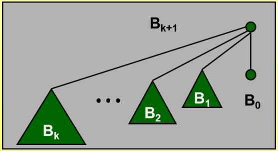
- Recursive definition: B_0 is a single node. B_k consists of 2 binomial trees B_{k-1} linked together, where the root of one subtree is the leftmost child of the other.



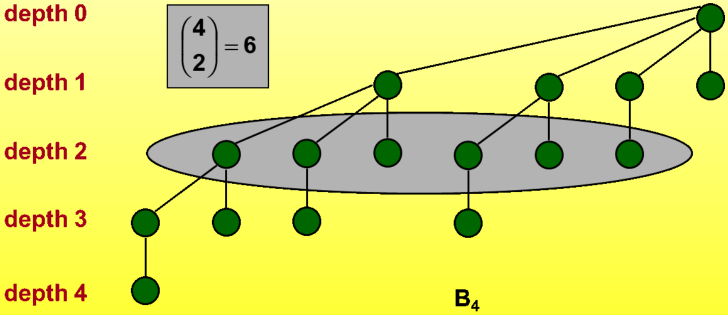


Useful Properties

- For order k binomial tree B_k
  1. Number of nodes $= 2^k$
  2. Height $= k$
  3. Degree of root $= k$
  4. Deleting root yields binomial trees $B_{k-1}, \ldots, B_0$
  5. B_k has $\binom{k}{i}$ nodes at depth i
- Proved by induction.

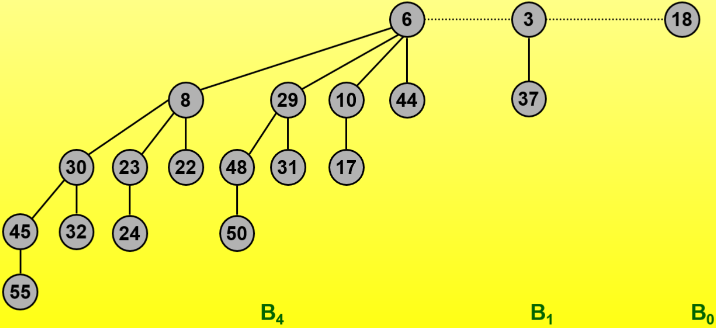


Useful Properties - Example



Binomial Heap: Overview

- Sequence of binomial trees that satisfy binomial heap property:
  1. Each tree is min-heap ordered
  2. 0 or 1 binomial tree of order k can be included.



Binomial Heap: Implementation

- Represent trees using left-child, right sibling pointers.
  Three links per node: *parent*, *left* (left-most child), *right* (right sibling).
- Roots of trees connected with singly linked list.
  Degrees of trees strictly increasing as we traverse the root list.

Binomial Heap: Implementation

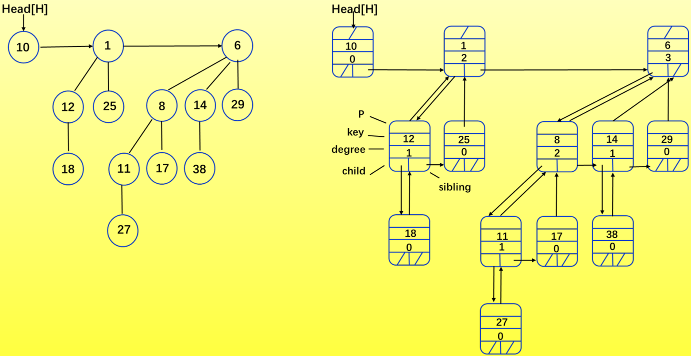
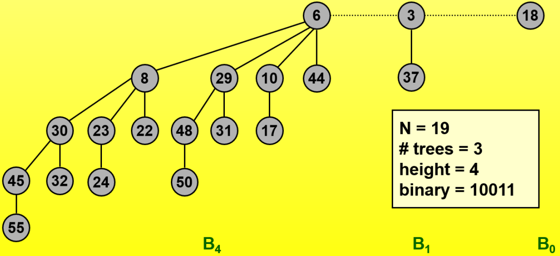


Figure: A binomial heap H and its more detailed representation. The heap consists of binmial tree B_0, B_2 and B_3 which have 1,4 and 8 nodes respectively.

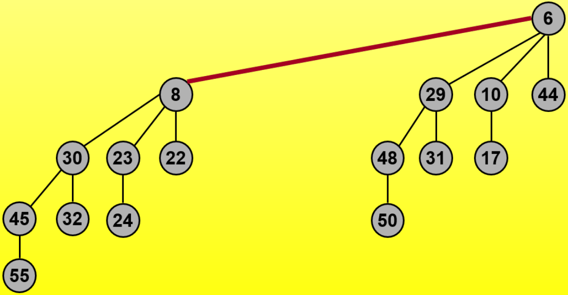
Binomial Heap: Properties

- Properties of N-node binomial heap
  1. Min key contained in root of B_0, B_1, ..., B_k
  2. Contains binomial tree B_i iff $b_i = 1$ where $b_n \cdot b_2 b_1 b_0$ is binary representation of $N = \sum_{i=0}^{\lfloor \log N \rfloor} b_i 2^i$.
  3. At most $\lfloor \log N \rfloor + 1$ binomial trees.
  4. Height $\leq \lfloor \log N \rfloor$

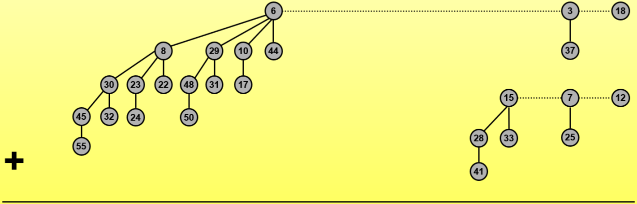


Binomial Heap: Union

- Create H that is union of heaps H' and H'' (in $O(1)$ time):
  1. "Mergeable heaps"
  2. Easy if H' and H'' are each an order k binomial tree.
     a. connect roots of H' and H''
     b. choose smaller key to be root of H



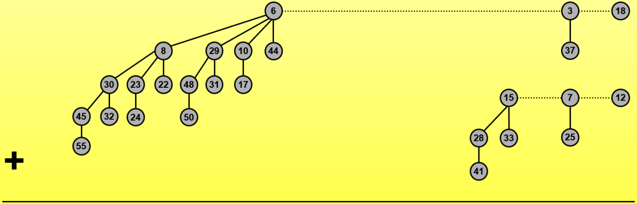
Binomial Heap: Union



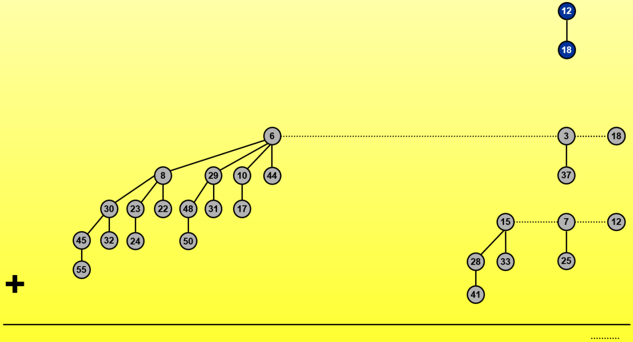
19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

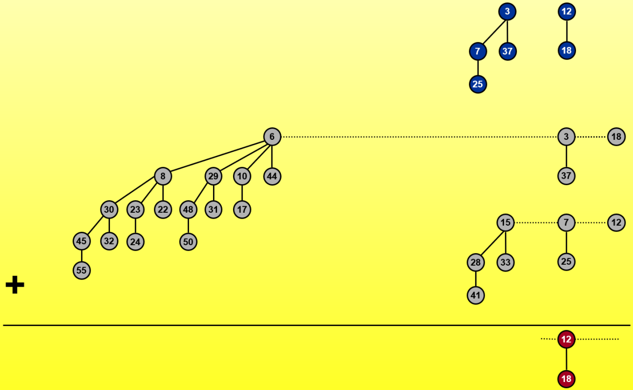
Binomial Heap: Union



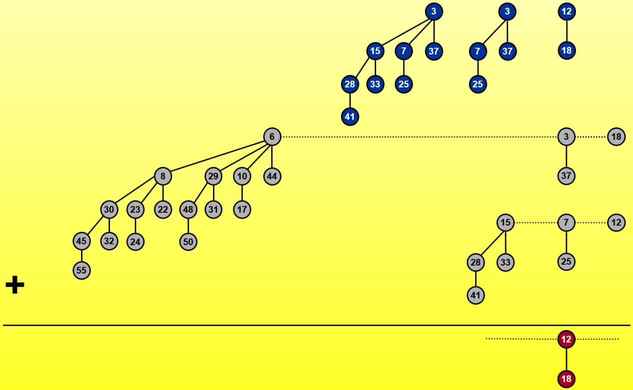
Binomial Heap: Union



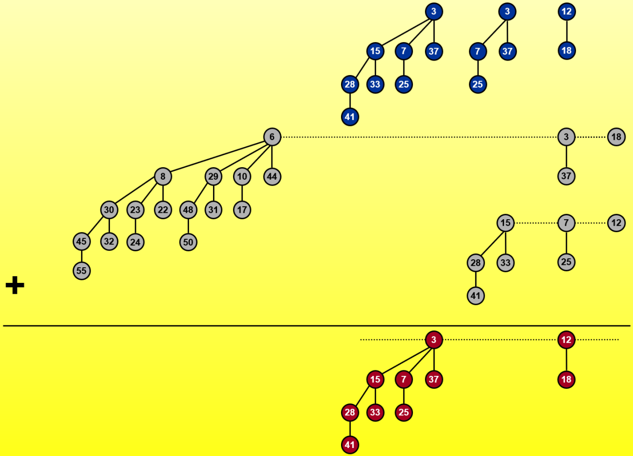
Binomial Heap: Union



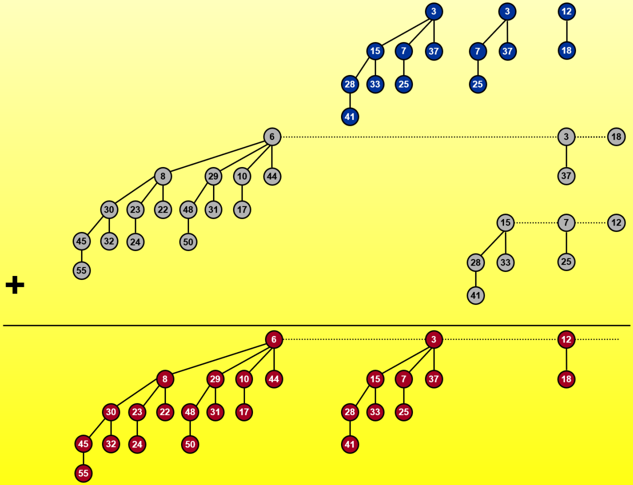
Binomial Heap: Union



Binomial Heap: Union



Binomial Heap: Union

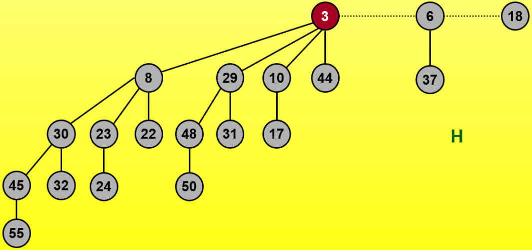


Analysis of Union

- Create heap H that is union of heaps H' and H"
  Analogous to binary addition.
- Running time: $O(\log N)$
  Proportional to number of trees in root lists
  $\lfloor \log N' \rfloor + 1 + \lfloor \log N'' \rfloor + 1 \leq 2(\lfloor \log N \rfloor + 1)$

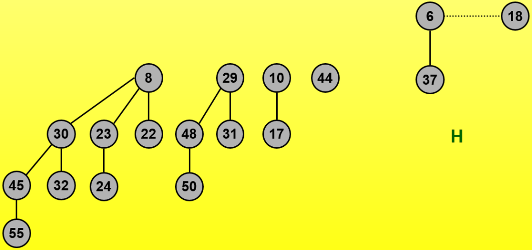
Binomial Heap: Delete Min

- Delete node with minimum key in binomial heap H:
  1. Find root x with min key in root list of H, and delete
  2. $H' \leftarrow$ broken binomial trees
  3. $H \leftarrow \textsc{Union}(H', H)$
- Running time: $O(\log N)$



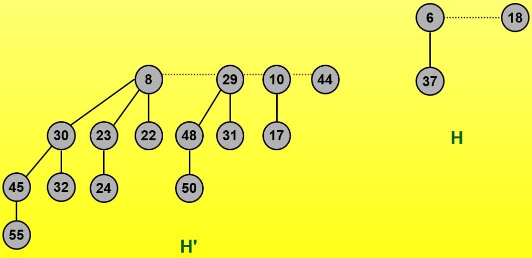
Binomial Heap: Delete Min

- Delete node with minimum key in binomial heap H:
  1. Find root x with min key in root list of H, and delete
  2. $H' \leftarrow$ broken binomial trees
  3. $H \leftarrow \textsc{Union}(H', H)$
- Running time: $O(\log N)$



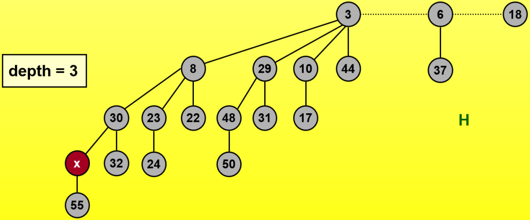
Binomial Heap: Delete Min

- Delete node with minimum key in binomial heap H:
  1. Find root x with min key in root list of H, and delete
  2. $H' \leftarrow$ broken binomial trees
  3. $H \leftarrow \textsc{Union}(H', H)$
- Running time: $O(\log N)$



Binomial Heap: Decrease Key

- Decrease key of node x in binomial heap H:
  1. Suppose x is in binomial tree B_k
  2. Bubble node x up the tree if x is too small
- Running time: $O(\log\ N)$
  Proportional to depth of node $x \leq \lfloor \log_2 N \rfloor$

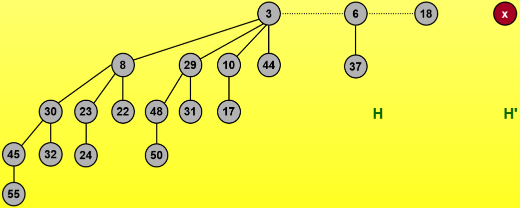


Binomial Heap: Delete

- Delete node x in binomial heap H:
  1. Decrease key of x to $-\infty$
  2. DELETEMIN
- Running time: $O(\log N)$

Binomial Heap: Insert

- Insert a new node x into binomial heap H
  1. $H' \leftarrow \textsc{MakeHeap}(x)$
  2. $H \leftarrow \textsc{Union}(H', H)$
- Running time: $O(\log N)$



Recall

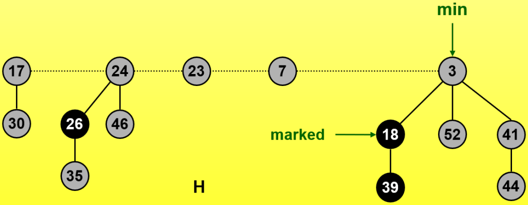
		Heaps			
Operation	**Linked List**	**Binary**	**Binomial**	**Fibonacci**	**Relaxed**
make-heap	1	1	1	1	1
insert	1	log N	log N	1	1
find-min	N	1	log N	1	1
delete-min	N	log N	log N	log N	log N
union	1	N	log N	1	1
decrease-key	1	log N	log N	1	1
delete	N	log N	log N	log N	log N
is-empty	1	1	1	1	1

Fibonacci Heaps: Overview

- Fibonacci heap history: Fredman and Tarjan (1986)
  1. Ingenious data structure and analysis
  2. Original motivation: $O(m + n \log n)$ shortest path algorithm, also led to faster algorithms for MST, weighted bipartite matching
  3. Still ahead of its time
- Fibonacci heap intuition:
  1. Similar to binomial heaps, but less structured
  2. Decrease-key and union run in $O(1)$ time (amortized)
  3. "Lazy" unions
- Fibonacci heaps are named after the Fibonacci numbers, which are used in their running time analysis.

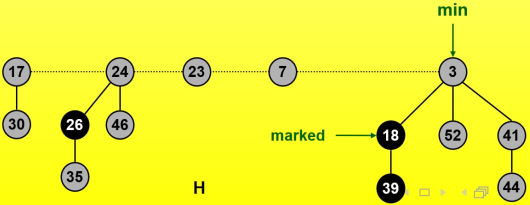
Fibonacci Heaps: Structure

- Fibonacci heap:
  Set of min-heap ordered trees



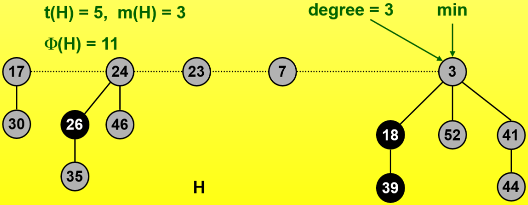
Fibonacci Heaps: Implementation

- Each node contains a pointer to its parent and a pointer to any one of its children. The children are linked together in a circular, doubly linked list:
  Can quickly splice off subtrees
- Roots of trees connected with circular doubly linked list:
  Fast union
- Pointer to root of tree with min element:
  Fast find-min



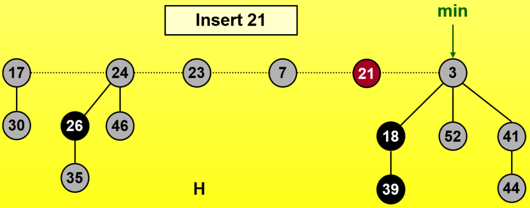
Fibonacci Heaps: Potential Function

- $Degree[x]$ = degree of node x
- $D(n)$ = max degree of any node in Fibonacci heap with n nodes
- $Mark[x]$ = mark of node x (black or gray)
- $t(H) = \#$ trees
- $m(H) = \#$ marked nodes
- $\Phi(H) = t(H) + 2m(H)$ = potential function



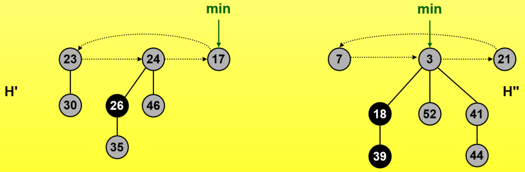
Fibonacci Heaps: Insert

- Insert:
  1. Create a new singleton tree
  2. Add to left of min pointer
  3. Update min pointer
- Running time: $O(1)$ amortized



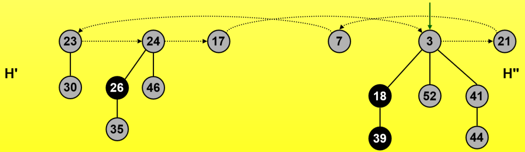
Fibonacci Heaps: Union

- Union:
  1. Concatenate two Fibonacci heaps
  2. Root lists are circular, doubly linked lists



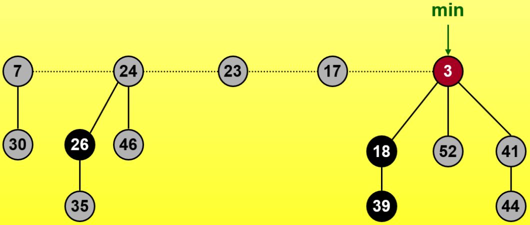
Fibonacci Heaps: Union

- Union:
  1. Concatenate two Fibonacci heaps
  2. Root lists are circular, doubly linked lists
- Concatenate the two root lists, and update the min pointer.
- Running time: $O(1)$ amortized



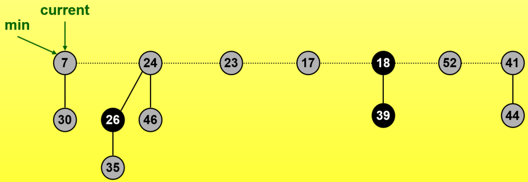
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



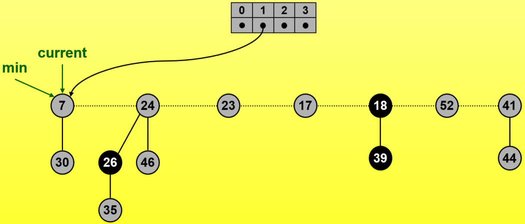
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



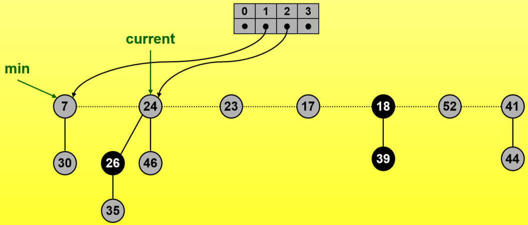
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



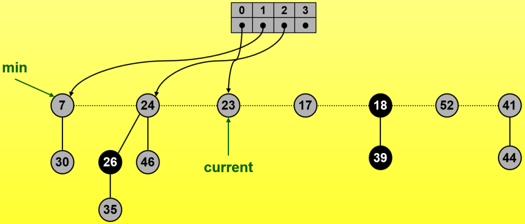
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



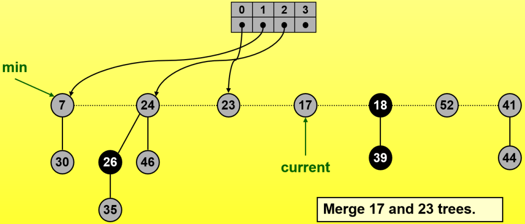
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



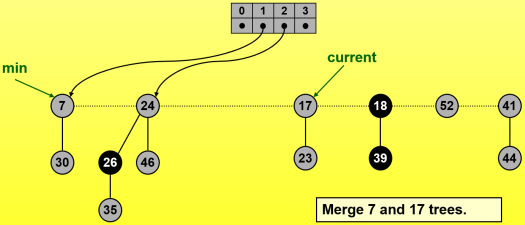
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



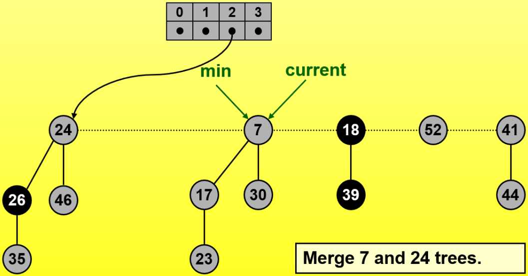
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



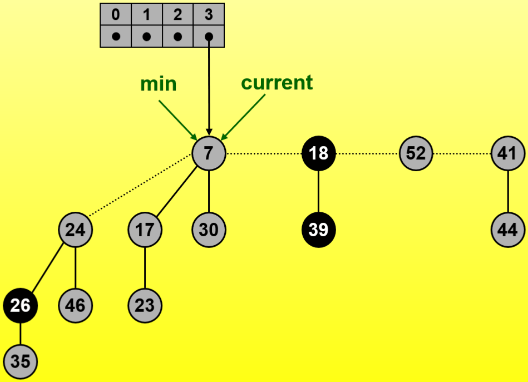
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



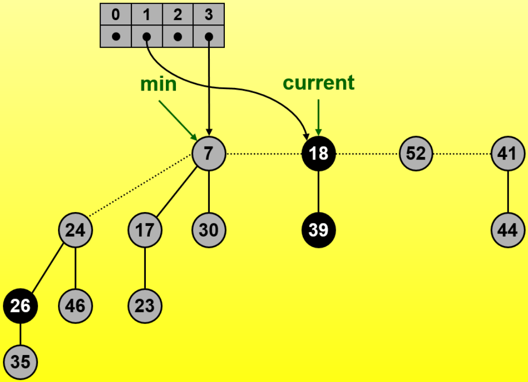
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



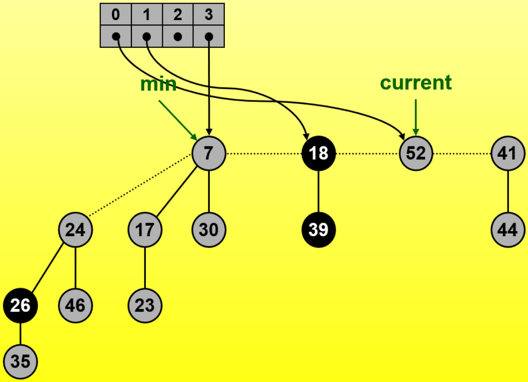
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



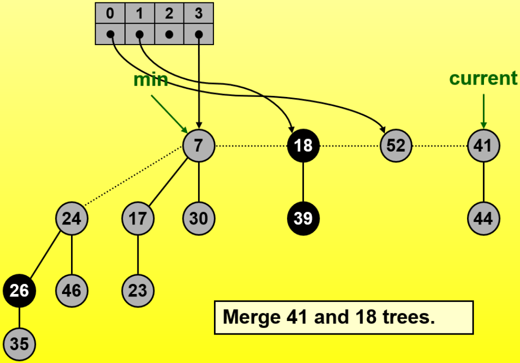
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



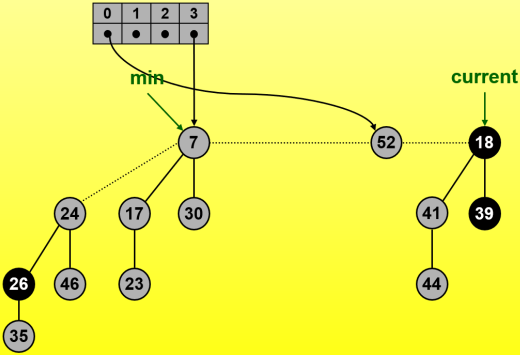
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



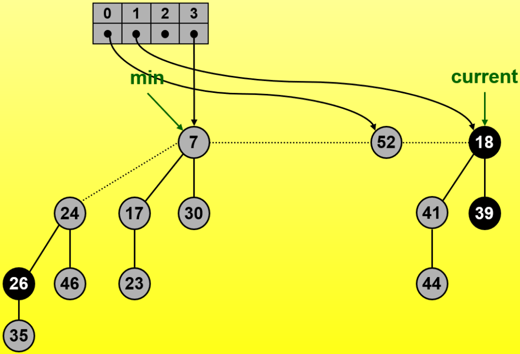
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



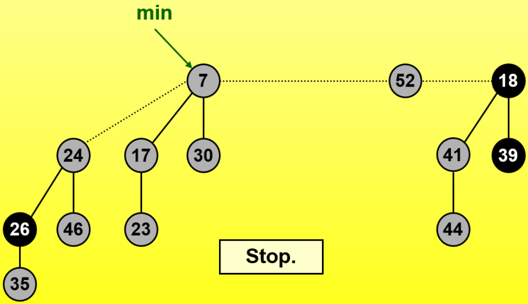
Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



Fibonacci Heaps: Delete Min

- Delete min and concatenate its children into root list
- Consolidate trees so that no two roots have same degree



Fibonacci Heaps: Delete Min Analysis

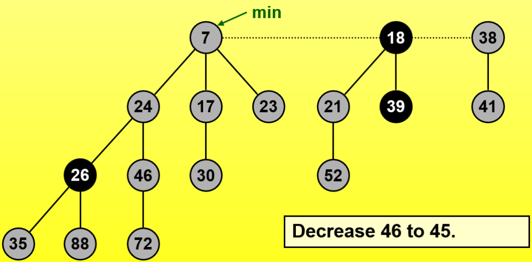
- Actual cost: $O(D(n) + t(H))$
  1. $O(D(n))$ work adding min's children into root list and updating min
  2. $O(D(n) + t(H))$ work consolidating trees
- Amortized cost: $O(D(n))$
  1. $t(H') \leq D(n) + 1$ since no two trees have same degree
  2. $\Delta\Phi(H) \leq D(n) + 1 - t(H)$

Fibonacci Heaps: Delete Min Analysis

- Is amortized cost of $O(D(n))$ good?
  1. Yes, if only Insert, Delete-min, and Union operations supported
     a. In this case, Fibonacci heap contains only binomial trees since we only merge trees of equal root degree
     b. This implies $D(n) \leq \lfloor \log_2 N \rfloor$
  2. Yes, if we support Decrease-key in clever way
     a. We'll show that $D(n) \leq \lfloor \log_\phi N \rfloor$ where ϕ is golden ratio
     b. Limiting ratio between successive Fibonacci numbers!

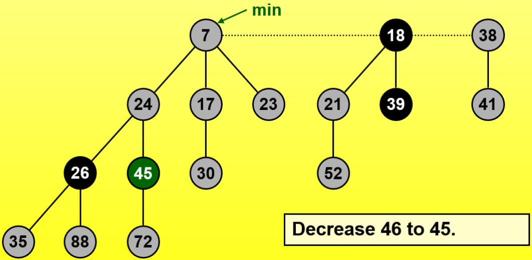
Fibonacci Heaps: Decrease Key

- **Case 0**: min-heap property not violated
  1. Decrease key of x to k
  2. Change heap min pointer if necessary



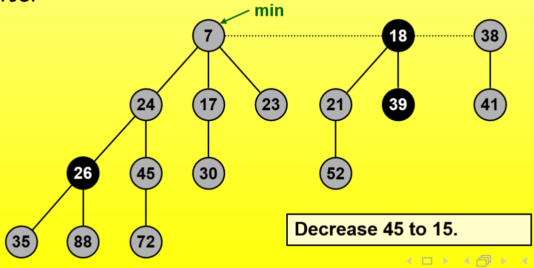
Fibonacci Heaps: Decrease Key

- **Case 0**: min-heap property not violated
  1. Decrease key of x to k
  2. Change heap min pointer if necessary



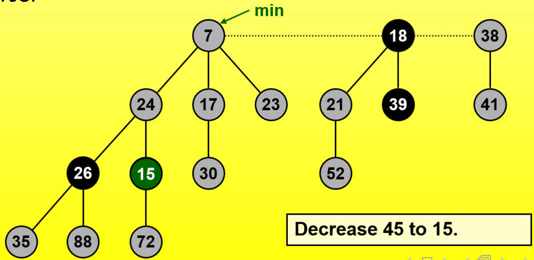
Fibonacci Heaps: Decrease Key

- **Case 1**: min-heap property violated; and parent of x is unmarked
  1. Decrease key of x to k
  2. Cut off link between x and its parent
  3. Mark parent
  4. Add tree rooted at x to root list, updating heap min pointer



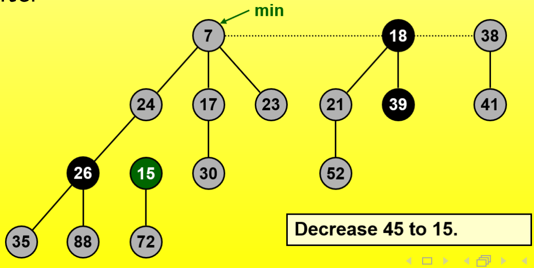
Fibonacci Heaps: Decrease Key

- **Case 1**:min-heap property violated; and parent of x is unmarked
  1. Decrease key of x to k
  2. Cut off link between x and its parent
  3. Mark parent
  4. Add tree rooted at x to root list, updating heap min pointer



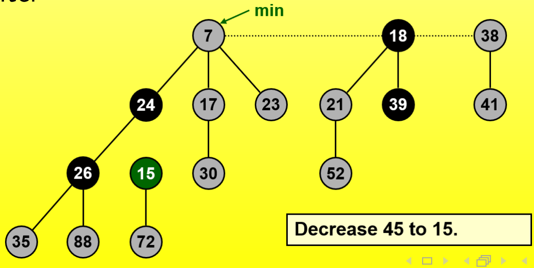
Fibonacci Heaps: Decrease Key

- **Case 1**:min-heap property violated; and parent of x is unmarked
  1. Decrease key of x to k
  2. Cut off link between x and its parent
  3. Mark parent
  4. Add tree rooted at x to root list, updating heap min pointer



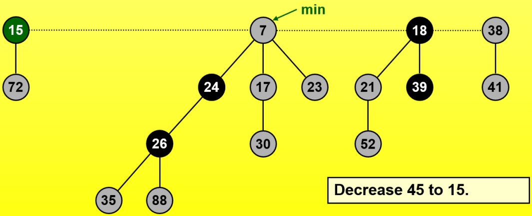
Fibonacci Heaps: Decrease Key

- **Case 1**:min-heap property violated; and parent of x is unmarked
  1. Decrease key of x to k
  2. Cut off link between x and its parent
  3. Mark parent
  4. Add tree rooted at x to root list, updating heap min pointer



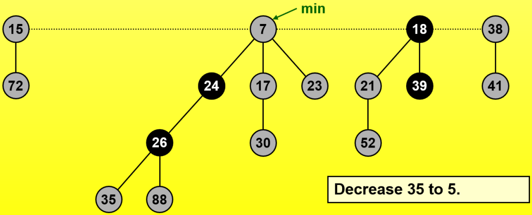
Fibonacci Heaps: Decrease Key

- **Case 1**:min-heap property violated; and parent of x is unmarked
  1. Decrease key of x to k
  2. Cut off link between x and its parent
  3. Mark parent
  4. Add tree rooted at x to root list, updating heap min pointer



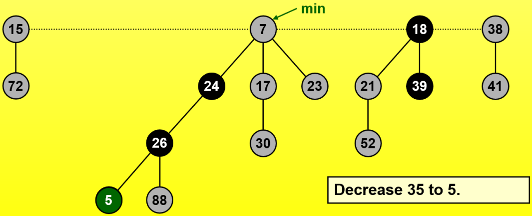
Fibonacci Heaps: Decrease Key

- **Case 2**:min-heap property violated; and parent of x is marked
  1. Decrease key of x to k
  2. Cut off link between x and its parent $p[x]$, and add x to root list
  3. Cut off link between $p[x]$ and $p[p[x]]$, add $p[x]$ to root list
     a. If $p[p[x]]$ unmarked, then mark it
     b. If $p[p[x]]$ marked, cut off $p[p[x]]$, unmark, and repeat



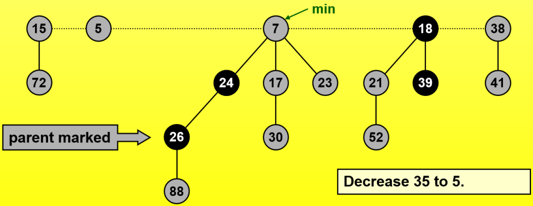
Fibonacci Heaps: Decrease Key

- **Case 2**:min-heap property violated; and parent of x is marked
  1. Decrease key of x to k
  2. Cut off link between x and its parent $p[x]$, and add x to root list
  3. Cut off link between $p[x]$ and $p[p[x]]$, add $p[x]$ to root list
     a. If $p[p[x]]$ unmarked, then mark it
     b. If $p[p[x]]$ marked, cut off $p[p[x]]$, unmark, and repeat



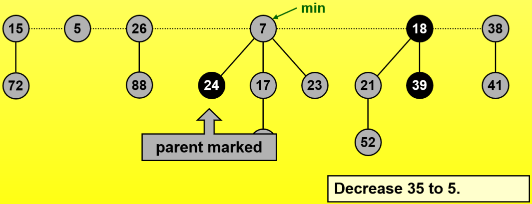
Fibonacci Heaps: Decrease Key

- **Case 2**:min-heap property violated; and parent of x is marked
  1. Decrease key of x to k
  2. Cut off link between x and its parent $p[x]$, and add x to root list
  3. Cut off link between $p[x]$ and $p[p[x]]$, add $p[x]$ to root list
     a. If $p[p[x]]$ unmarked, then mark it
     b. If $p[p[x]]$ marked, cut off $p[p[x]]$, unmark, and repeat



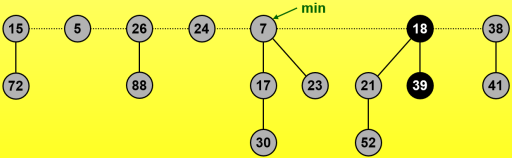
Fibonacci Heaps: Decrease Key

- **Case 2**:min-heap property violated; and parent of x is marked
  1. Decrease key of x to k
  2. Cut off link between x and its parent $p[x]$, and add x to root list
  3. Cut off link between $p[x]$ and $p[p[x]]$, add $p[x]$ to root list
     a. If $p[p[x]]$ unmarked, then mark it
     b. If $p[p[x]]$ marked, cut off $p[p[x]]$, unmark, and repeat



Fibonacci Heaps: Decrease Key

- **Case 2**: parent of x is marked
  1. Decrease key of x to k
  2. Cut off link between x and its parent $p[x]$, and add x to root list
  3. Cut off link between $p[x]$ and $p[p[x]]$, add $p[x]$ to root list
     a. If $p[p[x]]$ unmarked, then mark it
     b. If $p[p[x]]$ marked, cut off $p[p[x]]$, unmark, and repeat



Decrease 35 to 5.

Fibonacci Heaps: Decrease Key Analysis

- Actual cost: $O(c)$
  1. $O(1)$ time for decrease key
  2. $O(1)$ time for each of c cascading cuts, plus reinserting in root list
- Amortized cost: $O(1)$
  1. $t(H') = t(H) + c$
  2. $m(H') \leq m(H) - c + 2$
  3. $\Delta\Phi(H) \leq c + 2(-c + 2) = 4 - c$

Fibonacci Heaps: Delete

- Delete node x:
  1. Decrease key of x to $-\infty$
  2. Delete min element in heap
- Amortized cost: $O(D(n))$

Fibonacci Heaps: Bounding Max Degree

- **Key lemma**: In a Fibonacci heap with N nodes, the maximum degree of any node, denoted as $D(N)$, is at most $\log_\phi N$, where $\phi = \frac{(1+\sqrt{5})}{2}$.
- **Corollary**: Delete and Delete-min take $O(\log N)$ amortized time

Fibonacci Facts

- **Definition**: The Fibonacci sequence is

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k > 2 \end{cases}$$

- **Fact 1**: $F_{k+2} \geq \phi^k$

Fibonacci Facts

- **Definition**: The Fibonacci sequence is

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k > 2 \end{cases}$$

- **Fact 1**: $F_{k+2} \geq \phi^k$
  Proved by induction, and $\phi^2 = \phi + 1$.
- **Fact 2**: For $k \geq 0$, $F_{k+2} = 1 + \sum_{i=0}^{k} F_i = 2 + \sum_{i=2}^{k} F_i$

Proof of Key Lemma

- **Lemma**: Let x be a node with degree k, and let $y_1, \ldots, y_k$ denote the children of x in the order in which they were linked to x. Then:

$$degree(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i-2 & \text{if } i \geq 2 \end{cases}$$

- **Proof**:
  1. When y_i is linked to x, $y_1, \ldots, y_{i-1}$ already linked to x,
     $\Rightarrow degree(x) = i - 1$
     $\Rightarrow degree(y_i) = i - 1$ since we only link nodes of equal degree (in CONSOLIDATE)
  2. Since then, y_i has lost at most one child (or else, CASCADING-CUT will be triggered)
  3. Thus, $degree(y_i) = i - 1$ or $i - 2$

Proof of Key Lemma

- **Proof of Key Lemma:**:
  1. For any node x, we show that $size(x) \geq \phi^{degree(x)}$
     a. $size(x) = \#$ node in subtree rooted at x
     b. Taking base ϕ logs, $degree(x) \leq \log_\phi(size(x)) \leq \log_\phi N$
  2. Let s_k be min size of tree rooted at any degree k node
     a. Trivial to see that $s_0 = 1,\ s_1 = 2$
     b. s_k monotonically increases with k
  3. Let z be a degree k node and size(z)=s_k, and let $y_1, \ldots, y_k$ be children in order that they were linked to z

Proof of Key Lemma

- **Proof of Key Lemma:** :

4. Since $y_i.degree \geq i-2$ for $i \geq 2$, we have

$$\begin{aligned}
size(x) \geq s_k &\geq 2 + \sum_{i=2}^{k} s_{y_i.degree} \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} \qquad (since\ y_i.degree \geq i-2) \\
&\geq 2 + \sum_{i=2}^{k} F_i \qquad (prove\ s_k \geq F_{k+2} by induction) \\
&= F_{k+2} \geq \phi^k.
\end{aligned}$$

Data Structures for Disjoint Sets: Overview

- Some applications involve grouping n distinct elements into a collection of disjoint sets
- Two important operations are then finding which set a given element belongs to and uniting two sets
- This chapter explores methods for maintaining a data structure that supports these operations
- Application: connected components in an undirected graph, data clustering...

Disjoint-Set Operations

- Letting x denote an object, we wish to support the following operations:
  1. $\textsc{MakeSet}(x)$ creates a new set whose only member is x. We require that x not already be in some other set
  2. $\textsc{Union}(x, y)$ unites the dynamic sets that contain x and y, say S_x and S_y, into a new set that is the union of these two sets, then we remove sets S_x and S_y from S
  3. $\textsc{FindSet}(x)$ returns a pointer to the representative of the (unique) set containing x

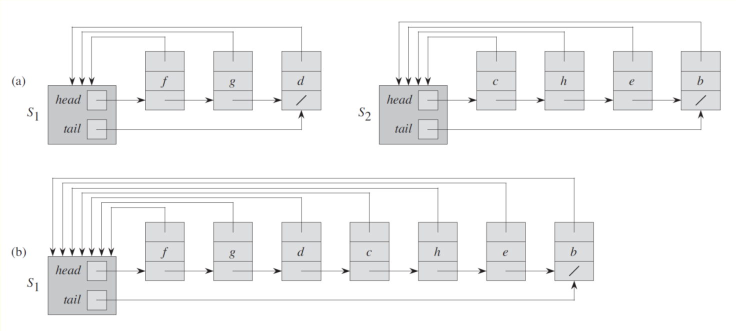
Running Time Analysis

- The running times of disjoint-set data structures shall be analyzed in terms of two parameters:
  1. n: the number of MAKESET operations
  2. m: the total number of MAKESET, UNION, and FINDSET operations
- The number of UNION operations is at most $n - 1$
- We have $m \geq n$

Linked-List Representation

- A simple way to implement a disjoint-set data structure is to represent each set by a linked list
- The first object in each linked list serves as its set's representative
- Each object in the linked list contains a set member, a pointer to the object containing the next set member, and a pointer back to the representative
- Each list maintains pointers *head*, to the representative, and *tail*, to the last object in the list

Linked-List - Example



- The result of $\text{UNION}(g, e)$, which appends the linked list containing e to the linked list containing g. The representative of the resulting set is f. The set object for es list, S_2, is destroyed

Running Time Analysis

- Both MAKESET and FINDSET only require $O(1)$ time
- **The worst case**: suppose there are objects x_1, x_2, ..., x_n, we first execute n MAKESET operations, then $n-1$ UNION operations: UNION$(x_2,\ x_1)$,...,UNION$(x_n,\ x_{n-1})$
  1. The n MAKESET operations takes $\Theta(n)$ time
  2. Because the i th UNION operation updates i objects, the total number of objects updated by all $n-1$ UNION operations is
  $$\sum_{i=1}^{n-1} i = \Theta\left(n^2\right)$$
  3. The amortized time of an operation is $\Theta(n)$

Smaller into Larger

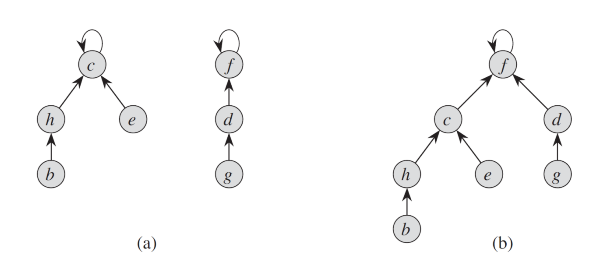
- **A weighted-union heuristic**: suppose that each list also includes the length of the list and that we always append the shorter list onto the longer, breaking ties arbitrarily
- **Theorem**: Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of m $\textsc{MakeSet}$, $\textsc{Union}$, and $\textsc{FindSet}$ operations, n of which are $\textsc{MakeSet}$ operations, takes $O(m + n \log n)$ time
- Proof?

Smaller into Larger

- **A weighted-union heuristic**: suppose that each list also includes the length of the list and that we always append the shorter list onto the longer, breaking ties arbitrarily
- **Theorem**: Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of m $\textsc{MakeSet}$, $\textsc{Union}$, and $\textsc{FindSet}$ operations, n of which are $\textsc{MakeSet}$ operations, takes $O\left(m + n\log\ n\right)$ time
- Proof?
  For any $k \leq n$, after an object x's pointer has been updated $\lceil \log k \rceil$ times, the resulting set must have at least k members. So, each element will at most be updated $\lceil \log n \rceil$ times in UNION operations.

Disjoint-Set Forests

- In a faster implementation of disjoint sets, we represent sets by rooted trees, with each node containing one member and each tree representing one set
- The straightforward algorithms that use this representation are no faster than ones that use the linked-list representation



Representing Sets as Trees

- MAKESET: create a tree with just one node
- FINDSET:follow parent pointers until we find the root of the tree. The nodes visited on this simple path toward the root constitute the find path
- UNION: cause the root of one tree to point to the root of the other

Heuristics to Improve the Running Time

- **Union by rank**: for each node, we maintain a rank, which is an upper bound on the height of the node. In union by rank, we make the root with smaller rank point to the root with larger rank during a UNION operation
- **Path compression**: we use it during FINDSET operations to make each node on the find path point directly to the root. Path compression does not change any ranks

Disjoint-Set Forests - Pseudocode I

MakeSet(x)

1: $p[x] \leftarrow x$
2: $rank[x] \leftarrow 0$

Union(x, y)

1: Link(FindSet(x), FindSet(y))

Disjoint-Set Forests - Pseudocode II

$\textsc{Link}(x, y)$

if $rank[x] > rank[y]$ **then**
    $p[y] \leftarrow x$
else
    $p[x] \leftarrow y$
    if $rank[x] = rank[y]$ **then**
        $rank[y] \leftarrow rank[y] + 1$
    end if
end if

$\textsc{FindSet}(x)$

if $x \neq p[x]$ **then**
    $p[x] \leftarrow \textsc{FindSet}(p[x])$
end if
return $p[x]$

Running Time Analysis

- **Theorem**: In general, amortized cost is $O(\alpha(n))$, where $\alpha(n)$ grows really, really, really slow
  proof: Really, really, really long
- In any conceivable application of a disjoint-set data structure, $\alpha(n) \leq 4$