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# Convolutional Neural Network for Fluid Dynamics

## *Thesis Project*

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# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Relevance . . . . .	2
1.2	Aim and Objectives . . . . .	2
<b>2</b>	<b>Preliminaries &amp; Related work</b>	<b>3</b>
2.1	Classic reduced order methods . . . . .	3
2.2	ML for reducing order . . . . .	3
2.2.1	Neural Network . . . . .	3
2.2.2	Convolutional Neural Network . . . . .	4
2.2.3	Autoencoder . . . . .	4
2.2.4	Physics informed Neural Networks . . . . .	5
2.3	Time evolution as time series forecasting . . . . .	7
2.3.1	Neural Network . . . . .	7
2.3.2	Recurrent Neural Network . . . . .	7
2.4	Other ML in CFD . . . . .	10
2.5	Blood Flow modelling . . . . .	10
2.5.1	Mathematical representation . . . . .	10
<b>3</b>	<b>Methodology</b>	<b>11</b>
3.1	Data Generation . . . . .	11
3.1.1	Numerical Method . . . . .	11
3.1.2	Geometries . . . . .	11
3.2	Network Architectures . . . . .	12
3.2.1	Auto-encoder . . . . .	12
3.2.2	Neural Network . . . . .	12
3.2.3	Recurrent Neural Network with GRUs . . . . .	12
3.3	Tuning, Training and Testing protocols . . . . .	12
<b>4</b>	<b>Results</b>	<b>13</b>
<b>5</b>	<b>Discussion</b>	<b>14</b>
<b>6</b>	<b>Conclusions and Recommendations</b>	<b>15</b>
	<b>Bibliography</b>	<b>16</b>

# 1 | Introduction

## 1.1 Relevance

## 1.2 Aim and Objectives

The aim of this study is to construct a single reduced order model for time-dependent incompressible flow that can account for different boundary conditions, material parameters and geometries (computational domains). As test case, blood flow during one heartbeat will be used. The objectives are high (need to specify?) accuracy, measured as difference between the predicted velocity field and simulated velocity field, and speed-up between the OpenFoam simulations and machine learning predictions.

## 2 | Preliminaries & Related work

In this chapter relevant theory and related research is discussed to set the framework for this study.

### 2.1 Classic reduced order methods

Besides simplified physics (or operational based reduction methods) most classic reduced order methods (ROMs) are projection-based:

- proper orthogonal decomposition (POD) methods
- reduced basis methods
  - Principal Component Analysis (PCA)
- balancing methods(?) [REF: <https://www.sciencedirect.com/science/article/pii/S0045794917313202>]

Con of using classic methods are they usually rely on linear basis functions constructed by Singular Value Decomposition (SVD) [REF: deep fluids] or eigenvalue decomposition (PCA). Also they depend on fixed computational domain (fixed geometry), although work has been done on moving objects in the domain [REF: <https://arxiv.org/abs/2106.02338>] and [REF: <http://grail.cs.washington.edu/projects/model-reduction/41-treuille.pdf>].

### 2.2 ML for reducing order

Instead of using classical reduced order methods, neural networks can be used to learn non-linear relations and lower dimension representations of physical simulations. Two types of network that can be used for this purpose are convolutional neural networks and autoencoders. These networks can also be physics informed or physics reinforced. All the above concepts will be discussed in this section.

#### 2.2.1 Neural Network

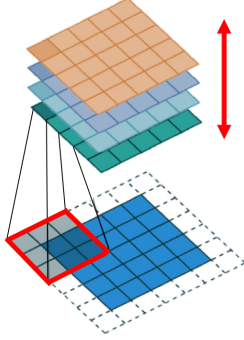
A neural network (NN) is a general machine learning method that is has many purposes, from regression and classification to embedding. A NN is built by three sets of neurons: input neurons, (multiple layers of) hidden neurons and output neurons. The inputs of a neuron are multiplied by a weight, summed and passed though an activation function to determine the output of a neuron. This activation function can be linear, but this would entail that multiple layers of a network can be collapsed back to one. Hence, to learn complex relations, non-linear activation functions are used. How the neurons are connected determines the architecture of a NN.

Choosing the architecture of a NN is a trade-off between making the NN complex enough such that the relation between input and output can be learned (underfitting) and keeping the NN simple enough to be able to generalize (overfitting). Generalizable refers to the ability of the NN to make predictions on input data that does not belong to the training data set, i.e. is not seen by the network before. There are multiple regularization techniques such as early stopping of training process, parameter norm penalties such as L1 and L2 regularization and adding drop-out probability to layers of neurons.

*For further discussion of how NN work, activation functions and regularization techniques the reader is referred to appendix ... or just not discussed in this work*

### 2.2.2 Convolutional Neural Network

A convolutional NN is characterized by convolution kernels that slide across the input features creating feature maps.



As depicted in figure 2.1 by the red square, a convolution kernel has a kernel size (in this case 3x3) on which it performs a filter operation. The kernel also has a stride, defined as the amount of spaces the kernel shifts per step. The depth of the kernel (or filter) is the amount of different filter operations it performs on the input data, and thus how many feature maps the layer creates. This is depicted in figure 2.1 by the red arrow. The different filters uncover relations in the input data. The filters are the weights of the network, and are learned during training phase. Using a CNN reduces the number of parameters to be learned in a network, as the neurons share weights, i.e. the same filter/kernel is applied to every input neuron to create one feature map.

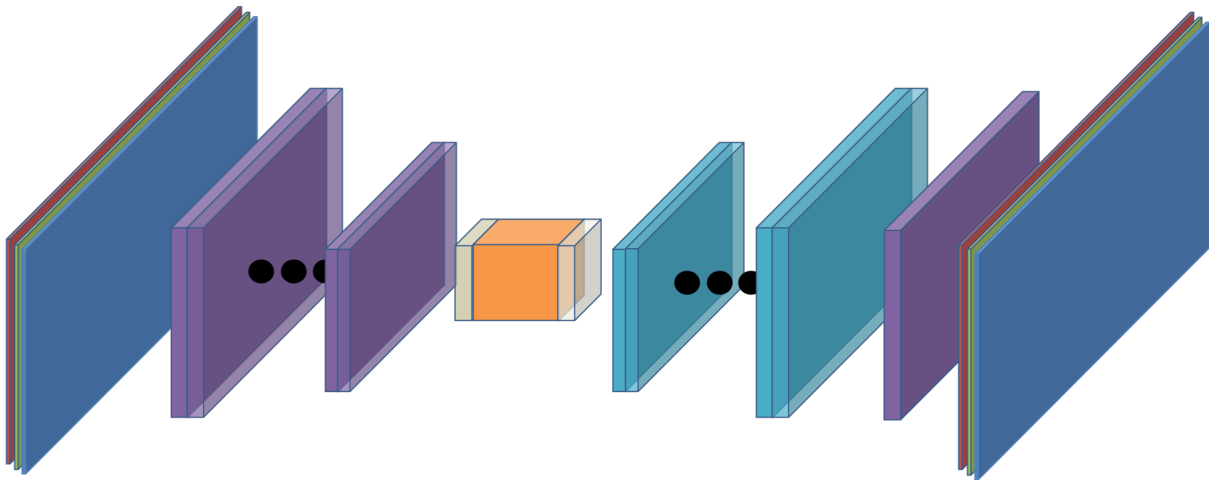
**Figure 2.1:** Convolution

A CNN can consist of multiple convolutional layers, each reducing the dimension of the input by kernel size - 1 along each axis. Further reduction of the dimension is achieved by adding Max Pooling layers, which is an operation that returns the max value within its pool (same as kernel). By reducing the dimension and thus the total number of weights, max pooling layers reduce the computational and memory usage of the CNN.

A CNN was used by Lopez-Martin et al. to find spatio-temporal relations in velocity fields generated by synthetic jet simulations [1].

### 2.2.3 Autoencoder

An autoencoder is a type over NN that consists of an encoder and decoder part. The encoder part of the network tries to find an embedding or lower dimension representation (depicted in orange in figure 2.2) of the input data. This is achieved by reducing the amount of neurons in each consecutive layer (purple part figure 2.2) until the amount of neurons is equal to the desired latent dimension size. The decoder part of the network (light blue in figure 2.2) then tries to decode the latent representation back to the full input dimensionality. This process is used in various applications, for example denoising input data.



**Figure 2.2:** Autoencoder structure for RGB image

Also for the purpose of creating reduced order models autoencoders have been used. Simpson et al used a simple autoencoder, consisting of layers of normal neuron, to find a reduced order representation of systems under forcing. It was noted that dynamics not captured in the training data were not captured by the produced ROM. Moreover, for high dimensionality input, a convolutional autoencoder was suggested to prevent training time and amount of training data issues [2].

A convolutional autoencoder (CAE) is an autoencoder network structure, using convolutional layers as explained in the sections above. Kim et al. and [3] Eichinger et al [4] used this architecture. Opposed to other papers reviewed, Eichinger et al. studied steady state fluid flow, using a binary and signed distance function representations of the input domain and retrieving the velocity field as output. They found a speedup in the order of 100 comparing their CAE with OpenFoam simulations [4].

Kim et al. used an CAE network to generate smoke and fluid simulations for graphical implementations (games, videos). Their approach led to significant speedup while maintaining accuracy for a variety of fluid behaviour. However, the ability to reconstruct physically accurate scenarios depended heavily on the how closely the scenarios matched the input training data [3].

Due to the depth of their networks, the vanishing gradient problem can occur. The large number of steps in backpropagation though the network results in the gradient becoming very small. Hence the update of the weights becomes very small. In other words, learning halts. To prevent this, both studies implemented residual connections, allowing information to pass certain layers of the network and thus increasing the gradient.

A (different) Kim et al. proposed using a shallow masked autoencoder instead of a deep convolutional autoencoder to improve efficiency. Shallow in this case means the encoder and decoder only consisted of one hidden layer. This was possible as they developed a hyper-reduction technique exploiting classic numerical methods for solving PDEs/ODEs. The hidden layer and output layer of the decoder were sparsely connected by multiplying the weight matrix of this connection by a mask matrix. The mask matrix consists of zeros and ones, and was constructed to reflect local connectivity as in the central difference scheme of the Finite Difference Method [5].

*However, using such a mask matrix makes the autoencoder problem (PDE/ODE and computational domain) specific.*

## 2.2.4 Physics informed Neural Networks

The idea of making neural networks for solving problems involving PDEs 'physics informed' was explored by Raissi et al. [6]. Information about the physics of the underlying problem can be inserted into the NN by constructing activation and loss functions specific for the underlying differential operator. All network types before mentioned thus can be physics informed, if the activation and/or loss function is tailored to the PDE.

Chen et al. created a neural network to predict the projection coefficients of the POD-Galerkin projection (PDNN), a neural network to predict the reduced order solution (PINN) and a neural network to predict the projection of high-fidelity solution on the reduced space (PRNN) by incorporating appropriate terms into the loss function [7].

To ensure conservation of mass thus non-divergence for incompressible fluid dynamics Kim et al. introduced a stream function based loss function in the decoder part of their network. In the stream loss function as described in equation 2.1,  $G(\mathbf{c})$  is the output of the network and  $\mathbf{u}_c$  is a simulation sample from the data set.

$$L_G(\mathbf{c}) = \lambda_u \|\mathbf{u}_c - \nabla \times G(\mathbf{c})\|_1 + \lambda_{\nabla \mathbf{u}} \|\nabla \mathbf{u}_c - \nabla(\nabla \times G(\mathbf{c}))\|_1 \quad (2.1)$$

Making the curl of the model output the reconstruction target, which is divergence free by construction ( $\nabla \cdot (\nabla \times G(\mathbf{c})) = 0$ ) and ensuring the derivatives also match.

Above approach of pushing a solution towards one that obeys the physical laws by modification of loss functions can be seen as soft physical constraining the network. Physical laws are not enforced at all times but are encouraged. Mohan et al. took a different approach, embedding hard physical constraints in the neural network architecture. This was done by adding non-trainable layers with physical input after a CAE structure. The decoder in this network has the vector potential as output. The non-trainable layers consists of a layer enforcing BCs with ghost cells, a layer which computes all spatial derivatives and a last layer to compute the curl on the vector potential field.

Physics can also be injected in a time-series forecasting networks. For example, by taking a reduced order representation, created either by classical methods or ML techniques, and concatenating this with computed hidden state vectors of a recurrent neural network (which will be explained in the following section) [8].

## 2.3 Time evolution as time series forecasting

Instead using Euler implicit method or other classical numerical methods for time evolution of a PDE solution, we can view this part of the problem as a time series forecasting problem. There are two machine learning techniques for time series forecasting, the previously discussed general NNs and Recurrent NNs. Recurrent NNs can be built with different unit types: standard units, long short-term memory (LSTM) units or Gated Recurrent Units (GRUs). (Also RNNs can have a specific architecture called an Encoder-Decoder network with or without attention mechanism.) All these techniques will be discussed in the following sections.

### 2.3.1 Neural Network

The most naive way of predicting future time steps with machine learning is using a standard neural network. With a NN, no assumption on the input data is made and there is no mechanism for 'memorizing' previous input data. However, this approach is still valuable to explore, as making no assumptions on the input data also means the network has all flexibility to learn undiscovered patterns and less complexity in the network is desirable as it is computationally and memory-wise more efficient.

Multiple studies were done using a NN for time evolution in combination with using ML techniques for reducing order. Kim et al. used an autoencoder to create a latent space representation of smoke and fluid simulations and thereafter a NN to find the subsequent latent representation. As input for the NN, they used the latent representation found by the autoencoder concatenated with a control vector difference between user input parameters. The output of the network is the difference between the current latent space representation and the next latent representation[3] .

Lopez-Martin et al. created one Deep NN with 3D convolutional layers to reduce dimensionality of velocity fields generated by a synthetic jet simulation. Three dimensions in this case are x,y and time. The final layers of the network were used for time evolution. This was achieved by adding a convolutional layer that controls the final depth dimension of the feature space to be the number of time steps to be predicted. This tensor is reshaped such that it can be broken into the number of time steps to be predicted vectors. These vectors are then used as input to the fully connected layers, which perform the time evolution [1].

### 2.3.2 Recurrent Neural Network

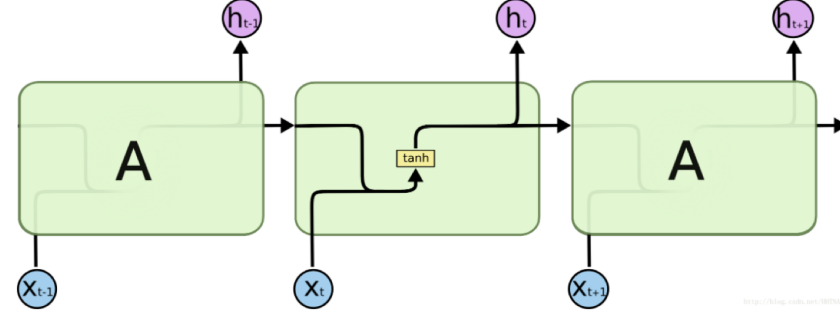
Recurrent Neural Networks have input, hidden and output neurons similar to a standard neural network. However, the neurons are divided in groups belonging to a fixed time step. The input neurons are connected to neurons in the hidden layer belonging to the same time step. Also the neurons within the hidden layer are fully connected to hidden neurons with the same assigned time step and one way connected to hidden neurons of the next time step. The output neurons are again only connected to hidden neurons belonging to the same time step.

*make image representation*



## Standard Unit

A standard unit or standard RNN cell has two inputs:  $x_t$ , the new part of a sequence and  $h_{t-1}$  the hidden state of the previous part of the sequence. These are concatenated, multiplied by the weight matrix, added with the bias vector and the results is scaled between  $[-1,1]$  by the hyperbolic tangent activation function. The outcome is a new hidden state:  $h_t = \tanh(W * x_t \hat{h}_{t-1} + b)$ , which will be used in the next time step. Note that the weight matrix is constant for each time step.



**Figure 2.3:** Standard unit RNN, figure from:

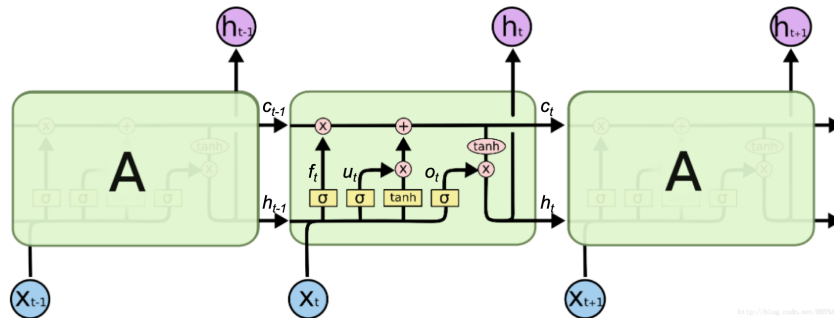
However, the vanishing gradient problem occurs when the sequence is very long. A large sequence length is similar to a deep NN, as the gradient is backpropagated many times through the network. Again, this leads to very small to no weight updates and learning halts.

## Long Short-Term Memory Unit

The LSTM unit was developed in ... to solve the vanishing gradient problem of the RNNs with standard units. The LSTM unit has three inputs: again the new part of the sequence  $x_t$  and hidden state of the previous part of the sequence  $h_{t-1}$ . In addition, there is a second state vector which is the cell state or LSTMs memory  $c_{t-1}$ . The cell state passes through the LSTM unit with less computation, making it easier to pass through unchanged. Similar to the idea of adding residual connections in deep NNs, this helps preserve the gradient. In the LSTM unit three gates are computed from the concatenation  $x = x_t \hat{h}_{t-1}$ , each having their own weight matrix:

- Forget gate  $f = \sigma(W_f x + b_f)$
- Update gate  $u = \sigma(W_u x + b_u)$
- Output gate  $o = \sigma(W_o x + b_o)$

Here,  $\sigma$  is the sigmoid function which returns values between 0 and 1, killing input close to 0 and letting input close to 1 pass nearly unchanged. The new cell state is computed by multiplying the previous cell state by what was learned to forget (the forget gate) and adding this to what was learned to be remembered (update gate times  $\tanh(W_c * x + b_c)$ ):  $c_t = f * c_{t-1} + u * \tanh(W_c * x + b_c)$ . The new hidden state is computed by the cell state scaled between  $[-1,1]$  by tanh function multiplied by what was learned to be exposed to the hidden state (output gate):  $h_t = o * \tanh(c_t)$ . This gives the ability to control what to forget from the cell state, what to store from the input in the cell state and what part of the cell state to expose to the hidden state at a given point in time.



**Figure 2.4:** LSTM unit RNN, figure from:

RNNs with LSTM units have been used in combination with classical reduced order methods for temporal evolution in simulations of physical phenomena. For example, Hu et al. [9] used POD and SVD together with an LSTM network for predicting flood induced conditions. They reported a maintained accuracy in comparison to the full model while CPU cost was reduced three orders of magnitude. Also RNNs with LSTM units have been used in combination with machine learning techniques for order reduction. Pawar et al. also merged a classical reduced order approach, POD with Galerkin projection, with LSTM network for time evolution in fluid mechanics simulations. Simpson et al. [2] trained an autoencoder for order reduction and LSTM network to predict the response of dynamical systems in the latent space based on forcing time histories.

### Gated Recurrent Unit

GRUs are an evolution of LSTM units and introduced quite recently in 2014 [REF: CHO 2014]. By dropping the output gate and only using two gates:

- Reset gate  $r = \sigma(W_r x + b_r)$
- Update gate  $z = \sigma(W_z x + b_z)$

The reset gate is similar to the forget gate in the LSTM unit and is to decide what to forget. The update gate is to decide what should be passed to the output. First a proposed new hidden state is computed by multiplying the old hidden state with the reset gate, concatenating this with the new part of the sequence and scaling between  $[-1,1]$  with the tanh function:  $\hat{h}_t = \tanh(Wx_r(r * h_{t-1}))$ . Thereafter, the actual new hidden state is computed by taking a linear sum between the previous hidden state and the proposed new hidden state, in which the reset gate determines how much of the hidden state is updated:  $h_t = (1 - z_t)h_{t-1} + z_t\hat{h}_t$ .

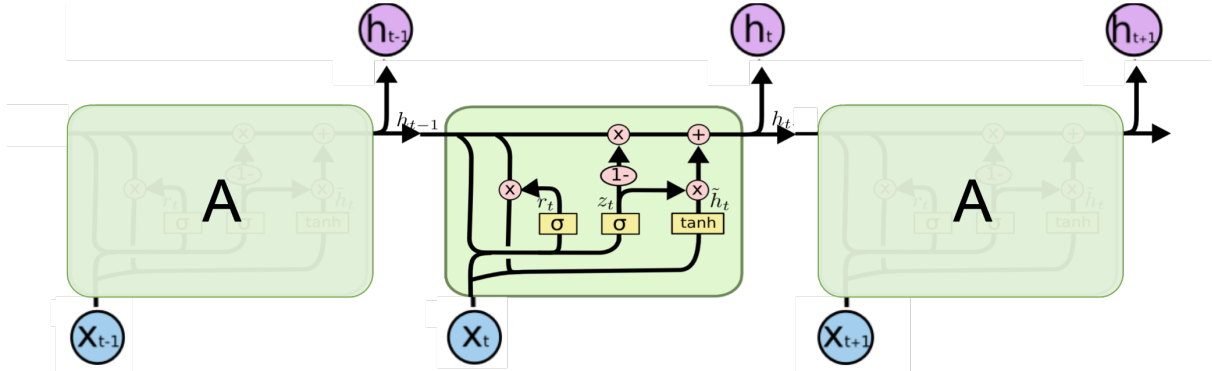


Figure 2.5: GRU RNN. figure from

With only two gates, there is no mechanism to control which parts of the state is exposed to the output, but the GRU has a less complex structure. This makes the units computationally more efficient while it was proven to have comparable performance to LSTM units [REF: Empirical Evaluation of Gated Recurrent Neural Networks on Sequence Modeling].

To the author's knowledge, an RNN with GRUs for time evolution has not been used in combination with classical or machine learning reduced order techniques. Hence, this is a novel approach which will be explored in this research. *no papers found, novel approach?*

(Encoder-decoder, attention mechanism)

## 2.4 Other ML in CFD

improve accuracy of coarse grid CFD simulation [REF: <https://www.pnas.org/content/pnas/118/21/e2101784118.full.pdf>]

## 2.5 Blood Flow modelling

### 2.5.1 Mathematical representation

To create a realistic synthetic data set, the flow of blood was modeled by Navier-Stokes equations as proposed in [10], adapted from 3D to 2D.

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial}{\partial t}(\mathbf{u}) + (\nabla \mathbf{u})\mathbf{u} + \nabla \frac{p}{\rho} - \nabla \cdot (2\nu D(\mathbf{u})) &= 0\end{aligned}\tag{2.2}$$

In which  $\mathbf{u}$  is the velocity,  $p$  is the mean normal stress ( $p = -\frac{1}{2}(tr\mathbf{T})$  with  $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$  the fluid stress),  $\nu$  is the kinematic viscosity ( $\nu = \frac{\mu}{\rho}$ , with  $\mu$  the dynamic viscosity) and  $\mathbf{D}$  the strain-rate tensor ( $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ )).

# 3 | Methodology

## 3.1 Data Generation

### 3.1.1 Numerical Method

To solve the mathematical problem stated in the previous section. OpenFoam software was used. In particular the IcoFoam solver, for incompressible, Newtonian fluids was implemented. The IcoFoam solver solves the following equation:

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial}{\partial t}(\mathbf{u}) + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\nu \nabla \mathbf{u}) = -\nabla p \quad (3.1)$$

In which  $\mathbf{u}$  is the velocity in  $\frac{m}{s}$  and  $p$  is the kinematic pressure in  $\frac{m^2}{s^2}$ . Taking  $\nu = \dots$  and  $p = \dots$  makes equations 2.2 and 3.1 NOT!! equivalent as shown below.

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \nabla \cdot \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} u_x & u_y \end{bmatrix} = \nabla \cdot \begin{bmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(u_x^2) + \frac{\partial}{\partial y}(u_x u_y) \\ \frac{\partial}{\partial y}(u_y^2) + \frac{\partial}{\partial x}(u_x u_y) \end{bmatrix} = \begin{bmatrix} 2u_x \frac{\partial u_x}{\partial x} + u_x \frac{\partial u_y}{\partial y} + u_y \frac{\partial u_x}{\partial y} \\ 2u_y \frac{\partial u_y}{\partial y} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial x} \end{bmatrix}$$

$$\nabla \cdot (\nu \nabla \mathbf{u}) = \nabla \cdot \left( \nu \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \right) = \nu \begin{bmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} \\ \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_x}{\partial x \partial y} \end{bmatrix}$$

$$(\nabla \mathbf{u}) \mathbf{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \end{bmatrix}$$

$$\nabla \cdot (2\nu \mathbf{D}(\mathbf{u})) = \nabla \cdot (2\nu * \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)) = \nabla \cdot \left( \nu \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} \end{bmatrix} \right) = \nabla \cdot \left( \nu \begin{bmatrix} 2\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & 2\frac{\partial u_y}{\partial y} \end{bmatrix} \right)$$

$$= \nu \begin{bmatrix} 2\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial y^2} \\ 2\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial x^2} \end{bmatrix}$$

### 3.1.2 Geometries

Four different geometries were created with the GMSH python library by the following steps:

1. define 2D mesh by creating points, lines and a surface in the xy-plane
2. extrude the surface in the z-direction by one cell (this is necessary as OpenFoam only takes 3D meshes)
3. add all surfaces to appropriate Physical Groups and name these groups
4. save the mesh as .msh2 file (this is compatible with gmshToFoam function)
5. in openfoam, run gmshToFoam on mesh file
6. set front and back patches to empty type in constant/boundary file, set inflow and outflow boundaries to patch type and set other boundaries to wall type
7. define all input parameters in the 0 folder
8. run icoFoam as solver
9. convert output to VTK files for visualization with Paraview by the "foamToVTK -allPatches" command

All code to create the meshes can be found in [REF APPENDIX MESH CODE].

## 3.2 Network Architectures

Taking into account the possibilities discussed in chapter 2, first of all a convolutional autoencoder was chosen for order reduction, given the high dimensionality of the input data and desired flexibility in the computational domain. Two network architectures are proposed: an autoencoder with a NN for time evolution and an autoencoder with a RNN using GRUs for time evolution.

*Secondly, a 3D convolutional network with dimensions  $(x,y,t)$  was explored.*

### 3.2.1 Auto-encoder

### 3.2.2 Neural Network

### 3.2.3 Recurrent Neural Network with GRUs

## 3.3 Tuning, Training and Testing protocols

## 4 | Results

## 5 | Discussion

## 6 | Conclusions and Recommendations



# Bibliography

- [1] Manuel Lopez-Martin, Soledad Le Clainche, and Belen Carro. “Model-free short-term fluid dynamics estimator with a deep 3D-convolutional neural network”. In: *Expert Systems with Applications* 177 (2021). ISSN: 09574174. DOI: 10.1016/j.eswa.2021.114924.
- [2] Thomas Simpson, Nikolaos Dervilis, and Eleni Chatzi. “Machine Learning Approach to Model Order Reduction of Nonlinear Systems via Autoencoder and LSTM Networks”. In: *Journal of Engineering Mechanics* 147 (10 2021). ISSN: 0733-9399. DOI: 10.1061/(asce)em.1943-7889.0001971.
- [3] Byungsoo Kim et al. “Deep Fluids: A Generative Network for Parameterized Fluid Simulations”. In: *Computer Graphics Forum* 38 (2 May 2019), pp. 59–70. ISSN: 14678659. DOI: 10.1111/cgf.13619.
- [4] Matthias Eichinger et al. *Technical Report Series Center for Data and Simulation Science Surrogate Convolutional Neural Network Models for Steady Computational Fluid Dynamics Simulations SURROGATE CONVOLUTIONAL NEURAL NETWORK MODELS 1 FOR STEADY COMPUTATIONAL FLUID DYNAMICS 2 SIMULATIONS*. 2016.
- [5] Youngkyu Kim et al. “A fast and accurate physics-informed neural network reduced order model with shallow masked autoencoder”. In: *Journal of Computational Physics* 451 (2022). ISSN: 10902716. DOI: 10.1016/j.jcp.2021.110841.
- [6] M. Raissi, P. Perdikaris, and G. E. Karniadakis. “Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations”. In: *Journal of Computational Physics* 378 (2019). ISSN: 10902716. DOI: 10.1016/j.jcp.2018.10.045.
- [7] Wenqian Chen et al. “Physics-informed machine learning for reduced-order modeling of nonlinear problems”. In: *Journal of Computational Physics* 446 (2021). ISSN: 10902716. DOI: 10.1016/j.jcp.2021.110666.
- [8] Suraj Pawar et al. “Model fusion with physics-guided machine learning: Projection-based reduced-order modeling”. In: *Physics of Fluids* 33 (6 2021). ISSN: 10897666. DOI: 10.1063/5.0053349.
- [9] R. Hu et al. “Rapid spatio-temporal flood prediction and uncertainty quantification using a deep learning method”. In: *Journal of Hydrology* 575 (Aug. 2019), pp. 911–920. ISSN: 00221694. DOI: 10.1016/j.jhydro.2019.05.087.
- [10] H. Švihlová et al. “Determination of pressure data from velocity data with a view toward its application in cardiovascular mechanics. Part 1. Theoretical considerations”. In: *International Journal of Engineering Science* 105 (2016). ISSN: 00207225. DOI: 10.1016/j.ijengsci.2015.11.002.
- [11] My Ha Dao. *Projection-Based Reduced Order Model for Simulations of Nonlinear Flows with Multiple Moving Objects*.
- [12] Amirhossein Arzani and Scott T.M. Dawson. *Data-driven cardiovascular flow modelling: Examples and opportunities*. 2021. DOI: 10.1098/rsif.2020.0802.
- [13] H. Švihlová et al. “Determination of pressure data from velocity data with a view towards its application in cardiovascular mechanics. Part 2. A study of aortic valve stenosis”. In: *International Journal of Engineering Science* 114 (2017). ISSN: 00207225. DOI: 10.1016/j.ijengsci.2017.01.002.
- [14] Amir Barati Farimani, Joseph Gomes, and Vijay S Pande. *Deep Learning the Physics of Transport Phenomena*. 2017.
- [15] M. Cheng et al. “An advanced hybrid deep adversarial autoencoder for parameterized nonlinear fluid flow modelling”. In: *Computer Methods in Applied Mechanics and Engineering* 372 (Dec. 2020). ISSN: 00457825. DOI: 10.1016/j.cma.2020.113375.