CS5020: Nonlinear Optimisation: Theory and Algorithms
Coding Exercise - 1 (5 Marks)

Exercise 1. Let $x = \begin{bmatrix} x(1) \\ x(2) \end{bmatrix}$. Plot the following function

$$f(x(1), x(2)) = 0.1 \times (x(1)-1)^4 + 0.9 \times (x(2))^4 + (x - \begin{bmatrix} 2 \\ 1 \end{bmatrix})^\top \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} (x - \begin{bmatrix} 2 \\ 1 \end{bmatrix}) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}^\top x.$$

Linear Regression. Let the dataset be given by $(a_i, b_i)_{i=1}^n$, where a_i, b_i are real numbers. The aim is to fit/approximate/express variable b as a linear function of variable a, i.e., $b \approx a \cdot x(1) + x(2)$. Using the data given, this can be posed as an optimisation problem, where we can reduce the total squared error. Towards this, let us define $e_i = a_i \cdot x(1) + x(2) - b_i$. The objective function is then $f(x(1), x(2)) = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (a_i \cdot x(1) + x(2) - b_i)$, and the optimisation problem is

$$\min_{x \in \mathbb{R}^2} \quad f(x)$$

Linear Regression With Positivity Constraints. Let us impose a positivity constraint, i.e., we wish to minimise $f(x)$, however, with the extra constraint that the error $e_i \geq 0$. This is given by optimisation problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x) \\ s.t \quad & a_1 \cdot x(1) + x(2) - b_1 \geq 0 \\ & a_2 \cdot x(1) + x(2) - b_2 \geq 0 \\ & \vdots \\ & a_n \cdot x(1) + x(2) - b_n \geq 0 \end{aligned}$$

Linear Regression With Negativity Constraints. Let us impose a positivity constraint, i.e., we wish to minimise $f(x)$, however, with the extra constraint that the error $e_i \geq 0$. This is given by optimisation problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x) \\ s.t \quad & a_1 \cdot x(1) + x(2) - b_1 \leq 0 \\ & a_2 \cdot x(1) + x(2) - b_2 \leq 0 \\ & \vdots \\ & a_n \cdot x(1) + x(2) - b_n \leq 0 \end{aligned}$$

Exercise 2. Generate 5 different datasets each of $n = 10$ points. For each of these 5 different datasets do the following (using matplotlib.pyplot):

(1) Plot the function $f(x(1), x(2))$ (using matplotlib library). This should be a 3-D plot.

(2) Plot the function $f(x(1), x(2))$ and the constraints for linear regression with positivity constraints.

(3) Plot the function $f(x(1), x(2))$ and the constraints for linear regression with negativity constraints.

The constraint regions should be shown in the form of shading (use fill_between from matplotlib.pyplot).

Example Dataset. As a concrete example, you can take one of the 5 datasets to be the one in the following table.

i	a_i	b_i
1	0	2
2	0.5	1
3	1	4
4	1.5	3
5	2	6
6	2.5	5
7	3	8
8	3.5	7
9	4	10
10	4.5	9